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Abstract

In this paper we consider holomorphic functions on the m-dimensional Lie ball $LB(0,1)$ which admit a square integrable extension on the Lie sphere. We then define orthogonal projections of this set onto suitable subsets of functions defined in lower dimensional spaces to obtain several Radon-type transforms. For all these transforms we provide the kernel and an integral representation, besides other properties. In particular, we introduce and study a generalization to the case of the Lie ball of the Szegő-Radon transform introduced in [2], and various types of Hua-Radon transforms.

Key words: Holomorphic functions, Monogenic functions, Lie ball, Lie sphere, Radon-type transforms.
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1 Introduction

In this paper we consider some transforms defined as orthogonal projections from modules of holomorphic functions in several variables onto suitable modules defined in lower dimensional spaces. For this reason we refer to these transforms as Radon-type transforms. We addressed this type of problem in our papers [2, 3]. In [2] we abstractly defined the Szegő-Radon transform as the orthogonal projection of a Hilbert module of left monogenic functions (i.e. nullsolutions of the Dirac operator) in the unit ball onto a suitable closed submodule of monogenic functions depending only on two variables. In [3] we define the Bargmann-Radon transform as the projection of a real Bargmann module (see [7]) on the closed submodule of monogenic functions spanned by the monogenic plane waves. In both cases, we prove that these projections can be written in integral form in terms of suitable kernels. Moreover, we have a characterization formula which gives the transform of a function in terms of its complex extension followed by its restriction to the nullcone in $\mathbb{C}^m$.

Monogenic functions in the unit ball are harmonic thus they admit holomorphic extension to the Lie ball $LB(0,1)$. Thus it is a natural question to ask if it possible to define Radon-type transforms in the Lie ball setting.

In this work, we consider the module $\mathcal{OL}^2(LB(0,1))$ of holomorphic functions on the unit Lie ball $LB(0,1)$ which admit a square integrable extension on the Lie sphere. Functions in $\mathcal{OL}^2(LB(0,1))$ possess useful properties, like the fact that they admit a Fischer decomposition.

The reproducing kernel for the module $\mathcal{OL}^2(LB(0,1))$ is the Cauchy-Hua kernel which is also considered in [5, 6, 10]. These two papers contain other useful results among which the study of spherical monogenics on the unit Lie sphere, see [10], that is a refinement of some results in [6] on spherical harmonics on the unit Lie sphere.
The Radon-type transforms considered in this paper are projections of $\mathcal{OL}^2(LB(0,1))$ onto suitable submodules of holomorphic functions defined on lower dimensional sets. Specifically, in Section 2 and 3 we provide some basic material on monogenic functions in general and on the Fischer decomposition of holomorphic functions on the Lie ball. In particular, we introduce the Cauchy-Hua kernel and study the link with the Szegő kernel which is the reproducing kernel for submodules of monogenic functions. The core of the paper is contained in the section 4 to 7 where we introduce the Radon-type transforms via suitable projections. For all these transforms we provide the kernel and an integral representation, besides other properties. In Section 4, we study generalizations to the Lie ball of the Szegő-Radon transform that we introduced in [2]. A transform that can be defined for a subclass of functions of several complex variables is the Hua-Radon transform introduced in Section 5. In this section we in fact consider orthogonal projection from a module of holomorphic functions on the m-dimensional Lie ball to holomorphic functions in the two-dimensional Lie ball. This transformation is completely geometrical and does not use any theory of monogenic functions. In Section 6 we consider the so-called polarized Hua-Radon transform whose definition is based on the decomposition of the Hua-Radon transform into two complementary and orthogonal pieces defined using monogenic functions. In this section, the techniques of hypercomplex analysis are crucial to obtain the kernel. Finally, in Section 7 we introduce the monogenic Hua-Radon transform which may be seen as an extension of both the extended Szegő-Radon and the polarized Hua-Radon transforms. Some extra work is needed to compute an orthogonal system of plane waves which includes both the orthogonal bases used to construct the modules associated with the extended Szegő-Radon and the polarized Hua-Radon transforms. Using this orthogonal system one can provide an expression for the reproducing kernel as a series. A closed form for this kernel seems hard to obtain and may be the object of future research.

2 Preliminary results

In this section we introduce the necessary notation and the preliminary results that will useful in the sequel. Some classical sources to have more information are the books [1], [4].
Let us consider m imaginary units $\underline{e}_1, \dots, \underline{e}_m$which satisfy the relations $\underline{e}_i\underline{e}_j + \underline{e}_j\underline{e}_i = -2\delta_{ij}$. We denote by $\mathbb{R}_m$ and by $\mathbb{C}_m$ the real and complex Clifford algebra, respectively, generated by $\underline{e}_1, \dots, \underline{e}_m$. An element x in the Clifford algebra $\mathbb{R}_m$ (or $\mathbb{C}_m$) is of the form $x = \sum_A \underline{e}_A x_A$ where $x_A \in \mathbb{R}$ (or $x_A \in \mathbb{C}$) $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi–index, $\underline{e}_A = \underline{e}_{i_1}\underline{e}_{i_2} \cdots \underline{e}_{i_r}$ and $\underline{e}_\emptyset = 1$.

The complex Clifford algebra $\mathbb{C}_m$ can be seen as the complexification of the real Clifford algebra $\mathbb{R}_m$, i.e. $\mathbb{C}_m = \mathbb{R}_m \oplus i\,\mathbb{R}_m$. Any complex Clifford number $c \in \mathbb{C}_m$ may be written as $c = a + ib$, where $a, b \in \mathbb{R}_m$

The so called 1-vectors are elements in $\mathbb{R}_m$ which are linear combinations with real coefficients of the elements $\underline{e}_i$, $i = 1, \dots, m$. The map given by $(x_1, x_2, \dots, x_m) \mapsto \underline{x} = x_1\underline{e}_1 + \dots + x_m\underline{e}_m$ allows to identify a 1-vector with an element in the Euclidean space. The norm of a 1-vector is defined as $|\underline{x}| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and the scalar product of $\underline{x}$ and $\underline{y} = y_1\underline{e}_1 + \cdots + y_m\underline{e}_m$ is

$$\langle \underline{x}, \underline{y} \rangle = x_1 y_1 + \cdots + x_m y_m.$$

In the sequel, we will denote by $B(0,1)$ the unit ball with center at the origin in $\mathbb{R}^m$ while the symbol $\mathbb{S}^{m-1}$ will denote its boundary, that is the sphere of unit 1-vectors in $\mathbb{R}^m$:

$$\mathbb{S}^{m-1} = \{\underline{x} = \underline{e}_1 x_1 + \ldots + \underline{e}_m x_m \ : \ x_1^2 + \ldots + x_m^2 = 1\},$$

whose area, denoted by A_m is given by

$$A_m = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})}.$$

Similarly to what has been done in the real setting, we can identify an element in $\mathbb{C}^m$ with a 1-vector with complex coefficients. The scalar product of two complex vectors $\underline{z} = z_1\underline{e}_1 + \cdots + z_m\underline{e}_m$, $z_\ell \in \mathbb{C}$ and $\underline{w} = w_1\underline{e}_1 + \cdots + w_m\underline{e}_m$, $w_\ell \in \mathbb{C}$ is

$$\langle \underline{z}, \underline{w} \rangle = \bar{z}_1 w_1 + \cdots + \bar{z}_m w_m.$$

Definition 2.1. *The Lie ball can be defined as*

$$LB(0,1) = \{\underline{z} = \underline{x} + i\underline{y} \in \mathbb{C}^m \mid S_{\underline{x},\underline{y}} \subset B(0,1)\}$$

where $S_{\underline{x},\underline{y}}$ is the codimension 2 sphere

$$S_{\underline{x},\underline{y}} = \{\underline{u} \in \mathbb{R}^m \mid |\underline{u} - \underline{x}| = |\underline{y}|, \quad \langle \underline{u} - \underline{x}, \underline{y} \rangle = 0\}.$$

Remark 2.2. Another way to introduce the Lie ball is to consider the Lie norm

$$L(\underline{z})^2 = L(\underline{x} + i\underline{y})^2 = \sup_{\underline{u} \in S_{\underline{x},\underline{y}}} |\underline{u}|^2 = |\underline{x}|^2 + |\underline{y}|^2 + 2|\underline{x} \wedge \underline{y}|$$

where $\underline{z} = \underline{x} + i\underline{y} \in \mathbb{C}^m$ so that

$$LB(0,1) = \{\underline{z} \in \mathbb{C}^m \mid L(\underline{z}) < 1\}.$$

The boundary $\partial LB(0,1)$ corresponds to the set of spheres $S_{\underline{x},\underline{y}} \subset \overline{B(0,1)}$ for which $S_{\underline{x},\underline{y}} \cap \mathbb{S}^{m-1} \neq \emptyset$, see e.g. [10]. The Lie sphere LS^{m-1} is the set of points $\underline{z} \in \mathbb{C}^m$ for which $S_{\underline{x},\underline{y}} \subset \mathbb{S}^{m-1}$, which allows to write it as

$$LS^{m-1} = \{\underline{\omega} e^{i\theta} \mid \underline{\omega} \in \mathbb{S}^{m-1}, \, \theta \in [0,\pi)\} = (\mathbb{S}^{m-1} \times \mathbb{S}^1)/\mathbb{Z}_2,$$

where the equivalence relation is given by $(\underline{\omega}, e^{i\theta}) \sim (-\underline{\omega}, -e^{i\theta})$. The Lie sphere LS^{m-1} is the Shilov boundary of the Lie ball, moreover, it is a minimal set for which

$$|f(\underline{z})| \leq \sup_{e^{i\theta}\underline{\omega} \in LS^{m-1}} |f(e^{i\theta}\underline{\omega})|, \ f \text{ holomorphic in } \overline{LB(0,1)}.$$

We now introduce an automorphism in $\mathbb{C}_m$ called Hermitian conjugation and defined for $\lambda, \mu \in \mathbb{C}_m$ by

$$(\lambda\mu)^\dagger = \mu^\dagger \lambda^\dagger, \quad (\mu_A \underline{e}_A)^\dagger = \mu_A^c \underline{e}_A^\dagger, \quad \underline{e}_j^\dagger = -\underline{e}_j, \quad j = 1, \ldots, m,$$

where μ_A^c stands for the complex conjugate of the complex number μ_A.
The following result is proved in [2]:

Proposition 2.3. *Let $\underline{t}, \underline{s} \in \mathbb{R}^n$ be such that $|\underline{t}| = |\underline{s}| = 1$ and $\langle \underline{t}, \underline{s} \rangle = 0$ and let $\underline{\tau} = \underline{t} + i\underline{s} \in \mathbb{C}^m$. Then $\underline{\tau}^\dagger = -\underline{t} + i\underline{s}$ and*

1. $\underline{\tau}\,\underline{\tau}^\dagger\underline{\tau} = 4\underline{\tau}$,

2. $\underline{\tau}^2 = 0$,

3. $\underline{\tau}^\dagger\,\underline{\tau} + \underline{\tau}\,\underline{\tau}^\dagger = 4$.

Remark 2.4. Let $\underline{z} \in \mathbb{C}^m$ and consider $\underline{\tau} = \underline{t} + i\underline{s} = \sum_{\ell=1}^m (t_\ell + is_\ell)\underline{e}_i$ as above. Then

$$\langle \underline{z}, \underline{\tau}\rangle^\dagger = (\sum_{\ell=1}^m \overline{z}_\ell(t_\ell + is_\ell))^\dagger = \sum_{\ell=1}^m z_\ell(t_\ell - is_\ell).$$

In particular, when $\underline{x} \in \mathbb{R}^m$ we have

$$\langle \underline{x}, \underline{\tau}\rangle^\dagger = \sum_{\ell=1}^m x_\ell(t_\ell - is_\ell)$$
$$\langle \underline{x}, \underline{\tau}^\dagger\rangle = \sum_{\ell=1}^m -x_\ell(t_\ell - is_\ell)$$

so that

$$\langle \underline{x}, \underline{\tau}\rangle^\dagger = -\langle \underline{x}, \underline{\tau}^\dagger\rangle. \tag{1}$$

To conclude the part on the preliminaries we recall the definition of monogenic functions.

Definition 2.5. *A function $f : \Omega \subseteq \mathbb{R}^m \to \mathbb{C}_m$ defined and continuously differentiable in the open set Ω is said to be (left) monogenic if it satisfies*

$$\partial_{\underline{x}} f(\underline{x}) = \sum_{j=1}^m \underline{e}_i \partial_{x_j} f(\underline{x}) = 0.$$

If $f : \Omega \subseteq \mathbb{C}^m \to \mathbb{C}_m$ is as above, we say that f is (left) monogenic in Ω if it is holomorphic and in the kernel of the complexified Dirac operator $\sum_{j=1}^m \underline{e}_i \partial_{z_j}$. We denote by $\mathcal{M}(\Omega)$ the right $\mathbb{C}_m$-module of (left) monogenic functions in Ω.

3 Spherical monogenics on the Lie sphere

In this section we recall some results on the module $\mathcal{ML}^2(B(0,1))$ from [2]. We then generalize this module of monogenic functions to the case of the Lie ball and we study the Fischer decomposition of its elements as well as a reproducing kernel, namely the Cauchy-Hua kernel.

Definition 3.1. *We will denote by $\mathcal{ML}^2(B(0,1))$ the right $\mathbb{C}_m$-module of monogenic functions $f : B(0,1) \to \mathbb{C}_m$ which extend to $\mathcal{L}^2(\mathbb{S}^{m-1})$ and such that*

$$\left[\int_{\mathbb{S}^{m-1}} f^\dagger(\underline{\omega}) f(\underline{\omega}) dS(\underline{\omega})\right]_0 < \infty.$$

We note that $\mathcal{ML}^2(B(0,1))$ is a closed submodule of $\mathcal{L}^2(B(0,1))$. It is also clear that $\mathcal{ML}^2(B(0,1))$ is a right submodule of the right $\mathbb{C}_m$-module of monogenic functions. It can be equipped with the inner product

$$\langle f, g\rangle_{\mathcal{ML}^2(B(0,1))} := \int_{\mathbb{S}^{m-1}} f^\dagger(\underline{\omega}) g(\underline{\omega}) dS(\underline{\omega}).$$

For the sake of simplicity, in the sequel we will denote this inner product by $\langle \cdot, \cdot \rangle_{\mathcal{ML}^2}$. Note that the norm of f is given by

$$\|f\|^2_{\mathcal{L}^2(\mathbb{S}^{m-1})} := \left[\int_{\mathbb{S}^{m-1}} f^\dagger(\underline{\omega}) f(\underline{\omega}) dS(\underline{\omega}) \right]_0 < \infty.$$

The following result is in [3], Section 4:

Proposition 3.2. *The reproducing kernel for $\mathcal{ML}^2(B(0,1))$ is the Szegő kernel*

$$S(\underline{x}, \underline{u}) = \frac{1 + \underline{x}\,\underline{u}}{(1 + |\underline{x}|^2\, |\underline{u}|^2 - 2\langle \underline{x}, \underline{u}\rangle)^{m/2}}$$

so that

$$f(\underline{x}) = \frac{1}{A_m} \int_{S^{m-1}} \frac{1 + \underline{x}\,\underline{u}}{(1 + |\underline{x}|^2\, |\underline{u}|^2 - 2\langle \underline{x}, \underline{u}\rangle)^{m/2}}\; f(\underline{u})\, d\underline{u}.$$

A monogenic function $f(\underline{x})$ admits a holomorphic extension $f(\underline{z})$ to the Lie ball $LB(0,1)$.

Definition 3.3. *By $\mathcal{OL}^2(LB(0,1))$ we denote the right $\mathbb{C}_m$-module of holomorphic functions $f : LB(0,1) \to \mathbb{C}_m$ whose boundary value $f(e^{i\theta}\underline{\omega})$ belongs to $\mathcal{L}^2(LS^{m-1})$ and such that*

$$\left[\int_{\mathbb{S}^{m-1}} \int_0^\pi f^\dagger(e^{i\theta}\underline{\omega}) f(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta \right]_0 < \infty.$$

The $\mathbb{C}_m$-module $\mathcal{OL}^2(LB(0,1))$ can be equipped with the inner product

$$\langle f, g\rangle_{\mathcal{OL}^2(LB(0,1))} = \int_{\mathbb{S}^{m-1}} \int_0^\pi f^\dagger(e^{i\theta}\underline{\omega}) g(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta.$$

For the sake of simplicity, this inner product will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{OL}^2}$.

Remark 3.4. We recall that a monogenic function f (over $\mathbb{R}^m$) admits an expansion of the form

$$f(\underline{x}) = \sum_{k=0}^{\infty} P_k[f](\underline{x}),$$

where $P_k[f]$ are spherical monogenics of degree k, i.e.

$$P_k[f](\lambda \underline{x}) = \lambda^k P_k(\underline{x}), \qquad \partial_{\underline{x}} P_k[f](\underline{x}) = 0.$$

We have that

$$P_k[f](\underline{x}) = \frac{1}{A_m} \int_{\mathbb{S}^{m-1}} C_k(\underline{x}, \underline{u})\, f(\underline{u})\, d\underline{u}$$

where $A_m = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})}$ and $C_k(\underline{x}, \underline{u})$ are zonal spherical monogenics (see [**?**]) of the form

$$\begin{aligned} C_k(\underline{x}, \underline{u}) : &= \frac{(|\underline{x}||\underline{y}|)^k}{(m-2)} \Big((k+m-2) C_k^{m/2-1}(t) + (m-2) \frac{\underline{x} \wedge \underline{u}}{|\underline{x}|\, |\underline{u}|} C_{k-1}^{m/2}(t) \Big) \\ &= (|\underline{x}||\underline{y}|)^k \Big(C_k^{m/2}(t) + \frac{\underline{x}\,\underline{u}}{|\underline{x}|\, |\underline{u}|} C_{k-1}^{m/2}(t) \Big), \end{aligned} \tag{2}$$

$C_k^{m/2}(t)$ are the Gegenbauer polynomials and $t = \langle \underline{x}, \underline{u}\rangle / |\underline{x}||\underline{u}|$.
Note that the Szegő kernel can be written as $S(\underline{x}, \underline{u}) = \sum_{k=0}^{\infty} C_k(\underline{x}, \underline{u})$.

Remark 3.5. Every function $f \in \mathcal{OL}^2(LB(0,1))$ can be decomposed into homogeneous polynomials

$$f(\underline{z}) = \sum_{k=0}^{\infty} R_k(\underline{z}), \qquad R_k \in \mathcal{P}_k, \tag{3}$$

where $\mathcal{P}_k$ denotes the set of polynomials in the variable $\underline{z}$ homogeneous of degree k, see [9] and [10]. The decomposition (3) converges in the set of holomorphic functions $\mathcal{O}(LB(0,1))$ moreover it is orthogonal, i.e., $\mathcal{P}_k \perp \mathcal{P}_\ell$ for $k \neq \ell$.
Conversely, given a series $\sum_{k=0}^{\infty} R_k(\underline{x})$ of homogeneous polynomials in $\underline{x}$ which converges in $B(0,1)$, its complex extension converges in $LB(0,1)$. This is due to the estimate

$$|R_k(\underline{z})| \leq c_k L(\underline{z})^k \sup_{|\underline{x}|=1} |R_k(\underline{x})|,$$

where c_k is a slowly increasing constant, see [10, Section 2].
This fact also implies that the complex extension $f(\underline{z})$ of the function $f(\underline{x})$ solution in $B(0,1)$ of $\partial_{\underline{x}}^s f(\underline{x}) = 0$ is generally holomorphic in the Lie ball and not beyond.

Remark 3.6. Every $R_z \in \mathcal{P}_k$ admits a monogenic decomposition, the so-called Fischer decomposition:

$$R_k(\underline{z}) = \sum_{\ell=0}^{k} \underline{z}^\ell P_{k-\ell}(\underline{z}), \tag{4}$$

where $\partial_{\underline{z}} P_{k-\ell}(\underline{z}) = 0$ and $P_{k-\ell} \in \mathcal{P}_{k-\ell}$ and, in general, for any $(\ell, k) \in \mathbb{N}^2$, $(\ell', k') \in \mathbb{N}^2$

$$\begin{aligned}\langle \underline{z}^{\ell'} P_{k'}(\underline{z}), \underline{z}^\ell P_k(\underline{z}) \rangle &= \int_{\mathbb{S}^{m-1}} \int_0^\pi P_{k'}^\dagger(\underline{\omega}) e^{-i(k'+\ell')\theta} (-\omega)^{\ell'} \underline{\omega}^\ell e^{i(k+\ell)\theta} P_k(\underline{\omega})\, dS(\underline{\omega})\, d\theta \\ &= \delta_{k,k'} \delta_{\ell,\ell'} \pi \int_{\mathbb{S}^{m-1}} P_{k'}^\dagger(\underline{\omega}) P_k(\underline{\omega})\, dS(\underline{\omega}),\end{aligned}$$

so the monogenic decomposition is orthogonal.

Taking into account these remarks we can prove the following:

Theorem 3.7 (Monogenic Fischer decomposition)**.** *Every function $f \in \mathcal{OL}^2(LB(0,1))$ admits an orthogonal Fischer decomposition of the form*

$$f(\underline{z}) = \sum_{k,\ell=0}^{\infty} \underline{z}^\ell P_{k,\ell}[f](\underline{z})$$

where

$$P_{k,\ell}[f](\underline{z}) = \frac{1}{\pi A_m} \int_{\mathbb{S}^{m-1}} \int_0^\pi C_k(\underline{z}, e^{-i\theta}\underline{\omega})(e^{i\theta}\underline{\omega})^{-\ell} f(e^{i\theta}\underline{\omega})\, d\theta\, dS(\underline{\omega}) \tag{5}$$

is an inner spherical monogenic of degree k and C_k is as in (2)*.*

Proof. The decomposition (4) is orthogonal and leads to the identity in $\mathcal{OL}^2(LB(0,1))$, see Remark 3.5. To show the assertion, we rewrite the integral on the right hand side of (5) using the orthogonality:

$$\begin{aligned}&\frac{1}{\pi A_m} \int_{\mathbb{S}^{m-1}} \int_0^\pi C_k(\underline{z}, e^{-i\theta}\underline{\omega})(e^{i\theta}\underline{\omega})^{-\ell} P_{k,\ell}[f](e^{i\theta}\underline{\omega})\, d\theta\, dS(\underline{\omega}) \\ &= \frac{1}{\pi A_m} \int_{\mathbb{S}^{m-1}} \int_0^\pi C_k(\underline{z}, \underline{\omega}) e^{-ik\theta} e^{ik\theta} P_{k,\ell}[f](\underline{\omega})\, d\theta\, dS(\underline{\omega}) \\ &= \frac{1}{A_m} \int_{\mathbb{S}^{m-1}} C_k(\underline{z}, \underline{\omega}) P_{k,\ell}[f](\underline{\omega})\, dS(\underline{\omega}) \\ &= P_{k,\ell}[f](\underline{z}),\end{aligned}$$

as stated. ☐

Proposition 3.8. *The reproducing kernel for $\mathcal{OL}^2(LB(0,1))$ is the Cauchy-Hua kernel*

$$\frac{1}{\pi A_m} H(\underline{z}, e^{-i\theta}\underline{\omega}) = \frac{1}{\pi A_m}(-(\underline{\omega} - e^{-i\theta}\underline{z})^2)^{-m/2}.$$

The Cauchy-Hua kernel can also be written in the following three forms:

$$\begin{aligned} \frac{1}{\pi A_m} H(\underline{z}, e^{-i\theta}\underline{\omega}) &= \sum_{k,\ell=0}^{\infty} \underline{z}^{\ell} C_k(\underline{z}, e^{-i\theta}\underline{\omega})(e^{-i\theta}\underline{\omega})^{-\ell} \\ &= \frac{1}{\pi A_m}(1 - e^{-2i\theta}\underline{z}^2 + 2e^{-i\theta}\langle \underline{\omega}, \underline{z}\rangle)^{-m/2} \\ &= \frac{1}{\pi A_m} \sum_{\ell=0}^{\infty} \underline{z}^{\ell} S(\underline{z}, e^{-i\theta}\underline{\omega})(e^{i\theta}\underline{\omega})^{-\ell} \end{aligned}$$

where $S(\underline{z}, e^{-i\theta}\underline{\omega})$ is the Szegő kernel.

Proof. The Cauchy-Hua kernel is the reproducing kernel for $\mathcal{OL}^2(LB(0,1))$, see [5], [6]. From Theorem 3.7 we deduce the monogenic decomposition of the kernel. This in turn leads to the expression in terms of the Szegő kernel. One may also proceed by direct computations from the equality

$$\sum_{\ell=0}^{\infty} \underline{z}^{\ell}(1 + \underline{z}e^{-i\theta}\underline{\omega})e^{-i\ell\theta}(-\underline{\omega})^{\ell} = 1$$

which can be easily verified. ☐

Remark 3.9. (1) The Cauchy-Hua kernel has an anti-holomorphic extension obtained by extending $-e^{-i\theta}\underline{\omega}$ to $\underline{u}^{\dagger} = -\sum_{j=1}^{m} \bar{u}_j \underline{e}_j$:

$$\frac{1}{(1 + \underline{z}^2(\underline{u}^{\dagger})^2 + 2\langle \underline{u}^{\dagger}, \underline{z}\rangle)^{m/2}} = \sum_{\ell=0}^{\infty} \underline{z}^{\ell} S(\underline{z}, -\underline{u}^{\dagger})(\underline{u}^{\dagger})^{\ell} = \sum_{k,\ell=0}^{\infty} \underline{z}^{\ell} C_k(\underline{z}, -\underline{u}^{\dagger})(\underline{u}^{\dagger})^{\ell}.$$

This is in fact a reproducing kernel for $\underline{z}, \underline{u} \in LB(0,1)$, however with an abuse of terminology we call it reproducing kernel also its version in Proposition 3.8, where the variable in the second component is $-e^{-i\theta}\underline{\omega}$ and belongs to LS^{m-1}.
(2) Note that the monogenic Szegő kernel $S(\underline{z}, \underline{u}^{\dagger})$ is the monogenic part of the Cauchy-Hua kernel $H(\underline{z}, -\underline{u}^{\dagger}) = (1 + \underline{z}^2(\underline{u}^{\dagger})^2 + 2\langle \underline{z}, \underline{u}^{\dagger}\rangle)^{-m/2}$.

4 The extended Szegő-Radon transform

The Szegő-Radon transform was originally defined in [2]. In this section we recall its definition and we then introduce and study the so-called extended Szegő-Radon transform.

Definition 4.1. *Let $\underline{t}, \underline{s} \in \mathbb{R}^m$ be such that $|\underline{t}| = |\underline{s}| = 1$ and $\langle \underline{t}, \underline{s}\rangle = 0$ and let $\underline{\tau} = \underline{t} + i\underline{s} \in \mathbb{C}^m$. By $\mathcal{ML}^2(\underline{\tau})$ we denote the completion of the right $\mathbb{C}_m$-Hilbert module consisting of all finite linear combinations of the form*

$$\sum_{\ell} \langle \underline{x}, \underline{\tau}\rangle^{\ell} \underline{\tau} a_{\ell}, \quad a_{\ell} \in \mathbb{C}_m.$$

Note that $\mathcal{ML}^2(\underline{\tau})$ is a closed submodule of $\mathcal{ML}^2(B(0,1))$. In the sequel, we will denote by $f_{\underline{\tau},\ell}(\underline{x})$ the plane wave monogenics

$$f_{\underline{\tau},\ell}(\underline{x}) = \langle \underline{x}, \underline{\tau} \rangle^\ell \underline{\tau}.$$

Definition 4.2. *The (polarized) Szegő–Radon transform is the orthogonal projection operator*

$$\mathcal{R}_{\underline{\tau}} : \mathcal{ML}^2(B(0,1)) \to \mathcal{ML}^2(\underline{\tau}).$$

Remark 4.3. The kernel of the Szegő-Radon transform is given by

$$\begin{aligned} K_{\underline{\tau}}(\underline{x}, \underline{u}) &= \frac{1}{A_m} \frac{\underline{\tau}\,\underline{\tau}^\dagger}{4} (1 + \langle \underline{x}, \underline{\tau} \rangle \langle \underline{u}, \underline{\tau}^\dagger \rangle)^{-m/2} \\ &= \frac{\underline{\tau}\,\underline{\tau}^\dagger}{4} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{\Gamma(\frac{m}{2} + \ell)}{2\pi^{m/2}\Gamma(\ell+1)} \langle \underline{x}, \underline{\tau} \rangle^\ell \langle \underline{u}, \underline{\tau}^\dagger \rangle^\ell \end{aligned}$$

because $f_{\underline{\tau},k}$ is orthogonal to $f_{\underline{\tau},\ell}$ for $k \neq \ell$ and

$$\langle f_{\underline{\tau},\ell}, f_{\underline{\tau},\ell} \rangle = \frac{2\pi^{m/2}\Gamma(\ell+1)}{\Gamma(\frac{m}{2} + \ell)} \underline{\tau}^\dagger \underline{\tau},$$

see Proposition 2.9 in [2] and Remark 3.7 in [3].

Next result corresponds to Theorem 2.19 in [2] (the first equality) and Theorem 4.5 in [3] (the second equality):

Theorem 4.4. *The Szegő-Radon transform of $f \in \mathcal{ML}^2(B(0,1))$ can be expressed as*

$$\mathcal{R}_{\underline{\tau}}[f](\underline{x}) := \int_{\mathbb{S}^{m-1}} K_{\underline{\tau}}(\underline{x}, \underline{\omega}) f(\underline{\omega}) dS(\underline{\omega}) = \frac{\underline{\tau}\,\underline{\tau}^\dagger}{4} f(-\frac{1}{2}\underline{\tau}^\dagger \langle \underline{x}, \underline{\tau} \rangle).$$

Our next goal is to extend this transform to $\mathcal{OL}^2(LB(0,1))$ and to this end, we consider the modified plane waves

$$f^s_{\underline{\tau},\ell}(\underline{z}) = \underline{z}^s \langle \underline{z}, \underline{\tau} \rangle^\ell \underline{\tau}.$$

Note that, from now on, the superscript s is a natural number which does not have to be confused with the 1-vector $\underline{s}$.

Definition 4.5. *For any given $\underline{\tau}$ and $s \in \mathbb{N}$, by $\mathcal{M}^s(\underline{\tau})$ we denote the completion of the right $\mathbb{C}_m$-Hilbert module consisting of all finite linear combinations of the form*

$$\sum_{s,\ell} \underline{z}^s \langle \underline{z}, \underline{\tau} \rangle^\ell \underline{\tau} a_\ell = \sum_{s,\ell} f^s_{\underline{\tau},\ell}(\underline{z}) a_\ell, \quad a_\ell \in \mathbb{C}_m.$$

Note that $\mathcal{M}^s(\underline{\tau})$ is a submodule of $\mathcal{OL}^2(LB(0,1))$.

Definition 4.6. *For any $s \in \mathbb{N}$, the extended Szegő-Radon transform is the orthogonal projection operator*

$$\mathcal{R}^s_{\underline{\tau}} : \mathcal{OL}^2(LB(0,1)) \to \mathcal{M}^s(\underline{\tau}).$$

To compute the kernel for this transform, we first prove the following result whose proof follows the one of Proposition 2.9 in [2]:

Lemma 4.7. *Let $\underline{t}, \underline{s} \in \mathbb{R}^m$ be such that $|\underline{t}| = |\underline{s}| = 1$ and $\langle \underline{t}, \underline{s} \rangle = 0$, let $\underline{\tau} = \underline{t} + i\underline{s} \in \mathbb{C}^m$ and $\omega \in \mathbb{S}^{m-1}$. Then for $k \neq \ell$*

$$\int_{\mathbb{S}^{m-1}} \langle \underline{\omega}, \underline{\tau}^\dagger \rangle^k \langle \underline{\omega}, \underline{\tau} \rangle^\ell dS(\underline{\omega}) = 0$$

and

$$\int_{\mathbb{S}^{m-1}} \langle \underline{\omega}, \underline{\tau}^\dagger \rangle^k \langle \underline{\omega}, \underline{\tau} \rangle^k dS(\underline{\omega}) = (-1)^k 2\pi^{\frac{m}{2}} \frac{\Gamma(k+1)}{\Gamma(m/2+k)}.$$

Proof. Up to a change of the basis, we may set $\underline{t} = \underline{e}_1$ and $\underline{s} = \underline{e}_2$. We consider for $n > 3$ the decomposition $\mathbb{R}^m = \mathbb{R}^2 \oplus \mathbb{R}^{m-2}$. Let θ be the angle between $\underline{\omega}$ and the projection of $\underline{\omega}$ on $\mathbb{R}^2$. Then the element $dS(\underline{\omega})$ can be written as

$$dS(\underline{\omega}) = \cos\varphi \, d\varphi \, dS(\underline{\omega}_{1,2}) \sin^{m-3}\varphi \, dS(\underline{\omega}_{m-3})$$

where $\underline{\omega} = \cos\varphi(\cos\psi \, \underline{e}_1 + \sin\psi \, \underline{e}_2) + \sin\varphi \underline{\eta}$, with $\varphi \in [0, \pi/2)$ and $\psi \in [0, 2\pi)$, so that in fact $dS(\underline{\omega}_{1,2}) := d\psi$, and $\underline{\eta} := \underline{\omega}_{m-3}$. Thus we can write where $\underline{\eta}$ is a suitable element in $\mathbb{S}^{m-3}$, $m > 3$. We also have $\underline{\tau} = \underline{e}_1 + i\underline{e}_2$ and $\underline{\tau}^\dagger = -\underline{e}_1 + i\underline{e}_2$ and so the scalar product in $\mathbb{C}^n$ of $\underline{\omega}$ and $\underline{\tau}$ is

$$\langle \underline{\omega}, \underline{\tau} \rangle = \cos\varphi(\cos\psi + i\sin\psi) = \cos\varphi e^{i\psi}$$
$$\langle \underline{\omega}, \underline{\tau}^\dagger \rangle = \cos\varphi(-\cos\psi + i\sin\psi) = -\cos\varphi e^{-i\psi}.$$

Then we have:

$$\begin{aligned} &= \int_{\mathbb{S}^{m-1}} \langle \underline{\omega}, \underline{\tau}^\dagger \rangle^k \langle \underline{\omega}, \underline{\tau} \rangle^\ell dS(\underline{\omega}) \\ &= \int_0^{\pi/2} \int_0^{2\pi} \int_{\mathbb{S}^{m-3}} (-1)^k (\cos\varphi)^{k+\ell+1} e^{i(\ell-k)\psi} (\sin\varphi)^{m-3} d\varphi \, d\psi \, dS(\underline{\eta}). \end{aligned}$$

The last integral vanishes for $k \neq \ell$. When $k = \ell$ with standard computations we have:

$$\int_0^{\pi/2} \int_0^{2\pi} \int_{\mathbb{S}^{m-3}} (-1)^k (\cos\varphi)^{2k+1} (\sin\varphi)^{m-3} d\varphi \, d\psi \, dS(\underline{\eta}) = (-1)^k 2\pi^{\frac{m}{2}} \frac{\Gamma(k+1)}{\Gamma(m/2+k)}.$$

□

Proposition 4.8. *Let $\underline{t}, \underline{s} \in \mathbb{R}^m$ be such that $|\underline{t}| = |\underline{s}| = 1$ and $\langle \underline{t}, \underline{s} \rangle = 0$ and let $\underline{\tau} = \underline{t} + i\underline{s} \in \mathbb{C}^m$. Let $s, s' \in \mathbb{N}$ be fixed. The functions $f^s_{\underline{\tau},\ell}(\underline{z})$ are such that*

$$\langle f^s_{\underline{\tau},\ell}, f^{s'}_{\underline{\tau},\ell'} \rangle_{\mathcal{OL}^2} = 0, \qquad \text{for } (\ell, s) \neq (\ell', s')$$
$$\langle f^s_{\underline{\tau},\ell}, f^s_{\underline{\tau},\ell} \rangle_{\mathcal{OL}^2} = 2\pi^{\frac{m}{2}+1} \underline{\tau}^\dagger \underline{\tau} \frac{\Gamma(\ell+1)}{\Gamma(\frac{m}{2}+\ell)}.$$

Proof. Using Lemma 4.7 and the fact that $\langle e^{i\theta}\underline{\omega}, \underline{\tau} \rangle^\dagger = \langle e^{-i\theta}\underline{\omega}, -\underline{\tau}^\dagger \rangle$, we have

$$\begin{aligned} \langle f^s_{\underline{\tau},\ell}, f^s_{\underline{\tau},\ell} \rangle &= \int_{\mathbb{S}^{m-1}} \int_0^\pi \langle e^{-i\theta}\underline{\omega}, -\underline{\tau}^\dagger \rangle^\ell (e^{i\theta}\underline{\omega})^{-s} (e^{i\theta}\underline{\omega})^s \langle e^{i\theta}\underline{\omega}, \underline{\tau} \rangle^\ell \underline{\tau}^\dagger \underline{\tau} dS(\underline{\omega}) \, d\theta \\ &= \underline{\tau}^\dagger \underline{\tau} \pi \int_{\mathbb{S}^{m-1}} \langle \underline{\omega}, -\underline{\tau}^\dagger \rangle^\ell \langle \underline{\omega}, \underline{\tau} \rangle^\ell dS(\underline{\omega}) \qquad (6) \\ &= 2\pi^{m/2+1} \underline{\tau}^\dagger \underline{\tau} \frac{\Gamma(\ell+1)}{\Gamma(\frac{m}{2}+\ell)}. \end{aligned}$$

Moreover, Lemma 4.7 also yields orthogonality for $(\ell, s) \neq (\ell', s')$, i.e.

$$\langle f^s_{\underline{\tau},\ell}, f^{s'}_{\underline{\tau},\ell'} \rangle = 0.$$

□

As a consequence we have:

Theorem 4.9. *The extended Szegő-Radon transform can be expressed as*

$$\mathcal{R}^s_{\underline{\tau}}[f](z) = \int_{\mathbb{S}^{m-1}} \int_0^\pi K^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) f(e^{i\theta}\underline{\omega})\, dS(\underline{\omega}) d\theta$$

where

$$K^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) = \frac{\underline{z}^s \underline{\tau}\,\underline{\tau}^\dagger}{4\pi} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{\Gamma(\frac{m}{2}+\ell)}{2\pi^{m/2}\Gamma(\ell+1)} \langle \underline{z}, \underline{\tau}\rangle^\ell \langle e^{-i\theta}\underline{\omega}, \underline{\tau}^\dagger\rangle^\ell (e^{i\theta}\underline{\omega})^{-s}.$$

Proof. To prove the statement, it is enough to show that the kernel $K^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega})$ reproduces the generators of $\mathcal{M}^s(\underline{\tau})$. In fact, we have:

$$\begin{aligned}
&\int_{\mathbb{S}^{m-1}} \int_0^\pi K^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) f^s_{\underline{\tau},\ell}(e^{i\theta}\underline{\omega}) \\
&= \frac{\underline{z}^s \langle \underline{z}, \underline{\tau}\rangle^\ell \underline{\tau}}{4\pi} \frac{\Gamma(\frac{m}{2}+\ell)}{2\pi^{m/2}\Gamma(\ell+1)} \int_{\mathbb{S}^{m-1}} \int_0^\pi \sum_{\ell=0}^{\infty} (-1)^\ell \langle e^{-i\theta}\underline{\omega}, \underline{\tau}^\dagger\rangle^\ell (e^{i\theta}\underline{\omega})^{-s} \underline{\tau}^\dagger f^s_{\underline{\tau},\ell}(e^{i\theta}\underline{\omega})\, dS(\underline{\omega}) d\theta \\
&= \underline{z}^s \langle \underline{z}, \underline{\tau}\rangle^\ell \frac{\underline{\tau}}{4\pi} \frac{\Gamma(\frac{m}{2}+\ell)}{2\pi^{m/2}\Gamma(\ell+1)} \langle f^s_{\underline{\tau},\ell}, f^s_{\underline{\tau},\ell}\rangle \\
&= \underline{z}^s \langle \underline{z}, \underline{\tau}\rangle^\ell \frac{\underline{\tau}\,\underline{\tau}^\dagger \underline{\tau}}{4} = \underline{z}^s \langle \underline{z}, \underline{\tau}\rangle^\ell \underline{\tau} = f^s_{\underline{\tau},\ell}(\underline{z}),
\end{aligned}$$

where we used (6). The statement follows. ☐

Remark 4.10. It is immediate to rewrite the above kernel in closed form:

$$K^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) = \frac{1}{\pi A_m} \underline{z}^s \frac{\underline{\tau}\,\underline{\tau}^\dagger}{4} (1 + \langle \underline{z}, \underline{\tau}\rangle \langle e^{-i\theta}\underline{\omega}, \underline{\tau}^\dagger\rangle)^{-m/2} (e^{i\theta}\underline{\omega})^{-s}.$$

We now prove the following result which shows that the extended Szegő-Radon transform $\mathcal{R}^s_{\underline{\tau}}$ is in fact the Szegő-Radon transform $\mathcal{R}_{\underline{\tau}}$ of the monogenic part f_s multiplied by $\underline{z}^s$:

Proposition 4.11. *Let $f \in \mathcal{OL}^2(LB(0,1))$ admit the monogenic decomposition*

$$f(\underline{u}) = \sum_{\ell=0}^{\infty} \underline{u}^\ell f_\ell(\underline{u})$$

with $\partial_{\underline{u}} f_\ell(\underline{u}) = 0$. Then:

$$\mathcal{R}^s_{\underline{\tau}}[f] = \underline{z}^s \mathcal{R}_{\underline{\tau}}[f_s].$$

Proof. We have the following equalities:

$$\begin{aligned}
\mathcal{R}^s_{\underline{\tau}}[f](\underline{z}) &= \int_{\mathbb{S}^{m-1}} \int_0^\pi K^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) f(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta \\
&= \int_{\mathbb{S}^{m-1}} \int_0^\pi K^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) (e^{i\theta})^s f_s(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta \\
&= \frac{1}{\pi A_m} \underline{z}^s \frac{\underline{\tau}\,\underline{\tau}^\dagger}{4} \int_{\mathbb{S}^{m-1}} \int_0^\pi (1 + \langle \underline{z}, \underline{\tau}\rangle \langle e^{-i\theta}\underline{\omega}, \underline{\tau}^\dagger\rangle)^{-m/2} f_s(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta \\
&= \frac{1}{A_m} \underline{z}^s \frac{\underline{\tau}\,\underline{\tau}^\dagger}{4} \int_{\mathbb{S}^{m-1}} (1 + \langle \underline{z}, \underline{\tau}\rangle \langle \underline{\omega}, \underline{\tau}^\dagger\rangle)^{-m/2} f_s(\underline{\omega})\, dS(\underline{\omega}) \\
&= \underline{z}^s \mathcal{R}_{\underline{\tau}}[f_s](\underline{z}),
\end{aligned}$$

and the statement follows. ☐

Let us set $\mathcal{M}(\underline{\tau}) = \oplus_{s\in\mathbb{N}}\mathcal{M}^s(\underline{\tau})$. Then $\mathcal{M}(\underline{\tau})$ is the closure of the span of all polynomials of the form

$$\underline{z}^s \langle \underline{z}, \underline{\tau}\rangle^k \underline{\tau}.$$

Note that these are not plane waves as they are defined on a two-dimensional complex subspace. We now introduce the following:

Definition 4.12. *The complete Szegő-Radon transform $\mathcal{R}_{\underline{\tau}}:\ \mathcal{OL}^2(LB(0,1)) \to \mathcal{M}(\underline{\tau})$ is defined by*

$$\mathcal{R}_{\underline{\tau}}[f] = \sum_{s=0}^{\infty} \mathcal{R}^s_{\underline{\tau}}[f],$$

for any $f \in \mathcal{OL}^2(LB(0,1))$.

Remark 4.13. It follows from the definition that the kernel of the complete Szegő-Radon transform on the whole $\mathcal{OL}^2(LB(0,1))$ is given by

$$\begin{aligned} K_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) &= \sum_{s=0}^{\infty} K^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) \\ &= \frac{1}{4\pi A_m}(\underline{\tau}\,\underline{\tau}^\dagger - \underline{z}\,\underline{\tau}\,\underline{\tau}^\dagger e^{-i\theta}\underline{\omega}) \frac{(1+\underline{z}^2 e^{-2i\theta})^{-1}}{1+\langle \underline{z}, \underline{\tau}\rangle\langle e^{-i\theta}\underline{\omega}, \underline{\tau}^\dagger\rangle)^{m/2}} \end{aligned}$$

where, as usual, $\underline{z}^2 = -\sum_{j=1}^m z_j^2$.

5 The Hua-Radon transform

In this section we discuss another transform that can be defined for functions in several complex variables, the so-called Hua-Radon transform, and to this end we need to define a suitable submodule of $\mathcal{OL}^2(LB(0,1))$:

Definition 5.1. *Let $\underline{t}$, $\underline{s} \in \mathbb{R}^m$, $\underline{\tau} = \underline{t} + i\underline{s}$, with $|\underline{t}| = |\underline{s}| = 1$, $\underline{t} \perp \underline{s}$, be fixed. By $\mathcal{OL}^2(\underline{\tau})$ we denote the closed submodule of $\mathcal{OL}^2(LB(0,1))$ consisting of holomorphic functions defined in the 2-dimensional Lie ball in the variables $\langle \underline{z}, \underline{s}\rangle$, $\langle \underline{z}, \underline{t}\rangle$ and with boundary values in $\mathcal{L}^2(LS^{m-1})$.*

We note that a function defined in the 2-dimensional Lie ball may be seen as a function of the variables $\langle \underline{z}, \underline{\tau}\rangle$, $\langle \underline{z}, \underline{\tau}^\dagger\rangle$:

$$f(\langle \underline{z}, \underline{\tau}\rangle, \langle \underline{z}, \underline{\tau}^\dagger\rangle), \qquad \tau = \underline{t} + i\underline{s},$$

that is holomorphic for

$$|\langle \underline{z}, \underline{\tau}\rangle| < 1, \qquad |\langle \underline{z}, \underline{\tau}^\dagger\rangle| < 1;$$

in other words, the 2-dimensional Lie ball is a 2-dimensional polydisc. A natural basis for $\mathcal{OL}^2(\underline{\tau})$ is given by the functions

$$f_{\underline{\tau},k,\ell}(\underline{z}) = \langle \underline{z}, \underline{\tau}\rangle^k \langle \underline{z}, \underline{\tau}^\dagger\rangle^\ell, \tag{7}$$

whose orthogonality properties are studied in the next result.

Proposition 5.2. *Let $\underline{t}$, $\underline{s} \in \mathbb{R}^m$ be such that $|\underline{t}| = |\underline{s}| = 1$ and $\langle \underline{t}, \underline{s} \rangle = 0$ and let $\underline{\tau} = \underline{t} + i\underline{s} \in \mathbb{C}^m$. The functions $f_{\underline{\tau},k,\ell}(\underline{z})$ are such that*

$$\langle f_{\underline{\tau},k,\ell}(\underline{z}), f_{\underline{\tau},k',\ell'}(\underline{z}) \rangle_{\mathcal{OL}^2} = 0, \qquad (k,\ell) \neq (k',\ell'),$$

and

$$\langle f_{\underline{\tau},k,\ell}(\underline{z}), f_{\underline{\tau},k,\ell}(\underline{z}) \rangle_{\mathcal{OL}^2} = 2\pi^{m/2+1} \frac{\Gamma(k+\ell+1)}{\Gamma(k+\ell+m/2)}.$$

Proof. It is easy to verify that for $(k,\ell) \neq (k',\ell')$ the functions $f_{\underline{\tau},k,\ell}(\underline{z})$, $f_{\underline{\tau},k',\ell'}$ are orthogonal with respect to the inner product in $\mathcal{OL}^2(LB(0,1))$, namely

$$\int_{\mathbb{S}^{m-1}} \int_0^{\pi} f^{\dagger}_{\underline{\tau},k,\ell}(e^{i\theta}\underline{\omega}) f_{\underline{\tau},k',\ell'}(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta = 0. \tag{8}$$

To prove the second equality, we set

$$\begin{aligned} I :&= \int_{\mathbb{S}^{m-1}} \int_0^{\pi} f^{\dagger}_{\underline{\tau},k,\ell}(e^{i\theta}\underline{\omega}) f_{\underline{\tau},k,\ell}(e^{i\theta}\underline{\omega}) dS(\underline{\omega})\, d\theta \\ &= (-1)^{k+\ell} \int_{\mathbb{S}^{m-1}} \int_0^{\pi} \langle \underline{\omega}, \underline{\tau}^{\dagger} \rangle^k \langle \underline{\omega}, \underline{\tau} \rangle^{\ell} e^{-i(k+\ell)\theta} e^{i(k+\ell)\theta} \langle \underline{\omega}, \underline{\tau} \rangle^k \langle \underline{\omega}, \underline{\tau}^{\dagger} \rangle^{\ell}\, dS(\underline{\omega})\, d\theta \\ &= (-1)^{k+\ell} \pi \int_{\mathbb{S}^{m-1}} \langle \underline{\omega}, \underline{\tau}^{\dagger} \rangle^{k+\ell} \langle \underline{\omega}, \underline{\tau} \rangle^{k+\ell} dS(\underline{\omega}). \end{aligned}$$

If we take $\underline{\tau} = \underline{e}_1 + i\underline{e}_2$ we obtain that the previous integral equals

$$I = \pi \int_{\mathbb{S}^{m-1}} (\omega_1^2 + \omega_2^2)^{k+\ell}\, dS(\underline{\omega}).$$

Using the polar decomposition and assuming $m \geq 3$, we set

$$\underline{\omega} = \cos\varphi(\cos\psi \underline{e}_1 + \sin\psi \underline{e}_2) + \sin\varphi \underline{\nu}, \qquad \varphi \in [0,\pi/2),\ \psi \in [0,2\pi),\ \underline{\nu} \in \mathbb{S}^{m-3}.$$

The surface element $dS(\underline{\omega})$ rewrites as

$$dS(\underline{\omega}) = \cos\varphi\, (\sin\varphi)^{m-3}\, d\varphi\, d\psi\, d\underline{\nu}.$$

We obtain

$$\begin{aligned} I &= \pi \int_0^{\pi/2} \int_0^{2\pi} \int_{\mathbb{S}^{m-3}} (\cos\varphi)^{2k+2\ell} (\sin\varphi)^{m-3} \cos\varphi\, d\psi\, d\varphi\, d\underline{\nu} \\ &= 2\pi^2 A_{m-2} \int_0^{2\pi} (\cos^2\varphi)^{k+\ell} (\sin\varphi)^{m-3}\, d(\sin\varphi) \\ &= 2\pi^2 A_{m-2} \int_0^1 (1-t^2)^{k+\ell} t^{m-3}\, dt \\ &= \pi^2 A_{m-2} \int_0^1 (1-s)^{k+\ell} s^{m/2-2}\, ds \\ &= \frac{2\pi^{m/2+1}}{\Gamma(m/2-1)} B(k+\ell, \frac{m}{2}-1) \\ &= 2\pi^{m/2+1} \frac{\Gamma(k+\ell+1)}{\Gamma(k+\ell+m/2)}, \end{aligned}$$

where we have set $t = \sin\varphi$ and then $s = t^2$ and this concludes the proof. □

Definition 5.3. *The Hua-Radon transform is the projection*

$$\mathcal{OL}^2(LB(0,1)) \to \mathcal{OL}^2(\underline{\tau}).$$

As a consequence of Proposition 5.2, we obtain that the kernel for the Hua-Radon transform is given by

$$\mathcal{K}_{\underline{\tau}}(z, e^{-i\theta}\underline{\omega}) = \frac{1}{2\pi^{m/2+1}} \sum_{k,\ell=0}^{\infty} \frac{\Gamma(k+\ell+m/2)}{\Gamma(k+\ell+1)} (-1)^{k+\ell} a^k b^\ell$$

where

$$a = \langle \underline{z}, \underline{\tau} \rangle \langle e^{-i\theta}\underline{\omega}, \underline{\tau}^\dagger \rangle, \qquad b = \langle \underline{z}, \underline{\tau}^\dagger \rangle \langle e^{-i\theta}\underline{\omega}, \underline{\tau} \rangle.$$

Remark 5.4. We can rewrite the kernel of the Hua-Radon transform as:

$$\begin{aligned}
\mathcal{K}_{\underline{\tau}}(z, e^{-i\theta}\underline{\omega}) &= \frac{1}{\pi A_m} \sum_{s=0}^{\infty} (-1)^s \frac{\Gamma(s+m/2)}{\Gamma(s+1)\Gamma(m/2)} \frac{a^{s+1} - b^{s+1}}{a-b} \\
&= \frac{1}{\pi A_m (a-b)} [a(1+a)^{-m/2} - b(1+b)^{-m/2}]
\end{aligned}$$

which, for $m = 2$ reduces to

$$\begin{aligned}
\mathcal{K}_{\underline{\tau}}(z, e^{-i\theta}\underline{\omega}) &= \frac{1}{2\pi^2} \frac{1}{(1+a)(1+b)} \\
&= \frac{1}{2\pi^2} \frac{1}{(1 + \langle \underline{z}, \underline{\tau} \rangle \langle e^{-i\theta}\underline{\omega}, \underline{\tau}^\dagger \rangle)(1 + \langle \underline{z}, \underline{\tau}^\dagger \rangle \langle e^{-i\theta}\underline{\omega}, \underline{\tau} \rangle)},
\end{aligned}$$

i.e. to the Cauchy-Hua kernel in $\mathbb{C}^2$.

6 The polarized Hua-Radon

The Hua-Radon transform deals with copies of $\mathbb{C}^2$ embedded in $\mathbb{C}^m$. We will show that using techniques typical from Clifford analysis we are able to construct a subspace $\mathfrak{M}(\underline{\tau})$ of $\mathcal{OL}^2(\underline{\tau})$ such that

$$\mathcal{OL}^2(\underline{\tau}) = \mathfrak{M}(\underline{\tau}) \oplus \mathfrak{M}(\underline{\tau}^\dagger), \qquad \mathfrak{M}(\underline{\tau}) \perp \mathfrak{M}(\underline{\tau}^\dagger).$$

In order to define $\mathfrak{M}(\underline{\tau})$, we introduce the functions

$$\begin{aligned}
\psi_{\underline{\tau},2s,k}(\underline{z}) &= \underline{\tau} \langle \underline{z}, \underline{\tau} \rangle^{s+k} \langle \underline{z}, \underline{\tau}^\dagger \rangle^s = \underline{\tau} f_{\underline{\tau},k+s,s}(\underline{z}) \\
\psi_{\underline{\tau},2s+1,k}(\underline{z}) &= \underline{\tau}^\dagger \underline{\tau} \langle \underline{z}, \underline{\tau} \rangle^{s+k+1} \langle \underline{z}, \underline{\tau}^\dagger \rangle^s = \underline{\tau}^\dagger \underline{\tau} f_{\underline{\tau},k+s+1,s}(\underline{z}).
\end{aligned}$$

where $f_{\underline{\tau},r,s}$ is as in (7). We have:

Proposition 6.1. *The function* $\psi_{\underline{\tau},0,k}(\underline{z}) = \underline{\tau} \langle \underline{z}, \underline{\tau} \rangle^k$ *is monogenic, i.e.* $\partial_{\underline{z}} \psi_{\underline{\tau},0,k}(\underline{z}) = 0$. *Moreover:*

$$\begin{aligned}
\partial_{\underline{z}} \psi_{\underline{\tau},2s+1,k}(\underline{z}) &= 4(s+k+1) \psi_{\underline{\tau},2s,k}(\underline{z}) \\
\partial_{\underline{z}} \psi_{\underline{\tau},2s+2,k}(\underline{z}) &= (s+1) \psi_{\underline{\tau},2s+1,k}(\underline{z}).
\end{aligned}$$

Proof. We proceed by direct computations. The first assertion is immediate. The other formulas follow from:

$$\begin{aligned}\partial_{\underline{z}}\psi_{\underline{T},2s+1,k}(\underline{z}) &= (s+k+1)\underline{T}\,\underline{T}^{\dagger}\underline{T}\langle\underline{z},\underline{T}\rangle^{s+k}\langle\underline{z},\underline{T}^{\dagger}\rangle^{s}\\ &= 4(s+k+1)\psi_{\underline{T},2s,k}(\underline{z})\\ \partial_{\underline{z}}\psi_{\underline{T},2s+2,k}(\underline{z}) &= (s+1)\underline{T}^{\dagger}\underline{T}\langle\underline{z},\underline{T}\rangle^{s+k+1}\langle\underline{z},\underline{T}^{\dagger}\rangle^{s}\\ &= (s+1)\psi_{\underline{T},2s+1,k}(\underline{z}).\end{aligned}$$

□

Definition 6.2. *Let $s\in\mathbb{N}_0$. We denote by $\mathfrak{M}^s(\underline{T})$ the completion of the right $\mathbb{C}_m$-module spanned by the set $\{\psi_{\underline{T},s,k},\ k\in\mathbb{N}\}$.*
We denote by $\mathcal{R}^{H,s}_{\underline{T}}$ the projection from $\mathcal{OL}^2(LB(0,1))$ to $\mathfrak{M}^s(\underline{T})$.

Proposition 6.3. *Let $s\in\mathbb{N}$. The following properties hold:*

1. *The map*
$$\partial_{\underline{z}}:\ \mathfrak{M}^s(\underline{T})\to\mathfrak{M}^{s-1}(\underline{T})$$
is an isomorphism.

2. *Every $f\in\mathfrak{M}^s(\underline{T})$ satisfies $\partial_{\underline{z}}^{s+1}f=0$.*

3. *For all $s,\ell\in\mathbb{N}$ we have*
$$\begin{aligned}&\mathfrak{M}^{2s}(\underline{T})\perp\mathfrak{M}^{2\ell+1}(\underline{T}),\\ &\mathfrak{M}^{2s}(\underline{T})\perp\mathfrak{M}^{2\ell}(\underline{T}),\quad s\neq\ell\\ &\mathfrak{M}^{2s+1}(\underline{T})\perp\mathfrak{M}^{2\ell+1}(\underline{T}),\quad s\neq\ell.\end{aligned}$$

Proof. To show the first assertion we use Proposition 6.1 which shows that indeed $\partial_{\underline{z}}:\ \mathfrak{M}^s(\underline{T})\to\mathfrak{M}^{s-1}(\underline{T})$. It is also clear that the kernel of $\partial_{\underline{z}}$ is trivial and that, given any $\psi_{\underline{T},s-1,k}$ it can be obtained as $\partial_{\underline{z}}(C\psi_{\underline{T},s,k})$ where C is a suitable constant depending on s,k.
To show the second assertion, we show that it holds for the generators. To this end, we compute:

$$\partial_{\underline{z}}^{s+1}(\psi_{\underline{T},s,k})=\partial_{\underline{z}}^{s}(C_1\psi_{\underline{T},s-1,k})=\partial_{\underline{z}}^{s-1}(C_2\psi_{\underline{T},s-2,k})=\ldots=\partial_{\underline{z}}(C_{s-1}\psi_{\underline{T},0,k})=0$$

by virtue of Proposition 6.1.
To prove the third assertion, we observe that $\psi^{\dagger}_{\underline{T},2s,k}\psi_{\underline{T},2s'+1,k'}=0$ since $(\underline{T}^{\dagger})^2=0$ so the orthogonality of $\mathfrak{M}^{2s}(\underline{T})$ and $\mathfrak{M}^{2s'+1}(\underline{T})$ follows immediately. Moreover, see Proposition 5.2, for $s\neq s'$ we have

$$\int_{\mathbb{S}^{m-1}}\int_0^{\pi}\psi^{\dagger}_{\underline{T},2s,k}(e^{i\theta}\underline{\omega})\,\psi^{\dagger}_{\underline{T},2s',k'}(e^{i\theta}\underline{\omega})\,dS(\underline{\omega})d\theta=0,$$

so that $\mathfrak{M}^{2s}(\underline{T})\perp\mathfrak{M}^{2s'}(\underline{T})$ and similarly $\mathfrak{M}^{2s+1}(\underline{T})\perp\mathfrak{M}^{2s'+1}(\underline{T})$. □

In view of this result, the direct sum $\oplus_{s=0}^{\infty}\mathfrak{M}^s(\underline{T})$ is also orthogonal, and this is the ground for the next definition:

Definition 6.4. *The orthogonal and direct sum*

$$\mathfrak{M}(\underline{T})=\oplus_{s=0}^{\infty}\mathfrak{M}^s(\underline{T})$$

is called the polarized Hua-Radon module.

We have the following result:

Theorem 6.5. *The following equality holds*

$$\mathcal{OL}^2(\underline{\tau}) = \mathfrak{M}(\underline{\tau}) \oplus \mathfrak{M}(\underline{\tau}^\dagger),$$

where the direct sum is also orthogonal.

Proof. The fact that $\underline{\tau}^2 = (\underline{\tau}^\dagger)^2 = 0$ immediately gives

$$\psi^\dagger_{\underline{\tau},2s,k}\psi_{\underline{\tau}^\dagger,2s',k'} = 0,$$
$$\psi^\dagger_{\underline{\tau},2s+1,k}\psi_{\underline{\tau}^\dagger,2s'+1,k'} = 0.$$

Moreover

$$\int_{\mathbb{S}^{m-1}}\int_0^\pi \psi^\dagger_{\underline{\tau},2s,k}(e^{i\theta}\underline{\omega})\psi_{\underline{\tau}^\dagger,2s'+1,k'}(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})d\theta = 0, \quad \forall k,k',s,s'.$$

In fact the integral equals

$$\int_{\mathbb{S}^{m-1}}\int_0^\pi \psi^\dagger_{\underline{\tau},2s,k}(e^{i\theta}\underline{\omega})\psi_{\underline{\tau}^\dagger,2s'+1,k'}(e^{i\theta}\underline{\omega})$$
$$= 4\underline{\tau}^\dagger \int_{\mathbb{S}^{m-1}}\int_0^\pi e^{i\theta(k+k'+1)}\langle\underline{\omega},\underline{\tau}^\dagger\rangle^{s+k+s'+k+1}\langle\underline{\omega},\underline{\tau}\rangle^{s+s'}\, dS(\underline{\omega})d\theta$$

and we can use Lemma 4.7. Now we note that $\mathcal{M}(\underline{\tau}) \oplus \mathcal{M}(\underline{\tau}^\dagger)$ is the closure of the span of elements of the form

$$\underline{\tau}\,\underline{\tau}^\dagger\langle\underline{z},\underline{\tau}\rangle^{s+k+1}\langle\underline{z},\underline{\tau}^\dagger\rangle^s, \quad \underline{\tau}\,\underline{\tau}^\dagger\langle\underline{z},\underline{\tau}\rangle^{s}\langle\underline{z},\underline{\tau}^\dagger\rangle^s \quad \underline{\tau}^\dagger\underline{\tau}\langle\underline{z},\underline{\tau}\rangle^{s+k+1}\langle\underline{z},\underline{\tau}^\dagger\rangle^s$$
$$\underline{\tau}^\dagger\underline{\tau}\langle\underline{z},\underline{\tau}\rangle^{s}\langle\underline{z},\underline{\tau}^\dagger\rangle^{k+s+1}, \quad \underline{\tau}^\dagger\underline{\tau}\langle\underline{z},\underline{\tau}\rangle^{s}\langle\underline{z},\underline{\tau}^\dagger\rangle^s \quad \underline{\tau}\,\underline{\tau}^\dagger\langle\underline{z},\underline{\tau}\rangle^{s}\langle\underline{z},\underline{\tau}^\dagger\rangle^{k+s+1}$$

which covers the whole set $\{f_{\underline{\tau},k,s}\}$, $k,s\in\mathbb{N}$ hence the statement follows. □

Definition 6.6. *The projection*

$$\mathcal{R}^H_{\underline{\tau}} :\ \mathcal{OL}^2(LB(0,1)) \to \mathfrak{M}(\underline{\tau})$$

is called polarized Hua-Radon transform.

Theorem 6.7. *The kernel of the polarized Hua-Radon transform is*

$$K_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) = \frac{1}{\pi A_m(a-b)}\left[a(1+a)^{-m/2} - b(1+b)^{-m/2}\right]$$
$$+ \frac{1}{\pi A_m}\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+m/2)}{\Gamma(2\ell+1)\Gamma(m/2)}a^\ell b^\ell,$$

where $a = \langle\underline{z},\underline{\tau}\rangle\langle e^{-i\theta}\underline{\omega},\underline{\tau}^\dagger\rangle$, $b = \langle\underline{z},\underline{\tau}^\dagger\rangle\langle e^{-i\theta}\underline{\omega},\underline{\tau}\rangle$.

Our next task is to compute the reproducing kernel $L_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega})$ of the polarized Hua-Radon transform $\mathcal{R}^H_{\underline{\tau}}$. To that end, we compute the kernel $L^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega})$ of the projection $\mathcal{R}^{H,s}_{\underline{\tau}}$. Then, by construction, the kernel for the polarized Hua-Radon transform equals

$$L_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) = \sum_{s=0}^{\infty} L^s_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega})$$

while the kernel of the Hua-Radon transform is decomposed as

$$\mathcal{K}_{\underline{\mathcal{I}}}(\underline{z}, e^{-i\theta}\underline{\omega}) = L_{\underline{\mathcal{I}}}(\underline{z}, e^{-i\theta}\underline{\omega}) + L_{\underline{\mathcal{I}}^\dagger}(\underline{z}, e^{-i\theta}\underline{\omega}).$$

This decomposition will be verified explicitly as a double check for the calculations. First we have:

Theorem 6.8. *The kernel for the polarized Hua-Radon transform is given by*

$$\begin{aligned} L_{\underline{\mathcal{I}}}(\underline{z}, e^{-i\theta}\underline{\omega}) &= \frac{1}{\pi A_m}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(k+2s+\frac{m}{2})}{\Gamma(k+2s+1)\Gamma(\frac{m}{2})} a^{k+s} b^s \\ &- \frac{1}{\pi A_m}\frac{\underline{\mathcal{I}}^\dagger\underline{\mathcal{I}}}{4}\sum_{s=0}^{\infty}\frac{\Gamma(2s+\frac{m}{2})}{\Gamma(2s+1)\Gamma(\frac{m}{2})} a^s b^s, \end{aligned} \tag{9}$$

where, as before, $a = \langle \underline{z}, \underline{\mathcal{I}}\rangle\langle e^{-i\theta}\underline{\omega}, \underline{\mathcal{I}}^\dagger\rangle$, $b = \langle \underline{z}, \underline{\mathcal{I}}^\dagger\rangle\langle e^{-i\theta}\underline{\omega}, \underline{\mathcal{I}}\rangle$.

Proof. To perform the computations, we consider two cases: in the even case we have

$$L_{\underline{\mathcal{I}}}^{2s}(\underline{z}, e^{-i\theta}\underline{\omega}) = \sum_{k=0}^{\infty}\frac{\underline{\mathcal{I}}\,\underline{\mathcal{I}}^\dagger}{4}\lambda_k^{2s} f_{\underline{\mathcal{I}},k+s,s}(\underline{z}) f_{\underline{\mathcal{I}}^\dagger,k+s,s}(e^{-i\theta}\underline{\omega})$$

where, according to Proposition 5.2:

$$\lambda_k^{2s} = (-1)^k \frac{1}{2\pi^{m/2+1}}\frac{\Gamma(k+2s+\frac{m}{2})}{\Gamma(k+2s+1)}.$$

Thus, in terms of the variables $a = \langle \underline{z}, \underline{\mathcal{I}}\rangle\langle e^{-i\theta}\underline{\omega}, \underline{\mathcal{I}}^\dagger\rangle$, $b = \langle \underline{z}, \underline{\mathcal{I}}^\dagger\rangle\langle e^{-i\theta}\underline{\omega}, \underline{\mathcal{I}}\rangle$ we have

$$L_{\underline{\mathcal{I}}}^{2s}(\underline{z}, e^{-i\theta}\underline{\omega}) = \frac{1}{\pi A_m}\frac{\underline{\mathcal{I}}\,\underline{\mathcal{I}}^\dagger}{4}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(k+2s+\frac{m}{2})}{\Gamma(k+2s+1)\Gamma(\frac{m}{2})}a^{k+s}b^s.$$

Similarly, in the odd case, we have

$$L_{\underline{\mathcal{I}}}^{2s+1}(\underline{z}, e^{-i\theta}\underline{\omega}) = \sum_{k=0}^{\infty}\frac{\underline{\mathcal{I}}\,\underline{\mathcal{I}}^\dagger}{4}\lambda_k^{2s+1} f_{\underline{\mathcal{I}},k+s+1,s}(\underline{z}) f_{\underline{\mathcal{I}}^\dagger,k+s+1,s}(e^{-i\theta}\underline{\omega})$$

where

$$\lambda_k^{2s+1} = (-1)^{k+1}\frac{1}{2\pi^{m/2+1}}\frac{\Gamma(k+2s+1+\frac{m}{2})}{\Gamma(k+2s+2)}$$

which yields, in terms of the variables a and b,

$$L_{\underline{\mathcal{I}}}^{2s+1}(\underline{z}, e^{-i\theta}\underline{\omega}) = \frac{1}{\pi A_m}\frac{\underline{\mathcal{I}}^\dagger\underline{\mathcal{I}}}{4}\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(k+2s+\frac{m}{2})}{\Gamma(k+2s+1)\Gamma(\frac{m}{2})}a^{k+s}b^s.$$

Using $\frac{1}{4}(\underline{\mathcal{I}}\,\underline{\mathcal{I}}^\dagger + \underline{\mathcal{I}}^\dagger\underline{\mathcal{I}}) = 1$ and

$$\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(k+2s+\frac{m}{2})}{\Gamma(k+2s+1)\Gamma(\frac{m}{2})}a^{k+s}b^s = \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(k+2s+\frac{m}{2})}{\Gamma(k+2s+1)\Gamma(\frac{m}{2})}a^{k+s}b^s - \frac{\Gamma(2s+\frac{m}{2})}{\Gamma(2s+1)\Gamma(\frac{m}{2})}a^s b^s$$

we obtain the result by adding the kernels for $\mathcal{R}_{\underline{\mathcal{I}}}^{H,s}$. □

For the sequel, it is useful to introduce the functions

$$\phi(t) = \sum_{s=0}^{\infty} \frac{\Gamma(2s + \frac{m}{2})}{\Gamma(2s + 1)\Gamma(\frac{m}{2})} t^s = {}_2F_1(\frac{m}{4}, \frac{m+2}{4}, \frac{1}{2}; t),$$

where ${}_2F_1$ is the hypergeometric function, and

$$\psi(t) = t \sum_{s=0}^{\infty} \frac{\Gamma(2s + 1 + \frac{m}{2})}{\Gamma(2s + 2)\Gamma(\frac{m}{2})} t^s.$$

We have:

Lemma 6.9. *We have the identity:*

$$\phi(a^2) - \frac{1}{a}\psi(a^2) = (1 + a)^{-m/2}. \tag{10}$$

Proof. The identity follows from the following computations:

$$\begin{aligned}
\phi(a^2) - \frac{1}{a}\psi(a^2) &= \sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell + m/2)}{\Gamma(2\ell + 1)\Gamma(m/2)} a^{2\ell} - \sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell + 1 + m/2)}{\Gamma(2\ell + 2)\Gamma(m/2)} a^{2\ell+1} \\
&= \sum_{j=0}^{\infty} \frac{\Gamma(j + m/2)}{\Gamma(j + 1)\Gamma(m/2)} (-a)^j. \\
&= (1 + a)^{-m/2}.
\end{aligned}$$

□

Remark 6.10. By some computations, not relevant for the present work, one may show that also the function ψ can be written in terms of the hypergeometric function ${}_2F_1$. Formula (10) is a functional equation satisfied by the hypergeometric function which is proved within the framework of hypercomplex analysis.

We now have to study the term (which is scalar):

$$\check{L}_{\underline{T}}(\underline{z}, e^{-i\theta}\underline{\omega}) = \frac{1}{\pi A_m} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k + 2s + \frac{m}{2})}{\Gamma(k + 2s + 1)\Gamma(\frac{m}{2})} a^{k+s} b^s.$$

Lemma 6.11. *In terms of the functions ϕ, ψ the function $\check{L}_{\underline{T}}(\underline{z}, e^{-i\theta}\underline{\omega})$ can be written as*

$$\begin{aligned}
\check{L}_{\underline{T}}(\underline{z}, e^{-i\theta}\underline{\omega}) &= \frac{1}{\pi A_m(a - b)} \left[a\phi(a^2) - b\phi(ab) - \psi(a^2) + \psi(ab)\right] \\
&= \frac{1}{\pi A_m(a - b)} \left[a \left(\phi(a^2) - \frac{1}{a}\psi(a^2)\right) - (b\phi(ab) - \psi(ab))\right].
\end{aligned}$$

Proof. We decompose $\check{L}_{\underline{T}}(\underline{z}, e^{-i\theta}\underline{\omega})$ into its even and odd parts as

$$\check{L}_{\underline{T}}(\underline{z}, e^{-i\theta}\underline{\omega}) = \check{L}_{\text{even}}(\underline{z}, e^{-i\theta}\underline{\omega}) + \check{L}_{\text{odd}}(\underline{z}, e^{-i\theta}\underline{\omega})$$

by taking k even and k odd respectively. We obtain, for $k=2j$:

$$\begin{aligned}
\check{L}_{\text{even}}(\underline{z},e^{-i\theta}\underline{\omega}) &= \frac{1}{\pi A_m}\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(2(j+s)+\frac{m}{2})}{\Gamma(2(j+s)+1)\Gamma(\frac{m}{2})}a^{2(j+s)}\left(\frac{b}{a}\right)^s \\
&= \frac{1}{\pi A_m}\sum_{s=0}^{\infty}\sum_{\ell=s}^{\infty}\frac{\Gamma(2\ell+\frac{m}{2})}{\Gamma(2\ell+1)\Gamma(\frac{m}{2})}a^{2\ell}\left(\frac{b}{a}\right)^s \\
&= \frac{1}{\pi A_m}\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+\frac{m}{2})}{\Gamma(2\ell+1)\Gamma(\frac{m}{2})}a^{2\ell}\sum_{s=0}^{\ell}\left(\frac{b}{a}\right)^s \\
&= \frac{1}{\pi A_m}\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+\frac{m}{2})}{\Gamma(2\ell+1)\Gamma(\frac{m}{2})}a^{\ell}\,\frac{a^{\ell+1}-b^{\ell+1}}{a-b} \\
&= \frac{1}{\pi A_m}\left[\frac{a}{a-b}\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+\frac{m}{2})}{\Gamma(2\ell+1)\Gamma(\frac{m}{2})}a^{2\ell}-\frac{b}{a-b}\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+\frac{m}{2})}{\Gamma(2\ell+1)\Gamma(\frac{m}{2})}(ab)^{\ell}\right] \\
&= \frac{1}{\pi A_m}\left[a\frac{\phi(a^2)}{a-b}-b\frac{\phi(ab)}{a-b}\right].
\end{aligned}$$

Moreover, for $k=2j+1$ with similar computations we have

$$\begin{aligned}
\check{L}_{\text{odd}}(\underline{z},e^{-i\theta}\underline{\omega}) &= -\frac{1}{\pi A_m}\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(2(j+s)+1+\frac{m}{2})}{\Gamma(2(j+s)+2)\Gamma(\frac{m}{2})}a^{2(j+s)+1}\left(\frac{b}{a}\right)^s \\
&= -\frac{1}{\pi A_m}\sum_{s=0}^{\infty}\sum_{\ell=s}^{\infty}\frac{\Gamma(2\ell+1+\frac{m}{2})}{\Gamma(2\ell+2)\Gamma(\frac{m}{2})}a^{2\ell+1}\left(\frac{b}{a}\right)^s \\
&= -\frac{1}{\pi A_m}\sum_{s=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+1+\frac{m}{2})}{\Gamma(2\ell+2)\Gamma(\frac{m}{2})}a^{\ell+1}\frac{a^{\ell+1}-b^{\ell+1}}{a-b} \\
&= -\frac{1}{\pi A_m}\left[\frac{\psi(a^2)}{a-b}-\frac{\psi(ab)}{a-b}\right],
\end{aligned}$$

and the assertion follows. □

Remark 6.12. The term $b\phi(b)-\psi(ab)$ can only be rewritten as

$$\begin{aligned}
b\phi(ab)-\psi(ab) &= b\left(\phi(ab)-\frac{1}{b}\psi(ab)\right) \\
&= b\left(\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+m/2)}{\Gamma(2\ell+1)\Gamma(m/2)}a^{\ell}b^{\ell}-a\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+1+m/2)}{\Gamma(2\ell+2)\Gamma(m/2)}a^{\ell}b^{\ell}\right).
\end{aligned}$$

Combining the foregoing results we arrive at

Theorem 6.13. *The kernel $L_{\underline{\tau}}$ for the polarized Hua-Radon transform is given by:*

$$\begin{aligned}
L_{\underline{\tau}}(\underline{z},e^{-i\theta}\underline{\omega}) = \frac{1}{\pi A_m(a-b)}&\left[a(1+a)^{-m/2}-b\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+m/2)}{\Gamma(2\ell+1)\Gamma(m/2)}a^{\ell}b^{\ell}\right. \\
&\left.+ab\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+1+m/2)}{\Gamma(2\ell+2)\Gamma(m/2)}a^{\ell}b^{\ell}\right]-\frac{1}{\pi A_m}\frac{\underline{\tau}^{\dagger}\underline{\tau}}{4}\sum_{\ell=0}^{\infty}\frac{\Gamma(2\ell+\frac{m}{2})}{\Gamma(2\ell+1)\Gamma\left(\frac{m}{2}\right)}a^{\ell}b^{\ell}.
\end{aligned}$$

Proof. Substitution of the previous Lemmas in Theorem 6.8. □

As a verification of all the results so far we now prove

Theorem 6.14. *The kernel for the Hua-Radon transform is given by*

$$\mathcal{K}_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) = L_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) + L_{\underline{\tau}^\dagger}(\underline{z}, e^{-i\theta}\underline{\omega}).$$

Proof. It is similar to the proof of Theorem 6.13. The kernel $L_{\underline{\tau}^\dagger}(\underline{z}, e^{-i\theta}\underline{\omega})$ is obtained from $L_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega})$ by intertwining $\underline{\tau}$, $\underline{\tau}^\dagger$ and a, b. This leads to

$$\begin{aligned} L_{\underline{\tau}^\dagger}(\underline{z}, e^{-i\theta}\underline{\omega}) = \frac{1}{\pi A_m(a-b)} &\left[-b(1+b)^{-m/2} + a\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+m/2)}{\Gamma(2\ell+1)\Gamma(m/2)} a^\ell b^\ell \right. \\ &\left. -ab\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+1+m/2)}{\Gamma(2\ell+2)\Gamma(m/2)} a^\ell b^\ell \right] - \frac{1}{\pi A_m}\frac{\underline{\tau}^\dagger\underline{\tau}}{4}\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+\frac{m}{2})}{\Gamma(2\ell+1)\Gamma\left(\frac{m}{2}\right)} a^\ell b^\ell. \end{aligned}$$

Hence we obtain

$$\begin{aligned} &L_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}) + L_{\underline{\tau}^\dagger}(\underline{z}, e^{-i\theta}\underline{\omega}) = \\ &= \frac{1}{\pi A_m(a-b)} \left[a(1+a)^{-m/2} - b\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+m/2)}{\Gamma(2\ell+1)\Gamma(m/2)} a^\ell b^\ell \right. \\ &\left. +ab\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+1+m/2)}{\Gamma(2\ell+2)\Gamma(m/2)} a^\ell b^\ell \right] - \frac{1}{\pi A_m}\frac{\underline{\tau}^\dagger\underline{\tau}}{4}\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+\frac{m}{2})}{\Gamma(2\ell+1)\Gamma\left(\frac{m}{2}\right)} a^\ell b^\ell \\ &+ \frac{1}{\pi A_m(a-b)} \left[-b(1+b)^{-m/2} + a\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+m/2)}{\Gamma(2\ell+1)\Gamma(m/2)} a^\ell b^\ell \right. \\ &\left. -ab\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+1+m/2)}{\Gamma(2\ell+2)\Gamma(m/2)} a^\ell b^\ell \right] - \frac{1}{\pi A_m}\frac{\underline{\tau}^\dagger\underline{\tau}}{4}\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+\frac{m}{2})}{\Gamma(2\ell+1)\Gamma\left(\frac{m}{2}\right)} a^\ell b^\ell \\ &= \frac{1}{\pi A_m(a-b)} \left[a(1+a)^{-m/2} - b(1+b)^{-m/2} \right] \\ &- \frac{1}{\pi A_m(a-b)}(b-a)\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+m/2)}{\Gamma(2\ell+1)\Gamma(m/2)} a^\ell b^\ell \\ &- \frac{1}{\pi A_m}\left(\frac{\underline{\tau}^\dagger\underline{\tau} + \underline{\tau}\underline{\tau}^\dagger}{4}\right)\sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell+m/2)}{\Gamma(2\ell+1)\Gamma(m/2)} a^\ell b^\ell \\ &= \mathcal{K}_{\underline{\tau}}(\underline{z}, e^{-i\theta}\underline{\omega}). \end{aligned}$$

□

7 The monogenic Hua-Radon transform

It is not immediate to see if there is a link between the extended Szegő-Radon transform and the polarized Hua-Radon transform. However, they admit a common extension that is obtained from the functions $\underline{z}^j\psi_{\underline{\tau},2s,k}(\underline{z})$ and $\underline{z}^j\psi_{\underline{\tau},2s+1,k}(\underline{z})$ which clearly include the basis of the modules $\mathcal{M}^s(\underline{\tau})$ and $\mathfrak{M}^s(\underline{\tau})$ used to define the extended Szegő-Radon transform and the polarized Hua-Radon transform, respectively. Unfortunately, the functions above do not form an orthogonal system. Thus we consider a suitable modification of these functions. To this end, let M denote the projection onto the monogenic part and define

$$\psi^j_{\underline{T},2s,k}(\underline{z}) = \underline{z}^j M[\psi_{\underline{T},2s,k}](\underline{z})$$
$$\psi^j_{\underline{T},2s+1,k}(\underline{z}) = \underline{z}^j M[\psi_{\underline{T},2s+1,k}](\underline{z}).$$

It is immediate that the degrees of homogeneity of $\psi^j_{\underline{T},2s,k}$, $\psi^j_{\underline{T},2s+1,k}$ are $j+k+2s$ and $j+k+2s+1$ respectively. Moreover with respect to the scalar product in $\mathcal{OL}^2(LB(0,1))$ we have:

Proposition 7.1. *Consider the family of functions* $\{\psi^j_{\underline{T},\alpha,k}, \quad \alpha, k \in \mathbb{N}\}$. *Then elements with different degrees of homogeneity are orthogonal and any two elements* $\psi^j_{\underline{T},\alpha,k}$, $\psi^{j'}_{\underline{T},\alpha',k'}$ *with* $j \neq j'$ *are orthogonal. Furthermore*

$$\langle \psi^j_{\underline{T},\alpha,k}, \psi^j_{\underline{T},\alpha',k'}\rangle = \langle M[\psi_{\underline{T},\alpha,k}], M[\psi_{\underline{T},\alpha',k'}]\rangle = \langle M[\psi_{\underline{T},\alpha,k}], \psi_{\underline{T},\alpha',k'}\rangle.$$

Lemma 7.2. *For suitable constants* $\mu_\ell = \mu_{\ell,\alpha,k}$, $\ell = 1, \ldots, \mu_\alpha$ *we have*

$$M[\psi_{\underline{T},\alpha,k}] = \psi_{\underline{T},\alpha,k} + \mu_1 \underline{z}\psi_{\underline{T},\alpha-1,k} + \cdots + \mu_\alpha \underline{z}^\alpha \psi_{\underline{T},0,k}.$$

Proof. The result directly follows from the formulas in Proposition 6.1. □

By virtue of this result, to compute $\langle \psi^j_{\underline{T},\alpha,k}, \psi^j_{\underline{T},\alpha',k'}\rangle$ it is enough to compute $\langle M[\psi_{\underline{T},\alpha,k}], \psi_{\underline{T},\alpha',k'}\rangle$. The following technical lemma will be useful in the sequel:

Lemma 7.3. *For* $\ell \geq 0$, $s > 0$ *and* $k' = k - 2s$

$$\int_{\mathbb{S}^{m-1}} (\psi_{\underline{T},\alpha-2\ell,k}(\underline{\omega}))^\dagger \psi_{\underline{T},\alpha+2s,k'}(\underline{\omega}) = 0,$$

Proof. Let α be even, then the statement follows from

$$\begin{aligned} \psi_{\underline{T},\alpha-2\ell,k}(\underline{\omega}) &= \underline{T}\langle \underline{\omega}, \underline{T}\rangle^k |\langle \underline{\omega}, \underline{T}\rangle|^{\alpha-2\ell} \\ \psi_{\underline{T},\alpha+2s,k-2s}(\underline{\omega}) &= \underline{T}\langle \underline{\omega}, \underline{T}\rangle^{k-2s} |\langle \underline{\omega}, \underline{T}\rangle|^{\alpha+2s}. \end{aligned} \tag{11}$$

The case α odd is based on similar computations. □

we can now prove the following result:

Proposition 7.4. *For* $\alpha' > \alpha$ *the following equality holds*

$$\langle \underline{z}^j \psi_{\underline{T},\alpha-j,k}, \psi_{\underline{T},\alpha',k'}\rangle = 0.$$

Proof. To prove the statement we consider two cases: j even and j odd.
(i) Case j even, i.e. $j = 2\ell$. Then, it can be easily verified that the inner product

$$\langle \underline{z}^j \psi_{\underline{T},\alpha-j,k}, \psi_{\underline{T},\alpha',k'}\rangle$$

equals, up to a constant,

$$\int_{\mathbb{S}^{m-1}} (\psi_{\underline{T},\alpha-j,k}(\underline{\omega}))^\dagger \psi_{\underline{T},\alpha',k'}(\underline{\omega})\, dS(\underline{\omega}).$$

When $\alpha' - \alpha$ is odd the integrand is identically zero since $\underline{\tau}^2 = (\underline{\tau}^\dagger)^2 = 0$, so we consider the case $\alpha' - \alpha = 2s$. We note that for $s > 0$, $\ell \geq 0$ and $k' = k - 2s$ we have

$$\int_{\mathbb{S}^{m-1}} (\psi_{\underline{\tau},\alpha-2\ell,k}(\underline{\omega}))^\dagger \psi_{\underline{\tau},\alpha+2s,k'}(\underline{\omega})\, dS(\underline{\omega}) = 0$$

by Lemma 7.3.
(ii) Case j odd, i.e. $j = 2\ell + 1$. Then we have to compute the integral

$$\int_{\mathbb{S}^{m-1}} (\psi_{\underline{\tau},\alpha-2\ell-1,k}(\underline{\omega}))^\dagger \underline{\omega} \psi_{\underline{\tau},\alpha',k'}(\underline{\omega})\, dS(\underline{\omega})$$

for $\alpha' > \alpha$ and $\alpha + k = \alpha' + k'$. By selecting suitable the basis, it suffices to take $\underline{\tau} = \underline{e}_1 + i\underline{e}_2$ and to write $\underline{\omega} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \underline{\omega}_\perp$, so that $\underline{\omega}_\perp$ can be neglected in the previous integral and we are led to compute

$$\int_{\mathbb{S}^{m-1}} (\psi_{\underline{\tau},\alpha-2\ell-1,k}(\underline{\omega}))^\dagger (\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2) \psi_{\underline{\tau},\alpha',k'}(\underline{\omega})\, dS(\underline{\omega}).$$

The above integral vanishes when $\alpha' > \alpha$. To prove this assertion we note that

$$(\underline{e}_1 \pm i\underline{e}_2)(\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2) = (\omega_1 \underline{e}_1 - \omega_2 \underline{e}_2)(\underline{e}_1 \mp i\underline{e}_2).$$

Thus if α is even and $\alpha' - \alpha$ is odd, we use the equality

$$\underline{\tau}^\dagger \underline{\tau}\, \underline{\omega}\, \underline{\tau}^\dagger \underline{\tau} = 0$$

while if α is odd we use $\underline{\tau}^\dagger \underline{\omega} \underline{\tau} = 0$. So we are reduced to the case $\alpha' = \alpha + 2s$, $s > 0$ and $k' = k - 2s$.
For α even, we use again formulas of the form (11)

$$\begin{aligned} \psi_{\underline{\tau},\alpha-2\ell-1,k}(\underline{\omega}) &= \underline{\tau}^\dagger \underline{\tau} \langle \underline{\omega}, \underline{\tau} \rangle^{k+1} |\langle \underline{\omega}, \underline{\tau} \rangle|^{\alpha-2\ell-1} \\ \psi_{\underline{\tau},\alpha+2s,k-2s}(\underline{\omega}) &= \underline{\tau} \langle \underline{\omega}, \underline{\tau} \rangle^{k-2s} |\langle \underline{\omega}, \underline{\tau} \rangle|^{\alpha+2s}. \end{aligned}$$

and the fact that $(\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2)\underline{\tau}\langle \underline{\omega}, \underline{\tau} \rangle^{k-2s}$ is spherical harmonic of degree $k - 2s + 1$ for $s > 0$ we have orthogonality.
Let now α odd. Formulas of the form (11) give

$$\begin{aligned} \psi_{\underline{\tau},\alpha-2\ell-1,k}(\underline{\omega}) &= \underline{\tau} \langle \underline{\omega}, \underline{\tau} \rangle^{k} |\langle \underline{\omega}, \underline{\tau} \rangle|^{\alpha-2\ell+1} \\ \psi_{\underline{\tau},\alpha+2s,k-2s}(\underline{\omega}) &= \underline{\tau}^\dagger \underline{\tau} \langle \underline{\omega}, \underline{\tau} \rangle^{k+1-2s} |\langle \underline{\omega}, \underline{\tau} \rangle|^{\alpha+2s-1} \end{aligned}$$

and we use them together with the fact that $\psi^\dagger_{\underline{\tau},\alpha-2\ell-1,k}(\underline{\omega})(\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2)$ is spherical harmonic of degree $k + 1$. The statement follows. □

Corollary 7.5. *For $\alpha' > \alpha$ the following equality holds*

$$\langle M[\psi_{\underline{\tau},\alpha,k}], \psi_{\underline{\tau},\alpha',k'} \rangle = 0.$$

Proof. To show the assertion, we use Lemma 7.2 to decompose $M[\psi_{\underline{\tau},\alpha,k}]$ in terms of $\psi_{\underline{\tau},\alpha-j,k}$, then the statement follows from Proposition 7.4. □

Definition 7.6. *By $\mathfrak{M}^{j,\alpha}(\underline{\tau})$ be the right $\mathbb{C}_m$-module spanned by $\{\psi^j_{\underline{\tau},\alpha,k},\ k \in \mathbb{N}\}$ and let*

$$\mathcal{R}^{j,\alpha}_{\underline{\tau}}:\ \mathcal{OL}^2(LB(0,1)) \to \mathfrak{M}^{j,\alpha}(\underline{\tau}).$$

The projection operator

$$\sum_{\alpha\in\mathbb{N}} \mathcal{R}^{j,\alpha}_{\underline{\tau}}:\ \mathcal{OL}^2(LB(0,1)) \to \oplus_{\alpha\in\mathbb{N}}\mathfrak{M}^{j,\alpha}(\underline{\tau}),$$

where $\oplus_{\alpha\in\mathbb{N}}\mathfrak{M}^{j,\alpha}(\underline{\tau})$ is a direct, orthogonal sum, is called monogenic Hua-Radon transform.

As in the case of the other transforms, our next task is to compute the kernel of this operator.

Theorem 7.7. *The monogenic kernel of the Hua-Radon transform is given by*

$$K^j(\underline{z}, e^{-i\theta}\underline{\omega}) = \sum_{\alpha\in\mathbb{N}} K^{j,\alpha}(\underline{z}, e^{-i\theta}\underline{\omega})$$

where

$$K^{j,\alpha}(\underline{z}, e^{-i\theta}\underline{\omega}) = \underline{z}^j L^\alpha(\underline{z}, e^{-i\theta}\underline{\omega})(\underline{\omega}e^{i\theta})^{-j}$$

and, for suitable coefficients λ^α_k

$$L^\alpha(\underline{z}, e^{-i\theta}\underline{\omega}) = \sum_{k=0}^{\infty} \lambda^\alpha_k M[\psi_{\underline{\tau},\alpha,k}(\underline{z})]M[\psi_{\underline{\tau},\alpha,k}(e^{i\theta\underline{\omega}})]^\dagger.$$

Proof. To prove the result, it is enough to show that $K^{j,\alpha}$ reproduces the elements of the basis, namely

$$\psi^j_{\underline{\tau},\alpha,k}(\underline{z}) = \int_{\mathbb{S}^{m-1}}\int_0^\pi K^{j,\alpha}(\underline{z}, e^{-i\theta}\underline{\omega})\psi^j_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta \tag{12}$$

and then the result follows using the orthogonality relations. Formula (12) is equivalent to

$$\int_{\mathbb{S}^{m-1}}\int_0^\pi L^\alpha(\underline{z}, e^{-i\theta}\underline{\omega})\, M[\psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})] = M[\psi_{\underline{\tau},\alpha,k}(\underline{z})],$$

or, using orthogonality, to

$$\lambda^\alpha_k M[\psi_{\underline{\tau},\alpha,k}(\underline{z})] = \int_{\mathbb{S}^{m-1}}\int_0^\pi M[\psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})]^\dagger M[\psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})]\, dS(\underline{\omega})\, d\theta = M[\psi_{\underline{\tau},\alpha,k}(\underline{z})].$$

We have

$$\begin{aligned}
\frac{\underline{\tau}^\dagger\underline{\tau}}{4}(\lambda^\alpha_k)^{-1} &= \int_{\mathbb{S}^{m-1}}\int_0^\pi M[\psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})]^\dagger M[\psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})]\, dS(\underline{\omega})\, d\theta \\
&= \int_{\mathbb{S}^{m-1}}\int_0^\pi M[\psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})]^\dagger \psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta \\
&= \sum_{j=0}^{\alpha}\mu_j \int_{\mathbb{S}^{m-1}}\int_0^\pi (\psi_{\underline{\tau},\alpha-j,k}(e^{i\theta}\underline{\omega}))^\dagger (e^{-i\theta}\underline{\omega})^{-j}\psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta.
\end{aligned}$$

Let us now set

$$\Phi_j = \int_{\mathbb{S}^{m-1}}\int_0^\pi (\psi_{\underline{\tau},\alpha-j,k}(e^{i\theta}\underline{\omega}))^\dagger (e^{-i\theta}\underline{\omega})^{-j}\psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})\, dS(\underline{\omega})\, d\theta.$$

First, we consider the case α even.
For $j = 2\ell$ we have

$$\Phi_{2\ell} = \pi \int_{\mathbb{S}^{m-1}} (\psi_{\underline{\tau},\alpha-2\ell,k}(\underline{\omega}))^\dagger (\underline{\omega})^{-j} \psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})\, dS(\underline{\omega}),$$

and since α is even:

$$\psi_{\underline{\tau},\alpha-2\ell,k}(\underline{\omega}) = \underline{\tau}\langle \underline{\omega}, \underline{\tau}\rangle^k |\langle \underline{\omega}, \underline{\tau}\rangle|^{\alpha-2\ell}, \qquad \psi_{\underline{\tau},\alpha,k}(\underline{\omega}) = \underline{\tau}\langle \underline{\omega}, \underline{\tau}\rangle^k |\langle \underline{\omega}, \underline{\tau}\rangle|^{\alpha}.$$

Thus we conclude that for $\alpha = 2s$:

$$\Phi_{2\ell} = \pi \underline{\tau}^\dagger \underline{\tau} \int_{\mathbb{S}^{m-1}} |\langle \underline{\omega}, \underline{\tau}\rangle|^{2(k-\ell+2s)}\, dS(\underline{\omega}).$$

Now let us set $\underline{\tau} = \underline{e}_1 + i\underline{e}_2$, $\langle \underline{\omega}, \underline{\tau}\rangle = \omega_1 + i\omega_2$ and $\underline{\omega} = \cos\varphi(\cos\psi e_1 + \sin\psi e_2) + \sin\phi \underline{\nu}$, $n = k - \ell + 2s$, with computations similar to those done in Section 5 we get

$$\begin{aligned}
\int_{\mathbb{S}^{m-1}} |\langle \underline{\omega}, \underline{\tau}\rangle|^{2n}\, dS(\underline{\omega}) &= \int_0^{\pi/2}\int_0^{2\pi}\int_{S^{m-3}} (\cos\varphi)^{2n}(\sin\varphi)^{m-3}\cos\varphi\, d\psi\, d\varphi\, d\underline{\nu} \\
&= 2\pi A_{m-2}\int_0^{\pi/2} (\cos\varphi)^{2n}(\sin\varphi)^{m-3} d(\sin\varphi) \\
&= 2\pi A_{m-2}\int_0^1 (1-t^2)n t^{m-3}\, dt = \pi A_{m-2} B(n, \frac{m}{2}-1),
\end{aligned}$$

so that

$$\Phi_{2\ell} = (-1)^\ell \pi^2 \frac{2\pi^{m/2-1}}{\Gamma(m/2-1)} \underline{\tau}^\dagger \underline{\tau} B(k-\ell+2s, \frac{m}{2}-1).$$

Let us now reason as before to treat the case $j = 2\ell+1$:

$$\begin{aligned}
\Phi_{2\ell+1} &= (-1)^{\ell+1}\pi \int_{S^{m-1}} (\psi_{\underline{\tau},\alpha-2\ell-1,k}(\underline{\omega}))^\dagger \underline{\omega} \\
&, \psi_{\underline{\tau},\alpha,k}(\underline{\omega})\, dS(\underline{\omega}) \\
&= (-1)^{\ell+1}\pi \int_{S^{m-1}} (\psi_{\underline{\tau},2s-2\ell-1,k}(\underline{\omega}))^\dagger (\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2)\, \psi_{\underline{\tau},2s,k}(\underline{\omega})\, dS(\underline{\omega})
\end{aligned}$$

where

$$\psi_{\underline{\tau},2s-2\ell-1,k}(\underline{\omega}) = \underline{\tau}^\dagger \underline{\tau}\langle \underline{\omega}, \underline{\tau}\rangle^{k+1} |\langle \underline{\omega}, \underline{\tau}\rangle|^{2s-2\ell-2}, \qquad \psi_{\underline{\tau},2s,k}(\underline{\omega}) = \underline{\tau}\langle \underline{\omega}, \underline{\tau}\rangle^k |\langle \underline{\omega}, \underline{\tau}\rangle|^{2s}.$$

Thus we have:

$$\Phi_{2\ell+1} = (-1)^{\ell+1}\pi \underline{\tau}^\dagger \underline{\tau} \int_{S^{m-1}} (\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2)\underline{\tau} |\langle \underline{\omega}, \underline{\tau}\rangle|^{2(2s+k-1-\ell)} (\langle \underline{\omega}, \underline{\tau}\rangle)^\dagger\, dS(\underline{\omega}).$$

Since

$$\underline{\tau}^\dagger \underline{\tau}(\omega_1 \underline{e}_1 + \omega_2 \underline{e}_2)\underline{\tau} = -\underline{\tau}^\dagger \underline{\tau}(1 - i\underline{e}_1\underline{e}_2)(\omega_1 + i\omega_2) = 4\underline{\tau}^\dagger \underline{\tau}\langle \underline{\omega}, \underline{\tau}\rangle,$$

we finally have

$$\Phi_{2\ell+1} = (-1)^{\ell+1} 4\pi \underline{\tau}^\dagger \underline{\tau} \int_{S^{m-1}} |\langle \underline{\omega}, \underline{\tau}\rangle|^{2(k+2s-\ell)}\, dS(\underline{\omega}) = (-1)^{\ell+1} \frac{8\pi^{m/2+1}}{\Gamma(m/2-1)} \underline{\tau}^\dagger \underline{\tau} B(n, \frac{m}{2}-1).$$

Summarizing, in the case α even we have

$$\int_{\mathbb{S}^{m-1}}\int_0^\pi M[\psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega})]^\dagger \psi_{\underline{\tau},\alpha,k}(e^{i\theta}\underline{\omega}) = \sum_{\ell=0}^{s}\mu_{2\ell}\Phi_{2\ell} + \sum_{\ell=0}^{s-1}\mu_{2\ell+1}\Phi_{2\ell+1}$$

$$= \frac{2\pi^{m/2+1}}{\Gamma\left(\frac{m}{2}-1\right)}\underline{\tau}^\dagger\underline{\tau}\left(\sum_{\ell=0}^{s}(-1)^\ell\mu_{2\ell}B(k-\ell+2s, m/2-1)\right.$$

$$\left. -4\sum_{\ell=0}^{s-1}(-1)^\ell\mu_{2\ell+1}B(k-\ell+2s, m/2-1)\right).$$

We conclude that

$$(\lambda_k^{2s})^{-1} = \frac{8\pi^{m/2+1}}{\Gamma(m/2-1)}\left(\sum_{\ell=0}^{s}(-1)^\ell\mu_{2\ell}B(k-\ell+2s, m/2-1)\right.$$

$$\left. -4\sum_{\ell=0}^{s-1}(-1)^\ell\mu_{2\ell+1}B(k-\ell+2s, m/2-1)\right).$$

The case α odd can be treated in a similar way.

□

References

[1] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman Res. Notes in Math., 76, 1982.

[2] F. Colombo, I. Sabadini, F. Sommen, *On the Szegő-Radon projection of monogenic functions*, Adv. Appl. Math., **74** (2016), 1–22.

[3] F. Colombo, I. Sabadini, F. Sommen, *On the Bargmann-Radon transform in the monogenic setting*, J. Geom. Phys. **120** (2017), 306–316.

[4] R. Delanghe, F. Sommen, V. Souček, Clifford Algebra and Spinor-valued Functions, Mathematics and Its Applications 53, Kluwer Academic Publishers Group, Dordrecht, 1992.

[5] L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in Classical Domains*, Moskow, 1959 (in Russian); Translation of Math. Monographs 6, AMS, Providence, Rhode Island, 1963.

[6] M. Morimoto, *Analytic functionals on the Lie sphere*, Tokyo J. Math., **3** (1980), 1–35.

[7] D. Pena Pena, I. Sabadini, F. Sommen, *Segal-Bargmann-Fock modules of monogenic functions*, J. Math. Phys. **58** (2017), 103507, 9 pp.

[8] I. Sabadini, F. Sommen, *A Fourier–Borel transform for monogenic functionals on the Lie ball*, J. Math. Soc. Japan, **68** (2016), 1487–1504.

[9] J. Siciak, *Holomorphic continuation of harmonic functions*, Ann. Polon. Math., **29** (1974), 67–73.

[10] F. Sommen, *Spherical monogenic on the Lie sphere*, J. Funct. Anal. **92**, 372–402 (1990).