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GLOBAL SURVIVAL OF BRANCHING RANDOM WALKS AND TREE-LIKE BRANCHING RANDOM WALKS

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Abstract. The local critical parameter λ_s of continuous-time branching random walks is completely understood and can be computed as a function of the reproduction rates. On the other hand, only for some classes of branching random walks it is known that the global critical parameter λ_w is a certain function of the reproduction rates, which we denote by $1/K_w$. We provide here new sufficient conditions which guarantee that the global critical parameter equals $1/K_w$. This result extends previously known results for branching random walks on multigraphs and general branching random walks. We show that these sufficient conditions are satisfied by periodic tree-like branching random walks. We also discuss the critical parameter and the critical behaviour of continuous-time branching processes in varying environment. So far, only examples where $\lambda_w = 1/K_w$ were known; here we provide an example where $\lambda_w > 1/K_w$.

Keywords: branching random walk, branching process, local survival, global survival, varying environment, tree-like, critical parameters, generating function.
AMS subject classification: 60J80, 60K35.

1. Introduction

The theory of time-homogeneous branching processes dates back to the work of Galton and Watson ([12]) and the characterization of survival of these processes is very simple: the expectation of the offspring distribution must be strictly larger than 1. One way to add complexity is to study the process on a spatial structure: the individuals live on the vertices of a graph $(X, E(X))$ and randomly reproduce; the offspring are placed onto neighbouring vertices. If we look at the trajectory of lineages, they can be seen as random walks, which branch whenever an individual has more than one child, whence the name Branching Random Walk (briefly, BRW) for the process. In a BRW survival can be global or local, meaning that with positive probability there will always be someone alive on the graph (global survival) or on a given vertex (local survival). Clearly local survival is more restrictive than global survival and both situations become more likely when individuals get more prolific. In continuous time an easy way to tune reproductions (and to have markovianity) is to fix $\lambda > 0$ and to attach to each particle living at x and to each oriented edge xy of the graph an exponential clock whose parameter is proportional to λ. The same is repeated for all particles, sites and edges. Whenever the clock rings, the corresponding particle at x (if still alive) places an offspring at y. The larger λ, the higher the probability of global and local survival will be.

To be more precise, the graph is endowed with a matrix $K = (k_{xy})_{x,y\in X}$ of nonnegative *rates*. Each particle has an exponential lifetime of mean 1, and individuals living at x place children at y, one at a time, at time intervals with $Exp(\lambda k_{xy})$ law (death and reproduction clocks being all independent). Starting with one particle at time 0, it is well-known that there exist two critical parameters, $\lambda_w \le \lambda_s$ such that for $0 \le \lambda < \lambda_w$ the BRW goes extinct almost surely; for $\lambda \in (\lambda_w, \lambda_s]$ there is local extinction but global survival; for $\lambda_s < \lambda$ there is local survival (when $\lambda = \lambda_w$, depending on the cases, there can be global extinction or global survival). The two parameters λ_s and λ_w are called local (or strong) and global (or weak) critical parameters, respectively. These parameters in principle depend on the starting vertex, but they are actually equal for all vertices in the same irreducible class (for more details, also on the critical cases, see Section 2).

The characterization of λ_s in terms of the matrix K has been known for quite a while. Indeed, Pemantle and Stacey proved ([20, Lemma 3.1]) that if the graph $(X, E(X))$ is irreducible, $K = Adj(X)$ (where by $Adj(X)$ we mean the adjacency matrix of $(X, E(X))$) and $M = \lim_{n\to\infty}(k_{xx}^{(2n)})^{1/2n}$, (where $k_{xx}^{(2n)}$ is the (x, x) element of the $2n$-th power of the matrix K), then $\lambda_s = 1/M$. This result has been extended to irreducible BRWs on multigraphs by [3, Theorem 3.1] and then to generic BRWs with rates K by [4, Theorem 4.1]. The behaviour at $\lambda = \lambda_s$ is also understood: [3, Theorem 3.5] and [4, Theorem 4.7] prove that there is almost sure extinction in continuous time, in the case of multigraphs and in general, respectively. The discrete-time case has been described in [25, Theorem 4.1]. The critical behaviour was also investigated independently, with different techniques, in [19].

The characterization of λ_w is more challenging and is the main aim of this paper. If $X = \mathbb{Z}^d$ and $K = Adj(X)$, then $\lambda_w = \lambda_s$: it is also said that there is no weak phase. The absence of weak phase can be found in many cases which, like $\mathbb{Z}^d$, are nonamenable. Nonamenability by itself is neither necessary nor sufficient for $\lambda_w = \lambda_s$, as was proven in [20]. Nevertheless adding some kind of regularity to the graph, like quasi-transitivity (see [22, Theorem 3.1]) or some more general regularity (see [3, Theorem 3.6]) turns nonamenability into an equivalent condition for the absence of weak phase. The presence of the weak phase was first observed on regular trees $\mathbb{T}_d$ (endowed with $K = Adj(\mathbb{T}_d)$) and in that case, $\lambda_w = 1/d$ was computed in [18] (note that vertex transitivity makes λ_w easy to determine, since the total progeny is a Galton-Watson process). When either the graph or K lack regularity, the characterization of λ_w is not obvious. For instance, on Galton-Watson trees, only some bounds for λ_w are known (see [20, 23]).

We note that it is possible to define λ_s using the entire sequence $\{k_{xx}^{(n)}\}_{n\in\mathbb{N}}$: indeed one has $\lambda_s = 1/\limsup_{n\to\infty}(k_{xx}^{(n)})^{1/n}$ ([4, Theorem 4.1]). This result has an intuitive explanation in the fact that the expected value of the cardinality of the set of n-th generation descendants living at x is $\lambda^n k_{xx}^{(n)}$. Then, moving to λ_w it would be natural to conjecture that $\lambda_w = 1/\limsup_{n\to\infty}(\sum_{y\in X} k_{xy}^{(n)})^{1/n}$. The first thing to note is that this conjecture has to be modified since in [4, Example 2] we have a BRW where $\lambda_w = 1/\liminf_{n\to\infty}(\sum_{y\in X} k_{xy}^{(n)})^{1/n}$. We denote by K_w the last limit and then look for conditions guaranteeing that $\lambda_w = 1/K_w$.

The main result of this paper, Theorem 3.2, states that for a generic continuous-time BRW, two uniformity conditions, (U1) and (U2), together are sufficient for $\lambda_w = 1/K_w$ (see Section 3 for the definition of these conditions). We mention here that an adjacency matrix always satisfies (U2) and, in the case of multigraphs with $K = Adj(X)$, it was already known that (U1) was a sufficient condition for $\lambda_w = 1/K_w$ ([3, Theorem 3.2]). For generic BRWs, the proof requires a new and different technique, which heavily relies on multidimensional generating functions and their fixed points (generating function techniques have proven to be excellent tools to identify the extinction probabilities of a BRW, see for instance [4, 13, 14]). So far in the general case it was proven that $\lambda_w = 1/K_w$ is true for BRWs which can be projected onto finite spaces, namely the $\mathcal{F}$-BRWs ([4, Proposition 4.5]). Theorem 3.2 extends this result, since $\mathcal{F}$-BRWs satisfy the two conditions (U1) and (U2), while there are examples of BRWs satisfying the uniformity conditions without being $\mathcal{F}$-BRWs (for instance, a periodic tree-like BRW). Moreover, again a uniformity request, more restrictive than (U1), had proven to be sufficient for $\lambda_w = 1/K_w$ ([4, Proposition 4.6]).

Here is an outline of the paper. In Section 2 we formally define the process, its local and global survival and the associated critical parameters. We also introduce the generating function G of the BRW and recall Theorem 2.3 which links global survival with some properties of G. In Section 3 we first prove Theorem 3.2 and then Theorem 3.4, which gives a sufficient condition for the uniformity condition (U1) to hold. Section 4 is devoted to examples where the equality $\lambda_w = 1/K_w$ holds. The first example is given by periodic tree-like BRWs, which we define in this paper, much in the spirit of [21]. In particular they are a family of self-similar BRWs, which can be neither quasi-transitive nor $\mathcal{F}$-BRWs. To figure an idea of the self-similarity we require, one can think of BRWs on tree-like graphs. Tree-like structures arise naturally in the context of complex networks. In particular, many social and information networks present a large-scale tree-like structure or a hierarchical structure (see [1], [10] and references therein). The global survival of this family of processes could not be treated with the previously known techniques. The second example is given by continuous-time branching processes in varying environment: namely branching processes where individuals breed accordingly to a Poisson process whose parameter depends on the generation. It suffices to interpret generations as space variables and BRW techniques apply. We also show that, even for such a particular law of the process, when $\lambda = \lambda_w$ still global extinction and global survival are both possible. Examples 4.4 and 4.5 show that (U1) is not necessary for $\lambda_w = 1/K_w$ (in the first case $\lambda_w < \lambda_s$, in the second case $\lambda_w = \lambda_s$). Example 4.6 shows that (U2) is not necessary for $\lambda_w = 1/K_w$. Finally Example 4.7 we construct an irreducible BRW where $\lambda_w > 1/K_w$. It is worth mentioning that, in the reducible case, it can even happen that $\lambda_w = 1/K_w$ if the process starts from certain vertices and $\lambda_w > 1/K_w$ if it starts from other vertices (see Example 4.7).

2. Basic definitions

2.1. **The Branching Random Walk and its survival.** Let us consider (X, K) where X is a countable (or finite) set and $K = (k_{xy})_{x,y\in X}$ is a matrix of nonnegative entries such that $\sum_{y\in X} k_{xy} < +\infty$ for all $x \in X$. The couple (X, K) identifies the BRW, that is a family of continuous-time processes, depending on a positive parameter λ, where each particle has an exponentially distributed random lifetime with parameter 1. When $\lambda > 0$ is fixed, for each particle alive at x, there is a clock which rings at $Exp(\lambda k_{xy})$-distributed intervals; each time the clock rings the particle places one newborn at y. We say that the λ-BRW has a death rate 1 and a reproduction rate λk_{xy} from x to y. All the particles behave independently. We observe (see [6, Remark 2.1]) that the assumption of a non-constant death rate does not represent a significant generalization. With a slight abuse of notation, when there is no ambiguity, we omit the dependence on λ and denote by BRW also the process with a fixed λ. It has to be mentioned that under the name BRW one can find, in the literature, several kinds of processes: for instance processes in discrete time, with no death, where parents randomly walk either before or after breeding, on continuous space, in random environment or with multiple types ([9, 11, 15, 16, 17] just to mention a few).

To the process (X, K) we associate a graph (X, E_K) where $(x, y) \in E_K$ if and only if $k_{xy} > 0$. We say that there is a path from x to y, and we write $x \to y$, if it is possible to find a finite sequence $\{x_i\}_{i=0}^n$ (where $n \in \mathbb{N}$) such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E_K$ for all $i = 0, \dots, n-1$. If $x \to y$ and $y \to x$ we write $x \rightleftharpoons y$. Observe that there is always a path of length 0 from x to itself. The equivalence class $[x]$ of x with respect to $\rightleftharpoons$ is called *irreducible class of x*. We say that the matrix K and the BRW (X, K) are *irreducible* if and only if the graph (X, E_K) is *connected*, otherwise we call it *reducible*. The irreducibility of K means that, in the BRW, the progeny of any particle can spread to any site of the graph.

Depending on the initial configuration $\eta_0 \in \mathbb{N}^X$, the BRW $\{\eta_t\}_{t\geq 0}$ can survive in different ways. We consider initial configurations with only one particle placed at a fixed site x: let $\mathbb{P}^x$ be the law of this process.

Definition 2.1.

1. *The process survives locally in $A \subseteq X$ starting from $x \in X$ if* $\mathbf{q}(x, A) := 1 - \mathbb{P}^x(\limsup_{t\to\infty} \sum_{y\in A} \eta_t(y) > 0) < 1$.
2. *The process survives globally starting from x if* $\bar{\mathbf{q}}(x) := \mathbf{q}(x, X) < 1$.

From now on, $\mathbf{q}(x, y)$ will be shorthand for $\mathbf{q}(x, \{y\})$. Often we will simply say that local survival occurs "starting from x" or "at x": in this case we mean that $q(x, x) < 1$. When there is no survival, we say that there is extinction and the fact that extinction occurs with probability 1 will be tacitly understood.

Given $x \in X$, two critical parameters are associated to the continuous-time BRW: the *global survival critical parameter* $\lambda_w(x)$ and the *local survival critical parameter* $\lambda_s(x)$. They are defined as

$$\lambda_w(x) \equiv \lambda_w(x; X, K) := \inf\Big\{\lambda > 0\colon\ \mathbb{P}^x\Big(\sum_{w\in X}\eta_t(w) > 0, \forall t\Big) > 0\Big\},$$
$$\lambda_s(x) \equiv \lambda_s(x; X, K) := \inf\{\lambda > 0\colon\ \mathbb{P}^x\big(\limsup_{t\to\infty}\eta_t(x) > 0\big) > 0\}.$$

By definition, starting with one particle at x, for $\lambda > \lambda_w(x)$ (respectively $\lambda > \lambda_s(x)$) we have *global* or *weak* (respectively *local* or *strong*) survival with positive probability; for $\lambda < \lambda_w(x)$ (respectively $\lambda \le \lambda_s(x)$) we have almost sure global extinction (respectively local extinction). When $\lambda = \lambda_w(x)$ there might be global survival (as in [4, Example 3]) or global extinction (as in the case of $\mathcal{F}$-BRWs, see Section 2.2 for details). The critical parameters depend only on $[x]$: in the irreducible case we will write λ_s and λ_w instead of $\lambda_s(x)$ and $\lambda_w(x)$ respectively.

We define recursively $k^{(n)}_{xy} := \sum_{w\in X} k^{(n-1)}_{xw}k_{wy}$ (where $k^{(0)}_{xy} := \delta_{xy}$); moreover we set $T^n_x := \sum_{y\in X} k^{(n)}_{xy}$ and $\phi^{(n)}_{xy} := \sum_{x_1,\ldots,x_{n-1}\in X\setminus\{y\}} k_{xx_1}k_{x_1x_2}\cdots k_{x_{n-1}y}$; by definition $\phi^0_{xy} := 0$ for all $x, y \in X$. Clearly $\lambda^n k^{(n)}_{xy}$ is the average size of the nth generation at y of the progeny of a particle living at x; $\lambda^n T^n_x$ is the average size of the nth generation of the whole progeny of a particle living at x. Finally, $\lambda^n\phi^{(n)}_{xy}$ is the analog of $k^{(n)}_{xy}$ concerning only paths reaching y for the first time at the n-th step.

We introduce the following geometrical parameters

$$K_s(x, y) \equiv K_s(x, y; X, K) := \limsup_n (k^{(n)}_{xy})^{1/n}, \qquad K_w(x) \equiv K_w(x; X, K) := \liminf_n (T^n_x)^{1/n}.$$

In the rest of the paper, whenever there is no ambiguity, we will omit the dependence on X and K. We recall that $\lambda_s(x) = 1/K_s(x, x)$ ([4, Theorem 3]). Supermultiplicative arguments imply that $K_s(x, x) = \lim_{n\to\infty}(k^{(dn)}_{xx})^{1/dn}$ for some $d \in \mathbb{N}$ hence, for all $x \in X$, we have that $K_s(x, x) \le K_w(x)$. As the critical parameters, also $K_w(x)$ and $K_s(x, y)$ depend only on the irreducible classes $[x]$ and $[y]$. For an irreducible BRW, we write $K_w := K_w(x, y)$ and $K_s := K_s(x)$ for all $x, y \in X$.

Not only $\lambda_s(x)$ and $K_s(x)$ are constant inside each irreducible class, but they also depend only on the restriction of the BRW to the irreducible class $[x]$ (that is, they are the same if computed for the original BRW or for its restriction to $[x]$). This is due to the fact that local survival takes into account paths starting from x and going back to x. That might not be true for $\lambda_w(x)$ and $K_w(x)$ since when we restrict the BRW to $[x]$ we might lose paths from x which exit $[x]$ (in general $\lambda_w(x)$ of the restricted BRW is not smaller than the corresponding parameter for the original BRW and the reversed inequality holds for $K_w(x)$).

Remark 2.2. *While nothing can be said, in general, about the relationship between $\lambda_s(x)$ and $\lambda_s(y)$ for $[x] \neq [y]$, it is always true that if $x \to y$ then $\lambda_w(x) \le \lambda_w(y)$ and $K_w(x) \ge K_w(y)$. One may wonder under which conditions the inequality may be reversed. Given $A \subseteq X$, if we know that the restriction of the BRW to $X \setminus A$ dies out for all $\lambda < \inf\{\lambda_w(y)\colon y \in A\}$, then $\lambda_w(x) \ge \inf\{\lambda_w(y)\colon y \in A\}$ for all $x \in X$; the arguments are similar to those used in the comparison between a BRW and the associated no-death BRW as in* [6, before Proposition 2.1] *or* [5, Section 3.2]. *Applications can be found in Section 4.2.*

The following power series can be useful to identify the critical parameters

$$H(x, y|\lambda) := \sum_{n=0}^{\infty} k^{(n)}_{xy}\lambda^n, \qquad \Theta(x|\lambda) := \sum_{n=0}^{\infty} T^n_x\lambda^n, \qquad \Phi(x, y|\lambda) := \sum_{n=1}^{\infty}\phi^{(n)}_{xy}\lambda^n.$$

Clearly $1/K_s(x, y)$ is the convergence radius of $H(x, y|\lambda)$ and for all $\lambda \in \mathbb{C}$ such that $|\lambda| < 1/\limsup_n(T^n_x)^{1/n}$ we have $\Theta(x|\lambda) = \sum_{y\in Y} H(x, y|\lambda)$. The following relations hold (provided that

λ is such that the involved series converge):

$$\begin{aligned} H(x,y|\lambda) &= \delta_{xy} + \lambda \sum_{w\in X} k_{xw} H(w,y|\lambda) \\ &= \delta_{xy} + \lambda \sum_{w\in X} H(x,w|\lambda) k_{wy} \\ &= \delta_{xy} + \Phi(x,y|\lambda) H(y,y|\lambda), \\ \Theta(x|\lambda) &= 1 + \lambda \sum_{w\in X} k_{xw}\Theta(w|\lambda), \\ \Phi(x,x|\lambda) &= \lambda \sum_{y\in X, y\neq x} k_{xy}\Phi(y,x|\lambda) + \lambda k_{xx}. \end{aligned} \tag{2.1}$$

Moreover if $x, y, w \in X$ are distinct vertices such that every path from x to y contains w then $\Phi(x,y|\lambda) = \Phi(x,w|\lambda)\Phi(w,y|\lambda)$. We note that, since

$$H(x,x|\lambda) = \frac{1}{1-\Phi(x,x|\lambda)}, \qquad \forall \lambda \in \mathbb{C} : |\lambda| < \lambda_s(x), \tag{2.2}$$

we have that $\lambda_s(x) = \max\{\lambda \geq 0 : \Phi(x,x|\lambda) \leq 1\}$ for all $x \in X$ (remember that $\Phi(x,x|\cdot)$ is left-continuous on $[0, \lambda_s(x)]$ and that $1/(1-\Phi(x,x|\lambda))$ has no analytic prolongation in $\lambda_s(x)$).

2.2. **Generating functions and projections.** It is well-know that each continuous-time BRW has a discrete-time counterpart which survives/dies if and only if the original BRW does (see for instance [25, Section 2.2] or [6, Section 2.2 and Remark 2.1]). In this sense the class of continuous-time BRWs can be considered as a subclass of discrete-time BRWs. More precisely, let us denote by $\mu_x(f)$ the probability that a particle living at x places exactly $f(y)$ offspring at site y, before its death. The generating function $G : [0,1]^X \to [0,1]^X$ of the corresponding discrete-time BRW has x coordinate given by

$$G(z|x) := \sum_{f\in\Psi} \mu_x(f) \prod_{y\in X} z(y)^{f(y)}, \tag{2.3}$$

where Ψ is the space of finitely supported functions in $\mathbb{N}^X$. This generating function has been introduced in [4, Section 3] (see also [5, 8, 25] for additional properties). In the case of the discrete-time counterpart of a continuous-time BRW, given $\mathbf{q} \in [0,1]^X$, the x-coordinate of $G(\mathbf{q})$ can be written as

$$G(\mathbf{q}|x) := \frac{1}{1+\lambda K(\mathbf{1}-\mathbf{q})(x)}$$

where $\mathbf{1}(x) = 1$ for all $x \in X$ while $K\mathbf{z}(x) = \sum_{y\in X} k_{xy}\mathbf{z}(y)$ for all $\mathbf{z} \in [0,1]^X$ and $x \in X$. Note that G is continuous with respect to the *pointwise convergence topology* of $[0,1]^X$ and nondecreasing with respect to the usual partial order of $[0,1]^X$ (see [4, Sections 2 and 3] for further details); every time we say that an element of $[0,1]^X$ is the smallest (respectively largest) among a set of points $\mathcal{A}$, we are also implying that it is comparable with every element of the specific set $\mathcal{A}$. We stress that $\mathbf{z} < \mathbf{w}$ means $\mathbf{z}(x) \leq \mathbf{w}(x)$ for all $x \in X$ and $\mathbf{z}(x_0) < \mathbf{w}(x_0)$ for some $x_0 \in X$. Moreover, G represents the 1-step reproductions; we denote by $G^{(n)}$ the generating function associated to the n-step reproductions, which is inductively defined as $G^{(n+1)}(\mathbf{z}) = G^{(n)}(G(\mathbf{z}))$, where $G^{(0)}$ is the identity. Extinction probabilities are fixed points of G and the smallest fixed point is $\bar{\mathbf{q}} = \lim_{n\to\infty} G^{(n)}(\mathbf{0})$: more generally, given a solution of $G(\mathbf{z}) \leq \mathbf{z}$ then $\mathbf{z} \geq \bar{\mathbf{q}}$.

Global survival can be characterized by using G according to the following theorem (for the proof see [25, Theorem 4.1] and [6, Theorem 3.1]; it is based on [4, Proposition 2.1]). For a short proof see [2, Theorem 2.2].

Theorem 2.3. *Consider a BRW and a fixed $x \in X$. The following assertions are equivalent:*

(1) $\bar{\mathbf{q}}(x) < 1$ (i.e. there is global survival starting from x);
(2) there exists $\mathbf{q} \in [0,1]^X$ such that $\mathbf{q}(x) < 1$ and $G(\mathbf{q}) \leq \mathbf{q}$;

(3) there exists $\mathbf{q} \in [0,1]^X$ *such that* $\mathbf{q}(x) < 1$ *and* $G(\mathbf{q}) = \mathbf{q}$.

If $\mathbf{q}$ *satisfies either* (2) *or* (3)*, then* $\mathbf{q} \geq \bar{\mathbf{q}}$*. Moreover, global survival starting from* x *implies that* $\liminf_{n\to\infty} \lambda^n \sum_{y\in X} k^{(n)}_{xy} > 0$ *(or, equivalently,*$\inf_{n\to\infty} \lambda^n \sum_{y\in X} k^{(n)}_{xy} > 0$*).*

As a consequence of this theorem we have that $\lambda_w(x) \geq 1/K_w(x)$; indeed if $\lambda < 1/K_w(x)$ then $\liminf_{n\to\infty} \lambda \sqrt[n]{\sum_{y\in X} k^{(n)}_{xy}} < 1$ hence $\liminf_{n\to\infty} \lambda^n \sum_{y\in X} k^{(n)}_{xy} = 0$. This implies immediately that if $K_w(x) = 0$ then there is extinction for every $\lambda > 0$ whence $\lambda_w(x) = +\infty$. We show that the natural conjecture $\lambda_w(x) = 1/K_w(x)$ is false since, even in the irreducible case, we can have a strict inequality $\lambda_w > 1/K_w$ (see Example 4.7); moreover, we also show that there are reducible BRWs where $\lambda_w(x) > 1/K_w(x)$ and $\lambda_w(y) = 1/K_w(y)$ for some $x, y \in X$ (even though $\lambda_w(x) = \lambda_w(y)$). In [4] the following useful characterization of λ_w has been proven

$$\lambda_w(x) = \inf\{\lambda \in \mathbb{R} \colon \exists \mathbf{v} \in l^\infty(X), \mathbf{v}(x) > 0, \lambda K\mathbf{v} \geq \mathbf{v}\}. \tag{2.4}$$

We recall here the concept of *projection of a BRW* for a continuous-time BRW. It first appeared in [3] for multigraphs, in [4] for continuous-time BRWs and [25] for generic discrete-time BRWs (in these papers it is called *local isomorphism*).

Definition 2.4. *A projection of a BRW* (X, K) *onto* $(Y, \widetilde{K})$ *is a surjective map* $g : X \to Y$ *such that* $\sum_{z\in g^{-1}(y)} k_{xz} = \widetilde{k}_{g(x)y}$ *for all* $x \in X$ *and* $y \in Y$*. If there exists a projection of* (X, K) *onto a finite* $(Y, \widetilde{K})$ *then* (X, K) *is called* $\mathcal{F}$*-BRW.*

This is a particular case of the general definition used for discrete-time processes (see for instance [25, Definition 4.2], [6, Definition 2.2] or [8, Definition 2.3] for the basic properties). The main idea is to label the points in X by using the alphabet Y in such a way that the total rate at which particles at x generate children placed in the set of vertices with "label" y, depends only on the labels y and $g(x)$. If $\{\eta_t\}_{t\geq 0}$ is a realization of the BRW (X, K) then $\{\sum_{z\in g^{-1}(\cdot)} \eta_t(z)\}_{t\geq 0}$ is a realization of the BRW $(Y, \widetilde{K})$.

In particular, there is global survival for (X, K) starting from x if and only if there is global survival for $(Y, \widetilde{K})$ starting from $g(x)$ (for any given $\lambda > 0$) which implies $\lambda_w(x; X, K) = \lambda_w(g(x); Y, \widetilde{K})$ for all $x \in X$ (see for instance [4, proof of Proposition 4.5] or [25, before Theorem 4.3]). On the other hand, $K_w(x; X, K) = K_w(g(x); Y, \widetilde{K})$ for all $x \in X$ since it is easy to prove, by induction on n, that $\sum_{z\in X} k^{(n)}_{xz} = \sum_{y\in Y} \widetilde{k}^{(n)}_{g(x)y}$ for all $n \in \mathbb{N}$, $x \in X$; clearly this also implies that $\lim_{n\to\infty} \sqrt[n]{\sum_{z\in X} k^{(n)}_{xz}}$ exists if and only if $\lim_{n\to\infty} \sqrt[n]{\sum_{y\in Y} \widetilde{k}^{(n)}_{g(x)y}}$ does. It is worth mentioning that when (X, K) is an $\mathcal{F}$-BRW, then $\lambda_w(x) = 1/K_w(x)$ and there is almost sure global extinction starting with one particle at x when $\lambda = \lambda_w(x)$ (see for instance [3, 4, 5, 25]).

We observe that $(x, y) \in E_K$ implies $(g(x), g(y)) \in E_{\widetilde{K}}$ but the converse is not true; thus, if (X, K) is irreducible then $(Y, \widetilde{K})$ is irreducible as well and the converse is not true in general. If (X, K) is projected onto $(Y, \widetilde{K})$ then, for all $\mathbf{q} \in [0,1]^Y$ and $x \in X$,

$$G_X(\mathbf{q} \circ g | x) = G_Y(\mathbf{q} | g(x)) \tag{2.5}$$

that is

$$\frac{1}{1 + \lambda K(\mathbf{1}_X - \mathbf{q} \circ g)(x)} = \frac{1}{1 + \lambda \widetilde{K}(\mathbf{1}_Y - \mathbf{q})(g(x))}$$

where $\mathbf{1}_X(x) = \mathbf{1}_Y(y) := 1$ for all $x \in X$ and $y \in Y$. Moreover the following relation between the probabilities of extinctions hold: $\bar{\mathbf{q}}_X = \bar{\mathbf{q}}_Y \circ g$.

3. Main results

We know that $1/K_w(x) = \lambda_w$, where $K_w(x) = \liminf_{n\to\infty} (\sum_{y\in X} k^{(n)}_{xy})^{1/n}$, holds in many cases (but not in general, according to Example 4.7); in order to find more powerful conditions for this equality to hold, it is natural to define the following.

Definition 3.1. *Given a BRW (X,K) and given $\varepsilon>0$, $x\in X$, we define $N_{x,\varepsilon}:=\{n\in\mathbb{N}\colon \sum_{y\in X}k^{(n)}_{xy}\geq (K_w(x)-\varepsilon)^n\}$ and $n_x(\varepsilon):=\min N_{x,\varepsilon}$. We say that*
(1) condition (U1) is satisfied if for all $\varepsilon>0$, $\sup_{x\in X} n_x(\varepsilon)<+\infty$;
(2) condition (U2) is satisfied if $\inf\{k_{xy}\colon x,y$ such that $k_{xy}>0\}>0$.

The simplest example of a BRW satisfying (U1) and (U2) is any (X,K) where X is finite.

Theorem 3.2. *Let (X,K) be an irreducible, continuous-time BRW such that $\sup_{x\in X}\sum_{y\in X}k_{xy}<+\infty$. If conditions (U1) and (U2) hold then $\lambda_w=1/K_w$.*

Proof. By irreducibility λ_w and K_w do not depend on $x\in X$. Fix $\lambda>1/K_w$: we want to prove that the λ-BRW survives. Choose $\varepsilon>0$ such that $\lambda\,(K_w-\varepsilon)>1+\varepsilon$ and let $n_x:=n_x(\varepsilon)$ for all $x\in X$. We study a discrete-time BRW $(X,\widehat{K})$ where $\hat{k}_{xy}=k^{(n_x)}_{xy}$, for all $x,y\in X$. This means that, in $(X,\widehat{K})$, the 1-step children of a particle living at x are the n_x-th generation descendants of the particle, in (X,K). Clearly if $(X,\widehat{K})$ survives, so does (X,K). The generating function of $(X,\widehat{K})$ is given by

$$\overline{G}(\mathbf{z}|x)=G^{(n_x)}(\mathbf{z}|x)$$

where G is the generating function of (X,K).

Let ν_x be the distribution of the total number of children of a particle at x in $(X,\widehat{K})$. Denote by $\widehat{G}_x$ the 1-dimensional generating function of ν_x which is given by $\widehat{G}_x(t)\equiv\overline{G}\,(t\mathbf{1}_X|x)$. Then the mean number of descendants of a particle at x in $(X,\widehat{K})$ is

$$\widehat{G}'_x(1)=\sum_{n=0}^{\infty}n\nu_x(n)=\lambda^{n_x}T_x^{n_x}>(\lambda(K_w-\varepsilon))^{n_x}\geq 1+\varepsilon.$$

Since (X,K) is a continuous-time BRW, then $G(\mathbf{z}|x)=1/(1+\lambda\sum_{y\in X}k_{xy}(1-\mathbf{z}(y)))$. We can determine the first and second moments of the number of n-th generation descendants of a particle at x, by means of G. Indeed let us denote these moments by $m_{n,x}$ and $m^{(2)}_{n,x}$ respectively. If $\mathbf{z}=t\mathbf{1_X}$ we have

$$m_{n,x}=\frac{d}{dt}G^{(n)}(t\mathbf{1_X}|x)\Big|_{t=1}=\lambda\sum_{y\in X}k_{xy}\frac{d}{dt}G^{(n-1)}(t\mathbf{1_X}|y)\Big|_{t=1}=\lambda\sum_{y\in X}k_{xy}m_{n-1,y}$$

and

$$\begin{aligned}m^{(2)}_{n,x}-m_{n,x}&=\frac{d^2}{dt^2}G^{(n)}(t\mathbf{1_X}|x)\Big|_{t=1}\\&=\sum_{y\in X}\lambda k_{xy}\frac{d^2}{dt^2}G^{(n-1)}(t\mathbf{1_X}|y)\Big|_{t=1}+2\left(\sum_{y\in X}\lambda k_{xy}\frac{d}{dt}G^{(n-1)}(t\mathbf{1_X}|y)\Big|_{t=1}\right)^2.\end{aligned}$$

We denote by $\xi_{n,x}:=(m^{(2)}_{n,x}-m_{n,x})/m^2_{n,x}$. Then for all $x\in X$,

$$\xi_{n,x}:=2+\frac{\sum_{y\in X}\lambda k_{xy}\frac{d^2}{dt^2}G^{(n-1)}(t\mathbf{1_X}|y)\Big|_{t=1}}{\left(\sum_{y\in X}\lambda k_{xy}\frac{d}{dt}G^{(n-1)}(t\mathbf{1_X}|y)\Big|_{t=1}\right)^2}=2+\frac{\sum_{y\in X}\lambda k_{xy}\xi_{n-1,y}m^2_{n-1,y}}{\left(\sum_{y\in X}\lambda k_{xy}m_{n-1,y}\right)^2}.\tag{3.6}$$

Define $\xi_n:=\sup_{x\in X}\xi_{n,x}$; a straightforward computation shows that $\xi_{1,x}=2=\xi_1$ for all $x\in X$. From equation (3.6) we have

$$\xi_n\leq 2+\xi_{n-1}\sup_{x\in X}\frac{\sum_{y\in X}\lambda k_{xy}m^2_{n-1,y}}{\left(\sum_{y\in X}\lambda k_{xy}m_{n-1,y}\right)^2}\leq 2+\xi_{n-1}\sup_{x\in X}\frac{\sum_{y\in X}\lambda k_{xy}m^2_{n-1,y}}{\left(\sum_{y\in X}\sqrt{\lambda k_{xy}\delta}\,m_{n-1,y}\right)^2}\leq 2+\frac{\xi_{n-1}}{\delta}$$

where $\delta := \lambda \inf\{k_{xy} \colon x, y \text{ such that } k_{xy} > 0\}$. Hence by induction

$$\xi_n \leq 2 \sum_{k=0}^{n-1} \left(\frac{1}{\delta}\right)^k,$$

which implies $\xi_{n_x} \leq M := 2\sum_{k=0}^{N-1} \left(\frac{1}{\delta}\right)^k$ where $N := \sup_{x \in X} n_x < +\infty$ by condition (U1).

By Theorem 2.3 in order to prove survival of $(X, \widehat{K})$ it is enough to prove that $\widehat{G}_x(1-t) \leq 1-t$, for some $t \in (0,1)$ and for all $x \in X$. Writing the Taylor expansion of $\widehat{G}_x$ at 1 and using the monotonicity of $\widehat{G}''_x(\cdot)$, we have

$$\begin{aligned}\widehat{G}_x(1-t) &\leq 1 - m_{n_x,x}t + \frac{t^2}{2}\widehat{G}''_x(1) \\ &= 1 - m_{n_x,x}t + \frac{t^2}{2}\left(m^{(2)}_{n_x,x} - m_{n_x,x}\right).\end{aligned}$$

Therefore, $\widehat{G}_x(1-t) \leq 1-t$ for all $x \in X$ if

$$t \leq 2\left(\sup_{x \in X} \frac{m^{(2)}_{n_x,x} - m_{n_x,x}}{m_{n_x,x} - 1}\right)^{-1}.$$

Since $m_{n_x,x} > 1 + \varepsilon$ for any $x \in X$ and

$$\frac{m^{(2)}_{n_x,x} - m_{n_x,x}}{m_{n_x,x} - 1} = \xi_{n_x,x} \frac{m^2_{n_x,x}}{m_{n_x,x} - 1}$$

we get

$$\frac{m^{(2)}_{n_x,x} - m_{n_x,x}}{m_{n_x,x} - 1} \leq M \frac{(\lambda M')^{2N}}{\varepsilon}$$

where $M' := \sup_{x \in X} \sum_{y \in X} k_{xy}$. Thus the constant solution is obtained by choosing a strictly positive $t \leq 2\varepsilon/(M(\lambda M')^{2N})$. □

We note that in the previous theorem irreducibility is not necessary, it suffices that (X, K) is such that $K_w(x)$ does not depend on x. In particular this is the case when $(X, \widetilde{K})$ can be projected onto an irreducible BRW, which leads to the following corollary.

Corollary 3.3. *Let (X, K) be a BRW which can be projected onto an irreducible BRW $(Y, \widetilde{K})$ which satisfies the hypotheses of Theorem 3.2. Then $\lambda_w(x; X, K)$ and $K_w(x; X, K)$ do not depend on $x \in X$ and $\lambda_w(X, K) = 1/K_w(X, K)$.*

Proof. The independence comes from the equalities $\lambda_w(x; X, K) = \lambda_w(g(x); Y, \widetilde{K})$ and $K_w(x; X, K) = K_w(g(x); Y, \widetilde{K})$ and the fact that the two parameters computed on Y do not depend on the vertex by irreducibility. It is enough now to apply Theorem 3.2 to $(Y, \widetilde{K})$. □

As we remarked before, any BRW with X finite satisfies (U1) and (U2), thus the previous corollary is a generalization, in the irreducible case, of the results for $\mathcal{F}$-BRWs ([4, Proposition 4.5] and [5, Corollary 4.10(2)]). Morevoer, our result applies to BRWs which do not satisfy (U2) but can be projected onto a BRW which satisfy it. As an example, consider a BRW on $\mathbb{N}$ such that $k_{n\,n+1} = 1 - 1/2^{n+1}$, $k_{n\,n} := 1/2^{n+1}$ and 0 otherwise. This is an irreducible BRW which satisfies (U1) but not (U2) and can be projected onto a one-point BRW with rate 1; thus Corollary 3.3 applies.

The fact that (U1) is satisfied is the most tricky condition to check in Theorem 3.2; it is therefore interesting to provide sufficient conditions on K under which condition (U_1) holds. This is what we do in the following theorem. A particularly nice application of this theorem, together with Theorem 3.2, is given in Section 4.1 where we study the global survival critical parameter of periodic tree-like BRWs.

Theorem 3.4. *Let (X, K) be a continuous-time BRW such that (U2) holds. Suppose there exist $x_0 \in X$, $Y \subseteq X$, $n_0 \in \mathbb{N}$ such that*

(1) *for all $x \in X$, $\min\{n \in \mathbb{N} \colon k^{(n)}_{xy} > 0$ for some $y \in Y\} \le n_0$;*

(2) *for all $y \in Y$, there exists an injective map $\varphi_y : X \to X$ such that $\varphi_y(x_0) = y$ and $k_{\varphi_y(x)\varphi_y(z)} \ge k_{xz}$ for all $x, z \in X$;*

then (U1) holds.

Proof. Let $\delta = \inf\{k_{xy} \colon x, y \text{ such that } k_{xy} > 0\} > 0$ and put $\gamma_{xy} = k_{xy}/\delta$ for all $x, y \in X$. Define $\widehat{T}^n_x = \sum_{w \in X} \gamma^{(n)}_{xw}$ and $\widehat{K}_w := \liminf_{n\to\infty} \sqrt[n]{\sum_{w\in X} \gamma^{(n)}_{xw}}$. Clearly $\gamma_{xy} \ge 1$ for all $x, y \in X$, $T^n_x = \delta^n \widehat{T}^n_x$ for all $x \in X$ and $n \in \mathbb{N}$, $K_w = \delta \widehat{K}_w$; moreover $n \mapsto \widehat{T}^n_x$ is nondecreasing for all $x \in X$.

Given $x \in X$, let $y(x) \in X$ be a vertex such that $k^{(m_x)}_{xy(x)} > 0$ and $m_x := \min\{n \in \mathbb{N} \colon k^{(n)}_{xy} > 0$ for some $y \in Y\}$. Note that, by the hypothesis (2), we can map x_0 to $y(x)$ and get

$$\widehat{T}^n_{x_0} \le \widehat{T}^n_{y(x)} \le \gamma^{(m_x)}_{xy(x)} \sum_{w\in X} \gamma^{(n)}_{y(x)w} \le \widehat{T}^{n+m_x}_x \le \widehat{T}^{n+n_0}_x, \tag{3.7}$$

for all $n \in \mathbb{N}$ (the second inequality is due to $\gamma^{(m_x)}_{xy(x)} \ge 1$ and the last one holds since $m_x \le n_0$). By irreducibility and the definition of $\widehat{K}_w$, given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\left(\widehat{T}^{n_1}_{x_0}\right)^{\frac{1}{n_1+n_0}} \ge \widehat{K}_w - \frac{\varepsilon}{\delta}. \tag{3.8}$$

Applying (3.7),

$$\left(\widehat{T}^{n_1+n_0}_x\right)^{\frac{1}{n_1+n_0}} \ge \left(\widehat{T}^{n_1}_{x_0}\right)^{\frac{1}{n_1+n_0}} \ge \widehat{K}_w - \frac{\varepsilon}{\delta}.$$

Now,

$$\left(T^{n_1+n_0}_x\right)^{\frac{1}{n_1+n_0}} = \delta \left(\widehat{T}^{n_1+n_0}_x\right)^{\frac{1}{n_1+n_0}} \ge K_w - \varepsilon$$

thus (U1) is satisfied. □

4. Examples

This section is mainly devoted to examples of BRWs where $\lambda_w = 1/K_w$. In Section 4.1 we define the family of periodic tree-like BRWs, where Theorem 3.2 applies. Next, in Section 4.2 we view continuous-time branching processes in varying environment as BRWs and determine their critical parameter. Finally, Section 4.3 provides two examples (Examples 4.4 and 4.5) showing that (U1) is not necessary for $\lambda_w = 1/K_w$, even when $K_w = \lim_{n\to\infty} \sqrt[n]{T^n_x}$, both in the case where there is a weak phase ($\lambda_w < \lambda_s$) and where there is not ($\lambda_w = \lambda_s$). Example 4.6 shows that even in the irreducible case, it can be that $\lim_{n\to\infty} \sqrt[n]{T^n_x}$ does not exist and yet $\lambda_w = 1/K_w$. Example 4.7 is the first example where $\lambda_w > 1/K_w$.

4.1. Periodic tree-like BRWs.

We describe the construction of a class of irreducible BRWs that we call *periodic tree-like BRWs*. Let (I, E_I) be an irreducible finite oriented graph (possibly with loops) and let $\{(B_i, K(i))\}_{i\in I}$ be a family of finite and irreducible BRWs. It might happen that even if $i \ne j$ then $(B_i, K(i))$ and $(B_j, K(j))$ are isomorphic BRWs. Denote by $\{\varphi_{ij}\}_{(i,j)\in E_I}$ a family of one-to-one maps from the domains $\mathrm{D}(\varphi_{ij}) =: B^-_{ij} \subseteq B_i$ onto the images $\mathrm{Im}(\varphi_{ij}) =: B^+_{ij} \subseteq B_j$. The main step of the construction is *attaching an isomorphic copy of B_i to an isomorphic copy of B_j*, that is, identifying a point $x \in B^-_{ij}$ with $\varphi(x) \in B^+_{ij}$ for all $x \in B^-_{ij}$ (each copy of B_i is equipped with the same family of rates $K(i)$).

We start by constructing recursively a labeled tree $(\mathcal{T}, E_{\mathcal{T}})$ which is going to be the skeleton of the BRW. Denote by $i_0 \in I$ the *root* of I and let $\mathcal{T}_0 := \{(0, i_0)\}$ and $E(0) := \emptyset$; the label of $(0, i_0)$ is the projection $\pi((0, i_0)) := i_0$. Suppose we defined $\mathcal{T}_0, \ldots, \mathcal{T}_n$ and the set of edges $E(n) \subseteq \bigcup_{i=1}^n \mathcal{T}_i \times \bigcup_{i=1}^n \mathcal{T}_i$ and suppose we defined the label $\pi(x)$ for all $x \in \bigcup_{i=1}^n \mathcal{T}_i$; then $\mathcal{T}_{n+1} :=$

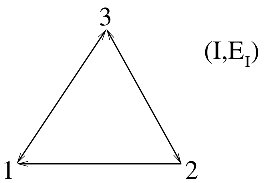

FIGURE 1. The graph (I, E_I).

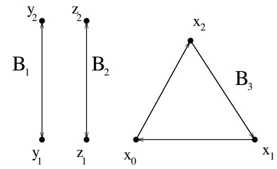

FIGURE 2. The pieces $\{B_i\}_{i\in I}$.

$\{(x,i)\colon x \in \mathcal{T}_n, (\pi(x), i) \in E_I\}$ and $E(n+1) := E(n) \cup \{(x,(x,i))\colon (x,i) \in \mathcal{T}_{n+1}\}$. Moreover $\pi((x,i)) := i$ for all $(x,i) \in \mathcal{T}_{n+1}$. Finally $\mathcal{T} := \bigcup_{n\in\mathbb{N}} \mathcal{T}_n$ and $E_{\mathcal{T}} := \bigcup_{n\in\mathbb{N}} E(n)$. Roughly speaking, to each point in $\mathcal{T}_n$ of label j, we attach the same number of edges exiting j in the graph (I, E_I) and we label the new endpoints accordingly; these new endpoints belong, by definition, to $\mathcal{T}_{n+1}$.

Note that, by construction, for every $i \in I$ there is an infinite number of vertices in $\mathcal{T}$ with label i; moreover, since (I, E_I) is connected and finite, the minimal number of steps required to go from i to i_0 is bounded from above by some n_1 with respect to i. The same bound holds for the minimal number of steps required to go from a point with label i in $\mathcal{T}$ to the closest point with label i_0.

We can construct now the periodic tree-like BRW. Let $\{(B_x, K(x))\}_{x\in\mathcal{T}}$ be a family of BRWs such that $(B_x, K(x))$ is an isomorphic copy of $(B_{\pi(x)}, K(\pi(x)))$ (we suppose that $B_x \cap B_y = \emptyset$ for all $x \neq y$). For every $(x,y) \in \mathcal{T}$, we attach B_x to B_y as described above. The resulting irreducible BRW is denoted by (X,K); note that, for this BRW, condition (U2) holds since it is constructed by means of a finite number of types of BRWs.

Let us choose x_0 in the *root set* $B_{(0,i_0)}$ where $(0, i_0)$ is the root of $\mathcal{T}$. We denote by Y the set of the copies of the vertex x_0 (collected inside the copies of the set $B_{(0,i_0)}$ inside X, namely $\{B_x\}_{x\in\mathcal{T}\colon \pi(x)=i_0}$). Given any graph, let $d(x,y)$ be the minimal number of steps required to go from x to y. Since $\{B_i\}_{i\in I}$ is a finite family of finite sets we have that $n_2 := \sup_{i\in I, x,y\in B_i} d(x,y) < +\infty$. Suppose we start from a vertex $x \in X$ and we want to reach the set Y; by construction, x belongs to a copy of B_{j_0} for some $j_0 \in I$. Let $\{j_0, \ldots, j_k \equiv i_0\}$ be the shortest path from j_0 to i_0 in (I, E_I); clearly, $k \leq n_1$. In order to reach Y from x it is enough to exit the copy of B_{j_0}, to cross a copy of $B_{j_1}, \ldots, B_{j_{k-1}}$ (in this order), to enter a copy of $B_{j_k} \equiv B_{i_0}$ and then to reach the copy of x_0 inside the last copy of B_{j_k}. Each one of these actions requires at most n_2 steps. This implies that the length of the shortest path from x to Y is at most $(n_1+1)n_2 =: n_0$. Then Theorem 3.4 applies, and since $\sup_{x\in X} \sum_{y\in Y} k_{xy} < +\infty$ we can apply Theorem 3.2 and get $\lambda_w = 1/K_w$.

Here is an explicit example of such a construction: define $I := \{1,2,3\}$, $i_0 := 3$ and consider the graph (I, E_I) pictured in Figure 1. The corresponding pieces are shown in Figure 2; Figure 3 explains how to join the pieces. The construction of the labeled tree $(\mathcal{T}, E_{\mathcal{T}})$ can be found in Figure 4 and the final graph associated to the periodic tree-like BRW is shown in Figure 5 where we denoted by x_0 the "actual x_0" contained in the root set and all its copies.

4.2. **Continuous-time branching process in varying environment.** Consider a continuous-time branching process where the breeding laws depend on the generation (while the death rate is always equal to 1); this is called a *branching process in varying environment* or *BPVE*. To be precise, pick a sequence $\{k_n\}_{n\in\mathbb{N}}$ of strictly positive real numbers. The reproduction rate of a particle of generation n is λk_n. By interpreting generations as space, the behavior of this process is equivalent to the global behavior of a BRW on $\mathbb{N}$ where $k_{nm} = k_n$ if $n = m-1$ and 0 otherwise.

We denote by $\lambda_w(n)$, as usual, the weak critical parameter of the associated BRW on $\mathbb{N}$ starting from n, which is the only critical parameter for this process ($\lambda_s(n) = +\infty$ for all n since there is clearly local extinction for every n). Being the rates strictly positive, there is always a positive probability of reaching n from 0, hence $\lambda_w(0) \leq \lambda_w(n)$. On the other hand in order to survive starting from 0 the process has to pass by n whence $\lambda_w(n) \leq \lambda_w(0)$. Thus $\lambda_w(n) = \lambda_w(0)$ for every

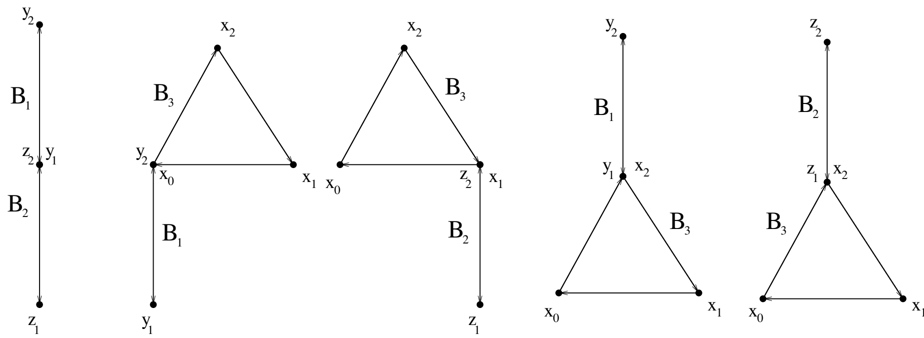

FIGURE 3. How to join pieces.

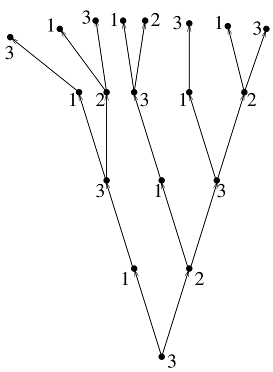

FIGURE 4. The labeled tree $(\mathcal{T}, E_{\mathcal{T}})$.

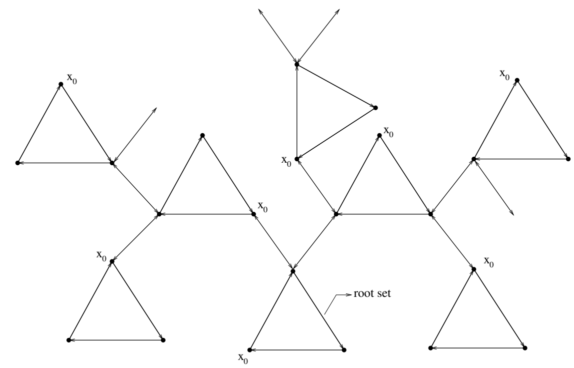

FIGURE 5. The final periodic tree-like BRW.

$n \in \mathbb{N}$. The generating function of the associated BRW is

$$G(\mathbf{q}|n) = \frac{1}{1 + \lambda k_n (1 - \mathbf{q}(n+1))}$$

for all $\mathbf{q} \in [0,1]^{\mathbb{N}}$. Easy computations show that $K_w = \liminf_{n\to\infty} \sqrt[n]{\prod_{i=0}^{n-1} k_i}$. We are going to prove in the next theorem that $\lambda_w = 1/K_w$. We point out that in this case if $K_w = +\infty$ then $\lambda_w = 0$ and there is survival for every $\lambda > 0$ (while it is always true that if $K_w = 0$ then $\lambda_w = +\infty$).

Theorem 4.1. *Let $\{X_t\}_{t\geq 0}$ be a continuous-time branching process in varying environment with reproduction rates $\{\lambda k_n\}_{n\in\mathbb{N}}$. Then $\lambda_w = 1/K_w$.*

Proof. We study the survival of the branching process by analyzing its associated continuous-time BRW. Assume that $K_w \in (0, \infty]$. Since $\lambda_w \geq 1/K_w$ for any BRW we just need to show that the reversed inequality holds. It follows from Theorem 2.3(2) that for this purpose it suffices to find, for any $\lambda > 1/K_w$ (where $1/K_w = 0$ if $K_w = +\infty$), a solution of $\lambda K\mathbf{v} \geq \mathbf{v}/(\mathbf{1} - \mathbf{v})$ where $\mathbf{v} \in [0,1]^{\mathbb{N}}$ such that $\mathbf{v} > \mathbf{0}$. Recall that $K\mathbf{v}(n) = k_n \mathbf{v}(n+1)$ for any n. Fix $\lambda > 1/K_w$ and choose $\rho \in (1/K_w, \lambda)$. Then define $\mathbf{v}(n) = t/(\rho^n \prod_{i=0}^{n-1} k_i)$ where $t \leq (1 - \frac{\rho}{\lambda})/M$ is fixed and M is an upper bound of the sequence $\{1/(\rho^n \prod_{i=0}^{n-1} k_i)\}_{n\in\mathbb{N}}$ (whose existence is guaranteed by the fact that $\mathbf{v}(n)/t$

converges to 0 as $n \to \infty$ by our choice of ρ). Now

$$\begin{aligned}\lambda K\mathbf{v}(n) &= \frac{\lambda k_n t}{\rho^{n+1}\prod_{i=0}^{n} k_i} = \frac{\lambda}{\rho}\cdot\frac{t}{\rho^n\prod_{i=0}^{n-1} k_i} \geq \frac{1}{1-tM}\cdot\frac{t}{\rho^n\prod_{i=0}^{n-1} k_i} \\ &\geq \frac{1}{1-t/(\rho^n\prod_{i=0}^{n-1} k_i)}\cdot\frac{t}{\rho^n\prod_{i=0}^{n-1} k_i} = \frac{\mathbf{v}(n)}{1-\mathbf{v}(n)}.\end{aligned} \tag{4.9}$$

□

Remark 4.2. *The identification of the critical parameter* $\lambda_w = 1/\liminf_{n\to\infty}\sqrt[n]{\prod_{i=0}^{n-1} k_i}$ *does not tell us anything about the critical behavior when* $\lambda = \lambda_w$*. There is not just one possible scenario: in some cases there might be extinction while in others there might be survival.*

Indeed, suppose that $\lim_{n\to\infty} k_n = k \in (0,+\infty)$ *and* $k \geq k_n$ *for every* $n \geq n_0$ *(for some* $n_0 \in \mathbb{N}$*). Then by a simple coupling argument when* $\lambda = \lambda_w = 1/k$ *the BRW starting from* n_0 *is stochastically bounded from above by a BRW with rightward constant rate* 1 *which is well-known to die out.*

Conversely, consider $k_n := (1+1/(n+1))^2$*; then* $\lambda_w = 1$*. Take* $\lambda = 1$ *and define* $\mathbf{v}(n) := 1/(n+2)$ *for all* $n \in \mathbb{N}$*. We claim that* $G(\mathbf{1}-\mathbf{v}) \leq \mathbf{1}-\mathbf{v}$*, that is* $\lambda K\mathbf{v} \geq \mathbf{v}/(\mathbf{1}-\mathbf{v})$*; indeed*

$$\begin{aligned}\lambda K\mathbf{v}(n) - \frac{\mathbf{v}(n)}{1-\mathbf{v}(n)} &= k_n\mathbf{v}(n+1) - \frac{1/(n+2)}{1-1/(n+2)} = \frac{(1+1/(n+1))^2}{n+3} - \frac{1}{n+1} \\ &= \frac{1}{n+1}\Big(\frac{(n+2)^2}{(n+1)(n+3)} - 1\Big) > 0.\end{aligned}$$

Hence by using $\mathbf{q} := \mathbf{1}-\mathbf{v}$*, according to Theorem 2.3 there is global survival starting from any* $n \in \mathbb{N}$*.*

4.3. **Other examples.** One of the main technical tools that we need in this section is the following theorem (see [7, Theorem 2.2]), which states that there cannot be a weak phase on slowly growing BRWs.

Theorem 4.3. *Let* (X,K) *be a continuous time non-oriented BRW and let* $x_0 \in X$*. Suppose that there exists* $\kappa \in (0,1]^X$ *and* $\{c_n\}_{n\in\mathbb{N}}$ *such that, for all* $n \in \mathbb{N}$

1. $\kappa(y)/\kappa(x_0) \leq c_n$ $\forall y \in B(x_0,n)$
2. $\kappa(x)k_{xy} = \kappa(y)k_{yx}$ $\forall x,y \in X$,

where $B(x,n)$ *is the ball of center* x *and radius* n *w.r.t. to the natural distance of the graph* (X,E_K)*. If* $\lim_{n\to\infty} c_n^{1/n} = 1$ *and* $\lim_{n\to\infty}|B(x_0,n)|^{1/n} = 1$*, then* $K_s(x_0,x_0) = K_w(x_0)$ *and there is no pure global survival starting from* x_0*. Moreover, in this case,* $\liminf_{n\to\infty}\sqrt[n]{T_x^n} = \limsup_{n\to\infty}\sqrt[n]{T_x^n}$*.*

In the proof of [7, Theorem 2.2] it was not explicitly mentioned that the $\lim_{n\to\infty}\sqrt[n]{T_x^n}$ exists, but it follows easily by noting that $\liminf_{n\to\infty}\sqrt[2n]{k_{x_0x_0}^{(2n)}} = \limsup_{n\to\infty}\sqrt[2n]{k_{x_0x_0}^{(2n)}}$.

The following is an example where condition (U1) is not satisfied, nevertheless $\lambda_w = 1/K_w < \lambda_s$ where, in this case, $K_w = \lim_{n\to\infty}\sqrt[n]{T_x^n}$.

Example 4.4. *Consider the irreducible continuous-time BRW on the graph obtained by identifying each vertex of* $\mathbb{T}_d$ $(d \geq 3)$ *with the vertex 0 of a copy of* $\mathbb{Z}$ *(each vertex is attached to a different copy of* $\mathbb{Z}$*) and let the rates matrix be the adjacency matrix of the graph. Denote this BRW by* (X,K_1)*. We claim that* (X,K_1) *can be projected into a BRW on* $\mathbb{Z}$ *with the following rates. Let* K *be defined by* $k_{00} := d, k_{n\,n+1} := 1 =: k_{n+1\,n}$ *and* 0 *otherwise. Denote this new BRW by* $(\mathbb{Z},K)$*.*

Since $k_{nm} = 1$ *for* $n \neq m$ *whenever* $|m-n| = 1$*, we conclude that* $\kappa(n)k_{nm} = \kappa(m)k_{mn}$ $\forall n,m \in \mathbb{N}$ *if and only if* $\kappa(n) = 1$ $\forall n$*. Then condition (2) in Theorem 4.3 is satisfied. Condition (1) in Theorem 4.3 is satisfied for any choice of* n_0 *by taking* $c_n = 1$ *for any* $n \in \mathbb{N}$*. Since* $c_n^{1/n} = 1$ *and* $|B(n_0,n)|^{1/n} \to 1$ *as* $n \to +\infty$ *for any choice of* n_0*, we conclude that* $K_s(\mathbb{Z},K) = K_w(\mathbb{Z},K) \geq 1/d$*, where* $K_w(\mathbb{Z},K) = \lim_{n\to\infty}\sqrt[n]{\sum_{y\in X} k_{n_0y}^{(n)}}$ *(the existence of the limit is guaranteed by Theorem 4.3).*

Since projecting a BRW does not modified the value of the critical weak parameter neither the value of K_w, we conclude that $\lambda_w(X, K_1) = \lambda_w(\mathbb{Z}, K) = 1/K_w(\mathbb{Z}, K) = 1/K_w(X, K_1) \le 1/d$.

Clearly, by going along a copy of $\mathbb{Z}$ in X at arbitrarily long distance from the junction with $\mathbb{T}_d$, we have that (U1) is not satisfied; more precisely, if $x(n)$ is a point in a copy of $\mathbb{Z}$ in X at distance n from the junction with $\mathbb{T}_d$, then $\sqrt[n]{\sum_{y\in X} k^{(n)}(x(n), y)} = 2 < d - \varepsilon \le K_w(\mathbb{Z}, K) - \varepsilon$ for all $\varepsilon \in (0, 1)$.

Finally, we recall that the critical strong parameter can change in a projection; indeed it is not difficult to see that $\lambda_s(X, K_1) = 1/(2\sqrt{d}) > 1/d \ge \lambda_w(X, K_1)$ for all $d \ge 3$. This can be proven by using the characterization $\lambda_s(X, K_1) = \max\{\lambda\colon \Phi(x, x|\lambda) \le 1\}$ where, by standard generating function computations (see equation (2.1) *or the proof of* [24, Lemma 1.24]*),*

$$\Phi(x, x|\lambda) = 1 - \frac{d-2}{d-1}\sqrt{1-4\lambda^2} - \frac{d}{2(d-1)}\sqrt{1-4\lambda^2 d},$$

x being a vertex in the tree $\mathbb{T}_d$.

In the following example, condition (U1) is not satisfied, nevertheless $\lambda_w = 1/K_w = \lambda_s$; as before $K_w = \lim_{n\to\infty} \sqrt[n]{T^n_x}$.

Example 4.5. *Consider the BRW obtained by attaching d copies of the graph $\mathbb{N}$ to a common origin 0 and by defining the rates according to the adjacency matrix as in the previous example. As before, this BRW can be projected onto $\mathbb{N}$ but we discuss this example without any projection. As before, the key for computing λ_w is to observe that Theorem 4.3 applies by taking $\kappa(x) = c_n = 1$ for all $n \in \mathbb{N}$ and every vertex x. This implies $\lambda_w = 1/K_w = 1/K_s = \lambda_s$ and $K_w = \lim_{n\to\infty} \sqrt[n]{\sum_{y\in X} k^{(n)}_{n_0 y}}$. What we have to do now is to compute K_s; we can do that by using the same technique as before. In this case*

$$\Phi(0, 0|\lambda) = d\frac{1-\sqrt{1-4\lambda^2}}{2}$$

which implies $1/K_s = \max\{\lambda\colon \Phi(0, 0|\lambda) \le 1\} = \sqrt{1/d - 1/d^2} = \sqrt{d-1}/d$. Thus $K_w = k_s = d/\sqrt{d-1}$; thus, as before, if $x(n)$ is a point in a copy of $\mathbb{Z}$ at distance n from the origin then $\sqrt[n]{\sum_{y\in X_1} k^{(n)}(x(n), y)} = 2 < K_w - \varepsilon$ for all $\varepsilon \in (0, K_w - 2)$. Whence (U1) does not hold.

The following is an example of an irreducible BRW on $\mathbb{N}$, where $\lambda_w = 1/K_w$ and $\lim_{n\to\infty} \sqrt[n]{T^n_x}$ does not exist. The idea is to pick outgoing rates which are either 1 or 2 (alternating long stretches of 1s and 2s in order to keep the sequence oscillating). Then we add rates from each n to 0, so that the BRW is irreducible. If these rates are small enough, their presence will neither affect λ_w nor K_w.

Example 4.6. *Consider the BRW on $X = \mathbb{N}$ with the following rates. Let K be defined by $k_{n n+1} := k_n \ge \delta > 0, k_{n0} := \varepsilon_n$ and 0 otherwise. Observe that $\{T^n_0/\delta^n\}_{n\in\mathbb{N}}$ is nondecreasing; indeed, since $k_x/\delta \ge 1$,*

$$\frac{T^n_0}{\delta^n} = \sum_{x\in\mathbb{N}} \frac{k^{(n)}_{0x}}{\delta^n} \le \sum_{x\in\mathbb{N}} \frac{k^{(n)}_{0x}}{\delta^n} \cdot \frac{k_x}{\delta} \le \sum_{y\in\mathbb{N}} \frac{k^{(n+1)}_{0y}}{\delta^{n+1}} \le \frac{T^{n+1}_0}{\delta^{n+1}}.$$

Choose the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ in such a way that $\beta := \sum_{i=0}^{+\infty} \varepsilon_i(\prod_{j=0}^{i-1} k_j)/\delta^{i+1} < 1$. Since $T^0_0 := 1$ and

$$\frac{T^n_0}{\delta^n} = \prod_{j=0}^{n-1} \frac{k_j}{\delta} + \sum_{i=0}^{n-2} \frac{\varepsilon_i}{\delta}\Big(\prod_{j=0}^{i-1} \frac{k_j}{\delta}\Big)\,\frac{T^{n-i-1}_0}{\delta^{n-i-1}} \le \prod_{j=0}^{n-1} \frac{k_j}{\delta} + \sum_{i=0}^{n-2} \frac{\varepsilon_i}{\delta}\Big(\prod_{j=0}^{i-1} \frac{k_j}{\delta}\Big)\,\frac{T^n_0}{\delta^n}$$

we get

$$\prod_{j=0}^{n-1} k_j \le T^n_0 \le \frac{\prod_{j=0}^{n-1} k_j}{1-\beta}. \tag{4.10}$$

Therefore

$$K_w = \liminf_{n\to\infty} (T_0^n)^{1/n} = \liminf_{n\to\infty} \left(\prod_{j=0}^{n-1} k_j\right)^{1/n}. \tag{4.11}$$

A straightforward computation shows that $\mathbf{v}(n) := 1/(\lambda^n \prod_{j=0}^{n-1} k_j)$ *is a solution in* $l^\infty(\mathbb{N})$ *with* $\mathbf{v} > 0$ *of* $\lambda K\mathbf{v} \geq \mathbf{v}$ *whenever* $\lambda > 1/K_w$*; equation (2.4) yields* $\lambda_w \leq 1/K_w$*. Since* $\lambda_w \geq 1/K_w$*, we may conclude that* $\lambda_w = 1/K_w$*. In [4], the following choice for the sequence* $\{k_n\}_{n\in\mathbb{N}}$ *is made. Define* $a_n := \lceil \log 2/\log(1+1/n)\rceil, b_n := \lceil \log 2/(\log 2 \log(2-1/n))\rceil$ *and* $\{c_n\}_{n\in\mathbb{N}}$ *recursively by* $c_1, c_{2r} = a_{2r}c_{2r-1}, c_{2r+1} = b_{2r+1}c_{2r}$ *for any* $r \geq 1$*. Let* $k_i := 1$ *if* $i \in (c_{2r-1}, c_{2r}]$ *(for some* $r \in \mathbb{N}$*) and* $k_i = 2$ *if* $i \in (c_{2r}, c_{2r-1}]$ *(for some* $r \in \mathbb{N}$*). It follows from this definition that* $\liminf_{n\to\infty} \left(\prod_{j=0}^{n-1} k_j\right)^{1/n} = 1$ *and that* $\limsup_{n\to\infty} \left(\prod_{j=0}^{n-1} k_j\right)^{1/n} = 2$*. Therefore,* $\liminf_{n\to\infty} (T_0^n)^{1/n} = 1$ *(note that in this last explicit example* $\sum_{y\in X} k_{xy}^{(n)} \in [1, 2^n]$ *for all* $x \in X$ *and* $n \geq 1$*, thus Theorem 3.2 applies).*

In the following example we have an irreducible BRW where $\lambda_w > 1/K_w$; moreover, we also provide an example of a reducible BRW where $\lambda_w(x) > 1/K_w(x)$ and $\lambda_w(y) = 1/K_w(y)$ for some $x, y \in X$ (nevertheless $\lambda_w(x) = \lambda_w(y)$ for all $x, y \in X$).

Example 4.7. *To avoid confusion in the notation, in this example we denote the rates by* $k_{x,y}$ *instead of* k_{xy}*. For simplicity we start with a reducible BRW on* $\mathbb{Z}$*: for all* $n \in \mathbb{N}$ *we take* $k_{n,n+1} \in \{1, 2\}$ *as in Example 4.6 in such a way that* $\liminf_{n\to\infty} \sqrt[n]{\prod_{i=0}^{n-1} k_{i,i+1}} = 1 < 2 = \limsup_{n\to\infty} \sqrt[n]{\prod_{i=0}^{n-1} k_{i,i+1}}$*, while* $k_{-n,-n-1} := 3 - k_{n,n+1}$ *for all* $n \in \mathbb{N}$*. Note that* $k_{-n,-n-1}k_{n,n+1} = 2$ *for all* $n \in \mathbb{N}$*; thus*

$$\prod_{i=0}^{n-1} k_{-i,-i-1} = \frac{2^n}{\prod_{i=0}^{n-1} k_{i,i+1}}.$$

Hence, $\liminf_{n\to\infty} \sqrt[n]{\prod_{i=0}^{n-1} k_{-i,-i-1}} = 1 < 2 = \limsup_{n\to\infty} \sqrt[n]{\prod_{i=0}^{n-1} k_{-i,-i-1}}$*. Applying Theorem 4.1 to the process restricted to* $\mathbb{N}$ *and to the process restricted to* $-\mathbb{N} := \{-n \colon n \in \mathbb{N}\}$*, we have that* $\lambda_w(n) = 1 = 1/K_w(n)$ *for all* $n \neq 0$*. According to Remark 2.2, on the one hand* $\lambda_w(0) \leq \lambda_w(1)$ *(since* $0 \to 1$*), on the other hand (by taking* $A := \mathbb{Z} \setminus \{0\}$ *in Remark 2.2)* $\lambda_w(0) \geq \lambda_w(1)$*; hence* $\lambda_w(0) = 1$*. In order to compute* $K_w(0)$ *we note that*

$$\sum_{i\in\mathbb{Z}} k_{0,i}^{(n)} = \prod_{i=0}^{n-1} k_{i,i+1} + \prod_{i=0}^{n-1} k_{-i,-i-1} = \prod_{i=0}^{n-1} k_{i,i+1} + \frac{2^n}{\prod_{i=0}^{n-1} k_{i,i+1}} \geq 2^{1+n/2},$$

whence $K_w(0) = \liminf_{n\to\infty} \sqrt[n]{\sum_{i\in\mathbb{Z}} k_{0,i}^{(n)}} \geq \sqrt{2}$*. This implies that* $\lambda_w(0) > 1/\sqrt{2} \geq 1/K_w(0)$*; thus we have a reducible example where* $\lambda_w(0) > 1/K_w(0)$ *and* $\lambda_w(n) = 1/K_w(n)$ *for all* $n \geq 1$*.*

Let us modify this BRW to make it irreducible, as we did in Example 4.6. We add $k_{n,0} := \varepsilon_n$*, such that* $\beta_+ := \sum_{i=0}^{+\infty} \varepsilon_i (\prod_{j=0}^{i-1} k_{j,j+1}) < 1/3$ *and* $\beta_- := \sum_{i=0}^{+\infty} \varepsilon_{-i} (\prod_{j=0}^{i-1} k_{-j,-j-1}) < 1/3$*. Note that, as in Example 4.6,* $\{T_0^n\}_{n\in\mathbb{N}}$ *is nondecreasing (in this case* $\delta = 1$*); moreover* $T_0^0 = 1$ *and*

$$\begin{aligned} T_0^n &= \prod_{j=0}^{n-1} k_{j,j+1} + \prod_{j=0}^{n-1} k_{-j,-j-1} + \sum_{i=0}^{n-2} \left[\varepsilon_i\Big(\prod_{j=0}^{i-1} k_{j,j+1}\Big) + \varepsilon_{-i}\Big(\prod_{j=0}^{i-1} k_{-j,-j-1}\Big)\right] T_0^{n-i-1} \\ &\leq \prod_{j=0}^{n-1} k_{j,j+1} + \prod_{j=0}^{n-1} k_{-j,-j-1} + \sum_{i=0}^{n-2} \left[\varepsilon_i\Big(\prod_{j=0}^{i-1} k_{j,j+1}\Big) + \varepsilon_{-i}\Big(\prod_{j=0}^{i-1} k_{-j,-j-1}\Big)\right] T_0^n. \end{aligned}$$

Hence

$$\prod_{j=0}^{n-1} k_{j,j+1} + \prod_{j=0}^{n-1} k_{-j,-j-1} \leq T_0^n \leq \frac{\prod_{j=0}^{n-1} k_{j,j+1} + \prod_{j=0}^{n-1} k_{-j,-j-1}}{1 - \beta_+ - \beta_-},$$

and $K_w = \liminf_{n\to\infty} \left(\prod_{j=0}^{n-1} k_{j,j+1} + \prod_{j=0}^{n-1} k_{-j,-j-1}\right)^{1/n} \geq \sqrt{2}$. *We prove that* $\lambda_w := \lambda_w(\mathbb{Z}, K) = 1$*; indeed, suppose, by contradiction, that* $\lambda_w < 1$. *For any fixed* $\lambda \in (\lambda_w, 1)$*, equation* (2.4) *guarantees the existence* $\mathbf{v} \in l^\infty(\mathbb{Z})$ *such that* $\mathbf{v}(0) > 0$ *and* $\lambda K\mathbf{v} \geq \mathbf{v}$. *In particular* $\mathbf{v}_+$ *(defined by* $\mathbf{v}_+(n) := \mathbf{v}(n)$ *for all* $n \in \mathbb{N}$*) satisfies* $\lambda K^+\mathbf{v}_+ \geq \mathbf{v}_+$ *where* $k^+_{0,1} = k_{0,1} + (3 - k_{0,1})\mathbf{v}(-1)/\mathbf{v}(1)$ *and* $k^+_{i,j} = k_{i,j}$ *for* $(i,j) \in \mathbb{N}^2 \setminus \{(0,1)\}$*; this would imply* $\lambda_w(\mathbb{N}, K^+) < 1$. *Similarly,* $\mathbf{v}_-$ *(defined by* $\mathbf{v}_-(n) := \mathbf{v}(-n)$ *for all* $n \in \mathbb{N}$*) satisfies* $\lambda K^-\mathbf{v}_- \geq \mathbf{v}_-$ *where* $k^-_{0,1} = k_{0,-1} + (3 - k_{0,-1})\mathbf{v}(1)/\mathbf{v}(-1)$ *and* $k^-_{i,j} = k_{-i,-j}$ *for* $(i,j) \in \mathbb{N}^2 \setminus \{(0,1)\}$*; this would imply* $\lambda_w(\mathbb{N}, K^-) < 1$. *Note that* $\min(k^+_{0,1}, k^-_{0,1}) \leq 3$. *Suppose, without loss of generality that* $k^+_{0,1} \leq 3$. *In this case,* $\sum_{i=0}^{+\infty} \varepsilon_i(\prod_{j=0}^{i-1} k^+_{j,j+1}) \leq (3/k_{0,1}) \sum_{i=0}^{+\infty} \varepsilon_i(\prod_{j=0}^{i-1} k_{j,j+1}) < 1$*, whence, by using the same above arguments,* $K_w(\mathbb{N}, K^+) = \liminf_{n\to\infty} \sqrt[n]{\prod_{j=0}^{n-1} k^+_{j,j+1}} = 1$*; thus* $K_w(\mathbb{N}, K^+) = 1 > \lambda_w(\mathbb{N}, K^+)$ *and this is a contradiction (as a consequence of Theorem 2.3).*

5. Final Remarks

As we already mentioned, the local critical value $\lambda_s(x)$ admits a complete and general description in terms of the matrix K, namely, $\lambda_s(x) = 1/\limsup_{n\to\infty} \sqrt[n]{k^{(n)}_{xx}}$. Moreover, at the global critical value there is always almost sure local extinction starting from x (see [3, 4, 25]). The global behavior is somehow more elusive. On the one hand, in [4] it has been shown that at the global critical value there might be almost sure global extinction starting from x as well as global survival (even in the irreducible case). On the other hand, a general description of $\lambda_w(x)$ is still missing even for irreducible BRWs except for some classes of processes (some of them described in this paper). The natural candidate, $1/\limsup_{n\to\infty} \sqrt[n]{\sum_{y\in X} k^{(n)}_{xy}}$, has been proven wrong in [4] (see also Example 4.6 for an irreducible BRW). The results in [4] suggested a new candidate, namely $1/\liminf_{n\to\infty} \sqrt[n]{\sum_{y\in X} k^{(n)}_{xy}}$, which coincides with $\lambda_w(x)$ in many cases, such as BRWs satisfying Theorem 3.2 and Examples 4.4 and 4.5. In Example 4.7 we showed that this cannot be a general characterization, not even for irreducible BRWs. Hence, even though general characterizations for $\lambda_w(x)$ in terms of functional inequalities are known (see equation (2.4)), the search for an explicit expression, similar to the one available for $\lambda_s(x)$, in the case of $\lambda_w(X)$ is still open.

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