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A GENERATING FUNCTION APPROACH TO BRANCHING RANDOM WALKS

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Abstract. It is well known that the behaviour of a branching process is completely described by the generating function of the offspring law and its fixed points. Branching random walks are a natural generalization of branching processes: a branching process can be seen as a one-dimensional branching random walk. We define a multidimensional generating function associated to a given branching random walk. The present paper investigates the similarities and the differences of the generating functions, their fixed points and the implications on the underlying stochastic process, between the one-dimensional (branching process) and the multidimensional case (branching random walk). In particular, we show that the generating function of a branching random walk can have uncountably many fixed points and a fixed point may not be an extinction probability, even in the irreducible case (extinction probabilities are always fixed points). Moreover, the generating function might not be a convex function. We also study how the behaviour of a branching random walk is affected by local modifications of the process. As a corollary, we describe a general procedure by which we can modify a continuous-time branching random walk which has a weak phase and turn it into a continuous-time branching random walk which has strong local survival for large or small values of the parameter and non-strong local survival for intermediate values of the parameter.

Keywords: branching random walk, branching process, strong local survival, generating function, fixed point, extinction probability.
AMS subject classification: 60J05, 60J80.

1. Introduction

A branching process, or Galton-Watson process (see [10]), is a process where a particle dies and gives birth to a random number of offspring, according to a given offspring law ρ ($\rho(n)$ being the probability of having exactly n children). Different particles breed independently, all according to ρ. Unless $\rho(0) = 0$, it is not completely trivial to tell whether the process survives (with positive probability) or it goes extinct (almost surely). This question can be answered by looking at the fixed points of the generating function $H(z) = \sum_{n=0}^{\infty} \rho(n) z^n$, which is defined for $z \in [0, 1]$. There is almost sure extinction if and only if H has only the fixed point $z = 1$. If there are two fixed points, namely $z = 1$ and $z = \bar{q} \in (0, 1)$, then there is extinction with probability $\bar{q}$ and survival with probability $1 - \bar{q}$. If we require that $\rho(1) < 1$ then the generating function H, being monotone and convex, has at most two fixed points, so this description settles all the possibilities for the branching process.

A branching random walk (BRW hereafter) is a process where particles are described by their location $x \in X$, where X is an at most countable set (X is usually interpreted as a spatial variable, but can also be seen as a "type", see for instance [13]). Particles at site $x \in X$ are replaced by a random number of children, which are placed at various locations on X. This class of processes (in continuous and discrete time) has been studied by many authors (see [1, 9, 12, 14, 15, 16, 17, 18, 20] just to mention a few); a survey on the subject can be found in [5].

The behaviour of a BRW is in general more complex than the one of a branching process: if we start with one particle at a given site x, only one of the following holds for the BRW: (1) it goes almost surely extinct, (2) it survives globally but not locally, (3) it survives globally and locally but with different probabilities (non-strong local survival), (4) it survives globally and locally with equal probability (strong local survival). We stress that there is no strong local survival when either there is non-strong local survival or almost sure local extinction.

Again, some answers can be obtained through the study of the multidimensional generating function G, defined on $[0,1]^X$, associated to the process. It is easy to note that all the extinction probabilities are fixed points of G, therefore if one proves that there is only one fixed point then there is almost sure extinction (the vector $\mathbf{1}$, defined as $\mathbf{1}(x) := 1$ for all $x \in X$, is always a fixed point). If there are at least two fixed points then there is global survival starting from some vertices and the extinction probability starting from x coincides with $\bar{\mathbf{q}}(x)$, where $\bar{\mathbf{q}}$ is the smallest fixed point (see [3, Corollary 2.2 and Section 3]). For a long time, it has been believed (see [21, Theorem 3]) that, in the irreducible case, no more than two fixed points were possible. This was disproved in [7], even though it remains true for irreducible BRWs on finite sets (see also [7, Corollary 3.1]). In this framework, two questions naturally arise: how many fixed points can the generating function of an irreducible BRW have? At least in the irreducible case, are all fixed points also extinction probabilities? Section 3 provides a negative answer to these questions: Examples 3.5 and 3.6 are a reducible and an irreducible BRWs respectively, where there are only two extinction probabilities but the set of fixed points is uncountable. We also show that the topological properties of the multidimensional G are different from the one-dimensional case: G may not be convex (Example 3.2). Moreover, the set $U_G := \{\mathbf{z} \in [0,1]^X : G(\mathbf{z}) \leq \mathbf{z}\}$ is not necessarily convex and its extremal points may be neither fixed points nor extremal points of $[0,1]^X$ (Examples 3.3 and 3.4).

Since extinction probabilities are fixed points of G, it is clear that if we know that G has only two fixed points and that the BRW survives locally, then there is strong local survival. Conversely, if there is non-strong local survival, then there must be at least three fixed points. Somehow related is the question of what happens if we modify locally (that is, on a fixed $A \subseteq X$) a given BRW: for instance if the original BRW has no strong local survival, what can be said about the modified BRW? Theorem 4.2 shows that there is global survival and no strong local survival in A in the original BRW if and only if there is global survival and no strong local survival in A in the modified BRW (regardless of the modifications that have been introduced in A). As a corollary we get that if the original BRW dies out locally in A and the modified BRW survives globally, then almost sure global extinction for the original one is equivalent to strong local survival in A for the modified BRW (Corollary 4.3). Moreover, for a fixed irreducible BRW, if there is global survival and no strong local survival in some $A \subseteq X$ then there is global survival and no strong local survival in all finite $B \subseteq X$.

From these results in discrete time, we are able to prove that, in continuous time, a modification of the BRW in a finite subset A, which lowers the weak critical parameter (something that can usually be achieved by adding a sufficiently fast reproduction rate at some site), implies that the weak and strong parameter of the modified BRW coincide. This allows us to describe a general method to produce examples such as Example 4.5, where the modified BRW has strong local survival for some values of the parameter below a threshold and above another threshold, but non-strong local survival for intermediate values of the parameter (Figures 4 and 5). This example was originally described in [7] but appears here with an easier proof and in a more general framework. Moreover, we prove that in general, a continuous-time BRW which is obtained by a local modification of another BRW, lowering its weak critical parameter, dies out globally at the weak critical parameter (which is not always true, see [3, Example 3]).

Here is the outline of the paper: in Section 2 we introduce the terminology, describe the most common types of BRWs and their features and define the multidimensional generating function associated to a BRW. Section 3 is devoted to the questions about the generating function, its fixed points and the extinction probabilities. In Section 4 we address the problem of the possible behaviour of modified BRWs. Section 5 contains the proofs of the results and the detailed computations for the examples.

2. Basic definitions and preliminaries

The most general example of a BRW lives in discrete time and it can be constructed easily as a process $\{\eta_n\}_{n\in\mathbb{N}}$ on a set X which is at most countable, where $\eta_n(x)$ is the number of particles alive at $x \in X$ at time n. The dynamics is decribed as follows: consider the (countable) measurable space $(S_X, 2^{S_X})$ where $S_X := \{f : X \to \mathbb{N}\colon \sum_y f(y) < \infty\}$ and let $\mu = \{\mu_x\}_{x\in X}$ be a family of probability measures on $(S_X, 2^{S_X})$. A particle of generation n at site $x \in X$ lives one unit of time; after that, a function $f \in S_X$ is chosen at random according to the law μ_x. This function describes the number of children and their positions, that is, the original particle is replaced by $f(y)$ particles at y, for all $y \in X$. The choice of f is independent for all breeding particles. The BRW is denoted by (X, μ). The total number of children associated to f is represented by the function $\mathcal{H} : S_X \to \mathbb{N}$ defined by $\mathcal{H}(f) := \sum_{y\in X} f(y)$; the associated law $\rho_x(\cdot) := \mu_x(\mathcal{H}^{-1}(\cdot))$ is the law of the random number of children of a particle living at x.

Some results rely on the *first-moment matrix* $M = (m_{xy})_{x,y\in X}$, where each entry $m_{xy} := \sum_{f\in S_X} f(y)\mu_x(f)$ represents the expected number of children that a particle living at x sends to y (briefly, the expected number of particles from x to y. For the sake of simplicity, we require that $\sup_{x\in X} \sum_{y\in X} m_{xy} < +\infty$. We denote by $\bar\rho_x := \sum_{n\geq 0} n\rho_x(n) \equiv \sum_{y\in X} m_{xy}$, which is the expected number of children of a particle living at x. Given a function f defined on X we denote by Mf the function $Mf(x) := \sum_{y\in X} m_{xy} f(y)$ whenever the right-hand side converges absolutely for all x.

If we observe the process at times $i \cdot n$ ($i \in \mathbb{N}$) we obtain a new BRW whose first-moment matrix is the nth power matrix M^n with entries $m^{(n)}_{xy}$. We define

$$M_s(x, y) := \limsup_{n\to\infty} \sqrt[n]{m^{(n)}_{xy}}, \quad M_w(x) := \liminf_{n\to\infty} \sqrt[n]{\sum_{y\in X} m^{(n)}_{xy}}, \qquad \forall x, y \in X; \tag{2.1}$$

see [2, 3] for some explicit computations and [23, Section 3.2] for the relation between $M_s(x, x)$ and some generating functions.

It is important to note that, for a generic BRW, the locations of the offsprings are not chosen independently but they are assigned by the chosen function $f \in S_X$. We denote by P the *diffusion matrix* with entries $p(x, y) = m_{xy}/\bar\rho_x$. In particular if $\bar\rho_x$ does not depend on $x \in X$, we have that $M_w(x) = \bar\rho$ for all $x \in X$ and $M_s(x, y) = \bar\rho \cdot \limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x, y)}$ (where the lim sup defines the *spectral radius* of P according to [22, Chapter I, Section 1.B]). When the offsprings are dispersed independently, they are placed according to P and the process is called *BRW with independent diffusion*: in this case

$$\mu_x(f) = \rho_x\left(\sum_y f(y)\right) \frac{(\sum_y f(y))!}{\prod_y f(y)!} \prod_y p(x, y)^{f(y)}, \quad \forall f \in S_X. \tag{2.2}$$

To a generic discrete-time BRW we associate a graph (X, E_μ) where $(x, y) \in E_\mu$ if and only if $m_{xy} > 0$. We say that there is a path from x to y, and we write $x \to y$, if it is possible to find a finite sequence $\{x_i\}_{i=0}^n$ (where $n \in \mathbb{N}$) such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E_\mu$ for all $i = 0, \ldots, n-1$ (observe that there is always a path of length 0 from x to itself). Whenever $x \to y$ and $y \to x$ we write $x \rightleftharpoons y$. The equivalence relation $\rightleftharpoons$ induces a partition of X: the class $[x]$ of x is called *irreducible class of* x. It is easy to show that if $x \rightleftharpoons x'$ and $y \rightleftharpoons y'$ then $M_s(x, y) = M_s(x', y')$ and $M_w(x) = M_w(x')$. Moreover, $m^{(n)}_{xx}$ and $M_s(x, x)$ depend only on the entries $(m_{ww'})_{w,w'\in[x]}$. If the graph (X, E_μ) is *connected* (that is, there is only one irreducible class) then we say that the first-moment matrix M is *irreducible*, otherwise we call it *reducible*; the same notation applies to the BRW. The irreducibility of M implies that the progeny of any particle can spread to any site of the graph. For an irreducible BRW, $M_s(x, y) = M_s$ and $M_w(x) = M_w$ for all $x, y \in X$.

We consider initial configurations with only one particle placed at a fixed site x and we denote by $\mathbb{P}^{\delta_x}$ the law of the corresponding process. The evolution of the process with more complex initial conditions can be obtained by superimposition. In the following, *wpp* is shorthand for "with positive probability" (although, when talking about survival, "wpp" will be usually tacitly understood). In order to avoid trivial situations where particles have one offspring almost surely, we assume henceforth the following.

Assumption 2.1. *For all $x \in X$ there is a vertex $y \rightleftharpoons x$ such that $\mu_y(f\colon \sum_{w\colon w \rightleftharpoons y} f(w) = 1) < 1$, that is, in every equivalence class (with respect to $\rightleftharpoons$) there is at least one vertex where a particle can have inside the class a number of children different from 1 wpp.*

We now distinguish between the possible behaviours of a BRW.

Definition 2.2.

(1) The process survives locally wpp in $A \subseteq X$ starting from $x \in X$ if $\mathbf{q}(x, A) := 1 - \mathbb{P}^{\delta_x}(\limsup_{n\to\infty} \sum_{y\in A} \eta_n(y) > 0) < 1$.
(2) The process survives globally wpp starting from x if $\bar{\mathbf{q}}(x) := \mathbf{q}(x, X) < 1$.
(3) There is strong local survival wpp in $A \subseteq X$ starting from $x \in X$ if $\mathbf{q}(x, A) = \bar{\mathbf{q}}(x) < 1$ and non-strong local survival wpp in A if $\bar{\mathbf{q}}(x) < \mathbf{q}(x, A) < 1$.
(4) The BRW is in a pure global survival phase starting from x if $\bar{\mathbf{q}}(x) < \mathbf{q}(x, x) = 1$ (where we write $\mathbf{q}(x, y)$ instead of $\mathbf{q}(x, \{y\})$ for all $x, y \in X$).

According to the previous definition, the probabilities of extinction in A starting from x are denoted by $\mathbf{q}(x, A)$, which depend on μ. When we need to stress this dependence, we write $\mathbf{q}^\mu(x, A)$. When $x = y$ we will simply say that local survival occurs "starting from x" or "at x'. When there is no survival wpp, we say that there is extinction and the fact that extinction occurs almost surely will be tacitly understood. There are many relations between $\bar{\mathbf{q}}(x)$ and $\mathbf{q}(x, y)$ and between $\mathbf{q}(w, x)$ and $\mathbf{q}(w, y)$ where $x, y, w \in X$ (see for instance Section 3 or [5, 23]).

Roughly speaking, strong local survival means that for almost all realizations the process either survives locally (hence globally) or it goes globally extinct. More precisely, there is strong survival at y starting from x if and only if the probability of local survival at y starting from x conditioned on global survival starting from x is 1.

We want to stress that $\bar{\mathbf{q}}(x) = \mathbf{q}(x, A)$ if and only if global survival from x is equivalent to strong local survival at A from x. On the other hand $\bar{\mathbf{q}}(x) < \mathbf{q}(x, A)$ if and only if there is global survival and no strong local survival at A from x (that is, either local extinction at A or non-strong local survival at A). Recall that no strong local survival in A from x means that either there is non-strong local survival in A from x or there is local extinction in A from x.

2.1. **Continuous-time Branching Random Walks.** In continuous time each particle has an exponentially distributed random lifetime with parameter 1 (death occurs at rate 1). During its lifetime each particle alive at x breeds into y according to the arrival times of its own Poisson process with parameter λk_{xy} (representing the reproduction rate), where $\lambda > 0$ and $K = (k_{xy})_{x,y\in X}$ is a nonnegative matrix. We denote by (X, K) the family of continuous-time BRWs (depending on $\lambda > 0$). It is not difficult to see that the introduction of a nonconstant death rate $\{d(x)\}_{x\in X}$ does not represent a significant generalization. Indeed one can study a new BRW with death rate 1 and reproduction rates $\{\lambda k_{xy}/d(x)\}_{x,y\in X}$; the two processes have the same behaviours in terms of survival and extinction ([7, Remark 2.1]).

To show that the class of continuous-time BRWs is "contained" into the class of discrete-time BRWs, we associate to a continuous-time BRW a discrete-time counterpart which takes into account all the offsprings of a particle before it dies. Thus, all results in discrete time concerning the probabilities of survival (local, strong local and global) extend smoothly to the continuous time setting. Conversely, each example in continuous-time induces an analogous example in discrete-time (just by using the discrete-time counterpart). In particular, by definition, a continuous-time BRW has some property if and only if its discrete-time counterpart has it. It is easy to show that μ_x

satisfies equation (2.2), where

$$\rho_x(i) = \frac{1}{1+\lambda k(x)} \left(\frac{\lambda k(x)}{1+\lambda k(x)}\right)^i, \qquad p(x,y) = \frac{k_{xy}}{k(x)}, \qquad k(x) := \sum_{y \in X} k_{xy}. \tag{2.3}$$

Clearly the discrete-time counterpart is a BRW with independent diffusion satisfying Assumption 2.1. Moreover $m_{xy} = \lambda k_{xy}$ and $\bar{\rho}_x = \lambda k(x)$.

Given $x \in X$, two critical parameters are associated to the continuous-time BRW: the *global survival critical parameter* $\lambda_w(x)$ and the *local survival critical parameter* $\lambda_s(x)$ defined as

$$\lambda_w(x) := \inf \Big\{\lambda > 0 \colon \mathbb{P}^{\delta_x}\Big(\sum_{w \in X} \eta_t(w) > 0, \forall t\Big) > 0\Big\},$$

$$\lambda_s(x) := \inf\{\lambda > 0 \colon \mathbb{P}^{\delta_x}\big(\limsup_{t \to \infty} \eta_t(x) > 0\big) > 0\}.$$

These values depend only on the irreducible class of x; in particular they are constant if the BRW is irreducible. The process is called *globally supercritical*, *critical* or *subcritical* if $\lambda > \lambda_w$, $\lambda = \lambda_w$ or $\lambda < \lambda_w$; an analogous definition is given for the local behaviour using λ_s instead of λ_w. Everytime the interval $(\lambda_w(x), \lambda_s(x))$ is not empty we say that there exists a *pure global survival phase* starting from x. No reasonable definition of a *strong local survival critical parameter* is possible (see [7]).

Given a continuous-time BRW (X, K), for all $x, y \in X$, we define

$$K_s(x,y) := \frac{M_s(x,y)}{\lambda} \equiv \limsup_{n \to \infty} \sqrt[n]{k_{xy}^{(n)}}, \quad K_w(x) := \frac{M_w(x)}{\lambda} \equiv \liminf_{n \to \infty} \sqrt[n]{\sum_{y \in X} k_{xy}^{(n)}},$$

where $M_s(x,y)$ and $M_w(x)$ are the corresponding parameters of the discrete-time counterpart. $K_s(x,y)$ and $K_w(x)$ depend only on the equivalence classes of x and y, hence if the BRW is irreducible, then they do not depend on $x, y \in X$.

Among continuous-time BRWs, two classes are worth mentioning: *site-breeding* BRWs (where $k(x)$ does not depend on $x \in X$) and *edge-breeding* BRWs (where $k_{xy} \in \mathbb{N}$, typically in a multigraph this is the number of edges from x to y).

2.2. **Infinite-dimensional generating function.** To the family $\{\mu_x\}_{x \in X}$, we associate a generating function $G : [0,1]^X \to [0,1]^X$, which can be considered as an infinite dimensional power series. More precisely, for all $\mathbf{z} \in [0,1]^X$, $G(\mathbf{z}) \in [0,1]^X$ is defined as the following weighted sum of (finite) products

$$G(\mathbf{z}|x) := \sum_{f \in S_X} \mu_x(f) \prod_{y \in X} \mathbf{z}(y)^{f(y)},$$

where $G(\mathbf{z}|x)$ is the x coordinate of $G(\mathbf{z})$. Note that if we have a realization $\{\eta_n\}_{n \in \mathbb{N}}$ of the BRW then $G(\mathbf{z}|x) = \mathbb{E}[\prod_{y \in X} \mathbf{z}(y)^{\eta_1(y)} | \eta_0 = \delta_x]$.

The family $\{\mu_x\}_{x \in X}$ is uniquely determined by G. Indeed fix a finite $X_0 \subseteq X$ and $x \in X$. For every $\mathbf{z}$ with support in X_0, we have $G(\mathbf{z}|x) = \sum_{f \in S_{X_0}} \mu_x(f) \prod_{y \in X_0} \mathbf{z}(y)^{f(y)}$ which can be identified with a power series with several variables (defined on $[0,1]^{X_0}$). Suppose that we have another generating function $\overline{G}$ (associated to $\{\overline{\mu}_x\}_{x \in X}$) such that $G = \overline{G}$. In particular $G(\mathbf{z}|x) = \overline{G}(\mathbf{z}|x)$ for every $\mathbf{z}$ with support in X_0. Thus $\mu_x(f) = \overline{\mu}_x(f)$ for all $f \in S_{X_0}$. Since $S_X = \bigcup_{\{X_0 \subseteq X \colon X_0 \text{ finite}\}} S_{X_0}$ we have that $\mu_x(f) = \overline{\mu}_x(f)$ for all $f \in S_X$.

Note that G is continuous with respect to the *pointwise convergence topology* of $[0,1]^X$ and nondecreasing with respect to the usual partial order of $[0,1]^X$ (see [3, Sections 2 and 3] for further details); everytime we say that an element of $[0,1]^X$ is the smallest (resp. largest) among a set of points $\mathcal{A}$, we are also implying that it is comparable with every element of the specific set $\mathcal{A}$. We stress that $\mathbf{z} < \mathbf{w}$ means $\mathbf{z}(x) \le \mathbf{w}(x)$ for all $x \in X$ and $\mathbf{z}(x_0) < \mathbf{w}(x_0)$ for some $x_0 \in X$. Moreover, G represents the 1-step reproductions; we denote by $G^{(n)}$ the generating function associated to the n-step reproductions, which is inductively defined as $G^{(n+1)}(\mathbf{z}) = G^{(n)}(G(\mathbf{z}))$, where $G^{(0)}$ is the identity. Extinction probabilities are fixed points of G and the smallest fixed point is $\bar{\mathbf{q}}$ (see Section 3 for details): more generally, given a solution of $G(\mathbf{z}) \le \mathbf{z}$ then $\mathbf{z} \ge \bar{\mathbf{q}}$.

When (X,μ) is a BRW with independent diffusion, we can compute explicitly G: indeed $G(\mathbf{z}|x) = \sum_{n\in\mathbb{N}} \rho_x(n)(P\mathbf{z}(x))^n$ where $P\mathbf{z}(x) = \sum_{y\in X} p(x,y)\mathbf{z}(y)$. If, in particular, $\rho_x(n) = \frac{1}{1+\bar\rho_x}(\frac{\bar\rho_x}{1+\bar\rho_x})^n$ (as in the discrete-time counterpart of a continuous-time BRW) then the previous expression becomes $G(\mathbf{z}|x) = (1+\bar\rho_x P(\mathbf{1}-\mathbf{z})(x))^{-1}$ or, in a more compact way,

$$G(\mathbf{z}) = \frac{\mathbf{1}}{\mathbf{1}+M(\mathbf{1}-\mathbf{z})} \tag{2.4}$$

where M is the first-moment matrix and $M\mathbf{v}(x) = \bar\rho_x P\mathbf{v}(x)$ (by definition of P).

2.3. **Projection of BRWs.** We introduce the concept of projection of a BRW onto another one (see also [5, 7] where this property is called *local isomorphism*).

Definition 2.3. *A BRW (X,μ) is projected onto a BRW (Y,ν) if there exists a surjective map $g : X \to Y$ such that $\nu_{g(x)}(\cdot) = \mu_x\left(\pi_g^{-1}(\cdot)\right)$, where $\pi_g : S_X \to S_Y$ is defined as $\pi_g(f)(y) = \sum_{z\in g^{-1}(y)} f(z)$ for all $f \in S_X$, $y \in Y$.*

Clearly, if (X,μ) is projected onto (Y,ν) then, for all $\mathbf{z} \in [0,1]^Y$ and $x \in X$,

$$G_X(\mathbf{z}\circ g|x) = G_Y(\mathbf{z}|g(x)). \tag{2.5}$$

Since μ is uniquely determined by G, equation (2.5) holds if and only if (X,μ) is projected onto (Y,ν) and g is the map in Definition 2.3. The rough idea behind this definition is to assign to every $x \in X$ a label ($g(x)$ drawn from Y) in such a way that, if $\{\eta_n\}_{n\in\mathbb{N}}$ is a realization of the BRW (X,μ) then the sum of the particles over all vertices with the same label, that is $\{\pi_g(\eta_n)\}_{n\in\mathbb{N}}$, is a realization of the BRW (Y,ν).

Note that equation (2.5) can be written as $G_X(\mathbf{z}\circ g) = G_Y(\mathbf{z})\circ g$ hence $G_X^{(n)}(\mathbf{z}\circ g) = G_Y^{(n)}(\mathbf{z})\circ g$ for all $n \in \mathbb{N}$. As a consequence, for the global extinction probabilities of these BRWs, we have $\bar{\mathbf{q}}_X = \bar{\mathbf{q}}_Y \circ g$; indeed $\mathbf{0}_X = \mathbf{0}_Y \circ g$, thus $\bar{\mathbf{q}}_X = \lim_{n\to\infty} G_X(\mathbf{0}_X) = \lim_{n\to\infty} G_Y(\mathbf{0}_X) = \bar{\mathbf{q}}_Y$.

A BRW which can be projected onto a BRW defined on a finite set, is called *$\mathcal{F}$-BRW* (see [7, Section 2.4]). To give an explicit example, consider a BRW with independent diffusion on a tree with two alternating degrees: this can be projected onto a BRW on a set of cardinality 2. Other examples are *quasitransitive BRWs* (see [7, Section 2.4] for the formal definition) where the action of the group of automorphisms (bijective maps preserving the reproduction laws) has a finite number of orbits. There are non-quasitransitive BRWs which are $\mathcal{F}$-BRWs (see [7, Figure 1]). More generally, let us define the map $\varpi_g : [0,1]^Y \to [0,1]^X$ by $\varpi_g(\mathbf{z}) = \mathbf{z}\circ g$; then $\varpi_g(F_{G_Y}) \subseteq F_{G_X}$, indeed, using equation (2.5), $G_x(\varpi_g(\mathbf{z})|x) = G_X(\mathbf{z}\circ g|x) = G_Y(\mathbf{z}|g(x)) = \mathbf{z}(g(x)) = \varpi_g(\mathbf{z})(x)$. In particular the set F_{G_X} is closed under the action of all maps ϖ_g for every projection g of (X,μ) onto itself. Moreover, it is easy to show that $\mathbf{q}^X(\cdot, g^{-1}(A)) = \varpi(\mathbf{q}^Y(\cdot, A))$ for all $A \subseteq X$.

Another example, is the case of BRWs where the laws of the offspring number $\rho_x = \rho$ is independent of $x \in X$; we call them *Branching Process*-like BRWs (or *BP*-like BRWs). In this case the BRW can be projected onto a BRW defined on a singleton $Y := \{y\}$, where the law of the number of children of each particle is ρ and $g(x) := y$ for all $x \in X$ (and this last BRW is actually a branching process). It is worth noting that in this case Assumption 2.1 is simply $\rho(1) < 1$. This kind of BRWs has been studied in [5, 7] where they are called *locally isomorphic to a branching process*. By using the equality $\bar{\mathbf{q}}_X = \bar{\mathbf{q}}_Y \circ g$ we have that $\bar{\mathbf{q}}$ is a constant vector $c\cdot\mathbf{1}$, where c is the smallest fixed point of the function $z \mapsto \sum_{i=0}^\infty \rho(z)z^i$.

2.4. **Conditions for survival/extinction.** We summarize here some conditions for survival and extinction in discrete and continuous time that we need in the rest of the paper. For the proofs and further results we refer, for instance, to [2, 3, 5, 23].

Theorem 2.4. *Let (X,μ) be a discrete-time BRW.*

(1) There is local survival starting from x if and only if $M_s(x,x) > 1$.

(2) There is global survival starting from x if and only if there exists $\mathbf{z} \in [0,1]^X$, $\mathbf{z}(x) < 1$ such that $G(\mathbf{z}|y) = \mathbf{z}(y)$, for all $y \in X$ (equivalently, such that $G(\mathbf{z}|y) \le \mathbf{z}(y)$, for all $y \in X$).
(3) If (X, μ) is an $\mathcal{F}$-BRW then there is global survival starting from x if and only if $M_w(x) > 1$.

Local survival depends only on the first-moment matrix while global survival, except for particular classes as explained in [7, Section 3.1], does not. Moreover, each solution $\mathbf{z}$ of the inequality in Theorem 2.4(2) satisfies $\mathbf{z} \ge \bar{\mathbf{q}}$, since the latter is the smallest among such solutions.

For a BRW with independent diffusion, from equation (2.4) and Theorem 2.4(2) we have that there is global survival starting from x, if and only if there exists $\mathbf{v} \in [0,1]^X$, $\mathbf{v}(x) > 0$ such that

$$M\mathbf{v} \ge \mathbf{v}/(\mathbf{1} - \mathbf{v}), \qquad (\text{equivalently, } M\mathbf{v} = \mathbf{v}/(\mathbf{1} - \mathbf{v})). \tag{2.6}$$

Remember that, for a continuous-time BRW, $M = \lambda K$. As before, each solution $\mathbf{v}$ of the previous inequality satisfies $\mathbf{v} \ge \mathbf{1} - \bar{\mathbf{q}}$, since the latter is the largest among such solutions. In the continuous-time case however, global and local survival are related to the critical values $\lambda_w(x)$ and $\lambda_s(x)$ so it is useful to be able to give some estimates.

Theorem 2.5. *Let (X, K) be a continuous-time BRW.*
(1) $\lambda_s(x) = 1/K_s(x,x)$ and if $\lambda = \lambda_s(x)$ then there is local extinction at x.
(2) $\lambda_w(x) \ge 1/K_w(x)$.
(3) If (X, K) is an $\mathcal{F}$-BRWs then $\lambda_w(x) = 1/K_w(x)$ and when $\lambda = \lambda_w(x)$ there is global extinction starting from x.

More conditions can be found for instance in [2, 3, 5]. In particular λ_w admits a characterization, in the spirit of equation (2.6), in terms of a system of functional inequalities (see [3, Theorem 4.2]) Even if there can be global survival when $\lambda = \lambda_w$ (see [3, Example 3], this is not true for a continuous-time $\mathcal{F}$-BRW. Indeed, in this case, $\lambda_w(x) = 1/K_s(x,x)$ and there is always global extinction starting from x when $\lambda = \lambda_w(x)$ (see [3, Theorems 4.7 and 4.8]).

So far all results describe conditions for extinction versus survival, that is, $\mathbf{q}(x, A) = 1$ versus $\mathbf{q}(x, A) < 1$. One could also investigate whether $\bar{\mathbf{q}}(x) = \mathbf{q}(x, A) < 1$ or $\bar{\mathbf{q}}(x) < \mathbf{q}(x, A) < 1$; to put it another way, what is the probability of local survival conditioned to global survival? Studying strong local survival is more complicated than working on local or global survival. Many properties which can be easily proven when studying local/global behaviour, do not hold for the strong local one. For instance, as we already observed, even the irreducible case, it is not possible give a reasonable definition of a critical parameter for strong local survival as we did for local and global survival. Moreover, in the irreducible case, local and global behaviours do not depend on the starting vertex (or, more generally, on the starting configuration as long as it is finite) but this is not true for strong local behaviour unless $\rho_x(0) > 0$ for all $x \in X$ (see Remark 3.1 below and [7, Example 4.3]).

Some conditions for strong local survival are achieved by using a generating function approach (see [7, Section 3.2], in particular Theorem 3.4 and Corollaries 3.1 and 3.1) and they are briefly discussed in Section 3. Among other results available in the literature, it is worth mentioning a characterization of strong local survival originally proven in [19, Theorem 2.1] and extended to a generic irreducible BRW in [7, Theorem 3.5]. Results on strong local survival for BRWs in random environment can be found, for instance, in [11].

3. Fixed points and extinction probabilities

Define $\mathbf{q}_n(x, A)$ as the probability of extinction in A no later than the n-th generation starting with one particle at x, namely $\mathbf{q}_n(x, A) = \mathbb{P}^{\delta_x}(\eta_k(y) = 0, \forall k \ge n, \forall y \in A)$. The sequence $\{\mathbf{q}_n(x, A)\}_{n \in \mathbb{N}}$ is nondecreasing and satisfies

$$\begin{cases} \mathbf{q}_n(\cdot, A) = G(\mathbf{q}_{n-1}(\cdot, A)), & \forall n \ge 1 \\ \mathbf{q}_0(x, A) = 0, & \forall x \in A. \end{cases} \tag{3.7}$$

Moreover, $\mathbf{q}_n(x, A)$ converges to $\mathbf{q}(x, A)$, which is the probability of local extinction in A starting with one particle at x (see Definition 2.2). Since G is continuous we have that $\mathbf{q}(\cdot, A) =$

$G(\mathbf{q}(\cdot, A))$, hence these extinction probabilities are fixed points of G, that is, elements of $F_G := \{\mathbf{z} \in [0,1]^X : G(\mathbf{z}) = \mathbf{z}\}$.

Note that $\mathbf{q}(\cdot, \emptyset) = \mathbf{1}$. Since $\bar{\mathbf{q}} = \lim_{n\to\infty} G^{(n)}(\mathbf{0})$ we have that $\bar{\mathbf{q}}$ is the smallest fixed point of G in $[0,1]^X$ (see [3, Corollary 2.2]); we stress here that $\bar{\mathbf{q}}$ is not only the smallest extinction probability vector, but the smallest among all fixed points; hence $\bar{\mathbf{q}} = \mathbf{1}$ if and only if F_G is a singleton. Using the same arguments, one can prove that $\bar{\mathbf{q}}$ is the smallest fixed point of $G^{(m)}$ for all $m \in \mathbb{N}$.

Note that $A \subseteq B \subseteq X$ implies $\mathbf{q}(\cdot, A) \ge \mathbf{q}(\cdot, B) \ge \bar{\mathbf{q}}$. From this we can derive trivial implications between local survival or extinction in A and B. In particular, strong local survival in A from x implies strong local survival in B from x; moreover, non-strong local survival in B from x implies either non-strong local survival in A from x or local extinction in A from x.

Since for all finite $A \subseteq X$ we have $\mathbf{q}(x, A) \ge 1 - \sum_{y\in A}(1 - \mathbf{q}(x, y))$ then, for any given finite $A \subseteq X$, $\mathbf{q}(x, A) = 1$ if and only if $\mathbf{q}(x, y) = 1$ for all $y \in A$.

If $x \to x'$ and $A \subseteq X$ then $\mathbf{q}(x', A) < 1$ implies $\mathbf{q}(x, A) < 1$; as a consequence, if $x \rightleftharpoons x'$ then $\mathbf{q}(x, A) < 1$ if and only if $\mathbf{q}(x', A) < 1$. Moreover if $y \rightleftharpoons y'$ we have $\mathbf{q}(x, y) = \mathbf{q}(x, y')$ for all $x \in X$. The main properties in the irreducible case are summarized in the following remark.

Remark 3.1. *In the irreducible case, for every $x \in X$ and $A \subseteq X$ finite and nonempty, we have $\mathbf{q}(x, A) = \mathbf{q}(x, x)$. Thus $\mathbf{q}(x, A) = \mathbf{q}(x, B)$ for every couple A, B of finite, nonempty subsets of X.*

If, in addition, $\rho_x(0) > 0$ for all $x \in X$, we have that if $\bar{\mathbf{q}}(x) = \mathbf{q}(x, A)$ for some $x \in X$ and a finite subset $A \subseteq X$ then $\bar{\mathbf{q}}(y) = \mathbf{q}(y, B)$ for all $y \in X$ and all (finite or infinite) subsets $B \subseteq X$ (hence, strong local survival is a common property of all subsets and all starting vertices, see Theorem 4.2). Clearly, this may not be true in the reducible case. Besides, if we drop the assumption $\rho_x(0) > 0$ for all $x \in X$, we might actually have $\bar{\mathbf{q}}(x) = \mathbf{q}(x, A) < 1$ and $\bar{\mathbf{q}}(y) < \mathbf{q}(y, A)$ for some $x, y \in X$ and a finite $A \subseteq X$ even when the BRW is irreducible (see [7, Example 4.3]*). Hence, in general, even for irreducible BRWs, strong local survival is not a common property of all vertices as local and global survival are.*

As we recalled in the introduction, the generating function G of a branching process has at most two fixed points in $[0,1]$, $\bar{q}$ and 1. This is still true for BRWs on finite sets X (see for instance [7, Corollary 3.1] or the proof of [21, Theorem 3] which is incorrect in the infinite case, but correct in the finite one). Moreover, for a branching process, G is strictly convex and U_G is closed, compact and convex (recall that U_G was defined in Section 1 as $\{\mathbf{z} \in [0,1]^X : G(\mathbf{z}) \le \mathbf{z}\}$). Let us denote by E_G the set of extinction probabilities: $E_G := \{\mathbf{q}(\cdot, A) : A \subseteq X\}$. For a branching process it is true that the extremal points of U_G are the fixed points $\bar{q}$ and 1 (where $\bar{q}$ may coincide with 1) and all fixed points are extinction probabilities: in short, $\mathrm{ext}(U_G) = F_G$ and $F_G = E_G$.

Some of these properties still hold in the general case, others do not, even when X is finite. It is clear that F_G and U_G are always closed and compact sets (with respect to the product topology of $[0,1]^X$), since they are closed subsets of the compact topological space $[0,1]^X$. We provide some counterexamples and conjectures on the other properties in the following sections.

3.1. **Convexity of G and U_G and extremal points.** Given any $\mathbf{w} \le \mathbf{z} \in [0,1]^X$ it is true that $t \mapsto G(\mathbf{w} + t(\mathbf{z} - \mathbf{w}))$ is convex, nevertheless G is not always a convex function, even when X is finite, as the following example shows.

Example 3.2. *Let $X = \{1, 2\}$ and $\mu_1 = \delta_{(1,1)}$, $\mu_2 = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(1,0)}$. Roughly speaking, every particle at 1 has one child at* 1 *and one at* 2 *almost surely, while every particle at 2 has one child at 1 with probability* 1/2 *and no children with probability* 1/2*. The generating function is*

$$G(x, y) = \begin{pmatrix} xy \\ (1 + x)/2 \end{pmatrix}$$

which is not convex. Nevertheless $U_G = \{(x, y) \in [0,1]^2 : 2y \ge x + 1\}$ is convex and $F_G = \{(0, 1/2), (1, 1)\}$. Clearly ext$(U_G) = F_G \cup \{(0, 1)\}$.

The following two examples show that not only U_G is not necessarily convex, but also its extremal points may not be elements of $F_G \cup \{0,1\}^X$.

Example 3.3. *Let $X = \{1,2\}$ and consider*

$$G(x,y) = \begin{pmatrix} (1+3y^2)/4 \\ (1+3x^2)/4 \end{pmatrix}$$

which corresponds to the process where each particle has no children with probability $1/4$ and 2 children on the other vertex with probability $3/4$. In this case F_G contains two vertices on the bisector (one of them is $(1,1)$ of course) while U_G is the intersection of $(1+3y^2)/4 \le x$ and $(1+3x^2)/4 \le y$ and the set of its extremal points is the whole boundary.

Example 3.4. *Take $X := \{1,2,3\}$, $\mu_1 = \delta_{(0,1,1)}$, $\mu_2 = \delta_{(1,2,1)}$ and $\mu_3 = \delta_{(1,1,0)}$. Roughly speaking every particle at j has two children: one in each point different from j. The generating function is*

$$G(x_1,x_2,x_3) = \begin{pmatrix} x_2x_3 \\ x_1x_3 \\ x_1x_2 \end{pmatrix}.$$

According to [5, Corollary 3.1] *for a finite-dimensional, irreducible BRW there are at most two solutions of $G(\mathbf{z}) \ge \mathbf{z}$ when $\mathbf{z} \ge \bar{\mathbf{q}}$, that is, $\bar{\mathbf{q}}$ and $\mathbf{1}$ (in this case the vertices $(1,1,1)$ and $(0,0,0)$, which are the only fixed points). It is easy to see that $(1/2,1/2,1)$ and $(1/2,1,1/2)$ are in U_G. The line connecting these points can be parametrized as $\mathbf{z}(t) := (1/2, 1/2+t/2, 1-t/2)$, $t \in [0,1]$ and $\mathbf{z}(t) \notin U_G$ for all $t \in (0,1)$ (since $G(\mathbf{z}(t)) \not\le \mathbf{z}(t)$ for all $t \in (0,1)$). Figures 1 and 2 show the shape of U_G as seen from the top (vertex $(1,1,1)$) and from the bottom (vertex $(0,0,0)$).*

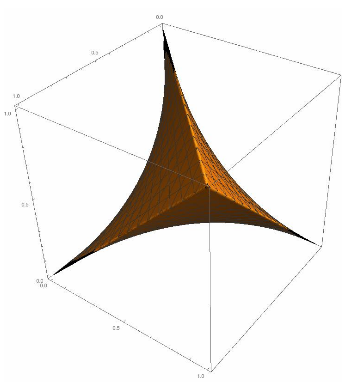

FIGURE 1. U_G from the top.

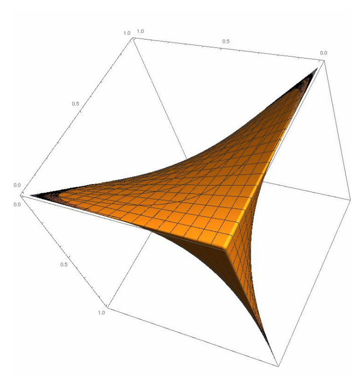

FIGURE 2. U_G from the bottom.

3.2. **How many fixed points does G have?** If a BRW is reducible, then there can be an infinite (even uncountable) number of fixed points. Consider this completely disconnected BRW: let $\{\rho_x\}_{x\in X}$ be an infinite collection of reproduction laws of supercritical branching processes, define the expected number of children $m_x := \sum_{n\in\mathbb{N}} n\rho_x(n) > 1$ and denote by $c_x < 1$ the extinction probability of the xth branching process. Clearly $G(\mathbf{z}|x) = \sum_{n\in\mathbb{N}} \rho_x(n)\mathbf{z}(x)^n$ and $F_G = \prod_{x\in X}\{c_x, 1\}$, which is uncountable. Moreover, every fixed point is an extinction probability, since for every $\mathbf{z} \in F_G$, $\mathbf{z} = \mathbf{q}(\cdot, A)$, where $A := \{x \in X : \mathbf{z}(x) < 1\}$.

Let us discuss the nontrivial case of an irreducible BRW. The generating function of an irreducible BRW has at most two fixed points, namely $\bar{\mathbf{q}}$ and $\mathbf{1}$, when X is finite. Since $E_G \subseteq F_G$, in order to find examples where $|F_G| \ge 3$, it suffices to find cases with $|E_G| \ge 3$. In particular, a BRW with non-strong local survival would do. In [7] two such examples were provided: [7, Examples 4.4 and 4.5] are irreducible BP-like BRWs with independent diffusion and non-strong local survival, thus with three different extinction probabilities.

It is worth mentioning that in the case of irreducible, quasitransitive BRWs, $\{\bar{\mathbf{q}}, \mathbf{1}\} = E_G$ (local survival starting from some $x \in X$ implies strong local survival starting from all $x \in X$). Thus $|E_G| = 2$ for irreducible, quasitransitive BRWs. The aforementioned examples in [7]) show that $\{\bar{\mathbf{q}}, \mathbf{1}\} \neq E_G$ (thus, non-strong local survival) is possible in the case of an irreducible $\mathcal{F}$-BRW. We recall that by [7, Theorem 3.4], for an $\mathcal{F}$-BRW, every fixed point $\mathbf{z}$ different from $\bar{\mathbf{q}}$ satisfies $\sup_{x \in X} \mathbf{z}(x) = 1$. In particular, if the BRW is irreducible either $\mathbf{q}(x,x) = \bar{\mathbf{q}}(x)$ for all $x \in X$ or $\sup_{x \in X} \mathbf{q}(x,x) = 1$.

These remarks do not settle the question of the possible cardinalities of F_G, even in the quasitransitive case, since, as we show in the following section, F_G can be much larger than E_G. Indeed Example 3.6 proves that, even for an $\mathcal{F}$-BRW, there may be an uncountable number of fixed points. It is an open question whether this also holds for some irreducible, quasitransitive BRW: we conjecture that the answer is positive (see Remark 3.7).

3.3. **Is every fixed point an extinction probability?** The answer is negative. We start with a reducible example and then we move to an irreducible example.

Example 3.5. *Consider a BRW on $\mathbb{N}$ where every particle at n has two children at $n+1$ with probability p and no children with probability $1-p$ ($p > 1/2$ to make it supercritical). This is a BP-like BRW; easy computations (see* [6, Proposition 4.33]*) show that $G(\mathbf{z}|n) = p\mathbf{z}(n+1))2 + 1 - p$ and $\bar{\mathbf{q}}(x) = (1-p)/p$ for every $x \in \mathbb{N}$. Moreover, due to the right drift, $\mathbf{q}(\cdot, A) = \mathbf{1}$ if A is finite and $\mathbf{q}(\cdot, A) = \bar{\mathbf{q}}$ if A is infinite. Every fixed point must satisfy $\bar{\mathbf{q}} \le \mathbf{z} \le \mathbf{1}$, thus $\mathbf{z}(0) \in [(1-p)/p, 1]$. Clearly if $\mathbf{z}(0) = (1-p)/p$ (resp. $\mathbf{z}(0) = 1$) we have $\mathbf{z} = (1-p)/p \cdot \mathbf{1}$ (resp. $\mathbf{z} = \mathbf{1}$). Fix $(1-p)/p < \mathbf{z}(0) < 1$; the equation $G(\mathbf{z}) = \mathbf{z}$ is equivalent to the recursive relation $\mathbf{z}(n+1) = \sqrt{(\mathbf{z}(n) - (1-p))/p}$. This defines a unique sequence $\mathbf{z}$ which is a fixed point. Indeed $(1-p)/p < \mathbf{z}(0) < 1$ and, by induction, if $(1-p)/p < \mathbf{z}(n) < 1$ then $(1-p)^2/p^2 < (\mathbf{z}(n) - (1-p))/p < 1$, thus $(1-p)/p < \mathbf{z}(n+1) < 1$. Obviously, all fixed points can be obtained by means of this procedure, hence the set F_G is uncountable while there are just two extinction probabilities.*

Example 3.6. *Consider the BRW on $\mathbb{N}$ where every particle at n has two children at $n+1$ with probability $p - \varepsilon$, one child at $\max(0, n-1)$ with probability ε and no children with probability $1-p$. We require that $2p - \varepsilon > 1$ for global survival, $\varepsilon > 0$ for irriducibility, $p < 1/\sqrt{2}$ and $\varepsilon(p-\varepsilon) \le 1/8$ for technical reasons (take for instance $p = 2/3$ and $\varepsilon \le 2/9$).*

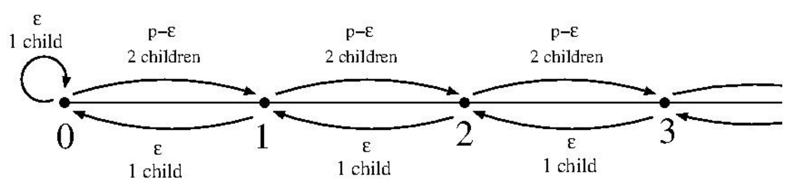

FIGURE 3. The irreducible BRW of Example 3.6.

This is an irreducible BP-like BRW (see Figure 3; according to Theorem 2.4, global and local survival depend on M_w and M_s. To compute these parameters we refer to ([2, 3] *and* [6, Section 4.6]*). In particular M_w is the expected number of children $2p - \varepsilon$. Moreover we have (see* [6, Proposition 4.33]*) $G(\mathbf{z}|n) = (p-\varepsilon)\mathbf{z}(n+1)^2 + \varepsilon\mathbf{z}(\max(0, n-1)) + 1 - p$ and $\bar{\mathbf{q}} = (1-p)/(p-\varepsilon) \cdot \mathbf{1}$. Besides, since $\varepsilon(p-\varepsilon) \le 1/8$ we have local extinction, that is, $M_s \le 1$ (for all the details, see Section 5). Hence for this BRW there is global survival but local extinction; thus $\mathbf{q}(\cdot, A) = \mathbf{1}$ if A is finite and, since the BRW must drift to the right in order to survive, $\mathbf{q}(\cdot, A) = \bar{\mathbf{q}}$ if A is infinite.*

We prove (see Section 5) that the equation $G(\mathbf{z}) = \mathbf{z}$, which is equivalent to the recursive equation $\mathbf{z}(n+1) = h\big(\mathbf{z}(n), \mathbf{z}(\max(0, n-1))\big)$ where $h(x,y) := \sqrt{\big(x - y\varepsilon - (1-p)\big)/(p-\varepsilon)}$, defines a unique fixed point for every $\mathbf{z}(0) \in [(1-p)/(p-\varepsilon), 1]$ (and every fixed point can be obtained this way). Thus the set of fixed points is uncountable but there are just two extinction probabilities.

We conjecture that the previous example extends to quasitransitive BRWs as the following remark suggests.

Remark 3.7. *Consider the BRW on $\mathbb{Z}$ where every particle at n has two children at $n+1$ with probability $p-\varepsilon$ (such that $2p-\varepsilon>1$), one child at $n-1$ with probability ε and no children with probability $1-p$: due to global survival, local extinction and the right drift we have just two extinction probabilities, namely $\mathbf{q}(\cdot, A)=\mathbf{1}$ if $\sup A$ is finite and $\mathbf{q}(\cdot,A)=\bar{\mathbf{q}}$ if $\sup A$ is infinite.*

Suppose that p and ε satisfy the assumptions of Example 3.6; in order to find an uncountable set of fixed points we can proceed as follows. Any fixed point $\mathbf{z}$ of Example 3.6, outside $\bar{\mathbf{q}}$ and $\mathbf{1}$, is a strictly increasing sequence $\{\mathbf{z}(n)\}_{n\in\mathbb{N}}$ converging to 1. The function ϕ_n mapping $\mathbf{z}(0)$ to $\mathbf{z}(n)$ is continuous, strictly increasing and maps $(1-p)/(p-\varepsilon)$ and 1 into themselves; thus ϕ_n is an invertible map from $[(1-p)/(p-\varepsilon),1]$ into itself. More precisely, ϕ_n can be obtained recursively as

$$\begin{cases}\phi_0(x):=x\\ \phi_1(x):=h(x,x)\\ \phi_{n+1}(x)=h\big(\phi_n(x),\phi_{n-1}(x)\big)\end{cases}$$

where $h(x,y):=\sqrt{\big(x-y\varepsilon-(1-p)\big)/(p-\varepsilon)}$ as in Example 3.6. Moreover $\{\phi_n(x)\}_{n\in\mathbb{N}}$ is strictly increasing for all $x\in\big((1-p)/(p-\varepsilon),1\big)$ and constant for all $x\in\{(1-p)/(p-\varepsilon),1\}$. Fix $\alpha\in\big((1-p)/(p-\varepsilon),1\big)$ and define $\mathbf{z}^{(n)}\in[0,1]^{\mathbb{Z}}$ as

$$\mathbf{z}^{(n)}(i):=\begin{cases}\phi_{n+i}(\phi_n^{-1}(\alpha)) & \text{if } i\geq -n,\\ 0 & \text{if } i<-n.\end{cases}$$

This is a left-translation of the fixed points of the previous example such that $\mathbf{z}^{(n)}(0)=\alpha$ for every $n\in\mathbb{N}$. We conjecture that the sequence $\{\mathbf{z}^{(n)}\}_{n\in\mathbb{N}}$ converges (pointwise) to some $\tilde{\mathbf{z}}\in\big((1-p)/(p-\varepsilon),1\big)^{\mathbb{Z}}$; more precisely we conjecture that $\{\mathbf{z}^{(n)}(i)\}_{n\in\mathbb{N}}$ is strictly increasing (resp. decreasing) when i is positive (resp. negative). If this holds, due to the continuity of the map $(x,y)\mapsto(p-\varepsilon)x^2+\varepsilon y+1-p$, then $\tilde{\mathbf{z}}(n)=(p-\varepsilon)\tilde{\mathbf{z}}(n+1)^2+\varepsilon\tilde{\mathbf{z}}(n-1)+1-p$ for every $i\in\mathbb{Z}$; whence, $\tilde{\mathbf{z}}$ is a (non constant) fixed point for the generating function of the quasitransitive BRW described above.

Let us summarize: we proved that, in the irreducible case,

X finite	$\Longrightarrow \{\bar{\mathbf{q}},\mathbf{1}\}=E_G=F_G$	[7, Corollary 3.1]
X infinite, (X,μ) quasitransitive	$\Longrightarrow \{\bar{\mathbf{q}},\mathbf{1}\}=E_G\,(\subsetneqq?)\,F_G$	[7, Corollary 3.2]
X infinite, (X,μ) $\mathcal{F}$-BRW	$\Longrightarrow \{\bar{\mathbf{q}},\mathbf{1}\}\subsetneqq E_G\subsetneqq F_G$	[7, Examples 4.4 and 4.5], Example 3.6,

where $\subsetneqq$ means there are cases where the inclusion is proper and cases where the equality holds. We point out here that the proper inclusion $\{\bar{\mathbf{q}},\mathbf{1}\}\neq E_G$ is equivalent to non-strong local survival (for some set A starting from some vertex x), while $\{\bar{\mathbf{q}},\mathbf{1}\}\neq F_G$ tells us nothing about strong local survival. We believe that following the ideas of Remark 3.7 one could obtain an example where $E_G\neq F_G$ for a quasitransitive BRW (hence $E_G\subsetneqq F_G$) but this exceeds the purpose of this paper.

4. Strong local survival and local modifications

We recall here the following theorem, (it is essentially [7, Theorem 3.3]). In the case of global survival, it gives equivalent conditions for strong local survival in terms of extinction probabilities .

Theorem 4.1. *For every nonempty subset $A\subseteq X$, the following assertions are equivalent.*

(1) $\mathbf{q}(x,A)=\bar{\mathbf{q}}(x)$*, for all* $x\in X$*;*
(2) $\mathbf{q}_0(x,A)\leq\bar{\mathbf{q}}(x)$*, for all* $x\in X$*;*
(3) for all $x\in X$, either $\bar{\mathbf{q}}(x)=1$ or the probability of visiting A at least once starting from x conditioned on global survival starting from x is 1;

(4) for all $x \in X$, either $\bar{\mathbf{q}}(x) = 1$ or the probability of local survival in A starting from x conditioned on global survival starting from x is 1 (strong local survival in A starting from x).
(5) For all $x \in X$ the probability of surviving globally starting from x without ever visiting A is 0.

This theorem implies that if there exists $x \in X$ such that $\mathbf{q}(x, A) > \bar{\mathbf{q}}(x)$ (that is, there is a positive probability of global survival and local extinction in A starting from x) then there exists $y \in X$ such that $\mathbf{q}_0(y, A) > \bar{\mathbf{q}}(y)$ (which implies that there is a positive probability that the BRW survives globally starting from y without ever visiting A, clearly $y \notin A$). Note that, $\mathbf{q}_0(x, A) > \bar{\mathbf{q}}(x)$ implies $\mathbf{q}(x, A) > \bar{\mathbf{q}}(x)$ but the converse is not true. Hence we have the following dichotomy: for every fixed nonempty A, either $\mathbf{q}(\cdot, A) = \bar{\mathbf{q}}(\cdot)$ or there is $x \in X \setminus A$ such that there is a positive probability of global survival starting from x without ever visiting A.

We note that there is no *a priori* order between the events $A_0 :=$"never visit A" and $GE :=$"global extinction". Nevertheless, Theorem 4.1 tells us that if $\mathbf{q}_0(\cdot, A) \le \bar{\mathbf{q}}(\cdot)$ then $\mathbb{P}^x(A_0 \setminus GE) = 0$ for all $x \in X$ (the converse is trivial).

From Theorem 4.1, which is stated for a single BRW, we derive Theorem 4.2 and its Corollaries 4.3 and 4.4 which give us information about the behaviour of a BRW after some modifications.

Theorem 4.2. *Consider two BRWs (X, μ) and (X, ν). Suppose that $A \subseteq X$ is a nonempty set such that $\mu_x = \nu_x$ for all $x \notin A$.*

(1) If we denote by $\mathbf{q}^\mu$ and $\mathbf{q}^\nu$ the extinction probabilities related to (X, μ) and (X, ν) respectively then we have that $\mathbf{q}_0^\mu(x, A) = \mathbf{q}_0^\nu(x, A)$ for all $x \in X$ and

$$\mathbf{q}^\mu(\cdot, A) = \bar{\mathbf{q}}^\mu(\cdot) \Longleftrightarrow \mathbf{q}^\nu(\cdot, A) = \bar{\mathbf{q}}^\nu(\cdot).$$

(2) If (X, μ) is irreducible and $B, C \subseteq X$ are two nonempty sets such that B is finite then

$$\mathbf{q}^\mu(\cdot, B) = \bar{\mathbf{q}}^\mu(\cdot) \Longrightarrow \mathbf{q}^\mu(\cdot, C) = \bar{\mathbf{q}}^\mu(\cdot).$$

As a consequence we have the following corollary.

Corollary 4.3. *Consider two BRWs (X, μ) and (X, ν). Suppose that $A \subseteq X$ is a nonempty set such that $\mu_x = \nu_x$ for all $x \notin A$.*

(1) Suppose that (X, μ) dies out locally in A from all $x \in X$ and (X, ν) survives globally from all $x \in X$; then

$$\bar{\mathbf{q}}^\mu(x) = 1 \text{ for all } x \in X \Longleftrightarrow \text{strong local survival for } (X, \nu) \text{ at } A \text{ from all } x \in X.$$

(2) If (X, μ) dies out globally from all $x \in X$ and (X, ν) survives globally from all $x \in X$ then there is strong local survival for (X, ν) in A from all $x \in X$.

The following corollary describes how a small and local modification can affect the phase diagram of a continuous-time BRW.

Corollary 4.4. *Let (X, K) and (X, K') two irreducible continuous-time BRWs such that $k_{xy} = k'_{xy}$ for all $x \in X \setminus A$ where A is a nonempty, finite set. Then the following are equivalent:*

(1) $\lambda'_w < \lambda_w$;
(2) $\lambda'_s < \lambda_w$;
(3) $\lambda'_w = \lambda'_s < \lambda_w$.

Moreover if one of the previous holds, for the BRW (X, K')

(i) if $\lambda \le \lambda'_w$ there is a.s. global and local extinction in every nonempty set B;
(ii) if $\lambda \in (\lambda'_w, \lambda_w)$ there is strong local survival in every nonempty set B;
(iii) if $\lambda = \lambda_w$ and the (X, K)-BRW dies out globally, then there is strong local survival in B for every nonempty set B, otherwise there is non-strong local survival in B for every nonempty finite set B;
(iv) if $\lambda \in (\lambda_w, \lambda_s]$ (when non empty) there is non-strong local survival in every nonempty finite set B;

(v) *if $\lambda > \lambda_s$ then local survival is strong (resp. non-strong) in a nonempty finite set B if and only if the same holds for (X, K).*

We already pointed out that at $\lambda = \lambda_w$, global survival is possible. This cannot happen if the process is a finite modification of another BRW, as in Corollary 4.4. An easy way to modify a BRW (X, K) in order to obtain $\lambda'_s < \lambda_w$, is to add a sufficiently rapid reproduction from y to y (for a fixed y).

We now apply Corollary 4.4 to the following example (see also [7, Example 4.2]) which can be discussed without using cumbersome arguments such as those contained in [2, Remark 3.2] and [7, Example 4.1].

Example 4.5. *Consider the edge-breeding continuous-time BRW on the homogeneous tree $\mathbb{T}_d$ with degree $d \geq 3$; in this case K is the adjacency matrix. It is easy to prove (see for instance* [7, Example 4.2]*) that $\lambda_w = 1/d < 1/2\sqrt{d-1} = \lambda_s$. If $\lambda \leq \lambda_w$ there is global extinction, if $\lambda > 1/2\sqrt{d-1}$ there is strong local survival (see* [7, Corollary 3.2]*) while if $\lambda \in (1/d, 1/2\sqrt{d-1}]$ the probability of global survival is positive and independent of the starting point and the probability of local survival in any finite $A \subseteq X$ is 0. The phase diagram is shown by Figure 4.*

Fix a vertex $y \in \mathbb{T}_d$ and denote by A the singleton $\{y\}$. Let us modify the BRW by adding a loop at y, that is, by considering a new matrix K' where all the entries are the same as those of K but $k'_{yy} > d$. Hence $\lambda'_s \leq 1/k'_{yy} < 1/d = \lambda_w$ and Corollary 4.4 applies. As a result, $\lambda'_s = \lambda'_w$ and we have the following behaviour for $(\mathbb{T}_d, K')$ (see Figure 5): if $\lambda < \lambda'_w$ there is global extinction, if $\lambda \in (\lambda'_w, 1/d]$ there is strong local survival, if $\lambda \in (1/d, 1/2\sqrt{d-1}]$ there is non-strong local survival and if $\lambda > 1/2\sqrt{d-1}$ there is strong local survival again.

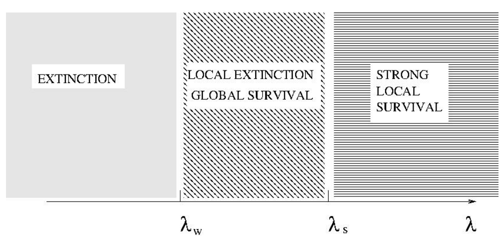

FIGURE 4. Phase diagram for $(\mathbb{T}_d, K)$.

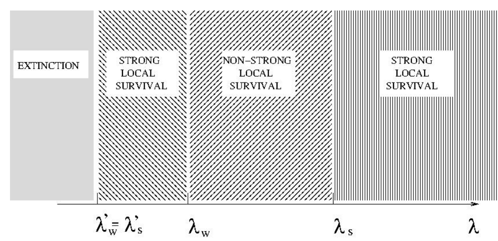

FIGURE 5. Phase diagram for $(\mathbb{T}_d, K')$.

We note that, as it always happens in a continuous-time BRW, $\mathbf{q}(\cdot, A)$ depends on λ since a continuous-time BRW actually is a family of processes indexed by λ. The function $\lambda \mapsto \mathbf{q}(\cdot, A)$ does not need to be continuous. Consider, for instance, the above edge-breeding BRW $(\mathbb{T}_d, K)$; if we look for the global extinction probability vector it is easy to show, by using equation (2.4) *and the equality $\bar{\mathbf{q}} = \lim_{n\to\infty} G^{(n)}(\mathbf{0})$, that $\bar{\mathbf{q}}(x) = \min(1, (d\lambda)^{-1})$ which is a nice continuous function. On the other hand, if we consider $\mathbf{q}(x, x)$ (where $x \in \mathbb{T}_d$) then it equals $\mathbf{1}$ in the interval $(0, 1/2\sqrt{d-1}]$ and $(d\lambda)^{-1} \cdot \mathbf{1}$ in the interval $(1/2\sqrt{d-1}, +\infty)$; thus there is a discontinuity at $1/2\sqrt{d-1}$.*

5. Proofs

Details on Remark 3.1. If the BRW is irreducible we have $\mathbf{q}(v, h) = \mathbf{q}(v, v)$ for all $v, h \in X$, which implies $\mathbf{q}(v, A) = \mathbf{q}(v, v) = \mathbf{q}(v, B)$ for all $v \in X$ and finite, nonempty sets A and B. Indeed if the process visits infinitely many times A starting from v then it visits infinitely many times at least a vertex $h \in A$ and, by irreducibility, it visits infinitely many times v. Similarly, if the process visits infinitely many times v starting from v then it visits infinitely many times any vertex $h \in A$.

If $\bar{\mathbf{q}}(x) = 1$ then $\mathbf{q}(y, B) = 1$ for all $y \in X$ and $B \subseteq X$ and there is nothing to prove. Suppose that $\bar{\mathbf{q}}(x) = \mathbf{q}(x, A) < 1$ and, by contradiction, $\bar{\mathbf{q}}(y) < \mathbf{q}(y, B)$ for some $x, y \in X$ and $A, B \subseteq X$ finite. We know that there is a positive probability that the process, starting from x has at least one descendant at y. There is also a positive probability that all the particles (except one at y) die and the progeny of the surviving particle survives globally but not locally in A. Thus, there is a positive probability, starting from x, of surviving globally but not locally in A and this is a contradiction. Hence $\bar{\mathbf{q}}(y) = \mathbf{q}(y, A)$ for all $y \in X$. But we proved above that, in the irreducible case, $\mathbf{q}(v, A) = \mathbf{q}(v, B)$ for all $v \in X$ and all finite nonempty subsets A and B, whence $\bar{\mathbf{q}}(y) = \mathbf{q}(y, A)$ for all $y \in X$ and every finite nonempty subset B. If B is infinite and $z \in B$ then $\bar{\mathbf{q}}(y) = \mathbf{q}(y, z) \geq \mathbf{q}(y, B) \geq \bar{\mathbf{q}}(y)$ for all $y \in X$. $\square$

Proof of Theorem 4.1. The equivalence between (1), (2), (3) and (4) was already proven in [7, Theorem 3.3]. Clearly, $\mathbb{P}^x(A_0 \setminus GE) = 0$ implies $\mathbf{q}_0(x, A) \leq \bar{\mathbf{q}}(x)$, hence (5) $\Longrightarrow$ (2). We prove now that (1) $\Longrightarrow$ (5). Indeed, define $A_n :=$"visit A at most n times". Hence, $A_{n+1} \supseteq A_n$ and $\bigcup_{n\in\mathbb{N}} A_n \supseteq GE$. Note that $\mathbf{q}(x, A) = \mathbb{P}^x\big(\bigcup_{n\in\mathbb{N}} A_n\big)$ and $\bar{\mathbf{q}}(x) = \mathbb{P}^x(GE)$. If $\mathbf{q}(x, A) = \bar{\mathbf{q}}(x)$ then $\mathbb{P}^x\big(\bigcup_{n\in\mathbb{N}} A_n \setminus GE\big) = 0$ which is equivalent to $\mathbb{P}^x(A_0 \setminus GE) = 0$ for all $n \in \mathbb{N}$. $\square$

Proof of Theorem 4.2. (1) We note that (X, μ) and (X, ν) have the same behaviour until they first hit A, hence $\mathbf{q}_0^\mu(x, A) = \mathbf{q}_0^\nu(x, A)$ for all $x \notin A$. If $x \in A$ then clearly $\mathbf{q}_0^\mu(x, A) = 0 = \mathbf{q}_0^\nu(x, A)$.

Suppose now that $\mathbf{q}^\mu(\cdot, A) \neq \bar{\mathbf{q}}^\mu(\cdot)$. Hence, according to Theorem 4.1 (see comments after its statement), there exists $x \in X \setminus A$ such that there is a positive probability of survival starting from x without ever visiting A. Since the two processes have the same behaviour until they first hit A, the same holds for (X, ν) and this implies that $\mathbf{q}^\nu(x, A) > \bar{\mathbf{q}}^\nu(x)$; thus $\mathbf{q}^\nu(\cdot, A) \neq \bar{\mathbf{q}}^\nu(\cdot)$.

(2) By Remark 3.1, when B and C are finite nonempty subsets, $\mathbf{q}^\mu(\cdot, B) = \mathbf{q}^\mu(\cdot, C)$, whence the implication is trivial.

Moreover, recall that $B \subseteq C$ implies $\mathbf{q}^\mu(\cdot, B) \geq \mathbf{q}^\mu(\cdot, C)$. hence, if C is infinite and $z \in C$ then, for all $y \in X$, Remark 3.1 yields

$$\bar{\mathbf{q}}^\mu(y) = \mathbf{q}^\mu(y, B) = \mathbf{q}^\mu(y, z) \geq \mathbf{q}^\mu(y, C) \geq \bar{\mathbf{q}}^\mu(y).$$

$\square$

Proof of Corollary 4.3. (1) According to the hypotheses $\mathbf{q}^\mu(\cdot, A) = \mathbf{1} > \bar{\mathbf{q}}^\nu(\cdot)$. Hence if $\bar{\mathbf{q}}^\mu(\cdot) = \mathbf{1} = \mathbf{q}^\mu(\cdot, A)$ then, according to Theorem 4.2(1), $\mathbf{q}^\nu(\cdot, A) = \bar{\mathbf{q}}^\nu(\cdot) < \mathbf{1}$, that is, there is strong local survival for (X, ν) in A from every $x \in X$. Conversely, $\mathbf{q}^\nu(\cdot, A) = \bar{\mathbf{q}}^\nu(\cdot) < \mathbf{1}$ implies, by Theorem 4.2(1), $\bar{\mathbf{q}}^\mu(\cdot) = \mathbf{q}^\mu(\cdot, A) = \mathbf{1}$, thus global extinction from every $x \in X$.

(2) If (X, μ) dies out globally from all $x \in X$ then it dies out locally in A from all $x \in X$ hence, from the previous part, there is strong local survival for (X, ν) in A from every $x \in X$.

$\square$

Proof of Corollary 4.4. Observe that the discrete-time counterparts of these continuous-time BRWs satisfy the hypotheses of Theorem 4.2, namely, their offspring distribution are the same outside A.

Clearly (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (2). We just need to prove that (1) $\Longrightarrow$ (3); more precisely, we prove that $\lambda'_w < \lambda_w \Longrightarrow \lambda'_w = \lambda'_s$. Take $\lambda \in (\lambda'_w, \lambda_w)$; the λ-(X, K') BRW survives globally, hence $\bar{\mathbf{q}}' < \mathbf{1}$. On the other hand, $\mathbf{1} = \bar{\mathbf{q}} = \mathbf{q}(\cdot, A)$ whence, according to Theorem 4.2(1), $\bar{\mathbf{q}}' = \mathbf{q}'(\cdot, A)$ which implies $\mathbf{q}'(\cdot, A) < \mathbf{1}$. If the λ-(X, K') BRW survives locally in the finite set A it means that it survives locally at a vertex $x \in A$ ($\Longleftrightarrow$ at every vertex, since the process is irreducible). This implies $\lambda \geq \lambda'_s$; thus $\lambda'_s = \lambda'_w$.

Note that in the discrete-time counterpart of a continuous-time BRW every particle at every vertex has a positive probability of dying without breeding; hence by Remark 3.1 strong local survival is a common property of all starting vertices.

We consider the following disjoint intervals for λ.

(i) Suppose that $\lambda < \lambda'_w$; by definition there is global, hence local, extinction. If $\lambda = \lambda'_w$ then, according to [3, Theorem 4.7] (see also [2, Theorem 3.5 and Section 4.2]), since $\lambda = \lambda'_w = \lambda'_s$

then the λ-(X, K') BRW dies out locally (at any finite set C), hence $\mathbf{q}'(\cdot, C) = \mathbf{1}$ (clearly, being $\lambda < \lambda_w$, $\mathbf{q}(\cdot, B) = \bar{\mathbf{q}} = \mathbf{1}$ for all $B \subseteq X$), using Theorem 4.2(1),

$$\mathbf{q}(\cdot, A) = \bar{\mathbf{q}} \Longrightarrow \bar{\mathbf{q}} = \mathbf{q}'(\cdot, A) = \mathbf{1}.$$

Since $\bar{\mathbf{q}} \le \mathbf{q}'(\cdot, B)$ for all B, we have $\mathbf{q}'(\cdot, B) = \mathbf{1}$.

(ii) $\lambda \in (\lambda'_w, \lambda_w)$. By definition, since $\lambda'_w = \lambda'_s$, there is global and local survival for the λ-(X, K') BRW. This implies that $\bar{\mathbf{q}}' \le \mathbf{q}'(\cdot, B) < \mathbf{1}$ for every set B. On the other hand, there is global and local extinction for the λ-(X, K) BRW which implies $\bar{\mathbf{q}} = \mathbf{q}(\cdot, B) = \mathbf{1}$. Again, according to Theorem 4.2(1), $\bar{\mathbf{q}}' = \mathbf{q}'(\cdot, A) < \mathbf{1}$, that is, strong local survival in A. Irreducibility implies $\bar{\mathbf{q}}' = \mathbf{q}'(\cdot, B)$ for every (finite or infinite) set B.

(iii) Clearly, since $\lambda = \lambda_w \le \lambda_s$, we have $\mathbf{q}(\cdot, B) = \mathbf{1}$ for all finite subsets B. Hence

$$\lambda - (X, K) \text{ survives globally} \Longleftrightarrow \bar{\mathbf{q}} < \mathbf{q}(\cdot, A)$$

that is, according to Theorem 4.2(1), if and only if $\bar{\mathbf{q}}' < \mathbf{q}'(\cdot, A)$. This, again, implies $\bar{\mathbf{q}}' < \mathbf{q}'(\cdot, B)$ for every nonempty finite subset B. If, on the other hand, $\lambda - (X, K)$ dies out globally, then $\bar{\mathbf{q}}' = \mathbf{q}'(\cdot, A)$ and $\bar{\mathbf{q}}' = \mathbf{q}'(\cdot, B)$ for every nonempty subset B.

(iv) $\lambda \in (\lambda_w, \lambda_s]$ (we suppose that the interval is nonempty, otherwise there is nothing to prove). Here we have $\bar{\mathbf{q}} < \mathbf{1} = \mathbf{q}(\cdot, B)$ for every finite subset B. Theorem 4.2(1) yields $\bar{\mathbf{q}}' < \mathbf{q}'(\cdot, A) < \mathbf{1}$ and, by irreducibility, $\bar{\mathbf{q}}' < \mathbf{q}'(\cdot, B) < \mathbf{1}$ for every finite, nonempty subset B.

(v) $\lambda > \lambda_s$. Now, $\mathbf{q}(\cdot, B) < \mathbf{1}$ and $\mathbf{q}'(\cdot, B) < \mathbf{1}$ for every nonempty $B \subset X$. Again, by Theorem 4.2(1), we have

$$\mathbf{q}^\mu(\cdot, A) = \bar{\mathbf{q}}^\mu(\cdot) \Longleftrightarrow \mathbf{q}^\nu(\cdot, A) = \bar{\mathbf{q}}^\nu(\cdot).$$

If B is finite then Theorem 4.2(2) yields the conclusion.

□

Details on Example 3.6. We are considering the BRW on $\mathbb{N}$ where every particle at n has two children at $n+1$ with probability $p - \varepsilon$, one child at $\max(0, n-1)$ with probability ε and no children with probability $1-p$. We fixed $p < 1/\sqrt{2}$ and $\varepsilon(p-\varepsilon) \le 1/8$. We know that $M_w = 2p - \varepsilon > 1$ and now we compute M_s. More precisely, we prove that, given $\varepsilon(p-\varepsilon) \le 1/8$, we have local extinction, that is, $M_s \le 1$. Indeed, $1/M_s = \max\{z \ge 0 \colon \Phi(x, x|z) \le 1\}$ where $\Phi(x, y|z) := \sum_{n=1}^\infty \phi^n(x, y) z^n$ and $\phi^n(x, y)$ is the expected progeny at y of a particle which is at x at time 0, along an n-step reproduction trail which hits y for the first time at step n (see [3, Section 2.2] and [23, Sec. 3.2]). It is easy to see that $\Phi(0, 0|z) = \varepsilon z + 2(p-\varepsilon) z \Phi(1, 0|z)$, $\Phi(1, 0|z) = \varepsilon z + 2(p-\varepsilon) z \Phi(2, 0|z)$ and $\Phi(2, 0|z) = (\Phi(1, 0|z))^2$. Solving the quadratic equation in $\Phi(1, 0|z)$ and choosing the solution which has a finite limit as $z \to 0$, we get that

$$\Phi(1, 0|z) = \frac{1 - \sqrt{1 - 8\varepsilon z^2 (p-\varepsilon)}}{4z(p-\varepsilon)},$$
$$\Phi(0, 0|z) = \varepsilon z + \frac{1 - \sqrt{1 - 8\varepsilon z^2 (p-\varepsilon)}}{2}.$$

Clearly $M_s \le 1$ if and only if $\Phi(x, x|1) \le 1$ which, in turn, is equivalent to $8\varepsilon(p-\varepsilon) \le 1$ and $2\varepsilon - 1 \le \sqrt{1 - 8\varepsilon(p-\varepsilon)}$. Note that $2p - \varepsilon > 1$ and $p < 1/\sqrt{2}$, hence $2\varepsilon - 1 < 4p - 3 < 2\sqrt{2} - 3 < 0$; thus $\Phi(x, x|1) < 1$ and $M_s \le 1$.

Let us compute the set of fixed points; we prove the existence of an uncountable number of fixed points. Clearly if $\mathbf{z}(0) = (1-p)/(p-\varepsilon)$ (resp. $\mathbf{z}(0) = 1$) we have $\mathbf{z} = (1-p)/(p-\varepsilon) \cdot \mathbf{1}$ (resp. $\mathbf{z} = \mathbf{1}$). This gives the two constant fixed points (the smallest one $\bar{\mathbf{q}}$ and the largest one): observe that these constants are the solutions of $J(x) = 0$, where $J(x) := (p-\varepsilon)x^2 - (1-\varepsilon)x + 1 - p$. Hence $J(x) < 0$ for all $x \in \big((1-p)/(p-\varepsilon), 1\big)$. Any other fixed point must satisfy $\bar{\mathbf{q}} < \mathbf{z} < \mathbf{1}$, thus $\mathbf{z}(0) \in \big((1-p)/(p-\varepsilon), 1\big)$. We prove by induction that, whenever we fix $\mathbf{z}(0) \in \big((1-p)/(p-\varepsilon), 1\big)$, then

$$(P_n) = \begin{cases} \mathbf{z}(n) > \mathbf{z}(n-1) \\ \mathbf{z}(n) \in \big((1-p)/(p-\varepsilon), 1\big) \\ 1-\mathbf{z}(n) > \frac{1-\mathbf{z}(n-1)}{2p} \end{cases}$$

hold for every $n \geq 1$. This will prove that any suitable choice of $\mathbf{z}(0)$ gives a fixed point. The previous conditions are clearly redundant but it is easier to proceed like this. Using the equation, $G(\mathbf{z}) = \mathbf{z}$, we have

$$\begin{cases} \mathbf{z}(1) = \sqrt{\frac{(1-\varepsilon)\mathbf{z}(0)-(1-p)}{p-\varepsilon}} \\ \mathbf{z}(n+1) = \sqrt{\frac{\mathbf{z}(n)-\varepsilon\mathbf{z}(n-1)-(1-p)}{p-\varepsilon}} \quad \text{if } n \geq 1. \end{cases} \tag{5.8}$$

Since $\mathbf{z}(0) \in \big((1-p)/(p-\varepsilon), 1\big)$ and $\mathbf{z}(1)^2 - \mathbf{z}(0)^2 = -J(\mathbf{z}(0))/(p-\varepsilon) > 0$ then $\mathbf{z}(1) > \mathbf{z}(0) > (1-p)/(p-\varepsilon)$. Clearly $\mathbf{z}(1) = \sqrt{\big((1-\varepsilon)\mathbf{z}(0) - (1-p\big)/(p-\varepsilon)} < \sqrt{\big(1-\varepsilon-(1-p)\big)/(p-\varepsilon)} < 1$. Moreover, using the previous inequality,

$$1-\mathbf{z}(1) = \frac{1-\frac{(1-\varepsilon)\mathbf{z}(0)-(1-p)}{p-\varepsilon}}{1+\sqrt{\frac{(1-\varepsilon)\mathbf{z}(0)-(1-p)}{p-\varepsilon}}} = \frac{1-\frac{(1-\varepsilon)\mathbf{z}(0)-(1-p)}{p-\varepsilon}}{1+\mathbf{z}(1)} > \frac{1-\mathbf{z}(0)}{2}\cdot\frac{1-\varepsilon}{p-\varepsilon} > \frac{1-\mathbf{z}(0)}{2p}$$

thus (P_1) holds.

Let us prove that $(P_n) \Longrightarrow (P_{n+1})$. Using $\mathbf{z}(n) > \mathbf{z}(n-1)$ we have $\mathbf{z}(n+1)^2 - \mathbf{z}(n)^2 > (\mathbf{z}(n) - \varepsilon\mathbf{z}(n) - (1-p))/(p-\varepsilon) - \mathbf{z}(n)^2 = -J(\mathbf{z}(n))/(p-\varepsilon) > 0$ where the last inequality comes from $\mathbf{z}(n) \in \big((1-p)/(p-\varepsilon), 1\big)$. Hence $\mathbf{z}(n+1) > \mathbf{z}(n) > (1-p)/(p-\varepsilon)$. On the other hand, using $\mathbf{z}(n) < 1$ and $1-\mathbf{z}(n-1) \leq 2p(1-\mathbf{z}(n))$,

$$\begin{aligned} 1-\mathbf{z}(n+1) &= \frac{1-\frac{\mathbf{z}(n)-\varepsilon\mathbf{z}(n-1)-(1-p)}{p-\varepsilon}}{1+\sqrt{\frac{\mathbf{z}(n)-\varepsilon\mathbf{z}(n-1)-(1-p)}{p-\varepsilon}}} = \frac{1-\mathbf{z}(n)-\varepsilon(1-\mathbf{z}(n-1))}{(p-\varepsilon)(1+\mathbf{z}(n+1))} \\ &> (1-\mathbf{z}(n))\frac{1-2p\varepsilon}{(p-\varepsilon)(1+\mathbf{z}(n+1))} = (\$) \end{aligned}$$

which implies $\mathbf{z}(n+1) < 1$ (since $p-\varepsilon > 1-p > 0$ whence $1-2p\varepsilon > 1-2p^2 > 0$ whenever $p < 1/\sqrt{2}$). Using this last inequality (and the bound $p < 1/\sqrt{2}$), we prove the last part of (P_{n+1}):

$$(\$) > (1-\mathbf{z}(n))\frac{1-2p\varepsilon}{2(p-\varepsilon)} = \frac{1-\mathbf{z}(n)}{2}\Big(\frac{\varepsilon(1-2p^2)}{p(p-\varepsilon)} + \frac{1}{p}\Big) > \frac{1-\mathbf{z}(n)}{2p}.$$

Hence the set of fixed points is uncountable. □

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