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**Beurling-Lax type theorems in the complex and
quaternionic setting: the half-space case**

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BEURLING-LAX TYPE THEOREMS IN THE COMPLEX AND QUATERNIONIC SETTING: THE HALF-SPACE CASE

DANIEL ALPAY AND IRENE SABADINI

ABSTRACT. We give a generalization of the Beurling-Lax theorem both in the complex and quaternionic settings. We consider in the first case functions meromorphic in the right complex half-plane, and functions slice hypermeromorphic in the right quaternionic half-space in the second case. In both settings we also discuss a unified framework, which includes both the disk and the half-plane for the complex case and the open unit ball and the half-space in the quaternionic setting.

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Key words: Beurling-Lax theorem, analytic functions in the unit disk, in the half-plane, slice hyperholomorphic functions in the quaternionic unit ball, in the quaternionic half space, de Branges Rovnyak spaces.

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1. INTRODUCTION

This paper mainly deals with a Beurling-Lax theorem for vector-valued functions meromorphic in the right open half-plane \mathbb{C}_r , and slice hypermeromorphic in the right half-space \mathbb{H}_+ in the quaternionic setting. For $\alpha \in \mathbb{C}$ we denote by R_α the resolvent-like

operator

$$R_\alpha f(z) = \begin{cases} \frac{f(z) - f(\alpha)}{z - \alpha}, & z \neq \alpha, \\ f'(\alpha), & z = \alpha, \end{cases}$$

where the (possibly vector-valued) function f is analytic in a neighborhood of α . The name comes from the resolvent identity

$$(1.1) \quad R_\alpha - R_\beta = (\alpha - \beta)R_\alpha R_\beta$$

which they satisfy, and which we use in the sequel.

It is useful to remark that $R_0^* = M_z$ (the operator of multiplication by z) in the Hardy space $\mathbf{H}_2(\mathbb{D})$ of the open unit disk \mathbb{D} and that

$$R_\alpha^* = (M_z + \bar{\alpha}I)^{-1}, \quad \alpha \in \mathbb{C}_r,$$

in the Hardy space $\mathbf{H}_2(\mathbb{C}_r)$ of the right half-plane (for the proof of this fact, one makes use of computations similar to those in the proof of Lemma 2.7).

The Beurling-Lax theorem gives a characterization of the M_z -invariant subspaces of the Hardy space, see [27, 28] and [33] for the vector-valued case. In the scalar case they are spaces of the form $j\mathbf{H}_2(\mathbb{D})$ in the disk case and $j\mathbf{H}_2(\mathbb{C}_r)$ in the half-plane case, where j is an inner function (meaning that the operator M_j of multiplication by j is an isometry from the corresponding Hardy space into itself). The orthogonal complement of such a space is therefore R_α -invariant (for appropriate choices of α) and has reproducing kernel

$$(1.2) \quad \frac{1 - j(z)\overline{j(w)}}{1 - z\bar{w}} \text{ (disk case) or } \frac{1 - j(z)\overline{j(w)}}{2\pi(z + \bar{w})} \text{ (half-plane case).}$$

These functions are still positive definite (in the open unit disk and the open right half-plane respectively) when j is assumed analytic and contractive in \mathbb{D} (respectively in \mathbb{C}_r), but not necessarily inner. Then the corresponding reproducing kernel Hilbert space is not included isometrically, but only contractively, inside the underlying Hardy space.

One of the purposes of this work is to characterize reproducing kernel Hilbert spaces with reproducing kernel of the form $\frac{1 - j(z)\overline{j(w)}}{2\pi(z + \bar{w})}$ for such j , and more generally Pontryagin spaces when moreover j is operator-valued. To put the study in perspective we begin this introduction with a few facts on Hardy spaces.

1.1. The case of Hardy spaces. We begin by recalling the following result:

Theorem 1.1. *The Hardy space $\mathbf{H}_2(\mathbb{D})$ is the reproducing kernel Hilbert space with reproducing kernel*

$$\frac{1}{1 - z\bar{w}}.$$

It is R_α invariant for $\alpha \in \mathbb{D}$, and the following identity holds:

$$(1.3) \quad \langle f, g \rangle + \alpha \langle R_\alpha f, g \rangle + \bar{\beta} \langle f, R_\beta g \rangle - (1 - \alpha\bar{\beta}) \langle R_\alpha f, R_\beta g \rangle - \overline{g(\beta)} f(\alpha) = 0,$$

where $f, g \in \mathbf{H}_2(\mathbb{D})$ and $\alpha, \beta \in \mathbb{D}$.

We will use (1.3) in the proof of Proposition 3.1 and for this reason we now give a quick proof of it. Note that in view of the resolvent identity (1.1), we have

$$R_0(I + \alpha R_\alpha) = R_\alpha, \quad \alpha \in \mathbb{D}.$$

The left hand-side of (1.3) may be rewritten as

$$\langle (I + \alpha R_\alpha)f, (I + \beta R_\beta)g \rangle - \langle R_0(I + \alpha R_\alpha)f, R_0(I + \beta R_\beta)g \rangle = \overline{g(\beta)}f(\alpha),$$

or, setting $F = (I + \alpha R_\alpha)f$ and $G = (I + \beta R_\beta)g$,

$$\langle F, G \rangle - \langle R_0F, R_0G \rangle = \overline{G(0)}F(0),$$

which is trivial in $\mathbf{H}_2(\mathbb{D})$.

In the case of the right half-plane \mathbb{C}_r we have:

Theorem 1.2. *The Hardy space $\mathbf{H}_2(\mathbb{C}_r)$ is the reproducing kernel Hilbert space with reproducing kernel*

$$\frac{1}{2\pi(z + \bar{w})}.$$

It is R_α invariant for $\alpha \in \mathbb{C}_r$, and the following identity holds:

$$(1.4) \quad \langle R_\alpha f, g \rangle + \langle f, R_\beta g \rangle + (\alpha + \bar{\beta})\langle R_\alpha f, R_\beta g \rangle + 2\pi\overline{g(\beta)}f(\alpha) = 0,$$

where $f, g \in \mathbf{H}_2(\mathbb{C}_r)$ and $\alpha, \beta \in \mathbb{C}_r$.

The proof that (1.4) holds in $\mathbf{H}_2(\mathbb{C}_r)$ will be used in the sequel, and thus it will be recalled in Section 2.4.

It is worthwhile to mention that an approach to generalized Beurling-Lax theorems was developed by de Branges and Rovnyak, see [22, 23, 24], and consists in leaving the realm of the Hardy space, but keeping equalities (1.3) or (1.4) (or, some variations of these), and then work in the setting of reproducing kernel spaces; see [16, 21, 37]. A related paper, which makes use of de Branges spaces and (1.4) and uses Riccati equations to consider the case of singular Gram matrices, is [26].

In another approach (see [24]) one considers inequality in (1.3), setting $f = g$ and $\alpha = \beta = 0$. More generally, setting $f = g$ and $\alpha = \beta$ in equalities (1.3) and (1.4), one can weaken the equalities to the requirements

$$(1.5) \quad \langle f, f \rangle + \alpha\langle R_\alpha f, f \rangle + \bar{\alpha}\langle f, R_\alpha f \rangle - (1 - |\alpha|^2)\langle R_\alpha f, R_\alpha f \rangle - |f(\alpha)|^2 \leq 0,$$

or

$$(1.6) \quad \langle R_\alpha f, f \rangle + \langle f, R_\alpha f \rangle + (2\operatorname{Re} \alpha)\langle R_\alpha f, R_\alpha f \rangle + |f(\alpha)|^2 \leq 0.$$

depending on the setting (disk or right half-plane). The corresponding Hilbert spaces are then contractively included inside the corresponding Hardy spaces. This will not be true anymore when one introduces indefinite metrics.

1.2. The case of Pontryagin spaces. To motivate our results and provide the setting to the paper we recall some results from [2, 10]. The various notions related to Pontryagin spaces are reviewed in Section 2.1, and a reader not familiar with the theory of indefinite inner product spaces can specialize the forthcoming discussion to the case of Hilbert spaces and positive definite functions.

Let \mathcal{C} and \mathcal{D} be two Pontryagin spaces with the same index of negativity. By $\mathbf{L}(\mathcal{D}, \mathcal{C})$ we will denote the set of continuous linear operators from \mathcal{D} to \mathcal{C} . A $\mathbf{L}(\mathcal{D}, \mathcal{C})$ -valued function S analytic in some open subset Ω of the open unit disk is called a generalized Schur function if the $\mathbf{L}(\mathcal{C}, \mathcal{C})$ -valued kernel

$$(1.7) \quad K_S(z, w) = \frac{I_{\mathcal{C}} - S(z)S(w)^*}{1 - z\bar{w}}, \quad z, w \in \Omega.$$

has a finite number of negative squares, say κ , in Ω . Assuming $0 \in \Omega$, it is proved in [10] that S is a generalized Schur function if and only if it can be written in the form

$$(1.8) \quad S(z) = D + zC(I_{\mathcal{P}} - zA)^{-1}B$$

where \mathcal{P} is a Pontryagin space with index of negativity κ and where the operator-matrix

$$(1.9) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{P} \oplus \mathcal{D} \longrightarrow \mathcal{P} \oplus \mathcal{C}$$

is coisometric. It follows that S has a unique meromorphic extension to \mathbb{D} , and for S so extended the kernel K_S has still κ negative squares for z, w in the domain of analyticity of S . We here recall that generalized Schur functions and related classes of vector-valued functions have been extensively studied by Krein and Langer; see for instance [30, 31, 32]. The reproducing kernel Pontryagin space $\mathcal{P}(S)$, with reproducing kernel K_S is R_0 -invariant and the coisometry property in (1.9) implies that

$$(1.10) \quad [R_0f, R_0f]_{\mathcal{P}} \leq [f, f]_{\mathcal{P}} - [f(0), f(0)]_{\mathcal{C}}, \quad \forall f \in \mathcal{P}.$$

Conversely, the following characterization of $\mathcal{P}(S)$ spaces was given in [10, Theorem 3.1.2, p. 85]:

Theorem 1.3. *Let \mathcal{C} be a Pontryagin space, and let Ω be an open subset of the open unit disk \mathbb{D} containing the origin. Let \mathcal{P} be a reproducing kernel Pontryagin space of \mathcal{C} -valued functions analytic in Ω , which is R_0 -invariant and such that (1.10) holds in \mathcal{P} . Then the elements of \mathcal{P} have a meromorphic extension to \mathbb{D} and there exists a Pontryagin space \mathcal{C}_1 with $\text{ind}_-(\mathcal{C}_1) = \text{ind}_-(\mathcal{C})$ and a function $S \in \mathcal{S}_{\kappa}(\mathcal{C}_1, \mathcal{C})$, with $\kappa = \text{ind}_-(\mathcal{P})$, such that the reproducing kernel of the space \mathcal{P} is of the form (1.7).*

Thus the function S is meromorphic in \mathbb{D} , with domain of analyticity $\Omega(S)$. Formula (1.7) means that elements of \mathcal{P} are restrictions to Ω of the elements of the reproducing kernel Hilbert space $\mathcal{P}(S)$ with reproducing kernel $K_S(z, w)$.

A version of Theorem 1.3 in the quaternionic setting, and for slice hyperholomorphic functions, was proved in [7, Theorem 7.1, p. 862]. The purpose of this note is to give a version of this result in the case of the right half-plane in the complex setting (see Theorem 2.5), and for the right half-space in the quaternionic setting (see Theorem 4.13). Finally we remark that inequality (1.10) can be set at an arbitrary point of the open unit disk \mathbb{D} as

$$(1.11) \quad [R_{\alpha}f, R_{\alpha}f]_{\mathcal{P}} \leq [(I_{\mathcal{P}} + \alpha R_{\alpha})f, (I_{\mathcal{P}} + \alpha R_{\alpha})f]_{\mathcal{P}} - [f(\alpha), f(\alpha)]_{\mathcal{C}}.$$

See [11, (3.6) in Theorem 3.4]. Furthermore, we will show that it is possible to use the setting developed in [13, 14, 15] to write both theorems 1.3 and 2.5 under a common setting.

The outline of the paper is as follows. It consists of three sections besides the introduction. The second section is devoted to the case of the right half-plane case \mathbb{C}_r , and is divided into four subsections. In the first two subsections we review briefly some notions on Pontryagin spaces and on operator-valued generalized Schur functions associated to the right half-plane. We then prove the counterpart of Theorem 1.3, and consider the particular case of spaces isometrically included in the Hardy space of the right half-plane. In the third section, divided into four subsections, we discuss the unified setting, to which we already alluded. We present the main aspects of this setting, discuss the Hardy space and the generalized Schur functions in this framework, and consider a general theorem, namely Theorem 3.2, which includes as particular cases Theorems 1.3 and 2.5. In the fourth and last section we study the counterpart of Theorem 1.3 in the setting of slice hyperholomorphic functions in the half-space case and we briefly discuss the unified setting. To keep the exposition self-contained, in this section we also provide the necessary background on slice hyperholomorphic functions.

2. THE COMPLEX-VALUED CASE

2.1. Pontryagin spaces and their operators. We begin this section by reviewing some basic facts on Pontryagin spaces. For more information we refer the reader to [20, 25, 29]. A vector space \mathcal{V} endowed with an Hermitian form $[\cdot, \cdot]$ is called a Pontryagin space if it can be written as

$$(2.1) \quad \mathcal{V} = \mathcal{V}_+[+] \mathcal{V}_-,$$

where:

- (a) The spaces $(\mathcal{V}, [\cdot, \cdot])$ and $(\mathcal{V}_-, -[\cdot, \cdot])$ are Hilbert spaces, and \mathcal{V}_- is finite dimensional.
- (b) $\mathcal{V}_+ \cap \mathcal{V}_- = \{0\}$ and

$$(2.2) \quad [v_+, v_-] = 0, \quad \forall v_+ \in \mathcal{V}_+ \quad \text{and} \quad v_- \in \mathcal{V}_-.$$

The decomposition (2.1) is called a fundamental decomposition; it is not unique (unless \mathcal{V} is a Hilbert space, or an anti Hilbert space), but all spaces \mathcal{V}_- appearing in a fundamental decompositions have the same dimension, called the index of negativity (or simply, the index) of the Pontryagin space. For a given fundamental decomposition, the map

$$(2.3) \quad \|v\| = [v_+, v_+] - [v_-, v_-]$$

is a norm and $(\mathcal{V}, \|\cdot\|)$ is a Hilbert space. All norms (2.3) are equivalent, and define the topology of the Pontryagin space. Continuity of linear operators is defined with respect to this topology.

Given two Pontryagin spaces $(\mathcal{V}_1, [\cdot, \cdot]_1)$ and $(\mathcal{V}_2, [\cdot, \cdot]_2)$, the adjoint of a continuous linear operator from \mathcal{V}_1 into \mathcal{V}_2 is defined by

$$(2.4) \quad [Av_1, v_2]_2 = [v_1, A^*v_2]_1, \quad \forall v_1 \in \mathcal{V}_1 \quad \text{and} \quad v_2 \in \mathcal{V}_2.$$

The following important results in the theory of linear operators in Pontryagin spaces are used in the sequel:

Theorem 2.1. *Let $(\mathcal{V}_1, [\cdot, \cdot]_1)$ and $(\mathcal{V}_2, [\cdot, \cdot]_2)$ be two Pontryagin spaces with same index of negativity, and let T be a densely defined contraction from $\mathcal{D}(T) \subset \mathcal{V}_1$ into \mathcal{V}_2 , meaning that*

$$[Tv_1, Tv_1]_2 \leq [v_1, v_1]_1, \quad \forall v_1 \in \mathcal{D}(T).$$

Then, T extends to an everywhere defined contraction, whose adjoint is also a contraction.

Theorem 2.2. *A contraction between two Pontryagin spaces with same index of negativity has a maximal strictly negative invariant subspace.*

We conclude this section with the notion of negative squares.

Definition 2.3. *Let $(\mathcal{C}, [\cdot, \cdot])$ be a Pontryagin space (the coefficient space). A $\mathbf{L}(\mathcal{C})$ -valued function $K(z, w)$ defined for z, w in a set Ω has a finite number, say κ , of negative squares in Ω if it is Hermitian,*

$$K(z, w) = K(w, z)^*, \quad \forall z, w \in \Omega,$$

and if the following condition holds: For every integer N and every choice of $w_1, \dots, w_N \in \Omega$ and $c_1, \dots, c_N \in \mathcal{C}$, the $N \times N$ Hermitian matrix with lk entry $[K(w_\ell, w_k)c_k, c_\ell]$ has at most κ strictly negative eigenvalues and exactly κ such eigenvalues for some choice of $N, w_1, \dots, w_N, c_1, \dots, c_N$.

Theorem 2.4. *There is a one-to-one correspondence between $\mathbf{L}(\mathcal{C}, \mathcal{C})$ -valued functions defined on Ω and having κ negative squares there and reproducing kernel Pontryagin spaces of index of negativity κ of functions defined on Ω .*

2.2. Generalized Schur functions. Let \mathcal{C} and \mathcal{D} be two Pontryagin spaces with the same index of negativity. A $\mathbf{L}(\mathcal{D}, \mathcal{C})$ -valued function S analytic in some open subset Ω of the open right half-plane \mathbb{C}_r is called a generalized Schur function if the $\mathbf{L}(\mathcal{C})$ -valued kernel

$$(2.5) \quad K_S(z, w) = \frac{I_{\mathcal{C}} - S(z)S(w)^*}{2\pi(z + \bar{w})}, \quad z, w \in \Omega.$$

has a finite number of negative squares, say κ , in Ω . Let $\alpha \in \Omega$. It follows from the analysis in [2] that a function S is a generalized Schur function of the right half-plane if and only if it can be written in the form

$$(2.6) \quad S(z) = H + \frac{z - \alpha}{z + \bar{\alpha}} G \left(I_{\mathcal{P}} - \frac{z - \alpha}{z + \bar{\alpha}} T \right)^{-1} F,$$

where \mathcal{P} is a Pontryagin space with index of negativity κ and where the operator-matrix

$$(2.7) \quad \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \mathcal{P} \oplus \mathcal{D} \longrightarrow \mathcal{P} \oplus \mathcal{C}$$

is coisometric. It follows that S has a unique meromorphic extension to \mathbb{C}_r , and for S so extended the kernel K_S has still κ negative squares for z, w in the domain of analyticity of S . It follows that the space $\mathcal{P}(S)$ is R_α -invariant and that

$$(2.8) \quad (2\operatorname{Re} \alpha)[R_\alpha f, R_\alpha f]_{\mathcal{P}(S)} + [R_\alpha f, f]_{\mathcal{P}(S)} + [f, R_\alpha f]_{\mathcal{P}(S)} + 2\pi[f(\alpha), f(\alpha)]_{\mathcal{C}} \leq 0$$

holds in $\mathcal{P}(S)$. Note that this inequality is the counterpart of (1.10), and appeared in [11, (3.7) in Theorem 3.4].

We refer to [19] for other results on realizations of Schur functions in a half-plane.

2.3. The structure theorem. The main result of this section is the following theorem, which is the counterpart of Theorem 1.3 in the case of the right half-space. To prove it, we follow closely the computations in [10]. A key tool in the arguments is inequality (2.8)

Theorem 2.5. *Let \mathcal{C} be a Pontryagin space, and let Ω be an open subset of \mathbb{C}_r . Let $\alpha \in \Omega$ be fixed. Let \mathcal{P} be a reproducing kernel Pontryagin space of \mathcal{C} -valued functions analytic in Ω , which is R_α -invariant and such that (2.8) holds in \mathcal{P} . Then every function of \mathcal{P} has a unique meromorphic extension to \mathbb{C}_r and there exists a Pontryagin space \mathcal{C}_1 with $\text{ind}_-(\mathcal{C}_1) = \text{ind}_-(\mathcal{C})$ and a function $S \in \mathcal{S}_\kappa(\mathcal{C}_1, \mathcal{C})$, with $\kappa = \text{ind}_-(\mathcal{P})$, such that the reproducing kernel of \mathcal{P} is of the form*

$$(2.9) \quad K_S(z, w) = \frac{I_{\mathcal{C}} - S(z)S(w)^*}{2\pi(z + \bar{w})}.$$

As for Theorem 1.3 we note that the function S is meromorphic in \mathbb{C}_r , with domain of analyticity $\Omega(S)$. Formula (2.9) means that elements of \mathcal{P} are restrictions to Ω of the elements of the reproducing kernel Hilbert space $\mathcal{P}(S)$ with reproducing kernel $K_S(z, w)$.

Proof of Theorem 2.5. We set

$$k = (2\text{Re } \alpha), \quad T = kR_\alpha + I_{\mathcal{P}}, \quad G = \sqrt{2\pi k}C_\alpha, \quad \text{and} \quad C = \begin{pmatrix} T \\ G \end{pmatrix},$$

where C_α denotes the point evaluation at α . After multiplying by 2π , inequality (2.8) may be rewritten as

$$(kR_\alpha + I_{\mathcal{P}})^*(kR_\alpha + I_{\mathcal{P}}) + 2\pi kC_\alpha^*C_\alpha \leq I_{\mathcal{P}},$$

that is,

$$(2.10) \quad I_{\mathcal{P}} - C^*C \geq 0.$$

By [10, Theorem 1.3.4 (1), p. 25] we have

$$(2.11) \quad \text{ind}_-(I_{\mathcal{P} \oplus \mathcal{C}} - CC^*) + \text{ind}_-(\mathcal{P}) = \text{ind}_-(I_{\mathcal{P}} - C^*C) + \text{ind}_-(\mathcal{P}) + \text{ind}_-(\mathcal{C}).$$

Using (2.10) we get:

$$\text{ind}_-(I_{\mathcal{P} \oplus \mathcal{C}} - CC^*) = \text{ind}_-(\mathcal{C}).$$

By the Bognár-Krámlí theorem, see [10, pp. 20-21], there exists a defect operator, that is there exists a Pontryagin space \mathcal{C}_1 with same negative index as \mathcal{C} and operators

$$(2.12) \quad F \in \mathbf{L}(\mathcal{C}_1, \mathcal{P}) \quad \text{and} \quad H \in \mathbf{L}(\mathcal{C}_1, \mathcal{C})$$

such that

$$(2.13) \quad I_{\mathcal{P} \oplus \mathcal{C}} - CC^* = \begin{pmatrix} F \\ H \end{pmatrix} \begin{pmatrix} F \\ H \end{pmatrix}^*.$$

It follows that the operator matrix

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} : \mathcal{P} \oplus \mathcal{C}_1 \rightarrow \mathcal{P} \oplus \mathcal{C}$$

is coisometric. We define S via (2.6) in a neighborhood of the point α . This formula defines a meromorphic function in \mathbb{C}_r , as is explained in [10] for the disk case. When \mathcal{C} is a Hilbert space, this follows from the fact that T is then a contraction and has a maximal strictly negative invariant subspace. The case of Pontryagin coefficient spaces is reduced

to the Hilbert space case using the Potapov-Ginzburg transform.

As in [10, Theorem 2.1.2 (1), p. 44] we have

$$\frac{I_{\mathcal{C}} - S(z)S(w)^*}{1 - \frac{z - \alpha}{z + \bar{\alpha}} \frac{\bar{w} - \bar{\alpha}}{\bar{w} + \alpha}} = G \left(I_{\mathcal{P}} - \frac{z - \alpha}{z + \bar{\alpha}} T \right)^{-1} \left(I_{\mathcal{P}} - \frac{w - \alpha}{w + \bar{\alpha}} T \right)^{-*} G^*,$$

which can be rewritten as

(2.14)

$$(z + \bar{\alpha})(\bar{w} + \alpha) \frac{I_{\mathcal{C}} - S(z)S(w)^*}{(z + \bar{w})(\alpha + \bar{\alpha})} = 2\pi k C_{\alpha} \left(I_{\mathcal{P}} - \frac{z - \alpha}{z + \bar{\alpha}} T \right)^{-1} \left(I_{\mathcal{P}} - \frac{w - \alpha}{w + \bar{\alpha}} T \right)^{-*} C_{\alpha}^*.$$

To conclude the proof we show that the point evaluation C_w is given by:

$$(2.15) \quad C_w = \frac{\alpha + \bar{\alpha}}{w + \bar{\alpha}} C_{\alpha} \left(I_{\mathcal{P}} - \frac{w - \alpha}{w + \bar{\alpha}} T \right)^{-1}.$$

To this end, we note that

$$(2.16) \quad (Tf)(z) = (kR_{\alpha}f)(z) + f(z) = \frac{(z + \bar{\alpha})f(z) - (\alpha + \bar{\alpha})f(\alpha)}{z - \alpha}.$$

Let now $h \in \mathcal{P}$ and set

$$\frac{\alpha + \bar{\alpha}}{w + \bar{\alpha}} \left(I_{\mathcal{P}} - \frac{w - \alpha}{w + \bar{\alpha}} T \right)^{-1} h = g$$

In view of (2.16) we have

$$h(z) = \frac{w + \bar{\alpha}}{\alpha + \bar{\alpha}} \left(g(z) - \frac{w - \alpha}{w + \bar{\alpha}} \frac{(z + \bar{\alpha})g(z) - (\alpha + \bar{\alpha})g(\alpha)}{z - \alpha} \right).$$

Setting $z = w$ in the above expression we obtain $h(w) = g(\alpha)$ so (2.15) is proved. Thus (2.14) can be rewritten as

$$\frac{I_{\mathcal{C}} - S(z)S(w)^*}{2\pi(z + \bar{w})} = C_z C_w^*,$$

and so $K_{\mathcal{S}}(z, w)$ is the reproducing kernel of \mathcal{P} . \square

Remark 2.6. When \mathcal{C} is separable the space \mathcal{P} is also separable since it consists of analytic functions, and \mathcal{C}_1 can be chosen separable.

2.4. The case of subspaces of the Hardy space. We now consider the special case where the space \mathcal{P} (which we now denote by \mathcal{H}) is a Hilbert space isometrically included in the Hardy space $\mathbf{H}_2(\mathbb{C}_r)$ of the right half-plane. The following lemma is proved in [11], and implies that (2.8) is in fact an equality in $\mathbf{H}_2(\mathbb{C}_r)$. Its proof is recalled for completeness.

Lemma 2.7. *Equality (1.4) holds in $\mathbf{H}_2(\mathbb{C}_r)$.*

Proof. We use the fact that $k(z, w) = \frac{1}{2\pi(z + \bar{w})}$ is the reproducing kernel of $\mathbf{H}_2(\mathbb{C}_r)$. We have for α, β, μ, ν in \mathbb{C}_r

$$R_{\alpha}k(\cdot, \mu) = -\frac{1}{\alpha + \bar{\mu}}k(\cdot, \mu) \quad \text{and} \quad R_{\alpha}k(\cdot, \nu) = -\frac{1}{\alpha + \bar{\nu}}k(\cdot, \nu).$$

It follows that

$$\begin{aligned}\langle R_\alpha k(\cdot, \mu), R_\beta k(\cdot, \nu) \rangle &= \frac{1}{(\alpha + \bar{\mu})(\bar{\beta} + \nu)} k(\nu, \mu) = \frac{1}{2\pi(\alpha + \bar{\mu})(\bar{\beta} + \nu)(\nu + \bar{\mu})}, \\ \langle R_\alpha k(\cdot, \mu), k(\cdot, \nu) \rangle &= -\frac{1}{\alpha + \bar{\mu}} k(\nu, \mu) = -\frac{1}{2\pi(\alpha + \bar{\mu})(\nu + \bar{\mu})}, \\ \langle k(\cdot, \mu), R_\beta k(\cdot, \nu) \rangle &= -\frac{1}{\bar{\beta} + \nu} k(\nu, \mu) = -\frac{1}{2\pi(\bar{\beta} + \nu)(\nu + \bar{\mu})}.\end{aligned}$$

Furthermore,

$$2\pi \overline{k(\beta, \nu)} k(\alpha, \mu) = \frac{2\pi}{(2\pi)^2} \frac{1}{(\alpha + \bar{\mu})(\bar{\beta} + \nu)} = \frac{1}{2\pi} \frac{1}{(\alpha + \bar{\mu})(\bar{\beta} + \nu)}.$$

Equality (1.4) for $f = k(\cdot, \mu)$ and $g = k(\cdot, \nu)$ is thus equivalent to

$$-\frac{1}{(\alpha + \bar{\mu})(\nu + \bar{\mu})} - \frac{1}{(\bar{\beta} + \nu)(\nu + \bar{\mu})} + \frac{\alpha + \bar{\beta}}{(\alpha + \bar{\mu})(\bar{\beta} + \nu)(\nu + \bar{\mu})} + \frac{1}{(\alpha + \bar{\mu})(\bar{\beta} + \nu)} = 0,$$

which is clearly an identity. Note also that the linear span of the reproducing kernels form a dense subset of $\mathbf{H}_2(\mathbb{C}_r)$. To prove (1.4) for all $f, g \in \mathbf{H}_2(\mathbb{C}_r)$ we remark that R_α and R_β are bounded there. This latter fact can be proved in two different ways. First, using the fact that $\mathbf{H}_2(\mathbb{C}_r)$ is isometrically included in the Lebesgue space $\mathbf{L}_2(\mathbb{R})$, writing

$$R_\alpha f(z) = \frac{f(z)}{z - \alpha} - \frac{f(\alpha)}{z - \alpha},$$

and using the Cauchy-Schwarz inequality. The second way (see [11]) consists in remarking that (1.4) implies, for f in the linear span of the reproducing kernels, that

$$(2\operatorname{Re} \alpha) \|R_\alpha f\|^2 \leq 2 \|R_\alpha f\| \|f\| + 2\pi \|C_\alpha\|^2 \|f\|^2.$$

This inequality implies that, on a dense set

$$\frac{\|R_\alpha f\|^2}{\|f\|^2} \leq \frac{1}{\operatorname{Re} \alpha} \left(\frac{\|R_\alpha f\|}{\|f\|} + \pi \|C_\alpha\|^2 \right),$$

which in turn, implies that R_α extends to a bounded operator in $\mathbf{H}_2(\mathbb{C}_r)$. \square

In the notation of Theorem 2.5 let us set $\mathcal{C} = \mathbb{C}$, and hence \mathcal{C}_1 is a Hilbert space. Thus there exists a function $S \in \mathcal{S}_0(\mathcal{C}_1, \mathbb{C})$ such that \mathcal{H} is the reproducing kernel Hilbert space with reproducing kernel $\frac{1 - S(z)S(w)^*}{2\pi(z + \bar{w})}$. For every $z \in \mathbb{C}$ the value $S(z)$ is a bounded operator (in fact a contraction) from \mathcal{C}_1 into \mathbb{C} , which (since \mathcal{C}_1 is separable; see Remark 2.6) we will write in matrix form as

$$(s_1(z) \quad s_2(z) \quad \cdots),$$

after choosing an orthonormal basis of \mathcal{C}_1 . By the properties of a vector-valued analytic function (see [36]) each of the functions s_j is analytic. Since

$$\frac{1}{2\pi(z + \bar{w})} = \frac{1 - S(z)S(w)^*}{2\pi(z + \bar{w})} + \frac{S(z)S(w)^*}{2\pi(z + \bar{w})}$$

and since \mathcal{H} is isometrically included in $\mathbf{H}_2(\mathbb{C}_r)$, we have

$$\left\langle \frac{1 - S(z)S(w)^*}{2\pi(z + \bar{w})}, \frac{S(z)S(v)^*}{2\pi(z + \bar{v})} \right\rangle_{\mathbf{H}_2(\mathbb{C}_r)} = 0, \quad \forall v, w \in \mathbb{C}_r.$$

Using the reproducing kernel property (or Cauchy's theorem) we have

$$\left\langle \frac{S(w)^*}{2\pi(z + \bar{w})}, \frac{S(v)^*}{2\pi(z + \bar{v})} \right\rangle_{\mathbf{H}_2(\mathbb{C}_r) \otimes \mathcal{C}_1} = \left\langle \frac{S(z)S(w)^*}{2\pi(z + \bar{w})}, \frac{S(z)S(v)^*}{2\pi(z + \bar{v})} \right\rangle_{\mathbf{H}_2(\mathbb{C}_r)}, \quad \forall v, w \in \mathbb{C}_r,$$

where we denote by $\mathbf{H}_2(\mathbb{C}_r) \otimes \mathcal{C}_1$ the Hardy space of \mathcal{C}_1 -valued functions. It follows that the operator of multiplication by S is an isometry from the closed linear span in $\mathbf{H}_2(\mathbb{C}_r) \otimes \mathcal{C}_1$ of the functions

$$z \mapsto \frac{S(w)^*}{2\pi(z + \bar{w})}, \quad w \in \mathbb{C}_r,$$

into $\mathbf{H}_2(\mathbb{C}_r)$. Let $j \in \mathbb{N}$ be such that $s_j \neq 0$. Then the above isometry property implies that the operator of multiplication by s_j is an isometry from $\mathbf{H}_2(\mathbb{C}_r)$ into itself. By the arguments in the scalar setting it follows that s_j is inner and hence all the other s_k , $k \neq j$ are identically equal to 0. Thus we can chose $\mathcal{C}_1 = \mathbb{C}$, and we obtain that \mathcal{H} is the reproducing kernel Hilbert space with reproducing kernel $\frac{1 - s_j(z)\overline{s_j(w)}}{2\pi(z + \bar{w})}$, which means that $\mathcal{H}^\perp = s_j \mathbf{H}_2(\mathbb{C}_r)$. We thus get back to the scalar version of the Beurling-Lax theorem.

More generally, consider a matrix $J \in \mathbb{C}^{n \times n}$, which is both self-adjoint and unitary: $J = J^* = J^{-1}$ (such a matrix is called a signature matrix). Define $\mathbf{H}_2(\mathbb{C}_r, J)$ to be the space $\mathbf{H}_2(\mathbb{C}_r)^n$ endowed with the form

$$[f, g]_J = \langle f, Jg \rangle_{\mathbf{H}_2(\mathbb{C}_r)^n}, \quad f, g \in \mathbf{H}_2(\mathbb{C}_r)^n.$$

As a side remark, note that $\mathbf{H}_2(\mathbb{C}_r, J)$ is a Krein space. The identity (1.4) holds in $\mathbf{H}_2(\mathbb{C}_r, J)$, when the coefficient space \mathbb{C}^n is endowed with the form

$$[c, d]_J = d^* J c, \quad c, d \in \mathbb{C}^n.$$

Theorem 2.5 gives then the characterization of spaces of the form $S\mathbf{H}_2(\mathbb{C}_r, J)$. See [17, 18] for related results.

3. A UNIFIED SETTING IN THE COMPLEX CASE

In this section we show how we can use a unified setting to treat both the case of the unit disk and the case of the right half-plane. We first briefly recall the setting developed in the papers [12, 14] and related papers.

3.1. A unified setting. The starting point is to consider an open connected subset Ω of the complex plane, and a pair of functions $a(z)$ and $b(z)$ analytic in Ω and such that the two sets

$$\Omega_+ = \{z \in \Omega ; |b(z)| < |a(z)|\} \quad \text{and} \quad \Omega_- = \{z \in \Omega ; |b(z)| > |a(z)|\}$$

are both nonempty. Then the "boundary" set

$$\Omega_0 = \{z \in \Omega ; |b(z)| = |a(z)|\}$$

is also nonempty.

We set

$$(3.17) \quad \rho(z, w) = a(z)\overline{a(w)} - b(z)\overline{b(w)}$$

and

$$\delta(z, w) = b(z)a(\alpha) - a(z)b(\alpha).$$

The representation of ρ is unique up to an hyperbolic translation, that is, if we have two different representations of ρ as (3.17),

$$\rho(z, w) = a(z)\overline{a(w)} - b(z)\overline{b(w)} = c(z)\overline{c(w)} - d(z)\overline{d(w)},$$

then

$$\begin{pmatrix} a(z) & b(z) \end{pmatrix} = \begin{pmatrix} c(z) & d(z) \end{pmatrix} U$$

where the matrix U is such that

$$U J_0 U^* = J_0, \quad \text{where } J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and in particular the sets Ω_+ , Ω_- and Ω_0 do not depend on the given representation of ρ . One introduces the resolvent-like operators

$$(R(a, b, \alpha)f)(z) = \frac{a(z)f(z) - a(\alpha)f(\alpha)}{a(\alpha)b(z) - b(\alpha)a(z)}.$$

We note (see [14, equation (3.14), p. 9]) that

$$(3.18) \quad a(\alpha)R(b, a, \alpha) + b(\alpha)R(a, b, \alpha) = -I.$$

The case $a(z) = 1$ and $b(z) = z$ corresponds to $\Omega_+ = \mathbb{D}$, while the case $a(z) = \sqrt{2\pi} \frac{z+1}{2}$ and $b(z) = \sqrt{2\pi} \frac{z-1}{2}$ corresponds to $\Omega_+ = \mathbb{C}_r$.

Functions of the form (3.17) seem to have been considered first in the papers [34] and [35]. We also refer to [9] for a recent application to the Schur algorithm, and to [1] for a sample application to interpolation.

3.2. The Hardy space. The function $\frac{1}{\rho(z, w)} = \frac{1}{a(z)\overline{a(w)} - b(z)\overline{b(w)}}$ is positive definite in Ω_+ . The associated reproducing kernel Hilbert space will be denoted by $\mathbf{H}_2(\rho)$. Let $\sigma(z) = \frac{b(z)}{a(z)}$. For a given representation $\rho(z, w) = a(z)\overline{a(w)} - b(z)\overline{b(w)}$ of ρ we have

$$\mathbf{H}_2(\rho) = \left\{ f(z) = \frac{F(\sigma(z))}{a(z)}, F \in \mathbf{H}_2(\mathbb{D}) \right\}$$

with norm $\|f\| = \|F\|$.

Proposition 3.1. *The equality*

$$(3.19) \quad \langle R(a, b, \alpha)f, R(a, b, \beta)g \rangle_{\mathbf{H}_2(\rho)} - \langle R(b, a, \alpha)f, R(b, a, \beta)g \rangle_{\mathbf{H}_2(\rho)} + \overline{g(\beta)}f(\alpha) = 0,$$

holds in $\mathbf{H}_2(\rho)$.

Proof. Let $f(z) = \frac{F(\sigma(z))}{a(z)} \in \mathbf{H}_2(\rho)$. We first note the formulas

$$(3.20) \quad (R(a, b, \alpha)f)(z) = \frac{(R_{\sigma(\alpha)}F)(\sigma(z))}{a(\alpha)a(z)}$$

$$(3.21) \quad (R(b, a, \alpha)f)(z) = -\frac{(R_{\sigma(\alpha)}zF)(\sigma(z))}{a(\alpha)a(z)}.$$

Therefore, with $g(z) = \frac{G(\sigma(z))}{a(z)}$ another element in $\mathbf{H}_2(\rho)$, and using the formula

$$(R_u(zf))(z) = f(z) + u(R_u f)(z),$$

we have

$$\begin{aligned} & \langle R(a, b, \alpha)f, R(a, b, \beta)g \rangle_{\mathbf{H}_2(\rho)} - \langle R(b, a, \alpha)f, R(b, a, \beta)g \rangle_{\mathbf{H}_2(\rho)} + \overline{g(\beta)}f(\alpha) = \\ & = \frac{1}{a(\alpha)a(\beta)} \langle R_{\sigma(\alpha)}F, R_{\sigma(\beta)}G \rangle_{\mathbf{H}_2(\mathbb{D})} - \\ & \quad - \frac{1}{a(\alpha)a(\beta)} \langle F + \sigma(\alpha)R_{\sigma(\alpha)}F, G + \sigma(\beta)R_{\sigma(\beta)}G \rangle_{\mathbf{H}_2(\mathbb{D})} + \overline{g(\beta)}f(\alpha), \end{aligned}$$

that is,

$$\langle R_{\sigma(\alpha)}F, R_{\sigma(\beta)}G \rangle_{\mathbf{H}_2(\mathbb{D})} - \langle F + \sigma(\alpha)R_{\sigma(\alpha)}F, G + \sigma(\beta)R_{\sigma(\beta)}G \rangle_{\mathbf{H}_2(\mathbb{D})} + \overline{G(\sigma(\beta))}F(\sigma(\alpha)),$$

which is equal to 0 by (1.3). \square

Both (1.3) and (1.4) are special cases of (3.19) (see [14, equation (4.2), p. 12] for the latter equation). We will weaken this equality to the requirement

$$(3.22) \quad \langle R(a, b, \alpha)f, R(a, b, \alpha)f \rangle - \langle R(b, a, \alpha)f, R(b, a, \alpha)f \rangle + |f(\alpha)|^2 \leq 0.$$

3.3. Generalized Schur functions. As in Section 2.2, we consider \mathcal{C} and \mathcal{D} two Pontryagin spaces with the same index of negativity. A $\mathbf{L}(\mathcal{D}, \mathcal{C})$ -valued function S analytic in some open subset of $\Omega_+ \subset \Omega$ is called a generalized Schur function if the $\mathbf{L}(\mathcal{C}, \mathcal{C})$ -valued kernel

$$(3.23) \quad K_S(z, w) = \frac{I_{\mathcal{C}} - S(z)S(w)^*}{a(z)\overline{a(w)} - b(z)\overline{b(w)}}, \quad z, w \in \Omega.$$

has a finite number of negative squares, say κ , in Ω . Let $\alpha \in \Omega$. A function S is a generalized Schur function if and only if it can be written in the form

$$(3.24) \quad S(z) = H + \frac{\sigma(z) - \sigma(\alpha)}{1 - \sigma(z)\overline{\sigma(\alpha)}} G \left(I_{\mathcal{P}} - \frac{\sigma(z) - \sigma(\alpha)}{1 - \sigma(z)\overline{\sigma(\alpha)}} T \right)^{-1} F,$$

where \mathcal{P} is a Pontryagin space with index of negativity κ and where the operator-matrix (2.7) is coisometric. See e.g. [2].

Still from [2] we have that the corresponding reproducing kernel Pontryagin space is $R(a, b, \alpha)$ -invariant and that the inequality

$$(3.25) \quad [R(a, b, \alpha)f, R(a, b, \alpha)f]_{\mathcal{P}} - [R(b, a, \alpha)f, R(b, a, \alpha)f]_{\mathcal{P}} + [f(\alpha), f(\alpha)]_{\mathcal{C}} \leq 0$$

holds in it.

3.4. The structure theorem. The following theorem contains as special cases Theorems 1.3 and 2.5. Note that its proof relies on the proof of Theorem 1.3.

Theorem 3.2. *Let \mathcal{C} be a Pontryagin space, and let Ω be an open subset of the open complex plane. Let \mathcal{P} be a reproducing kernel Pontryagin space of \mathcal{C} -valued functions analytic in Ω , which is R_0 -invariant and such that (3.25) holds in \mathcal{P} . Then every elements of \mathcal{P} has a unique meromorphic extension to Ω_+ , and there exists a Pontryagin space \mathcal{C}_1 with $\text{ind}_-(\mathcal{C}_1) = \text{ind}_-(\mathcal{C})$ and a function $S \in \mathcal{S}_\kappa(\mathcal{C}_1, \mathcal{C})$, with $\kappa = \text{ind}_-(\mathcal{P})$, such that the reproducing kernel of the space \mathcal{P} is of the form (3.23).*

Proof. We proceed in a number of steps.

STEP 1: (3.25) can be rewritten as

$$(3.26) \quad \left(\frac{(|a(\alpha)|^2 - |b(\alpha)|^2)R(a, b, \alpha) - \overline{b(\alpha)}I_{\mathcal{P}}}{a(\alpha)} \right)^* \left(\frac{(|a(\alpha)|^2 - |b(\alpha)|^2)R(a, b, \alpha) - \overline{b(\alpha)}I_{\mathcal{P}}}{a(\alpha)} \right) + (|a(\alpha)|^2 - |b(\alpha)|^2)C_\alpha^*C_\alpha \leq I_{\mathcal{P}}.$$

Recall that $a(\alpha) \neq 0$ since $\alpha \in \Omega_+$. Taking into account (3.18) we rewrite (3.25) as

$$R(a, b, \alpha)^*R(a, b, \alpha) - \frac{1}{|a(\alpha)|^2}(I_{\mathcal{P}} + b(\alpha)R(a, b, \alpha))^*(I_{\mathcal{P}} + b(\alpha)R(a, b, \alpha)) + C_\alpha^*C_\alpha \leq 0,$$

that is, (3.26).

We now proceed as in the proof of Theorem 2.5 with now

$$T = \frac{(|a(\alpha)|^2 - |b(\alpha)|^2)R(a, b, \alpha) - \overline{b(\alpha)}I_{\mathcal{P}}}{a(\alpha)}, \quad G = \sqrt{(|a(\alpha)|^2 - |b(\alpha)|^2)}C_\alpha, \quad \text{and } C = \begin{pmatrix} T \\ G \end{pmatrix}.$$

We define operators F, H via (2.13) and set $\sigma(z) = \frac{b(z)}{a(z)}$.

STEP 2: *It holds that*

$$(3.27) \quad (1 - \sigma(z)\overline{\sigma(\alpha)})I_{\mathcal{P}} - (\sigma(z) - \sigma(\alpha))T = \frac{\rho(\alpha, \alpha)}{|a(\alpha)|^2} \left(I_{\mathcal{P}} - \frac{\delta(z, \alpha)}{a(z)}R(a, b, \alpha) \right)$$

$$(3.28) \quad a(\alpha) \left(R(b, a, \alpha) + \frac{b(z)}{a(z)}R(a, b, \alpha) \right) = -I_{\mathcal{P}} + \frac{\delta(z, \alpha)}{a(z)}R(a, b, \alpha).$$

Indeed, we have

$$\begin{aligned} (1 - \sigma(z)\overline{\sigma(\alpha)})I_{\mathcal{P}} - (\sigma(z) - \sigma(\alpha))T &= \\ &= \left(1 - \frac{b(z)\overline{b(\alpha)}}{a(z)\overline{a(\alpha)}} \right) I_{\mathcal{P}} - \left(\frac{b(z)}{a(z)} - \frac{b(\alpha)}{a(\alpha)} \right) \left(\frac{(|a(\alpha)|^2 - |b(\alpha)|^2)R(a, b, \alpha) - \overline{b(\alpha)}I_{\mathcal{P}}}{a(\alpha)} \right) \\ &= \frac{\rho(\alpha, \alpha)}{|a(\alpha)|^2} \left(I_{\mathcal{P}} - \frac{\delta(z, \alpha)}{a(z)}R(a, b, \alpha) \right), \end{aligned}$$

which proves (3.27). On the other hand,

$$\begin{aligned} a(\alpha) \left(R(b, a, \alpha) + \frac{b(z)}{a(z)} R(a, b, \alpha) \right) &= a(\alpha) R(b, a, \alpha) + a(\alpha) \frac{b(z)}{a(z)} R(a, b, \alpha) \\ &= -I_{\mathcal{P}} - b(\alpha) R(a, b, \alpha) + a(\alpha) \frac{b(z)}{a(z)} R(a, b, \alpha) \\ &= -I_{\mathcal{P}} + \frac{\delta(z, \alpha)}{a(z)} R(a, b, \alpha), \end{aligned}$$

which is (3.28).

STEP 3: Let S be given by (3.24). Then,

$$(3.29) \quad S(z) = H - \frac{|a(\alpha)|^2 (b(z)a(\alpha) - a(z)b(\alpha))}{a(\alpha)^2 \rho(\alpha, \alpha)} G(a(z)R(b, a, \alpha) + b(z)R(a, b, \alpha))^{-1} F,$$

and

$$(3.30) \quad \frac{I_{\mathcal{C}} - S(z)S(w)^*}{a(z)\overline{a(w)} - b(z)\overline{b(w)}} = C_{\alpha} (a(z)R(b, a, \alpha) + b(z)R(a, b, \alpha))^{-1} (a(w)R(b, a, \alpha) + b(w)R(a, b, \alpha))^{-*} C_{\alpha}^*.$$

It follows from (3.27)-(3.28) that

$$(1 - \sigma(z)\overline{\sigma(\alpha)})I_{\mathcal{P}} - (\sigma(z) - \sigma(\alpha))T = -\frac{\rho(\alpha, \alpha)a(\alpha)}{|a(\alpha)|^2 a(z)} (a(z)R(b, a, \alpha) + b(z)R(a, b, \alpha)).$$

We now plug this expression in (3.24), taking into account that

$$\sigma(z) - \sigma(\alpha) = \frac{b(z)a(\alpha) - a(z)b(\alpha)}{a(z)a(\alpha)}$$

to get (3.29).

We now prove (3.30). From the similar formula for $a(z) = 1$ and $b(z) = z$ (see [10,]) we have, with

$$b_{\sigma}(z) = \frac{\sigma(z) - \sigma(\alpha)}{1 - \sigma(z)\overline{\sigma(\alpha)}},$$

$$\frac{I_{\mathcal{C}} - S(z)S(w)^*}{1 - b_{\sigma}(z)\overline{b_{\sigma}(w)}} = G(I_{\mathcal{P}} - b_{\sigma}(z)T)^{-1} (I_{\mathcal{P}} - b_{\sigma}(w)T)^{-*} G^*,$$

that is,

$$\begin{aligned} (1 - \sigma(z)\overline{\sigma(\alpha)}) \frac{I_{\mathcal{C}} - S(z)S(w)^*}{(1 - |\sigma(\alpha)|^2)(1 - \sigma(z)\overline{\sigma(w)})} (1 - \sigma(\alpha)\overline{\sigma(w)}) &= \\ = (1 - \sigma(z)\overline{\sigma(\alpha)}) G \left((1 - \sigma(z)\overline{\sigma(\alpha)}) I_{\mathcal{P}} - (\sigma(z) - \sigma(\alpha)) T \right)^{-1} \times \\ \times \left((1 - \sigma(w)\overline{\sigma(\alpha)}) I_{\mathcal{P}} - (\sigma(w) - \sigma(\alpha)) T \right)^{-*} G^* (1 - \sigma(\alpha)\overline{\sigma(w)}). \end{aligned}$$

Using the definition of G and (3.27)-(3.28) this last equality is equivalent to:

$$\begin{aligned} \frac{I_{\mathcal{C}} - S(z)S(w)^*}{(1 - |\sigma(\alpha)|^2)(1 - \sigma(z)\overline{\sigma(w)})} &= \\ &= \frac{|a(\alpha)|^2 a(z)}{a(\alpha)\rho(\alpha, \alpha)} C_{\alpha} (a(z)R(b, a, \alpha) + b(z)R(a, b, \alpha))^{-1} \times \\ &\quad \times (a(w)R(b, a, \alpha) + b(w)R(a, b, \alpha))^{-*} C_{\alpha}^* \frac{|a(\alpha)|^2 \overline{a(w)}}{a(\alpha)\rho(\alpha, \alpha)} \rho(\alpha, \alpha), \end{aligned}$$

from which the result follows since

$$\frac{a(z)\overline{a(w)}}{1 - |\rho(\alpha)|^2} = \frac{|a(\alpha)|^2 a(z)}{a(\alpha)\rho(\alpha, \alpha)} \frac{|a(\alpha)|^2 \overline{a(w)}}{a(\alpha)\rho(\alpha, \alpha)} \rho(\alpha, \alpha).$$

STEP 4: With S as in the previous step, the reproducing kernel of \mathcal{P} is equal to

$$\frac{I_{\mathcal{C}} - S(z)S(w)^*}{a(z)\overline{a(w)} - b(z)\overline{b(w)}}.$$

We verify that

$$(3.31) \quad C_w = -C_{\alpha} (a(w)R(b, a, \alpha) + b(w)R(a, b, \alpha))^{-1}$$

(note that (3.31) holds for $w = \alpha$ in view of (3.18)). We write

$$-(a(w)R(b, a, \alpha) + b(w)R(a, b, \alpha))^{-1} h = g.$$

Then

$$\begin{aligned} -h(z) &= ((a(w)R(b, a, \alpha) + b(w)R(a, b, \alpha)) g)(z) \\ &= \frac{a(w)b(z) - b(w)a(z)}{b(\alpha)a(z) - a(\alpha)b(z)} g(z) + \frac{a(w)b(\alpha) - b(w)a(\alpha)}{a(\alpha)b(z) - b(\alpha)a(z)} g(\alpha). \end{aligned}$$

Thus $-h(w) = g(\alpha)$ and (3.31) is proved. Thus the reproducing kernel can be written as $C_z C_w^*$, which ends the proof. \square

Remark 3.3. Since (3.19) holds in $\mathbf{H}_2(\rho)$ we obtain that the orthogonal of the space \mathcal{P} is of the form $S\mathbf{H}_2(\rho)$.

4. THE QUATERNIONIC-VALUED CASE

In this part we consider the case of slice hyperholomorphic functions. The results for quaternionic Pontryagin spaces corresponding to the results in Section 2.1 can be found in [4, 5, 7, 8], and are not repeated, but we provide precise references. The counterpart of Theorem 1.3 and of Theorem 2.6 have been proved in [7, Theorem 7.1, p. 862] and in [3, Theorem 7.2, p. 122], respectively. Here we mainly consider the counterpart of Theorem 2.5 for the half-space case. The Beurling-Lax theorem for slice hyperholomorphic functions on the open unit ball of the quaternions is discussed in Section 8.4 of [4].

We begin this section by providing some basic fact about slice hyperholomorphic functions with values in a Banach or Hilbert space. For more information, we refer the reader to [4].

4.1. Slice hyperholomorphic functions. We will denote by \mathbb{H} the skew field of quaternions. It contains elements of the form $p = x_0 + x_1i + x_2j + x_3k$ where $x_\ell \in \mathbb{R}$, and i, j, k are imaginary units such that $i^2 = j^2 = -1$, $ij = -ji$ and $k = ij$. The conjugate of p is denoted by \bar{p} and $\bar{p} = x_0 - x_1i - x_2j - x_3k$. Note that $p\bar{p} = \bar{p}p = |p|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$. The set \mathbb{S} of quaternions p such that $p^2 = -1$ consists of purely imaginary quaternions, namely quaternions of the form $p = x_1i + x_2j + x_3k$, with $|p| = 1$. It is a 2-dimensional sphere in \mathbb{H} identified with the Euclidean spaces \mathbb{R}^4 .

Let $I \in \mathbb{S}$; then the set of elements of the form $x + Iy$ is a complex plane denoted by \mathbb{C}_I . Every nonreal quaternion p belongs to a unique complex plane \mathbb{C}_I where I is determined by its imaginary part, normalized.

By \mathbb{H}_+ we denote the open half-space

$$\mathbb{H}_+ = \{p \in \mathbb{H} ; \operatorname{Re} p > 0\},$$

which intersects the positive real axis.

The counterpart of Schur functions in the slice hyperholomorphic setting were introduced in [3] and further studied in [4]. Here we give the following definition of slice hyperholomorphic functions (equivalent to the one given in [3]):

Definition 4.1. *Given be a two sided quaternionic Banach (or Hilbert) space \mathcal{X} , a real differentiable function $f : \Omega \subseteq \mathbb{H} \rightarrow \mathcal{X}$ is (weakly) slice hyperholomorphic if and only if $\frac{1}{2}(\partial_x + I\partial_y)f_I(x + Iy) = 0$ for all $I \in \mathbb{S}$.*

Remark 4.2. If, under the same hypothesis, one imposes $\frac{1}{2}\partial_x f_I(x + Iy) + \frac{1}{2}\partial_y f_I(x + Iy)I = 0$ for all $I \in \mathbb{S}$ the function f is said to be right (weakly) slice hyperholomorphic.

In particular, when a function f defined on Ω is quaternionic valued, we say that it is slice hyperholomorphic if and only if $\frac{1}{2}(\partial_x + I\partial_y)f_I(x + Iy) = 0$ for all $I \in \mathbb{S}$.

Remark 4.3. Given a two-sided quaternionic Hilbert space \mathcal{X} and a \mathcal{X} -valued function f slice hyperholomorphic in a neighborhood of $\alpha \in \mathbb{R}$, then f can be written as a convergent power series

$$f(p) = \sum_{n=0}^{\infty} (p - \alpha)^n f_n,$$

where the coefficients $f_n \in \mathcal{X}$.

In the sequel, we will consider open sets Ω which are axially symmetric slice domains (in short, s-domains).

Definition 4.4. *Let $\Omega \subseteq \mathbb{H}$. We say that Ω is axially symmetric if whenever $p = x_0 + Iy_0$ belongs to Ω also all the elements of the form $x_0 + Jy_0$, $J \in \mathbb{S}$ belongs to Ω .*

Ω is said to be a slice domain if it is a connected open set whose intersection with any complex plane \mathbb{C}_I is connected.

Given $p = x_0 + Iy_0$ the set of elements of the form $x_0 + Jy_0$, $J \in \mathbb{S}$ is a 2-dimensional sphere denoted by $[p]$. The sphere $[p]$ contains elements of the form $q^{-1}pq$ for $q \neq 0$.

Remark 4.5. The Identity Principle, see [3, 4], implies that two slice hyperholomorphic functions defined on an s-domain and \mathcal{X} -valued coincide if their restrictions to the real axis coincide. Moreover, any real analytic function $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathcal{X}$ can be extended to a function, denoted by $\operatorname{ext}(f)$, which is slice hyperholomorphic on a suitable axially symmetric s-domain Ω containing $[a, b]$. In fact, for any $x_0 \in [a, b]$ the function f can be

written as $f(x) = \sum_{n \geq 0} x^n f_n$, $f_n \in \mathcal{X}$, for x such that $|x - x_0| < \varepsilon$ and thus $(\text{ext } f)(p) = \sum_{n \geq 0} p^n f_n$ converges and defines a slice hyperholomorphic function for $|p - x_0| < \varepsilon_{x_0}$. Thus we can set $B(x_0, \varepsilon_{x_0}) = \{p \in \mathbb{H} : |p - x_0| < \varepsilon_{x_0}\}$ and $\Omega = \cup_{x_0 \in [a, b]} B(x_0, \varepsilon_{x_0})$.

In particular, let $f : \rho_S(A) \cap \mathbb{R} \rightarrow \mathcal{X}$ be the function $f(x) = (I - xA)^{-1}$, where $\rho_S(A)$ denotes the S-resolvent of A . Then

$$p^{-1} S_R^{-1}(p^{-1}, A) = (I - \bar{p}A)(I - 2\text{Re}(p)A + |p|^2 A^2)^{-1}$$

is the unique slice hyperholomorphic extension to $\rho_S(A)$. This extension will be denoted by $(I - pA)^{-\star}$.

The pointwise multiplication of two slice hyperholomorphic functions is not, in general hyperholomorphic, so we introduce the following notion of multiplication:

Definition 4.6. *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f, g : \Omega \rightarrow \mathcal{X}$ be slice hyperholomorphic functions with values in a two sided quaternionic Banach algebra \mathcal{X} . Let $f(x + Iy) = \alpha(x, y) + I\beta(x, y)$, $g(x + Iy) = \gamma(x, y) + I\delta(x, y)$. Then we define*

$$(4.1) \quad (f \star g)(x + Iy) := (\alpha\gamma - \beta\delta)(x, y) + I(\alpha\delta + \beta\gamma)(x, y).$$

It can be verified that $f \star g$ is slice hyperholomorphic. In a similar manner, one can define a multiplication, denoted by \star_r , between right slice hyperholomorphic functions.

In the sequel, we will make use of the following result, see [3, 4]:

Proposition 4.7. *Let A be a bounded linear operator from a right-sided quaternionic Banach \mathcal{X} space into itself, and let G be a bounded linear operator from \mathcal{X} into \mathcal{Y} , where \mathcal{Y} is a two sided quaternionic Banach space. The slice hyperholomorphic extension of $G(I - xA)^{-1}$, $1/x \in \sigma_S(A) \cap \mathbb{R}$, is*

$$(G - \bar{p}GA)(I - 2\text{Re}(p)A + |p|^2 A^2)^{-1}.$$

With an abuse of notation, we will write $G \star (I - pA)^{-\star}$ meaning the expression $(G - \bar{p}GA)(I - 2\text{Re}(p)A + |p|^2 A^2)^{-1}$.

Remark 4.8. The composition $f \circ g$ of two slice hyperholomorphic functions is not, in general, slice hyperholomorphic unless additional hypothesis are assumed. We say that a slice hyperholomorphic function is *quaternionic intrinsic* if it is quaternionic valued and, for every $I \in \mathbb{S}$, it takes elements belonging to \mathbb{C}_I to that same complex plane. The composition of two slice hyperholomorphic functions $f \circ g$, when defined, is slice hyperholomorphic when g is quaternionic intrinsic.

In particular, the composition with the quaternionic counterpart of the operator R_α will not be hyperholomorphic, unless $\alpha \in \mathbb{R}$.

In this setting, R_α is defined as

$$(4.2) \quad R_\alpha f(p) = (p - \alpha)^{-1}(f(p) - f(\alpha)) \stackrel{\text{def.}}{=} \begin{cases} \sum_{n=1}^{\infty} (p - \alpha)^{n-1} f_n, & p \neq \alpha, \\ f_1, & p = \alpha, \end{cases}$$

where $f(p) = \sum_{n=0}^{\infty} (p - \alpha)^n f_n$.

We end this part by recalling the notion of slice hypermeromorphic functions:

Definition 4.9. Let \mathcal{X} be a two-sided quaternionic Banach space. We say that a function $f : \Omega \rightarrow \mathcal{X}$ is (weakly) slice hypermeromorphic if for any Λ in the dual of \mathcal{X} , the function $\Lambda f : \Omega \rightarrow \mathbb{H}$ is slice hypermeromorphic in Ω .

Note that the previous definition means that Λf is slice hyperholomorphic in an open set Ω' , where the points belonging to $\Omega \setminus \Omega'$ are the poles of Λf and $(\Omega \setminus \Omega') \cap \mathbb{C}_I$ has no point limit in $\Omega \cap \mathbb{C}_I$ for $I \in \mathbb{S}$.

4.2. The Hardy space of the open half-space \mathbb{H}_+ . In this subsection we recall the definition of the Hardy space of the half space \mathbb{H}_+ .

Definition 4.10. We define $\mathbf{H}_2(\mathbb{H}_+)$ as the space of slice hyperholomorphic functions on \mathbb{H}_+ such that

$$(4.3) \quad \sup_{I \in \mathbb{S}} \int_{-\infty}^{+\infty} |f(Iy)|^2 dy < \infty.$$

Let us consider the function

$$(4.4) \quad k(p, q) = (\bar{p} + \bar{q})(|p|^2 + 2\operatorname{Re}(p)\bar{q} + \bar{q}^2)^{-1} = (|q|^2 + 2\operatorname{Re}(q)p + p^2)^{-1}(p + q)$$

which is slice hyperholomorphic in p and \bar{q} on the left and on the right, respectively in its domain of definition. Note that we can write

$$k(p, q) = (p + \bar{q})^{-\star}$$

where the \star -inverse is computed with respect to p .

We have:

Proposition 4.11. The kernel $\frac{1}{2\pi}k(p, q)$ is reproducing, i.e. for any $f \in \mathbf{H}_2(\mathbb{H}_+)$

$$f(p) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(p, Iy) f(Iy) dy.$$

The $\mathbf{L}(\mathcal{D}, \mathcal{C})$ -valued function S slice hypermeromorphic in an axially symmetric s-domain Ω which intersects the positive real line belongs to the class $\mathcal{S}_\kappa(\Omega)$ if the kernel

$$K_S(p, q) = k(p, q)I_C - S(p) \star k(p, q)I_C \star_r S(q)^*$$

has κ negative squares in Ω , where $k(p, q)$ is defined in (4.4).

4.3. Generalized Schur functions. Let \mathcal{C} and \mathcal{D} be a pair of two-sided quaternionic Pontryagin spaces with the same index of negativity. The $\mathbf{L}(\mathcal{D}, \mathcal{C})$ -valued function S analytic in some open, axially symmetric, subset Ω of the open right half-space \mathbb{H}_+ is called a generalized Schur function if the $\mathbf{L}(\mathcal{C}, \mathcal{C})$ -valued kernel $K_S(p, q)$ solution of the equation

$$(4.5) \quad 2\pi(pK_S(p, q) + K_S(p, q)^*\bar{q}) = I_C - S(p)S(q)^*, \quad p, q \in \Omega$$

has a finite number of negative squares, say κ , in Ω . Let $\alpha \in \Omega \cap [0, \infty)$. It is proved in [3] that a function S is a generalized Schur function of the right half-plane if and only if it can be written in the form

$$(4.6) \quad S(p) = H - \frac{p - \alpha}{p + \alpha} G \star \left(I_{\mathcal{P}} - \frac{p - \alpha}{p + \alpha} T \right)^{-\star} F.$$

where \mathcal{P} is a Pontryagin space with index of negativity κ and where the operator-matrix

$$(4.7) \quad \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \mathcal{P} \oplus \mathcal{D} \longrightarrow \mathcal{P} \oplus \mathcal{C}$$

is coisometric. It follows that S has a unique slice hyperholomorphic extension to \mathbb{H}_+ , and for S so extended the kernel K_S has still κ negative squares for p, q in the domain of slice hyperholomorphicity of S . It follows that the space $\mathcal{P}(S)$ is R_α -invariant.

We note that equation (4.6) gives the (unique) slice hypermeromorphic extension of

$$S(x) = H - \frac{x - \alpha}{x + \alpha} G \left(I_{\mathcal{P}} - \frac{x - \alpha}{x + \alpha} T \right)^{-1} F$$

from a real neighborhood $(\alpha - \eta, \alpha + \eta)$ to the open right half-space.

4.4. The structure theorem. We begin by proving that Lemma 2.7 can be generalized to this setting in fact we have:

Lemma 4.12. *Let $\alpha, \beta \in \mathbb{R}^+$. Then equality (1.4) holds in $\mathbf{H}_2(\mathbb{H}_+)$.*

Proof. Also in the quaternionic setting, we use the fact that $\frac{1}{2\pi}k(p, q) = \frac{1}{2\pi}(p + \bar{q})^{-*}$, where $k(p, q)$ is as in (4.4), is the reproducing kernel of $\mathbf{H}_2(\mathbb{H}_+)$ and we prove the equality for $k(\cdot, \mu), k(\cdot, \nu)$. Let us consider $\mu \in \mathbb{H}_+$ and $\alpha \in \mathbb{R}^+$. Then

$$k(\alpha, \mu) = (|\mu|^2 + \operatorname{Re}(\mu)\alpha + \alpha^2)^{-1}(\alpha + \mu) = (\alpha + \mu)^{-1},$$

moreover

$$\begin{aligned} R_\alpha k(p, \mu) &= (p - \alpha)^{-1}(k(p, \mu) - k(\alpha, \mu)) \\ &= (p - \alpha)^{-1}((|\mu|^2 + \operatorname{Re}(\mu)p + p^2)^{-1}(p + \mu) - (\alpha + \mu)^{-1}) \\ &= (p - \alpha)^{-1}(|\mu|^2 + \operatorname{Re}(\mu)p + p^2)^{-1}((p + \mu)(\alpha + \bar{\mu}) - (|\mu|^2 + \operatorname{Re}(\mu)p + p^2))(\alpha + \bar{\mu})^{-1} \\ &= (|\mu|^2 + \operatorname{Re}(\mu)p + p^2)^{-1}(p - \alpha)^{-1}(-p(p - \alpha) - (p - \alpha)\bar{\mu})(\alpha + \bar{\mu})^{-1} \\ &= (|\mu|^2 + \operatorname{Re}(\mu)p + p^2)^{-1}(-p - \bar{\mu})(\alpha + \bar{\mu})^{-1} \\ &= -k(p, \mu)(\alpha + \bar{\mu})^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \langle R_\alpha \frac{1}{2\pi}k(\cdot, \mu), \frac{1}{2\pi}k(\cdot, \nu) \rangle &= \langle -\frac{1}{2\pi}k(\cdot, \mu)(\alpha + \bar{\mu})^{-1}, \frac{1}{2\pi}k(\cdot, \nu) \rangle \\ &= -\frac{1}{2\pi}k(\nu, \mu)(\alpha + \bar{\mu})^{-1}, \\ \frac{1}{2\pi}k(\cdot, \mu), \langle R_\beta \frac{1}{2\pi}k(\cdot, \nu) \rangle &= \langle \frac{1}{2\pi}k(\cdot, \mu), -\frac{1}{2\pi}k(\cdot, \nu)(\beta + \bar{\nu})^{-1} \rangle \\ &= -(\beta + \nu)^{-1} \frac{1}{2\pi}k(\nu, \mu), \\ \langle R_\alpha \frac{1}{2\pi}k(\cdot, \mu), \langle R_\beta \frac{1}{2\pi}k(\cdot, \nu) \rangle \rangle &= \langle -\frac{1}{2\pi}k(\cdot, \mu)(\alpha + \bar{\mu})^{-1}, -\frac{1}{2\pi}k(\cdot, \nu)(\beta + \bar{\nu})^{-1} \rangle \\ &= (\beta + \nu)^{-1} \frac{1}{2\pi}k(\nu, \mu)(\alpha + \bar{\mu})^{-1}, \end{aligned}$$

furthermore

$$2\pi \overline{k(\beta, \nu)} k(\alpha, \mu) = \frac{1}{2\pi} \overline{(\beta + \bar{\nu})^{-1}} (\alpha + \bar{\mu})^{-1} = (\beta + \nu)^{-1} (\alpha + \bar{\mu})^{-1}.$$

Let us now compute the left-hand side of (1.4), neglecting everywhere the factor $1/2\pi$ (i.e. multiplying (1.4) by 2π):

$$\begin{aligned}
& 2\pi(\langle R_\alpha f, g \rangle + \langle f, R_\beta g \rangle + (\alpha + \bar{\beta})\langle R_\alpha f, R_\beta g \rangle + 2\pi\overline{g(\beta)}f(\alpha)) = \\
& = -k(\nu, \mu)(\alpha + \bar{\mu})^{-1} - (\beta + \nu)^{-1}k(\nu, \mu) + (\alpha + \beta)(\beta + \nu)^{-1}k(\nu, \mu)(\alpha + \bar{\mu})^{-1} \\
& \quad + (\beta + \nu)^{-1}(\alpha + \bar{\mu})^{-1} = \\
& = (\beta + \nu)^{-1}[-(\beta + \nu)k(\nu, \mu) - k(\nu, \mu)(\alpha + \bar{\mu}) + (\alpha + \beta)k(\nu, \mu) + 1](\alpha + \bar{\mu})^{-1} = \\
& = (\beta + \nu)^{-1}[-\nu k(\nu, \mu) - k(\nu, \mu)\bar{\mu} + 1](\alpha + \bar{\mu})^{-1}.
\end{aligned}$$

Proposition 4.7 in [3] yields $-\nu k(\nu, \mu) - k(\nu, \mu)\bar{\mu} + 1 = 0$ and thus the equality (1.4) holds. The fact that (1.4) holds in $\mathbf{H}_2(\mathbb{H}_+)$ follows from the fact that the linear span of the reproducing kernels form a dense subset of $\mathbf{H}_2(\mathbb{H}_+)$ and from the fact that R_α, R_β are bounded operators (see the proof of Lemma 2.7). \square

Theorem 4.13. *Let $\Omega \subset \mathbb{H}_+$ be a s -domain, and let $\alpha \in \mathbb{R} \cap \Omega$ and let \mathcal{C} be a two-sided quaternionic Pontryagin space. Let \mathcal{P} be a right-sided reproducing kernel Pontryagin space of \mathcal{C} -valued functions slice hyperholomorphic in Ω , which is R_α -invariant and such that inequality (2.8) holds in \mathcal{P} . Then the functions of \mathcal{P} have a slice hypermeromorphic extension to \mathbb{H}_+ and there exists a quaternionic two-sided Pontryagin space \mathcal{C}_1 with $\text{ind}_-(\mathcal{C}_1) = \text{ind}_-(\mathcal{C})$ and a function $S \in \mathcal{S}_\kappa(\mathcal{C}_1, \mathcal{C})$, with $\kappa = \text{ind}_-(\mathcal{P})$, such that the reproducing kernel of \mathcal{P} is given by*

$$(4.8) \quad 2\pi(pK_S(p, q) + K_S(p, q)\bar{q}) = I_{\mathcal{C}} - S(p)S(q)^*$$

Proof. The proof follows that Theorem 3.2, where now α is real and $z = t$ and $w = s$ are assumed real. Equality (2.11) is still valid here, and so is the factorization (2.13), see [7, Proof of Theorem 7.1, STEP 3 and (7.4), p. 862]. We conclude that for $t, s \in \Omega \cap \mathbb{R}$,

$$\frac{I_{\mathcal{C}} - S(t)S(s)^*}{2\pi(t + s)} = C_t C_s^*.$$

The result follows by slice hyperholomorphic extension of these operator-valued functions; see Remark 4.5. \square

Remark 4.14. As in the complex case, but now for real and positive α and β , we have (1.4) in the Hardy space of the half-space, that is (2.8) holds as an equality there. As in Section 2.4 we obtain that the orthogonal of the space \mathcal{P} is equal to $M_S \mathbf{H}_2(\mathbb{H}_+)$, where now M_S is the operator of \star -multiplication by S on the left. A special case was considered in [6]. Here too, one can consider the spaces $\mathbf{H}_2(\mathbb{H}_+, J)$, where J is a signature matrix. Now J is assumed to have real, rather than complex or quaternionic, coefficients.

Remark 4.15. One can define a unified setting as in the complex plane. Because of the problems arising with composition operators, it is necessary to restrict oneself with functions a and b in the class of intrinsic functions (see Remark 4.8); for functions slice hyperholomorphic in a neighborhood of the origin, this means that their developments in powers of p have only real coefficients.

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