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TEACHING MATHEMATICS WITH AUTOMATIC SYMBOLIC COMPUTATION

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Abstract. We discuss how the concept of problem solving is central in current research in mathematics education, and how the difficulties of making computations by hand and the time spent in teaching computational skills can move the focus away from the main issues. We describe an experiment performed with students in Mathematical Engineering which shows how the use of automatic symbolic computation can dramatically improve the teaching of both abstract mathematics and the problem solving skills.

1. Introduction

"Maths is one of the world's great systems of problem solving, which has empowered so much of human endeavor, particularly recently. Calculating is one piece of maths. It's part of the maths problem-solving process."[13]. This is a quotation from an interview to Conrad Wolfram, strategic director of Wolfram Research and founder of `computerbasedmath.org`, a project to build a completely new math curriculum for primary and secondary schools with computer-based computation at its heart. The topic mentioned by Wolfram is central in the current discussion in mathematics education. Problem solving refers to the generic activity of finding the solution of any kind of problem, not necessarily of mathematical type.

In this paper we discuss the huge possibilities offered by symbolic computation software to mathematics education, with a particular focus to advanced courses at University level. After an introduction on the history of mathematics education in Section 2, we discuss the issue of *problem solving* in Section 3. In Section 4 we introduce the use of symbolic computation in the framework of problem solving, and in Section 5 we present a teaching experiment performed at the Politecnico di Milano. A summary of our results and the students' feedback is illustrated in Section 6.

2. Some history

Mathematics education (also called *didactics of mathematics*) has now reached the status of an autonomous and recognized science; it presents the features of a discipline in its own right, as Romberg mentions in [24]. Great merit in the foundation of this discipline and its emergence as an independent theory goes to the French School, on which we will not dwell here. We would, however, at least set the above mentioned topic in the frame of current research in Mathematics Education.

To this purpose, we refer to the analysis of D'Amore[9]. He calls Didactics A the initial phase (1960-1980), Didactics B the intermediate stage (1980-2000) and Didactics C the current research. More precisely, D'Amore calls Didactics A (represented e.g. by Dienes[10] and Papy) the understanding of the teaching of mathematics as docendi ars, that is the disclosure of the mathematical content, in which the focus is on the teaching phase: the good teacher achieves its goal of an effective acquisition of the topic by students by transmitting the subject through interesting lessons, engaging activities, appealing didactic situations...[1].

A good teacher of type A has strong communication skills and ability to arouse interest, and can thus transfer the knowledge to the students. However, studies carried out since the 80s on cognitive transfer have shown that this *teaching artist* does not always get the expected result of an effective learning. Armella Moreno[19] writes "Teaching, as a simple process of education, under assumptions on the capacity of the student to absorb what you tell clearly, is not a theory, it is an illusion".

The failing results of Didactics A, focused only on Knowledge and the transmission of its contents, have expanded the scope of the research studies and made Didactics B blossom. D'Amore defines Didactics B as the epistemology of mathematics learning, namely as empirical research: it investigates the methods of construction of mathematical knowledge of the student. And, referring to Vergnaud [28] and Kilpatrick [17], D'Amore finds the basics of this epistemology in constructivism. The cognitive process is not passive acquisition, but active construction by the learner, who, in continuous interaction and adaptation to the environment, processes independently the information and the experiences. Hence the centrality of the learner in research on Didactics B. This kind of research submits to critical analysis both the way the student learns and the interactions and dynamics that take place

[1]In a nutshell, the main idea of Didactics A is: the better the teacher is in attracting the attention of the student and in making the concepts clear, the better the student will understand and learn.

in the classroom. Then, mathematics education becomes "a science that is concerned with the production and communication of mathematical knowledge and with whatever is peculiar of this production and communication"[2]. This science performs a systemic analysis of the components of the *didactic triangle* (teacher, student, Knowledge): it finds the ways and the conditions of the spread of mathematical knowledge; it spells out the consequences of such spread on both the student and the knowledge itself; it studies the institutions where the transmission of knowledge occurs. And it is designed to optimize scholastic achievements in mathematics: it considers how the student builds his knowledge in order to organize the best possible school situations.

The current trend of mathematics education is the epistemology of the teacher, i.e. his formation, his function and his beliefs. Didactics A analyzes the Knowledge and the modes of transmission of its contents, Didactics B has broadened the scope of the research to all the variables that determine the success or failure of the learning process. Together they opened the way "to create good learning situations"[5], i.e. "to translate the teaching effort into a true and conscious learning"[9]. Thus, didactics A and B led to the beginning of a new phase of research in mathematics education, which D'Amore calls Didactics C.

In fact, it remains to analyze the third element of the *didactic triangle*. Until a few years ago, nobody ever wondered whether and to what extent the teacher affects the learning process. For example, do his beliefs have an effect on the contents to be transmitted and on the way in which the pupil makes them its own? Consider then the effects of what Brousseau called *didactic contract*[3]: "In a teaching situation, prepared and carried out by a teacher, the student has usually the task of solving a (mathematical) problem provided by the teacher, but the access to this task is done through an interpretation of the questions asked, of the information provided and of the obligations imposed by the way the teacher teaches. The didactic contract consists both in the peculiar habits of the teacher that the pupil expects, and in the behaviour of the pupil expected by the teacher".

Also, research has revealed, through the concept of *didactic transposition*, the complex transition from the mathematical knowledge to the knowledge to be taught and then to the knowledge taught. The teacher adapts the

[2]Brousseau [6], and with him the French School, is the first to highlight the limits of Teaching A, paving the way for a new conception of mathematics education.
[3]this concept is introduced by Brousseau in [2, 3, 4] and then further developed by Chevellard, see [8]

mathematical knowledge (Savoir Savant[4]) to transform it into what Chevellard calls "knowledge to be taught"[8], which is what the pupils actually learn.[5]

And then again, one can cite as additional factors that influence the teaching-learning phenomenon: the choice of teaching situations[6]; the progressive adoption of the mathematical concepts by the individual through the passage from the images of these concepts to the models and the consequent formation of misconceptions[7]; various types of barriers to the learning of mathematics (ontogenetic, didactic and epistemological)[8]; the difficulties related to the use of a specific language, as in mathematics, and its various registers (mathematics as a language in itself, therefore equipped with its own syntax, semantics and pragmatics; the necessity of the teacher to explain mathematical concepts in a language understandable to the students and therefore the use of some kind of jargon; the easy confusion between common language and mathematical language; problems arising in the attempt to translate the definition of a mathematical object into a representation of the same object in some other semiotic system)[9, cap. 5, p. 77-93].

As explained above, we have made these brief and sketchy outline of research in mathematics education to delve more easily and more consciously in the theme of *problem solving*, considered by many scholars the priority in this discipline.

3. Problem solving

Surely, Wolfram is not the first to define mathematics as "one of the world's great systems of problem solving". Many great mathematicians have declared it to be, among other things, the art of *solving problems*. In 1980, Halmos writes: "What is really math? Axioms (like the parallel postulate)? Theorems (as the fundamental theorem of Algebra)? Proofs (as Gödel's)?

[4]this is how Chevelard calls the academic mathematical knowledge

[5]Particularly explanatory of this concept are the words of D'Amore: "the didactic transposition consists, from the point of view of the teacher, in building his own lessons drawing from the source of knowledge, taking into account the orientations provided by institutions and programs (knowledge to teach), to adapt them to his own classroom: the level of the students, the pursued goals. The didactic transposition consists in extracting an element of knowledge from its context (academic, social, ...) in order to adapt it to the peculiar, unique context of the class."[9, p. 40]

[6]Concerning the situation theories, we refer to [5]

[7]We will not enter the delicate problem of misconceptions, we refer to [9, p. 53-72] and [31, p. 87-90]

[8]The obstacle theory, introduced in [1] and developed in [20], is currently considered of fundamental importance in mathematics education. The obstacles can be related to the student, to the choices of the teacher or may be intrisic of the topic.

Definitions (as Menger's definition of dimension)? Theories (as category theory)? Formulas (as Cauchy's integral formula)? Methods (as the method of successive approximations)? Surely, mathematics could not exist without these ingredients, they are all essential. Still, a plausible viewpoint is that: none of them is at the core of the discipline, the main reason of existence for the mathematician is solve problems, mathematics is really about problems and solutions"[14]. And before him David Hilbert: "As long as a branch of science offers an abundance of problems, it is alive. A lack of problems foreshadows the extinction or the arrest of independent development. Just as every human undertaking pursues certain goals, so also mathematical research requires its problems. It is through the solution of the problems that the researcher tests himself; he finds new methods and new perspectives, and wins a wider and freer horizon."[15]. George Polya [21] extends the centrality of the problems in mathematics also to the teaching practice: "Solving problems means finding a way out of a difficulty, a way to get around an obstacle, to achieve a goal that is not readily accessible. Solving problems is a specific feature of intelligence, and the intelligence is the specific gift of mankind: you can consider the problem-solving activities as the most peculiar feature of the human race..... Then, a math teacher has a great opportunity. Obviously, *if he will use his lesson hours to have his students perform calculations, he will end up crushing their interest, slowing down their mental development and wasting the opportunities that present themselves.* Instead, if he arouses the curiosity of the students proposing problems of difficulty proportionate to their knowledge and he helps them to solve the problems by asking appropriate questions, he will be able to inspire in them a taste for original reasoning".

Following Polya, many researcher in mathematics education have reiterated and elaborated the importance of proposing challenging problems and situations taken from real life to the students. Unfortunately, however, the teaching in our schools mostly compresses the students in the mere execution of repetitive task. Usually teachers, when facing a new topic in the mathematics program, show on the blackboard the procedure and then require the students to solve similar problems in the same way. It is then necessary here to introduce the distinction between *problem* and *exercise.* The student carries out an *exercise* when he merely applies rules previously outlined and exhaustively covered by the teacher. Therefore, the solution is just the repetition of procedures already seen[9].

[9] For a discussion of this trend in Italy we refer to [30, 31]

To define a *problem* we use some quotations. Duncker, a member of Gestalt psychology[10], writes "a problem arises when a living being has a goal and is unable to reach it" [11]. Lester, an educator, writes: "A problem is a task for which: the individual or group that confronts it wants or needs to find a solution; there is no procedure immediately accessible that guarantees or determines a complete solution; the individual or group must make an effort to find a solution"[18]. Finally, we mention D'Amore, who prefers the term *problematic situation*, which "can create a problem or exercise according to the teaching situation" and which he defines as: "learning situation that involves the resolution of a problem, but designed in such a way that students cannot resolve the matter by simple repetition or application of knowledge or skills acquired"[9, p. 95-96].

Hence a profound difference between a teaching activity focused on the solution of exercises and one based instead on the solution of problems. In the first one, students just have to solve exercises assigned by the teacher to check their level of learning as a result of his explanations. In the other one, the teacher chooses to submit to the students problematic situations which require them to engage in a productive process. Each person, in such process of problem solving, is influenced by cognitive[11], metacognitive[12] and emotional factors[13]. The topic that mostly interests us now, however, is the repertoire of strategies that come into play when solving a problem, what Polya has precisely defined as *heuristic*: "The purpose of heuristic is to study methods and rules of invention and discovery"[21]. In the activity of problem solving, the issue of methods of finding a solution is fundamental. The *reproductive* thought, typical of this activity, comes to life and grows right in

[10]This discipline has given many contributions to the theory of problem solving, although it mainly refers to the definition of productive thinking, as opposed to reproductive thinking. Gestalt psychologists, in fact, aim "to establish the phenomenology of these processes and the characteristics that set them apart from those purely reproductive, to identify the conditions that favor them and those that hinder them, to locate the decisive moments of the process, when it creates a flash of understanding"[16, p. 36].

[11]As an example, depending on the knowledge regarding the topic where the problem under study belongs, the learner can judge the activity as a real problem or as a simple exercise. And, to address it, he uses strategies that he knows (heuristics) and he takes continuously decisions about how to manage the resources available[27]. Everything happens always within his *belief systems*[25], that is the personal beliefs about oneself, on the problem under consideration, on the goals set in carrying out this activity, on mathematics in general.

[12]The studies of Flavell[12], Brown-Walter[7] and Schoenfeld[26] identified two main characteristics of metacognition. On one hand, the awareness considered as knowledge that the subject has both of himself as a learner and of the resources at his disposal. On the other hand, control understood as pondered and prudent management of his cognitive capabilities.

[13]"The same thought has its origin not from another thought, but from the sphere of the motivations of our consciousness, which contains our passions and our needs, our interests and impulses, our affections and our emotions"[29]

the solution process. Dunker, back in 1935, made an enlighting distinction between solving algorithm, which leads to the solution of a problem, and a heuristic method, which instead is a way to build a solving algorithm[11].

The first is a formula - broadly understood as an instruction, a recipe - that leads to the solution of the problem. The second is a non-specialist strategy that helps to address the problem with the aim of getting to a solution. Dunker analyzes some heuristic methods in his view particularly effective: the *analysis of the goal* (what does the problem really require? what should I achieve?), the *analysis of the situation or of the data* (what are the data available to me? what can I use?) and the *analysis of the conflict* (what are the obstacles between me and the solution? how do I manipulate the data or items that I have in order to reach it?). If we want to immerse ourselves in the specific field of mathematics, a teacher can e.g. explain a proof, leading the students to use the *analysis of the goal* heuristic. He makes the statement explicit and encourages the students to ponder about questions such as: what are we required to achieve? what do we have to prove exactly? which steps lead to the formulation of the statement? what elements, data, theories do we need? The students are asked to reconstruct the proof starting from the goal, that is the statement. If, instead, the teacher wants them to address the opposite task - starting from the premises, and then developing the proof step by step until they reach the statement - then he directs the class to start from the data, that is to use the heuristic method of *analysis of the situation*. In both cases, it is clear that the mental processes that unfold towards the solution are ignited and nurtured by the questions that the solver arises.

As we have already mentioned, the first researcher to make a systematic study of these issues, specifically addressed to mathematics education, was Polya [22, 23, 21]. We do not enter into the details of his treatises, nor we discuss the many researchers who, because of the huge impetus provided by Polya on research about problem solving, have studied the most effective heuristic strategies concerning the solution of mathematical problems. We only recall, and then pursue, a fundamental question now widespread in this field of research that we have experienced in 20 years of university teaching. If the math teacher spends his lesson hours to make his students perform calculations, he will eventually dampen their interest and stop their mental development. He will play in full his role as an educator, stimulating and improving their reasoning skills if, instead, "he awakens the curiosity of pupils proposing problems of difficulty in proportion to their knowledge and help them find the solution them with appropriate questions".

4. Symbolic computations as a teaching tool

The solution of a problem is usually addressed with mathematics through the following steps:

(1) Creation of a mathematical model, that is translation of the real world into equations, starting from general laws and constitutive relations:
   (a) By general laws we mean e.g. the laws of Mechanics, that is laws that describe fundamental relationships, as the conservation of energy and momentum.
   (b) The constitutive relations depend on the particular problem addressed, e.g. Fourier's law for the heat flow.
(2) Study of the generic features of the mathematical model, e.g. to establish if it represents a well posed problem.
(3) Search for a technique to solve the equations. This usually consists in the creation of some algorithms that convert the input data into output data.
(4) Applications of the algorithms developed at step 3. This step usually consists in computations, which can be of two different types:
   (a) Symbolic computations.
   (b) Numerical computations.
(5) Interpretation of the mathematical results as answers to the real life problem.

The major part of mathematics education in primary schools consists in step 4b. This is a task that any pocket calculator can perform much more efficiently than any human being. In secondary schools, and often also in University courses, mathematics consists essetially in step 4a. Clearly, this step could be only performed by pen and paper until a few years ago. On the other hand, starting from the sixties, the development of Macsyma at the MIT introduced the automatic symbolic computation. Nowadays Maxima[14], a code distributed under a GPL licence, has taken the place of Macsyma. In 1980 the University of Waterloo (ON, Canada) developed Maple[15], and in 1988 Stephen Wolfram introduced Mathematica[16]. These softwares, we will provide examples with Mathematica, but our ideas apply to all of them, can perform tasks 4a and 4b in a very efficient, fast and accurate manner. It seems therefore a good idea to focus mathematics education, in particular in a University curriculum, on the remaining steps.

[14] `http://maxima.sourceforge.net`
[15] `http://www.maplesoft.com/products/maple/`
[16] `http://www.wolfram.com`

The fact that all the time spent in teaching computations (both symbolic and numeric) shifts the focus of mathematics away from the most important and creative part of problem solving is only one drawback of the *pen and paper* teaching of mathematics. It is our opinion that the major drawback consists in the fact that the computations can become often so compicated (by which we mean that they not only require a lot of time to be performed, but also they absorb all the attention of the student) that the true nature of the problem remains hidden behind. In order to exemplify this idea, we performed an experiment with some students in Mathematical Engineering at the Politecnico di Milano. Some students of a class in Calculus in many variables have attended an extracurricular brief course in Mathematica. During this course some problems have been addressed by leaving to the computer the task of performing complicated computations. In the next section we describe in detail one of these problems, which makes it very clear how the use of Mathematica can provide a much better understanding of mathematics. We chose this particular example because it is quite clear that in this case the computations by hand cannot be used to fully address the problem, since they would require an enormous amount of time to be performed. On the other hand, the same computations only require a few seconds to the computer, so the students can appreciate the real issue at hand.

5. An experiment with the implicit function theorem

We consider a classic problem of calculus in several variables, the implicit function theorem. Needless to say, this theorem has a very large number of applications, but the statement of the theorem sounds very abstract and unrelated to real life problems. We start with the statement of the theorem in the simplest version:

Theorem 1. *Let $A \subset \mathbb{R}^2$ be an open set, letf $: A \to \mathbb{R}$ be a continuously differentiable function. Let $(x_0, y_0) \in A$ be a zero of f. If $\frac{\partial f(x_0,y_0)}{\partial y} \neq 0$, then there exist neighborhoods $U(x_0)$ and $U(y_0)$ and a unique function $y : U(x_0) \to U(y_0)$ such that $y(x_0) = y_0$, $f(x, y(x)) = 0$ for all $x \in U(x_0)$. Furthermore, the function y is continuously differentiable and*

$$y'(x) = -\left(\frac{\partial f(x, y(x))}{\partial y}\right)^{-1} \frac{\partial f(x, y(x))}{\partial x}\,. \tag{5.1}$$

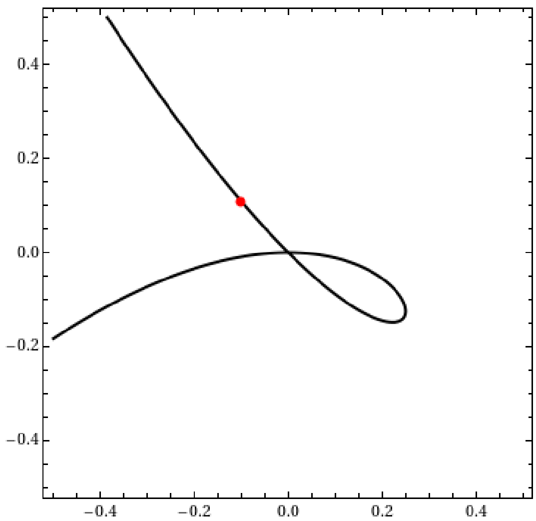
If additionally the function f is continuously differentiable k times, then so is the function y. If f is analytical, then so is y, Although the implicit function theorem guarantees the existence of the function y, it does not provide an algorithm to compute it explicitly, and in general such function cannot even be represented in terms of elementary functions, even in the simplest cases of f. On the other hand, when the function f (and therefore y) is analytical, the implicit function theorem provides an algorithm for the computation of the Taylor series

$$y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!}(x - x_0^k) \,.$$

More precisely, the first derivative is provided directly by (5.1), while the following derivatives can be computed recursively by differentiating (5.1). It is quite clear that this computation, although theoretically possible, is totally unfeasible even when f has a relatively simple analytical expression, due to the fact that the repeated differentiation of (5.1) rapidly generates a very complicated expression. As an example, consider the function

$$f(x, y) = x^3 + xy + y^2 \,.$$

The following picture (also obtained with Mathematica) displays the approximate zero level set of the function f in the square $\left[-\frac{1}{2}, \frac{1}{2}\right]^2$:



The point $(x_0, y_0) = \left(-\frac{1}{5}, \frac{1}{50}(5 + 3\sqrt{5})\right)$, marked in red in the figure, satisfies the assumptions of the implicit function theorem. One easily computes

$$y'(x) = -\frac{3x^2 + y(x)}{x + 2y(x)} \,, \tag{5.2}$$

so that $y'(x_0) = -\frac{1}{2} - \frac{11}{6\sqrt{5}}$. By differentiating (5.2), substituting (5.2) into the result and simplifying the resulting expression one can compute

$$y''(x) = \frac{2\left(\left(x-12x^2\right)y(x)+(1-12x)y(x)^2-9x^4\right)}{(2y(x)+x)^3},$$

so that $y''(x_0) = \frac{16\sqrt{5}}{27}$. One could, in principle, iterate the same procedure as many times as one wishes, and therefore compute all the derivatives of $y^{(k)}(x_0)$. On the other hand it is quite clear that such procedure, albeit possible, becomes rather cumbersome even for the third derivative in the case of a very simply function f.

As a consequence, one is led to teach a fundamental theorem of Mathematical Analysis without being able to present a satisfactory example in full, and most importantly without being able to show one of the important consequences of the theorem. This problem can be very easily circumvented by using Mathematica as a teaching tool.

Here are the third and fourth derivatives of the function y, computed with just a few lines of Mathematica code:

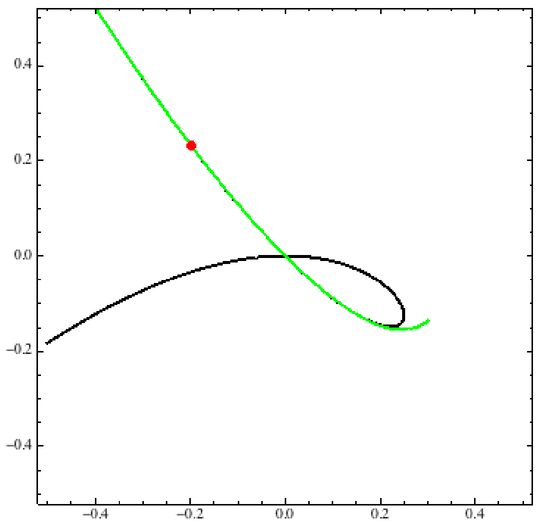
$$y^{(3)} = -\frac{1}{(2y(x)+x)^5}\left(\begin{array}{l} 6\left(\left(72x^2-10x+1\right)x^2y(x)+\left(72x^2+6x+1\right)\right. \\ \left. xy(x)^2+\left(54x^2-9x+1\right)x^4+32xy(x)^3+16y(x)^4\right) \end{array}\right),$$

$$y^{(4)} = -\frac{1}{(2y(x)+x)^7}\left(\begin{array}{l} 24\left(2\left(240x^2-40x+1\right)xy(x)^3\right. \\ +\left(240x^2-40x+1\right)y(x)^4+6\left(108x^2+15x-4\right) \\ x^3y(x)^2+\left(648x^3-150x^2+16x-1\right)x^3y(x)+ \\ \left.\left(405x^3-108x^2+15x-1\right)x^5\right) \end{array}\right).$$

And here is the Taylor expansion of order 12 of the function $y(x)$:

$$\begin{aligned} T(x) \quad = \quad & \frac{1}{50}\left(5+3\sqrt{5}\right)+\left(-\frac{1}{2}-\frac{11}{6\sqrt{5}}\right)\left(x+\frac{1}{5}\right)+\frac{8}{27}\sqrt{5}\left(x+\frac{1}{5}\right)^2 \\ & +\frac{35}{243}\sqrt{5}\left(x+\frac{1}{5}\right)^3+\frac{325\sqrt{5}}{2187}\left(x+\frac{1}{5}\right)^4+\frac{3875\sqrt{5}}{19683}\left(x+\frac{1}{5}\right)^5 \\ & +\frac{17500\sqrt{5}}{59049}\left(x+\frac{1}{5}\right)^6+\frac{256250\sqrt{5}}{531441}\left(x+\frac{1}{5}\right)^7+\frac{3953125\sqrt{5}}{4782969}\left(x+\frac{1}{5}\right)^8 \\ & +\frac{189921875\sqrt{5}}{129140163}\left(x+\frac{1}{5}\right)^9+\frac{3128125000\sqrt{5}}{1162261467}\left(x+\frac{1}{5}\right)^{10} \\ & +\frac{52660156250\sqrt{5}}{10460353203}\left(x+\frac{1}{5}\right)^{11}+\frac{902128906250\sqrt{5}}{94143178827}\left(x+\frac{1}{5}\right)^{12}. \end{aligned}$$

Needless to say, this kind of result is virtually impossible to obtain by hand. What is interesting is that we can use the Taylor expansion to plot an approximation of the function defined implicitly and compare it with the level set of the function $f(x,y)$:



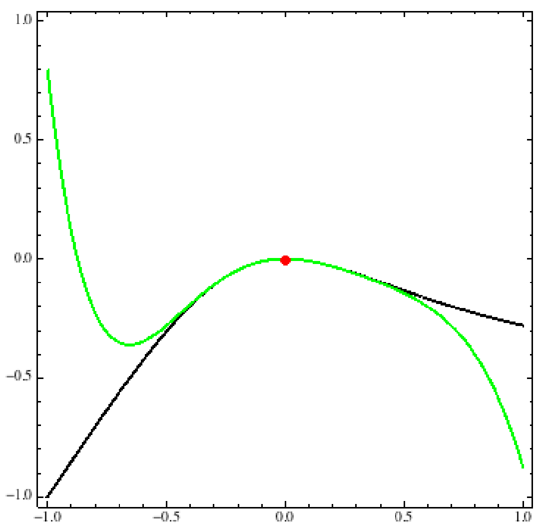
To make the experiment more interesting let us try a trascendent function, e.g.

$$f(x,y) = x^2e^y + ye^x \qquad \text{with} \qquad (x_0, y_0) = (0,0).$$

The 6th order Taylor expansion of the function defined implicitly at (x_0, y_0) is:

$$T(x) = -x^2 + x^3 + \frac{x^4}{2} - \frac{11}{6}x^5 + \frac{11}{24}x^6\,, \tag{5.3}$$

and the following picture displays the approximate zero level set of $f(x, y)$ (black) and the graph of $T(x)$ (green):



One may think that the computation of (5.3) is not a big deal; in fact, all the coefficients look quite simple. But how have we computed that? For brevity, we show here only the third derivative:

$$
\begin{aligned}
y'''(x) \quad = \quad & -\frac{1}{\left(x^2e^{y(x)}+e^x\right)^5}\Big(4x^7e^{5y(x)}+x^5e^{4y(x)+x}\left(x^3y(x)-6(x-1)x+16\right) \\
& +e^{5x}y(x)+x^3e^{3y(x)+2x}\left(3x^3y(x)^2-2(x(x+9)-3)xy(x)-6((x-7)x-2)\right)- \\
& e^{y(x)+4x}(y(x)(xy(x)(xy(x)+6x-6)+2((x-9)x+3))-6x+6)+ \\
& xe^{2y(x)+3x}\left(xy(x)\left(xy(x)(2xy(x)-3(x+4))-6\left(x^2-6\right)\right)+6(x(x+5)-4)\right)\Big)\ , \qquad (5.4)
\end{aligned}
$$

and we leave to the reader imagination to figure out how the sixth derivative looks like, although we mention that to print it explicitly would take approximately 40 lines similar to (5.4). Our point is that, although the Taylor polynomial of the function $y(x)$ looks very simple, in order to compute it one has to go through a very complicated computation, so complicated that not even the most courageous and determined student could perform it. But this computation is just the mean, not the end of the example. In fact the main result of this example should be to show the student how a function defined implicitly can be locally approximated by a polynomial.

6. Students' feedback

We wish to underline that our experiment was done with a group of students with a very good mathematical background in an advanced mathematical analysis course, while typical issues in mathematics education are usually discussed in the primary or secondary school context. We have discussed with the students who joined this experiment some of the topics addressed in this paper, in particular how the teaching of the mathematics should be oriented to the solution of problems, not to computation algorithms, and how cumbersome computations can come in the way of providing good examples of important theorems. We have shown that, fortunately, there is an easy way of getting rid of such computations, i.e. the use of software for symbolic computation. Software like Mathematica can relieve both the teacher and the students of the burden of computations, and can allow the focus of the teaching to be on the main issues. In particular, we have described in this paper one of many topics we covered with the students in Mathematical Engineering. In all such topics, good examples of the theory could be provided in a clean and effective way through Mathematica. But this is not the only positive outcome of these experiments. The students felt that the use of the software made the topics much more appealing, because it made much easier to see the importance of the mathematical theory and also because programming is more fun than making computations by hand. More importantly, the students appreciated being challenged by *problems to solve*. In fact, the algorithm to compute the Taylor series explained above (and many other developed during the course) was not made available to them, but they were led to develop it through careful suggestions, so that we made possible to teach mathematics through *problem solving* and not through *exercises.* We feel that this kind of experiments should be pursued further teaching mathematics at advanced level.

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