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FREE VIBRATIONS IN SPACE OF THE SINGLE MODE FOR THE KIRCHHOFF STRING

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Abstract. We study a single mode for the Kirchhoff string vibrating in space. In 3D a single mode is generally almost periodic in contrast to the 2D periodic case. In order to show a complete geometrical description of a single mode we prove some monotonicity properties of the almost periods of the solution, with respect to the mechanical energy and the momentum. As a consequence of these properties, we observe that a planar single mode in 3D is always unstable, while it is known that a single mode in 2D is stable (under a suitable definition of stability), if the energy is small.

1. **Introduction and main results.** The Kirchhoff equation is a classical nonlinear model for a vibrating string, that allows the definition of single or multiple modes, formally developing the solution by Fourier series depending on time. Several papers study the stability of a single mode for the planar case, see for example [4], [9], [11], showing stability (under various definitions) for single modes of low energy, and instability for higher energies.

The Kirchhoff model can be extended to the motion of a string in space, with few variations of the general properties of a solution. On the contrary, the study of a single mode in 3D is quite different. The main purpose of the present paper is to supply a qualitative geometrical description of a single mode in 3D, with respect to the mechanical energy and the angular momentum, which are natural invariants for the motion. As a consequence we show that: first, a single mode in 3D is generally almost periodic and not periodic in time, second, a planar mode, intended as a special case of a 3D simple mode, is always unstable. I found few references about single modes and no ones about stability (see for example [3]) even though a 3D frame is more realistic than a 2D one, and an elementary study needs only classical tools.

Let us first recall the definition of a single mode for the Kirchhoff string equation in space.

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The vector $\mathbf{u}(x,t) = \begin{bmatrix} v(x,t) \\ w(x,t) \end{bmatrix}$ describes the position of a string of length π, moving in space (x, v, w) under conditions

$$\begin{aligned} &\mathbf{u}_{tt} - (a + b\int_0^\pi \parallel \mathbf{u}_x \parallel^2 \ dx)\mathbf{u}_{xx} = \mathbf{0}, \qquad a \geq 0,\, b > 0, \\ &\mathbf{u}(x,0) = \mathbf{u}_0 \sin(nx), \quad \mathbf{u}_t(x,0) = \mathbf{u}_1 \sin(nx), \\ &\mathbf{u}(0,t) = \mathbf{u}(\pi,t) = 0, \end{aligned} \tag{1}$$

where $\parallel \mathbf{u}_x \parallel^2 = \mathbf{u}_x \cdot \mathbf{u}_x$, as usual.

The solution of system (1) is $\mathbf{u} = \mathbf{u}_n(t)\sin(nx)$, where

$$\mathbf{u}_n(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$$

is a solution of the autonomous system

$$\begin{cases} v'' + (\alpha + \beta(v^2 + w^2))v = 0 \\ w'' + (\alpha + \beta(v^2 + w^2))w = 0, \end{cases} \tag{2}$$

where $\alpha = n^2 a, \quad \beta = n^4 b$.

The position vector $\mathbf{u}_n(t)$ describes the trajectory in the plane (v, w) of the point of the string of coordinate $x = \frac{\pi}{2n}$. The system (2) is a system with two degrees of freedom, whose invariants are the mechanical energy E and the angular momentum k

$$E = \frac{1}{2}(v'^2 + w'^2 + \alpha(v^2 + w^2) + \frac{\beta}{2}(v^2 + w^2)^2), \qquad k = vw' - v'w.$$

When $k = 0$, $\mathbf{u}_n(t)$ has the same direction as $\mathbf{u}_n'(t)$ for any t. Therefore we have a planar motion, which we can suppose, under a suitable change of coordinates, to happen in the plane (x, v) i. e. $w \equiv 0$. The system (2) becomes a single equation

$$v'' + (\alpha + \beta v^2)v = 0. \tag{3}$$

Studying (3) is a simple exercise: all its solutions are periodic, with period T_0 depending on E.

In Theorem 1.1 we prove that the period is decreasing with respect to E, as a collateral result of the 3D case.

Papers [4], [11] study the stability of a planar single mode as a particular case of a two-mode. It is worth noting that a planar two-mode depends on a system formally similar to (2), but substantially different, because it lacks the second invariant k.

If instead $k \neq 0$, a system such as (2) is classically studied using polar coordinates (see for example [6], [10]), and its solutions are generally only almost periodic with two periods T_1, T_2.

More precisely, let $v = \rho\cos\theta$, $w = \rho\sin\theta$. The dynamical relations say

$$\rho\rho' = vv' + ww', \quad \rho^2\theta' = vw' - wv', \quad \parallel u' \parallel^2 = v'^2 + w'^2 = \rho'^2 + (\rho\theta')^2. \tag{4}$$

Differentiating the first equation in (4), and using the third equation, we obtain $\rho\rho'' - (\rho\theta')^2 = vv'' + ww''$, while the second equation becomes $\rho^2\theta' = k$. If $k \neq 0$, then it follows $\rho \neq 0$, and the system(2) is equivalent to

$$\begin{cases} \rho'' = \frac{k^2}{\rho^3} - \rho(\alpha + \beta\rho^2) \\ \theta' = \frac{k}{\rho^2}. \end{cases} \tag{5}$$

For every fixed k, the first equation in (5) is a system with one degree of freedom, its invariant is (obviously!) the mechanical energy of (2)

$$E(\rho, \rho') = \frac{1}{2}(\rho'^2 + \frac{k^2}{\rho^2} + \alpha\rho^2 + \frac{1}{2}\beta\rho^4) \tag{6}$$

and it can be studied using the "effective" potential energy

$$U_k(\rho) = \frac{1}{2}(\frac{k^2}{\rho^2} + \alpha\rho^2 + \frac{1}{2}\beta\rho^4) \tag{7}$$

(precisely $U_k(\rho)$ is equal to the potential energy of (2) plus the rotational part of the kinetic energy).
The function $U_k(\rho)$ is strictly convex for $\rho > 0$, and has a minimum in ρ_0, unique solution of $\frac{k^2}{\rho^4} = (\alpha + \beta\rho^2)$.
So the first equation in (5) has an equilibrium point in ρ_0, corresponding to the minimal energy $E_0 = U_k(\rho_0)$ and all its other orbits are closed cycles with $\rho_1(E, k) \leq \rho \leq \rho_2(E, k)$, where ρ_1, ρ_2 are the positive solution of $U_k(\rho) = E$, $E > E_0$.
The period of the periodic solutions $\rho = \rho(t)$ depends on E and k, and its formula is

$$T(E, k) = 2\int_{\rho_1(E,k)}^{\rho_2(E,k)} \frac{d\rho}{\sqrt{2(E - U_k(\rho))}}. \tag{8}$$

Let us now define, for $E > E_0$,

$$\bar{\theta}(E, k) = \int_0^{T(E,k)} \theta'(t)\, dt = \theta(T) - \theta(0). \tag{9}$$

Since $\rho(t)$ is periodic, we can suppose without loss of generality that $\rho(0) = \rho_2 > \rho_1$. Then, geometrically, $\bar{\theta}$ is the angle covered by the point $\mathbf{u}_n(t)$, departing at $t = 0$ at the maximum distance from the origin, to come back to the same distance $|\mathbf{u}_n(0)| = |\mathbf{u}_n(T)| = \rho_2$.

In the linear case, it is $|\bar{\theta}| = \pi$, corresponding to half oscillation. Then in a full oscillation the trajectory of the point (v, w) closes itself , and it is well known that it is an ellipse with center in the origin.

As another example, the Kepler first law of the planetary motion corresponds to a system with two degrees of freedom, with mechanical energy and angular momentum as its invariants, with $U_k(\rho) = \frac{k^2}{2\rho^2} - \frac{g}{\rho}$. For this system it is $\bar{\theta} = 2\pi$ (also in this case the trajectory is an ellipse, but with a focus in the origin instead of the center).

In general, knowing $\bar{\theta}$ provides a useful geometrical information that allows us to describe the trajectory, and to select between periodic and not periodic motions.

The first equation in (5) corresponds, for every fixed k, to an Hamiltonian planar system as in the examples in [5], [7], [2]. Unfortunately the Chicone's Theorem about the monotonicity of T with respect to E does not apply to our system and verifying the weaker criteria in [7], [2] is not so easy (see the Remark at the end of this section). Besides, writing down an analytical formula for the integral (8), which involves elliptic integrals, we are able to state the following results about the properties of $T(E, K)$, obtaining also the monotonicity with respect to the momentum k, and as a limit case for $k \to 0$, the result for the planar motion:

Theorem 1.1. *Let $T(E, k)$ be the oscillatory period in (8), $k \neq 0$: then*

1. $\frac{\partial}{\partial E} T(E, k) < 0.$
2. *Let $T_0(E)$ be the period of the planar solutions in (3):*
   then $\lim_{k \to 0} 2T(E, k) = T_0(E)$, $\frac{d}{dE} T_0(E) < 0.$
3. $\frac{\partial}{\partial k} T(E, k) < 0.$
4. $\lim_{E \to +\infty} T(E, k) = 0.$

Section 3 is dedicated to the proof of this theorem.

As observed before, if $\beta = 0$ (linear case) all the solutions of (2) are periodic, with minimal period independent of E, k, and equal to $\frac{2\pi}{\sqrt{\alpha}}$; passing in polar coordinates is useless, but, by the way, the period T of $\rho(t)$ is half the period of v, w.

The techniques introduced for the monotonicity of $T(E, k)$ apply also on $\bar{\theta}$: in the following Theorem 1.2 (for the proof see section 4) we prove that, in the nonlinear case, we have always $|\bar{\theta}| < \pi$ for every $k \neq 0$, $E > E_0$.

This is a crucial result, because it means that the angle covered in a full oscillation is $< 2\pi$, and the orbit can never close in a single oscillation, as it happens in the linear case.

More precisely, because of the second equation in (5), we obtain

$$\bar{\theta}(E, k) = 2 \int_0^{T(E,k)/2} \theta'(t)\, dt = 2k \int_{\rho_1(E,k)}^{\rho_2(E,k)} \frac{1}{\rho^2} \frac{d\rho}{\sqrt{2(E - U_k(\rho))}}. \tag{10}$$

The sign of $\bar{\theta}$ depends only on the sign of k, so it suffices to study $\bar{\theta}$ only for $k > 0$. The integral in (10) is elliptic too, and its direct analysis enables us to prove the following properties:

Theorem 1.2. *Let $\bar{\theta}(E, k)$ be the the angle spanned in half an oscillation in (9), $k > 0$: then*

1. $\frac{\partial}{\partial E} \bar{\theta}(E, k) > 0.$
2. $\lim_{E \to +\infty} \bar{\theta}(E, k) = \pi$ *for every fixed $k > 0$.*
3. *$\bar{\theta}(E, k) < \pi$ for every $E > E_0$, that is: the orbit can never be closed in a single full oscillation.*
4. $\lim_{k \to 0^+} \bar{\theta}(E, k) = \pi$ *for every fixed $E > E_0$.*
5. *$\bar{\theta}(E, k) > \sqrt{\frac{2}{3}}\pi$ for every $E > E_0$.*
6. $\frac{\partial}{\partial k} \bar{\theta}(E, k) < 0.$

The properties of T and $\bar{\theta}$ impact also on the properties of the solution's periods. The second equation in (5) implies that $\theta'(t)$ is periodic, with period T. If we introduce the mean value of $\theta'(t)$ in a period

$$M(E,k) = \frac{1}{T(E,k)} \int_0^{T(E,k)} \theta'(t)\, dt = \frac{\bar{\theta}(E,k)}{T(E,k)},$$

then $\theta'(t) - M$ has null mean value on a period, and then $\theta(t) = g(t) + Mt$, where $g(t)$ is periodic of period T. Coming back to (v,w), we obtain:

$$v(t) = \rho(t)\cos(g(t) + Mt), \quad w(t) = \rho(t)\sin(g(t) + Mt),$$

which implies that $v(t)$, $w(t)$ are almost periodic, with periods $T_1 = T(E,k)$ and $T_2 = 2\pi/M(E,k)$ (see also [10] for a more detailed exposition).

We recall that $v(t)$, $w(t)$ are periodic if and only if $\frac{T_1}{T_2}$ is rational, that is

$$\frac{T_1}{T_2} = \frac{\bar{\theta}(E,k)}{2\pi} = \frac{p}{q}. \tag{11}$$

The monotonicity properties of $\bar{\theta}(E,k)$ imply that there are infinite periodic and non periodic motions.

Finally the monotonicity properties of T and $\bar{\theta}$ imply monotonicity properties on T_1 and T_2 too.
As a consequence of Theorems 1.1, 1.2 we have

Corollary 1. :

1. *Both the maps* $(E,k) \mapsto (T(E,k), \bar{\theta}(E,k))$, $(E,k) \mapsto (T_1(E,k), T_2(E,k))$ *are locally invertible.*

2. $T_1 = T(E,k)$ *is decreasing with respect to both* E *and* k.

3. $T_2 = 2\pi \dfrac{T(E,k)}{\bar{\theta}(E,k)}$ *is decreasing with respect to* E *and increasing with respect to* k.

Proof. Let $\frac{\partial(T,\bar{\theta})}{\partial(E,k)}$ be the Jacobian matrix of $(E,k) \mapsto (T,\bar{\theta})$, then $\det \frac{\partial(T,\bar{\theta})}{\partial(E,k)} > 0$, $\det \frac{\partial(T_1,T_2)}{\partial(E,k)} = -\frac{2\pi T(E,k)}{\bar{\theta}^2(E,k)} \det \frac{\partial(T,\bar{\theta})}{\partial(E,k)} < 0$.

The second item is already proved in Theorem 1.1, and the monotonicity with respect to E of T_2 is an immediate consequence of Theorems 1.1, 1.2.

Instead of proving directly that $\frac{\partial T_2}{\partial k}$ is positive, is more convenient to study the inverse map. Being $\frac{\partial(T_1,T_2)}{\partial(E,k)}$ invertible, we have $\frac{\partial k}{\partial T_2} = (\det \frac{\partial(T_1,T_2)}{\partial(E,k)})^{-1} \frac{\partial T_1}{\partial E} > 0$. □

Remark 1. The period function of the solutions of Hamiltonian systems on the plane, with a center in the origin is extensively studied in many papers, under different hypothesis on the form and the regularity of the Hamiltonian.

Our first equation in (5) corresponds, for every fixed k, to a planar system with Hamiltonian $H(\rho,\rho') = \frac{1}{2}\rho'^2 + U_k(\rho)$, where U_k is defined in (7), $U_k \in \mathcal{C}^\infty(0,+\infty)$. Being its minimum $E_0 > 0$, the known sufficient conditions for the period's monotonicity have to be slightly modified. Let us set

$$W(\rho) = \frac{U_k(\rho) - E_0}{(U_k'(\rho))^2},$$

then W can be extended to a $\mathcal{C}^\infty(0, +\infty)$ bounded strictly positive function. The Chicone's sufficient condition ([5]) would request $W(\rho)$ concave, in order to imply T decreasing with respect to E. Unfortunately $W(\rho)$ changes concavity on $(0, +\infty)$.

The weaker sufficient condition in Theorem 1 in [2], with $\mathcal{C}^1$ regularity on the potential, is, as signaled, a generalization of a sufficient condition already present in [7]. Both these conditions can be written for our more regular potential $U_k(\rho)$ as follows:

The period $T(E, k)$ is strictly decreasing with respect to E if

$$W'(\rho)|_{\rho=\rho_2} < W'(\rho)|_{\rho=\rho_1},$$

where ρ_1, ρ_2 are the positive solutions of $U_k(\rho) = E$, $E > E_0$.

In my opinion this last condition could be satisfied, but its proof isn't simpler than the direct one that we have shown.

This paper is organized as follows: Section 2 is dedicated to the qualitative description of single motions, Section 3 to the properties of $T(E, k)$ and Section 4 to the properties of $\bar{\theta}(E, k)$. The Appendix contains all the calculations about the elliptic integral involved in the formula of $\bar{\theta}$ studied in Section 4.

2. Description of a simple mode, periodic solution and instability.

Let $\mathbf{u}_n(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$ be a solution of (2).

In the nonlinear case we have three type of motions:

- A planar motion has $\mathbf{u}_n(t)$ and $\mathbf{u}_n'(t)$ with the same constant direction (i. e. momentum $k = 0$). Its trajectory on the plane (v, w) is a segment centered in the origin, with the direction of $\mathbf{u}$.
- A mixed motion ($k \neq 0$, $E > E_0$) has its trajectory included in a circular ring $\rho_1 \leq \rho \leq \rho_2$, with $\rho_1 > 0$, that is essentially different from the elliptic trajectory of the linear case.
- The trajectory of a circular motion in (v, w), as the name says, is the circle $\rho = \rho_0$, corresponding to the minimal energy E_0.

Let us shorten with T and $\bar{\theta}$ the oscillatory period $T = T(E, k)$ and the angle $\bar{\theta} = \bar{\theta}(E, k)$ and assume, without lack of information, $k > 0$.

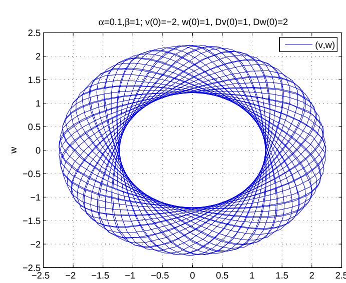

Figure 1. Mixed simple mode

An half oscillation for a mixed motion corresponds to a rotation of $\bar{\theta} < \pi$: let us suppose, for the sake of simplicity, $\rho = \rho_2$, $\theta = 0$ at $t = 0$ and $k > 0$. Then, at $t = nT$ we have $\rho = \rho_2$ for every $n \in \mathbb{N}$: the corresponding points on the circle $\rho = \rho_2$ have arguments $\theta = \theta(nT)$.

If $\frac{\bar{\theta}}{2\pi} = \frac{p}{q}$ (p, q being relatively prime integers) such points on the circle are exactly q, and the minimal angular distance between two points is exactly $\frac{2\pi}{q}$. Then the trajectory in (v, w) is a closed curve that seems a lace centerpiece, and the solution is periodic with minimal period $\tau = qT$, while in a period τ the point $\mathbf{u}_n(t)$ turn around the origin exactly p times (see (11)).

Otherwise the points with $\theta = \theta(nT)$ are dense on $\rho = \rho_2$, and the trajectory is dense in the ring $\rho_1 \leq \rho \leq \rho_2$.

Let us define $\theta_d = 2\pi - 2\bar{\theta}$ the delay angle, that is the angle missing for closing the "near ellipse" in a single full oscillation.

If the solution is periodic, we will note that θ_d is a multiple (sometimes coincident) of $\frac{2\pi}{q}$.

2.1. **Instability of the planar single motions.** We have pointed out in the introduction that a planar single mode is always periodic, and we have found out in the second statement of Theorem 1.1 that its period is strictly decreasing with respect to E. Then all the non null solutions of the autonomous system (3) are orbitally stable, but not Liapunov-stable, because of a slight change in the energy affecting the period.

If our scope is not only studying the autonomous system (3), but considering a single mode as a particular solution of the Kirchhoff equation in 2D, the approach is less simple (see [9], [4], [11]).

If we consider instead a single planar mode in a 3D context, we can easily deduce that it is always orbitally unstable. It is enough to study only the projection of the orbits of the system (2) on the plane (v, w) (the orbits lying in $\mathbb{R}^4$).

If we perturb a planar solution, introducing a small momentum k, the delay angle of the perturbed mixed solution will be small, by the fourth statement of the Theorem 1.2, and for t small, the perturbed mixed solution will be close to the planar one.

Increasing the time, the trajectory of the mixed solution will spread over the whole ring $\rho_1 \leq \rho \leq \rho_2$ (where ρ_1 will be near to 0, and ρ_2 will be near to the ρ_{max} of the planar solution). Then, without any doubts, the planar solution will be unstable.

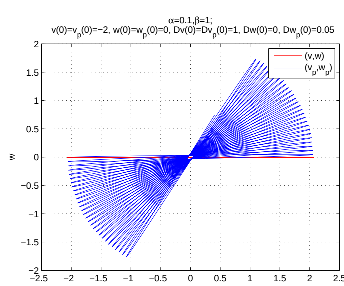

FIGURE 2. Instability of a planar mode

A very challenging open problem would be to discuss the stability of an almost periodic single mode as a particular solution of the Kirchhoff equation.

Also if we follow the path of [4], we have to consider a single mode as a particular case of a double mode. Indeed a double mode for the 3D Kirchhoff equation is solution of an autonomous system of eighth order with three invariants (the energy and two momenta) and a direct study seems quite impossible.

2.2. **Periodic mixed simple modes.** The third an fifth items of the Theorem 1.2 set an upper and lower bound on $\bar{\theta}$, that implies a bound also on the ratio $\dfrac{\bar{\theta}(E,k)}{2\pi} = \dfrac{p}{q}$, precisely it is

$$\frac{1}{\sqrt{6}} < \frac{p}{q} < \frac{1}{2}. \tag{12}$$

This relation implies some geometrical information about the trajectory in the plane (v,w): for example the motion with the minimal number of points on the circle $\rho = \rho_2$ corresponds to the ratio $3/7$, that is $\bar{\theta} = \frac{6}{7}\pi$, $\theta_d = \frac{2}{7}\pi$. The following motions with few points on the circle $\rho = \rho_2$ correspond respectively to $\frac{p}{q}$ equal to $\frac{4}{9}$, $\frac{5}{11}$ (see the figure), in all this cases it is $\theta_d = \frac{2\pi}{q}$.

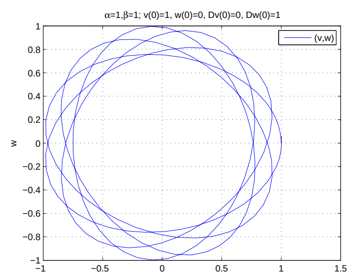

FIGURE 3. Periodic mixed mode

Generalizing, if $q \geq 7$ is odd, then $p = \frac{q-1}{2}$ is relatively prime with q, and $\theta_d = \frac{2\pi}{q}$. Let us suppose $\mathbf{u}(t_0) = \rho_2$: the pattern of the sequence of the points $\mathbf{u}(t_0 + jT)$, $\quad j \in I\!N$ on the maximal circle $\rho = \rho_2$ is essentially the same.

For $\frac{p}{q} = \frac{5}{12}$, that respects the (12), we can observe that $\theta_d = \frac{\pi}{3}$ is two times the minimal angular distance of two points on the maximal circle $\rho = \rho_2$; then the trajectory's geometry is different. The same happens when 4 divides $q \geq 12$ and $p = q/2 - 1$. Then p and q are relatively prime, and $\theta_d = 2\frac{2\pi}{q}$.

If q is even but 4 does not divide q , q and $q/2 - 1$ are both even and therefore not relatively prime, but q and $p = q/2 - 2$ are relatively prime. If $q \geq 22$, (12) is satisfied, and $\theta_d = 4\frac{2\pi}{q}$ is four time the minimal angular distance of two points on the maximal circle.

Let us increase q: we can easily observe that the previous three structures don't exhaust all the admissible cases for $\frac{p}{q}$ and the admissible periodic motions can assume more different configurations, according with the bounds in (12):

at $q = 17$ we have two motions, one with $\frac{p}{q} = \frac{8}{17}$ with the same geometry of the previous ones with an odd q, and another corresponding to $\frac{p}{q} = \frac{7}{17}$ with θ_d thrice the minimal distance of two points on the maximal circle.

For example at $q = 100$ (this one chosen as a value easy to verify) four ratios satisfy (12) and have factors relatively prime: precisely $\frac{41}{100}$, $\frac{43}{100}$, $\frac{47}{100}$, $\frac{49}{100}$, and so on.

A monotonic dependence of the period τ of a periodic mixed mode on the energy E cannot be expected, because τ depends on $T(E,k)$, that is monotonic, but also on the denominator of the ratio $\frac{\bar{\theta}(E,k)}{2\pi} = \frac{p}{q}$.

Nevertheless it is possible to prove the qualitative property that, if the energy goes to infinity, τ goes to infinity too.

Theorem 2.1. *Let $\tau = qT(E,k)$ be the period of a periodic mixed solution, where* $\dfrac{\bar{\theta}(E,k)}{2\pi} = \dfrac{p}{q}$ *(p, q relatively prime integer): then*
$\lim_{E\to+\infty} \tau = +\infty$ *for every fixed $k > 0$.*

The proof of this Theorem is at the end of Section 4.

At last we note that every periodic mixed solution will be orbitally unstable: a small perturbation on the initial data that changes E or k, involve small changes on ρ_1, ρ_2; but the solution, due to the continuity of $\bar{\theta}$ with respect to E, k, can become almost periodic, and its trajectory dense in the ring.

2.3. **Circular and planar motions.** The angular velocity of a circular motion of angular momentum k is constant: $\theta' = \frac{k}{\rho_0^2}$. Being ρ_0 the unique solution of $\frac{k^2}{\rho^4} = (\alpha + \beta\rho^2)$ then $|\theta'| = \sqrt{\alpha + \beta\rho_0^2}$, and the minimal period of a circular motion of momentum k is $T_{circ}(k) = \frac{2\pi}{\sqrt{\alpha+\beta\rho_0^2}}$. It is worth noting that $T_{circ}(k)$ is always less than $\frac{2\pi}{\sqrt{\alpha}}$, the constant period of a linear simple mode ($\beta = 0$).

We will observe that $T_{circ}(k) = \lim_{E\to E_0} T_2(E,k) := T_2(E_0,k)$, where $E_0 = U_k(\rho_0)$ (see the Remarks at the end of Section 3 and after the proof of the fifth item of Theorem 1.2).

Planar motions are always periodic, with period $T_0(E) = \lim_{k\to 0} 2T(E,k)$ (see Theorem 1.1). Being $\lim_{k\to 0} |\bar{\theta}(E,k)| = \pi$, due to the fourth item of Theorem 1.2, it follows also that

$T_0(E) = \lim_{k\to 0} T_2(E,k) := T_2(E,0)$.

For every fixed E we may compare the circular motion of energy E with the planar motion with the same energy: due to the monotonicity of T_2 with respect to k (Corollary 1), it follows that the period of the planar motion is smaller than the period of the circular one, and, by the way, of the period of a linear single mode with the same coefficient α, and $\beta = 0$.

3. Monotonicity properties of T(E,k).

Some numerical experiments on system (2) (such in the figures of the previous section), let us think about properties of monotonicity of the oscillatory period $T(E,k)$ and the angle $\bar{\theta}(E,k)$. Theorem 1.1 is the answer to our first guess.

Proof of Theorem 1.1. First, in order to avoid too many parameters, we set in (2) $\alpha = \nu$ and $\beta = 1$, which is a classical notation, if you are interested only in the non linear case (the parameter ν is related to the pretension of the string; moreover it is always possible to transform the general system (2) to an equivalent system with $\beta = 1$ posing $\nu = \alpha/\beta$, $\mathbf{u}_n(t) = \mathbf{u}(\sqrt{\beta}t)$).

Then we have

$$T(E,k) = 2\int_{\rho_1}^{\rho_2} \frac{d\rho}{\sqrt{2E - (\frac{k^2}{\rho^2} + \nu\rho^2 + \frac{1}{2}\rho^4))}},$$

that is an improper integral, being ρ_1 and ρ_2 roots of the denominator.
By letting, in this sections, $a = \rho_1^2$, $b = \rho_2^2$, and performing the change $z = \rho^2$, we have

$$\begin{aligned} T(E,k) &= \sqrt{2}\int_a^b \frac{dz}{\sqrt{-(z^3 + 2\nu z^2 - 4Ez + 2k^2)}} \\ &= \sqrt{2}\int_a^b \frac{dz}{\sqrt{-(z-a)(z-b)(z-c)}}. \end{aligned} \tag{13}$$

For $E > E_0$ (minimal energy for k fixed), the equation $z^3 + 2\nu z^2 - 4Ez + 2k^2 = 0$ has two positive roots a and b, and a negative one c, being $2k^2 = -abc$. The integral (13) is an elliptic integral and it is equal to (see for reference [1], page 597, or the Appendix, for a detailed calculation)

$$T(E,k) = \frac{2\sqrt{2}}{\sqrt{b-c}}\mathrm{K}(m), \tag{14}$$

where

$$m = \frac{b-a}{b-c}, \qquad \mathrm{K}(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}},$$

being a, b, c, m functions of E, k and $\mathrm{K}(m)$ a complete elliptic integral of the first kind.

Even though a, b, c can be explicitly calculated, as roots of a cubic equation, their formulas are unmanageable. It is more convenient to derive all their properties from the relations:

$$a + b + c = -2\nu, \tag{15}$$

$$ab + ac + bc = -4E, \tag{16}$$

$$abc = -2k^2. \tag{17}$$

The main idea of the proofs about the sign of the derivatives of T, $\bar{\theta}$ is that, using a, b, c instead of E, k, ν, all is reduced to study the sign of homogeneous polynomials in a, b, c under the conditions $c < 0 < a < b$.

Now, symbolic calculus programs as Mathematica or the Matlab Symbolic Toolbox are very efficient in manipulating polynomials, or differentiating and so on. The proof of this first item can also be followed, in my opinion, with paper and pencil, but I suggest the reader to use such programs for checking the soundness of the elementary, but tedious algebraic formulas involved. This will become more and more convenient for the subsequent derivatives, where the expressions of interest are longer.

1. We derive (15), (16), (17) with respect to E, keeping in mind that ν is constant, and k is independent of E and we obtain

$$\begin{aligned}
&a_E + b_E + c_E = 0,\\
&(b+c)a_E + (a+c)b_E + (a+b)c_E = -4,\\
&bca_E + acb_E + abc_E = 0,
\end{aligned}$$

and then

$$\begin{aligned}
a_E &= \tfrac{-4a}{(a-c)(b-a)} < 0,\\
b_E &= \tfrac{4b}{(b-a)(b-c)} > 0,\\
c_E &= \tfrac{4c}{(a-c)(b-c)} < 0.
\end{aligned} \tag{18}$$

Moreover, we have

$$\frac{\partial m}{\partial E} = \frac{\partial}{\partial E}\frac{b-a}{b-c} = 4\frac{a(b-c)^2 + b(a-c)^2 + c(a-b)^2}{(b-c)^3(b-a)(a-c)}. \tag{19}$$

The expression

$$a(b-c)^2 + b(a-c)^2 + c(a-b)^2 = 8\nu E + 18k^2$$

is positive. Then $m(E,k)$ is increasing with respect to E.

Is worth noting that $m(E_0,k) = 0$, and $\lim_{E\to+\infty} m(E,k) = \frac{1}{2}$, being, for the (15), $m = \frac{b-a}{2b+2\nu-a}$; then $\mathrm{K}(m)$ is an analytic function on $[0,1/2]$ an we can write down the derivative with respect to E of (14) for $E > E_0$ $(a < b)$, as a function of a, b, c and m

$$\frac{\partial}{\partial E}T(E,k) = 2\sqrt{2}\{\mathrm{K}(m)\frac{\partial}{\partial E}\frac{1}{\sqrt{b-c}} + \frac{1}{\sqrt{b-c}}\frac{d}{dm}\mathrm{K}(m)\frac{\partial m}{\partial E}\}. \tag{20}$$

We need now some elementary formulas about $\mathrm{K}(m)$. If we define

$$S(m) = \int_0^{\pi/2}\frac{\sin^2\theta\, d\theta}{\sqrt{1-m\sin^2\theta}}, \quad C(m) = \int_0^{\pi/2}\frac{\cos^2\theta\, d\theta}{\sqrt{1-m\sin^2\theta}}, \tag{21}$$

then the following relation hold: $S(m) + C(m) \equiv \mathrm{K}(m)$ (obviously) and

$$\varepsilon(m) := S(m) - C(m) \geq 0. \tag{22}$$

In order to prove that $\varepsilon(m) \geq 0$ it suffices changing the variable $\tau = \theta - \pi/4$, and verify that the even part of the new integrand function is non negative. That implies

$$\mathrm{K}(m) = 2C(m) + \varepsilon(m) \geq 2C(m). \tag{23}$$

In addition we have, integrating by parts,

$$\frac{d}{dm}\mathrm{K}(m) = \frac{1}{2}\int_0^{\pi/2}\frac{\sin^2\theta\, d\theta}{(\sqrt{1-m\sin^2\theta})^3} = \frac{1}{2(1-m)}C(m). \tag{24}$$

Now we substitute (18), (19), (24) in (20) and write down:

$$\frac{\partial}{\partial E}T(E,k) = \frac{-4\sqrt{2}\{(a-c)(b(a-c)-c(b-a))\mathrm{K}(m)-(a(b-c)^2+b(a-c)^2+c(b-a)^2)C(m)\}}{(b-a)(a-c)^2(b-c)^{5/2}}$$

that is, using (23)

$$\begin{aligned}&\frac{\partial}{\partial E}T(E,k)\\ =&\frac{-4\sqrt{2}\{(b-a)(3c^2+4E)C(m)+(a-c)(b(a-c)-c(b-a))\varepsilon(m)\}}{(b-a)(a-c)^2(b-c)^{5/2}}\leq 0.\end{aligned} \tag{25}$$

Actually it is

$$\begin{aligned}&2(a-c)(b(a-c)-c(b-a))-(a(b-c)^2+b(a-c)^2+c(b-a)^2)\\ =&(b-a)(3c^2-ab-ac-bc)=(b-a)(3c^2+4E)>0,\end{aligned}$$

and also the coefficient of $\varepsilon(m)$ is positive; then the first statement in Theorem 1 is proved.

2. Now we consider the period of the solutions of equation (3)

$$v''+(\nu+v^2)v=0.$$

Its potential is $U(v)=\frac{1}{2}v^2(\nu+\frac{v^2}{2})$, that is an even convex function, with a minimum in $v=0$. Therefore the orbits of the system in the phase plane (v,v') are closed orbits. Let $\pm v(E)$ be the solutions of the equation $U(v)=E$, for every $E>0$, then the solutions of (3) corresponding to the level of mechanical energy E have period

$$T_0(E)=2\int_{-v(E)}^{v(E)}\frac{dv}{\sqrt{2(E-U(v))}}=4\sqrt{2}\int_0^{v(E)}\frac{dv}{\sqrt{-(v^4+2\nu v^2-4E)}},$$

being the last an elliptic integral.

If we perform the change of variable $z=v^2$, it is easy to see the connection between $T_0(E)$ and $T(E,k)$. We have indeed

$$T_0(E)=2\sqrt{2}\int_0^{b_0}\frac{dz}{\sqrt{-z(z-b_0)(z-c_0)}}=2\,T(E,0), \tag{26}$$

being $b_0>0$ and $c_0<0$ the solutions of the equation $z^2+2\nu z-4E=0$. If we extend the definition of $T(E,k)$ as in (13), to $k=0$, then $a_0=0$, b_0, c_0 are the solutions of $z^3+2\nu z^2-4Ez+2k^2=0$ for $k=0$ and $T(E,0)$ satisfies (26).

Note that the factor 2 in the previous relation is well justified; in fact $T_0(E)$ is the period related to a full planar oscillation, while $T(E,0)$ is the period of $\rho(t)$, related to an half oscillation. Moreover we observe that the proof of the monotonicity of $T(E,k)$, performed before, was based on the relations on the roots of the cubic equation, relations that hold also for $k=0$. Then also the second statement in Theorem 1 is proved.

3. Now let us prove the third statement in Theorem 1, the general scheme being the same of the first statement.

It is worth noting that E and k are not totally independent. For every fixed momentum k, the mechanical energy E has to be $\geq E_0$, the energy associated to the circular motion $\rho = \rho_0$ with momentum k in the plane (v, w); for every fixed energy E, $|k|$ has to be $\leq k_{max}(E)$, where $k_{max}(E)$ is the momentum of a circular motion, counterclockwise, of energy E. Roughly speaking, a circular motion is the motion with minimal energy for fixed momentum and viceversa, maximal momentum for fixed mechanical energy.

Let us suppose now E constant, and $|k| < k_{max}(E)$. Before deriving $T(E, k)$ in (14) with respect to k, we derive (15), (16) and (17):

$$
\begin{aligned}
&a_k + b_k + c_k = 0,\\
&(b+c)a_k + (a+c)b_k + (a+b)c_k = 0,\\
&bca_k + acb_k + abc_k = -4k,
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
a_k &= \tfrac{4k}{(a-c)(b-a)} > 0,\\
b_k &= \tfrac{-4k}{(b-a)(b-c)} < 0,\\
c_k &= \tfrac{-4k}{(a-c)(b-c)} < 0.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\frac{\partial m}{\partial k} &= \frac{\partial}{\partial k}\frac{b-a}{b-c} = -8k\frac{(a^2+b^2+c^2-ab-ac-bc)}{(b-c)^3(b-a)(a-c)}\\
&= -\,32k\frac{(\nu^2+3E)}{(b-c)^3(b-a)(a-c)} < 0.
\end{aligned}
\tag{27}
$$

Using the previous relations, (23) for K and (24) for $\frac{d\mathrm{K}}{dm}$,

$$
\begin{aligned}
\frac{\partial}{\partial k}T(E,k) &= 2\sqrt{2}\{\mathrm{K}(m)\frac{\partial}{\partial k}\frac{1}{\sqrt{b-c}} + \frac{1}{\sqrt{b-c}}\frac{d}{dm}\mathrm{K}(m)\frac{\partial m}{\partial k}\} =\\
&= \frac{4\sqrt{2}k\{(a-c)(3a+2\nu)\mathrm{K}(m) - 8(\nu^2+3E)C(m)\}}{(b-a)(a-c)^2(b-c)^{5/2}} =\\
&= \frac{-\sqrt{2}k\{8(b-a)(a+b-2c)C(m) - 4(a-c)(3a+2\nu)\varepsilon(m)\}}{(b-a)(a-c)^2(b-c)^{5/2}}.
\end{aligned}
\tag{28}
$$

The coefficient of $C(m)$ in the (28) is negative, while the coefficient of $\varepsilon(m)$ is positive.

But $\varepsilon(m) \geq 0$ is small, with $\varepsilon(0) = 0$, and its derivative is

$$
\varepsilon'(m) = \frac{1}{2}\int_0^{\pi/2}\frac{\sin^2\theta(\sin^2\theta - \cos^2\theta)\,d\theta}{(\sqrt{1-m\sin^2\theta})^3} \geq 0
$$

(we can verify the positivity with he same argument used for the positivity of $\varepsilon(m)$ itself).

Moreover

$$
\varepsilon'(m) \leq \frac{1}{2}\int_0^{\pi/2}\frac{\sin^2\theta\,d\theta}{(\sqrt{1-m\sin^2\theta})^3} = \frac{1}{2(1-m)}C(m)
$$

(see (24)).

Then we have

$$\varepsilon(m) = m\varepsilon'(\xi) < m\frac{1}{2(1-\xi)}C(\xi) < mC(1/2) < 1.04\frac{\pi}{4}m,$$

where $0 < \xi < m < \frac{1}{2}$, $C(1/2) \simeq 0.8472$,[1] and $4C(1/2)/\pi < 1.04$.

Being $C(0) = \frac{\pi}{4}$, we have the following inequality regarding the {} bracket in (28):

$$\begin{aligned} &8(b-a)(a+b-2c)C(m) - 4(a-c)(3a+2\nu)\varepsilon(m) \\ >&\pi\frac{b-a}{b-c}\{2(b-c)(a+b-2c) - 1.04(a-c)(3a+2\nu)\}. \end{aligned}$$

Let us substitute $c = -2\nu - a - b$; then

$$\begin{aligned} &2(b-c)(a+b-2c) - 1.04(a-c)(3a+2\nu) \\ =&-6a^2/25 + 372ab/25 + 48a\nu/5 + 12b^2 + 648b\nu/25 + 296\nu^2/25, \end{aligned}$$

that is definitively positive, being $a \leq b$.

Then, also the third item in Theorem 1 is proved.

4. Now we know that $T(E,k)$ is decreasing with respect to k. In order to prove the last statement in Theorem 1, we may observe that

$$T(E,k) = \frac{2\sqrt{2}}{\sqrt{b-c}}\mathrm{K}(m) \leq T(E,0) = \frac{2\sqrt{2}}{\sqrt{b_0-c_0}}\mathrm{K}(m_0) \leq \frac{2\sqrt{2}}{\sqrt{b_0-c_0}}\mathrm{K}(1/2),$$

where $T(E,0)$, was introduced in (26), and $m_0 = \frac{b_0}{b_0-c_0}$. Being b_0, c_0 roots of $z^2 + 2\nu z - 4E = 0$, $b_0 - c_0 = 2\sqrt{\nu^2 + 4E}$, Then

$$\lim_{E\to+\infty} T(E,k) \leq \lim_{E\to+\infty} T(E,0) = 0.$$

□

Remark 2. The oscillatory period $T(E,k)$ is naturally defined for $E > E_0$, being $\rho = \rho_0$ constant if $E = E_0$.

Nevertheless it could be useful to define the limit period $T(E_0, k_{max})$, where a mixed motion becomes a circular one. Let us shorten $k_{max} = k_{max}(E_0)$: then we define $T(E_0, k_{max}) := \lim_{(E,k)\to(E_0,k_{max})} T(E,k)$.

Let us set $z_0 = \rho_0^2$; we have by the (15) (for $E = E_0$): $(b-c) = 2\nu + 3z_0$, $\mathrm{K}(0) = \pi/2$. Moreover z_0 is a double root of $z^3 + 2\nu z^2 - 4E_0 z + 2k_{max}^2 = 0$, that implies $3z_0^2 + 4\nu z_0 - 4E_0 = 0$,

$$z_0 = \frac{2}{3}(\sqrt{\nu^2 + 3E_0} - \nu), \tag{29}$$

and then

$$T(E_0, k_{max}) = \frac{2\pi}{\sqrt{4\nu + 6z_0}} = \frac{\pi}{\sqrt[4]{\nu^2 + 3E_0}}. \tag{30}$$

[1] See (70) in the Appendix. $mC(m) = \mathrm{E}(m) - (1-m)\mathrm{K}(m)$

4. **Properties of $\bar{\theta}(E,k)$.** As we said at the beginning of the previous section, some numerical experiments make us conjecture that the angle $\bar{\theta}(E,k)$ could also be increasing with respect to E.

As $T(E,k)$, $\bar{\theta}(E,k)$ is an elliptic integral, whose formula is reduced to a sum of a complete elliptic integral of the first kind, a complete elliptic integral of the third kind and an immediate integral (see the Appendix (67)).

We remind the reader about the canonical forms of the elliptic integrals, for $-\pi/2 < \psi < \pi/2$:

$$\mathrm{F}(\psi|m) = \int_0^\psi \frac{d\theta}{\sqrt{1-m\sin^2\theta}}, \quad \mathrm{E}(\psi|m) = \int_0^\psi \sqrt{1-m\sin^2\theta}\,d\theta.$$

These are respectively incomplete elliptic integrals of first and second kind, while $\mathrm{K}(m) = \mathrm{F}(\pi/2|m)$, $\mathrm{E}(m) = \mathrm{E}(\pi/2|m)$ are the complete ones and the Heuman's Lambda Function $\Lambda_0(\psi|m)$ is

$$\Lambda_0(\psi|m) = \frac{2}{\pi}(\mathrm{K}(m)\mathrm{E}(\psi|1-m) - (\mathrm{K}(m)-\mathrm{E}(m))\mathrm{F}(\psi|1-m)).$$

Let us define also some expression, depending on E, k:

$$E_a = E - \frac{k^2}{a} = \frac{1}{4}(-ab-ac+bc) = \frac{1}{4}((b-a)(c-a)-a^2) < 0, \tag{31}$$

$$E_b = E - \frac{k^2}{b} = \frac{1}{4}(-ab+ac-bc) = \frac{1}{4}((b-a)(b-c)-b^2), \tag{32}$$

$$E_c = E - \frac{k^2}{c} = \frac{1}{4}(+ab-ac-bc) = \frac{1}{4}((a-c)(b-c)-c^2) > 0, \tag{33}$$

$$\sin\psi = \frac{E_b}{-E_a}. \tag{34}$$

We perform on the integral (10) the same substitutions as in the previous section and, applying (67), it follows

$$\begin{aligned}\bar{\theta}(E,k) &= \sqrt{2}k\int_a^b \frac{1}{z}\frac{dz}{\sqrt{-(z-a)(z-b)(z-c)}}\\ &= \sqrt{2}k\{\frac{-a}{2E_a}\frac{1}{\sqrt{b-c}}\mathrm{K}(m) + \frac{\pi}{2\sqrt{2}|k|}(1+\Lambda_0(\psi|m))\}.\end{aligned}$$

the sign of $\bar{\theta}(E,k)$ depending only on the sign of the momentum k. Let us suppose, without lack of information, to be $k>0$, and let us use the definition of $T(E,k)$ in (14). Then we have

$$\bar{\theta}(E,k) = \frac{ka}{-4E_a}T(E,k) + \frac{\pi}{2}(1+\Lambda_0(\psi|m)). \tag{35}$$

We have listed the properties of $\bar{\theta}(E,k)$ in the Theorem 1.2:

Proof of Theorem 1.2. **1.** A closed expression for $\frac{d}{dE}\bar{\theta}(E,k)$ can be written down. It is huge, and it needs some attention in order to avoid trivial errors even if, at the end, can be quite simplified. As observed in the previous section, a symbolic calculus program could rather help.

Referring to the (35), we have already derived some useful formulas. We list some other relations

$$\frac{\partial}{\partial E}\frac{a}{-4E_a} = -\frac{k^2+\frac{a^3}{4}}{(b-a)(a-c)E_a^2}, \tag{36}$$

$$\cos\psi = \frac{k\sqrt{b-a}}{-\sqrt{2}E_a}, \quad \sqrt{1-(1-m)\sin^2\psi} = \sqrt{m}\frac{E_c}{-E_a}, \tag{37}$$

$$\frac{\partial\psi}{\partial E} = \frac{1}{\cos\psi}\,\frac{k^2(2a^2+2b^2-ab+4E)}{(b-a)(a-c)(b-c)E_a^2}. \tag{38}$$

We can write down the expression of the derivative of $\bar{\theta}(E,k)$:

$$\frac{\partial}{\partial E}\bar{\theta}(E,k) = kT(E,k)\frac{\partial}{\partial E}\frac{a}{-4E_a} + k\frac{a}{-4E_a}\frac{\partial}{\partial E}T(E,k)$$
$$+\frac{\pi}{2}\frac{\partial}{\partial\psi}\Lambda_0(\psi|m)\,\frac{\partial\psi}{\partial E} + \frac{\pi}{2}\frac{\partial}{\partial m}\Lambda_0(\psi|m)\,\frac{\partial m}{\partial E}.$$

Because of the length, we split this expression in addenda.

1.
$$kT(E,k)\frac{\partial}{\partial E}\frac{a}{-4E_a} = -k\frac{2\sqrt{2}}{\sqrt{b-c}}\mathrm{K}(m)\frac{k^2+\frac{a^3}{4}}{(b-a)(a-c)E_a^2},$$

for the (14), (36)

2.
$$k\frac{a}{-4E_a}\frac{\partial}{\partial E}T(E,k)$$
$$= k\frac{a}{-4E_a}\,\frac{-4\sqrt{2}(a-c)(b(a-c)-c(b-a))\mathrm{K}(m)+4\sqrt{2}(8\nu E+18k^2)C(m)}{(b-a)(a-c)^2(b-c)^{5/2}},$$

for the (25).

3.
$$\frac{\pi}{2}\frac{\partial}{\partial\psi}\Lambda_0(\psi|m)\,\frac{\partial\psi}{\partial E}$$
$$= (E_a^2C(m)+(a-c)k^2\frac{\mathrm{K}(m)}{2})\frac{\sqrt{2}k(2a^2+2b^2-ab+4E)}{E_a^2E_c\sqrt{b-c}\,(b-a)(a-c)(b-c)},$$

for the (68), (37), (38).

4.
$$\frac{\pi}{2}\frac{\partial}{\partial m}\Lambda_0(\psi|m)\,\frac{\partial m}{\partial E} = -S(m)\frac{\sqrt{2}kE_b(8\nu E+18k^2)}{-E_aE_c\sqrt{b-c}\,(b-a)(a-c)(b-c)^2},$$

for the (34), (37), (69), (19).

A convenient positive commune factor could be:

$$\frac{\sqrt{2}k}{E_a^2E_c(b-c)^{5/2}(a-c)^2(b-a)}.$$

Every observation on the sign of our derivative could be done on the expression:

$$
\begin{aligned}
&-2(a-c)(b-c)^2E_c(k^2+\frac{a^3}{4})\mathrm{K}(m)\\
&+aE_aE_c(a-c)(b(a-c)-c(b-a))\mathrm{K}(m)-aE_aE_c(8\nu E+18k^2)C(m)\\
&+(E_a^2C(m)+(a-c)k^2\frac{\mathrm{K}(m)}{2})(2a^2+2b^2-ab+4E)(a-c)(b-c)\\
&+S(m)E_aE_b(8\nu E+18k^2)(a-c). \qquad (39)
\end{aligned}
$$

As in the previous section we use the function $\varepsilon(m)$ definite in (22); we recall the relations

$$
\mathrm{K}(m)=2C(m)+\varepsilon(m), \quad S(m)=C(m)+\varepsilon(m)
$$

and we can rewrite (39) in a more convenient form:

$$
\begin{aligned}
&\{-4(a-c)(b-c)^2E_c(k^2+\frac{a^3}{4})\\
&+2aE_aE_c(a-c)(b(a-c)-c(b-a))-aE_aE_c(8\nu E+18k^2)\\
&+(E_a^2+(a-c)k^2)(2a^2+2b^2-ab+4E)(a-c)(b-c)\\
&+E_aE_b(8\nu E+18k^2)(a-c)\}C(m)+\{-2(a-c)(b-c)^2E_c(k^2+\frac{a^3}{4})\\
&+aE_aE_c(a-c)(b(a-c)-c(b-a))+\frac{k^2}{2}(2a^2+2b^2-ab+4E)(a-c)^2(b-c)\\
&+E_aE_b(8\nu E+18k^2)(a-c)\}\varepsilon(m)\\
&\\
&=E_a^2E_c\{8(b-a)(a+b-2c)C(m)-4(a-c)(3a+2\nu)\varepsilon(m)\}, \qquad (40)
\end{aligned}
$$

where the coefficient of $C(m)$ is ≥ 0 and that of $\varepsilon(m)$ is negative.

We observe that the $\{\}$ bracket in the last line of (40) is the same of (28). Moreover it is also

$$
\frac{\partial}{\partial E}\bar{\theta}(E,k)=-\frac{\partial}{\partial k}T(E,k)>0.
$$

That complete the proof of the first item of the Theorem 1.2.

2. We know already that, if $E\to+\infty$, then $a\to 0$, $b\to+\infty$, $E_b\sim E$, for every fixed k. In order to prove the second item in this Theorem we need to list also some other asymptotic behavior:

$$
a\sim\frac{k^2}{2E}, \quad b\sim 2\sqrt{E}, \quad c\sim -2\sqrt{E}, \qquad (41)
$$

$$
E_a\sim -E, \quad E_b\sim E, \quad E_c\sim E, \quad (E\to+\infty). \qquad (42)
$$

From (15) follows $c\sim -b$, from (17) $a=\frac{2k^2}{-bc}\sim\frac{2k^2}{b^2}$ and from (16) $-4E\sim -b^2$. From the first of (41) follows that $E_a=E-\frac{k^2}{a}\sim E-2E=-E$; the other two in (42) are straightforward.

We remind that $m\to 1/2$, $\sin\psi=\frac{E_b}{-E_a}\to 1$, $\psi\to\pi/2$.

It is well known that $\Lambda_0(\pi/2|m)\equiv 1, \quad \forall m, \quad 0<m<1$ (Legendre's Relation), then, for every fixed k , the second item of our Theorem is easily proved.

3. The upper bound for $\bar{\theta}(E,k)$ follows immediately from the previous two statements.

4. Now we use the ideas in the previous Section, where we defined $T(E,0)$: in this context $a_0 = 0$, b_0, c_0 are the solution of $z^3 + 2\nu z^2 - 4Ez = 0$, $E_a = \frac{1}{4}(-ab - ac + bc) \to \frac{1}{4}b_0c_0$ and $E_b = \frac{1}{4}(-ab + ac - bc) \to -\frac{1}{4}b_0c_0 > 0$, if $k \to 0$, for every fixed $E > E_0$. As in the previous item $\psi \to \pi/2$, then the first term in $\bar{\theta}(E,k)$ goes to 0, and the second to π.

5. In order to prove this item, we use the ideas in the final Remark in the previous Section: The limit for $E \to E_0$ of the first term in (35) is

$$\frac{k_{max} z_0}{-4E_a} T(E_0, k_{max}),$$

where $E_a = E_b = \frac{-z_0^2}{4}$ for $E = E_0$. It is also $\psi = -\pi/2$ for $E = E_0$.

Being Λ_0 odd with respect to ψ, the Legendre's Relation implies $\Lambda_0(-\pi/2|m) \equiv -1,\ \forall m, \quad 0 < m < 1$. It is certainly known, and in any case easy to show, that $\lim_{(\psi,m)\to(\pm\pi/2,0)} \Lambda_0(\psi|m) = \pm 1$.

Then the second term in (35) goes to 0 for $E \to E_0$.

Using (17) on k_{max} and (30), we obtain finally

$$\bar{\theta}(E_0, k_{max}) := \\ \frac{k_{max}}{z_0} T(E_0, k_{max}) = \frac{\sqrt{z_0^2(\nu + z_0)}}{z_0} \frac{2\pi}{\sqrt{4\nu + 6z_0}} = \sqrt{\frac{\nu + z_0}{\nu + \frac{3}{2}z_0}}\pi \geq \sqrt{\frac{2}{3}}\pi. \tag{43}$$

Being $\bar{\theta}$ increasing with respect to E, this property yields also a global lower bound for $\bar{\theta}$.

Remark 3. We observe that the angular velocity of a circular motion is constant ($\theta' = \frac{k}{\rho_0^2}$, $z_0 = \rho_0^2$), then this limit is coherent with the definition of $\bar{\theta}$ for a circular motion. I have chosen the [8] formula for the elliptic integral of the third kind, instead of that in [1], just to show more easily this property. The (43) shows also an explicit formula for $\bar{\theta}(E_0, k_{max})$, as we have found an explicit formula for $T(E_0, k_{max})$ in the final Remark of the previous section.

Being $T(E_0, k_{max})$ the time needed to cover the angle $\bar{\theta}(E_0, k_{max})$ with constant angular velocity, the time corresponding to the angle 2π , that is the minimal period of a circular solution, coincides with $T_2(E_0, k_{max})$, defined as follows (see (11)).

$$T_2(E_0, k_{max}) := 2\pi \frac{T(E_0, k_{max})}{\bar{\theta}(E_0, k_{max})} = 2\pi \frac{1}{\sqrt{\nu + z_0}} = \frac{2\pi}{\sqrt{\frac{1}{3}\nu + \frac{2}{3}\sqrt{\nu^2 + 3E_0}}}. \tag{44}$$

The last expression in (44) yields also an explicit formula depending on E_0, ν, due to (29).

6. Regarding the derivative with respect to k, at last, we follow the same scheme of the first item of this Theorem; in this case the help of a symbolic calculus program is highly recommended:

$$\begin{aligned}\frac{\partial}{\partial k}\frac{ka}{-4E_a} &= \frac{a}{-4E_a} + \frac{k^2(2a^2+4E)}{4E_a^2(b-a)(a-c)} \\ &= \frac{-a^4b+a^3b^2-a^4c-a^3bc+a^3c^2+3ab^2c^2}{16(b-a)(a-c)E_a^2},\end{aligned} \tag{45}$$

$$\frac{\partial \psi}{\partial k} = \frac{\sqrt{2}(a^3b-2a^2b^2+ab^3+a^3c+b^3c-a^2c^2-b^2c^2)}{-2E_a(b-a)^{3/2}(a-c)(b-c)}. \tag{46}$$

Now our derivative of $\bar{\theta}(E,k)$ is:

$$\frac{\partial}{\partial k}\bar{\theta}(E,k) = T(E,k)\frac{\partial}{\partial k}\frac{ka}{-4E_a} + \frac{ka}{-4E_a}\frac{\partial}{\partial k}T(E,k)$$

$$+\frac{\pi}{2}\frac{\partial}{\partial \psi}\Lambda_0(\psi|m)\,\frac{\partial \psi}{\partial k} + \frac{\pi}{2}\frac{\partial}{\partial m}\Lambda_0(\psi|m)\,\frac{\partial m}{\partial k}.$$

Because of the length, we split also this expression in addenda, representing $\mathrm{K}(m)$, $S(m)$ in terms of $C(m)$, $\varepsilon(m)$, (see (22) (23)).

1.

$$T(E,k)\frac{\partial}{\partial k}\frac{ka}{-4E_a} = \frac{\sqrt{2}(-a^4b+a^3b^2-a^4c-a^3bc+a^3c^2+3ab^2c^2)}{8(b-a)(a-c)(b-c)^{1/2}E_a^2}(2C(m)+\varepsilon(m)),$$

for the (14), (45).

2.

$$\frac{ka}{-4E_a}\frac{\partial}{\partial k}T(E,k) = \frac{-\sqrt{2}k^2a\{8(b-a)(a+b-2c)C(m)-4(a-c)(3a+2\nu)\varepsilon(m)\}}{-4E_a(b-a)(a-c)^2(b-c)^{5/2}},$$

for the (28).

3.

$$\frac{\pi}{2}\frac{\partial}{\partial \psi}\Lambda_0(\psi|m)\,\frac{\partial \psi}{\partial k} = \sqrt{2}((E_a^2+(a-c)k^2)C(m)+(a-c)k^2\varepsilon(m)/2)*$$

$$*\frac{a^3b-2a^2b^2+ab^3+a^3c+b^3c-a^2c^2-b^2c^2}{2E_a^2E_c(b-a)(a-c)(b-c)^{3/2}},$$

for the (68), (46).

4.

$$\frac{\pi}{2}\frac{\partial}{\partial m}\Lambda_0(\psi|m)\,\frac{\partial m}{\partial k} = (C(m)+\varepsilon(m))\frac{8\sqrt{2}k^2E_b(\nu^2+3E)}{-E_aE_c(b-a)(a-c)(b-c)^{5/2}},$$

for the (69), (27).

A convenient positive commune factor could be :

$$\frac{\sqrt{2}}{8E_a^2E_c(b-a)(a-c)^2(b-c)^{5/2}}.$$

Let us substitute now $k^2 = -abc/2$, then the numerator of our derivative will be:

$$
\begin{aligned}
&(-a^4b+a^3b^2-a^4c-a^3bc+a^3c^2+3ab^2c^2)(a-c)(b-c)^2E_c(2C(m)+\varepsilon(m))\\
&-E_aE_ca^2bc\{8(b-a)(a+b-2c)C(m)-4(a-c)(3a+2\nu)\varepsilon(m)\}\\
&+4(a-c)(b-c)((E_a^2+(a-c)k^2)C(m)+(a-c)k^2\varepsilon(m)/2)*\\
&*(a^3b-2a^2b^2+ab^3+a^3c+b^3c-a^2c^2-b^2c^2)\\
&+32abc(a-c)E_aE_b(\nu^2+3E)(C(m)+\varepsilon(m))=\\
\\
&=-16(b-a)E_a^2E_c(4E(a+b-2c)-c(c^2-ab))C(m)+\\
&16a(a-c)E_a^2E_c(b^2-ab+c^2-ac)\varepsilon(m)
\end{aligned}
$$

and the derivative itself will be

$$
\begin{aligned}
\frac{\partial}{\partial k}\bar{\theta}(E,k)=&\frac{-2\sqrt{2}(b-a)(4E(a+b-2c)-c(c^2-ab))C(m)}{(b-a)(a-c)^2(b-c)^{5/2}}\\
&+\frac{2\sqrt{2}a(a-c)(b^2-ab+c^2-ac)\varepsilon(m)}{(b-a)(a-c)^2(b-c)^{5/2}}
\end{aligned}
\tag{47}
$$

where the coefficient of $C(m)$ is negative, and that of $\varepsilon(m)$ is positive. Now we use the same argument presented already in the Theorem 1.1 for the k derivative of $T(E,k)$, using the relations $C(m)\geq\frac{\pi}{4}$, $\varepsilon(m)\leq 1.04\frac{\pi}{4}m$.

Then the opposite of the numerator in (47) is bigger than

$$
\{(b-c)(4E(a+b-2c)-c(c^2-ab))-1.04a(a-c)(b^2-ab+c^2-ac)\}\frac{\pi}{4}m.
$$

If we substitute $c=-a-b-2\nu$ in this last $\{\}$ bracket we obtain a very long expression, that is, fortunately, certainly positive.

That complete the proof of the Theorem 1.2. □

Proof of Theorem 2.1. Let us consider the delay angle $\theta_d=2(\pi-\bar{\theta}(E,k))$ defined in Section 2. As shown in that Section, the minimal angular distance of two points $\mathbf{u}(t)$ on the circle $\rho=\rho_2$ is $\frac{2\pi}{q}$, and θ_d is not necessary equal to $\frac{2\pi}{q}$, but is however its multiple; then it has to be, for every fixed k,

$$
\frac{2\pi}{q}\leq\theta_d\to 0,\quad E\to+\infty.
$$

Then $q\to+\infty$, by the second statement in Theorem 1.2.

Being $T(E,k)\to 0$, that is not sufficient to prove the thesis. Instead of proving the limit $\lim_{E\to+\infty}qT(E,k)$ to be infinity, we prove that

$$
\lim_{E\to+\infty}\frac{\theta_d}{2\pi T(E,k)}\geq\frac{1}{qT(E,k)}\to 0.
$$

For $E\to+\infty$, we have

$$
\begin{aligned}
\frac{\theta_d}{2\pi T(E,k)}=&\frac{\pi-\bar{\theta}(E,k)}{\pi T(E,k)}=\frac{\frac{\pi}{2}-\frac{\pi}{2}\Lambda_0(\psi|m)-\frac{ka}{-4E_a}T(E,k)}{\pi T(E,k)}\\
=&\frac{\frac{\pi}{2}-\frac{\pi}{2}\Lambda_0(\psi|m)}{\pi T(E,k)}-\frac{ka}{-4\pi E_a}
\end{aligned}
\tag{48}
$$

and the second term in the previous sum goes to zero. In order to evaluate the first term, we need some complementary formulas:

$$\frac{\pi}{2}-\frac{\pi}{2}\Lambda_0(\psi|m)\sim\sqrt{2}(\mathrm{E}(\frac{1}{2})-\frac{1}{2}\mathrm{K}(\frac{1}{2}))(\frac{\pi}{2}-\psi)\simeq 0.5991(\frac{\pi}{2}-\psi). \tag{49}$$

In order to prove (49) it is enough to observe that

$$\frac{\pi}{2}-\frac{\pi}{2}\Lambda_0(\psi|m)=\frac{\pi}{2}(\Lambda_0(\frac{\pi}{2}|m)-\Lambda_0(\psi|m))=$$
$$\int_{\psi}^{\frac{\pi}{2}}\frac{\mathrm{E}(m)-(1-m)\mathrm{K}(m)\sin^2\theta}{\sqrt{1-(1-m)\sin^2\theta}}\,d\theta=$$
$$=\frac{\mathrm{E}(m)-(1-m)\mathrm{K}(m)\sin^2\xi}{\sqrt{1-(1-m)\sin^2\xi}}(\frac{\pi}{2}-\psi),\qquad \psi\le\xi\le\frac{\pi}{2}.$$

Being $\psi\to\frac{\pi}{2}$, $m\to\frac{1}{2}$ and $\sqrt{2}(\mathrm{E}(\frac{1}{2})-\frac{1}{2}\mathrm{K}(\frac{1}{2}))\simeq 0.5991>0$, the (49) follows.
Now we need to evaluate

$$(\frac{\pi}{2}-\psi)\sim\frac{1}{2}k\sqrt{2}E^{-3/4}. \tag{50}$$

We refer to an elementary asymptotic formula: $\frac{\pi}{2}-\arcsin x\sim\sqrt{1-x}$ for $x\to 1^-$. Indeed , using (34), (41), (42) and the equality $2(E_a+E_b)=-ab$, we obtain

$$\frac{\pi}{2}-\psi=\frac{\pi}{2}-\arcsin\frac{E_b}{-E_a}\sim\sqrt{1-\frac{E_b}{-E_a}}=\sqrt{\frac{ab}{-2E_a}}\sim\frac{\sqrt{2}}{2}kE^{-3/4}.$$

Finally, always by (41), we have

$$T(E,k)=2\sqrt{2}\frac{\mathrm{K}(m)}{\sqrt{b-c}}\sim\sqrt{2}\mathrm{K}(\frac{1}{2})E^{-1/4}. \tag{51}$$

Putting together (49), (50) and (51) we have finally

$$\frac{\frac{\pi}{2}-\frac{\pi}{2}\Lambda_0(\psi|m)}{\pi T(E,k)}\sim\mathrm{const}\,kE^{-1/2}\to 0.$$

Then this property is also proved. □

5. **Appendix.** While the formulas on the elliptic integral in Section 3 are easy to find, this is not longer true for the elliptic integral in Section 4. We used some suggestions in [1], page 601 and we try to show a elementary exposition of the various substitution involved.

We need to transform the integrals

$$\int_a^b\frac{dx}{\sqrt{-(x-a)(x-b)(x-c)}}=\int_a^b\frac{dx}{y}, \tag{52}$$

$$\int_a^b\frac{dx}{x\sqrt{-(x-a)(x-b)(x-c)}}=\int_a^b\frac{1}{x}\frac{dx}{y}, \tag{53}$$

which are improper integrals, where $y=\sqrt{-(x-a)(x-b)(x-c)}$ and $c<0<a<b$, to more manageable ordinary integrals.

Then we perform a first change of variable:

$$w = \frac{x-c}{(x-a)(b-x)}.$$

In fact $w = w(x)$ is a positive, convex function in (a,b), with minimum value w_{min}, corresponding to $x = x_{min}$. For every $w \geq w_{min}$ we solve the equation

$$wx^2 - ((a+b)w - 1)x + abw - c = 0, \tag{54}$$

which solutions are

$$x_{\pm} = \frac{(a+b)w-1}{2w} \pm \frac{\sqrt{W}}{w},$$

being $W = \frac{1}{4}((a+b)w-1)^2 - (abw^2 - cw)$.
If we differentiate the (54) with respect to x we obtain

$$w'(x) = \frac{\pm 2\sqrt{W(w(x))}}{(x-a)(b-x)},$$

with the - sign if $a < x < x_{min}$ and the + one if $x_{min} < x < b$.
Now we perform the second substitution $z = z(w)$, where

$$z^2 = \frac{W}{w}. \tag{55}$$

(55) corresponds to the equation

$$(b-a)^2w^2 - (2(a+b) - 4c + 4z^2)w + 1 = 0. \tag{56}$$

As before, we solve (56) and obtain

$$w_{\pm} = \frac{(z^2+a-c) + (z^2+b-c) \pm 2Z}{(b-a)^2} = (\frac{\sqrt{z^2+b-c} \pm \sqrt{z^2+a-c}}{b-a})^2, \tag{57}$$

where $Z(z) = \sqrt{(z^2+a-c)(z^2+b-c)}$.
Only $w_+ \geq w_{min} = (\frac{\sqrt{b-c}+\sqrt{a-c}}{b-a})^2$. Keeping in mind that we are interested only in the + sign, we differentiate (56) with respect to w, and obtain, using (57),

$$z'(w) = \frac{Z(z(w))}{2\sqrt{w}\sqrt{W(w)}}.$$

Now we are able to transform both integrals (52), (53).
First $dx = x'(w)dw = x'(w)w'(z)dz = \frac{1}{w'(x)}\frac{1}{z'(w)}dz$, that is

$$dx = \pm(x-a)(b-x)\sqrt{\frac{x-c}{(x-a)(b-x)}}\frac{dz}{Z} = \pm\frac{y}{Z}dz,$$
$$\frac{dx}{y} = \pm\frac{dz}{Z},$$

with the - sign if $a < x < x_{min}$ and the + one if $x_{min} < x < b$.

At last we have, for (52),

$$\int_a^b \frac{dx}{y} = \int_a^{x_{min}} \frac{dx}{y} + \int_{x_{min}}^b \frac{dx}{y} = \int_{+\infty}^{w_{min}} ...dw + \int_{w_{min}}^{+\infty} ...dw =$$
$$= \int_{+\infty}^0 \frac{-dz}{Z} + \int_0^{+\infty} \frac{dz}{Z} = 2\int_0^{+\infty} \frac{dz}{Z}.$$

Let us set $z = \sqrt{a-c}\tan\theta$, $m = \frac{b-a}{b-c}$, then we obtain

$$\int_a^b \frac{dx}{y} = 2\int_0^{+\infty} \frac{dz}{Z} = \frac{2}{\sqrt{b-c}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}. \tag{58}$$

We apply now the same scheme to the integral (53), and we have

$$\int_a^b \frac{1}{x}\frac{dx}{y} = \int_{+\infty}^0 \frac{1}{x_-(w_+(z))}\frac{-dz}{Z} + \int_0^{+\infty} \frac{1}{x_+(w_+(z))}\frac{dz}{Z}$$
$$= \int_0^{+\infty} (\frac{1}{x_-(w_+(z))} + \frac{1}{x_+(w_+(z))})\frac{dz}{Z}.$$

x_+, x_- are solutions of the quadratic equation (54), as w_+, w_- are solutions of (56). Then we have

$$\frac{1}{x_-} + \frac{1}{x_+} = \frac{x_- + x_+}{x_- x_+} = \frac{(a+b)w_+ - 1}{abw_+ - c} = \frac{((a+b)w_+ - 1)(abw_- - c)}{(abw_+ - c)(abw_- - c)}$$
$$= \frac{(a+b)abw_+w_- - c(a+b)w_+ - abw_- + c}{(ab)^2 w_+ w_- - abc(w_+ + w_-) + c^2}$$
$$= \frac{(a+b)ab - (ab+ac+bc)(a+b-2c+2z^2) + 2Z(-ac-bc+ab) + c(b-a)^2}{(ab)^2 - 2abc(a+b-2c+2z^2) + c^2(b-a)^2}$$
$$= \frac{2c(ac-ab+bc) - 2(ab+ac+bc)z^2 + 2(-ac-bc+ab)Z}{(ac-ab+bc)^2 - 4abcz^2}.$$

We note that the connection between the roots of the cubic equation a,b,c and the mechanical energy and momentum E, k was already described in (16) , (17) , precisely being

$E = -\frac{1}{4}(ab+ac+bc)$, $k^2 = -\frac{1}{2}abc$.

In order to shorten the algebraic expressions [2], we define now:

$$E_a = E - \frac{k^2}{a} = \frac{1}{4}(-ab-ac+bc) = \frac{1}{4}((b-a)(c-a) - a^2) < 0, \tag{59}$$
$$E_b = E - \frac{k^2}{b} = \frac{1}{4}(-ab+ac-bc) = \frac{1}{4}((b-a)(b-c) - b^2), \tag{60}$$
$$E_c = E - \frac{k^2}{c} = \frac{1}{4}(+ab-ac-bc) = \frac{1}{4}((a-c)(b-c) - c^2 > 0. \tag{61}$$

Then we have

$$\int_a^b \frac{1}{x}\frac{dx}{y} = \int_0^{+\infty} \frac{-cE_c + Ez^2}{2E_c^2 + k^2z^2}\frac{dz}{Z} + \int_0^{+\infty} \frac{E_c}{2E_c^2 + k^2z^2}\, dz. \tag{62}$$

[2] Definitions anticipated in (31), (32), (33)

The second integral in (62) is immediate. Then we perform on the first integral the same change of variable $z = \sqrt{a-c}\tan\theta$ as in (58), and we obtain

$$\int_0^{+\infty} \frac{-cE_c + Ez^2}{2E_c^2 + k^2z^2}\frac{dz}{Z}$$
$$= \frac{1}{\sqrt{b-c}}\int_0^{\pi/2} \frac{-cE_c + ((a-c)E + cE_c)\sin^2\theta}{2E_c^2 + (k^2(a-c) - 2E_c^2)\sin^2\theta}\frac{d\theta}{\sqrt{1-m\sin^2\theta}}.$$

Now we manipulate the algebraic expressions in a, b, c observing that

$$(a-c)E + cE_c = aE_a,$$
$$k^2(a-c) - 2E_c^2 = -2E_a^2,$$

and we decompose the following function in the sum

$$\frac{-cE_c + aE_a\sin^2\theta}{2Ec^2 - 2Ea^2\sin^2\theta} = \frac{-a}{2E_a} + \frac{1}{2}(\frac{a}{E_a} - \frac{c}{E_c})\frac{1}{(1-n\sin^2\theta)},$$

where

$$n = \frac{E_a^2}{E_c^2}.$$

Then the integral (59)can be decomposed as follows

$$\int_a^b \frac{1}{x}\frac{dx}{y} = \frac{-a}{2E_a}\frac{1}{\sqrt{b-c}}\mathrm{K}(m) + \frac{1}{2\sqrt{b-c}}(\frac{a}{E_a} - \frac{c}{E_c})\Pi(n|m) + \frac{\pi}{2\sqrt{2}|k|}, \tag{63}$$

where

$$\Pi(n|m) = \int_0^{\pi/2} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-m\sin^2\theta}}$$

is a complete elliptic integral of the third kind.

We have not finished yet, because some work of polishing can be done on $\Pi(n|m)$. First we ought to show that $m \le n < 1$ (circular case, see for reference [8] , [1]). $|E_a| < E_c$ follows immediately from (59), (61), $n - m \ge 0$ is equivalent to

$$(b-c)E_a^2 - (b-a)E_c^2 = (a-c)E_b^2 \ge 0.$$

Let us now introduce the Heuman's Lambda Function $\Lambda_0(\psi|m)$

$$\Lambda_0(\psi|m) = \frac{2}{\pi}(\mathrm{K}(m)\mathrm{E}(\psi|1-m) - (\mathrm{K}(m) - \mathrm{E}(m))\mathrm{F}(\psi|1-m)),$$

where $\mathrm{E}(m) = \int_0^{\pi/2}\sqrt{1-m\sin^2\theta}\,d\theta$ is a complete elliptic integral of second kind, and

$$\mathrm{F}(\psi|m) = \int_0^{\psi} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}, \quad \mathrm{E}(\psi|m) = \int_0^{\psi}\sqrt{1-m\sin^2\theta}\,d\theta$$

are respectively incomplete elliptic integrals of first and second kind.

Both [8] , [1] show formulas for $\Pi(n|m)$; we have chosen the one in [8] [3]:

[3]Actually this formula holds for $n > m$, but for our integral, we can extend by continuity the final expression also for the only value of E where $m = n$, that is where $E_b = 0$. We point out also that [8] shows a nice elementary proof of this formula.

$$\Pi(n|m) = \sqrt{\frac{n}{(1-n)(n-m)}}\,\frac{\pi}{2}\Lambda_0(\psi|m), \quad \sin^2\psi = \frac{n-m}{n(1-n)}. \tag{64}$$

We verify the following relations:

$$\begin{aligned}
&\sin\psi = \frac{|E_b|}{|E_a|}, \qquad (65)\\
&\frac{n}{(1-n)(n-m)} = \frac{2(b-c)E_a^2E_c^2}{k^2(a-c)^2E_b^2},\\
&\frac{a}{E_a} - \frac{c}{E_c} = -\frac{(a-c)E_b}{E_aE_c},\\
&-\frac{1}{2}\frac{(a-c)E_b}{E_aE_c}\sqrt{\frac{2(b-c)E_a^2E_c^2}{k^2(a-c)^2E_b^2}} = \frac{\sqrt{b-c}}{\sqrt{2}\,|k|}\mathrm{sign}E_b,
\end{aligned}$$

we put together (63), (64) and the last three relations and we obtain

$$\int_a^b \frac{1}{x}\frac{dx}{y} = \frac{-a}{2E_a}\frac{1}{\sqrt{b-c}}\mathrm{K}(m) + \frac{\pi}{2\sqrt{2}|k|}(1 + (\mathrm{sign}E_b)\,\Lambda_0(\psi|m)). \tag{66}$$

Finally we observe that it is customary extend the definition of $\mathrm{F}(\psi|m)$, $\mathrm{E}(\psi|m)$, $\Lambda_0(\psi|m)$ to $-\pi/2 < \psi < \pi/2$. If we modify slightly to (65), putting

$$\sin\psi = \frac{E_b}{-E_a},$$

which has the same sign as E_b, we get that (66) become

$$\int_a^b \frac{1}{x}\frac{dx}{y} = \frac{-a}{2E_a}\frac{1}{\sqrt{b-c}}\mathrm{K}(m) + \frac{\pi}{2\sqrt{2}|k|}(1 + \Lambda_0(\psi|m)). \tag{67}$$

Complements about $\Lambda_0(\psi|m)$

The Heumann Lambda Function $\Lambda_0(\psi|m)$ is increasing in ψ, $\forall\psi$, decreasing in m for $\psi > 0$. The explicit formulas for the derivatives are well known (see for example Mathematica); they are

$$\begin{aligned}
\frac{\pi}{2}\frac{\partial}{\partial\psi}\Lambda_0(\psi|m) &= \frac{\mathrm{E}(m) - (1-m)\mathrm{K}(m)\sin^2\psi}{\sqrt{1-(1-m)\sin^2\psi}}\\
&= \frac{mC(m) + (1-m)\mathrm{K}(m)\cos^2\psi}{\sqrt{1-(1-m)\sin^2\psi}} \geq 0,
\end{aligned} \tag{68}$$

$$\begin{aligned}
\frac{\pi}{2}\frac{\partial}{\partial m}\Lambda_0(\psi|m) &= \frac{(\mathrm{E}(m) - \mathrm{K}(m))\sin(2\psi)}{4m\sqrt{1-(1-m)\sin^2\psi}}\\
&= -\frac{S(m)}{2}\frac{\sin\psi\cos\psi}{\sqrt{1-(1-m)\sin^2\psi}},
\end{aligned} \tag{69}$$

where the functions $C(m)$, $S(m)$ were defined in (21).

The connections between $C(m)$, $S(m)$ and the elliptic integrals $\mathrm{K}(m)$, $\mathrm{E}(m)$ are immediate:

$$
\begin{aligned}
&mS(m) = \mathrm{K}(m) - \mathrm{E}(m), \quad mC(m) = \mathrm{E}(m) - (1-m)\mathrm{K}(m), \\
&S(m) + C(m) = \mathrm{K}(m).
\end{aligned} \tag{70}
$$

For the readers convenience we give a short proof.
An alternative expression for Λ_0 is

$$
\frac{\pi}{2}\Lambda_0(\psi|m) = \int_0^{\psi} \frac{\mathrm{E}(m) - (1-m)\mathrm{K}(m)\sin^2\theta}{\sqrt{1-(1-m)\sin^2\theta}}\, d\theta,
$$

then the proof of (68) is straightforward. More work is needed for (69).

$$
\frac{\pi}{2}\frac{\partial}{\partial m}\Lambda_0(\psi|m) = \int_0^{\psi} \frac{N(m,\theta)}{2\sqrt{(1-(1-m)\sin^2\theta)^3}} d\theta, \tag{71}
$$

where N is the numerator of the integrand function

$$
\begin{aligned}
N = 2(1-(1-m)\sin^2\theta)(\frac{d}{dm}\mathrm{E}(m) + \mathrm{K}(m)\sin^2\theta - (1-m)\,\frac{d}{dm}\mathrm{K}(m)\sin^2\theta) \\
-(\mathrm{E}(m) - (1-m)\mathrm{K}(m)\sin^2\theta)\sin^2\theta.
\end{aligned}
$$

We recall that $\frac{d}{dm}\mathrm{K}(m) = \frac{1}{2(1-m)}C(m)$ as showed in (24), while $\frac{d}{dm}\mathrm{E}(m) = -\frac{1}{2}S(m)$ and that (70) hold. Fortunately it is possible write down N using only $S(m)$:

$$
N = -S(m)(1 - 2\sin^2\theta + (1-m)\sin^4\theta),
$$

then

$$
\frac{\pi}{2}\frac{\partial}{\partial m}\Lambda_0(\psi|m) = -S(m)\int_0^{\psi} \frac{1-2\sin^2\theta + (1-m)\sin^4\theta}{2\sqrt{(1-(1-m)\sin^2\theta)^3}} d\theta.
$$

Being

$$
\frac{\partial}{\partial\theta}\frac{\sin\theta\cos\theta}{\sqrt{1-(1-m)\sin^2\theta}} = \frac{1-2\sin^2\theta+(1-m)\sin^4\theta}{\sqrt{(1-(1-m)\sin^2\theta)^3}},
$$

we obtain finally the closed formula (69) for the m- partial derivative of Λ_0.

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