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Optimal Investment, Stochastic Labor Income and Retirement*

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Abstract

We address the optimal consumption-investment-retirement problem considering stochastic labor income. We study the Merton problem assuming that the agent has to take four different decisions: the retirement date which is irreversible; the labor and the consumption rate and the portfolio decision before retirement. After retirement the agent only chooses the portfolio and the consumption rate. We confirm some classical results and we show that labor, portfolio and retirement decisions interact in a complex way depending on the spanning opportunities.

Keywords: Intertemporal optimal consumption and portfolio, labor income, retirement.

JEL Codes: G11, G23, E21.

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1 Introduction

We analyze the intertemporal optimal investment-consumption problem for an agent choosing also the leisure rate during his working life and the retirement date. Before retirement he has to choose the investment strategy, the consumption rate and the leisure rate earning a stochastic wage; after retirement he fully enjoys leisure, he cannot work (or go back to work), he only defines the consumption rate and the portfolio of risky assets. We confirm some classical results and we show that labor, portfolio and retirement decisions interact in a complex way depending on the spanning opportunities.

Since the seminal contributions [Merton (1969), Merton (1971)], several papers have investigated optimal consumption and portfolio choices in an intertemporal setting. This paper is related to two independent strands of this literature adapting the Merton model to the real life.

The first group of papers analyzes optimal consumption and portfolio choices when the agent invests his wealth in financial assets and works earning a stochastic wage, see [Bodie et al. (1992), Bodie et al. (2004), Duffie et al (1997)], and [Henderson (2005), Koo (1998)]. [Duffie et al (1997), Henderson (2005)] and [Koo (1998)] consider a stochastic labor income when the agent does not choose the labor rate. Labor income is non tradable, i.e., markets are not complete and labor income is driven by a Brownian motion that does not enter the stochastic differential equations for asset prices. In this context the agent uses financial assets to hedge the risk that comes from the stochastic wage. As a general result we have that the optimal portfolio is made up of two parts: the component obtained through the classical optimization problem with only financial wealth and the hedging component.

This class of models captures the interaction between financial wealth and labor income. The analysis points out an important result - the use of financial assets to hedge labor income risk - but leaves aside two peculiarities of the agent's intertemporal choices: flexibility on the participation to the labor market during the active life and on the retirement decision. In [Bodie et al. (1992)] the authors have addressed the first issue showing that labor rate flexibility induces the agent to invest more in risky assets at the beginning of his life: the opportunity to choose the labor effort in the future works as an insurance opportunity against adverse investment outcomes, see also [Bodie et al. (2004)] in the case of a fixed retirement date and [Viceira (2001)] assuming a random retirement date and no flexibility in the labor supply.

A fixed or a random retirement date are not plausible assumptions to model agent's choices. As a matter of fact, modern welfare programs allow the agent to choose the retirement date under constraints concerning the age or the wealth accumulated in retirement plans. Flexibility in the retirement choice renders the optimal consumption-investment problem much more complicated: it becomes a variational inequality problem. The literature has addressed the problem assuming a deterministic (constant) wage with a zero-one labor rate (after and before retirement), see [Farhi and Panageas (2007)], a deterministic (constant) wage with an endogenous choice in the labor rate before retirement, see [Choi and Shim (2006), Choi et al. (2008), Zhang (2010)], or with a stochastic wage but with no flexibility on the labor market before retirement, see [Dybvig and Liu (2010)]. The analysis with a deterministic wage shows that the agent optimally retires when the wealth reaches a critical level, the bound-

ary is increasing in the wage.

This paper aims to fill the gap between these two strands of literature: we consider a stochastic labor income and we allow the agent to choose both the leisure-labor rate during his working life and the retirement date. The optimal investment-consumption problem is solved assuming a borrowing constraint. Our setting is the one of [Dybvig and Liu (2010)] with flexibility in the labor choice. As in [Bodie et al. (2004), Dybvig and Liu (2010)], we consider a stochastic income driven by the same factors as asset prices (complete markets). We develop our analysis in a general setting with many financial assets deriving a variational inequality for the dual problem associated with the optimization problem. Then, assuming a single risky asset and a lognormal process for the wage we derive the solution of the problem. In this setting we also consider the case of incomplete markets and we extend the analysis considering a fixed labor supply. We show that flexibility in the labor supply affects in a complex way both the portfolio and the retirement decision.

With a single risky asset we confirm results already obtained in the optimal consumption literature and we obtain several interesting results. As in large part of the literature with fixed labor supply and no retirement option, a more risk averse agent prefers a smoother consumption path, he consumes less today, he dislikes risk and therefore he invests less in the risky asset and works more to insure his financial investment. A permanent wage shock leads the agent to consume more and to work more, the effect on the optimal portfolio depends on the correlation between wage and asset return. As we move from perfect negative correlation between wage and risky asset to perfect positive correlation both labor supply and risky portfolio decrease because there are less insurance opportunities and consumption goes up. In a market with restricted spanning opportunities the agent acts in a myopic way consuming a lot.

For a large set of parameters the wealth to retire threshold is first decreasing in risk aversion and wage and then increasing. The result comes from the interaction between the working opportunity option and the endogenous labor supply/portfolio choice with insurance and disutility effects. In an incomplete market setting, the agent consumes less, works more and detains less of the risky financial asset with respect to the complete market setting. He also decides to retire earlier. As far as labor flexibility is concerned, we show that it helps to dampen the effects of parameter changes on agent's choices.

The paper is organized as follows. In Section 2 we introduce the financial market model and the optimal investment-consumption and retirement problem. In Section 3 we address the problem via a duality approach. In Section 4 we introduce the Bellman equation and the free boundary problem associated with the optimal investment-consumption and retirement problem. Section 5 develops the numerical analysis. Section 6 is devoted to the analysis of our problem with incomplete markets. In Section 7 we extend our analysis to the case of a fixed labor supply. Finally, Section 8 concludes.

2 The Optimal Consumption-Investment Problem

We consider a continuous time economy. The infinite horizon life of the agent is divided in two parts: before retirement and after retirement. In the first part of his life the agent consumes, invests in financial markets and chooses the labor supply rate earning an exogenous stochastic income, after retirement he does not work, he only consumes and invests his wealth in financial markets. After retirement the agent cannot go back to work.

There are $N + 1$ assets: the risk-free asset with a constant instantaneous interest rate r and N risky assets. Asset prices $\mathbf{S}(t)$ evolve as

$$dS_n(t) = S_n(t) \left(b_n dt + \sum_{k=1}^N \sigma_{nk} dB_k(t) \right), \quad S_n(0) = S_{n0}, \quad n = 1, \dots, N, \quad (1)$$

where \mathbf{b} is the N dimensional vector of constant drifts of the risky asset prices, $\boldsymbol{\sigma}$ is the $N \times N$ invertible constant volatility matrix; $\mathbf{B}(t)$ is a N -vector of Brownian motions on the probability space (Ω, \mathcal{F}, P) . By $\mathcal{F} = \{\mathcal{F}_t\}_{t=0}^{\infty}$ we denote the augmentation under P of the natural filtration generated by $\mathbf{B}(t)$. In bold face we denote vectors or matrices of constants and of Brownian motions.

The labor income process $Y(t) = Y(\mathbf{S}(t), t)$ (wage) is exogenous and stochastic. Its evolution is fully described by the risk factors affecting asset prices. More specifically, we assume

$$dY(t) = \mu_1(t, Y)dt + \boldsymbol{\mu}_2^\top(t, Y)d\mathbf{B}(t), \quad Y(0) = Y_0 > 0, \quad (2)$$

where μ_1 is a scalar function and $\boldsymbol{\mu}_2$ is a N -vector of functions, such that $Y(t) > 0, \forall t > 0$. Labor income is not an additional source of risk, i.e., it is tradable, and therefore markets are complete. $Y(t)$ is progressively measurable with respect to \mathcal{F} and satisfies the condition $\int_0^t Y(s)ds < \infty, \forall t \geq 0$ a.s..

We denote by $c(t)$, $\boldsymbol{\theta}(t)$ and $l(t)$ consumption, risky asset portfolio and leisure processes, respectively. We make the following assumptions on these processes:

- $c(t)$ is a non negative progressively process measurable with respect to \mathcal{F} , such that $\int_0^t c(s)ds < \infty, \forall t \geq 0$ a.s.,
- $\theta_n(t)$ represents the amount of wealth invested in the n -th risky asset at time $t, n = 1, \dots, N$, it is a progressively adapted process with respect to \mathcal{F} , such that $\int_0^t |\boldsymbol{\theta}(s)|^2 ds < \infty, \forall t \geq 0$ a.s.,
- $l(t) \in [0, 1]$ is the rate of leisure at time $t, l(t)$ is a process measurable with respect to \mathcal{F} .

Let $\tau \geq 0$ be the retirement date chosen by the agent, i.e., $\tau \in \mathcal{S}$, where \mathcal{S} denotes the set of all \mathcal{F} -stopping time. The retirement decision is irreversible: during his working life the agent chooses the leisure rate, after retirement he fully enjoys leisure. At time t the agent is endowed with one unit of “time”, before retirement he allocates his time between labor ($1 - l(t)$) and leisure ($l(t)$) with the constraint that he has to work at least $1 - L$, after retirement he only enjoys leisure:

$$0 \leq l(t) \leq L < 1 \text{ if } 0 \leq t < \tau, \quad l(t) = 1 \text{ if } t \geq \tau.$$

Let $W(0) = W_0$ be the initial wealth. The consumption-investment-leisure strategy $(c, l, \boldsymbol{\theta})$ is admissible if it satisfies the above technical conditions and the wealth process $W(t)$ satisfies the dynamic budget constraint

$$dW(t) = [(1 - l(t))Y(t) - c(t)] dt + \boldsymbol{\theta}^\top(t) (\mathbf{b} dt + \boldsymbol{\sigma} d\mathbf{B}(t)) + \left(W(t) - \sum_{n=1}^N \theta_n(t) \right) r dt \quad (3)$$

where $(1 - l(t))Y(t)dt$ is the labor income received before retirement, i.e., for $0 \leq t < \tau$, $\boldsymbol{\theta}^\top(t) (\mathbf{b} dt + \boldsymbol{\sigma} d\mathbf{B}(t))$ refers to the financial wealth evolution and $\left(W(t) - \sum_{n=1}^N \theta_n(t) \right) r dt$ to the wealth invested in the money market (risk-free bond).

We impose a no borrowing condition

$$W(t) \geq 0, \quad \forall t \geq 0 \quad (4)$$

that is the agent cannot borrow money in the risk-free market against future income.

The agent maximizes the expected utility over the infinite horizon: preferences are time separable with exponential discounting. The instantaneous utility at time t is a function of consumption and leisure, i.e., $u(c(t), l(t))$. Therefore, given the initial wealth W_0 and the labor income process $Y(t)$, we look for an admissible process triplet $(c, l, \boldsymbol{\theta})$ and a stopping time $\tau \in \mathcal{S}$ that maximize

$$E \left[\int_0^{+\infty} e^{-\beta t} u(c(t), l(t)) dt \right], \quad (5)$$

subject to the budget constraint (3) and the no borrowing condition (4). β is the subjective discount rate and the utility function is given by

$$u(c, l) = \frac{1}{\alpha} \frac{(l^{1-\alpha} c^\alpha)^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad 0 < \alpha < 1. \quad (6)$$

The infinite horizon is a strong assumption that allows us to handle analytically the problem; to make the setting more plausible, we may allow for a hazard rate of mortality as in [Dybvig and Liu (2010)] adding a positive coefficient to the discount factor β .

Given the initial wealth W_0 and the labor income Y_0 , the agent maximizes the expected utility (5):

$$\begin{aligned} \mathcal{J}(W_0, Y_0; c, l, \boldsymbol{\theta}, \tau) &:= E \left[\int_0^{+\infty} e^{-\beta t} u(c(t), l(t)) dt \right] \\ &= E \left[\int_0^\tau e^{-\beta t} u(c(t), l(t)) dt + \int_\tau^{+\infty} e^{-\beta t} u(c(t), 1) dt \right] \end{aligned}$$

acting on $(c, l, \boldsymbol{\theta}, \tau)$ with $\tau \in \mathcal{S}$ and subject to the budget constraint (3) and the no borrowing condition (4). The value function is defined as

$$\mathcal{V}(W_0, Y_0) := \sup_{(c, l, \boldsymbol{\theta}, \tau) \in \mathcal{A}} \mathcal{J}(W_0, Y_0; c, l, \boldsymbol{\theta}, \tau), \quad (7)$$

where \mathcal{A} denotes the set of admissible strategies, i.e., processes $(c, l, \boldsymbol{\theta}, \tau)$ that satisfy (3) and (4), $\tau \in \mathcal{S}$, such that $\mathcal{J}(W_0, Y_0; c, l, \boldsymbol{\theta}, \tau) < +\infty$.

3 A dual approach

In this section we solve our problem through a duality approach, see, for instance, [Choi et al. (2008), Dybvig and Liu (2010), Farhi and Panageas (2007), He and Pages (1993)] and [Karatzas and Wang (2000)].

More in details, [He and Pages (1993)] solves via the duality method the optimal consumption-portfolio problem with a stochastic tradable labor income, [Karatzas and Wang (2000)] addresses a mixed optimal stopping/control problem: the agent chooses the portfolio, the consumption rate and the horizon of the problem deciding the final date as a stopping time. [Choi et al. (2008), Dybvig and Liu (2010), Farhi and Panageas (2007)] adapt these techniques to address the optimal consumption-portfolio-leisure problem allowing the agent to choose the retirement date. In this paper we follow closely [Choi et al. (2008), He and Pages (1993)]. For the proofs, we refer to Appendix B.

Let us define the market price of risk $\Theta := \sigma^{-1}(\mathbf{b} - r\mathbf{1})$, where $\mathbf{1}$ is the N -vector $(1, 1, \dots, 1)^\top$, the state-price-density process (or pricing kernel)

$$H(t) := e^{-(r+\frac{1}{2}|\Theta|^2)t - \Theta^\top \mathbf{B}(t)} = e^{-rt} e^{-\Theta^\top \mathbf{B}(t) - \frac{1}{2}|\Theta|^2 t}$$

and, for a given $t > 0$, the equivalent risk neutral martingale measure

$$\bar{P}(A) := E \left[e^{-\Theta^\top \mathbf{B}(t) - \frac{1}{2}|\Theta|^2 t} \mathbf{1}_A \right] \quad \forall A \in \mathcal{F}_t.$$

Under the risk neutral density we can redefine the budget and the no borrowing constraint as static constraints. Following [Choi et al. (2008)], by the optional sampling theorem we have that the budget constraint becomes

$$E \left[\int_0^\tau H(t) (c(t) + Y(t)(l(t) - 1)) dt + H(\tau)W(\tau) \right] \leq W_0 \quad (8)$$

and the no borrowing constraint ($W(t) \geq 0, \forall t, 0 \leq t \leq \tau$) is equivalent to

$$E_t \left[\int_t^\tau \frac{H(s)}{H(t)} (c(s) + Y(s)(l(s) - 1)) ds + \frac{H(\tau)}{H(t)} W(\tau) \right] \geq 0 \quad \forall t : 0 \leq t \leq \tau. \quad (9)$$

Note that the no borrowing constraint is never binding after retirement. The objective function can be rewritten as

$$\begin{aligned} \mathcal{J}(W_0, Y_0; c, l, \theta, \tau) &= E \left[\int_0^\tau e^{-\beta t} u(c(t), l(t)) dt + e^{-\beta \tau} \int_\tau^{+\infty} e^{-\beta(t-\tau)} u(c(t), 1) dt \right] \\ &= E \left[\int_0^\tau e^{-\beta t} u(c(t), l(t)) dt + e^{-\beta \tau} U(W(\tau)) \right], \end{aligned}$$

where

$$U(W(\tau)) := \sup_{(c, \theta)} E \left[\int_\tau^{+\infty} e^{-\beta(t-\tau)} u(c(t), 1) dt \right] \quad (10)$$

is the optimal expected utility attainable at time τ with wealth $W(\tau)$: for $t \geq \tau$ the agent solves the classical optimal consumption-portfolio problem with initial wealth $W(\tau)$ subject to the budget constraint. $U(W(\tau))$ is the indirect utility function associated with the maximization problem at time τ .

Duality between the primal consumption-portfolio problem and the shadow price problem allows us to establish an existence result of the individual intertemporal problem when asset prices follow an Itô process. Then the dual problem can be solved via dynamic programming techniques (Hamilton-Jacobi-Bellman equation) when asset prices follow a Markov diffusion process.

We define the convex conjugate of the utility function $u(c, l)$:

$$\tilde{u}(z, Y) := \max_{\substack{c \geq 0 \\ 0 \leq l \leq L}} u(c, l) - (c + Yl)z. \quad (11)$$

In a similar way, we also define the convex conjugate of U :

$$\tilde{U}(z) := \sup_{w \geq 0} U(w) - wz. \quad (12)$$

If I denotes the inverse of U' , i.e., $I(z) = \inf\{w : U'(w) = z\}$, then we also have $\tilde{U}(z) = U(I(z)) - zI(z)$.

Let \hat{c} and \hat{l} be the pair of processes that provides a solution to (11), then the following proposition holds true.

Proposition 3.1. *Let*

$$\tilde{z} = \left(\frac{\alpha}{1-\alpha} Y \right)^{\alpha(1-\gamma)-1} L^{-\gamma}. \quad (13)$$

If $z \geq \tilde{z}$, then

$$\tilde{u}(z, Y) = \frac{1}{\alpha} \frac{(\hat{l}^{1-\alpha} \hat{c}^\alpha)^{1-\gamma}}{1-\gamma} - (\hat{c} + Y\hat{l})z, \quad (14)$$

where

$$\hat{c} = \frac{\alpha}{1-\alpha} Y \hat{l} \quad \text{and} \quad \hat{l} = z^{-\frac{1}{\gamma}} \left(\frac{\alpha}{1-\alpha} Y \right)^{\frac{\alpha(1-\gamma)-1}{\gamma}}. \quad (15)$$

If $z < \tilde{z}$, then

$$\tilde{u}(z, Y) = \frac{(\hat{c}^\alpha L^{1-\alpha})^{(1-\gamma)}}{\alpha(1-\gamma)} - (\hat{c} + YL)z, \quad (16)$$

where

$$\hat{c} = \left(z L^{(\alpha-1)(1-\gamma)} \right)^{\frac{1}{\alpha(1-\gamma)-1}}. \quad (17)$$

Let $\lambda > 0$ be a Lagrange Multiplier. We consider a non-increasing process $D(t) > 0$ with $D(0) = 1$ to take into account the no bankruptcy constraint, as in [Choi et al. (2008), He and Pages (1993)], i.e., $D(t)$ is the integral of the shadow price of the no bankruptcy constraint. Then, the following proposition holds.

Proposition 3.2. *Let*

$$\begin{aligned} \tilde{V}(\lambda, D, \tau, Y_0) := & E \left[\int_0^\tau e^{-\beta t} (\tilde{u}(\lambda D(t) e^{\beta t} H(t), Y(t)) + \lambda Y(t) D(t) e^{\beta t} H(t)) dt \right. \\ & \left. + e^{-\beta \tau} \tilde{U}(\lambda D(\tau) e^{\beta \tau} H(\tau)) \right], \end{aligned} \quad (18)$$

then

$$\mathcal{J}(W_0, Y_0; c, l, \theta, \tau) \leq \tilde{V}(\lambda, D, \tau, Y_0) + \lambda W_0. \quad (19)$$

For a fixed stopping time $\tau \in \mathcal{S}$ and initial wealth W_0 , we denote by \mathcal{A}_τ the set of admissible processes $(c, l, \boldsymbol{\theta})$. Thus we define

$$V_\tau(W_0, Y_0) := \sup_{(c, l, \boldsymbol{\theta}) \in \mathcal{A}_\tau} \mathcal{J}(W_0, Y_0; c, l, \boldsymbol{\theta}, \tau). \quad (20)$$

From Proposition 3.2, we obtain

$$V_\tau(W_0, Y_0) \leq \inf_{\lambda > 0, D(t) > 0} \left[\tilde{V}(\lambda, D, \tau, Y_0) + \lambda W_0 \right]. \quad (21)$$

and the following result holds true.

Proposition 3.3. *Equality in (21) holds if all the following conditions are satisfied:*

- for $0 \leq t < \tau$

$$\frac{\partial u}{\partial c}(c(t), l(t)) = \lambda D(t) e^{\beta t} H(t) \quad \text{and} \quad \frac{\partial u}{\partial l}(c(t), l(t)) = \lambda Y(t) D(t) e^{\beta t} H(t) \quad (22)$$

if $l(t) \leq L$, otherwise

$$\frac{\partial u}{\partial c}(c(t), L) = \lambda D(t) e^{\beta t} H(t) \quad \text{and} \quad l(t) = L; \quad (23)$$

- it holds

$$\begin{aligned} W(\tau) &= I(\lambda D(\tau) e^{\beta \tau} H(\tau)), \\ E \left[\int_0^\tau H(s)(c(s) + (l(s) - 1)Y(s)) ds + H(\tau)W(\tau) \right] &= W_0; \end{aligned}$$

- for any $t \in [0, \tau)$ such that $D(t)$ is not constant, i.e., $dD(t) \neq 0$,

$$E_t \left[\int_t^\tau \frac{H(s)}{H(t)} (c(s) + (l(s) - 1)Y(s)) ds + \frac{H(\tau)}{H(t)} W(\tau) \right] = 0. \quad (24)$$

Optimality conditions in (22-23) provide a relation between the optimal consumption, the leisure process and the couple $(\lambda, D(t))$. Thus, once the optimal multiplier λ^* and the non-increasing process $D^*(t)$ are computed, we can use these conditions to obtain the optimal processes $c^*(t)$ and $l^*(t)$, as discussed in Appendix A.

In order to compute the optimal couple $(\lambda^*, D^*(t))$, we proceed as follows. Let

$$\tilde{\mathcal{V}}(\lambda, Y_0) := \sup_{\tau \in [0, +\infty]} \inf_{D(t) > 0} \tilde{V}(\lambda, D, \tau, Y_0).$$

Since (7) and (20) imply

$$\mathcal{V}(W_0, Y_0) = \sup_{\tau \in [0, +\infty]} V_\tau(W_0, Y_0),$$

then, under the conditions of Proposition 3.3, it holds

$$\begin{aligned} \mathcal{V}(W_0, Y_0) &= \sup_{\tau \in [0, +\infty]} \inf_{\lambda > 0, D(t) > 0} \left[\tilde{V}(\lambda, D, \tau, Y_0) + \lambda W_0 \right] \\ &= \inf_{\lambda > 0} \left[\tilde{\mathcal{V}}(\lambda, Y_0) + \lambda W_0 \right]. \end{aligned} \quad (25)$$

Let us define the process

$$z(t) = \lambda D(t)e^{\beta t} H(t), \quad z(0) = \lambda \quad (26)$$

and

$$\phi(t, z, y) := \sup_{\tau > t} \inf_{D(t) > 0} E \left[\int_t^\tau e^{-\beta s} \{ \tilde{u}(z(s), Y(s)) + Y(s)z(s) \} ds + e^{-\beta \tau} \tilde{U}(z(\tau)) \right] \Big| z(t) = z, Y(t) = y. \quad (27)$$

Since

$$\phi(0, \lambda, Y_0) = \tilde{V}(\lambda, Y_0) = \sup_{\tau \in [0, +\infty]} \inf_{D(t) > 0} \tilde{V}(\lambda, D, \tau, Y_0), \quad (28)$$

once ϕ is computed, we can use (25) to obtain the value function $\mathcal{V}(W_0, Y_0)$ and thus the optimal couple $(\lambda^*, D^*(t))$ and the optimal strategies, as shown in Appendix A.

4 The Bellman equation and the free boundary problem

To address the optimization problem and to find out $\phi(\cdot)$, we start from the dual function \tilde{U} . After retirement the optimal consumption-investment problem is a classical Merton problem, considering the utility function (6), we have

$$U(W(\tau)) = \sup_{(c, \theta)} E \left[\int_\tau^{+\infty} e^{-\beta(t-\tau)} \frac{c^{\alpha(1-\gamma)}}{\alpha(1-\gamma)} dt \right],$$

subject to the dynamic budget constraint (3) without labor income, i.e., $l(t) = 1 \forall t \geq \tau$.

It follows from standard results (see [Karatzas and Shreve (1998), Chapter 3]) that

$$U(w) = \left(\frac{1}{\xi} \right)^\Gamma \frac{w^{1-\Gamma}}{1-\Gamma}$$

where $\Gamma = 1 - \alpha(1 - \gamma)$ and $\xi = \frac{\Gamma-1}{\Gamma} \left(r + \frac{\Theta^2}{2\Gamma} \right) + \frac{\beta}{\Gamma}$.

In order to guarantee that the function U is well defined, we assume $\xi > 0$. Thus, since U is known analytically, we can also compute \tilde{U} in (12):

$$\tilde{U}(z) = \frac{\Gamma}{\xi(1-\Gamma)} z^{\frac{\Gamma-1}{\Gamma}}.$$

Since by Itô's formula we have

$$\frac{dz(t)}{z(t)} = \frac{dD(t)}{D(t)} + (\beta - r)dt - \Theta^\top(t)d\mathbf{B}(t), \quad z(0) = \lambda,$$

in order to derive the Bellman equation, we follow [He and Pages (1993)] controlling the decreasing process D . Assume that D is absolutely continuous with

respect to t , i.e., there exists $\psi \geq 0$ such that $dD(t) = -D(t)\psi(t)dt$. If ϕ is twice differentiable with respect to z and y , then the Bellman equation becomes

$$\min_{\psi \geq 0} \left\{ e^{-\beta t}(\tilde{u}(z, y) + zy) - \psi(t)z \frac{\partial \phi}{\partial z} + \mathcal{L}_t \phi \right\} = 0, \quad (29)$$

where

$$\mathcal{L}_t \phi = \frac{\partial \phi}{\partial t} + (\beta - r)z \frac{\partial \phi}{\partial z} + \mu_1 \frac{\partial \phi}{\partial y} + \frac{1}{2} \Theta^2 z^2 \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{2} \mu_2^2 \frac{\partial^2 \phi}{\partial y^2} - \Theta^\top \mu_2 z \frac{\partial^2 \phi}{\partial z \partial y}$$

is the differential operator for the dual indirect utility function. Condition (29) can be rewritten compactly as

$$\min \left\{ \mathcal{L}_t \phi + e^{-\beta t}(\tilde{u}(z, y) + zy), -\frac{\partial \phi}{\partial z} \right\} = 0. \quad (30)$$

Equation (30) divides the time-state space of the differential equation in two different regions:

$$\Omega_1 = \left\{ (t, z, y) \text{ s.t. } \frac{\partial \phi}{\partial z} < 0 \right\}, \quad \Omega_2 = \left\{ (t, z, y) \text{ s.t. } \frac{\partial \phi}{\partial z} = 0 \right\}.$$

Let $\hat{z}(t, y) > 0$ be the manifold that separates the space $\{z \geq 0\}$ into two halves Ω_1 and Ω_2 , with Ω_1 being the lower half and Ω_2 the upper one. The Bellman principle suggests that if $(t, z(t), y(t)) \in \Omega_1$ then D must be constant at time t , while D should jump if $(t, z(t), y(t)) \in \Omega_2$ until z reaches the critical boundary $\hat{z}(t, y(t))$. Thus, if $(0, z(0), y(0))$ starts in the no-jump region Ω_1 , then the process D will decrease only at time t such that $(t, z(t), y(t))$ hits the critical boundary, at that time the borrowing constraint is binding and the wealth becomes zero. For further details see [He and Pages (1993)] and Appendix A.

In order to solve our optimal stopping time problem, we need to consider another free boundary associated with the retirement decision. We define \bar{z} as the manifold that divides the state price in the pre-retirement region and the post-retirement region. Thus when the boundary is touched, the retirement is optimal. As a consequence $\phi(t, z, y) = e^{-\beta t} \bar{U}(z)$ in the set

$$\{(t, z, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (\mathbb{R}^+ \setminus \{0\}) : z(t) \leq \bar{z}(t, y)\},$$

i.e., in this region ϕ does not depend on the wage y and it is equal to the dual function associated to the post-retirement utility. Note that $\bar{z} < \hat{z}$, since we are looking for a retirement strategy under the no-borrowing condition, which holds only in Ω_1 . Finally, we recall that the value of the dual function \tilde{u} depends on the fixed manifold \tilde{z} (see Proposition 3.1), which satisfies the condition $\bar{z} \leq \tilde{z}$, since this manifold belongs to the pre-retirement region.

Thus the optimal stopping time, consumption, investment and leisure problem can be rewritten as a free boundary value problem, considering the smooth pasting conditions along the manifold \hat{z} and \bar{z} , as in [Choi et al. (2008)] and [Farhi and Panageas (2007)]: find the free boundaries $\bar{z}(t, y) : \mathbb{R}^+ \times (\mathbb{R}^+ \setminus \{0\}) \rightarrow \mathbb{R}^+$ and $\hat{z}(t, y) : \mathbb{R}^+ \times (\mathbb{R}^+ \setminus \{0\}) \rightarrow \mathbb{R}^+$ and the function

$$\phi \in \mathbf{C}^1(\mathbb{R}^+ \times \mathbb{R}^+ \times (\mathbb{R}^+ \setminus \{0\})) \cap \mathbf{C}^2(\mathbb{R}^+ \times (\mathbb{R}^+ \setminus \{\bar{z}\}) \times (\mathbb{R}^+ \setminus \{0\}))$$

such that

$$\begin{aligned}
\frac{\partial \phi}{\partial z}(t, z, y) &= 0 \quad \text{if } z \geq \widehat{z}(t, y), \quad \frac{\partial \phi}{\partial z}(t, z, y) \leq 0 \quad \text{otherwise,} \\
\mathcal{L}_t \phi(t, z, y) + e^{-\beta t} (\tilde{u}(z, y) + zy) &\geq 0 \quad \text{if } z \geq \widehat{z}(t, y), \\
\mathcal{L}_t \phi(t, z, y) + e^{-\beta t} (\tilde{u}(z, y) + zy) &= 0 \quad \text{if } \bar{z}(t, y) < z \leq \widehat{z}(t, y), \\
\mathcal{L}_t \phi(t, z, y) + e^{-\beta t} (\tilde{u}(z, y) + zy) &\leq 0 \quad \text{if } 0 < z \leq \bar{z}(t, y), \\
\phi(t, z, y) &\geq e^{-\beta t} \tilde{U}(z) \quad \text{if } z > \bar{z}(t, y), \quad \phi(t, z, y) = e^{-\beta t} \tilde{U}(z) \quad \text{otherwise,}
\end{aligned} \tag{31}$$

Theorem 1 (Verification Theorem). *If $(\widehat{z}(t, y), \bar{z}(t, y), \phi(t, y, z))$ is a solution of the variational inequality (31), then $\phi(t, y, z)$ satisfies (27).*

Proof. This result is mainly an application of the Dynkin's formula, see also [Choi et al. (2008)][Theorem 5.1] and [He and Pages (1993)][Theorem 3]. \square

Once the above free boundary value problem is solved, the optimal retirement, consumption, investment and leisure processes can be computed as in Appendix A.

4.1 Solving the free boundary value problem

As in [Choi et al. (2008), Farhi and Panageas (2007)], we can eliminate the time-dependence of ϕ guessing a solution of the form

$$\phi(t, z, y) = e^{-\beta t} \widehat{\phi}(z, y).$$

Thus (30) becomes

$$\min \left\{ \mathcal{L} \widehat{\phi}(z, y) + \tilde{u}(z, y) + zy, -\frac{\partial \widehat{\phi}}{\partial z}(z, y) \right\} = 0, \tag{32}$$

with

$$\mathcal{L} \widehat{\phi} = -\beta \widehat{\phi} + (\beta - r)z \frac{\partial \widehat{\phi}}{\partial z} + \mu_1 \frac{\partial \widehat{\phi}}{\partial y} + \frac{1}{2} \Theta^2 z^2 \frac{\partial^2 \widehat{\phi}}{\partial z^2} + \frac{1}{2} \mu_2^2 \frac{\partial^2 \widehat{\phi}}{\partial y^2} - \Theta^\top \mu_2 z \frac{\partial^2 \widehat{\phi}}{\partial z \partial y}. \tag{33}$$

Notice that this implies that the free boundaries \bar{z} and \widehat{z} do not depend on the time, and, instead of (31), we can solve a time-independent problem: find the free boundaries $\bar{z}(y) : \mathbb{R}^+ \setminus \{0\} \rightarrow \mathbb{R}^+$ and $\widehat{z}(y) : \mathbb{R}^+ \setminus \{0\} \rightarrow \mathbb{R}^+$ and the function $\widehat{\phi} \in \mathbf{C}^1(\mathbb{R}^+ \times (\mathbb{R}^+ \setminus \{0\})) \cap \mathbf{C}^2((\mathbb{R}^+ \setminus \{\bar{z}\}) \times (\mathbb{R}^+ \setminus \{0\}))$ such that

$$\begin{aligned}
\frac{\partial \widehat{\phi}}{\partial z}(z, y) &= 0 \quad \text{if } z \geq \widehat{z}(y), \quad \frac{\partial \widehat{\phi}}{\partial z}(z, y) \leq 0 \quad \text{otherwise,} \\
\mathcal{L} \widehat{\phi}(z, y) + \tilde{u}(z, y) + zy &\geq 0 \quad \text{if } z \geq \widehat{z}(y), \\
\mathcal{L} \widehat{\phi}(z, y) + \tilde{u}(z, y) + zy &= 0 \quad \text{if } \bar{z}(y) < z \leq \widehat{z}(y), \\
\widehat{\phi}(z, y) &\geq \tilde{U}(z) \quad \text{if } z > \bar{z}(y), \quad \widehat{\phi}(z, y) = \tilde{U}(z) \quad \text{otherwise.}
\end{aligned}$$

In the following, we assume that there exists a unique solution $(\widehat{z}(y), \bar{z}(y), \widehat{\phi}(z, y))$, $\bar{z}(y) < z < \widehat{z}(y)$, $y \in \mathbb{R}^+ \setminus \{0\}$, of the above variational problem. Existence and uniqueness can be addressed via viscosity solution methods, see for example

[Ceci and Bassan (2004), Pham (1998), Pham (2009)]; following this approach, one should prove that the value function (27) is a viscosity solution of the variational problem associated with the Hamilton-Jacobi-Bellman problem (31). In what follows we concentrate on agent's behaviour dealing directly with the numerical approximation of the solution.

To solve the PDE, we recall that the free boundary \widehat{z} is defined by

$$\frac{\partial \widehat{\phi}}{\partial z}(\widehat{z}(y), y) = 0 \quad \text{and} \quad \frac{\partial^2 \widehat{\phi}}{\partial z^2}(\widehat{z}(y), y) = 0,$$

and \bar{z} by

$$\widehat{\phi}(\bar{z}(y), y) = \widetilde{U}(\bar{z}(y)) \quad \text{and} \quad \frac{\partial \widehat{\phi}}{\partial z}(\bar{z}(y), y) = \widetilde{U}'(\bar{z}(y)).$$

Thus, considering the regularity of $\widehat{\phi}$, we have to solve the following free-boundary problem

$$\mathcal{L}\widehat{\phi}(z, y) + \widetilde{u}(z, y) + zy = 0 \quad \text{if } \bar{z} < z < \widehat{z}, \quad (34)$$

with the boundary conditions

$$\frac{\partial \widehat{\phi}}{\partial z}(z, y) = 0 \quad \text{and} \quad \frac{\partial \widehat{\phi}}{\partial y}(z, y) = 0 \quad \text{if } z = \widehat{z}(y), \quad (35)$$

$$\frac{\partial \widehat{\phi}}{\partial z}(z, y) = \widetilde{U}'(z) \quad \text{and} \quad \frac{\partial \widehat{\phi}}{\partial y}(z, y) = \widetilde{U}'(z) \frac{\partial \bar{z}}{\partial y} \quad \text{if } z = \bar{z}(y), \quad (36)$$

$$\widehat{\phi}(z, y) = \widetilde{U}(z) \quad \text{if } y \rightarrow 0, \quad (37)$$

$$\frac{\partial \widehat{\phi}}{\partial y}(z, y) = 0 \quad \text{if } y \rightarrow +\infty, \quad (38)$$

and the free boundary conditions (necessary to compute \bar{z} and \widehat{z})

$$\frac{\partial^2 \widehat{\phi}}{\partial z^2}(\widehat{z}(y), y) = 0 \quad \text{and} \quad \widehat{\phi}(\bar{z}(y), y) = \widetilde{U}(\bar{z}(y)). \quad (39)$$

Conditions (35) and (36) are smooth-pasting conditions. Condition (37) comes from the assumption that nobody works for free, i.e., wage equal to 0 implies immediate retirement. Finally, Condition (38) is imposed assuming that when the wage is large enough, then small changes in the wage do not affect the agent's strategy. The two last conditions are necessary for the numerical solution of the variational problem, which we are dealing with in the next section.

5 Numerical analysis

In this section we present the numerical technique and the results of numerical experiments concerning the solution of the free boundary problem assuming a lognormal process for labor income:

$$dY(t) = M_1 Y(t) dt + \mathbf{M}_2^\top Y(t) d\mathbf{B}(t),$$

i.e., $\mu_1 = M_1 Y(t)$ and $\mu_2 = M_2 Y(t)$ in (2). Thus, (33) becomes

$$\mathcal{L}\hat{\phi}(z, y) = -\beta\hat{\phi} + (\beta - r)z \frac{\partial \hat{\phi}}{\partial z} + M_1 y \frac{\partial \hat{\phi}}{\partial y} + \frac{\Theta^2 z^2}{2} \frac{\partial^2 \hat{\phi}}{\partial z^2} + \frac{M_2^2 y^2}{2} \frac{\partial^2 \hat{\phi}}{\partial y^2} - \Theta^\top M_2 z y \frac{\partial^2 \hat{\phi}}{\partial z \partial y}.$$

The code for the numerical analysis is written in MATLAB. All the computations have been performed using MATLAB R2009B. More precisely, we consider a finite element iterative algorithm performing the following steps: given a guess domain,

1. create a triangular mesh using the DISTMESH generator [Person (2004)];
2. solve the PDE (34) with boundary conditions (35)-(38) by a finite element method with piecewise continuous polynomial functions of degree 2;
3. compute new free boundaries using conditions (39), see [Crank (1984), Section 8.2.2] for further details;
4. if the distance between the new free-boundaries and the previous ones does not fall below a constant tolerance, return to step 1.

In our simulations, we consider a mesh with approximately 16,000 elements, and we truncate the domain to $[y_{\min}, y_{\max}]$, with $y_{\min} = 0.1$, $y_{\max} = 8$. The fixed tolerance is 0.001, and the distance between the new free-boundaries (say \bar{z}_N and \hat{z}_N) and the old ones (\bar{z}_O and \hat{z}_O) is computed as

$$\frac{1}{16} \sum_{i=1}^8 |\bar{z}_N(i) - \bar{z}_O(i)| + |\hat{z}_N(i) - \hat{z}_O(i)|,$$

i.e., it is computed in the points with ordinate $y = i$, $i = 1, \dots, y_{\max}$.

Our experiments show that the optimal strategies computed with this numerical procedure depend on the number of elements of the mesh: for example, moving from 16,000 to 20,000 elements could lead to a variation of $\pm 10\%$ of the numerical approximation of the optimal strategies. However with 16,000 elements we obtain numerical solutions which are accurate and stable enough to study the optimal strategies in a reasonable computational time.

As shown in Appendix A, we have to compute the optimal strategy as the derivative of $\hat{\phi}$. To this end, we consider piecewise quadratic finite elements. To obtain the values reported in the following tables and figures, we compute the optimal initial consumption, leisure rate, portfolio and wealth to retire threshold for initial wage $Y_0 = \{0.5, 1, 1.5, \dots, 5.5, 6\}$ and for $\gamma = \{2.5, 3, 3.5, 4, \dots, 9, 9.5, 10\}$. If necessary, we address approximation errors by smoothing the numerical approximation of these strategies using a spline interpolation technique.

We deal with the following set of parameters: risk-free rate $r = 0.02$, a single risky asset with drift $b = 0.07$ and volatility $\sigma = 0.15$, the discount factor is $\beta = 0.015$, the weight determining utility of consumption and leisure is $\alpha = 0.6$. The labor income is assumed to follow a lognormal process

$$dY(t) = 0.05Y(t)dt + M_2 Y(t)dB(t), \quad Y(0) = Y_0,$$

with $M_1 = 0.05$. Moreover we set $S_0 = 1$ and $L = 0.8$, i.e., during the working life the agent has to work at least $1/5$ of the day. Risk aversion γ and initial

wage Y_0 vary between 3 and 10 and between 1 and 6, respectively. We are in a complete market setting, i.e., the Brownian motion driving the wage is the one of the asset. We consider the case of perfect positive or negative correlation with volatility of wage M_2 equal to -0.1 , -0.05 , 0.05 and 0.1 . Thus, we assume that the financial asset is riskier than labor income with a higher premium: the drift and the volatility of the asset price are smaller than those of the asset price. In the first two cases, correlation is negative and therefore labor provides insurance to risky asset holding reducing the variance of wealth, in the others labor and the risky asset are the same asset with different volatility (labor is safer than the security). Note that the agent cannot sell short leisure.

The analysis is organized as follows. First, we discuss how consumption, leisure, portfolio, retirement threshold vary with the risk aversion coefficient and with the initial wage Y_0 , then we discuss the role of correlation and of volatility.

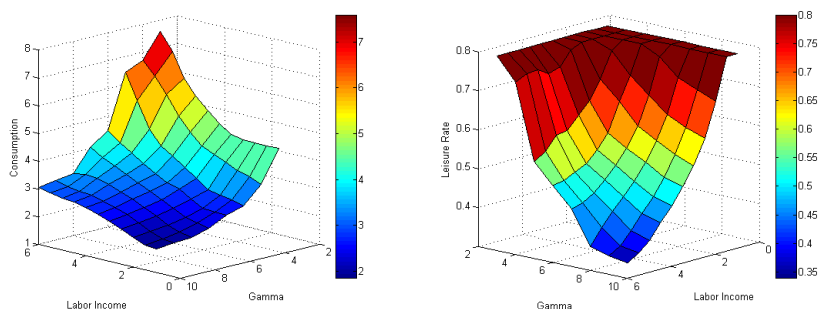


Figure 1: Optimal Consumption and Leisure Rate. $M_2 = 0.1$.

We compute numerically the initial consumption, initial leisure rate, the initial portfolio and the wealth to retire threshold. Initial wealth is $W_0 = 1$. In Figure 1 we plot optimal consumption and optimal leisure for $M_2 = 0.1$. Numerical experiments show that the consumption and the leisure rate decrease as risk aversion goes up. Instead, as the initial wage Y_0 goes up we observe that the consumption increases and that the leisure rate decreases. This pattern is confirmed for all the values of M_2 considered in our analysis (for positive and negative perfect correlation).

These numerical experiments confirm some theoretical results established in different settings and provide a theoretical explanation to some empirical evidence. The negative relation between consumption and risk aversion coefficient confirms the theoretical prediction contained in the literature mainly with a fixed labor supply, e.g. see [Deaton (1992)]: as the relative risk aversion coefficient goes up we have that the saving rate increases - the agent chooses a smoother consumption rate - and the actual consumption rate decreases. As far as the reaction to the wage rate is concerned, we have that consumption is increasing in this variable. Actually an increase of the initial wage Y_0 represents an increase of the expected wage for the future and therefore is permanent. Our analysis confirms that the agent consumes more when he faces a non temporary increase in wage. This effect is associated with the labor supply of the agent: as the wage goes up the agent decides to work more, a result that has been observed empirically in many papers, e.g. see [Abowd and Card (1989),

Domeij and Floden (2006)]. Labor supply is also increasing in risk aversion as observed in [Tallarini (2000)] and in [Marcet et al. (2007)] with incomplete markets: labor activity has a low volatility, provides (perfect) insurance to the financial asset and therefore a more risk averse agent works more. This result holds true both with negative and positive correlation between wage and risky asset. Finally, confirming classical results, see [Henderson (2005), Merton (1969), Merton (1971)], the optimal portfolio decreases as the risk aversion coefficient goes up, see Figure 2.

The rationale for these effects is quite simple. A more risk averse agent prefers a smother consumption path and aims to reduce volatility: he consumes less today, he invests less in the risky asset and more in the risk-free asset, he works more to insure his financial investment because wage is safer than the financial asset ($|M_2| < \sigma$). This interpretation holds true independently of the correlation between wage and financial asset.

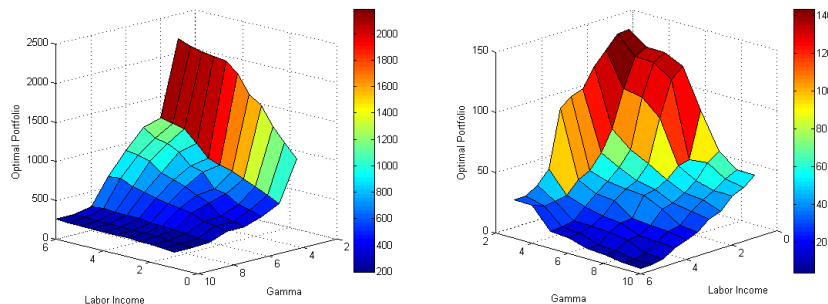


Figure 2: Optimal Portfolio. $M_2 = -0.1$ (left) and $M_2 = 0.1$ (right).

In Figure 2 we show the optimal portfolio for $M_2 = \pm 0.1$. As observed above, the optimal portfolio strategy is decreasing with respect to γ in both cases. As Y_0 goes up, the behavior of the optimal portfolio depends on the correlation: decreasing in case of perfect positive correlation, increasing otherwise. A decreasing path is also observed for $M_2 = 0.05$, while for $M_2 = -0.05$, in some cases the optimal portfolio is first increasing and then decreasing, see Table 2. The rationale is that with negative correlation the financial asset provides insurance to labor income, this is not the case when correlation is positive. As discussed above, the agent works more when the wage goes up, as a consequence to reduce the wealth volatility he holds more of the financial asset in case of negative correlation and less in case of positive correlation.

In Figure 3, we report the wealth to retire threshold for $M_2 = 0.1$. To fully understand the optimal retirement decision we have to evaluate the working life option (expected wage): leaving aside the insurance effect on the risky portfolio, i.e., wage and asset price are correlated, we have that the value of the option increases in Y_0 , M_1 and decreases in M_2 . Numerical experiments show that the wealth to retire is first decreasing and then increasing in the initial wage Y_0 , see also Tables 1 and 2. As far as risk aversion is concerned, for a low level of initial wage the threshold is decreasing in γ , for a high enough initial wage the behavior is first decreasing and then increasing in γ , see Figure 3. The result on risk aversion is different from the one obtained in [Choi et al. (2008), Farhi and Panageas (2007)] for a constant wage allowing the agent to borrow

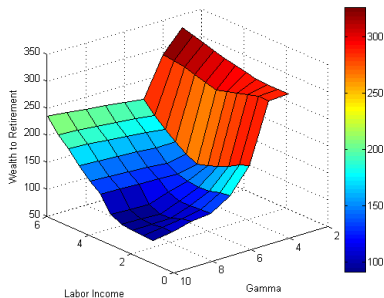


Figure 3: Wealth to Retire. $M_2 = 0.1$.

against the value of future income: they show that the threshold is increasing in the coefficient of relative risk aversion. The rationale is that the non retirement condition offers an insurance opportunity to the agent (working life option) and therefore a strongly risk averse agent abandons the working life only when he reaches a high enough wealth.

Let us consider the wealth to retire shape as risk aversion changes in our setting. The argument described above in favor of higher threshold as risk aversion goes up also works when the wage is stochastic but there are other mechanisms at work. As risk aversion goes up, the agent works more and detains a smaller portfolio of the risky asset, these choices induce disutility of labor and a smaller insurance need. These effects lead the agent to retire early (when a small level of wealth is reached). Note that the value of the working option is high when Y_0 is large. Our analysis suggests that for a small Y_0 this second argument prevails and the threshold is decreasing in risk aversion, for a large enough Y_0 the first effect prevails for a high risk aversion.

As far as an initial wage increase is concerned, the agent works more and this leads to a higher disutility and to the decision to retire earlier with a smaller wealth, see [Matsuyama (2008)] for a similar effect. On the other hand a higher Y_0 renders more valuable the working option inducing an increasing wealth to retire threshold. Figure 3 shows that the first effect prevails for a very low risk aversion (the working option has a little value), for a high enough risk aversion the second becomes predominant, and the U-shaped pattern becomes less evident.

It is interesting to compare the agent's choices as the volatility of the wage and the correlation of the wage with the risky asset price change. In Tables 1 and 2 we report the optimal consumption, leisure rate, portfolio and wealth to retire threshold as M_2 goes from -0.1 to 0.1 for different values of Y_0 with $\gamma = 3$ and $\gamma = 6$, respectively. Note that we have both a variance effect (riskiness of labor decreases going from -0.1 to -0.05 and increases going from 0.05 to 0.1) and a correlation effect.

We observe that as M_2 goes up consumption increases, except from $M_2 = -0.05$ to $M_2 = 0.05$ for a low level of wage. The optimal portfolio is always decreasing and the leisure rate is increasing. The wealth to retire instead is hump shaped with the maximum in $M_2 = -0.05$. Similar results hold true for other values of γ .

These results are puzzling and deserve a careful analysis. As M_2 goes from

Table 1: Optimal Strategies. $\gamma = 3$.

Y_0	1.5	2.5	3.5	4.5	1.5	2.5	3.5	4.5
M_2	Consumption				Leisure Rate			
-0.1	2.5844	2.6888	2.9409	3.1447	0.8000	0.7170	0.5602	0.4659
-0.05	4.7479	4.7496	4.7621	4.9143	0.8000	0.8000	0.8000	0.7280
0.05	4.0666	4.2060	4.4300	5.2838	0.8000	0.8000	0.8000	0.7828
0.1	4.3712	4.4020	4.7631	5.6438	0.8000	0.8000	0.8000	0.8000
M_2	Portfolio				Wealth to Retire			
-0.1	1173.5	1577.9	1920.8	2067.0	397.24	394.43	395.12	397.21
-0.05	826.96	919.21	994.05	1055.4	474.51	426.45	457.56	463.74
0.05	122.42	99.48	92.36	90.31	354.20	341.44	332.16	340.73
0.1	111.45	82.36	77.63	56.21	288.45	290.34	298.18	308.44

Table 2: Optimal Strategies. $\gamma = 6$.

Y_0	1.5	2.5	3.5	4.5	1.5	2.5	3.5	4.5
M_2	Consumption				Leisure Rate			
-0.1	2.3209	2.5276	2.8262	3.0419	0.8000	0.6740	0.5383	0.4506
-0.05	2.4638	2.6303	2.9420	3.1233	0.8000	0.7014	0.5604	0.4627
0.05	2.1992	2.6676	3.0305	3.4362	0.8000	0.7114	0.5772	0.5091
0.1	2.4467	2.7232	3.1173	3.5770	0.8000	0.7262	0.5938	0.5299
M_2	Portfolio				Wealth to Retire			
-0.1	344.73	504.53	630.48	735.63	162.79	145.77	134.17	162.33
-0.05	245.47	258.44	261.02	118.14	184.30	169.58	158.17	169.39
0.05	135.15	85.76	45.91	21.47	160.91	140.83	132.81	161.82
0.1	124.76	40.82	36.08	19.07	127.26	119.84	129.51	160.65

−0.05 to 0.1 insurance opportunities shrink because labor stops to provide insurance to the financial asset and becomes riskier. As a consequence, the agent works less and invests less in the financial asset. This result is confirmed isolating the correlation effect, i.e., comparing the case $M_2 = -0.1$ to $M_2 = 0.1$ and $M_2 = -0.05$ to $M_2 = 0.05$. On this phenomenon see also [Bodie et al. (1992), Dybvig and Liu (2010), Guiso et al. (1996), Heaton and Lucas (2000a)] and [Henderson (2005), Viceira (2001)]. The rationale is simple; passing from a perfect negative correlation to a perfect positive correlation with an increasing volatility, labor doesn’t provide anymore insurance to financial risk and becomes riskier, therefore the agent decides to invest less in the risky asset. Also the labor supply decreases as the risky investment is smaller. The variance argument induces a different effect in case of perfect negative correlation: going from $M_2 = -0.05$ to $M_2 = -0.1$ we have that labor income is more risky and the agent reacts taking more risk both in the financial market and in the labor market, on this effect see [Floden (2006), Parker et al. (2005)]. The rationale is that in case of negative correlation as wage volatility goes up the agent detains a larger portfolio of the financial asset in a significant way and therefore he has to work more to “insure” his investment.

The behavior of consumption is more intriguing. It seems that when correlation is negative the agent reacts to a smaller wage volatility consuming more, instead when correlation is positive the agent reacts to a higher volatil-

ity consuming more. Note that as correlation goes from negative to positive, consumption goes up when risk aversion is high enough. It seems that when financial and insurance opportunities become the same and riskier the agent reacts consuming more today.

Going from the negative correlation to the positive correlation case (from $M_2 = -0.1$ to $M_2 = 0.1$, as well as from $M_2 = -0.05$ to $M_2 = 0.05$), the wealth necessary to retire decreases: the volatility of the wage ($|M_2|$) is the same, but in the negatively correlated case the agent chooses to retire only with a high value of wealth because the working opportunity has a great value as it reduces financial risk. The threshold is hump shaped with a maximum in $M_2 = -0.05$, actually for this parameter (small volatility and perfect negative correlation with the risky asset) the working life option has the highest value and therefore the agent waits to reach a high level of wealth before retirement.

6 Incomplete Markets

In this section we deal with our problem assuming incomplete markets, i.e., considering the single risky asset case, the wage evolves as

$$dY(t) = \mu_1(t, Y(t))dt + \mu_2(t, Y(t))d\tilde{B}(t).$$

We denote by ρ the correlation between $\tilde{B}(t)$ and the Brownian motion $B(t)$ introduced in (1). We stress that $\rho = 1$ (-1) corresponds to the perfectly positive (negative) correlated setting. In this case, applying the Ito's Lemma as in Section 4, we obtain

$$\begin{aligned} \mathcal{L}_t \phi &= \frac{\partial \phi}{\partial t} + (\beta - r)z \frac{\partial \phi}{\partial z} + \mu_1 \frac{\partial \phi}{\partial y} + \frac{1}{2} \Theta^2 z^2 \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{2} \mu_2^2 \frac{\partial^2 \phi}{\partial y^2} - \rho \Theta \mu_2 z \frac{\partial^2 \phi}{\partial z \partial y} \\ &+ \frac{1}{2} \sqrt{1 - \rho^2} \mu_2 z \frac{\partial^2 \phi}{\partial z \partial y} - (1 - \rho^2) \mu_2^2 \frac{\left(\frac{\partial^2 \phi}{\partial z \partial y} \right)^2}{\frac{\partial^2 \phi}{\partial z^2}}, \end{aligned}$$

and thus (31) becomes

$$\begin{aligned} \mathcal{L} \hat{\phi} &= -\beta \hat{\phi} + (\beta - r)z \frac{\partial \hat{\phi}}{\partial z} + \mu_1 \frac{\partial \hat{\phi}}{\partial y} + \frac{1}{2} \Theta^2 z^2 \frac{\partial^2 \hat{\phi}}{\partial z^2} + \frac{1}{2} \mu_2^2 \frac{\partial^2 \hat{\phi}}{\partial y^2} - \rho \Theta \mu_2 z \frac{\partial^2 \hat{\phi}}{\partial z \partial y} \\ &+ \frac{1}{2} \sqrt{1 - \rho^2} \mu_2 z \frac{\partial^2 \hat{\phi}}{\partial z \partial y} - (1 - \rho^2) \mu_2^2 \frac{\left(\frac{\partial^2 \hat{\phi}}{\partial z \partial y} \right)^2}{\frac{\partial^2 \hat{\phi}}{\partial z^2}}. \end{aligned}$$

In Table 3 we assume again a lognormal process for the labor income, considering the same parameters as in Section 5, and we report the optimal consumption, leisure rate, portfolio and wealth to retire for $\gamma = 7$ and $M_2 = 0.1$.

Comparing row $\rho = -0.5$ to row $\rho = -1$ and row $\rho = 0.5$ to row $\rho = 1$ in Table 3, we observe that in an incomplete market setting the agent consumes less, works more and detains less of the risky financial asset with respect to the complete market setting. This observation holds true independently of the sign of the correlation between the risky asset and the wage. The result on consumption confirms what has been observed in [Aiyagari (1994), Huggett (1997)],

Table 3: Optimal Strategies. $\gamma = 7$ and $M_2 = 0.1$.

Y_0	1.5	2.5	3.5	4.5	1.5	2.5	3.5	4.5
ρ	Consumption				Leisure Rate			
-1	2.2632	2.4864	2.7903	3.0345	0.8000	0.6630	0.5315	0.4501
-0.5	2.2630	2.4762	2.7070	2.9633	0.8000	0.6603	0.5156	0.4390
0.5	1.8854	2.3173	2.5164	2.7123	0.8000	0.6176	0.5608	0.4952
1	2.2131	2.5681	2.9688	3.3744	0.8000	0.6848	0.5655	0.4999
ρ	Portfolio				Wealth to Retire			
-1	336.53	423.08	474.69	546.59	143.11	132.60	132.86	163.62
-0.5	217.63	228.51	198.69	188.23	133.20	116.70	132.22	163.36
0.5	43.83	10.73	8.56	4.52	102.55	101.56	121.97	150.81
1	55.18	31.54	29.19	18.02	109.30	107.11	131.55	163.46

Marcet et al. (2007)]. As far as the optimal portfolio is concerned our result confirms the analysis contained in [Heaton and Lucas (2000), Henderson (2005)]. For the same reason also the wealth to retire threshold is smaller in an incomplete than in a complete market setting: as the labor market doesn't provide perfect insurance to the risky asset, the agent decides to retire earlier.

We cannot interpret our results only along the market completeness/incompleteness dimension, we have also to consider the volatility of labor income per se. From the above Table we observe that all the variables have a U -shaped pattern: the minimum for consumption, leisure, risky asset portfolio and wealth to retire threshold is obtained in the incomplete market case with positive correlation ($\rho = 0.5$), that is the worst case from an insurance point of view.

7 Fixed vs. flexible labor supply

In this section we assume a fixed leisure rate L_F , i.e., $l(t) = L_F$ before retirement and $l(t) = 1$ after. The convex conjugate of the utility function (11) becomes

$$\tilde{u}(z, Y) = \max_{c \geq 0} u(c, L_F) - (c + YL_F)z,$$

and Proposition 3.1 is replaced by

$$\tilde{u}(z, Y) = \frac{(\hat{c}^\alpha L_F^{1-\alpha})^{(1-\gamma)}}{\alpha(1-\gamma)} - (\hat{c} + YL_F)z,$$

where

$$\hat{c} = \left(zL_F^{(\alpha-1)(1-\gamma)} \right)^{\frac{1}{\alpha(1-\gamma)-1}}.$$

Thus, once the free boundary value problem is solved considering the above convex conjugate utility function in the r.h.s. of the partial differential equation (33), the optimal consumption strategy is computed as in Appendix A, i.e.,

- for $0 \leq t < \tau^*$

$$c^*(t) = \left(z^*(t)L_F^{(\alpha-1)(1-\gamma)} \right)^{\frac{1}{\alpha(1-\gamma)-1}} \quad \text{and} \quad l^*(t) = L_F;$$

- for $t \geq \tau^*$

$$c^*(t) = z^*(t)^{\frac{1}{\alpha(1-\gamma)-1}} \quad \text{and} \quad l^*(t) = 1.$$

In Tables 4 and 5 we report the optimal consumption and portfolio for different values of γ and of the initial wage Y_0 , considering the perfect positively correlated case, i.e., $M_2 = 0.05$ and $M_2 = 0.1$, and a fixed leisure rate equal to $L_F = 0.6$.

Table 4: Optimal Strategies. $M_2 = 0.05$.

Y_0	1.5	2.5	3.5	4.5	1.5	2.5	3.5	4.5
γ	Consumption				Portfolio			
3	3.9721	4.0909	4.1648	4.4028	317.47	178.12	153.21	66.67
4	3.2700	3.3230	3.4202	3.6209	199.41	142.91	54.20	30.19
5	3.1002	3.1713	3.2685	3.4471	174.42	126.99	34.93	23.00
6	2.7773	2.8228	2.9085	3.0529	162.09	92.45	25.39	16.14
7	2.5699	2.5990	2.6690	2.8003	153.66	70.90	23.19	12.25

Table 5: Optimal Strategies. $M_2 = 0.1$.

Y_0	1.5	2.5	3.5	4.5	1.5	2.5	3.5	4.5
γ	Consumption				Portfolio			
3	4.9159	5.1020	5.4048	6.7869	315.73	127.32	83.45	28.22
4	3.8356	3.9562	4.1193	4.8718	230.13	81.99	52.91	18.56
5	3.2193	3.3115	3.4083	3.8612	172.32	59.96	38.71	16.14
6	2.6304	2.8080	2.9642	3.3082	158.48	51.10	33.42	15.54
7	2.6003	2.6644	2.7094	2.9094	114.68	48.89	25.79	14.61

We observe that the behavior of the optimal strategies as the parameters change is the same as in the case in which the agent is free to choose his leisure rate, i.e., both consumption and portfolio are decreasing in the risk aversion coefficient and increasing in the wage rate.

Comparing the case in which the agent is free to choose the leisure rate with the case in which the leisure rate is fixed, i.e., comparing Tables 1-3 with Tables 4-5, we observe that, as the risk aversion decreases or the wage rate increases, the increase in consumption with a fixed labor supply is smaller than that observed with a flexible labor rate. This result confirms that labor flexibility provides insurance and allows the agent to consume more.

It is difficult to compare the consumption rate in the two settings because the labor supply is different. However, there is some evidence that flexibility leads to a higher consumption rate and a higher investment in the risky asset, specially when the flexible leisure rate is smaller than L_F , confirming results provided in [Floden (2006), Low (2002), Benitez-Silva (2003)].

8 Conclusions

In this article we have presented an analysis of optimal consumption, labor supply and portfolio when the agent is free to choose when to retire. The wage

is assumed to be stochastic, and we have considered the case of perfect (and non perfect) positive and negative correlation between the wage and the risky asset. In the lognormal case, we have provided a detailed analysis of the agent's strategies, considering different values of the risk aversion parameter, as well as of the volatility of the wage.

We confirm several results already shown in a simpler setting and we provide some new insights. We confirm classical results on wage, risk aversion and consumption, labor supply, portfolio. We shed light on the complexity of the retirement decision that is not only driven by insurance motivations (working opportunity option) but also by portfolio and labor supply decisions. We show that the agent may react to a shrink in insurance opportunities in a myopic way consuming more. Finally we show that benefits of market completeness and of labor flexibility are relevant also with a stochastic wage and flexible retirement date.

The framework is rich enough to address several interesting issues. Future research includes optimal choices assuming a life annuity and social security design.

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A Optimal Retirement, Consumption, Investment and Leisure Processes

Once $\phi(t, z, y)$ and the free boundaries $\hat{z}(y)$ and $\bar{z}(y)$ are computed, from (28) we obtain

$$\tilde{V}(\lambda, Y_0) = \phi(0, \lambda, Y_0)$$

and the value function is given by (25)

$$\begin{aligned}\mathcal{V}(W_0, Y_0) &= \inf_{\lambda > 0} \left[\tilde{V}(\lambda, Y_0) + \lambda W_0 \right] \\ &= \tilde{V}(\lambda^*, Y_0) + \lambda^* W_0 = \phi(0, \lambda^*, Y_0) + \lambda^* W_0.\end{aligned}\quad (40)$$

From (18) and (25) we also obtain

$$\begin{aligned}\mathcal{V}(W_0, Y_0) &= \sup_{\tau \in [0, +\infty]} \inf_{D(t) > 0} \left[\tilde{V}(\lambda^*, D, \tau, Y_0) \right] + \lambda^* W_0 \\ &= \tilde{V}(\lambda^*, D^*, \tau^*, Y_0) + \lambda^* W_0,\end{aligned}$$

where we denote with τ^* the optimal retirement time. As in [He and Pages (1993)] and [Farhi and Panageas (2007)], we obtain the following characterization for the process D^* (and thus z^*) and for the optimal stopping time τ^* .

Let us define $\hat{z}^*(t) = \hat{z}(Y(t))$ and $\bar{z}^*(t) = \bar{z}(Y(t))$; D^* decreases only when z^* touch the barrier \hat{z}^* , otherwise it remains constant; thus, since $D^*(0) = 1$, it holds

$$D^*(t) = \min \left(1, \inf_{0 \leq s \leq t} \frac{\hat{z}^*(s)}{\lambda^* e^{\beta s} H(s)} \right),$$

and, from (26), we can define the optimal process

$$z^*(t) = \lambda^* D^*(t) e^{\beta t} H(t), \quad z(0) = \lambda^*.$$

Moreover, Theorem 1 implies

$$\tau^* = \inf \{ t \geq 0 \text{ such that } z^*(t) = \bar{z}^*(t) \}.$$

Considering (22-23), the optimal strategies of consumption (c^*) and leisure (l^*) are given by Proposition 3.1: for $0 \leq t < \tau^*$, if $z^* \geq \bar{z}$

$$c^*(t) = \left(\frac{\alpha}{1-\alpha} Y(t) \right)^{\frac{(\alpha-1)(1-\gamma)}{\gamma}} z^*(t)^{-\frac{1}{\gamma}} \quad \text{and} \quad l^*(t) = \left(\frac{\alpha}{1-\alpha} Y(t) \right)^{\frac{(\alpha-1)(1-\gamma)}{\gamma} - 1} z^*(t)^{-\frac{1}{\gamma}},$$

otherwise

$$c^*(t) = \left(z^*(t) L^{(\alpha-1)(1-\gamma)} \right)^{\frac{1}{\alpha(1-\gamma)-1}} \quad \text{and} \quad l^*(t) = L.$$

For $t \geq \tau^*$ it holds

$$\frac{\partial u}{\partial c}(c^*(t), 1) = z^*(t) \quad \text{and} \quad l^*(t) = 1,$$

and thus

$$c^*(t) = z^*(t)^{\frac{1}{\alpha(1-\gamma)-1}} \quad \text{and} \quad l^*(t) = 1.$$

Moreover, considering (12), the wealth at retirement satisfies

$$\tilde{U}(z^*(\tau^*)) = U(W^*(\tau^*)) - W^*(\tau^*) z^*(\tau^*)$$

and thus

$$W^*(\tau^*) = -\tilde{U}'(z^*(\tau^*)).$$

Moreover, due to (22-23), we also have

$$W^*(\tau^*) = I(z^*(\tau^*)).$$

In order to compute the optimal portfolio strategy, we first of all need to define the optimal wealth. Before retirement, i.e., for $0 < t < \tau^*$, it holds

$$W^*(t) = -\frac{\partial}{\partial z}\phi(0, z^*(t), Y(t)); \quad (41)$$

in fact, since for any t such that $0 < t < \tau^*$ (25) implies

$$\begin{aligned} \mathcal{V}(W^*(t), Y(t)) &= \inf_{\lambda > 0} \left[\tilde{V}(\lambda, Y(t)) + \lambda W^*(t) \right] \\ &= \tilde{V}(z^*(t), Y(t)) + z^*(t)W^*(t) = \phi(0, z^*(t), Y(t)) + z^*(t)W^*(t), \end{aligned} \quad (42)$$

and thus, differentiating (42) with respect to z , we obtain condition (41).

Remark 2. Since $z^* \in \Omega_1$, then $\frac{\partial \phi}{\partial z} < 0$, i.e., Condition (41) implies a positive wealth according to the no-borrowing constraint.

Once the optimal wealth is computed, given $W(0)$ and condition (3), the optimal portfolio θ^* is such that

$$\begin{aligned} &\int_0^t \theta^{*\top}(s) (\mathbf{b} ds + \sigma d\mathbf{B}(s)) + \int_0^t \left(W^*(s) - \sum_{n=1}^N \theta_n^*(s) \right) r ds \\ &= W^*(t) - W(0) - \int_0^t [(1 - l^*(s))Y(s) - c^*(s)] ds. \end{aligned}$$

We can split the optimal portfolio strategy before retirement in two components, θ_M^* and θ_H^* : the first component is the classical Merton investment strategy, which does not depend on the stochastic labor income, θ_H^* can be interpreted as intertemporal hedging demand and is hedging the inflow of stochastic labor income. Thus

$$\begin{aligned} \theta_t^* &= \theta_M^* + \theta_H^* \\ &= -\sigma^{-1} \Theta \frac{\partial_W \mathcal{V}(W(t), Y(t))}{\partial_{WW} \mathcal{V}(W(t), Y(t))} - \sigma^{-1} \mu_2(Y(t), t) \frac{\partial_{WY} \mathcal{V}(W(t), Y(t))}{\partial_{WW} \mathcal{V}(W(t), Y(t))} \\ &= -\sigma^{-1} \Theta \frac{z^*(t)}{\partial_W z^*(t)} - \sigma^{-1} \mu_2(Y(t), t) \frac{\partial_Y z^*(t)}{\partial_W z^*(t)} \end{aligned}$$

because of (42).

The optimal portfolio and wealth after retirement, i.e., $t \geq \tau^*$, are computed according to a classical Merton problem (see Section 4), i.e.,

$$\theta^*(t) = -\sigma^{-1} \Theta \frac{U'(W^*(t))}{U''(W^*(t))},$$

and $W^*(t)$ is equal to

$$W^*(\tau^*) - \int_{\tau^*}^t c^*(s) ds + \int_{\tau^*}^t \theta^{*\top}(s) (\mathbf{b} ds + \sigma d\mathbf{B}(s)) + \int_{\tau^*}^t \left(W^*(s) - \sum_{n=1}^N \theta_n^*(s) \right) r ds,$$

see [Choi et al. (2008)] and [Karatzas and Shreve (1998), Section 3] for further details.

B Proofs

B.1 Proposition 3.1

Proof. Necessary and sufficient conditions for the unconstrained optimal problem (11) are

$$\frac{\partial}{\partial c} u(c, l)|_{(c, l) = (\widehat{c}, \widehat{l})} = z, \quad \text{i.e.,} \quad \widehat{l}(t)^{(1-\gamma)(1-\alpha)-1} \widehat{c}(t)^{\alpha(1-\gamma)-1} = z(t), \quad (43)$$

and

$$\frac{\partial}{\partial l} u(c, l)|_{(c, l) = (\widehat{c}, \widehat{l})} = zY, \quad \text{i.e.,} \quad \widehat{l}(t)^{(1-\gamma)(1-\alpha)-1} \widehat{c}(t)^{\alpha(1-\gamma)} = \frac{\alpha}{1-\alpha} z(t)Y(t). \quad (44)$$

By the hypotheses on the utility function, positivity constraints on c and l are always satisfied, instead we have to take care of the constraint from above on l .

Let \tilde{z} be defined as in (13). It is easy to show that if $z \geq \tilde{z}$, then \widehat{l} and \widehat{c} in (15) are obtained from first order conditions (43)-(44) and are admissible, i.e., $\widehat{c} \geq 0$ and $0 \leq \widehat{l} \leq L$. Moreover, $\tilde{u}(z, Y)$ defined in (14) is the optimal convex conjugate utility. Instead, if $z < \tilde{z}$, then the optimal leisure rate is $\widehat{l} = L$ and condition (43) leads to the optimal consumption (17) and to the optimal convex conjugate utility (16). \square

B.2 Proposition 3.2

Proof. Considering (11) and (12), it holds

$$\begin{aligned} \mathcal{J}(W_0, Y_0; c, l, \theta, \tau) &= E \left[\int_0^\tau e^{-\beta t} \{ u(c(t), l(t)) - \lambda D(t) e^{\beta t} H(t) (c(t) + Y(t)l(t)) \} dt \right. \\ &\quad \left. + e^{-\beta \tau} \{ U(W(\tau)) - \lambda D(\tau) e^{\beta \tau} H(\tau) W(\tau) \} \right] \\ &\quad + \lambda E \left[\int_0^\tau D(t) H(t) (c(t) + Y(t)l(t)) dt + D(\tau) H(\tau) W(\tau) \right] \\ &\leq E \left[\int_0^\tau e^{-\beta t} \tilde{u}(\lambda D(t) e^{\beta t} H(t), Y(t)) dt + e^{-\beta \tau} \tilde{U}(\lambda D(\tau) e^{\beta \tau} H(\tau)) \right] \\ &\quad + \lambda E \left[\int_0^\tau D(t) H(t) (c(t) + Y(t)l(t)) dt + D(\tau) H(\tau) W(\tau) \right]. \end{aligned} \quad (45)$$

Exploiting the constraints (8), (9) and being $D(t)$ non increasing such that $D(0) = 1$ we obtain

$$\begin{aligned} &E \left[\int_0^\tau D(t) H(t) (c(t) + Y(t)l(t)) dt + D(\tau) H(\tau) W(\tau) \right] \\ &= E \left[\int_0^\tau D(t) H(t) (c(t) + (l(t) - 1)Y(t)) dt + D(\tau) H(\tau) W(\tau) + \int_0^\tau Y(t) D(t) H(t) dt \right] \\ &= E \left[\int_0^\tau Y(t) D(t) H(t) dt \right] + E \left[H(\tau) W(\tau) + \int_0^\tau H(t) (c(t) + (l(t) - 1)Y(t)) dt \right] \\ &\quad + E \left[\int_0^\tau H(t) E_t \left[\frac{H(\tau)}{H(t)} W(\tau) + \int_t^\tau \frac{H(s)}{H(t)} (c(s) + (l(s) - 1)Y(s)) ds \right] dD(t) \right] \\ &\leq E \left[\int_0^\tau Y(t) D(t) H(t) dt \right] + W_0. \end{aligned} \quad (46)$$

Thus inequality (19) holds true. \square

B.3 Proposition 3.3

Proof. Equality in (21) holds if and only if equalities in (45) and (46) hold true. Let us start with (45): this equality holds if

$$\begin{aligned} u(c(t), l(t)) - \lambda D(t)e^{\beta t}H(t)(c(t) + Y(t)l(t)) &= \tilde{u}(\lambda D(t)e^{\beta t}H(t), Y(t)) \\ U(W(\tau)) - \lambda D(\tau)e^{\beta \tau}H(\tau)W(\tau) &= \tilde{U}(\lambda D(\tau)e^{\beta \tau}H(\tau)). \end{aligned}$$

By Proposition 3.1, this is equivalent to set for $0 \leq t < \tau$

$$\frac{\partial u}{\partial c}(c(t), l(t)) = \lambda D(t)e^{\beta t}H(t) \quad \text{and} \quad \frac{\partial u}{\partial l}(c(t), l(t)) = \lambda Y(t)D(t)e^{\beta t}H(t)$$

if $l(t) \leq L$, otherwise

$$\frac{\partial u}{\partial c}(c(t), L) = \lambda D(t)e^{\beta t}H(t) \quad \text{and} \quad l(t) = L;$$

and, reasoning as above for the utility function U , to impose

$$U'(W(\tau)) = \lambda D(\tau)e^{\beta \tau}H(\tau),$$

i.e., $W(\tau) = I(\lambda D(\tau)e^{\beta \tau}H(\tau))$.

Now we deal with inequality (46): it becomes an equality if

$$E \left[H(\tau)W(\tau) + \int_0^\tau H(t)(c(t) + (l(t) - 1)Y(t))dt \right] = W_0, \quad (47)$$

$$E \left[\int_0^\tau H(t)E_t \left[\frac{H(\tau)}{H(t)}W(\tau) + \int_t^\tau \frac{H(s)}{H(t)}(c(s) + (l(s) - 1)Y(s))ds \right] dD(t) \right] = 0, \quad (48)$$

and the last condition is equivalent to (24).

Notice that (47) and (48) imply (8) and (9), respectively. See [Choi et al. (2008)] for further details. \square