Information gain in quantum continual measurements

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Abstract

Inspired by works on information transmission through quantum channels, we propose the use of a couple of mutual entropies to quantify the efficiency of continual measurement schemes in extracting information on the measured quantum system. Properties of these measures of information are studied and bounds on them are derived.

1 Quantum measurements and entropies

We speak of quantum continual measurements when a quantum system is taken under observation with continuity in time and the output is not a single random variable, but rather a stochastic process [1, 2]. The aim of this paper is to quantify, by means of entropic quantities, the effectiveness of a continual measurement in extracting information from the underlying quantum system.

Various types of entropies and bounds on informational quantities can be introduced and studied in connection with continual measurements [3–5]. In particular, in Ref. [5] the point of view was the one of information transmission: the quantum system is a channel in which some information is encoded at an initial time; the continual measurement represents the decoding apparatus. In this paper, instead, we consider the quantum system in itself, not as a transmission channel, and we propose and study a couple of mutual entropies giving two indexes of how good is the continual measurement in extracting information about the quantum system.

1.1 Algebras, states, entropies

From now on $\mathcal{H}$ will be a separable complex Hilbert space, the space where our quantum system lives.

1.1.1 Von Neumann algebras and normal states

A normal state on $\mathcal{L}(\mathcal{H})$ (bounded linear operators on $\mathcal{H}$) is identified with a statistical operator, $\mathcal{T}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ are the trace-class and the space of the statistical operators on $\mathcal{H}$, respectively.

Let $(\Omega, \mathcal{F}, Q)$ be a measure space, where Q is a σ-finite measure. We consider the W^*-algebras $L^\infty(\Omega, \mathcal{F}, Q)$ and $L^\infty\big(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H})\big) \simeq L^\infty(\Omega, \mathcal{F}, Q) \otimes \mathcal{L}(\mathcal{H})$. Let us note that a normal state on $L^\infty(\Omega, \mathcal{F}, Q)$ is a probability density with respect to Q, while a normal state σ on $L^\infty\big(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H})\big)$ is a measurable function $\omega \mapsto \sigma(\omega) \in \mathcal{T}(\mathcal{H})$, $\sigma(\omega) \geq 0$, such that $\operatorname{Tr}\{\sigma(\omega)\}$ is a probability density with respect to Q.

1.1.2 Relative entropy

The general definition of the relative entropy $S(\Sigma|\Pi)$ for two states Σ and Π is given in [6]; here we give only some particular cases of the general definition.

Let us consider two quantum states σ, $\tau \in \mathcal{S}(\mathcal{H})$ and two classical states q_k on $L^\infty(\Omega, \mathcal{F}, Q)$ (two probability densities with respect to Q). The von Neumann entropy, the quantum relative entropy and the classical one are

$$S_{\mathrm{q}}(\tau) := -\operatorname{Tr}\{\tau \ln \tau\}, \qquad S_{\mathrm{q}}(\sigma\|\tau) = \operatorname{Tr}\{\sigma(\ln\sigma - \ln\tau)\}, \tag{1}$$

$$S_{\mathrm{c}}(q_1\|q_2) = \int_\Omega Q(\mathrm{d}\omega)\, q_1(\omega) \ln \frac{q_1(\omega)}{q_2(\omega)}\,. \tag{2}$$

Let us consider now two normal states σ_k on $L^\infty\big(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H})\big)$ and set $q_k(\omega) := \operatorname{Tr}\{\sigma_k(\omega)\}$, $\varrho_k(\omega) := \sigma_k(\omega)/q_k(\omega)$ (these definitions hold where the denominators do not vanish and are completed arbitrarily where the denominators vanish). Then, the relative entropy is

$$\begin{aligned} S(\sigma_1\|\sigma_2) &= \int_\Omega Q(\mathrm{d}\omega) \operatorname{Tr}\left\{\sigma_1(\omega)\big(\ln\sigma_1(\omega) - \ln\sigma_2(\omega)\big)\right\} \\ &= S_{\mathrm{c}}(q_1\|q_2) + \int_\Omega Q(\mathrm{d}\omega)\, q_1(\omega) S_{\mathrm{q}}\big(\varrho_1(\omega)\|\varrho_2(\omega)\big). \end{aligned} \tag{3}$$

We are using a subscript "c" for classical entropies, a subscript "q" for purely quantum ones and no subscript for general entropies, eventually of a mixed character. Having used the natural logarithm in these definitions, the entropies are in *nats*. To obtain entropies in *bits* one has to divide by $\ln 2$.

The following result is very useful ([6] Corollary 5.20 and Eq. (5.22)).

Proposition 1. *Let $\Pi_1 \otimes \Pi_2$ and Σ_{12} be normal states of the tensor product von Neumann algebra $\mathcal{M}_1 \otimes \mathcal{M}_2$ and let $\Sigma_i = \Sigma_{12}\big|_{\mathcal{M}_i}$, $i = 1, 2$. Then,*

$$\begin{aligned} S(\Sigma_{12}\|\Pi_1 \otimes \Pi_2) &= S(\Sigma_1\|\Pi_1) + S(\Sigma_{12}\|\Sigma_1 \otimes \Pi_2) \\ &= S(\Sigma_1\|\Pi_1) + S(\Sigma_2\|\Pi_2) + S(\Sigma_{12}\|\Sigma_1 \otimes \Sigma_2) \end{aligned} \tag{4}$$

The quantity $S(\Sigma_{12}\|\Sigma_1 \otimes \Sigma_2)$ is the relative entropy of a state with respect to its marginals; this is what we call *mutual entropy*.

1.2 Instruments and channels

1.2.1 Channels

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two W^*-algebras. A linear map Λ^* from $\mathcal{M}_2$ to $\mathcal{M}_1$ is said to be a *channel* ([6] p. 137) if it is completely positive, unital (i.e. identity preserving) and normal (or, equivalently, weakly* continuous).

Due to the equivalence [7] of w*-continuity and existence of a preadjoint Λ, a *channel* is equivalently defined by: Λ is a norm-one, completely positive linear map from the predual $\mathcal{M}_{1*}$ to the predual $\mathcal{M}_{2*}$. Let us note also that Λ maps normal states on $\mathcal{M}_1$ into normal states on $\mathcal{M}_2$.

A key result which follows from the convexity properties of the relative entropy is *Uhlmann monotonicity theorem* ([6], Theor. 1.5 p. 21), which implies that channels decrease the relative entropy.

Theorem 2. *If Σ and Π are two normal states on $\mathcal{M}_1$ and Λ is a channel from $\mathcal{M}_{1*} \to \mathcal{M}_{2*}$, then $S(\Sigma\|\Pi) \geq S(\Lambda[\Sigma]\|\Lambda[\Pi])$.*

1.2.2 Instruments and POV measures

The notion of instrument is central in quantum measurement theory; an instrument gives the probabilities and the state changes [8, 9].

Let $(\Omega, \mathcal{F})$ be a measurable space. An *instrument* $\mathcal{I}$ is a map valued measure such that (i) $\mathcal{I}(F)$ is a completely positive, linear, bounded operator on $\mathcal{T}(\mathcal{H})$, $\forall F \in \mathcal{F}$, (ii) $\mathcal{I}(\Omega)$ is trace preserving, (iii) for every countable family $\{F_i\}$ of disjoint sets in $\mathcal{F}$ one has $\sum_i \operatorname{Tr}\{a\,\mathcal{I}(F_i)[\rho]\} = \operatorname{Tr}\{a\,\mathcal{I}(\bigcup_i F_i)[\rho]\}$, $\forall \rho \in \mathcal{T}(\mathcal{H})$, $\forall a \in \mathcal{L}(\mathcal{H})$.

The map $F \mapsto \mathcal{I}(F)^*[\mathbb{1}]$ turns out to be a positive operator valued (POV) measure (the observable associated with the instrument $\mathcal{I}$). For every $\rho \in \mathcal{S}(\mathcal{H})$ the map $F \mapsto P_\rho(F) := \operatorname{Tr}\{\mathcal{I}(F)[\rho]\}$ is a probability measure: the probability that the result of the measurement be in F when the pre-measurement state is ρ. Moreover, given the result F, the post-measurement state is $\big(P_\rho(F)\big)^{-1}\mathcal{I}(F)[\rho]$.

1.2.3 The instrument as a channel

Given an instrument $\mathcal{I}$ with value space $(\Omega, \mathcal{F})$ it is always possible to find a σ-finite measure on $(\Omega, \mathcal{F})$ (or even a probability measure), such that all the probabilities P_ρ, $\rho \in \mathcal{S}(\mathcal{H})$, are absolutely continuous with respect to Q.

Theorem 3 ([10], Theorem 2)**.** *Let $\mathcal{I}$ be an instrument on the trace-class of a complex separable Hilbert space $\mathcal{H}$ with value space $(\Omega, \mathcal{F})$ and let Q be a σ-finite measure on $(\Omega, \mathcal{F})$ such that $\operatorname{Tr}\{\mathcal{I}(\bullet)[\rho]\} \ll Q$, $\forall \rho \in \mathcal{S}(\mathcal{H})$. Then, there exists a unique channel $\Lambda_\mathcal{I}$ from $\mathcal{T}(\mathcal{H})$ into $L^1\big(\Omega, \mathcal{F}, Q; \mathcal{T}(\mathcal{H})\big)$ such that*

$$\mathbb{E}_Q\left[f \operatorname{Tr}\{a\,\Lambda_\mathcal{I}[\rho]\}\right] = \int_\Omega f(\omega) \operatorname{Tr}\{a\,\mathcal{I}(\mathrm{d}\omega)[\rho]\} \tag{5}$$
$$\forall \rho \in \mathcal{T}(\mathcal{H}), \quad \forall a \in \mathcal{L}(\mathcal{H}), \quad \forall f \in L^\infty(\Omega, \mathcal{F}, Q).$$

Viceversa, a channel Λ from $\mathcal{T}(\mathcal{H})$ into $L^1\big(\Omega, \mathcal{F}, Q; \mathcal{T}(\mathcal{H})\big)$ defines a unique instrument $\mathcal{I}$ by

$$\mathcal{I}(F)[\rho] = \mathbb{E}_Q\left[1_F \Lambda[\rho]\right], \qquad \forall \rho \in \mathcal{T}(\mathcal{H}), \quad \forall F \in \mathcal{F}\,. \tag{6}$$

1.2.4 A posteriori states

When $\rho \in \mathcal{S}(\mathcal{H})$, then $\Lambda_{\mathcal{I}}[\rho]$ is a normal state on $L^\infty\big(\Omega,\mathcal{F},Q;\mathcal{L}(\mathcal{H})\big)$. Let us normalize the positive trace-class operators $\Lambda_{\mathcal{I}}[\rho](\omega)$ by setting

$$\pi_\rho(\omega) := \begin{cases} \left(\mathrm{Tr}_{\mathcal{H}}\left\{\Lambda_{\mathcal{I}}[\rho](\omega)\right\}\right)^{-1}\Lambda_{\mathcal{I}}[\rho](\omega) & \text{if } \mathrm{Tr}_{\mathcal{H}}\left\{\Lambda_{\mathcal{I}}[\rho](\omega)\right\} > 0 \\ \tilde{\rho} \quad \big(\tilde{\rho} \in \mathcal{S}(\mathcal{H}),\ \text{fixed}\big) & \text{if } \mathrm{Tr}_{\mathcal{H}}\left\{\Lambda_{\mathcal{I}}[\rho](\omega)\right\} = 0 \end{cases} \tag{7}$$

Then, we have

$$\int_F \pi_\rho(\omega) P_\rho(\mathrm{d}\omega) = \mathcal{I}(F)[\rho]\,, \quad \forall F \in \mathcal{F}\,, \qquad \text{(Bochner integral).} \tag{8}$$

According to Ozawa [11], π_ρ is a family *of a posteriori states* for the instrument $\mathcal{I}$ and the pre-measurement state ρ. The interpretation is that $\pi_\rho(\omega)$ is the state just after the measurement to be attributed to the quantum system if the result of the measurement has been exactly ω.

Let us note that $p_\rho := \mathrm{Tr}\left\{\Lambda_{\mathcal{I}}[\rho]\right\}$ and $\overline{\pi}_\rho := \int_\Omega P_\rho(\mathrm{d}\omega)\pi_\rho(\omega) = \mathcal{I}(\Omega)[\rho]$ are the marginals of the state $\Lambda_{\mathcal{I}}[\rho]$ on the algebras $L^\infty(\Omega,\mathcal{F},Q)$ and $\mathcal{L}(\mathcal{H})$, respectively. Then, $S\left(\Lambda_{\mathcal{I}}[\rho]\| p_\rho\overline{\pi}_\rho\right)$ is a first example of a mutual entropy. From Eqs. (3) and (1) we get

$$S\left(\Lambda_{\mathcal{I}}[\rho]\| p_\rho\overline{\pi}_\rho\right) = \int_\Omega S_{\mathrm{q}}\big(\pi_\rho(\omega)\|\overline{\pi}_\rho\big)P_\rho(\mathrm{d}\omega) = S_{\mathrm{q}}(\overline{\pi}_\rho) - \int_\Omega S_{\mathrm{q}}\big(\pi_\rho(\omega)\big)P_\rho(\mathrm{d}\omega). \tag{9}$$

Quantities like this one are used in quantum information transmission and are known as Holevo capacities or χ-quantities [12–14]; Eq. (9) gives the χ-quantity of the *ensemble* of states $\{P_\rho, \pi_\rho\}$.

2 Continual measurements

Quantum continual measurement theory can be formulated in different equivalent ways. To construct our entropic measures of efficiency, we need two approaches to continual measurements: the one based on positive operator valued measures, instruments, quantum channels [1,5,15] and the one based on classical stochastic differential equations (SDE's), known also as quantum trajectory theory [2,4,16].

The SDE approach to continual measurements is based on a couple of stochastic equations, a linear one for random trace-class operators and a non-linear one for random statistical operators. The two equations are linked by a change of normalization and a change of probability measure. Both equations have a Hilbert space formulation, particularly suited for numerical computations. We shall use a simplified version of SDE's for continual measurements as presented in [2].

2.1 The linear equation

Let $H(t), L_l(t), R_j(t), V_k^r(t), J_k(t)$ be bounded operators on $\mathcal{H}$; their time dependence is taken to be continuous from the left and with limits from the right in the strong topology. The indices k, l, j take a finite number of values; the index r can take infinitely many values, but in this case the series $\sum_r V_k^r(t)^* V_k^r(t)$ is strongly convergent. Let the operator $H(t)$ be self-adjoint, $H(t) = H(t)^*$, and let us define ($\forall \rho \in \mathcal{T}(\mathcal{H})$)

$$\mathcal{J}_k(t)[\rho] := \sum_r V_k^r(t)\rho V_k^r(t)^* \,, \qquad J_k(t) := \mathcal{J}_k(t)^*[\mathbb{1}] = \sum_r V_k^r(t)^* V_k^r(t) \,, \tag{10}$$

$$\mathcal{L}(t) := \mathcal{L}_0(t) + \mathcal{L}_1(t) + \mathcal{L}_2(t), \tag{11}$$

$$\mathcal{L}_0(t)[\rho] := -\mathrm{i}[H(t),\rho] + \sum_l \left(L_l(t)\rho L_l(t)^* - \frac{1}{2}\left\{L_l(t)^* L_l(t), \rho\right\}\right), \tag{12}$$

$$\mathcal{L}_1(t)[\rho] := \sum_j \left(R_j(t)\rho R_j(t)^* - \frac{1}{2}\left\{R_j(t)^* R_j(t), \rho\right\}\right), \tag{13}$$

$$\mathcal{L}_2(t)[\rho] := \sum_k \left(\mathcal{J}_k(t)[\rho] - \frac{1}{2}\left\{J_k(t), \rho\right\}\right). \tag{14}$$

By [,] we denote the commutator and by { , } the anticommutator.

Then, we introduce a probability space $(\Omega, \mathcal{F}, Q)$ where the Poisson processes $N_k(t)$, of intensity λ_k, and the standard (continuous) Wiener processes $W_j(t)$ are defined. All the processes are assumed to be independent from the other ones. We introduce also the two-times natural filtration of such processes:

$$\mathcal{F}_t^s = \sigma\{W_j(u) - W_j(s),\, N_k(v) - N_k(s),\, u, v \in [s,t],\, j,k = 1, \ldots\}. \tag{15}$$

Having all these ingredients, we can introduce the linear equation of continual measurement theory, for the a trace-class valued process σ_t:

$$\begin{aligned} \mathrm{d}\sigma_t = \mathcal{L}(t)[\sigma_{t\_}]\mathrm{d}t &+ \sum_j \left(R_j(t)\sigma_{t\_} + \sigma_{t\_} R_j(t)^*\right)\mathrm{d}W_j(t) \\ &+ \sum_k \left(\frac{1}{\lambda_k}\mathcal{J}_k(t)[\sigma_{t\_}] - \sigma_{t\_}\right)\left(\mathrm{d}N_k(t) - \lambda_k \mathrm{d}t\right). \end{aligned} \tag{16}$$

The initial condition is taken to be a non-random statistical operator: $\sigma_0 \equiv \sigma_{0\_} \in \mathcal{S}(\mathcal{H})$.

The notation $\sigma_{t\_}$ means that, in case there is a jump in the noise at time t, the value just before the jump $\sigma_{t\_}$ of σ has to be taken. More precisely, if the augmented natural filtration of the noises is considered, the solution can be taken to be continuous from the right and with limits from the left and $\sigma_{t\_}$ is just the limit from the left. We prefer not to add the null sets to the natural filtration and by σ_t we mean some $\mathcal{F}_t^0$-adapted version of the solution.

Properties of the solution. Let us consider now, for $0 \leq s \leq t$, the von Neumann algebra $L^\infty\big(\Omega, \mathcal{F}_t^s, Q; \mathcal{L}(\mathcal{H})\big) \simeq L^\infty(\Omega, \mathcal{F}_t^s, Q) \otimes \mathcal{L}(\mathcal{H})$ (cf. Section 1.1.1) and let us give a name to the set of normal states on this algebra:

$$\mathcal{S}_t^s := \left\{ \tau \in L^1\big(\Omega, \mathcal{F}_t^s, Q; \mathcal{T}(\mathcal{H})\big) : \tau(\omega) \geq 0\,,\ \int_\Omega \mathrm{Tr}\{\tau(\omega)\}\, Q(\mathrm{d}\omega) = 1 \right\}. \tag{17}$$

- First of all, it is possible to prove that $\sigma_t \in \mathcal{S}_t^0$; we can say that the solution at time t of Eq. (16) is a kind of quantum/classical state.
- The marginals of σ_t are (cf. Section 1.2.4):

- The probability density $p_t := \mathrm{Tr}\{\sigma_t\}$. The probability measure $p_t(\omega)Q(\mathrm{d}\omega)$ will be the physical probability.
- The *a priori state at time t* $\eta_t := \mathbb{E}_Q[\sigma_t] \in \mathcal{S}(\mathcal{H})$. This is the state to be attributed at time t to the system when no selection is done and the result of the measurement has not been taken into account.

- Moreover, we define the random *a posteriori state at time t* $\rho_t = \dfrac{1}{p_t}\,\sigma_t$. This is the state to be attributed at time t to the system known the result of the measurement up to t.

Note that $p_0 = 1$, $\rho_0 = \sigma_0 = \eta_0$, $\eta_{t_-} = \eta_t = \eta_{t_+}$.

2.2 Physical probabilities

A very important property of Eq. (16) is that p_t is a mean one Q-martingale, which implies that

$$P_t(\mathrm{d}\omega) := p_t(\omega)Q(\mathrm{d}\omega)\big|_{\mathcal{F}_t^0} \tag{18}$$

is a consistent family of probabilities, i.e., if $0 \le t < T$, $P_T(F) = P_t(F)$, $\forall F \in \mathcal{F}_t^0$. These are taken as physical probabilities.

From Eq. (16) we have that p_t satisfies the Doléans equation

$$\mathrm{d}p_t = p_{t_-}\Big\{\sum_j m_j(t)\,\mathrm{d}W_j(t) + \sum_k \left(\frac{\mu_k(t)}{\lambda_k} - 1\right)\big(\mathrm{d}N_k(t) - \lambda_k\,\mathrm{d}t\big)\Big\}, \tag{19}$$

where

$$m_j(t) = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\rho_{t_-}\right\}, \qquad \mu_k(t) = \mathrm{Tr}\left\{J_k(t)\rho_{t_-}\right\}. \tag{20}$$

The solution of this equation, with $p_0 = 1$, is

$$p_t = \exp\Big\{\sum_j\left[\int_0^t m_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t m_j(s)^2\,\mathrm{d}s\right] + \sum_k\left[\int_0^t \ln\frac{\mu_k(s)}{\lambda_k}\,\mathrm{d}N_k(s) + \int_0^t (\lambda_k - \mu_k(s))\,\mathrm{d}s\right]\Big\}. \tag{21}$$

Remark 1.

1. The output of the continual measurement is the set of processes $W_j(t)$, $N_k(t)$, $0 \le t \le T$, under the physical probability P_T; T is a completely arbitrary large time. By the consistency of the probabilities (18), P_T can be substituted by P_t in any expectation involving $\mathcal{F}_t^0$-measurable random variables (for $t < T$).

2. By Girsanov theorem and its generalizations for situations with jumps, we have that, under the physical probability, the processes

$$\widehat{W}_j(t) = W_j(t) - \int_0^t m_j(s)\,\mathrm{d}s \tag{22}$$

are independent, standard Wiener processes and $N_k(t)$ is a counting process of stochastic intensity $\mu_k(t)\mathrm{d}t$.

3. Expressions for the moments of the outputs can be given; in particular we have the mean values
$$\mathbb{E}_{P_t}\left[W_j(t)\right] = \int_0^t n_j(s)\,\mathrm{d}s\,, \qquad \mathbb{E}_{P_t}\left[N_k(t)\right] = \int_0^t \nu_k(s)\,\mathrm{d}s\,, \tag{23}$$
where
$$n_j(t) = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\eta_t\right\} = \mathbb{E}_{P_t}[m_j(t)], \tag{24a}$$
$$\nu_k(t) = \mathrm{Tr}\left\{J_k(t)\eta_t\right\} = \mathbb{E}_{P_t}[\mu_k(t)]. \tag{24b}$$

2.3 The non-linear SDE

Under the physical law P_T, the a posteriori states ρ_t satisfy the non-linear SDE
$$\mathrm{d}\rho_t = \mathcal{L}(t)[\rho_{t_-}]\mathrm{d}t + \sum_j \big(R_j(t)\rho_{t_-} + \rho_{t_-}R_j(t)^* - m_j(t)\rho_{t_-}\big)\,\mathrm{d}\widehat{W}_j(t) + \sum_k \left(\frac{1}{\mu_k(t)}\,\mathcal{J}_k(t)[\rho_{t_-}] - \rho_{t_-}\right)\left(\mathrm{d}N_k(t) - \mu_k(t)\mathrm{d}t\right). \tag{25}$$

Let us stress that for the a priori states we have
$$\eta_t = \mathbb{E}_Q[\sigma_t] = \mathbb{E}_{P_t}[\rho_t] \tag{26}$$
and that they satisfy the *master equation*
$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta_t = \mathcal{L}(t)[\eta_t]. \tag{27}$$

2.4 The fundamental matrix and the instruments

To apply the notions of Section 1 to continual measurements, we need to see how such a theory is connected to instruments and channels [2–5]. This is done by introducing the fundamental matrix Λ_t^s of (16). This operator is defined by stipulating that $\Lambda_t^s[|u_i\rangle\langle u_j|]$ satisfies (16) with initial condition $\Lambda_s^s[|u_i\rangle\langle u_j|] = |u_i\rangle\langle u_j|$, where $\{u_i\ i = 1, \ldots\}$ is a c.o.n.s. in $\mathcal{H}$. It turns out that Λ_t^s is a channel from $\mathcal{T}(\mathcal{H})$ into $L^1\big(\Omega, \mathcal{F}_t^s, Q; \mathcal{T}(\mathcal{H})\big)$, or, by trivial ampliation, from $L^1\big(\Omega, \mathcal{F}_s^r, Q; \mathcal{T}(\mathcal{H})\big)$ into $L^1\big(\Omega, \mathcal{F}_t^r, Q; \mathcal{T}(\mathcal{H})\big)$, $0 \le r \le s \le t$. Then, we have
$$\Lambda_t^s[\sigma_s] = \sigma_t\,, \qquad \Lambda_t^s = \Lambda_t^u \circ \Lambda_u^s\,, \quad 0 \le s \le u \le t\,. \tag{28}$$
The instrument associated to this channel is
$$\mathcal{I}_t^s(F)[\rho] = \mathbb{E}_Q\left[1_F \Lambda_t^s[\rho]\right] \equiv \int_F \Lambda_t^s(\omega)[\rho]Q(\mathrm{d}\omega), \qquad \forall F \in \mathcal{F}_t^s. \tag{29}$$
The time evolution of the quantum states is the one generated by $\mathcal{L}(t)$ and we have
$$\mathcal{U}(t,s)[\rho] = \mathcal{I}_t^s(\Omega)[\rho] = \mathbb{E}_Q\left[\Lambda_t^s[\rho]\right], \tag{30}$$
$$\mathcal{U}(t,s)[\eta_s] = \eta_t\,, \qquad \mathcal{U}(t,s) = \mathcal{U}(t,u)\circ\mathcal{U}(u,s)\,, \quad 0 \le s \le u \le t\,. \tag{31}$$

According to the definitions of Section 1.2.4, the random statistical operator ρ_t is the a posteriori state for the instrument $\mathcal{I}_t^0$ and the pre-measurement state $\rho_0 \equiv \eta_0$.

Another important property is

$$\mathbb{E}_Q[\sigma_t|\mathcal{F}_t^s] = \Lambda_t^s[\eta_s] \in \mathcal{S}_t^s \,. \tag{32}$$

Indeed, by the first of (28) and the fact that Λ_t^s is $\mathcal{F}_t^s$-measurable, we have $\mathbb{E}_Q[\sigma_t|\mathcal{F}_t^s] = \Lambda_t^s\big[\,\mathbb{E}_Q[\sigma_s|\mathcal{F}_t^s]\big]$. By the fact that all the noises have independent increments, we have that σ_s is independent from $\mathcal{F}_t^s$ and $\mathbb{E}_Q[\sigma_s|\mathcal{F}_t^s] = \mathbb{E}_Q[\sigma_s] = \eta_s$. This gives Eq. (32).

3 Mutual entropies and information gains

3.1 The information embedded in the a posteriori states

The quantity σ_t is a state on $L^\infty\big(\Omega, \mathcal{F}_t^0, Q; \mathcal{L}(\mathcal{H})\big) = L^\infty(\Omega, \mathcal{F}_t^0, Q) \otimes \mathcal{L}(\mathcal{H})$ and its marginals on $L^\infty(\Omega, \mathcal{F}_t^0, Q)$ and $\mathcal{L}(\mathcal{H})$ are p_t and η_t, respectively. The mutual entropy $S(\sigma_t\|p_t\eta_t)$ is the "information" contained in the joint state with respect to the product of these marginals; more explicitly we have (compare with (9))

$$S(\sigma_t\|p_t\eta_t) = \int_\Omega P_t(\mathrm{d}\omega)\, \mathrm{Tr}\left\{\rho_t(\omega)\big(\ln\rho_t(\omega) - \ln\eta_t\big)\right\}$$

and we can write

$$S(\sigma_t\|p_t\eta_t) = \mathbb{E}_{P_t}[S_\mathrm{q}(\rho_t\|\eta_t)] = S_\mathrm{q}(\eta_t) - \mathbb{E}_{P_t}[S_\mathrm{q}(\rho_t)]. \tag{33}$$

This mutual entropy is a sort of quantum information embedded by the measurement in the a posteriori states. When the measurement is not informative, we have $\rho_t(\omega) = \eta_t$ and $S(\sigma_t\|p_t\eta_t) = 0$. It is zero also if for any reason it happens that η_t is a pure state. For instance, if $\mathcal{U}(t,0)$ has a unique equilibrium state which is pure, then $\lim_{t\to+\infty} S(\sigma_t\|p_t\eta_t) = 0$ even if the measurement is "good".

Let us note that from Eq. (33) we have the bound

$$S(\sigma_t\|p_t\eta_t) \leq S_\mathrm{q}(\eta_t). \tag{34}$$

When the von Neumann entropy of the a priori state is not zero, an instantaneous index of "goodness" of the measurement could be $S(\sigma_t\|p_t\eta_t)\big/S_\mathrm{q}(\eta_t)$, while a "cumulative" index could be $\int_0^T \frac{S(\sigma_t\|p_t\eta_t)}{S_\mathrm{q}(\eta_t)}\,\mathrm{d}t$.

3.2 A classical continual information gain

3.2.1 Product densities

Let us consider any time s in the time interval $(0,t)$ and let us decompose the von Neumann algebra $L^\infty(\Omega, \mathcal{F}_t^0, Q)$ as $L^\infty(\Omega, \mathcal{F}_t^0, Q) = L^\infty(\Omega, \mathcal{F}_s^0, Q) \otimes L^\infty(\Omega, \mathcal{F}_t^s, Q)$. Now, the density p_t can be seen as a state on $L^\infty(\Omega, \mathcal{F}_t^0, Q)$ and we can consider its marginals p_s^0 and p_t^s on the two factors $L^\infty(\Omega, \mathcal{F}_s^0, Q)$ and $L^\infty(\Omega, \mathcal{F}_t^s, Q)$, respectively. These marginals are given by

$$p_s^0 = \mathbb{E}_Q[p_t|\mathcal{F}_s^0], \qquad p_t^s = \mathbb{E}_Q[p_t|\mathcal{F}_t^s]. \tag{35}$$

By using the fact that $\{p_t,\, t\geq 0\}$ is a martingale and by taking the trace of Eq. (32), we get

$$p_s^0 = p_s\,, \qquad p_t^s = \mathrm{Tr}\{\Lambda_t^s[\eta_s]\}\,. \tag{36}$$

By comparing the last equality with $p_t = \mathrm{Tr}\{\sigma_t\} = \mathrm{Tr}\{\Lambda_t^0[\eta_0]\}$, we see that p_t^s is similar to p_t, but with s as initial time, instead of 0, and with η_s as initial state, instead of η_0. By this remark and Eq. (21), we get

$$p_t^s = \exp\Bigg\{\sum_j \bigg[\int_s^t m_j(u;s)\,\mathrm{d}W_j(u) - \frac{1}{2}\int_s^t m_j(u;s)^2\,\mathrm{d}u\bigg] + \sum_k \bigg[\int_s^t \ln\frac{\mu_k(u;s)}{\lambda_k}\,\mathrm{d}N_k(u) + \int_s^t (\lambda_k - \mu_k(u;s))\,\mathrm{d}u\bigg]\Bigg\}, \tag{37}$$

where

$$m_j(t;s) = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\rho_{t_-}^s\right\}, \qquad \mu_k(t;s) = \mathrm{Tr}\left\{J_k(t)\rho_{t_-}^s\right\}, \tag{38}$$

$$\rho_t^s = \frac{1}{p_t^s}\,\Lambda_t^s[\eta_s]. \tag{39}$$

The random state ρ_t^s is the a posteriori state for the instrument $\mathcal{I}_t^s$ and the pre-measurement state η_s; it satisfy the non-linear SDE (25).

Then, we can consider the mutual entropy $S_\mathrm{c}(p_t\|p_s^0p_t^s)$. But the significance of this quantity is dubious, because the time s is completely arbitrary and, moreover, we could divide the time interval in more pieces. For instance, we can take the decomposition $L^\infty(\Omega,\mathcal{F}_t^0,Q) = L^\infty(\Omega,\mathcal{F}_r^0,Q)\otimes L^\infty(\Omega,\mathcal{F}_s^r,Q)\otimes L^\infty(\Omega,\mathcal{F}_t^s,Q)$ and we recognize that $p_r^0p_s^rp_t^s$ is the product of the marginals of p_t related to this decomposition. Taking a finer generic partition of $(0,t)$ with $t_0=0$ and $t_n=t$, we recognize that $\prod_{j=1}^n p_{t_j}^{t_{j-1}}$ is again a product of marginals of p_t. To eliminate arbitrariness, let us consider finer and finer partitions and let us go to a continuous product of marginals.

Let us note that we have

$$\lim_{s\uparrow t} m_j(t;s) = n_j(t), \qquad \lim_{s\uparrow t}\mu_k(t;s) = \nu_k(t), \qquad \text{a.s.}$$

Then, for an infinitesimal interval we get

$$p_{s+\mathrm{d}s}^s = \exp\Bigg\{\sum_j \bigg[n_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\,n_j(s)^2\mathrm{d}s\bigg] + \sum_k \bigg[\frac{\nu_k(s)}{\lambda_k}\mathrm{d}N_k(s) + (\lambda_k - \nu_k(s))\,\mathrm{d}s\bigg]\Bigg\} \tag{40}$$

and, so, the following density q_t is the continuous product of marginals of p_t:

$$q_t = \exp\Bigg\{\sum_j \bigg[\int_0^t n_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t n_j(s)^2\,\mathrm{d}s\bigg] + \sum_k \bigg[\int_0^t \ln\frac{\nu_k(s)}{\lambda_k}\,\mathrm{d}N_k(s) + \int_0^t (\lambda_k - \nu_k(s))\,\mathrm{d}s\bigg]\Bigg\}. \tag{41}$$

Notice that $n_j(t)$ and $\nu_k(t)$ are deterministic functions. Under the probability $q_T(\omega)Q(\mathrm{d}\omega)$, the processes $W_j(t) - \int_0^t n_j(s)\,\mathrm{d}s$ are independent, standard Wiener processes and $N_k(t)$ is a Poisson process of time dependent intensity $\nu_k(t)$.

Under $q_T(\omega)Q(\mathrm{d}\omega)$, the processes W_j, N_k have independent increments as under Q (so they can be interpreted as noises), but the means have been changed and made equal to the means they have under P_T.

The fact that it is possible to consider a "continuous product of marginals" is not so unexpected; indeed, the theory of continual measurements is connected to infinite divisibility [15].

We have already seen that the marginals of p_t with respect to the decomposition of the time interval $(0,t)$ into $(0,s)$ and (s,t) are $p_s^0 = p_s$ and p_t^s given by Eq. (37). The analogous marginals for q_t are $q_s^0 = q_s$ and

$$q_t^s = \exp\biggl\{\sum_j \biggl[\int_t^u n_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_t^u n_j(s)^2\,\mathrm{d}s\biggr] \\ + \sum_k \biggl[\int_t^u \ln\frac{\nu_k(s)}{\lambda_k}\,\mathrm{d}N_k(s) + \int_t^u (\lambda_k - \nu_k(s))\,\mathrm{d}s\biggr]\biggr\} = \frac{q_t}{q_s}\,. \qquad (42)$$

3.2.2 The classical mutual entropy $S_\mathrm{c}(p_t\|q_t)$

The density q_t is no more dependent on some arbitrary choice of intermediate times and the measure $q_T(\omega)Q(\mathrm{d}\omega)$ has a distinguished role and can be considered as a reference measure. So, we can introduce the relative entropy

$$S_\mathrm{c}(p_t\|q_t) = \mathbb{E}_{P_t}\left[\ln\frac{p_t}{q_t}\right].$$

Being q_t a product of marginals of p_t, this quantity is a mutual entropy and, being q_t the finest product of marginals, we can interprete $S_\mathrm{c}(p_t\|q_t)$ as a measure of the classical information on the measured system extracted in the time interval $(0,t)$. Other reasons can be given to reinforce this interpretation.

By Eqs. (21), (37), (41), (42) we have $p_t^0 = p_t$, $q_t^0 = q_t$, $q_u = q_t q_u^t$. By Proposition 1 or by direct computation, we get

$$S_\mathrm{c}(p_t\|q_t) - S_\mathrm{c}(p_s\|q_s) = S_\mathrm{c}(p_t\|p_s q_t^s), \qquad 0 \leq s \leq t\,. \qquad (43)$$

Firstly, by the positivity of relative entropies, this equation says that

$$0 \leq S_\mathrm{c}(p_s\|q_s) \leq S_\mathrm{c}(p_t\|q_t), \qquad (44)$$

i.e. that $S_\mathrm{c}(p_t\|q_t)$ is non negative and not decreasing in time, as should be for a measure of an information gain in time. Moreover, the increment of information in the time interval (s,t) can be written as

$$S_\mathrm{c}(p_t\|p_s q_t^s) = \mathbb{E}_Q\left[p_s\,\mathbb{E}_Q\left[\frac{p_t}{p_s}\ln\frac{p_t/p_s}{q_t/q_s}\middle|\mathcal{F}_s^0\right]\right]. \qquad (45)$$

This expression can be interpreted as a *conditional relative entropy* ([17] pp. 22–23). The quantity $\mathbb{E}_Q\left[\frac{p_t}{p_s}\ln\frac{p_t/p_s}{q_t/q_s}\middle|\mathcal{F}_s^0\right]$ has the same structure as $S_\mathrm{c}(p_t\|q_t)$, but it refers to the interval (s,t) and it is constructed with the conditional densities. We can say that Eq. (43) expresses in a consistent way a kind of "additivity property" of our measure of information.

Having the explicit exponential forms of the densities p_t and q_t, we can compute the explicit expression of the information gain.

Proposition 4. *The explicit expression of the classical mutual entropy $S_{\rm c}(p_t\|q_t)$ is*

$$S_{\rm c}(p_t\|q_t) = \frac{1}{2}\sum_j \int_0^t \mathrm{Var}_{P_t}[m_j(s)]\mathrm{d}s + \sum_k \int_0^t \mathbb{E}_{P_t}\left[\mu_k(s)\ln\frac{\mu_k(s)}{\nu_k(s)}\right]\mathrm{d}s \tag{46}$$

Proof. By Eqs. (21) and (41) we get

$$\begin{aligned}
\ln\frac{p_t}{q_t} &= \sum_j \left[\int_0^t \big(m_j(s)-n_j(s)\big)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t \big(m_j(s)^2 - n_j(s)^2\big)\mathrm{d}s\right] \\
&+ \sum_k \left[\int_0^t \ln\frac{\mu_k(s)}{\nu_k(s)}\big(\mathrm{d}N_k(s) - \lambda_k\,\mathrm{d}s\big) + \int_0^t \left(\lambda_k \ln\frac{\mu_k(s)}{\nu_k(s)} - \mu_k(s) + \nu_k(s)\right)\mathrm{d}s\right] \\
&= \sum_j \left[\int_0^t \big(m_j(s)-n_j(s)\big)\big(\mathrm{d}W_j(s) - m_j(s)\mathrm{d}s\big) + \frac{1}{2}\int_0^t \big(m_j(s)-n_j(s)\big)^2\mathrm{d}s\right] \\
&\qquad + \sum_k \left[\int_0^t \ln\frac{\mu_k(s)}{\nu_k(s)}\big(\mathrm{d}N_k(s) - \mu_k(s)\,\mathrm{d}s\big)\right. \\
&\qquad\qquad \left. + \int_0^t \mu_k(s)\left(\frac{\nu_k(s)}{\mu_k(s)} - \ln\frac{\nu_k(s)}{\mu_k(s)} - 1\right)\mathrm{d}s\right].
\end{aligned}$$

By point 2 in Remark 1, the first term in the j sum and the first term in the k sum have zero mean under P_T (or under P_t, by consistency). Therefore, Eq. (46) follows by taking the P_t-mean of $\ln p_t/q_t$ and by taking into account Eqs. (24). □

Remark 2.

1. By (24b) and Jensen inequality applied to the convex function $x\ln x$, we have that both integrands in formula (46) are non-negative and, so, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}S_{\rm c}(p_t\|q_t) = \frac{1}{2}\sum_j \mathrm{Var}_{P_t}[m_j(t)] + \sum_k \mathbb{E}_{P_t}\left[\mu_k(t)\ln\frac{\mu_k(t)}{\nu_k(t)}\right] \geq 0. \tag{47}$$

   The positivity of this time derivative follows also from Eq. (44).

2. By the properties of relative entropy $\mathbb{E}_{P_T}[S_{\rm q}(\rho_t\|\eta_t)] = 0$ is equivalent to $\rho_t = \eta_t$, P_T-a.s. By Eqs. (24), (47), this last relation implies the vanishing of the quantity (47). So, we have

$$\mathbb{E}_{P_T}[S_{\rm q}(\rho_t\|\eta_t)] = 0 \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}S_{\rm c}(p_t\|q_t) = 0\,. \tag{48}$$

3. From Eqs. (20), (24), (47) we see that
   - if $R_j(t) + R_j(t)^* \propto \mathbb{1}$, then $\mathrm{Var}_{P_t}[m_j(t)] = 0$,
   - if $J_k(t) \propto \mathbb{1}$, then $\ln\frac{\mu_k(t)}{\nu_k(t)} = 0$.

   This says that when both conditions hold for all j and k, no information is extracted from the system, whatever the initial state is.

3.3 A quantum/classical mutual entropy

The two mutual entropies introduced in Sections 3.1 and 3.2.2 can be obtained from a unique mutual entropy

$$S(\sigma_t \| q_t\eta_t) = \int_\Omega Q(\mathrm{d}\omega)\,\mathrm{Tr}\left\{\sigma_t(\omega)\big(\ln\sigma_t(\omega) - \ln q_t(\omega)\eta_t\big)\right\}. \tag{49}$$

Indeed, by Proposition 1 or by direct computation, we get

$$S(\sigma_t \| q_t\eta_t) = S(\sigma_t \| p_t\eta_t) + S_\mathrm{c}(p_t\|q_t) = \mathbb{E}_{P_t}[S_\mathrm{q}(\rho_t\|\eta_t)] + S_\mathrm{c}(p_t\|q_t). \tag{50}$$

4 An upper bound on the increments of $S_\mathrm{c}(p_t\|q_t)$

4.1 The main bound

By Proposition 1 and Eqs. (21), (37), (41), (42), the increment of information in the time interval (t,u) can be expressed as

$$S_\mathrm{c}(p_u\|q_u) - S_\mathrm{c}(p_t\|q_t) = S_\mathrm{c}(p_u\|p_tp_u^t) + S_\mathrm{c}(p_u^t\|q_u^t). \tag{51}$$

Lemma 5. *For $0 \le t \le u$, we have the bound*

$$0 \le S_\mathrm{c}\left(p_u\|p_tp_u^t\right) \le \mathbb{E}_{P_u}\left[S_\mathrm{q}\left(\rho_t\|\eta_t\right) - S_\mathrm{q}\left(\rho_u\|\rho_u^t\right)\right]. \tag{52}$$

Proof. Consider the mutual entropy $S(\sigma_t\|p_t\eta_t)$ introduced in Section 3.1 and apply to both states the channel Λ_u^t. By Theorem 2 and the definition (39) we get the inequality

$$\begin{aligned}\mathbb{E}_{P_t}\left[S_\mathrm{q}\left(\rho_t\|\eta_t\right)\right] &= S\left(\sigma_t\|p_t\eta_t\right) \ge S\left(\Lambda_u^t[\sigma_t]\|\Lambda_u^t[p_t\eta_t]\right) = S\left(\sigma_u\|p_t\sigma_u^t\right)\\ &= S\left(p_u\rho_u\|p_tp_u^t\rho_u^t\right) = \mathbb{E}_{P_u}\left[\mathrm{Tr}\left\{\rho_u\left(\ln p_u + \ln\rho_u - \ln\left(p_tp_u^t\right) - \ln\rho_u^t\right)\right\}\right]\\ &= S_\mathrm{c}\left(p_u\|p_tp_u^t\right) + \mathbb{E}_{P_u}\left[S_\mathrm{q}\left(\rho_u\|\rho_u^t\right)\right],\end{aligned}$$

and this gives (52). □

Apart from the different notations, Eq. (52) is the bound (29) in Ref. [5].

From Eqs. (40) and (42) we get immediately

$$\lim_{u\downarrow t}\frac{S_\mathrm{c}\left(p_u^t\|q_u^t\right)}{u-t} = 0. \tag{53}$$

Then, the second summand in the expression (51) of the increment of information becomes negligible with respect to the first when $u \downarrow t$. Therefore, from Lemma 5 we have immediately the following theorem.

Theorem 6 (The bound on the derivative of $S_\mathrm{c}(p_t\|q_t)$)**.** *The following bound holds:*

$$\begin{aligned}0 \le \frac{\mathrm{d}}{\mathrm{d}t}\,S_\mathrm{c}(p_t\|q_t) &\le -\frac{\mathrm{d}}{\mathrm{d}u}\,\mathbb{E}_{P_T}[S_\mathrm{q}(\rho_u\|\rho_u^t)]\Big|_{u=t^+}\\ &\equiv \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}_{P_T}[S_\mathrm{q}(\rho_t)] - \frac{\mathrm{d}}{\mathrm{d}u}\,\mathbb{E}_{P_T}[S_\mathrm{q}(\rho_u^t)]\Big|_{u=t^+}.\end{aligned} \tag{54}$$

Remark 3. We already saw in Remark 2 that $\mathbb{E}_{P_T}[S_\mathrm{q}(\rho_t\|\eta_t)] = 0$ is equivalent to $\rho_t = \eta_t$, P_T-a.s.; but this implies $\rho_u = \rho_u^t$, P_T-a.s., because in this case these two quantities, which satisfy the same equation, have the same initial condition at time t. Therefore we have $\mathbb{E}_{P_T}[S_\mathrm{q}(\rho_u\|\rho_u^t)] = 0$, $\forall u \geq t$, and

$$\mathbb{E}_{P_T}[S_\mathrm{q}(\rho_t\|\eta_t)] = 0 \quad \Rightarrow \quad -\frac{\mathrm{d}}{\mathrm{d}u}\, \mathbb{E}_{P_T}[S_\mathrm{q}(\rho_u\|\rho_u^t)]\Big|_{u=t^+} = 0. \tag{55}$$

4.2 Explicit computation of the bound

All the derivatives can be elaborated and from Eq. (54) we get the following explicit form of the difference between the bound and the time derivative in which we are interested in.

Proposition 7. *By computation of all the terms appearing in Eq.* (54) *we get*

$$\begin{aligned}
0 \leq{}& \frac{\mathrm{d}}{\mathrm{d}t}\, \mathbb{E}_{P_T}[S_\mathrm{q}(\rho_t)] - \frac{\mathrm{d}}{\mathrm{d}u}\, \mathbb{E}_{P_T}[S_\mathrm{q}(\rho_u^t)]\Big|_{u=t^+} - \frac{\mathrm{d}}{\mathrm{d}t}\, S_\mathrm{c}(p_t\|q_t) \\
={}& \sum_k \mathbb{E}_{P_T}\big[\, \mathrm{Tr}\left\{ J_k(t)\rho_t\,(\ln\rho_t - \ln\eta_t) - \mathcal{J}_k(t)[\rho_t]\,(\ln\mathcal{J}_k(t)[\rho_t] - \ln\mathcal{J}_k(t)[\eta_t])\,\right\}\big] \\
&+ \frac{1}{2}\sum_j \mathbb{E}_{P_T}\Bigg[\int_0^{+\infty}\mathrm{d}u\,\mathrm{Tr}\Bigg\{\frac{\eta_t}{u+\eta_t}\,(R_j(t)+R_j(t)^*)\,\frac{\eta_t}{u+\eta_t}\,(R_j(t)+R_j(t)^*) \\
&\qquad - \frac{\rho_t}{u+\rho_t}\,(R_j(t)+R_j(t)^*)\,\frac{\rho_t}{u+\rho_t}\,(R_j(t)+R_j(t)^*)\Bigg\}\Bigg] \\
&\qquad + \sum_l \mathbb{E}_{P_T}\left[\mathrm{Tr}\left\{L_l(t)\eta_t\big[L_l(t)^*,\,\ln\eta_t\big] - L_l(t)\rho_t\big[L_l(t)^*,\,\ln\rho_t\big]\right\}\right].
\end{aligned} \tag{56}$$

Proof. Let us start with the term $\frac{\mathrm{d}}{\mathrm{d}u}\, \mathbb{E}_{P_T}[S_\mathrm{q}(\rho_u^t)]\Big|_{u=t^+}$. By recalling that ρ_u^t satisfies in u the non-linear SDE with initial condition η_t at $u = t$ and that $\eta_t + \mathcal{L}(t)[\eta_t]\mathrm{d}t = \eta_{t+\mathrm{d}t}$, we get

$$\rho_{t+\mathrm{d}t}^t - \eta_{t+\mathrm{d}t} = \sum_j A_j(t)\mathrm{d}\check{W}_j(t) + \sum_k (\tau_k(t) - \eta_t)\,(\mathrm{d}N_k(t) - \nu_k(t)\mathrm{d}t)\,,$$

where

$$A_j(t) := R_j(t)\eta_t + \eta_t R_j(t)^* - n_j(t)\eta_t\,, \qquad \tau_k(t) := \frac{1}{\nu_k(t)}\,\mathcal{J}_k(t)[\eta_t],$$

$$\mathrm{d}\check{W}_j(t) := \mathrm{d}W_j(t) - n_j(t)\mathrm{d}t\,.$$

By setting also

$$B(t) := -\frac{1}{2}\sum_k\{J_k(t) - \nu_k(t), \eta_t\} + \mathcal{L}_0(t)[\eta_t] + \mathcal{L}_1(t)[\eta_t],$$

we can write

$$\eta_{t+\mathrm{d}t} = \left(1 - \sum_k \nu_k(t)\mathrm{d}t\right)\eta_t + \sum_k \nu_k(t)\tau_k(t)\mathrm{d}t + B(t)\mathrm{d}t\,.$$

Moreover, by the properties of the increments of the counting processes, we have

$$\rho^t_{t+\mathrm{d}t}\mathrm{d}N_k(t) = \tau_k(t)\mathrm{d}N_k(t),$$

$$\begin{aligned}&\left(1-\sum_k \mathrm{d}N_k(t)\right)\rho^t_{t+\mathrm{d}t}\\ &\qquad = \left(1-\sum_k \mathrm{d}N_k(t)\right)\left(\eta_t + B(t)\mathrm{d}t + \sum_j A_j(t)\,\mathrm{d}\check{W}_j(t)\right).\end{aligned}$$

By putting these things all together and by using the rules of stochastic calculus, we get

$$\begin{aligned}\rho^t_{t+\mathrm{d}t}\ln\rho^t_{t+\mathrm{d}t} - \eta_t\ln\eta_t &= \sum_k \big[\tau_k(t)\ln\tau_k(t) - \eta_t\ln\eta_t\big]\mathrm{d}N_k(t)\\ +\,\eta_t\left[\ln\left(\eta_t + B(t)\mathrm{d}t + \sum_j A_j(t)\,\mathrm{d}\check{W}_j(t)\right) - \ln\eta_t\right] &+ \sum_j A_j(t)\ln\eta_t\,\mathrm{d}\check{W}_j(t)\\ +\,B(t)\mathrm{d}t\,\ln\eta_t + \sum_j A_j(t)&\left[\ln\big(\eta_t + A_j(t)\,\mathrm{d}\check{W}_j(t)\big) - \ln\eta_t\right]\mathrm{d}\check{W}_j(t).\end{aligned}$$

It exists a nearly obvious and very useful integral representation of the logarithm of an operator ([6] p. 51):

$$\ln A = \int_0^{+\infty}\left(\frac{1}{1+t} - \frac{1}{t+A}\right)\mathrm{d}t.$$

By iterating this formula we get also

$$\begin{aligned}\ln(A+B) - \ln A &= \int_0^{+\infty}\frac{1}{t+A}\,B\,\frac{1}{t+A+B}\,\mathrm{d}t\\ &= \int_0^{+\infty}\frac{1}{t+A}\,B\,\frac{1}{t+A}\left(1 - B\,\frac{1}{t+A+B}\right)\mathrm{d}t\,.\end{aligned}$$

These two formulae and stochastic calculus rules allow to write

$$\begin{aligned}\mathbb{E}_{P_T}\left[\mathrm{Tr}\left\{\rho^t_{t+\mathrm{d}t}\ln\rho^t_{t+\mathrm{d}t} - \eta_t\ln\eta_t\right\}\right] &= \sum_k \left[S_{\mathrm{q}}(\eta_t) - S_{\mathrm{q}}\big(\tau_k(t)\big)\right]\nu_k(t)\mathrm{d}t\\ -\,\mathrm{d}t\sum_j\int_0^{+\infty}\mathrm{d}u\,\mathrm{Tr}&\left\{\frac{\eta_t}{(u+\eta_t)^2}\,A_j(t)\,\frac{1}{u+\eta_t}\,A_j(t)\right\}\\ +\,\mathrm{d}t\sum_j\int_0^{+\infty}\mathrm{d}u\,\mathrm{Tr}&\left\{\frac{1}{u+\eta_t}\,A_j(t)\,\frac{1}{u+\eta_t}\,A_j(t)\right\}\\ &+\mathrm{d}t\,\mathrm{Tr}\left\{B(t)\left(\ln\eta_t + \int_0^{+\infty}\frac{\eta_t}{(u+\eta_t)^2}\,\mathrm{d}u\right)\right\}.\end{aligned}$$

By computing the integral we get

$$\operatorname{Tr}\left\{B(t)\left(\ln\eta_t+\int_0^{+\infty}\frac{\eta_t}{(u+\eta_t)^2}\,\mathrm{d}u\right)\right\}=\operatorname{Tr}\left\{B(t)\left(\ln\eta_t+\mathbb{1}\right)\right\}$$

$$=\operatorname{Tr}\left\{B(t)\ln\eta_t\right\}=\sum_k\operatorname{Tr}\left\{\left[\nu_k(t)-J_k(t)\right]\eta_t\ln\eta_t\right\}$$

$$+\sum_j\operatorname{Tr}\left\{\left[R_j(t)\eta_tR_j(t)^*-R_j(t)^*R_j(t)\eta_t\right]\ln\eta_t\right\}$$

$$+\sum_l\operatorname{Tr}\left\{\left[L_l(t)\eta_tL_l(t)^*-L_l(t)^*L_l(t)\eta_t\right]\ln\eta_t\right\}$$

and by using the integration by parts with $\frac{1}{(u+\eta_t)^2}=-\frac{\mathrm{d}}{\mathrm{d}u}\frac{1}{u+\eta_t}$ we have also

$$\sum_j\int_0^{+\infty}\mathrm{d}u\operatorname{Tr}\left\{\frac{1}{u+\eta_t}\,A_j(t)\,\frac{1}{u+\eta_t}\,A_j(t)-\frac{\eta_t}{(u+\eta_t)^2}\,A_j(t)\,\frac{1}{u+\eta_t}\,A_j(t)\right\}$$

$$=\sum_j\int_0^{+\infty}\mathrm{d}u\;\operatorname{Tr}\left\{A_j(t)\,\frac{1}{(u+\eta_t)^2}\,A_j(t)\,\frac{u}{u+\eta_t}\right\}$$

$$=\sum_j\int_0^{+\infty}\mathrm{d}u\;\operatorname{Tr}\left\{A_j(t)\,\frac{\eta_t}{(u+\eta_t)^2}\,A_j(t)\,\frac{1}{u+\eta_t}\right\}$$

$$=\frac{1}{2}\sum_j\int_0^{+\infty}\mathrm{d}u\operatorname{Tr}\left\{\frac{1}{u+\eta_t}\,A_j(t)\,\frac{1}{u+\eta_t}\,A_j(t)\right\}.$$

From the previous formulae we get

$$-\frac{\mathrm{d}}{\mathrm{d}u}\,\mathbb{E}_{P_T}[S_{\mathrm{q}}(\rho_u^t)]\Big|_{u=t^+}=\sum_k\operatorname{Tr}\left\{\mathcal{J}_k(t)[\eta_t]\ln\frac{\mathcal{J}_k(t)[\eta_t]}{\nu_k(t)}-J_k(t)\eta_t\ln\eta_t\right\}$$

$$+\sum_j\operatorname{Tr}\left\{R_j(t)\eta_t\big[R_j(t)^*,\,\ln\eta_t\big]\right\}+\sum_l\operatorname{Tr}\left\{L_l(t)\eta_t\big[L_l(t)^*,\,\ln\eta_t\big]\right\}$$

$$+\frac{1}{2}\sum_j\Bigg(\int_0^{+\infty}\mathrm{d}u\operatorname{Tr}\Bigg\{R_j(t)^*\frac{\eta_t}{u+\eta_t}R_j(t)^*\frac{\eta_t}{u+\eta_t}+\frac{\eta_t}{u+\eta_t}R_j(t)\frac{\eta_t}{u+\eta_t}R_j(t)$$

$$+\frac{2}{u+\eta_t}R_j(t)\frac{{\eta_t}^2}{u+\eta_t}R_j(t)^*\Bigg\}-n_j(t)^2\Bigg).$$

But we have

$$\operatorname{Tr}\left\{R_j(t)\eta_t\big[R_j(t)^*,\,\ln\eta_t\big]\right\}+\int_0^{+\infty}\mathrm{d}u\operatorname{Tr}\left\{\frac{1}{u+\eta_t}R_j(t)\frac{{\eta_t}^2}{u+\eta_t}R_j(t)^*\right\}$$

$$=\int_0^{+\infty}\mathrm{d}u\operatorname{Tr}\Bigg\{\frac{1}{u+\eta_t}R_j(t)\frac{{\eta_t}^2}{u+\eta_t}R_j(t)^*-\frac{1}{u+\eta_t}R_j(t)\eta_tR_j(t)^*$$

$$+\frac{1}{u+\eta_t}R_j(t)\frac{\eta_t}{u+\eta_t}R_j(t)^*\Bigg\}$$

$$=\int_0^{+\infty}\mathrm{d}u\operatorname{Tr}\left\{-\frac{u}{u+\eta_t}R_j(t)\frac{\eta_t}{u+\eta_t}R_j(t)^*+\frac{1}{u+\eta_t}R_j(t)\frac{\eta_t}{u+\eta_t}R_j(t)^*\right\}$$

$$=\int_0^{+\infty}\mathrm{d}u\operatorname{Tr}\left\{\frac{\eta_t}{u+\eta_t}R_j(t)\frac{\eta_t}{u+\eta_t}R_j(t)^*\right\}.$$

Then, we have the final expression

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}u}\,\mathbb{E}_{P_T}[S_{\mathrm{q}}(\rho_u^t)]\Big|_{u=t^+} &= \sum_k \mathrm{Tr}\left\{\mathcal{J}_k(t)[\eta_t]\ln\frac{\mathcal{J}_k(t)[\eta_t]}{\nu_k(t)} - J_k(t)\eta_t\ln\eta_t\right\} \\
&+\frac{1}{2}\sum_j\Bigg(\int_0^{+\infty}\mathrm{d}u\,\mathrm{Tr}\left\{\frac{\eta_t}{u+\eta_t}\left(R_j(t)+R_j(t)^*\right)\frac{\eta_t}{u+\eta_t}\left(R_j(t)+R_j(t)^*\right)\right\} \\
&\qquad - n_j(t)^2\Bigg) + \sum_l \mathrm{Tr}\left\{L_l(t)\eta_t\left[L_l(t)^*,\,\ln\eta_t\right]\right\}. \qquad (57)
\end{aligned}
$$

Analogously we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}_{P_T}[S_{\mathrm{q}}(\rho_t)] &= -\sum_k \mathbb{E}_{P_T}\left[\mathrm{Tr}\left\{\mathcal{J}_k(t)[\rho_t]\ln\frac{\mathcal{J}_k(t)[\rho_t]}{\mu_k(t)} - J_k(t)\rho_t\ln\rho_t\right\}\right] \\
&-\frac{1}{2}\sum_j\mathbb{E}_{P_T}\Bigg[\int_0^{+\infty}\mathrm{d}u\,\mathrm{Tr}\left\{\frac{\rho_t}{u+\rho_t}\left(R_j(t)+R_j(t)^*\right)\frac{\rho_t}{u+\rho_t}\left(R_j(t)+R_j(t)^*\right)\right\} \\
&\qquad - m_j(t)^2\Bigg] - \sum_l \mathbb{E}_{P_T}\left[\mathrm{Tr}\left\{L_l(t)\rho_t\left[L_l(t)^*,\,\ln\rho_t\right]\right\}\right]. \qquad (58)
\end{aligned}
$$

By (57), (58), (47) we get the statement of the Proposition. □

Corollary 8. *A sufficient condition to have the equality in the main bound*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\,S_{\mathrm{c}}(p_t\|q_t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}_{P_T}[S_{\mathrm{q}}(\rho_t)] - \frac{\mathrm{d}}{\mathrm{d}u}\,\mathbb{E}_{P_T}[S_{\mathrm{q}}(\rho_u^t)]\Big|_{u=t^+} \qquad (59)
$$

is to have P_T-a.s. in ω ($T\geq t$), $\forall r,k,j,l$,

$$
[V_k^r(t),\rho_t(\omega)]=0\,,\quad [R_j(t)+R_j(t)^*,\rho_t(\omega)]=0\,,\quad [L_l(t),\rho_t(\omega)]=0\,. \qquad (60)
$$

Proof. By the commutation relations (60) we get

$$
\begin{aligned}
&J_k(t)\rho_t\left(\ln\rho_t-\ln\eta_t\right) - \mathcal{J}_k(t)[\rho_t]\left(\ln\mathcal{J}_k(t)[\rho_t]-\ln\mathcal{J}_k(t)[\eta_t]\right) \\
&\qquad = J_k(t)\rho_t\left(\ln\rho_t-\ln\eta_t-\ln J_k(t)\rho_t+\ln J_k(t)\eta_t\right)=0
\end{aligned}
$$

and the first term in Eq. (56) vanishes.

By Eq. (60) also the last term in Eq. (56) is zero because it explicitly involves vanishing commutators.

Finally, let us consider one of the terms in the j-sum. We have

$$
\begin{aligned}
&\mathbb{E}_{P_T}\left[\int_0^{+\infty}\mathrm{d}u\,\mathrm{Tr}\left\{-R_j(t)^*\frac{\rho_t}{u+\rho_t}R_j(t)^*\frac{\rho_t}{u+\rho_t}\right\}\right] \\
&\qquad = -\,\mathbb{E}_{P_T}\left[\int_0^{+\infty}\mathrm{d}u\,\mathrm{Tr}\left\{R_j(t)^{*2}\frac{{\rho_t}^2}{(u+\rho_t)^2}\right\}\right] \\
&\qquad = -\,\mathbb{E}_{P_T}\left[\mathrm{Tr}\left\{R_j(t)^{*2}\rho_t\right\}\right] = -\,\mathrm{Tr}\left\{R_j(t)^{*2}\eta_t\right\}.
\end{aligned}
$$

The opposite result comes out from the corresponding term with η and the j-sum vanishes too. □

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