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"Francesco Brioschi"
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Boccellari, T.; Tomarelli, F.

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Piazza Leonardo da Vinci, 32 - 20133 Milano (Italy)

Generic uniqueness of minimizer for Blake & Zisserman functional

Tommaso Boccellari - Franco Tomarelli
Politecnico di Milano - Dipartimento di Matematica "Francesco Brioschi"

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ABSTRACT: Blake-Zisserman functional $F^g_{\alpha,\beta}$ *achieves a finite minimum for any pair of real numbers* α, β *such that* $0 < \beta \leq \alpha \leq 2\beta$ *and any* $g \in L^2(0,1)$.
Uniqueness of minimizer does not hold in general. Nevertheless, in the $1D$ *case uniqueness of minimizer is a generic property for* $F^g_{\alpha,\beta}$ *in the sense that it holds true for almost all gray levels data* g *and parameters* α, β*: we prove that, whenever* $\frac{\alpha}{\beta} \notin \mathbb{Q}$*, the minimizer is unique for any* g *belonging to a* G_δ *subset of* $L^2(0,1)$ *dependent on* α *and* β.

Contents

1 Introduction

Image segmentation plays an important role in computer vision and in the understanding of biological vision. The first variational model for image segmentation was proposed by D.Mumford & J.Shah [21], [22] and studied by

several authors ([16], [17], [18], [20]). Blake & Zisserman showed some inconvenient related to Mumford & Shah approach and introduced an alternative way ([3]) to translate image segmentation task into a variational formulation which actually is a free gradient discontinuity problem. Blake & Zisserman approach was studied in [5], [6], [7], [8], [9], [10], [13], [12], [15].
Here we focus the question of uniqueness restricting the analysis to the 1 dimensional Blake & Zisserman functional $F^g_{\alpha,\beta}$ defined as follows.
Given $g \in L^2(0,1)$, $\alpha, \beta \in \mathbb{R}$ and $u \in \mathcal{H}^2$ we set $F^g_{\alpha,\beta} : \mathcal{H}^2 \to [0,+\infty)$

$$F^g_{\alpha,\beta}(u) = \int_0^1 |\ddot{u}(x)|^2 \, dx + \int_0^1 |u(x) - g(x)|^2 \, dx + \alpha \sharp (S_u) + \beta \sharp (S_{\dot{u}} \backslash S_u). \quad (1.1)$$

Here and in the sequel for all $u \in L^2(0,1)$, $\dot{u}$ denotes the absolutely continuous part of the distributional derivative u' of u, $\ddot{u}$ denotes the absolutely continuous part of $(\dot{u})'$, $S_u \subseteq (0,1)$ denotes the approximate discontinuity set ([1]) of u and $S_{\dot{u}} \subseteq (0,1)$ the approximate discontinuity set of $\dot{u}$, $\mathcal{H}^2$ denotes the set of $v \in L^2(0,1)$ such that S_v and $S_{\dot{v}}$ are finite sets and $v \in H^2(I)$ for any interval $I \subseteq (0,1) \setminus (S_v \cup S_{\dot{v}})$, eventually $\sharp$ denotes the counting measure. We will call singular set of u the set $S_u \cup S_{\dot{u}}$. We set

$$m^g(\alpha, \beta) = \inf\{F^g_{\alpha,\beta}(u) \quad \forall u \in \mathcal{H}^2\},$$

$$\operatorname{argmin} F^g_{\alpha,\beta} = \{u \in \mathcal{H}^2 : \quad F^g_{\alpha,\beta}(u) = m^g(\alpha, \beta)\}.$$

We recall that $\operatorname{argmin} F^g_{\alpha,\beta} \neq \emptyset$ whenever the two following conditions are satisfied ([15]):

$$0 < \beta \leq \alpha \leq 2\beta \quad (1.2)$$

$$g \in L^2(0,1). \quad (1.3)$$

Nevertheless minimizers are not unique in general. In [4] Section 3 we exhibit examples of $g \in L^2(0,1)$ and α, β fulfilling (1.2) such that $F^g_{\alpha,\beta}$ has more than one minimizer (see Counterexample 3.1, 3.2, 3.3 of [4]). Moreover we give an example of a non empty open subset $\mathcal{N} \subseteq L^2(0,1)$ such that for any $g \in \mathcal{N}$ there are α and β satisfying (1.2) and $\sharp (\operatorname{argmin} F^g_{\alpha,\beta}) \geq 2$ (see Counterexample 3.4 of [4]). Anyway $m^g(\alpha, \beta)$ continuously depends on g, α, β ([4], Theorem. 2.3).
The main result of this paper is the following:

Theorem 1.1 *For any α and β with $0 < \beta \leq \alpha \leq 2\beta$ and $\frac{\alpha}{\beta} \notin \mathbb{Q}$, there is a G_δ set (countable intersection of dense open sets) $E_{\alpha,\beta} \subseteq L^2(0,1)$ such that for any $g \in E_{\alpha,\beta}$ we have $\sharp (\operatorname{argmin} F^g_{\alpha,\beta}) = 1$.*

As usual we denote by G_δ the intersection of at most countably many dense open sets. Since the complement in $L^2(0,1)$ of a G_δ subset and the complement in $\mathbb{R}^2$ of the set $\{(\alpha,\beta)\in\mathbb{R}^2:\ \alpha/\beta\notin\mathbb{Q}\}$ are sets of first category, Theorem 1.1 says that uniqueness for minimizers of $F^g_{\alpha,\beta}$ is a generic property. The whole picture we obtain about generic uniqueness and counterexamples is coherent with the presence of instable patterns, each of them corresponding to a bifurcation of optimal segmentation under variation of parameters: this fact is natural since suitable combinations of alfa and beta are related to contrast threshold, crease detection, "luminance sensitivity", resistance to noise and double-edge detection (see [BZ]).
The absolutely continuous part of functional (1.1) will be denoted by

$$\mathcal{F}^g(u)=\int_0^1 |\ddot{u}(x)|^2\,dx+\int_0^1 |u(x)-g(x)|^2\,dx. \tag{1.4}$$

Uniqueness of solution and its coincidence with the datum are shown in case of piecewise affine datum g, under suitable smallness assumption on α, β (Lemma 3.7). In the general case the discussion about uniqueness of minimizers is outlined as follows. We identify partitions $\{q_i\}_{i=0}^{\mathsf{Q}}$ of the interval $(0,1)$ with vectors $\mathbf{q}=(q_i)_{i=1}^{\mathsf{Q}}$ such that $0<q_1<\ldots<q_{\mathsf{Q}}<1$. In case of a partition associated to a singular set ($\{q_i\}_{i=0}^{\mathsf{Q}}=S_u\cup S_{\dot{u}}$) the vector $\mathbf{q}$ is called location of the singular set and the ordered attribute of belonging to S_u or $S_{\dot{u}}$ is called quality (see Definition 3.1). Theorem 3.2 states that if u is a minimizer of $F^g_{\alpha,\beta}$ then it is the *unique minimizer* of $F^g_{\alpha,\beta}(w)$ among w in $\mathcal{H}^2$ fulfilling $S_w=S_u$ and $S_{\dot{w}}=S_{\dot{u}}$; Euler equations (recalled in Theorem 2.1) may lack uniqueness (see Counterexamples 3.1-3.4 in [4]), moreover even with fixed singular set the whole system of Euler equations is overdetermined. For these reasons we introduce Problem 3.3, related to a selection of Euler equations, where we prescribe two parameters (still called location and quality, see (3.3)) associated to suitable transition conditions: this is motivated by the fact that in case of minimizers the two notions of location and quality for function and Problem 3.3 coincide). Theorems 3.8, 3.9, 3.11 prove that system (3.2) has unique solution b which depends linearly on g and has energy $\mathcal{F}^g(b)$ quadratic on g. Theorem 3.14 shows analytic dependence of energy (1.4) with respect to location of singular set (varying on open cells of CW structure induced by piecewise affine datum g). Lemma 3.19 shows continuous dependence for solution b of Problem 3.3 with respect to perturbations of the singular set of a piecewise affine datum g. In Section 4 we introduce two auxiliary problems: Problem 4.2, which is equivalent to minimization of functional (1.1) in case of continuous piecewise affine datum g, and Problem 4.4 which is related to a different selection of Euler equations, in such a way that common solutions of Problems 3.3 and 4.4 fulfill the whole system of

Euler equations (i)-(iv),(vi) in Theorem 2.1. In Section 5 we introduce the excess functional E which vanishes only on common solutions of both Problems 3.3 and 4.4; by exploiting this tool, for suitable integers m and n, we define subsets of $\mathbb{R}^m \times \mathbb{R}^n$ measuring how many pairs $(g, \mathbf{t})$ exist such that g is a continuous piecewise affine function with no more than m creases and $\mathbf{t} \in \mathbb{R}^n$ is the ordered singular set of a solution of Problem 4.2 with datum g: we prove (Theorem 5.4) that these subsets are finite CW complexes of dimension m. In Section 6 we prove that the set of all affine data related to suitably refined partitions and exhibiting non uniqueness of minimizer with different arrangement and same prescribed cardinality of singular set has null m dimensional Lebesgue measure (Theorem 6.4) where $m = \mathsf{Q} + 2$ and Q is the cardinality of the partition.
In Section 7 the main result (Theorem 1.1) is deduced as a consequence of the following intermediate claim (Theorem 7.2): for any α, β fulfilling (1.2) and α/β irrational, the set of data g with uniqueness of minimizer for $F^g_{\alpha,\beta}$ is dense in L^2. Theorem 7.2 is achieved by exploiting several technical steps proven in Sections 3 - 7: the idea is to show that, for sufficiently fine partitions $\mathbf{q} = (q_i)_{i=1}^{\mathsf{Q}}$ of $(0, 1)$, the set of continuous piecewise linear functions g associated to $\mathbf{q}$ such that $F^g_{\alpha,\beta}$ has more than one minimizer is small; here small means that its $(\mathsf{Q} + 2)$-dimensional Lebesgue measure is zero, after identification of continuous piecewise linear functions associated to the partition $\mathbf{q}$ and the euclidean space $\mathbb{R}^{\mathsf{Q}+2}$.
We emphasize that, with continuous piecewise affine datum g, jump and crease points of minimizers are not necessarily localized among those of g (see Section 4 of [4]): hence the techniques used for proving the generic uniqueness for Mumford-Shah functional in [2] cannot be applied to Blake-Zisserman functional. For this reason we follow a different strategy, by carefully exploiting some intersection properties between real analytic varieties.

2 Euler equations

In this section we recall the whole set of Euler equations and the compliance identity for minimizers of the functional $F^g_{\alpha,\beta}$ (Theorems 2.1, 2.1 of [4]). For the multidimensional situation ($n \geq 2$) we refer to [7], [10] and [12].

Theorem 2.1 (Euler equations) *If* (1.2) *and* (1.3) *hold true then every* *u which minimizes* (1.1) *in* $\mathcal{H}^2$ *is also a solution of the following system:*

$$\begin{cases} (i) & u'''' + u = g & \text{on } (0,1) \setminus (S_{\dot u} \cup S_u) \\ (ii) & \ddot u_+ = \ddot u_- = 0 & \text{on } S_{\dot u} \cup S_u \cup \{0,1\} \\ (iii) & \dddot u_+ = \dddot u_- = 0 & \text{on } S_u \cup \{0,1\} \\ (iv) & \dddot u_+ = \dddot u_- & \text{on } S_{\dot u} \\ (v) & \frac{1}{2}(u_+ + u_-) = g & \text{on } S_u \cap \{\text{continuity points of } g\} \end{cases}$$

In (ii) *and* (iii) *we conventionally set* $\ddot u_-(0) = \ddot u_+(1) = 0 = \dddot u_+(1) = \dddot u_-(0)$. *If, in addition to* (1.2) *and* (1.3), g *is continuous piecewise affine then* (iii),(iv) *improve as follows*

$$(vi) \qquad \dddot u_+ = \dddot u_- = 0 \qquad \text{on } (S_u \cup S_{\dot u} \cup \{0,1\}) \setminus S_{\dot g}$$

If, in addition to (1.2) *and* (1.3), $\alpha = \beta$ *then* (iii),(iv) *improve as follows*

$$(vii) \qquad \dddot u_+ = \dddot u_- = 0 \qquad \text{on } S_u \cup S_{\dot u} \cup \{0,1\}\,.$$

By summarizing:

$$(viii) \qquad \ddot u \in H^2(0,1) \quad \text{and} \quad (\ddot u)'' + u = g \ \text{ in } \mathcal{D}'(0,1).$$

Proof. Properties (i)-(v), (vii), $(viii)$ are proven in [4] Section 2 Theorem 2.1. Property (vi) is a straightforward consequence of (iii) and of (3.24) which will be proved in Lemma 3.18. □

Theorem 2.2 (Compliance identity) *Assume* (1.2) *and* (1.3). *Then we have, for any* $u \in \mathcal{H}^2$ *fulfilling the Euler equations* (i)-(iv) *of Theorem 2.1:*

$$\mathcal{F}^g(u) = \int_0^1 (g^2 - gu)\,dx, \qquad \int_0^1 |\ddot u|^2\,dx = \int_0^1 (gu - u^2)\,dx \tag{2.1}$$

and

$$F^g_{\alpha,\beta}(u) \;=\; \int_0^1 (g^2 - gu)\,dx \;+\; \alpha\,\sharp\,(S_u) \;+\; \beta\,\sharp\,(S_{\dot u} \setminus S_u). \tag{2.2}$$

In particular any u *minimizing* $F^g_{\alpha,\beta}$ *over* $\mathcal{H}^2$ *fulfills* (2.1) *and* (2.2).

Theorem 2.3 *For any possibly discontinuous piecewise affine function* g *with* $S_g \cup S_{\dot g} \neq \varnothing$ *we introduce the subset* $\mathcal{S}[g]$ *of* $\mathcal{H}^2$ *as follows:*
$v \in \mathcal{S}[g]$ *if and only if, either*

$$(i) \begin{cases} \sharp\,(S_{\dot v} \setminus S_v) \;<\; \sharp\,(S_{\dot g} \setminus S_g) \\ \sharp\,(S_v) \;<\; \sharp\,(S_g) + \sharp\,(S_{\dot g} \setminus S_g) - \sharp\,(S_{\dot v} \setminus S_v)\,, \end{cases}$$

or

$$(ii)\begin{cases} \sharp\,(S_v) \;<\; \sharp\,(S_g) \\ \sharp\,(S_{\dot v}\setminus S_v) \;<\; \sharp\,(S_{\dot g}\setminus S_g) + 2(\sharp\,(S_g) - \sharp\,(S_v)). \end{cases}$$

Then $\mathcal{S}[g] \neq \varnothing$ *and*

$$\inf_{v\in\mathcal{S}[g]} \mathcal{F}^g(v) \;>\; 0. \tag{2.3}$$

These results are proven in [4] Section 2.

3 Notation and preliminary results

In this section we fix the notation used throughout the following sections in the proof of generic uniqueness and show some preliminary results.

$\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $r > 0$ let $B_r(x) = \{y \in \mathbb{R}^n : \quad |y - x| < r\}$, for any $A, B \subseteq \mathbb{R}^n$ let $\mathrm{dist}(A, B) = \inf\{|a - b| \;\; a \in A,\; b \in B\}$, ∂A denotes the topological boundary of A. We denote by $L^2(0,1)$ the space of all measurable square integrable real valued functions and by $S_u \subseteq (0,1)$ the approximate discontinuity set of u whenever $u \in L^2(0,1)$; we set

$$H^k(a,b) = \{v \in L^2(a,b) \colon \quad v^{(h)} \in L^2(a,b),\; 0 \le h \le k\},$$

$$\mathcal{H}^2 = \{v \in L^2 \colon \quad S_v \cup S_{\dot v} \text{ is finite, } v \in H^2(I)\;\forall \text{ interval } I \subseteq (0,1)\backslash(S_v \cup S_{\dot v})\}.$$

For any $u \in \mathcal{H}^2$ we call *jump points* the elements of $S_u \subseteq (0,1)$ and *crease points* the elements of $(S_{\dot u} \setminus S_u) \subseteq (0,1)$.

Let $\Omega_{\mathsf{j},\mathsf{c}}$ be the set of all $\binom{\mathsf{j}+\mathsf{c}}{\mathsf{j}}$ orderings of $\mathsf{j}+\mathsf{c}$ distinct points in $(0,1)$ such that j among them are (undistinguished jump points) marked with J, and c among them are (undistinguished crease points) marked with C: for any $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$, $\mathsf{T} = \mathsf{j}+\mathsf{c}$, and $l = 1, ..., \mathsf{T}$, we set $\sigma_l = \mathsf{J}$ if the l-th point is a jump point, $\sigma_l = \mathsf{C}$ if the l-th point is a crease point, moreover we set $\sigma_0 = \mathsf{J}$ and $\sigma_{\mathsf{T}+1} = \mathsf{J}$ by convention.

In this way each element of $u \in \mathcal{H}^2$ with j jump points and c crease points singles out exactly one element of $\Omega_{\mathsf{j},\mathsf{c}}$, while several functions in $\mathcal{H}^2$ (with the same ordering of jumps and creases) may correspond to one single element of $\Omega_{\mathsf{j},\mathsf{c}}$.

We are interested in elements of $\mathcal{H}^2$ with fixed numbers j of jump points and c of crease points whose location is free in $(0,1)$: for any integer T (representing the sum $\mathsf{T} = \mathsf{j}+\mathsf{c}$) we introduce the open connected subset A_{T} of $(0,1)^{\mathsf{T}}$ as

$$A_{\mathsf{T}} = \{\mathbf{t} = (t_1, ..., t_{\mathsf{T}}) \in (0,1)^{\mathsf{T}} \colon \quad t_1 < ... < t_{\mathsf{T}}\}.$$

A_{T} is identified in a natural way with the set of partitions of $(0,1)$ with cardinality T. Abusing notation, whenever needed, we write $a \in \mathbf{t}$ to mean $a \in \{t_i\}_{i=1}^{\mathsf{T}}$ while $t_0 = 0$, $t_{\mathsf{T}+1} = 1$ are always understood.
We observe that each $u \in \mathcal{H}^2$ uniquely defines a pair

$$(\mathbf{t}, \sigma) = (\mathbf{t}(u), \sigma(u)) \in A_{\sharp(S_u \cup S_{\dot{u}})} \times \Omega_{\sharp(S_u), \sharp(S_u \setminus S_{\dot{u}})}$$

such that

$$S_u \cup S_{\dot{u}} = \mathbf{t} = \{t_l(u)\}_{l=1}^{\sharp(S_u \cup S_{\dot{u}})}$$

and $t_l(u)$ is a crease point if $\sigma_l(u) = \mathsf{C}$ and a jump point if $\sigma_l(u) = \mathsf{J}$.

Definition 3.1 *For any function $u \in \mathcal{H}^2$ we call:*

- *location of u: the vector $\mathbf{t} = \mathbf{t}(u) \in A_{\mathsf{T}}$ associated to $S_u \cup S_{\dot{u}}$,*
- *quality of u: the element $\sigma = \sigma(u) \in \Omega_{\mathsf{j},\mathsf{c}}$ describing the ordered kind of singularity associated to $S_u \cup S_{\dot{u}}$,*
- *arrangement of u: the pair $(\mathbf{t}(u), \sigma(u))$ location of u and quality of u.*

At first we deal with minimizers of $F^g_{\alpha,\beta}$ with exactly j jump points and c crease points with prescribed arrangement. Then we will examine candidate minimizers having arrangement compatible with (possibly different) a prescribed arrangement.

Theorem 3.2 *Assume $u \in \operatorname{argmin} F^g_{\alpha,\beta}$. Define*

$$\mathcal{H}^2_u = \{v \in \mathcal{H}^2 : \quad \mathbf{t}(v) = \mathbf{t}(u),\ \sigma(v) = \sigma(u)\},$$

then uniqueness hold true on $\mathcal{H}^2_u$:

$$\operatorname*{argmin}_{v \in \mathcal{H}^2_u} F^g_{\alpha,\beta} = \{u\}.$$

Moreover, for any fixed $w \in \mathcal{H}^2$ with $S_w \subseteq S_u$ and $S_{\dot{w}} \subseteq S_u \cup S_{\dot{u}}$ there is a convex neighborhood $U = U(w)$ of 0 in $\mathbb{R}$ such that the Euler equations (i)-(iv) of Theorem 2.1 are not satisfied by any $u + \lambda w$ with $\lambda \in U \setminus \{0\}$.

Proof. If $w \in \mathcal{H}^2$ and $S_w \nsubseteq S_u$ or $S_{\dot{w}} \nsubseteq S_u \cup S_{\dot{u}}$ then $u + \lambda w \notin \mathcal{H}^2_u$ for any $\lambda \in \mathbb{R} \setminus \{0\}$.
In order to perform variations in $\mathcal{H}^2_u$ we have to test only functions $w \in \mathcal{H}^2$ with $S_w \subseteq S_u$ and $S_{\dot{w}} \subseteq S_u \cup S_{\dot{u}}$. Fix a w fulfilling these properties, then there is a finite (possibly empty) set $P = P(w) \subseteq \mathbb{R} \setminus \{0\}$ such that

$$u + \lambda w \notin \mathcal{H}^2_u \qquad \forall \lambda \in P$$

because of possible cancellations in $S_{\dot{u}+\lambda\dot{w}} \cup S_{u+\lambda w}$. Nevertheless $0 \notin P$ and

$$u + \lambda w \in \mathcal{H}^2_u \qquad \forall \lambda \text{ in the open set } \mathbb{R} \setminus P.$$

Set $\varphi(\lambda) = \mathcal{F}^g(u+\lambda w)$. $\mathcal{F}^g$ is strictly convex in $\mathcal{H}^2$, hence φ is strictly convex in $\mathbb{R}$, hence φ is strictly convex in the maximal open interval $U = U(w)$ of $\mathbb{R} \setminus P$ containing 0. Then 0 belongs to the interior of U, hence 0 is the unique minimum point of φ in $\mathbb{R} \setminus P$, say

$$\mathcal{F}^g(u) = \min_{\lambda \in \mathbb{R}\setminus P} \mathcal{F}^g(u + \lambda w) = \min_{\lambda \in \mathbb{R}\setminus P} \varphi(\lambda) = \varphi(0).$$

Since $u \in \operatorname{argmin} F^g_{\alpha,\beta}$ and

$$F^g_{\alpha,\beta}(u + \lambda w) = \mathcal{F}^g(u + \lambda w) + \alpha \sharp (S_u) + \beta \sharp (S_{\dot{u}} \setminus S_u) \quad \forall \lambda \in \mathbb{R} \setminus P$$

then

$$\min_{\lambda \in \mathbb{R}\setminus P} F^g_{\alpha,\beta}(u + \lambda w) = \min_{\lambda \in \mathbb{R}\setminus P} \left(\varphi(\lambda) + \alpha \sharp (S_u) + \beta \sharp (S_{\dot{u}} \setminus S_u)\right) = F^g_{\alpha,\beta}(u),$$

and the first part of the thesis is achieved.
For any fixed $w \in \mathcal{H}^2_u$ let $P = P(w)$ and $U = U(w)$ be defined as above and $\lambda \in \mathbb{R} \setminus P$: then $\psi(\lambda) := F^g_{\alpha,\beta}(u + \lambda w) = \varphi(\lambda) + \alpha \sharp (S_u) + \beta \sharp (S_{\dot{u}} \setminus S_u)$ for any $\lambda \in \mathbb{R} \setminus P$ since $\sharp (S_{u+\lambda w}) = \sharp (S_u)$ and $\sharp (S_{\dot{u}+\lambda\dot{w}} \setminus S_{u+\lambda w}) = \sharp (S_{\dot{u}} \setminus S_u)$ for any $\lambda \in \mathbb{R} \setminus P$.
The previous argument entails that ψ is strictly convex in $U = U(w)$ and $\psi(\lambda)$ has unique strict minimizer at $\lambda = 0$ with respect to U hence there are finite values of

$$\psi'_{\pm}(\lambda) \neq 0 \qquad \forall \lambda \in U \setminus \{0\}. \tag{3.1}$$

From (3.1) we deduce the second part of the thesis. Arguing by contradiction, we assume that for some fixed $w \in \mathcal{H}^2_u$ and $\lambda \in U(w) \setminus \{0\}$ the function $u + \lambda w$ fulfills the Euler equations (i)-(iv) of Theorem 2.1. Then $u + \lambda w \in H^4((0, 1) \setminus (S_u \cup S_{\dot{u}}))$ and by labelling t_l, $l = 1, ..., \mathsf{T}$, the ordered finite set $S_u \cup S_{\dot{u}}$, and $t_0 = 0$, $t_{\mathsf{T}+1} = 1$, we deduce the existence of $\psi'(\lambda)$:

$$\begin{aligned}
\psi'(\lambda) = \tfrac{d}{d\lambda} &\left(\textstyle\int_0^1 (\ddot{u}(x) + \lambda \ddot{w}(x))^2 \, dx + \int_0^1 (u(x) + \lambda w(x) - g(x))^2 \, dx \right) = \\
&2 \textstyle\int_0^1 \Big(\ddot{u}(x) + \lambda \ddot{w}(x) \Big) \ddot{w}(x) \, dx + 2 \int_0^1 \Big(u(x) + \lambda w(x) - g(x) \Big) w(x) \, dx = \\
&-2 \textstyle\int_0^1 \Big(\dddot{u}(x) + \lambda \dddot{w}(x) \Big) \dot{w}(x) \, dx + 2 \int_0^1 \Big(u(x) + \lambda w(x) - g(x) \Big) w(x) \, dx + \\
&\sum_{l=0}^{\mathsf{T}} \Big(\big(\ddot{u}(t_{l+1}) + \lambda \ddot{w}(t_{l+1}) \big) \dot{w}(t_{l+1}) - \big(\ddot{u}(t_l) + \lambda \ddot{w}(t_l) \big) \dot{w}(t_l) \Big) =
\end{aligned}$$

$$
\begin{aligned}
&2\int_0^1 \Big(\ddddot{u}(x) + \lambda \ddddot{w}(x)\Big) w(x)\,dx + 2\int_0^1 \Big(u(x) + \lambda w(x) - g(x)\Big) w(x)\,dx + \\
&\big(\ddot{u}_-(1) + \lambda \ddot{w}_-(1)\big)\dot{w}_-(1) - \big(\ddot{u}_+(0) + \lambda \ddot{w}_+(0)\big)\dot{w}_+(0) + \\
&\sum_{t \in S_u \cup S_{\dot{u}}} \Big(\big(\ddot{u}_-(t) + \lambda \ddot{w}_-(t)\big)\dot{w}_-(t) - \big(\ddot{u}_+(t) + \lambda \ddot{w}_+(t)\big)\dot{w}_+(t)\Big) + \\
&-\sum_{l=0}^{\mathsf{T}} \Big(\big(\dddot{u}(t_{l+1}) + \lambda \dddot{w}(t_{l+1})\big) w(t_{l+1}) - \big(\dddot{u}(t_l) + \lambda \dddot{w}(t_l)\big) w(t_l)\Big) = \\
&2\int_0^1 \Big(\big(\ddddot{u}(x) + \lambda \ddddot{w}(x)\big) + \big(u(x) + \lambda w(x)\big) - g(x)\Big) w(x)\,dx + \\
&\big(\ddot{u}_-(1) + \lambda \ddot{w}_-(1)\big)\dot{w}_-(1) - \big(\ddot{u}_+(0) + \lambda \ddot{w}_+(0)\big)\dot{w}_+(0) + \\
&\qquad \sum_{t \in S_u \cup S_{\dot{u}}} \Big(\big(\ddot{u}_-(t) + \lambda \ddot{w}_-(t)\big)\dot{w}_-(t) - \big(\ddot{u}_+(t) + \lambda \ddot{w}_+(t)\big)\dot{w}_+(t)\Big) + \\
&\qquad \big(\dddot{u}_+(0) + \lambda \dddot{w}_+(0)\big) w_+(0) - \big(\dddot{u}_-(1) + \lambda \dddot{w}_-(1)\big) w_-(1) + \\
&\qquad \sum_{t \in S_u \cup S_{\dot{u}}} \Big(\big(\dddot{u}_+(t) + \lambda \dddot{w}_+(t)\big) w_+(t) - \big(\dddot{u}_-(t) + \lambda \dddot{w}_-(t)\big) w_-(t)\Big).
\end{aligned}
$$

Since $u + \lambda w$ satisfies the Euler equations (i)-(iv) of Theorem 2.1, by substitution in the above identity we obtain the existence of $\psi'(\lambda)$ for the chosen $\lambda \in U \setminus \{0\}$ and $\psi'(\lambda) = 0$, which contradicts (3.1). □

We introduce and study the following auxiliary problem in order to overcome the possible lack of uniqueness of the solutions of Euler equations.

Problem 3.3 *Given* $\mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ *and* $g \in L^2(0,1)$, *find* $b \in \mathcal{H}^2(0,1)$ *s.t.* $b = b_l$ *on* (t_l, t_{l+1}) *where*

$$
\left.
\begin{array}{lll}
(i) & b_l'''' + b_l = g & \text{on } (t_l, t_{l+1}) \text{ for } l = 0, 1, ..., \mathsf{T} \\
(ii) & b_l''(t_l) = b_l''(t_{l+1}) = 0 & \text{for } l = 0, 1, ..., \mathsf{T} \\
(iii) & b_l'''(t_l) = 0 & \text{if } l = 0 \text{ or } (\sigma_l = \mathsf{J},\ l \in \{1, ..., \mathsf{T}\}) \\
(iv) & b_l'''(t_{l+1}) = 0 & \text{if } l = \mathsf{T} \text{ or } (\sigma_{l+1} = \mathsf{J},\ l \in \{1, ..., \mathsf{T}\}) \\
(v) & b_{l-1}'''(t_l) = b_l'''(t_l) & \text{if } l \in \{1, ..., \mathsf{T}\} \text{ and } \sigma_l = \mathsf{C} \\
(vi) & b_{l-1}(t_l) = b_l(t_l) & \text{if } l \in \{1, ..., \mathsf{T}\} \text{ and } \sigma_l = \mathsf{C}
\end{array}
\right\} \tag{3.2}
$$

$\mathbf{t}$ *and* σ *are called respectively location and quality of Problem 3.3.* (3.3)

We emphasize that any solution b of Problem 3.3 is neither forced to have a jump at t_l when $\sigma_l = \mathsf{J}$, nor to have a crease when $\sigma_l = \mathsf{C}$ (though this may happen at some or every t_l). Nevertheless location and quality of the solution b are compatible with location and quality of Problem 3.3 in the following sense: $\mathbf{t}(b) \subseteq \mathbf{t}$, $S_b \subseteq \{t_i\colon\ \sigma_i = \mathsf{J}\}$ and $S_{\dot{b}} \setminus S_b \subseteq \{t_i\colon\ \sigma_i = \mathsf{C}\}$.

For any choice of $\mathbf{t} \in A_{\mathsf{T}}$ and of $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ with $\mathsf{j}+\mathsf{c} = \mathsf{T}$, Problem 3.3 amounts for $\mathsf{T}+1$ fourth order O.D.E.s linked by $4(\mathsf{T}+1)$ boundary conditions: in fact (ii) contains $2(\mathsf{T}+1)$ conditions, (iii) and (iv) together contains $2(\mathsf{j}+1)$ conditions, (v) and (vi) together contains $2\mathsf{c}$ conditions. Problem 3.3 is not linear in $\mathbf{t}$, nevertheless b has a nice dependence on $\mathbf{t}$ as we will show in Theorem 3.14.
A priori it is not obvious wether Problem 3.3 has a solution or not for any choice of g, σ, $\mathbf{t}$. Anyway for any fixed $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ and $g \in L^2(0,1)$ we will show (Lemma 3.6) the existence and the uniqueness of a solution for sufficiently many choices of $\mathbf{t} \in A_{\mathsf{T}}$ in order to continue the analysis (actually for any $\mathbf{t} \in A_{\mathsf{T}}$ by Theorem 3.8).

Remark 3.4 *Obviously system* (3.2) *splits in several uncoupled systems at each point* t_l *such that* $\sigma_l = \mathsf{J}$. *Coupling do act only at each* t_l *s.t.* $\sigma_l = \mathsf{C}$

We show that the differential system (3.2) can be replaced by an algebraic linear system whose block structure is fully described by the following preliminary lemma where the uncoupling of (3.2) at points of quality J (jump) is emphasized: the related decomposition (3.5) of b will be exploited with several different choices of d_l in Lemmas 3.11, 3.19, 3.20.

Lemma 3.5 *Fix* $\mathsf{T},\mathsf{j},\mathsf{c} \in \{0,1,2,...\}$, $\mathsf{T} = \mathsf{j}+\mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$, $g \in L^2(0,1)$ *and a solution* d_l *of*

$$d_l'''' + d_l = g \text{ on } (t_l, t_{l+1}) \text{ for any } l \in \{0,...,\mathsf{T}\}. \tag{3.4}$$

Then: $\mathsf{T}+1 \geq \sharp\,(S_b \cup S_{\dot{b}} \cup \{0,1\})$ *and*
1. any solution of Problem 3.3, if it exists, has the form

$$b_l = d_l + \sum_{i=1}^{4} \mathbf{c}_{l,i} w_i \qquad \forall l \in \{0,...,\mathsf{T}\} \tag{3.5}$$

where

$$\begin{aligned} w_1 &= \exp(-x/\sqrt{2})\cos(x/\sqrt{2}) & \qquad w_2 &= \exp(x/\sqrt{2})\cos(x/\sqrt{2}) \\ w_3 &= \exp(-x/\sqrt{2})\sin(x/\sqrt{2}) & \qquad w_4 &= \exp(x/\sqrt{2})\sin(x/\sqrt{2}) \end{aligned} \tag{3.6}$$

are four linearly independent solutions of the homogeneous equation $w''''+w = 0$ *and* $\mathbf{c}_{l,i}$ *are real numbers such that*
2. $\mathbf{c} = (\mathbf{c}_{0,1}, \mathbf{c}_{0,2}, \mathbf{c}_{0,3}, \mathbf{c}_{0,4}, ..., \mathbf{c}_{l,i}, ..., \mathbf{c}_{\mathsf{T},1}, \mathbf{c}_{\mathsf{T},2}, \mathbf{c}_{\mathsf{T},3}, \mathbf{c}_{\mathsf{T},4}) \in \mathbb{R}^{4(\mathsf{T}+1)}$ *is the solution of a linear system*

$$\mathbb{U}\,\mathbf{c} = \mathbf{a} \tag{3.7}$$

obtained by evaluating some derivatives of the sum (3.5) *at* $(\mathsf{T}+2)$ *points* $\{0, t_1, ..., t_{\mathsf{T}}, 1\}$ *associated to partition* $\mathbf{t}$*;*

3. $\mathbf{a} = \mathbf{a}[g, \mathbf{t}, \sigma] \in \mathbb{R}^{4(\mathsf{T}+1)}$ *depends on* $\mathbf{t}, \sigma$ *and on* g *(only through* d_l*) and has the form*

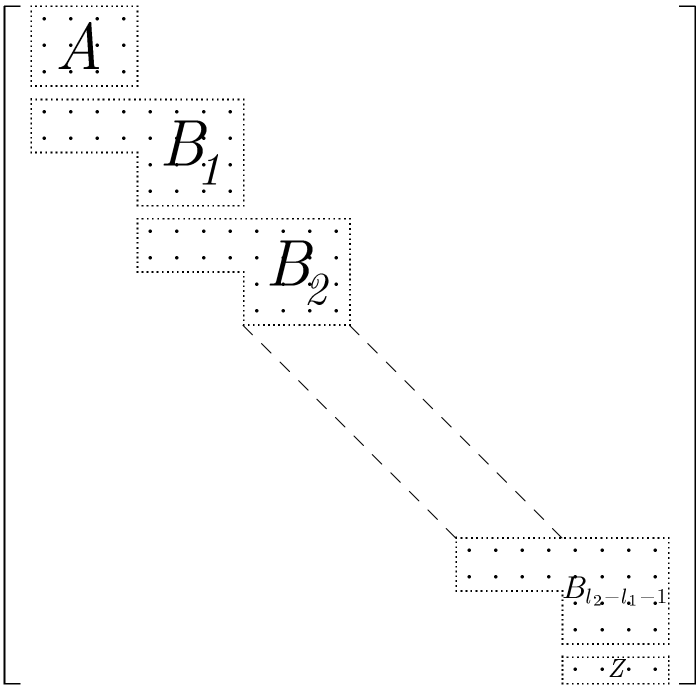
$$\mathbf{a}_{4l+i} = \begin{cases} d_l'''(t_l) - d_{l-1}'''(t_l) & \text{if } i = 1 \text{ and } \sigma_l = \mathsf{C} \\ -d_l'''(t_l) & \text{if } i = 1 \text{ and } \sigma_l = \mathsf{J} \\ -d_l''(t_l) & \text{if } i = 2, \\ -d_l''(t_{l+1}) & \text{if } i = 3, \\ d_{l+1}(t_{l+1}) - d_l(t_{l+1}) & \text{if } i = 4 \text{ and } \sigma_{l+1} = \mathsf{C} \\ -d_l'''(t_{l+1}) & \text{if } i = 4 \text{ and } \sigma_{l+1} = \mathsf{J} \end{cases} \quad \forall l \in \{0, ..., \mathsf{T}\}; \quad (3.8)$$

4. $\mathbb{U} = \mathbb{U}[\mathbf{t}, \sigma]$ *is a* $4(\mathsf{T}+1) \times 4(\mathsf{T}+1)$ *matrix depending only on* σ *and on* $\mathbf{t}$ *through values of* $\{w_i, w_i'', w_i'''\}_{i=1}^4$ *at* $\mathbf{t}$ *(*$\mathbb{U}$ *is a real analytic function of* $\mathbf{t}$ *for any* σ*). Moreover* $\mathbb{U}$ *is a square block diagonal matrix: each square block* $\mathbb{U}_{l_1,l_2}$ *(related to an uncoupled subsystem) is identified by two consecutive jump points* t_{l_1}*,* t_{l_2} *and possible intermediated creases (recall that* $\sigma_0 = \sigma_{\mathsf{T}+1} = \mathsf{J}$*)*

$$l_1, l_2 \in \{0, ..., T+1\}: \begin{cases} \sigma_{l_1} = \sigma_{l_2} = \mathsf{J} \\ \sigma_l = \mathsf{C} & \text{for } l \in \{l_1+1, ..., l_2-1\}, \end{cases}$$

so that each square block $\mathbb{U}_{l_1,l_2}$ *of* $\mathbb{U}$ *takes the form*

$\mathbb{U}_{l_1,l_2}[\mathbf{t}, \sigma] =$



(3.9)

where

$$A = \begin{bmatrix} w_1'''(t_{l_1}) & w_2'''(t_{l_1}) & w_3'''(t_{l_1}) & w_4'''(t_{l_1}) \\ w_1''(t_{l_1}) & w_2''(t_{l_1}) & w_3''(t_{l_1}) & w_4''(t_{l_1}) \\ w_1''(t_{l_1+1}) & w_2''(t_{l_1+1}) & w_3''(t_{l_1+1}) & w_4''(t_{l_1+1}) \end{bmatrix}$$

$$B_j =$$

$$\begin{bmatrix} w_1(t_{l_1+j}) & w_2(t_{l_1+j}) & w_3(t_{l_1+j}) & w_4(t_{l_1+j}) & -w_1(t_{l_1+j}) & -w_2(t_{l_1+j}) & -w_3(t_{l_1+j}) & -w_4(t_{l_1+j}) \\ w_1'''(t_{l_1+j}) & w_2'''(t_{l_1+j}) & w_3'''(t_{l_1+j}) & w_4'''(t_{l_1+j}) & -w_1'''(t_{l_1+j}) & -w_2'''(t_{l_1+j}) & -w_3'''(t_{l_1+j}) & -w_4'''(t_{l_1+j}) \\ & & & & w_1''(t_{l_1+j}) & w_2''(t_{l_1+j}) & w_3''(t_{l_1+j}) & w_4''(t_{l_1+j}) \\ & & & & w_1''(t_{l_1+j+1}) & w_2''(t_{l_1+j+1}) & w_3''(t_{l_1+j+1}) & w_4''(t_{l_1+j+1}) \end{bmatrix}$$

for $j \in \{1, ..., l_2 - l_1 - 1\}$

$$Z = \begin{bmatrix} w_1'''(t_{l_2}) & w_2'''(t_{l_2}) & w_3'''(t_{l_2}) & w_4'''(t_{l_2}) \end{bmatrix}.$$

Proof. Claim about cardinality of singular set and Statement 1 are straightforward. Statement 2-4 are deduced by evaluation of $(3.2)(ii-vi)$ as follows. Condition $(3.2)(ii)$ gives

$$\sum_{i=1}^{4} \mathbf{c}_{l,i} w_i''(t_l) = -d_l''(t_l), \qquad \sum_{i=1}^{4} \mathbf{c}_{l,i} w_i''(t_{l+1}) = -d_l''(t_{l+1}) \qquad \forall l \in \{0, ..., \mathsf{T}\}.$$

Conditions $(3.2)(iii)$ and $(3.2)(iv)$ give

$$\begin{cases} \sum_{i=1}^{4} \mathbf{c}_{l,i} w_i'''(t_l) = -d_l'''(t_l) & \text{if } l = 0 \text{ or } (\sigma_l = \mathsf{J},\ l \in \{1, ..., \mathsf{T}\}) \\ \sum_{i=1}^{4} \mathbf{c}_{l,i} w_i'''(t_{l+1}) = -d_l'''(t_{l+1}) & \text{if } l = \mathsf{T} \text{ or } (\sigma_{l+1} = \mathsf{J},\ l \in \{1, ..., \mathsf{T}\}). \end{cases}$$

Condition $(3.2)(v)$ gives

$$d_{l-1}'''(t_l) + \sum_{i=1}^{4} \mathbf{c}_{l-1,i} w_i'''(t_l) - \left(d_l'''(t_l) + \sum_{i=1}^{4} \mathbf{c}_{l,i} w_i'''(t_l) \right) = 0$$
$$\text{if } l \in \{1, ..., \mathsf{T}\} \text{ and } \sigma_l = \mathsf{C},$$

hence

$$\sum_{i=1}^{4} \mathbf{c}_{l-1,i} w_i'''(t_l) - \sum_{i=1}^{4} \mathbf{c}_{l,i} w_i'''(t_l) = d_l'''(t_l) - d_{l-1}'''(t_l) \quad \text{if } l \in \{1, ..., \mathsf{T}\} \text{ and } \sigma_l = \mathsf{C}.$$

Condition (3.2)(vi) gives

$$\sum_{i=1}^{4}\mathbf{c}_{l-1,i}w_i(t_l)-\sum_{i=1}^{4}\mathbf{c}_{l,i}w_i(t_l)=d_l(t_l)-d_{l-1}(t_l)\quad \text{if } l\in\{1,...,\mathsf{T}\} \text{ and } \sigma_l=\mathsf{C}. \ \square$$

Lemma 3.6 *For any* $\mathsf{T},\mathsf{j},\mathsf{c}\in\{0,1,2,...\}$, $\mathsf{T}=\mathsf{j}+\mathsf{c}$, $\sigma\in\Omega_{\mathsf{j},\mathsf{c}}$ *and* $g\in L^2(0,1)$, *the set*

$$\{\mathbf{t}\in A_{\mathsf{T}}:\quad \textit{Problem 3.3 is uniquely solvable}\}$$

is independent of g.
Then we can define

$$\mathcal{A}[\sigma]=\{\mathbf{t}\in A_{\mathsf{T}}:\quad \textit{Problem 3.3 is uniquely solvable}\}.$$

For any j, c *and* $\sigma\in\Omega_{\mathsf{j},\mathsf{c}}$ *the set* $\mathcal{A}[\sigma]$ *is an open set.*
Moreover

$$\begin{cases} \textit{for every } g\in L^2(0,1),\ \alpha,\ \beta \textit{ with } (1.2) \textit{ and } u\in \operatorname{argmin} F^g_{\alpha,\beta}, \\ \qquad \textit{we have}\quad \mathbf{t}(u)\in\mathcal{A}[\sigma(u)]. \end{cases} \tag{3.10}$$

In particular

$$\mathcal{A}[\sigma(u)]\neq\varnothing\quad \forall g\in L^2(0,1),\ \alpha,\ \beta \textit{ with } (1.2) \textit{ and } u\in\operatorname{argmin}F^g_{\alpha,\beta}. \tag{3.11}$$

Proof. Fix g, σ, $\mathbf{t}$ such that Problem 3.3 has a solution. We choose $d=(d_0,...,d_{\mathsf{T}})$ of Lemma 3.5 as a fixed particular solution of the differential equations (3.2.i) (without imposing (3.2.ii)-(3.2.vi)) as follows:

$$\text{e.g.}\quad d''''+d=g\quad \text{on } (0,1),\quad d(0)=d'(0)=d''(0)=d'''(0)=0 \tag{3.12}$$

$$\text{in particular}\quad d_l''''+d_l=g\quad \text{on } (t_l,t_{l+1}) \text{ for } l=0,1,...,\mathsf{T}. \tag{3.13}$$

Problem 3.3 is uniquely solvable for any g if and only if the matrix $\mathbb{U}[\mathbf{t},\sigma]$ of Lemma 3.5 is an invertible matrix. Then $\mathcal{A}[\sigma]$ is an open subset of A_{T} since $\mathcal{A}[\sigma]=\{\mathbf{t}\in A_{\mathsf{T}}:\quad \det(\mathbb{U}[\mathbf{t},\sigma])\neq 0\}$. Condition $\det(\mathbb{U})\neq 0$ does not depends on g since w_i solve the homogeneous equation. Hence $\mathcal{A}[\sigma]$ is independent of g.
Eventually we show (3.10) and (3.11).
For any $g\in L^2(0,1)$ and $u\in\operatorname{argmin}F^g_{\alpha,\beta}$ (which is a non empty set) then $\mathbf{t}(u)\in\mathcal{A}[\sigma(u)]$, we define $b_l=u$ on (t_l,t_{l+1}) for any $l=0,...,\mathsf{T}$. By Euler equations (i)-(iv) of Theorem 2.1, we obtain a solution of Problem 3.3 with $\sigma=\sigma(u)$ and $\mathbf{t}=\mathbf{t}(u)$. Once $\sigma=\sigma(u)$ and $\mathbf{t}=\mathbf{t}(u)$ are fixed as above,

uniqueness property would fail if and only if $\mathbb{U}[\mathbf{t},\sigma]$ is not invertible; in this case all solutions of Problem 3.3 could be expressed as follows

$$\left.\begin{array}{l} u \;+\; \lambda \sum_{l=0}^{\mathsf{T}} \sum_{i=1}^{4} \mathbf{e}_{l,i}\, w_i\, \chi_{(t_l,t_{l+1})} \\ \forall\lambda\in\mathbb{R},\quad \forall\mathbf{e}=(\mathbf{e}_{l,i})_{l,i}\in\mathbb{R}^{4(\mathsf{T}+1)}\ \text{ with }\ \mathbb{U}\,\mathbf{e}=\mathbf{0}. \end{array}\right\} \tag{3.14}$$

But (3.14) would be a violation of the last statement of Theorem 3.2: in fact w_i in C^∞, $i=1,...,4$ entail that $u+\lambda\sum_{l=0}^{\mathsf{T}}\sum_{i=1}^{4}\mathbf{e}_{l,i}w_i\chi_{(t_l,t_{l+1})}\in\mathcal{H}^2_u$ for any λ in a small neighborhood of $0\in\mathbb{R}$. □

Lemma 3.7 *For any piecewise affine (possibly discontinuous) function g*

$$\exists\delta>0:\ \operatorname{argmin} F^g_{\alpha,\beta}=\{g\}\ \ \forall\alpha,\beta \ \textit{s.t.}\ (1.2)\ \textit{and}\ \alpha\,\sharp\,(S_g)+\beta\,\sharp\,(S_{\dot g}\setminus S_g)<\delta.$$

Proof. Set $\mathsf{j}=\sharp\,(S_g)$, $\mathsf{c}=\sharp\,(S_{\dot g}\setminus S_g)$. For any $v\in\mathcal{H}^2$, by setting $\mathsf{s}=\sharp\,(S_v)$ and $\mathsf{p}=\sharp\,(S_{\dot v}\setminus S_v)$, at least one of the following (mutually exclusive) five cases may occur:

$$1)\ \begin{cases}\mathsf{s}\ge\mathsf{j}\\ \mathsf{p}\ge\mathsf{c}\end{cases}\qquad 2)\ \begin{cases}\mathsf{p}<\mathsf{c}\\ \mathsf{s}\ge\mathsf{j}+\mathsf{c}-\mathsf{p}\end{cases}\qquad 3)\ \begin{cases}\mathsf{p}<\mathsf{c}\\ \mathsf{s}<\mathsf{j}+\mathsf{c}-\mathsf{p}\end{cases}$$

$$4)\ \begin{cases}\mathsf{s}<\mathsf{j}\\ \mathsf{p}\ge\mathsf{c}+2(\mathsf{j}-\mathsf{s})\end{cases}\qquad 5)\ \begin{cases}\mathsf{s}<\mathsf{j}\\ \mathsf{p}<\mathsf{c}+2(\mathsf{j}-\mathsf{s})\end{cases}$$

(in fact either $\mathsf{s}\ge\mathsf{j}$ or $\mathsf{s}<\mathsf{j}$; if $\mathsf{s}\ge\mathsf{j}$ then either $\mathsf{s}\ge\mathsf{j}+\mathsf{c}-\mathsf{p}$ (and this always occurs if $\mathsf{p}\ge\mathsf{c}$) or $\mathsf{s}<\mathsf{j}+\mathsf{c}-\mathsf{p}$; if $\mathsf{s}<\mathsf{j}$ then only one among cases 4 and 5 may occur). We show that $F^g_{\alpha,\beta}(v)>F^g_{\alpha,\beta}(g)$ for any α, β satisfying (1.2) and $v\in\mathcal{H}^2$, $v\neq g$ in each one of the five cases.
Case 1) then $F^g_{\alpha,\beta}(v)>F^g_{\alpha,\beta}(g)$ since $\mathcal{F}^g(v)>0=\mathcal{F}^g(g)$ and $\alpha\,\mathsf{s}+\beta\,\mathsf{p}\ge\alpha\,\mathsf{j}+\beta\,\mathsf{c}$.
Case 2) $F^g_{\alpha,\beta}(v)>F^g_{\alpha,\beta}(g)$, since $\mathcal{F}^g(v)>0$ hence

$$\begin{aligned}F^g_{\alpha,\beta}(v)=\mathcal{F}^g(v)+\alpha\,\mathsf{s}+\beta\,\mathsf{p}>\alpha\,\mathsf{j}+\alpha\,\mathsf{c}-\alpha\,\mathsf{p}+\beta\,\mathsf{p}=\\ \alpha\,\mathsf{j}+\beta\,\mathsf{c}+(\alpha-\beta)\,(\mathsf{c}-\mathsf{p})\ge\alpha\,\mathsf{j}+\beta\,\mathsf{c}=F^g_{\alpha,\beta}(g).\end{aligned}$$

Case 4) $F^g_{\alpha,\beta}(v)>F^g_{\alpha,\beta}(g)$, since $\mathcal{F}^g(v)>0$ hence

$$\begin{aligned}F^g_{\alpha,\beta}(v)=\mathcal{F}^g(v)+\alpha\,\mathsf{s}+\beta\,\mathsf{p}>\alpha\,\mathsf{s}+\beta\,\mathsf{c}+2\beta\,\mathsf{j}-2\beta\,\mathsf{s}=\\ \alpha\,\mathsf{j}+\beta\,\mathsf{c}+(2\beta-\alpha)\,(\mathsf{j}-\mathsf{s})\ge\alpha\,\mathsf{j}+\beta\,\mathsf{c}=F^g_{\alpha,\beta}(g).\end{aligned}$$

About cases 3) and 5) we observe that $S_g \cup S_{\dot{g}} \neq \varnothing$, then by Theorem 2.3

$$0 < \delta = \min\{\mathcal{F}^g(v) \colon \; v \in \mathcal{H}^2 \; \text{ belonging to cases 3 and 5}\},$$

hence in 3) and 5), for any α, β satisfying (1.2) and, in addition, so small that $\alpha\,\mathsf{j} + \beta\,\mathsf{c} < \delta$, we have $F^g_{\alpha,\beta}(v) > F^g_{\alpha,\beta}(g)$ for any v in cases 3) and 5). $\square$
We know that the set $\mathcal{A}[\sigma]$ is never empty, now we show its coincidence with the whole A_{T}: by exploiting the property that $\mathcal{A}[\sigma]$ is independent on the datum g, we choose piecewise affine g with quality σ for $S_g \cup S_{\dot{g}}$, then we prove that g itself is the unique minimizer for $F^g_{\alpha,\beta}$ provided α and β are suitably small.

Theorem 3.8 *Problem 3.3 admits unique solution, that is* $\mathcal{A}[\sigma] = A_{\mathsf{T}}$ *for any* $\mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ *and* $g \in L^2(0,1)$.

Proof. Fix $\mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ and $\mathbf{t} \in A_{\mathsf{T}}$. By Lemma 3.6, $\mathcal{A}[\sigma]$ is independent of g. Then in the definition of $\mathcal{A}[\sigma]$ we choose a piecewise affine (possibly discontinuous) g such that $\sigma(g) = \sigma$ and $\mathbf{t}(g) = \mathbf{t}$. Lemma 3.7 together with Theorem 2.1 and the second claim in Theorem 3.2 entail that, for any fixed σ, $\mathbf{t}$ and piecewise affine g, Problem 3.3 admits a unique solution: in fact any solution different from g must be of the form (3.14) with g plugged in place of u and a suitable choice $\widetilde{\lambda} \neq 0$ plugged in place of λ; then $g + \lambda \sum_{l=0}^{\mathsf{T}} \sum_{i=1}^{4} e_{l,i}\, w_i\, \chi_{(t_l, t_{l+1})}$ would be a solution for any $\lambda \in \mathbb{R}$ by linearity of conditions (3.3.ii)-(3.3.vi) which contradicts the second statement of Theorem 3.2. $\square$

Theorem 3.9 *If we fix a piecewise affine (possibly discontinuous) function* g *and label its location by* $\mathbf{q} = S_g \cup S_{\dot{g}}$ *and its quality by* $\sigma = \sigma(g)$, *then Problem 3.3 with data* $\mathbf{q}$, σ *and* g *admits* g *itself as unique solution:* $g = b[g, \mathbf{q}, \sigma]$.
The same property holds true for Problem 3.3 with any data $\widetilde{\mathbf{q}}$, $\widetilde{\sigma}$, g *suth that the arrangement* $(S_g \cup S_{\dot{g}}, \sigma(g))$ *is compatible with* $(\widetilde{\mathbf{q}}, \widetilde{\sigma})$ *i.e.:* $S_g \cup S_{\dot{g}} \subseteq \widetilde{\mathbf{q}}$ *and qualities* $\widetilde{\sigma}$ *and* $\sigma(g)$ *coincides on common points.*

Proof. The fact that g is a solution is trivial since $\ddot{g} \equiv 0$. Uniqueness statement follows by Theorem 2.1, Lemma 3.7 and the fact that Euler equations are independent on α, β. $\square$
Theorem 3.8 allow the introduction of the following basic notation about solution of Problem 3.3 and its related energy.

Definition 3.10 *For any* $\mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ *and* $g \in L^2(0,1)$, *set*

1. $b = b[g, \mathbf{t}, \sigma]$ *is the unique function* $b = b(x) \in \mathcal{H}^2$ *piecewise defined by the solutions* $\{b_l = b_l[g, \mathbf{t}, \sigma] \in H^2(t_l, t_{l+1})\}_{l=0}^{\mathsf{T}}$ *of Problem 3.3. The dependence on right hand side* g*, location* $\mathbf{t}$ *and quality* σ *will be dropped whenever there is no risk of confusion. For any* $l \in \{0, ..., \mathsf{T}\}$ *we denote by* $b'_l, b''_l, ..., b_l^{(r)}$ *the first , second, ..., r-th distributional derivative in* (t_l, t_{l+1}) *of* b_l *with respect to* x*. Notice that* $b'_l = \dot{b}_l$*,* $b''_l = \ddot{b}_l$*, ..., but* b *and* b'' *may be different from* $\dot{b}$ *and* $\ddot{b}$ *due to singular part at* t_l*.*

2. $\mathbb{F}(g, \mathbf{t}, \sigma)$ *is the absolutely continuous part* $\mathcal{F}^g$ *of* $F^g_{\alpha,\beta}$ *evaluated at* $b[g, \mathbf{t}, \sigma]$*:*

$$\mathbb{F}(g, \mathbf{t}, \sigma) = \mathcal{F}^g(b[g, \mathbf{t}, \sigma]), \tag{3.15}$$

$$\mathbb{F}(\cdot, \cdot, \sigma) : L^2(0,1) \times A_{\mathsf{T}} \to \mathbb{R}.$$

In the following proposition we list some properties of b and $\mathbb{F}$.

Theorem 3.11 *Fix* $\mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$*,* $\mathsf{T} = \mathsf{j} + \mathsf{c}$ *and* $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$*, then*

1. *the map* $g \mapsto b(g, \mathbf{t}, \sigma)$ *is linear in* $g \in L^2(0,1)$ *for any* $\mathbf{t} \in A_{\mathsf{T}}$*,*
   in particular $g \equiv 0$ *entails* $b \equiv 0$*;*
   the map $g \mapsto \mathbb{F}(g, \mathbf{t}, \sigma)$ *is* 2*-homogeneous with respect to* $g \in L^2(0,1)$ *for any* $\mathbf{t} \in A_{\mathsf{T}}$*;*

2. *the map* $b_l(\cdot, \mathbf{t}, \sigma) : L^2(0,1) \to H^2(t_l, t_{l+1})$*, say* $g \mapsto b_l(g, \mathbf{t}, \sigma)$
   is continuous from $L^2(0,1)$ *to* $H^2(t_l, t_{l+1})$ *where both spaces are endowed with the strong topology, for any* $\mathbf{t} \in A_{\mathsf{T}}$ *and* $l = 0, ..., \mathsf{T}$*;*
   the map $b_l(\cdot, \cdot, \sigma) : L^2(0,1) \times A_{\mathsf{T}} \to L^\infty(0,1)$*, say* $(g, \mathbf{t}) \mapsto b_l(g, \mathbf{t}, \sigma)$
   is continuous from $L^2(0,1)$ *times* A_{T} *endowed with the product topology (strong* $L^2(0,1)$ *times Euclidean topology of* $\mathbb{R}^{\mathsf{T}}$*) to* $L^\infty(0,1)$ *endowed with the strong topology;*

3. *the map* $\mathbb{F}(\cdot, \cdot, \sigma) : L^2(0,1) \times A_{\mathsf{T}} \to \mathbb{R}$
   is continuous on $L^2(0,1)$ *times* A_{T} *endowed with the product topology (strong* $L^2(0,1)$ *times Euclidean topology of* $\mathbb{R}^{\mathsf{T}}$*);*

4. *for any* $g \in L^2(0,1)$ *and* $u \in \operatorname{argmin} F^g_{\alpha,\beta}$*, if* u *has* j *jump points,* c *crease points and quality* σ*, the function* $\mathbf{t} \longmapsto \mathbb{F}(g, \mathbf{t}, \sigma)$ *achieves its minimum with respect to* $\mathbf{t}$ *in* A_{k} *at* $\mathbf{t}(u) = (t_1(u), ..., t_{\mathsf{k}}(u))$*. Moreover* $S_u = \{t_l(u) : \ \sigma_l = \mathsf{J}\}$*,* $S_{\dot{u}} \setminus S_u = \{t_l(u) : \ \sigma_l = \mathsf{C}\}$*, and* $b = u$ *is the only admissible minimizer of* $\mathcal{F}$ *in* $\mathcal{H}^2_u$*.*

Proof. Statement 1 follows by linearity in g of resolvent operator for Problem 3.3 with prescribed arrangement $(\mathbf{t}, \sigma)$, by (3.2.(i)) and compliance identity (Theorem 2.2).
Choose $d = (d_0, ..., d_{\mathsf{T}})$ fulfilling (3.12), hence (3.4) is trivially fulfilled and $d \in H^4(0,1) \subseteq C^3(0,1)$, the map $g \longmapsto d$ is linear continuous from $L^2(0,1)$ to $H^4(0,1)$ and well defined by Theorem 3.8.
The function b takes the form (3.5) with $\mathbf{c} = \mathbb{U}^{-1}\mathbf{a}$, (3.8), (3.9) hold true and both invertible matrix $\mathbb{U}$ and vector $\mathbf{a}$ are analytic functions of $\mathbf{t}$ since all entries of $\mathbb{U}$ and $\mathbf{a}$ are linear functions of $w_i(t_l)$, $w_i''(t_l)$ and $w_i'''(t_l)$; hence statement 2 holds true.
Statement 3 follows by statement 2 and

$$\mathbb{F}(g, \mathbf{t}, \sigma) = \sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \left(\left| (\ddot{b}_l[g, \mathbf{t}, \sigma](x) \right|^2 + |(b_l[g, \mathbf{t}, \sigma](x) - g(x)|^2 \right) dx.$$

Statement 4 follows by Theorems 2.1 and 3.2. $\square$
In Sections 3,4,5 and 6 we denote by $\mathbf{q} = (q_i)_{i=1}^{\mathsf{Q}}$ the location (and by $\{q_i\}_{i=1}^{\mathsf{Q}}$ the related partition) associated to crease points $S_{\dot{g}}$ of continuous piecewise affine datum g and we denote by $\mathbf{t} = (t_i)_{i=1}^{\mathsf{T}}$ the location associated to the singular set $S_v \cup S_{\dot{v}}$ of the competing functions $v \in \mathcal{H}^2$. The location $\mathbf{t}$ of singular set of solution of Problem 3.3 and the location $\mathbf{q}$ (singular set of the datum) may be different. Abusing notation, whenever needed, we write $x \in \mathbf{q}$ to mean $x \in \{q_i\}_{i=1}^{\mathsf{Q}}$, while $q_0 = 0$, $q_{\mathsf{Q}+1} = 1$ are always understood.
Each location $\mathbf{q} = (q_i)_{i=1}^{\mathsf{Q}}$ induces a decomposition of $[0,1]^{\mathsf{T}}$ in cubes, this naturally gives to $[0,1]^{\mathsf{T}}$ a finite CW complex structure. For any $d \in \{1, ..., \mathsf{T}\}$, a *$d$-dimensional open cell* W of $[0,1]^{\mathsf{T}}$ is a d-dimensional open face of a cube $\prod_{k=1}^{d} [q_{i_k}, q_{i_k+1}]$, a *0-dimensional open cell* W of $[0,1]^{\mathsf{T}}$ is a point $(q_{i_1}, ..., q_{i_{\mathsf{T}}})$.
For any $i, d \in \{1, ..., \mathsf{T}\}$, any $\mathbf{t} \in [0,1]^{\mathsf{T}}$, and any d-dimensional open cell W of $[0,1]^{\mathsf{T}}$, we say that t_i is a free coordinate in W if and only if $t_i \notin \mathbf{q}$ for any $\mathbf{t} \in W$. Clearly a 0-dimensional cell of $[0,1]^{\mathsf{T}}$ has no free coordinates.
The set $A_{\mathsf{T}} \subseteq [0,1]^{\mathsf{T}}$ is an open subset of a finite CW complex, with an abuse of language we introduce the following definition.

Definition 3.12 *Whenever U is an open d-dimensional cell of $[0,1]^{\mathsf{T}}$ such that $U \cap A_{\mathsf{T}} \neq \emptyset$, we call d-dimensional open cell of the CW structure induced on A_{T} by $[0,1]^{\mathsf{T}}$ also the set $W = U \cap A_{\mathsf{T}}$.*
The free coordinates of $W = U \cap A_{\mathsf{T}}$ are exactly the same free coordinates of U but they may have different range when $W \subsetneq U$.

A short summary of what is needed to know about CW complexes can be found in the Appendix A.

Fig.1 provides a simple low dimensional visualization of cells in A_{T}.

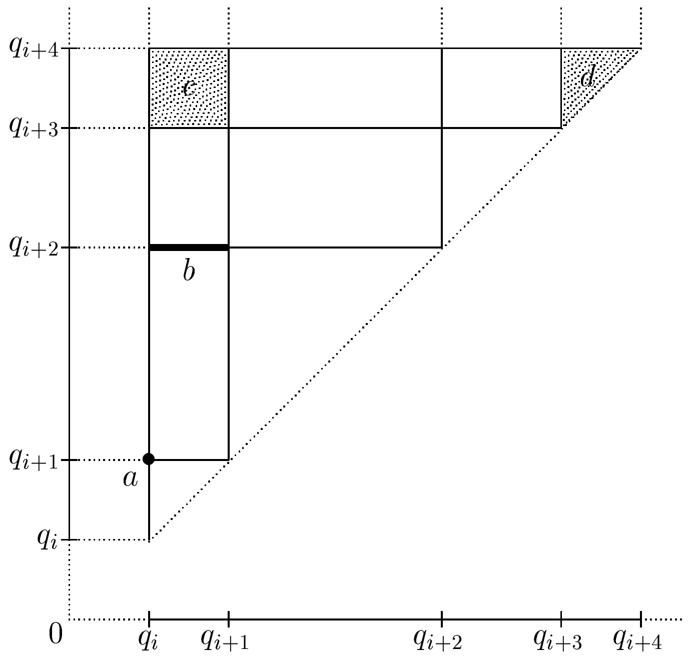


Figure 1

a: 0 cell, no free coordinates;
b: 1 cell, t_1 free coordinate, $q_i < t_1 < q_{i+1}$;
c: 2 cell, t_1, t_2 free coordinates, $q_i < t_1 < q_{i+1}$, $q_{i+3} < t_2 < q_{i+4}$;
d: 2 cell (abusing language), t_1, t_2 free coordinates.

For any location $\mathbf{q} = (q_i)_{i=1}^{\mathsf{Q}}$ we identify the space $\mathbb{R}^{\mathsf{Q}+2}$ with the space $\mathbb{A}_{\mathbf{q}}$ of continuous piecewise affine functions g with crease points at $\mathbf{q}$. Precisely for any $\mathbf{g} = (\mathbf{g}_0, \mathbf{g}_1, ..., \mathbf{g}_{\mathsf{Q}+1}) \in \mathbb{R}^{\mathsf{Q}+2}$ the identification between the vector parameter $\mathbf{g}$ and the function $g \in L^2(0,1)$ is given by:

$$\begin{cases} g(x) = \sum_{i=1}^{\mathsf{Q}+1} (\mathbf{g}_i(x - q_{i-1}) + \mathbf{z}_{i-1}) \chi_{[q_{i-1}, q_i)}(x) \quad \text{where} \\ \mathbf{z}_0 = \mathbf{g}_0 \\ \mathbf{z}_l = \mathbf{g}_l(q_l - q_{l-1}) + \mathbf{z}_{l-1} \qquad \text{for } l \in \{1, ..., \mathsf{Q}\}. \end{cases} \tag{3.16}$$

Notice that (3.16) induces a linear and injective identification between $\mathbf{g}$ and g, hence $\mathbb{A}_{\mathbf{q}}$ is a vector space of dimension $\mathsf{Q} + 2$:

$$\mathbf{q} \in A_{\mathsf{Q}}\,, \qquad \mathbf{g} \in \mathbb{R}^{\mathsf{Q}+2}\,, \qquad g \in \mathbb{A}_{\mathbf{q}} \simeq \mathbb{R}^{\mathsf{Q}+2}. \tag{3.17}$$

Restrictions of $\mathbb{F}(\cdot, \cdot, \sigma)$ and $b_l^{(r)}(\cdot, \cdot, \sigma)$ to $\mathbb{A}_{\mathbf{q}} \times A_{\mathsf{T}}$ play a fundamental role in the following: the restriction of both $\mathbb{F}$ and of $b_l^{(r)}$ to $\mathbb{A}_{\mathbf{q}} \times A_{\mathsf{T}}$ can be considered as functions defined on $\mathbb{R}^{\mathsf{Q}+2} \times A_{\mathsf{T}}$ through the identification between $\mathbb{A}_{\mathbf{q}}$ and $\mathbb{R}^{\mathsf{Q}+2}$described by (3.16) and (3.17). Actually (abusing notation) we specialize Definition 3.10 when g belongs to $\mathbb{A}_{\mathbf{q}}$, as follows.

Definition 3.13 *For any* $\mathsf{Q}, \mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ *and* $\mathbf{g} \in \mathbb{R}^{\mathsf{Q}+2}$, *define*

$$b(\cdot, \cdot, \sigma) : \mathbb{R}^{\mathsf{Q}+2} \times A_{\mathsf{T}} \to L^2(0, 1) \quad by \quad b[\mathbf{g}, \mathbf{t}, \sigma](x) = b[g, \mathbf{t}, \sigma](x) \tag{3.18}$$

$$\mathbb{F}(\cdot, \cdot, \sigma) : \mathbb{R}^{\mathsf{Q}+2} \times A_{\mathsf{T}} \to \mathbb{R} \quad by \quad \mathbb{F}(\mathbf{g}, \mathbf{t}, \sigma) = \mathbb{F}(g, \mathbf{t}, \sigma) \tag{3.19}$$

where the right-hand sides are given by Definition 3.10 and $g \in \mathbb{A}_{\mathbf{q}}$ *is associated by* (3.16) *and* (3.17) *to vector* $\mathbf{g}$ *and singular set location* $\mathbf{q}$.

We are going to show that both (3.18) and (3.19) are polynomials in $\mathbf{g}$ with coefficients which are continuous functions of $\mathbf{t} \in A_{\mathsf{T}}$ and their restrictions to $\mathbb{R}^{\mathsf{Q}+2} \times W$ are real analytic functions of $\mathbf{g}$ and t_l for any open cell W of the CW structure induced by $\mathbf{q}$ on A_{T} and any t_l free coordinate in the open cell W.

Theorem 3.14 *Fix* $\mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$ *and* $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$, *then*

1. *The map* $\mathbf{g} \mapsto b[\mathbf{g}, \mathbf{t}, \sigma](x)$ *is a linear function of* $\mathsf{Q} + 2$ *variables* $\mathbf{g}$ *for any* $\mathbf{t} \in A_{\mathsf{T}}$;
   the map $\mathbf{g} \mapsto \mathbb{F}(\mathbf{g}, \mathbf{t}, \sigma)$ *is a* 2*-homogeneous polynomial of* $\mathsf{Q}+2$ *variables (the coordinates* $\mathbf{g}_i$ *of* $\mathbf{g} \in \mathbb{R}^{\mathsf{Q}+2}$*);*

2. *for any* $\mathsf{Q}, r \in \{0, 1, 2, ...\}$, $\mathbf{q} \in A_{\mathsf{Q}}$ *and any open cell* W *of the CW structure induced by* $\mathbf{q}$ *on* A_{T}, *the restrictions to* $\mathbb{A}_{\mathbf{q}} \times W$ *of* $b_l^{(r)}$ *and of* $\mathbb{F}$ *(e.g. functions* (3.18) *and* (3.19)*) are real analytic functions of* $\mathbf{g}$ *and* t_j *where* t_j *is a free coordinate of the open cell* W.

Proof. Statement 1 follows by Theorem 3.11(1) and identifications (3.16) and (3.17). Statement 2 follows by the same argument used in the proof of point 2 of Theorem 3.11 and the simple remark that the map $g \longmapsto d$ appearing in the proof is real analytic in the free coordinates of W whenever $g \in \mathbb{A}_{\mathbf{q}}$. $\square$

Theorem 3.14 allows us to introduce the following notation.

Definition 3.15 *For any* $\mathsf{Q}, \mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ *and* $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^{\mathsf{Q}+2}$, *referring to* (3.16), (3.17) *and Definition 3.13, set*

$$\frac{\partial \mathbb{F}}{\partial \mathbf{f}} = \lim_{\varepsilon \to 0} \frac{\mathbb{F}(\mathbf{g} + \varepsilon \mathbf{f}, \mathbf{t}, \sigma) - \mathbb{F}(\mathbf{g}, \mathbf{t}, \sigma)}{\varepsilon} \tag{3.20}$$

and

$$\frac{\partial b_l^{(r)}}{\partial \mathbf{f}} = \lim_{\varepsilon \to 0} \frac{b_l^{(r)}(\mathbf{g} + \varepsilon \mathbf{f}, \mathbf{t}, \sigma) - b_l^{(r)}(\mathbf{g}, \mathbf{t}, \sigma)}{\varepsilon}, \qquad (3.21)$$

moreover, for any open cell W of the CW structure induced by $\mathbf{q}$ on A_{T}, and any a free coordinate t_i of W we denote by $\frac{\partial \mathbb{F}}{\partial t_i}$ and by $\frac{\partial b_l^{(r)}}{\partial t_i}$ the directional derivatives of $\mathbb{F}$ and of $b_l^{(r)}$ with respect to the free coordinate t_i of $\mathbf{t}$.

We are going to evaluate several derivatives of $\mathbb{F}$.

Lemma 3.16 *For any $\mathsf{Q}, \mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ and $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^{\mathsf{Q}+2}$, referring to (3.16),(3.17) and Definition 3.13, the explicit representation of mixed directional derivatives with respect to the first variable is:*

$$\frac{\partial^2 \mathbb{F}(\mathbf{g}, \mathbf{t}, \sigma)}{\partial \mathbf{f} \partial \mathbf{h}} =$$

$$2 \int_0^1 \ddot{b}[f, \mathbf{t}, \omega](x) \ddot{b}[g, \mathbf{t}, \sigma](x) + (b[f, \mathbf{t}, \sigma](x) - f(x))(b[g, \mathbf{t}, \sigma](x) - g(x))\, dx.$$

Proof. Since $b_l[\mathbf{g}, \mathbf{t}, \sigma]$ and $b_l''[\mathbf{g}, \mathbf{t}, \sigma]$ are linear in $\mathbf{g}$ and $b_l'' = \ddot{b}_l$ in (t_l, t_{l+1}) then the following equalities hold true: $\frac{\partial g}{\partial \mathbf{f}} = f$ and

$$\frac{\partial(b_l[\mathbf{g}, \mathbf{t}, \sigma] - g)}{\partial \mathbf{f}} = b_l[f, \mathbf{t}, \sigma] - f, \qquad \frac{\partial(b_l[\mathbf{g}, \mathbf{t}, \sigma] - g)}{\partial \mathbf{h}} = b_l[h, \mathbf{t}, \sigma] - h,$$

$$\frac{\partial \ddot{b}_l[\mathbf{g}, \mathbf{t}, \sigma]}{\partial \mathbf{f}} = \ddot{b}_l[f, \mathbf{t}, \sigma], \qquad \frac{\partial \ddot{b}_l[\mathbf{g}, \mathbf{t}, \sigma]}{\partial \mathbf{h}} = \ddot{b}_l[h, \mathbf{t}, \sigma],$$

$$\frac{\partial^2 \ddot{b}_l[\mathbf{g}, \mathbf{t}, \sigma]}{\partial \mathbf{f} \partial \mathbf{h}} \equiv \frac{\partial^2 (b_l[\mathbf{g}, \mathbf{t}, \sigma] - g)}{\partial \mathbf{f} \partial \mathbf{h}} \equiv 0.$$

By Theorem 3.14(2), derivatives with respect to $\mathbf{f}$, $\mathbf{h}$ commute with the integration in x. Then

$$\frac{\partial^2 \mathbb{F}(\mathbf{g}, \mathbf{t}, \sigma)}{\partial \mathbf{f} \partial \mathbf{h}} = \sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \frac{\partial^2}{\partial \mathbf{f} \partial \mathbf{h}} \Big((\ddot{b}_l[\mathbf{g}, \mathbf{t}, \sigma](x))^2 + (b_l[\mathbf{g}, \mathbf{t}, \sigma](x) - g(x))^2 \Big)\, dx$$

$$= 2 \sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \Bigg(\frac{\partial^2 \ddot{b}_l[\mathbf{g}, \mathbf{t}, \sigma]}{\partial \mathbf{f} \partial \mathbf{h}} b_l''[g, \mathbf{t}] + \frac{\partial \ddot{b}_l[\mathbf{g}, \mathbf{t}, \sigma]}{\partial \mathbf{f}} \frac{\partial \ddot{b}_l[\mathbf{g}, \mathbf{t}, \sigma]}{\partial \mathbf{h}} +$$

$$\frac{\partial^2 (b_l[\mathbf{g}, \mathbf{t}, \sigma] - g)}{\partial \mathbf{f} \partial \mathbf{h}} (b_l[\mathbf{g}, \mathbf{t}, \sigma] - g) + \frac{\partial(b_l[\mathbf{g}, \mathbf{t}, \omega] - g)}{\partial \mathbf{f}} \frac{\partial(b_l[\mathbf{g}, \mathbf{t}, \sigma] - g)}{\partial \mathbf{h}} \Bigg)\, dx =$$

$$2 \int_0^1 \ddot{b}[f, \mathbf{t}, \sigma] \ddot{b}[h, \mathbf{t}, \sigma] + (b[f, \mathbf{t}, \sigma] - f)(b[h, \mathbf{t}, \sigma] - h)\, dx. \quad \square$$

Lemma 3.17 *Fix* $\mathsf{Q}, \mathsf{T}, \mathsf{j}, \mathsf{c}, d \in \{0,1,2,...\}$, $\mathsf{Q} \geq d > 0$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$, *any open d-cell W of the CW structure induced by* $\mathbf{q}$ *on* A_{T}, *and any free coordinate* t_i *of* W.
Then the derivative of $\mathbb{F}$ *with respect to* t_i *exists in* $W \subseteq A_{\mathsf{T}}$ *due to Theorem 3.14, moreover referring to* (3.16), (3.17) *and Definition 3.13, for any* $g \in \mathbb{A}_{\mathbf{q}}$ *we have*

$$\frac{\partial \mathbb{F}(\mathbf{g}, \mathbf{t}, \sigma)}{\partial t_i} = (b_{i-1}(t_i) - b_i(t_i))(b_{i-1}(t_i) + b_i(t_i) - 2g(t_i)) \\ -2b_i'''(t_i)(b_i'(t_i) - b_{i-1}'(t_i)), \tag{3.22}$$

where $b_i'''(t_i) = b_{i-1}'''(t_i)$ *is understood.*

Proof. The variables $\mathbf{g}$, $\mathbf{t}$ and ω in the argument of $b_l^{(r)}$ are understood.
We exploit the facts: g and $\mathbf{g}$ depend on $\mathbf{q}$ but not on $\mathbf{t}$; $\left(\frac{\partial b_l'}{\partial t_i}\right)' = \frac{\partial b_l''}{\partial t_i}$ thank to Theorem 3.14(2); the integrand $b[\mathbf{g}, \mathbf{t}, \sigma](x)$ analytically depends on free coordinate t_i by Theorem 3.14(2), hence $\frac{\partial}{\partial t_i}$ commutes with integration; Theorem 2.1(i)-(iv).

$$\frac{\partial \mathbb{F}(\mathbf{g}, \mathbf{t}, \sigma)}{\partial t_i} = \frac{\partial}{\partial t_i}\left(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \left((b_l'')^2 + (b_l - g)^2\right) dx\right) =$$

$$(b_{i-1}''(t_i))^2 + (b_{i-1}(t_i) - g(t_i))^2 - (b_i''(t_i))^2 - (b_i(t_i) - g(t_i))^2$$

$$+ 2\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} \left(b_l'' \frac{\partial b_l''}{\partial t_i} + (b_l - g)\frac{\partial b_l}{\partial t_i}\right) dx$$

$$= (b_{i-1}(t_i) - b_i(t_i))(b_{i-1}(t_i) + b_i(t_i) - 2g(t_i))$$

$$+ 2\sum_{l=0}^{\mathsf{T}}\left(-\int_{t_l}^{t_{l+1}} b_l''' \frac{\partial b_l'}{\partial t_i}\, dx + \left[b_l'' \frac{\partial b_l'}{\partial t_i}\right]_{t_l}^{t_{l+1}} + \int_{t_l}^{t_{l+1}} (b_l - g)\frac{\partial b_l}{\partial t_i}\, dx\right)$$

$$= (b_{i-1}(t_i) - b_i(t_i))(b_{i-1}(t_i) + b_i(t_i) - 2g(t_i))$$

$$+ 2\sum_{l=0}^{\mathsf{T}}\left(\int_{t_l}^{t_{l+1}} (b_l'''' + b_l - g)\frac{\partial b_l}{\partial t_i}\, dx - \left[b_l''' \frac{\partial b_l}{\partial t_i}\right]_{t_l}^{t_{l+1}}\right)$$

$$= (b_{i-1}(t_i) - b_i(t_i))(b_{i-1}(t_i) + b_i(t_i) - 2g(t_i)) - 2\sum_{l=0}^{\mathsf{T}}\left[b_l''' \frac{\partial b_l}{\partial t_i}\right]_{t_l}^{t_{l+1}}$$

$$= (b_{i-1}(t_i) - b_i(t_i))(b_{i-1}(t_i) + b_i(t_i) - 2g(t_i)) -$$

$$2\sum_{l=0}^{\mathsf{T}}\left(b_l'''(t_{l+1})\frac{\partial b_l}{\partial t_i}(t_{l+1}) - b_l'''(t_l)\frac{\partial b_l}{\partial t_i}(t_l)\right) =$$

$$= (b_{i-1}(t_i) - b_i(t_i))(b_{i-1}(t_i) + b_i(t_i) - 2g(t_i)) - 2\sum_{l=1}^{\mathsf{T}} \left(b'''_{l-1}(t_l) \frac{\partial b_{l-1}}{\partial t_i}(t_l) - b'''_l(t_l) \frac{\partial b_l}{\partial t_i}(t_l) \right)$$

$$= (b_{i-1}(t_i) - b_i(t_i))(b_{i-1}(t_i) + b_i(t_i) - 2g(t_i)) - 2\sum_{l=1}^{\mathsf{T}} b'''_l(t_l) \left(\frac{\partial b_{l-1}}{\partial t_i}(t_l) - \frac{\partial b_l}{\partial t_i}(t_l) \right).$$

If $\omega_l = \mathsf{J}$ then $b'''_{l-1}(t_l) = b'''_l(t_l) = 0$.
If $\omega_l = \mathsf{C}$ we define the function $\mathbf{t} \mapsto \varphi_l(\mathbf{t}) = b_{l-1}(t_l) - b_l(t_l)$; notice that $\varphi_l(\mathbf{t}) \equiv 0$ for any $\mathbf{t}$ due to (3.2.(vi)). By performing carefully derivative of $b_l[\mathbf{g}, \mathbf{t}, \sigma](x)$, since $\partial (b_i(t_i)) / \partial t_i = b'_i(t_i) + (\partial b_i / \partial t_i)(t_i)$ we get

$$0 = \frac{\partial \varphi_l}{\partial t_i}(\mathbf{t}) = \begin{cases} \frac{\partial b_{l-1}}{\partial t_i}(t_l) - \frac{\partial b_l}{\partial t_i}(t_l) & \text{if } l \neq i \\ \frac{\partial b_{i-1}}{\partial t_i}(t_i) - \frac{\partial b_i}{\partial t_i}(t_i) + b'_{i-1}(t_i) - b'_i(t_i) & \text{if } l = i. \end{cases}$$

by substitution we get (3.22). □

Lemma 3.18 *For any* $\mathsf{Q} \in \{0, 1, 2, ...\}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $g \in \mathbb{A}_{\mathbf{q}}$ *and* $u \in \operatorname{argmin} F^g_{\alpha,\beta}$, *we have:*

$$\frac{1}{2}(u_+(t) + u_-(t)) = g(t) \text{ for any } t \in S_u \tag{3.23}$$

and

$$\dddot{u}_\pm(t) = 0 \text{ for any } t \in S_{\dot{u}} \setminus S_{\dot{g}} \tag{3.24}$$

Proof. By Euler equations (i)-(iv) of Theorem 2.1 u is a solution of Problem 3.3, $\mathbb{F}(g, \cdot, \sigma(u))$ achieves its minimum at $\mathbf{t}(u)$ and $\partial (\mathbb{F}(\mathbf{g}, \mathbf{t}(u), \sigma(u))) / \partial t_i = 0$. Then by (3.2.$(iii)$), (3.22) we deduce (3.23) when $t_i \in S_u$ (property already proven in a different way in Theorem 2.1(v)); $b_i(t_i) = b_{i-1}(t_i)$, $b'_i(t_i) = b'_{i-1}(t_i)$ and (3.22) entails (3.24) when $t_i \in S_{\dot{u}} \setminus (S_u \cup S_{\dot{g}})$. □

In the following technical lemma we show that for any location $\mathbf{t}$ and quality ω, if g is continuous piecewise affine with either only one ramp or only one jump (such that $S_g \cup S_{\dot{g}}$ is contained in $\{t_i\}_{i=0}^{\mathsf{T}}$ and has quality compatible with ω) and h is a continuous ramp having singular set close to the singular set of g, then the solution $b[h, \mathbf{t}, \omega]$ of Problem 3.3 with data h, $\mathbf{t}$ and ω is close to g in the piecewise C^3 lagrangian norm.

Lemma 3.19 *Fix* $\mathsf{T}, \mathsf{j}, \mathsf{c}, l_1, l_2, k \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\omega \in \Omega_{\mathsf{j},\mathsf{c}}$ *such that*

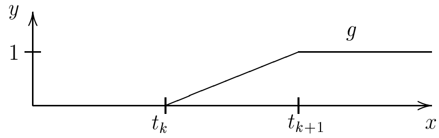
$$l_1 < l_2, \qquad 0 \leq l_1 \leq k \leq l_2 \leq \mathsf{T} + 1,$$

$$\omega_{l_1} = \omega_{l_2} = \mathsf{J}, \qquad \omega_l = \mathsf{C} \quad \forall l \in \{l_1+1, ..., l_2-1\}.$$

We study perturbations of location $\mathbf{t}$ *and quality* ω *of singular set for two kinds of datum* g *in Problem 3.3:*

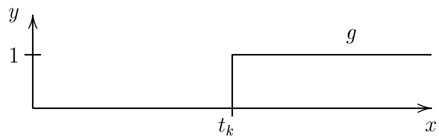
$$\text{either } g(x) = \frac{x-t_k}{t_{k+1}-t_k}\chi_{[t_k,t_{k+1}]}(x) + \chi_{(t_{k+1},1]}(x) \text{ for } k \neq l_1,\, l_2 \tag{3.25}$$

that is



$$\text{or } g(x) = \chi_{(t_k,1]}(x) \text{ for } k = l_1,\, l_2 \tag{3.26}$$

that is



Perturbation of datum is chosen as follows

$$h(x) = h[r,s](x) = \frac{x-r}{s-r}\chi_{[r,s]}(x) + \chi_{(s,1]}(x) \text{ with } \begin{cases} t_k \le r < s \le t_{k+1} & \text{if } k \neq l_1,\, l_2, \\ t_{l_1} \le r < s \le t_{l_2} & \text{if } k = l_1,\, l_2. \end{cases}$$

Then there is $C > 0$ *depending only on* $\mathbf{t}$ *s.t. (referring to Definition 3.10):*
if $k \neq l_1,\, l_2$, *then*

$$\sum_{a=0}^{3} \left\| \frac{d^a}{dx^a}\left(b_l[h,\mathbf{t},\omega] - g\right)\right\|_{L^\infty(t_l,t_{l+1})} + \ |b_l'''[h,\mathbf{t},\omega](t_l)| \ \le$$
$$C\ (|r-t_k| + |s-t_{k+1}|) \quad \text{for } \ 0 \le l \le \mathsf{T}, \tag{3.27}$$

$$\left|b_l'[h,\mathbf{t},\omega](t_l) - b_{l-1}'[h,\mathbf{t},\omega](t_l)\right| \le$$
$$\begin{cases} C\ (|r-t_k| + |s-t_{k+1}|) & \text{if } l \notin \{k, k+1\} \\ C & \text{if } l \in \{k, k+1\}, \end{cases} \tag{3.28}$$

$$\text{the map } (r,s) \mapsto b_k'''[h[r,s],\mathbf{t},\omega](t_k) \text{ is analytic,} \tag{3.29}$$

$$\begin{gathered}\text{the map } (r,s) \mapsto b_k'''[h[r,s],\mathbf{t},\omega](t_k) \text{ is not identically zero} \\ \text{on } \{(r,s) \in [t_k,t_{k+1}] \times [t_k,t_{k+1}] : r < s\} \\ \text{and vanishes of order } 1 \text{ as } (r,s) \to (t_k,t_{k+1}); \end{gathered} \tag{3.30}$$

if $k = l_1,\, l_2$, *then*

$$\sum_{a=0}^{3} \left\| \frac{d^a}{dx^a}\left(b_l[h,\mathbf{t},\omega] - g\right)\right\|_{L^\infty(t_l,t_{l+1})} + \ |b_l'''[h,\mathbf{t},\omega](t_l)| \ \le$$
$$C\ (|r-t_k| + |s-t_k|) \quad \text{for } \ 0 \le l \le \mathsf{T}, \tag{3.31}$$

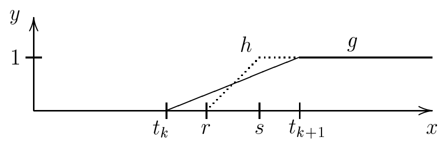
$$\left|b_l'[h,\mathbf{t},\omega](t_l) - b_{l-1}'[h,\mathbf{t},\omega](t_l)\right| \leq \\ C\ (|r-t_k|+|s-t_k|)\ \textit{if}\ l\in\{l_1+1,...,l_2-1\}. \quad (3.32)$$

□

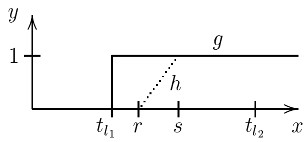
We emphasize that $g = b[g,\mathbf{t},\omega]$ holds true in the above statement (due to Theorem 3.9), hence (3.27) and (3.31) express also continuous dependence of b with respect to the perturbation h, i.e.

$$\|b_l[h,\mathbf{t},\omega] - b_l[g,\mathbf{t},\omega]\|_{L^\infty(t_l,t_{l+1})} \leq C\ (|r-t_k|+|s-t_{k+1}|)\,.$$

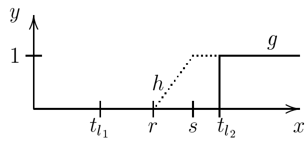
Proof. The following three pictures represent the three admissible cases for datum g together with the associated perturbations h



for $k \neq l_1,\ l_2$,



for $k = l_1$,



for $k = l_2$.

The unique solution of the Cauchy problem

$$u'''' + u = g \text{ on } (x_1,x_2), \qquad u(x_1) = u'(x_1) = u''(x_1) = u'''(x_1) = 0$$

can be represented through $u(x) = \int_{x_1}^x W(x-y)g(y)\,dy$ where

$$W(x) = \frac{\sqrt{2}}{4}\,(w_1(x) - w_2(t) + w_3(x) + w_4(x)) = \sum_{i=0}^{+\infty}\frac{(-1)^i}{(3+4i)!}x^{3+4i}, \quad (3.33)$$

$$\begin{array}{ll} w_1 = \exp(-x/\sqrt{2})\cos(x/\sqrt{2}) & w_2 = \exp(x/\sqrt{2})\cos(x/\sqrt{2}) \\ w_3 = \exp(-x/\sqrt{2})\sin(x/\sqrt{2}) & w_4 = \exp(x/\sqrt{2})\sin(x/\sqrt{2}). \end{array} \quad (3.34)$$

We set $j = k$ if $k < l_2$, $j = k-1$ if $k = l_2$. Then, by $g'' = 0$ on $[t_l, t_{l+1}]$ for any $l = 0,...,\mathsf{T}$ and Remark 3.9, we define d_l by

$$d_l[h,\mathbf{t},\omega](x) = \begin{cases} 0 & \forall x\in[t_l,t_{l+1}] \quad \text{if } l<j, \\ g(x) + \int_{t_j}^x W(x-y)(h(y)-g(y))\,dy & \forall x\in[t_j,t_{j+1}] \quad \text{if } l=j, \\ 1 & \forall x\in[t_l,t_{l+1}] \quad \text{if } l>j. \end{cases} \quad (3.35)$$

If b solves Problem 3.3 with data $\mathbf{t}$, ω and h then d_l fulfills (3.4) of Lemma 3.5. We choose decomposition (3.5) with this choice of d_l. We denote by

$d_l^{(a)}$ and $W^{(a)}$ the a-th distributional derivative in (t_l, t_{l+1}) of d_l and of W respectively. For $a = 1, 2, 3$ we have

$$d_j^{(a)}[h, \mathbf{t}, \omega](x) = $$
$$\int_{t_j}^{x} W^{(a)}(x-y)(h(y)-g(y))\, dy + \begin{cases} 1/\left(t_{k+1}-t_k\right) & \text{if } k \neq l_1,\, l_2 \text{ and } a = 1 \\ 0 & \text{if } k = l_1,\, l_2 \text{ or } a \neq 1. \end{cases}$$

For $a = 0, 1, 2, 3$ we estimate $\left|\int_{t_j}^{x} W^{(a)}(x-y)(h(y)-g(y))\, dy\right|$ uniformly on $x \in (t_j, t_{j+1})$. To this aim we observe that $\max\limits_{a=\{0,1,2,3\}} \left\|W^{(a)}\right\|_{L^\infty(0,1)} < +\infty$. If $k \neq l_1, l_2$, by (3.35), $g' = (t_{k+1} - t_k)^{-1}$ and $g'' = g''' = 0$ in $[t_k, t_{k+1}]$ we get

$$\|h-g\|_{L^\infty(0,1)} \leq \frac{\max\left\{r-t_k, t_{k+1}-s\right\}}{t_{k+1}-t_k},$$

then we can choose $C_0 = C_0(\mathbf{t}, W) + \infty$ such that

$$\|h-g\|_{L^\infty(0,1)} \leq C_0\, \left(|r-t_k| + |s-t_{k+1}|\right),$$

$$\begin{aligned} |d_k[h,\mathbf{t},\omega](x) - g(x)| &\leq C_0\, \left(|r-t_k| + |s-t_{k+1}|\right), \\ \left|d_k'[h,\mathbf{t},\omega](x) - \tfrac{1}{t_{k+1}-t_k}\right| &\leq C_0\, \left(|r-t_k| + |s-t_{k+1}|\right), \\ \left|d_k^{(a)}[h,\mathbf{t},\omega](x)\right| &\leq C_0\, \left(|r-t_k| + |s-t_{k+1}|\right) \quad \text{for } a = 2, 3. \end{aligned} \tag{3.36}$$

If $k = l_1$ or $k = l_2$ we have $\|h-g\|_{L^\infty(0,1)} = 1$, $spt(h-g) \subseteq [t_{l_1}, s] \cup [r, t_{l_2}]$ then by (3.35), in $[t_{l_1}, t_{l_2}]$ we have either $g = 0$ or $g = 1$ and $g' = g'' = g''' = 0$, and we can choose $C_1 = C_1(\mathbf{t}, W) < +\infty$ such that

$$\left|\int_{t_j}^{x} |W(x-y)(h(y)-g(y))|\, dy\right| \leq C_1\, \left(|r-t_k| + |s-t_k|\right),$$

$$\begin{aligned} |d_j[h,\mathbf{t},\omega](x) - g(x)| &\leq C_1\, \left(|r-t_k| + |s-t_k|\right), \\ \left|d_j^{(a)}[h,\mathbf{t},\omega](x)\right| &\leq C_1\, \left(|r-t_k| + |s-t_k|\right) \quad \text{for } a = 1, 2, 3. \end{aligned} \tag{3.37}$$

Since $\omega_{l_1} = \omega_{l_2} = \mathsf{J}$, by Remark 3.4 system (3.2) splits into three separate systems which give b on $[0, t_{l_1}]$, on $[t_{l_1}, t_{l_2}]$ and on $[t_{l_2}, 1]$ respectively. Since $h = 0$ on $[0, t_{l_1}]$ we have $b = 0$ on $[0, t_{l_1}]$, since $h = 1$ on $[t_{l_2}, 1]$ we have $b = 1$ on $[t_{l_2}, 1]$, then we have to study b only on the interval $[t_{l_1}, t_{l_2}]$ that is the subsystem

$$\mathbb{V}\boldsymbol{\gamma} = \boldsymbol{\alpha} \tag{3.38}$$

of system (3.7) corresponding to the $4(l_2-l_1)\times 4(l_2-l_1)$ square diagonal block $\mathbb{V}=\mathbb{U}_{l_1,l_2}$ of the matrix $\mathbb{U}$ of Lemma 3.5.
Hence, by denoting $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^{4(l_2-l_1)}$, (3.3), (3.36) and (3.37) entail the existence of a positive constant $C_2=C_2(\mathbf{t},W)<+\infty$ such that

$$\|\boldsymbol{\alpha}[h,\mathbf{t},\omega]\|\le\begin{cases} C_2\ (|r-t_k|+|s-t_{k+1}|) & \text{if } k\neq l_1,\ l_2\\ C_2\ (|r-t_k|+|s-t_k|) & \text{if } k=l_1,\ l_2\end{cases}$$

then there is $C_3=C_3(\mathbf{t},w_i)<+\infty$ (since $\boldsymbol{\gamma}=\mathbb{V}^{-1}\,\boldsymbol{\alpha}$ and the matrix $\mathbb{V}$ depends only on fixed data $\mathbf{t}$ and w_i) such that

$$\|\boldsymbol{\gamma}[h,\mathbf{t},\omega]\|=\begin{cases} C_3\ (|r-t_k|+|s-t_{k+1}|) & \text{if } k\neq l_1,\ l_2\\ C_3\ (|r-t_k|+|s-t_k|) & \text{if } k=l_1,\ l_2.\end{cases}$$

Statements (3.27), (3.28), (3.31), (3.32) follow. We are left to prove (3.29), (3.30). For any $l=0,...,\mathsf{T}$ choose d_l and b_l as in (3.35), (3.5).
The vector $\boldsymbol{\alpha}$ is an analytic function of (r,s) since (3.8) entails that $\boldsymbol{\alpha}$ depends on (r,s) only through $d_k^{(a)}(t_k)$ and $d_k^{(a)}(t_{k+1})$ for $a=0,1,2,3$, hence $\boldsymbol{\gamma}=\mathbb{V}^{-1}\,\boldsymbol{\alpha}$ is an analytic function of (r,s). Moreover, for $a=0,1,2,3$:

$$d_k^{(a)}[h[r,s],\mathbf{t},\omega](t_k)=0,$$

$$d_k^{(a)}[h[r,s],\mathbf{t},\omega](t_{k+1})=\int_{t_k}^{t_{k+1}}W^{(a)}(t_{k+1}-y)h[r,s](y)\,dy=$$
$$\int_r^s W^{(a)}(t_{k+1}-y)\frac{y-r}{s-r}\,dy+\int_s^{t_{k+1}}W^{(a)}(t_{k+1}-y)\,dy.$$

This identities together with (3.5) proves (3.29). We prove (3.30) first by showing that the partial derivative of $b_k'''[h[r,s],\mathbf{t},\omega](t_k)$ with respect to r is not identically zero, then exploiting $b_k'''[h[t_k,t_{k+1}],\mathbf{t},\omega]=h'''[t_k,t_{k+1}]_+(t_k)=0$ due to $b_k=h[t_k,t_{k+1}]$ on (t_k,t_{k+1}) and $h[t_k,t_{k+1}]$ is linear on (t_k,t_{k+1}).
To this aim we set $\varphi(x)=\frac{x-t_{k+1}}{(t_{k+1}-t_k)^2}\chi_{(t_k,t_{k+1})}(x)$ and claim

$$\left.\frac{\partial}{\partial r}b_k'''[h[r,s],\mathbf{t},\omega](t_k)\right|_{r=t_k,\ s=t_{k+1}}=b_k'''[\varphi,\mathbf{t},\omega](t_k). \tag{3.39}$$

By assuming (3.39) and arguing by contradiction assume $b_k'''[\varphi,\mathbf{t},\omega](t_k)=0$. By (3.2.$v$) and t_k crease point for $b[\varphi,\mathbf{t},\omega]$ we get $b_{k-1}'''[\varphi,\mathbf{t},\omega](t_k)=0$. Hence $g=0$ in $[t_k,t_{k+1}]$ and Theorem 3.9 together entail

$$\begin{cases} b_{k-1}[\varphi,\mathbf{t},\omega](x)=0 & x\in(t_{k-1},t_k),\\ b_k[\varphi,\mathbf{t},\omega](x)=\varphi & x\in(t_k,t_{k+1}).\end{cases}$$

But $\varphi_+(t_k) = -1/(t_{k+1} - t_k)$ entails $b'''_k[\varphi, \mathbf{t}, \omega](t_k) \neq 0$, then by (3.39) we get (3.30).
Now we prove the claimed equality (3.39). To this aim we prove:

$$\frac{\partial}{\partial r}\boldsymbol{\alpha}[h[r,s],\mathbf{t},\omega]\Big|_{r=t_k,\ s=t_{k+1}} = \boldsymbol{\alpha}[\varphi,\mathbf{t},\omega]\ . \tag{3.40}$$

By substituting $r = t_k$, $s = t_{k+1}$ in

$$\frac{\partial}{\partial r} d_k^{(a)}[h[r,s],\mathbf{t},\omega](t_{k+1}) = \frac{\partial}{\partial r}\left(\int_r^s W^{(a)}(t_{k+1}-y)\frac{y-r}{s-r}\,dy + \int_s^{t_{k+1}} W^{(a)}(t_{k+1}-y)\,dy\right) = \int_r^s W^{(a)}(t_{k+1}-y)\frac{y-s}{(s-r)^2}dy$$

we get

$$\frac{\partial}{\partial r} d_k^{(a)}[h[t_k,t_{k+1}],\mathbf{t},\omega](t_{k+1}) = \int_{t_k}^{t_{k+1}} W^{(a)}(t_{k+1}-y)\varphi(y)dy = d_k^{(a)}[\varphi,\mathbf{t},\omega](t_{k+1}). \tag{3.41}$$

We have $d_k^{(a)}[h[r,s],\mathbf{t},\omega](t_k) = 0$ for $a = 0,2,3$ then

$$\frac{\partial}{\partial r} d_k^{(a)}[h[r,s],\mathbf{t},\omega](t_k)\Big|_{r=t_k,\ s=t_{k+1}} = 0 \qquad \text{for } a = 0,2,3. \tag{3.42}$$

Equality (3.40) follows by (3.41) for any entry of $\boldsymbol{\alpha}[h[r,s],\mathbf{t},\omega]$ of type

$$\left.\begin{array}{ll} 1) & d'''_{k+1}[h[r,s],\mathbf{t},\omega](t_{k+1}) - d'''_k[h[r,s],\mathbf{t},\omega](t_{k+1}) = -d'''_k[h[r,s],\mathbf{t},\omega](t_{k+1}) \\ 2) & -d''_k[h[r,s],\mathbf{t},\omega](t_{k+1}) \\ 3) & d_{k+1}[h[r,s],\mathbf{t},\omega](t_{k+1}) - d_k[h[r,s],\mathbf{t},\omega](t_{k+1}) = 1 - d_k[h[r,s],\mathbf{t},\omega](t_{k+1}), \end{array}\right\} \tag{3.43}$$

for any other entry (3.40) is a trivial consequence of (3.42) since both sides of the equality are zero.
By (3.40), $\boldsymbol{\gamma} = \mathbb{V}^{-1}\,\boldsymbol{\alpha}$ and $\mathbb{V}$ independent of r, s

$$\frac{\partial}{\partial r}\boldsymbol{\gamma}[h[r,s],\mathbf{t},\omega]\Big|_{r=t_k,\ s=t_{k+1}} = \boldsymbol{\gamma}[\varphi,\mathbf{t},\omega]. \tag{3.44}$$

Eventually by (3.5), (3.42), (3.44) and $d_k'''[\varphi, \mathbf{t}, \omega](t_k) = 0$ we have

$$\frac{\partial}{\partial r} b_k'''[h[r,s], \mathbf{t}, \omega](t_k)\big|_{r=t_k,\ s=t_{k+1}} =$$

$$\frac{\partial}{\partial r} d_k''[h[r,s], \mathbf{t}, \omega](t_k)\big|_{r=t_k,\ s=t_{k+1}} + \sum_{i=1}^{4} \frac{\partial}{\partial r} \boldsymbol{\gamma}_{k,i}[h[r,s], \mathbf{t}, \omega](t_k)\big|_{r=t_k,\ s=t_{k+1}} \dddot{w}_i(t_k)$$

$$= \sum_{i=1}^{4} \boldsymbol{\gamma}_{k,i}[\varphi, \mathbf{t}, \omega](t_k) \dddot{w}_i(t_k) = b_k'''[\varphi, \mathbf{t}, \omega](t_k),$$

say (3.39). □

In the following lemma we show that for suitable step datum $\chi_{(a,1]}$ with jump in the interval (t_k, t_{k+1}) the value $b_k[\chi_{(a,1]}, \mathbf{t}, \omega](t_k)$ is not zero.

Lemma 3.20 *Fix* $\mathsf{T}, \mathsf{m}, \mathsf{n} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{m} + \mathsf{n}$, $\widetilde{\mathbf{t}} \in A_{\mathsf{T}}$ *and* $\omega \in \Omega_{\mathsf{m},\mathsf{n}}$. *For any* $k \in \{1, ..., \mathsf{T}\}$ *we set* $\vartheta_k : [\widetilde{t}_k, \widetilde{t}_{k+1}] \to \mathbb{R}$ *by*

$$\vartheta_k(a) = b_k[\chi_{(a,1]}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_k) \qquad \forall a \in [\widetilde{t}_k, \widetilde{t}_{k+1}],$$

where $b = b[\chi_{(a,1]}, \widetilde{\mathbf{t}}, \omega]$ *is the unique solution of Problem 3.3. Then*

1. ϑ_k *is an analytic function with respect to* $a \in (\widetilde{t}_k, \widetilde{t}_{k+1})$ *and is continuous with respect to* $a \in [\widetilde{t}_k, \widetilde{t}_{k+1}]$,
2. *for any* $\varepsilon \in \Big(0, \mathrm{dist}(\widetilde{\mathbf{t}}, \partial A_{\mathsf{T}})\Big)$ *there is* $a \in (\widetilde{t}_k + \varepsilon, \widetilde{t}_{k+1} - \varepsilon)$ *such that* $\vartheta_k(a) \neq 0$. *Here* ∂A_{T} *is the topological boundary of* A_{T} *in* $\mathbb{R}^{\mathsf{T}}$.

Proof. Throughout the proof we write ϑ in place of ϑ_k since k is fixed. Referring to (3.33) we define d_l by

$$d_l[\chi_{(a,1]}, \widetilde{\mathbf{t}}, \omega](x) = \begin{cases} 0 & \forall x \in [\widetilde{t}_l, \widetilde{t}_{l+1}] \quad \text{if } l < k, \\ \int_{\widetilde{t}_l}^{x} W(x-y)\chi_{(a,1]}(y)\, dy & \forall x \in [\widetilde{t}_k, \widetilde{t}_{k+1}] \quad \text{if } l = k, \\ 1 & \forall x \in [\widetilde{t}_l, \widetilde{t}_{l+1}] \quad \text{if } l > k, \end{cases}$$

hence d_l fulfills (3.4) and we choose the decomposition (3.5) of $b[\chi_{(a,1]}, \widetilde{\mathbf{t}}, \omega]$ related to this choice of d_l.

By Lemma 3.5 and Theorem 3.8 it is enough to prove that both $d_l^{(r)}[\chi_{(a,1]}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_l)$ and $d_l^{(r)}[\chi_{(a,1]}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{l+1})$ are analytic functions of a on $(\widetilde{t}_k, \widetilde{t}_{k+1})$ and continuous functions of a on $[\widetilde{t}_k, \widetilde{t}_{k+1}]$, $r = 0, 1, 2, 3$. If $l \neq k$ then this fact is straightforward, if $l = k$ then this fact follows by direct computation:

$$d_k^{(r)}[\chi_{(a,1]}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_k) = 0, \qquad d_k^{(r)}[\chi_{(a,1]}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{k+1}) = \int_a^{\widetilde{t}_{k+1}} W^{(r)}(\widetilde{t}_{k+1} - y)\, dy.$$

Then statement 1 is proven.
Statement 2 will follow by the first statement if we show that

$$\vartheta(\widetilde{t}_k) \neq 0, \tag{3.45}$$

since (3.45) together with statement 1 entails that the analytic function ϑ may have only isolated zeros in $(\widetilde{t}_k, \widetilde{t}_{k+1})$.
If $\omega_k = \mathsf{J}$ then by Theorem 3.9 $\vartheta(\widetilde{t}_k) = b_k[\chi_{(\widetilde{t}_k,1]}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_k) = \chi_{(\widetilde{t}_k,1]}(\widetilde{t}_{k+}) = 1$.
If $\omega_k = \mathsf{C}$ then the following longer analysis is required to show (3.45).
By recalling the convention $\omega_0 = \omega_{\mathsf{T}+1} = \mathsf{J}$ we denote by l_1, l_2 the unique pair of integers fulfilling

$$\begin{cases} 0 \leq l_1 < k < l_2 \leq \mathsf{T}+1, \\ \omega_{l_1} = \omega_{l_2} = \mathsf{J}, \qquad \omega_l = \mathsf{C} \quad \forall l \in \{l_1+1, ..., l_2-1\}, \end{cases}$$

we define a $4(l_2 - l_1)$-dimensional row vector $\mathbf{v}$ by

$$\mathbf{v}_l = \begin{cases} w_i(t_k) & \text{if } l = 4\,(k-1-l_1)+i \text{ for } i = 1, ..., 4, \\ 0 & \text{otherwise (say } l \neq 4\,(k-1-l_1)+i). \end{cases} \tag{3.46}$$

Notice that $\mathbf{v}$ has only four non trivial entries coincident with the left half of the first line of block B_{k-l_1}.
We make a new choice of d_l by

$$d_l[\chi_{(\widetilde{t}_k,1]}, \widetilde{\mathbf{t}}, \omega](x) = \begin{cases} 0 & \text{if } l < k \\ 1 & \text{if } l \geq k \end{cases} \qquad \forall x \in [\widetilde{t}_l, \widetilde{t}_{l+1}], \tag{3.47}$$

hence d_l fulfills (3.4) and we choose the decomposition (3.5) of $b[\chi_{(a,1]}, \widetilde{\mathbf{t}}, \omega]$ with this choice of d_l.
Since $\omega_{l_1} = \omega_{l_2} = \mathsf{J}$ by Remark 3.4 system (3.2) splits into three separate systems which give b on $[0, \widetilde{t}_{l_1}]$, on $[\widetilde{t}_{l_1}, \widetilde{t}_{l_2}]$ and on $[\widetilde{t}_{l_2}, 1]$ respectively. Since $h = 0$ on $[0, \widetilde{t}_{l_1}]$ we have $b \equiv 0$ on $[0, \widetilde{t}_{l_1}]$, since $h = 1$ on $[\widetilde{t}_{l_2}, 1]$ we have $b \equiv 1$ on $[\widetilde{t}_{l_2}, 1]$, then, by Lemma 3.5, we have to study b only on the interval $[\widetilde{t}_{l_1}, \widetilde{t}_{l_2}]$ that is the subsystem

$$\mathbb{V}\,\boldsymbol{\gamma} = \boldsymbol{\alpha} \tag{3.48}$$

of system (3.7) corresponding to the diagonal block $\mathbb{V} \stackrel{\text{def}}{=} \mathbb{U}_{l_1,l_2}$ of $\mathbb{U}$ (see (3.8),(3.9)) with $b_l = d_l + \sum_{i=1}^{4} \boldsymbol{\gamma}_{l,i} w_i$ and d_l defined by (3.47). System (3.48) is an algebraic system: $4(l_2 - l_1)$ algebraic equations, $4(l_2 - l_1)$ unknowns

$\boldsymbol{\gamma} = (\boldsymbol{\gamma}_{l,i}) \in \mathbb{R}^{4(l_2-l_1)}$ with $l = 1, ..., (l_2 - l_1 + 1)$ and $i = 1, ..., 4$; the matrix $\mathbb{V} = \mathbb{V}[\widetilde{\mathbf{t}}, \omega] = [\mathbb{V}_{i,j}]_{i,j=1}^{4(l_2-l_1)}$ is invertible by Lemma 3.5 and Theorem 3.8; here $\boldsymbol{\alpha} = \boldsymbol{\alpha}[\chi_{(\widetilde{t}_k,1]}, \mathbf{t}, \omega]$ replaces $\mathbf{a}$ in (3.8) with $\boldsymbol{\alpha}_l = \mathbf{a}_{l+4l_1}$.
Vector $\boldsymbol{\alpha}$ has only one non zero entry i.e.

$$\boldsymbol{\alpha}_{4(k-l_1)} = d_k[\chi_{(\widetilde{t}_k,1]}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_k) - d_{k-1}[\chi_{(\widetilde{t}_k,1]}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_k) = 1. \tag{3.49}$$

Arguing by contradiction assume that $\vartheta(\widetilde{t}_k) = b_k[\chi_{(\widetilde{t}_k,1]}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_k) = 0$. Then by $\omega_k = \mathsf{C}$, (3.5) and (3.47) we deduce that the unique solution $\boldsymbol{\gamma}$ of (3.48) fulfills the following relationship, where the common dependence on $[\chi_{(\widetilde{t}_k,1]}, \widetilde{\mathbf{t}}, \omega]$ is always understood:

$$0 = b_k(\widetilde{t}_k) = b_{k-1}(\widetilde{t}_k) = d_{k-1}(\widetilde{t}_k) + \sum_{i=1}^{4} \boldsymbol{\gamma}_{k-1,i} w_i(\widetilde{t}_k) = \sum_{i=1}^{4} \boldsymbol{\gamma}_{k-1,i} w_i(\widetilde{t}_k) \tag{3.50}$$

Hence, due to (3.46), (3.48) and (3.50), the $(4(l_2-l_1)+1)$ dimensional vector $[\boldsymbol{\gamma}, -1]$, fulfills the linear system

$$\begin{bmatrix} \mathbb{V} & \boldsymbol{\alpha} \\ \mathbf{v} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma} \\ -1 \end{bmatrix} = 0. \tag{3.51}$$

Equation (3.51) entails $\det \begin{bmatrix} \mathbb{V} & \boldsymbol{\alpha} \\ \mathbf{v} & 0 \end{bmatrix} = 0$ then, since $\det(\mathbb{V}) \neq 0$, $\mathbf{v} \neq \mathbf{0}$, and α has only one non zero component given by (3.49), we get

> $\mathbf{v}$ can be uniquely written as a non trivial linear combination of the $4(k - l_1) - 1$ rows of $\mathbb{V}$ different from the $4(k - l_1)$-th row whose coefficient vector is denoted by $\boldsymbol{\rho}$: $\mathbf{v}_l = \sum_{j \neq 4(k-l_1)} \rho_j \, \mathbb{V}_{j,l}$. (3.52)

We consider two possibilities for coefficient $\boldsymbol{\rho}_{4(k-l_1)+1}$ (related to the row below the one with unique non trivial component of $\boldsymbol{\alpha}$): both possibilities leads to a contradiction.

- If $\boldsymbol{\rho}_{4(k-l_1)+1} \neq 0$ we choose a $4(l_2 - l_1 - k)$ square matrix by selecting $4(l_2 - k)$ square diagonal SE block of (3.9) and decomposing it as follows:

$$\mathbf{u}_2 = \begin{bmatrix} \mathbb{V}_{4(k-l_1)+1,4(k-l_1)+1} & \cdot & \cdot & \cdot & \mathbb{V}_{4(k-l_1)+1,4(l_2-l_1)} \end{bmatrix}$$

$$\mathbb{U}_2 = \begin{bmatrix} \mathbb{V}_{4(k-l_1)+2,4(k-l_1)+1} & \cdot & \cdot & \cdot & \mathbb{V}_{4(k-l_1)+2,4(l_2-l_1)} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \mathbb{V}_{4(l_2-l_1),4(k-l_1)+1} & \cdot & \cdot & \cdot & \mathbb{V}_{4(l_2-l_1),4(l_2-l_1)} \end{bmatrix}.$$

By (3.46) we have $\begin{bmatrix} \mathbf{v}_{4(k-l_1)+1} & \cdot & \cdot & \cdot & \mathbf{v}_{4(l_2-l_1)} \end{bmatrix} = \mathbf{0}$, hence (3.52) entails

$$0 = \det \begin{bmatrix} \boldsymbol{\rho}_{4(k-l_1)+1}\, \mathbf{u}_2 \\ \mathbb{U}_2 \end{bmatrix} = \boldsymbol{\rho}_{4(k-l_1)+1} \det \begin{bmatrix} \mathbf{u}_2 \\ \mathbb{U}_2 \end{bmatrix}. \tag{3.53}$$

Since $\begin{bmatrix} \mathbf{u}_2 \\ \mathbb{U}_2 \end{bmatrix}$ is the NW square diagonal block of the invertible matrix of coefficients of the linear system (3.2.ii)-(3.2.vi) obtained by solving Problem 3.3 on the interval $[\widetilde{t}_k, 1]$ with arrangement data as follows:

$$\begin{cases} \mathsf{S} = \mathsf{T} - k, \; \mathsf{j} = 0, \; \mathsf{c} = \mathsf{S} \\ \mathbf{t} \text{ such that } t_l = \widetilde{t}_{l+k} \qquad \forall l \in \{1, ..., \mathsf{S}\} \\ \sigma \text{ such that } \sigma_l = \omega_{l+k} \qquad \forall l \in \{1, ..., \mathsf{S}\}. \end{cases}$$

Then Lemma 3.5 and Theorem 3.8 entail $\det \begin{bmatrix} \mathbf{u}_2 \\ \mathbb{U}_2 \end{bmatrix} \neq 0$ contradicting (3.53).

- If $\boldsymbol{\rho}_{4(k-l_1)+1} = 0$ we choose a $4(k-l_1)$ square block by taking the $4(k-l_1)$ square diagonal NW block of (3.9) and decomposing it as follows:

$$\mathbb{U}_1 = \begin{bmatrix} \mathbb{V}_{1,1} & \cdot & \cdot & \cdot & \mathbb{V}_{1,4(k-l_1)} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \mathbb{V}_{4(k-l_1)-1,1} & \cdot & \cdot & \cdot & \mathbb{V}_{4(k-l_1)-1,4(k-l_1)} \end{bmatrix}$$
$$\mathbf{u}_1 = \begin{bmatrix} \mathbb{V}_{4(k-l_1),1} & \cdot & \cdot & \cdot & \mathbb{V}_{4(k-l_1),4(k-l_1)} \end{bmatrix}.$$

Definition (3.46) entails

$$\mathbf{u}_1 = \begin{bmatrix} \mathbf{v}_1 & \cdot & \cdot & \cdot & \mathbf{v}_{4(k-l_1)} \end{bmatrix} \neq \mathbf{0}, \quad \begin{bmatrix} \mathbf{v}_{4(k-l_1)+1} & \cdot & \cdot & \cdot & \mathbf{v}_{4(l_2-l_1)} \end{bmatrix} = \mathbf{0}.$$

Hence properties $\det(\mathbb{V}) \neq 0$ and (3.52) entail that

$\mathbf{u}_1$ can be uniquely written as a non trivial linear combination of the rows of $\mathbb{U}_1$. (3.54)

hence (3.54) entails

$$\det \begin{bmatrix} \mathbb{U}_1 \\ \mathbf{u}_1 \end{bmatrix} = 0. \tag{3.55}$$

Though $\begin{bmatrix} \mathbb{U}_1 \\ \mathbf{u}_1 \end{bmatrix}$ is a square NW block of $\mathbb{V}$, equation (3.55) does not entail an immediate contradiction with $\det(\mathbb{V}) \neq 0$, since $\mathbb{V}$ is not a square block diagonal matrix with NW minor given by $\begin{bmatrix} \mathbb{U}_1 \\ \mathbf{u}_1 \end{bmatrix}$.

We consider a symmetric arrangement of creases in $[\widetilde{t}_k, 2\widetilde{t}_k - \widetilde{t}_{l_1}]$:

$$\begin{cases} (i) & \mathsf{S} = 2(k - l_1) - 1,\ \mathsf{c} = \mathsf{S},\ \mathsf{j} = 0 \\ (ii) & \mathbf{t} \text{ such that } t_l = \begin{cases} \widetilde{t}_{l+l_1} & \text{if } l = 0, ..., k - l_1 \\ 2\widetilde{t}_k - \widetilde{t}_{2k-l_1-l} & \text{if } l = k - l_1 + 1, ..., \mathsf{S} + 1, \end{cases} \\ (iii) & \sigma \text{ such that } \sigma_l = \mathsf{C} \text{ for any } l = 1, ..., \mathsf{S}, \end{cases} \tag{3.56}$$

and the following differential problem with arrangement (3.56)

$$\left.\begin{array}{lll} (i) & z_l'''' + z_l = g & \text{on } (t_l, t_{l+1}) \text{ for } l = 0, 1, ..., \mathsf{S} \\ (ii) & z_l''(t_l) = z_l''(t_{l+1}) = 0 & \text{for } l = 0, 1, ..., \mathsf{S} \\ (iii) & z_l'''(t_l) = 0 & \text{for } l = 0, 1, ..., \mathsf{S} \\ (iv) & z_l'''(t_{l+1}) = 0 & \text{for } l = 0, 1, ..., \mathsf{S} \\ (v) & z_{l-1}'''(t_l) = z_l'''(t_l) & \text{for } l = 0, 1, ..., \mathsf{S} \\ (vi) & z_{l-1}(t_l) = z_l(t_l) & \text{for } l = 0, 1, ..., \mathsf{S}, \end{array}\right\} \tag{3.57}$$

we also denote by $\mathbb{W} = \mathbb{W}[\mathbf{t}, \sigma] = [\mathbb{W}_{i,j}]_{i,j=1}^{4(\mathsf{S}+1)}$ the invertible matrix of coefficients of the algebraic linear system related to (3.57) by the same construction made in Lemma 3.5.

If the arrangement of $(\mathbf{t}, \sigma)$ fulfills (3.56) we get the following identity for the $4(k - l_1)$ square diagonal NW block of $\mathbb{W}$:

$$\begin{bmatrix} \mathbb{U}_1 \\ \mathbf{u}_1 \end{bmatrix} = \begin{bmatrix} \mathbb{W}_{1,1} & \cdots & \mathbb{W}_{1,4(k-l_1)} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \mathbb{W}_{4(k-l_1)-1,1} & \cdots & \mathbb{W}_{4(k-l_1)-1,4(k-l_1)} \\ \mathbb{W}_{4(k-l_1),1} & \cdots & \mathbb{W}_{4(k-l_1),4(k-l_1)} \end{bmatrix}.$$

We select the $4(\mathsf{S} - l_1 - k + 1)$ SE square diagonal block and substitute its first row with the one above, by setting:

$$\mathbf{m} = \begin{bmatrix} \mathbb{W}_{4(k-l_1),4(k-l_1)+1} & \cdots & \mathbb{W}_{4(k-l_1),4(\mathsf{S}+1)} \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} \mathbb{W}_{4(k-l_1)+2,4(k-l_1)+1} & \cdots & \mathbb{W}_{4(k-l_1)+2,4(\mathsf{S}+1)} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \mathbb{W}_{4(\mathsf{S}-l_1),4(k-l_1)+1} & \cdots & \mathbb{W}_{4(\mathsf{S}+1),4(\mathsf{S}+1)} \end{bmatrix}.$$

Theorem 3.8 applied to Problem (3.57) entails $\det(\mathbb{W}) \neq 0$ then, by (3.55), $\mathbf{m}$ is not a linear combination of the rows of $\mathbb{M}$ that is

$$\det \begin{bmatrix} \mathbf{m} \\ \mathbb{M} \end{bmatrix} \neq 0. \tag{3.58}$$

We introduce the two following problems

Given $f \in L^2(\widetilde{t}_{l_1}, \widetilde{t}_k)$ find φ such that

$$\left.\begin{array}{lll} (i) & \varphi_l'''' + \varphi_l = f & \text{on } (t_l, t_{l+1}) \text{ for } l = 0, ..., k-l_1-1 \\ (ii) & \varphi_l''(t_l) = \varphi_l''(t_{l+1}) = 0 & \text{for } l = 0, ..., k-l_1-1 \\ (iii) & \varphi_{l_1}'''(t_0) = 0 & \\ (iv) & \varphi_{l-1}'''(t_l) = \varphi_l'''(t_l) & \text{for } l = 1, ..., k-l_1-1 \\ (v) & \varphi_{k-l_1-1}(t_{k-l_1}) = 0 & \\ (vi) & \varphi_{l-1}(t_l) = \varphi_l(t_l) & \text{for } l = 1, ..., k-l_1-1\,. \end{array}\right\} \tag{3.59}$$

Given $h \in L^2(\widetilde{t}_k, 2\widetilde{t}_k - \widetilde{t}_{l_1})$ find ψ such that

$$\left.\begin{array}{lll} (i) & \psi_l'''' + \psi_l = h & \text{on } (t_l, t_{l+1}) \text{ for } l = k-l_1, ..., \mathsf{S} \\ (ii) & \psi_l''(t_l) = \psi_l''(t_{l+1}) = 0 & \text{for } l = k-l_1, ..., \mathsf{S} \\ (iii) & \psi_{\mathsf{S}}'''(t_{\mathsf{S}+1}) = 0 & \\ (iv) & \psi_{l-1}'''(t_l) = \psi_l'''(t_l) & \text{for } l = k-l_1+1, ..., \mathsf{S} \\ (v) & \psi_{k-l_1}(t_{k-l_1}) = 0 & \\ (vi) & \psi_{l-1}(t_l) = \psi_l(t_l) & \text{for } l = k-l_1+1, ..., \mathsf{S}. \end{array}\right\} \tag{3.60}$$

Problems (3.59) (3.60) are slight modification of Problem 3.2 and their solutions have the same value at $\widetilde{t}_k = t_{k-l_1}$: if $l = k - l_1$ then (v) reads $\varphi_{k-l_1-1}(\widetilde{t}_k) = 0 = \psi_{k-l_1}(\widetilde{t}_k)$. Notice that matrices $\begin{bmatrix} \mathbb{U}_1 \\ \mathbf{u}_1 \end{bmatrix}$ and $\mathbb{V}$ play the same role respectively in Problems 3.59 and 3.3 while matrices $\begin{bmatrix} \mathbf{m} \\ \mathbb{M} \end{bmatrix}$ and $\mathbb{V}$ play the same role respectively in Problems 3.60 and 3.3. Then

$$\left\{\begin{array}{l} \text{existence and uniqueness of solutions} \\ \qquad\qquad\qquad \text{of Problem 3.59 depends only on } \begin{bmatrix} \mathbb{U}_1 \\ \mathbf{u}_1 \end{bmatrix}, \\ \text{existence and uniqueness of solutions} \\ \qquad\qquad\qquad \text{of Problem 3.60 depends only on } \begin{bmatrix} \mathbf{m} \\ \mathbb{M} \end{bmatrix}. \end{array}\right. \tag{3.61}$$

Consider the affine map $\mu : [\widetilde{t}_{l_1}, \widetilde{t}_k] \to [\widetilde{t}_k, 2\widetilde{t}_k - \widetilde{t}_{l_1}]$ defined by $\mu(t) = 2\widetilde{t}_k - t$. Observe that any solution φ of Problem 3.59 with datum f gives a solution $\psi(t) = \varphi(\mu^{-1}(t))$ of Problem 3.60 with datum $h = f(\mu^{-1}(t))$, and that any solution ψ of Problem 3.60 with datum h gives a solution $\varphi(t) = \psi(\mu(t))$ of Problem 3.59 with datum $f = h(\mu(t))$. By (3.61) either $\begin{bmatrix} \mathbb{U}_1 \\ \mathbf{u}_1 \end{bmatrix}$ and $\begin{bmatrix} \mathbf{m} \\ \mathbb{M} \end{bmatrix}$ are both non singular, or $\begin{bmatrix} \mathbb{U}_1 \\ \mathbf{u}_1 \end{bmatrix}$ and $\begin{bmatrix} \mathbf{m} \\ \mathbb{M} \end{bmatrix}$ are both singular, hence there is a contradiction between (3.55) and (3.58). $\square$

4 An auxiliary variational problem

We have already noticed that jump and crease points of $\operatorname{argmin} F^g_{\alpha,\beta}$ are not necessarily localized among those of g even if it is a continuous and piecewise affine function (see Section 4 of [4]). In this section we develop some technical tools enabling us to overcome this difficulty. At first we introduce a problem which is equivalent to the minimization of Blake-Zisserman functional in case of continuous piecewise affine datum g.

Definition 4.1 *For any* $\mathsf{Q} \in \{0, 1, 2, ...\}$ *and* $\mathbf{q} = (q_i)_{i=1}^{\mathsf{Q}} \in A_{\mathsf{Q}}$ *let*

$$\mathcal{K}_{\mathbf{q}} = \{\, u \in \mathcal{H}^2 : \ \begin{array}{ll} \ddot{u}(t^+) = \ddot{u}(t^-) = 0 & \forall t \in S_u \cup S_{\dot{u}} \quad \textit{and} \\ \dddot{u}(t^+) = \dddot{u}(t^-) = 0 & \forall t \in (S_u \cup S_{\dot{u}}) \setminus \{q_i\}_{i=1}^{\mathsf{Q}} \,\}. \end{array}$$

Problem 4.2 *Given* $\mathsf{Q} \in \{0, 1, 2, ...\}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $g \in L^2(0, 1)$ *and* α, β *satisfying* (1.2)*, minimize the functional* $F^g_{\alpha,\beta}$ *on* $\mathcal{K}_{\mathbf{q}}$.

Remark 4.3 *If* $g \in \mathbb{A}_{\mathbf{q}}$ *(continuous piecewise affine function with crease points at* $\mathbf{q}$*) in addition to usual assumptions* (1.2) *and* (1.3)*, then the set of solutions of Problem 4.2 coincide with the set of minimizers of Blake-Zisserman functional* $F^g_{\alpha,\beta}$ *with the same data* α, β, g. *This is true because* $F^g_{\alpha,\beta}$ *admits minimizers over* $\mathcal{H}^2$ *and they must belong to* $\mathcal{K}_{\mathbf{q}}$ *due to* (*ii*) *and* (*vi*)*, of Theorem 2.1.*

Motivated by this remark, from now on, we focus the multiplicity of solutions of Problem 4.2. We introduce the following problem in order to study elements of $\mathcal{K}_{\mathbf{q}} \cap \operatorname{argmin} F^g_{\alpha,\beta}$ having location and quality compatible with suitable location and quality a priori prescribed with at most j jump points and c crease points. Analysis made in Section 3 and Remark 4.3 suggest to look for solutions γ of the following problem.

Problem 4.4 *Given* $\mathsf{Q}, \mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ *and* $g \in L^2(0,1)$, *find* $\gamma \in \mathcal{H}^2(0,1)$ *s.t.* $\gamma = \gamma_l$ *on* (t_l, t_{l+1}) *where*

$$\left.\begin{array}{lll}
(i) & \gamma_l'''' + \gamma_l = g & \text{on } (t_l, t_{l+1}) \text{ for } l = 0, 1, ..., \mathsf{T} \\
(ii) & \gamma_l''(t_l) = \gamma_l''(t_{l+1}) = 0 & \text{for } l = 0, 1, ..., \mathsf{T} \\
(iii) & \gamma_l'''(t_l) = 0 & \text{if either } l = 0, \text{ or } l = 1, ..., \mathsf{T} \text{ s.t. } \sigma_l = \mathsf{J}, \\
 & & \qquad \text{or } l = 1, ..., \mathsf{T} \text{ s.t. } t_l \notin \mathbf{q} \\
(iv) & \gamma_l'''(t_{l+1}) = 0 & \text{if either } l = \mathsf{T}, \text{ or } l = 1, ..., \mathsf{T} \text{ s.t. } \sigma_{l+1} = \mathsf{J}, \\
 & & \qquad \text{or } l = 1, ..., \mathsf{T} \text{ s.t. } t_{l+1} \notin \mathbf{q} \\
(v) & \gamma_{l-1}'''(t_l) = \gamma_l'''(t_l) & \text{if } l = 1, ..., \mathsf{T} \text{ and } \sigma_l = \mathsf{C} \text{ and } t_l \in \mathbf{q} \\
(vi) & \gamma_{l-1}(t_l) = \gamma_l(t_l) & \text{if } l = 1, ..., \mathsf{T} \text{ and } \sigma_l = \mathsf{C} \text{ and } t_l \in \mathbf{q}
\end{array}\right\} \quad (4.1)$$

$$\mathbf{t} \text{ and } \sigma \text{ are called respectively location and quality of Problem 4.4.} \quad (4.2)$$

We emphasize that γ could be discontinuous at some t_l if $\sigma_l = \mathsf{C}$ and $t_l \notin \mathbf{q}$.

Theorem 4.5 *For any* $\mathsf{Q}, \mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ *and* $g \in L^2(0,1)$, *Problem 4.4 admits unique solution.*

Proof. Consider the quality ω defined by $\omega_l = \sigma_l$ if $t_l \in \mathbf{q}$ and $\omega_l = \mathsf{J}$ otherwise. Problem 4.4 is equivalent to Problem 3.3 with datum g, quality ω and location $\mathbf{t}$, then the thesis follows by Theorem 3.8. □

Remark 4.6 *Location and quality of the solution* γ *of Problem 4.4 are compatible with location and quality* (4.2) *of Problem 4.4, say*

$$S_\gamma \subseteq \{t_i: \quad \sigma_i = \mathsf{J}\}, \qquad S_{\dot\gamma} \setminus S_\gamma \subseteq \{t_i: \quad \sigma_i = \mathsf{C}\}.$$

Remark 4.7 *We notice that, when* $g \in \mathbb{A}_{\mathbf{q}}$, *the relationship between Problem 4.2 and Problem 4.4 is analogous to relationship between minimization of* $F^g_{\alpha,\beta}$ *and Problem 3.3: any solution* u *of Problem 4.2 solves Problem 4.4 with the same location* $\mathbf{t} = \mathbf{t}(u)$ *and quality* $\sigma = \sigma(u)$.

Definition 4.8 *For any* $\mathsf{Q}, \mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$ *and* $g \in L^2(0,1)$, *set:*

1. $\gamma = \gamma[g, \mathbf{t}, \mathbf{q}, \sigma]$ *is the unique function* $\gamma = \gamma(x) \in \mathcal{H}^2$ *piecewise defined by the solutions* $\{\gamma_l = \gamma_l[g, \mathbf{t}, \mathbf{q}, \sigma] \in H^2(t_l, t_{l+1})\}_{l=0}^{\mathsf{T}}$ *of system* (4.1). *Parameters* g, $\mathbf{t}$, $\mathbf{q}$, σ *will be dropped whenever there is no risk of confusion. For any* $l \in \{0, ..., \mathsf{T}\}$ *we denote by* $\gamma_l', \gamma_l'', ..., \gamma_l^{(r)}$ *the first, second, ..., r-th distributional derivative in* (t_l, t_{l+1}) *of* γ_l *with respect to* x. *Notice that* $\gamma_l' = \dot\gamma_l$, $\gamma_l'' = \ddot\gamma_l$, ..., *but* γ' *and* γ'' *may be different from* $\dot\gamma$ *and* $\ddot\gamma$ *due to singular part at* t_l.

2. $\mathfrak{F}(g, \mathbf{t}, \mathbf{q}, \sigma)$ *is the absolutely continuous part* $\mathcal{F}^g$ *of* $F^g_{\alpha,\beta}$ *evaluated at the solution* $\gamma[g, \mathbf{t}, \mathbf{q}, \sigma]$ *of Problem 4.4*

$$\mathfrak{F}(g, \mathbf{t}, \mathbf{q}, \sigma) = \mathcal{F}^g(\gamma[g, \mathbf{t}, \mathbf{q}, \sigma]) \tag{4.3}$$

$$\mathfrak{F}(\cdot, \cdot, \mathbf{q}, \sigma) : L^2(0,1) \times A_{\mathsf{T}} \to \mathbb{R}$$

3. *If in addition* g *is continuous piecewise affine with location* $\mathbf{q}$ *(i.e.* $g \in \mathbb{A}_{\mathbf{q}}$*) and the vector* $\mathbf{g}$ *is associated to* g *by* (3.16) *and* (3.17)*, set:*

$$\gamma(\cdot, \cdot, \mathbf{q}, \sigma) : \mathbb{R}^{\mathsf{Q}+2} \times A_{\mathsf{T}} \to L^2(0,1) \quad by$$

$$\gamma[\mathbf{g}, \mathbf{t}, \mathbf{q}, \sigma](x) = \gamma[g, \mathbf{t}, \mathbf{q}, \sigma](x) \tag{4.4}$$

$$\mathfrak{F}(\cdot, \cdot, \mathbf{q}, \sigma) : \mathbb{R}^{\mathsf{Q}+2} \times A_{\mathsf{T}} \to \mathbb{R} \quad by \quad \mathfrak{F}(\mathbf{g}, \mathbf{t}, \mathbf{q}, \sigma) = \mathfrak{F}(g, \mathbf{t}, \mathbf{q}, \sigma). \tag{4.5}$$

We emphasize that Definition 3.10 and Definition 3.13 depends on the location and quality of Problem 3.3 while Definition 4.8 depends not only on the location and quality (4.2) of Problem 4.4 but also on vector $\mathbf{q}$ (coincident with location of g in case 3).

Proposition 4.9 *Fix* $\mathsf{Q}, \mathsf{T}, \mathsf{j}, \mathsf{c}, r \in \{0, 1, 2, ...\}$*,* $\mathsf{T} = \mathsf{j} + \mathsf{c}$*,* $\mathbf{q} \in A_{\mathsf{Q}}$ *and* $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$*, then:*

1. *the map* $g \mapsto \gamma(g, \mathbf{t}, \mathbf{q}, \sigma)$ *is linear in* $g \in L^2(0,1)$ *for any* $\mathbf{t} \in A_{\mathsf{T}}$*,*
   in particular $g \equiv 0$ *entails* $\gamma \equiv 0$*;*
   the map $g \mapsto \mathfrak{F}(g, \mathbf{t}, \mathbf{q}, \sigma)$ *is continuous and* 2*-homogeneous with respect to* $g \in L^2(0,1)$ *for any* $\mathbf{t} \in A_{\mathsf{T}}$*;*

2. *for any piecewise affine function* $g \in \mathbb{A}_{\mathbf{q}}$ *and any solution* u *of Problem 4.4 such that* u *has* j *jump points,* c *crease points and quality* σ*, the map* $\mathbf{t} \longmapsto \mathfrak{F}(g, \mathbf{t}, \mathbf{q}, \sigma)$ *achieves its minimum with respect to* $\mathbf{t}$ *in* A_{T} *at* $\mathbf{t}(u) = (t_1(u), ..., t_{\mathsf{T}}(u))$*. Moreover* $S_u = \{t_l(u) : \quad \sigma_l = \mathsf{J}\}$*,* $S_{\dot u} \setminus S_u = \{t_l(u) : \quad \sigma_l = \mathsf{C}\}$ *and* $\gamma = u$ *is the unique minimizer of* $\mathcal{F}$ *in* $\mathcal{H}^2_u$*;*

3. *for any piecewise affine function* g *the function* $\gamma = g$ *solves Problem 4.4 with data* g*,* $\mathbf{t} = \mathbf{t}(g)$*,* $\mathbf{q} = \mathbf{q}(g)$ *and* $\sigma = \sigma(g)$*;*

4. *for any open cell* W *of the CW structure induced by* $\mathbf{q}$ *on* A_{T}*, the restriction to* $\mathbb{A}_{\mathbf{q}} \times W$ *of* $\gamma_l^{(r)}[\cdot, \cdot, \mathbf{q}, \sigma](t_l)$ *and of* $\gamma_l^{(r)}[\cdot, \cdot, \mathbf{q}, \sigma](t_{l+1})$ *(e.g. evaluations at* t_l*,* t_{l+1} *of functions* (4.4) *and their* r*-th derivatives with respect to* x*) are real analytic functions of* $\mathbf{g}$ *and* t_j *where* t_j *is a free coordinate of the open cell* W*;*

5. *for any open cell W of the CW structure induced by $\mathbf{q}$ on A_{T}, the restriction to $\mathbb{A}_{\mathbf{q}} \times W$ of $\mathfrak{F}(\cdot,\cdot,\mathbf{q},\sigma)$ (e.g. function (4.5)) is real analytic functions of $\mathbf{g}$ and t_j where t_j is a free coordinate of the open cell W.*

Proof. Consider the quality ω defined by $\omega_l = \mathsf{J}$ if $t_l \notin \mathbf{q}$ and $\omega_l = \sigma_l$ otherwise. Since $\mathfrak{F}(\cdot,\cdot,\mathbf{q},\sigma)$ is the restriction of $\mathbb{F}(\cdot,\cdot,\omega)$ to $L^2(0,1) \times A_{\mathsf{T}}$ and $\gamma^{(r)}[\cdot,\cdot,\mathbf{q},\sigma] = b^{(r)}[\cdot,\cdot,\omega]$ the proposition follows from the analogous results about $\mathbb{F}$ and b: Theorem 3.11, Lemma 3.7, Theorem 3.14. □

5 CW structure of the set of data with vanishing excess $\{E = \mathbf{0}\}$

We introduce the excess functional E to represent the deviation realized by solution of Problem 4.4 from (expected for minimizers) vanishing values of suitable weights. Excess E is given in Definition 5.1 in such a way that the set $\{E = \mathbf{0}\}$ select all data for Problem 3.3 whose related solution fulfills the whole set of Euler conditions (i)-(vi) of Theorem 2.1.
Euler conditions (i)-(vi) of Theorem 2.1 altogether form an overdetermined differential system: for this reason we introduced Problems 3.3 and 4.4 (each of them contains only part of these conditions) and showed that both have unique solution for any choice of the arrangement. If the evaluation of the excess E on the solution γ of Problem 4.4 vanishes then γ is also a solution of Problem 3.3, more precisely such γ fulfills all Euler conditions (i)-(vi).

Definition 5.1 *For any $\mathsf{Q},\mathsf{T},\mathsf{j},\mathsf{c} \in \{0,1,2,...\}$, $\mathsf{T} = \mathsf{j}+\mathsf{c}$, $\mathbf{t} \in A_{\mathsf{T}}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$, and $g \in \mathbb{A}_{\mathbf{q}}$ we define $E : \mathbb{A}_{\mathbf{q}} \times A_{\mathsf{T}} \times A_{\mathsf{Q}} \times \Omega_{\mathsf{j},\mathsf{c}} \to \mathbb{R}^{\mathsf{T}}$ by*

$$E(g,\mathbf{t},\mathbf{q},\sigma) = \big(E_1(g,\mathbf{t},\mathbf{q},\sigma),...,E_{\mathsf{T}}(g,\mathbf{t},\mathbf{q},\sigma)\big)$$

where

$$E_l(g,\mathbf{t},\mathbf{q},\sigma) = \begin{cases} \gamma_{l-1}[g,\mathbf{t},\mathbf{q},\sigma](t_l) + \gamma_l[g,\mathbf{t},\mathbf{q},\sigma](t_l) - 2g(t_l) & \text{if } \sigma_l = \mathsf{J} \text{ and } t_l \notin \mathbf{q}, \\ \gamma_l[g,\mathbf{t},\mathbf{q},\sigma](t_l) - \gamma_{l-1}[g,\mathbf{t},\mathbf{q},\sigma](t_l) & \text{if } \sigma_l = \mathsf{C}, \\ 0 & \text{otherwise,} \end{cases}$$

and $\gamma = \gamma[g,\mathbf{t},\mathbf{q},\sigma]$ is the solution of Problem 4.4.

Notice that if W is a cell of the CW structure induced on A_{T} by $\mathbf{q}$ and t_l is not a free coordinate of W, then $t_l \in \mathbf{q}$. Since $t_l \in \mathbf{q}$ entails either $\sigma_l = \mathsf{C}$ or $E_l = 0$ we get

$$E_l = 0 \qquad \forall l \text{ such that } t_l \text{ is not a free coordinate of the cell } W. \tag{5.1}$$

Notice that Proposition 4.9(4) entails (via identifications (3.16) and (3.17) between $g \in \mathbb{A}_{\mathbf{q}}$ and $\mathbf{g} \in \mathbb{R}^{\mathsf{Q}+2}$) that the restriction of the function $E(\cdot,\cdot,\mathbf{q},\sigma)$ to $\mathbb{A}_{\mathbf{q}} \times W$ is an analytic function of $\mathbf{g}$, $\mathbf{t}$, for any open d-dimensional cell $W \subseteq A_{\mathsf{T}}$ of the CW decomposition induced on A_{T} by $\mathbf{q}$. Moreover, referring to Definition 4.1, (3.23) together with Theorem 2.1(vi) entail

$$E(g,\mathbf{t}(u),\mathbf{q},\sigma(u)) = \mathbf{0} \qquad \forall \mathbf{g}, \quad \forall u \in \operatorname{argmin} F^g_{\alpha,\beta} \subseteq \mathcal{K}_{\mathbf{q}}. \tag{5.2}$$

In this section (still referring to identifications (3.16) and (3.17) between $g \in \mathbb{A}_{\mathbf{q}}$ and $\mathbf{g} \in \mathbb{R}^{\mathsf{Q}+2}$) we study the CW structure of the set

$$\{E = \mathbf{0}\} := \{(\mathbf{g},\mathbf{t}) \in \mathbb{A}_{\mathbf{q}} \times W \colon \quad E(g,\mathbf{t},\mathbf{q},\sigma) = \mathbf{0}\} \tag{5.3}$$

in a small neighborhood of a fixed point $\widetilde{\mathbf{t}} \in A_{\mathsf{T}}$ when the location $\mathbf{q}$ appearing in the definition of E is suitably fine. Toward this aim we introduce the definition of exhaustive sequence of partitions where, as usual, we identify partitions with vectors.

Definition 5.2 *A sequence of partitions $\{\mathbf{q}_m\}_{m\geq 0}$ is called exhaustive if*

$$\mathbf{q}_m \subset \mathbf{q}_{m+1} \text{ for any } m \geq 0, \qquad \bigcup_{m\geq 0} \mathbf{q}_m \text{ is dense in } (0,1).$$

Lemma 5.3 *Fix $\mathsf{T},\mathsf{j},\mathsf{c} \in \{0,1,2,...\}$, $\mathsf{T} = \mathsf{j}+\mathsf{c}$, $\widetilde{\mathbf{t}} \in A_{\mathsf{T}}$ and $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$. Then $\exists \varepsilon \in (0, \operatorname{dist}(\widetilde{\mathbf{t}}, \partial A_{\mathsf{T}})/2)$ s.t. for all exhaustive sequence of partitions $\{\mathbf{q}_m\}_{m\geq 0}$ $\exists \overline{m}\colon\ \forall m > \overline{m}$*

$$\forall l \in \{1,...,\mathsf{T}\} \quad \exists i = i(l),\ \exists q_i := q_{i(l)} \in \mathbf{q}_m \text{ such that}$$
$$[q_{i-1}, q_i] \subseteq (t_l, t_{l+1}) \quad \forall \mathbf{t}\colon \left|\mathbf{t} - \widetilde{\mathbf{t}}\right|_{\mathbb{R}^{\mathsf{T}}} < \varepsilon \tag{5.4}$$

and, by setting ${}_jh(x) = (x - q_{j-1})\chi_{[q_{j-1},q_j]}(x) + (q_j - q_{j-1})\chi_{(q_j,1]}(x)$ for any $j \in \{1,...,\mathsf{Q}_m+1\}$ and $\mathsf{Q}_m = \dim \mathbf{q}_m$, we have

$$\gamma_l[h,\mathbf{t},\mathbf{q}_m,\sigma] = \mathbf{h}_0 + \sum_{j=1}^{\mathsf{Q}_m+1} \gamma_l[{}_jh,\mathbf{t},\mathbf{q}_m,\sigma]\,\mathbf{h}_j \quad \forall h \in \mathbb{A}_{\mathbf{q}_m}, \forall l \in \{1,...,\mathsf{T}\}, \tag{5.5}$$

say the solution $\gamma_l[{}_jh,\mathbf{t},\mathbf{q}_m,\sigma]$ of Problem 4.4 is the coefficient of $\mathbf{h}_j$ (through the identifications between $h \in \mathbb{A}_{\mathbf{q}_m}$ and $\mathbf{h} \in \mathbb{R}^{\mathsf{Q}_m+2}$, see (3.16), (3.17)*) in the linear combination* (5.5) *representing γ_l, and*

$$\gamma_k[{}_{i(l)}h,\mathbf{t},\mathbf{q}_m,\sigma] \equiv 0 \text{ if } k < l \text{ and } (\sigma_l = \mathsf{J} \text{ or } t_l \notin \mathbf{q}_m), \tag{5.6}$$

$$\gamma_l[{}_{i(l)}h,\mathbf{t},\mathbf{q}_m,\sigma](t_l) \neq 0. \tag{5.7}$$

Proof. Statement (5.4) follows from Definition 5.2.
Statement (5.5) follows by Proposition 4.9(1) via identification (3.16) which now reads as follows

$$\begin{cases} \mathbf{z}_0 = \mathbf{h}_0, \qquad \mathbf{z}_l = \mathbf{h}_l(q_l - q_{l-1}) + \mathbf{z}_{l-1}, \\ h(x) = \sum_{j=1}^{\mathsf{Q}_m+1} (\mathbf{h}_j(x - q_{j-1}) + \mathbf{z}_{j-1})\chi_{[q_{j-1},q_j)}(x) \; = \; \sum_{j=0}^{\mathsf{Q}_m+1} {}_j h \; \mathbf{h}_j \; . \end{cases} \tag{5.8}$$

Statement (5.6) follows by (4.1) and Proposition 4.9(1).
Referring to Problem 3.3, we define $b_l^\omega : (\widetilde{t}_l, \widetilde{t}_{l+1}) \to \mathbb{R}$ by

$$b_l^\omega(a) = b_l[\chi_{(a,1)}, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_l).$$

Lemma 3.20 entails that b_l^ω is a (not identically zero) real analytic function with respect to $a \in (\widetilde{t}_l, \widetilde{t}_{l+1})$ for any $\omega \in \Omega_{\mathsf{T}} = \bigcup_{\mathsf{m}+\mathsf{n}=\mathsf{T}} \Omega_{\mathsf{m},\mathsf{n}}$.
Since Ω_{T} is a finite set we have that $\bigcup_{\omega\in\Omega_{\mathsf{T}}} \{x \in (\widetilde{t}_l, \widetilde{t}_{l+1}) : \; b_l^\omega(x) = 0\}$ is a discrete set hence we can choose

$$a_l \in (\widetilde{t}_l, \widetilde{t}_{l+1})\colon \quad b_l^\omega(a_l) \neq 0 \qquad \forall \omega \in \Omega_{\mathsf{T}}.$$

Continuity of $b_l[\chi_{(a_l,1)}, \mathbf{t}, \omega](t_l)$ with respect to t_l (Theorem 3.11(2)) entails

$$\exists c, \varepsilon_l > 0\colon \quad \left|b_l[\chi_{(a_l,1)}, \mathbf{t}, \omega](t_l)\right| > c \quad \forall \omega \in \Omega_{\mathsf{T}}, \; \forall \mathbf{t} \in A_{\mathsf{T}} : \left|\mathbf{t} - \widetilde{\mathbf{t}}\right| < 2\varepsilon_l.$$

For any $a, b \in [0, 1]$ with $a < b$, set

$$h(x) = \frac{x-a}{b-a}\chi_{[a,b]}(x) + \chi_{(b,1]}(x),$$

Continuity of $b_l[g, \mathbf{t}, \omega](t_l)$ with respect to g (Theorem 3.11(2)) entails, for the same ε_l choosen before,

$$\exists \delta > 0 : \begin{cases} \operatorname{dist}(a_l, \{t_l\}_{l=0}^{\mathsf{T}+1}) > \delta \qquad \forall \mathbf{t} \in A_{\mathsf{T}}, \; \left|\mathbf{t} - \widetilde{\mathbf{t}}\right| < \varepsilon_l, \\ \mid b_l[h, \mathbf{t}, \omega](t_l)\mid > \frac{c}{2} \qquad \forall a, b \in (a_l - \delta, a_l + \delta). \end{cases}$$

By exploiting linearity of $b_l[g, \mathbf{t}, \omega](t_l)$ with respect to g (Theorem 3.11(1)) we have for any l

$$\exists \varepsilon_l, \delta > 0 : \begin{cases} \operatorname{dist}(a_l, \{t_l\}_{l=0}^{\mathsf{T}+1}) > \delta \qquad \forall \mathbf{t} \in A_{\mathsf{T}}, \; \left|\mathbf{t} - \widetilde{\mathbf{t}}\right| < \varepsilon_l, \\ \mid b_l[(b-a)h, \mathbf{t}, \omega](t_l)\mid > \frac{c}{2}(b-a) \qquad \forall a, b \in (a_l - \delta, a_l + \delta). \end{cases} \tag{5.9}$$

For any $l \in \{1, ..., \mathsf{T}\}$ fix ε_l and δ as in (5.9), then by (5.4) we can choose index m_l such that partition $\mathbf{q}_{m_l}$ in the given sequence has components q_{i-1}, $q_i \in (a_l - \delta, a_l + \delta)$ and set

$$\varepsilon = \min \{\varepsilon_l \ \ \forall l \in \{1, ..., \mathsf{T}\}\} > 0, \qquad \overline{m} = \max \{m_l \ \ \forall l \in \{1, ..., \mathsf{T}\}\} < +\infty.$$

For any $m \geq \overline{m}$ and $\left\| \mathbf{t} - \widetilde{\mathbf{t}} \right\|_{\mathbb{R}^{\mathsf{T}}} < \varepsilon$, define ω by $\omega_l = \sigma_l$ if $t_l \in \mathbf{q}$ and $\omega_l = \mathsf{J}$ otherwise. Since $\gamma_l[{}_ih, \mathbf{t}, \mathbf{q}_m, \sigma] = b_l[{}_ih, \mathbf{t}, \omega]$, thesis (5.7) follows by applying (5.9) to $b_l[{}_ih, \mathbf{t}, \omega]$. □

Theorem 5.4 *Fix* $\mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\widetilde{\mathbf{t}} \in A_{\mathsf{T}}$, $\sigma \in \Omega_{\mathsf{j},\mathsf{c}}$. *Then* $\forall \varepsilon$ *s.t.* $0 < \varepsilon < \frac{1}{2}\mathrm{dist}(\widetilde{\mathbf{t}}, \partial A_{\mathsf{T}})$ *and* $\forall$ *exhaustive family of partitions* $\{\mathbf{q}_m\}_{m \geq 0}$ $\exists \widetilde{m}$ *such that: for any* $\mathbf{q}_m$ *with* $m \geq \widetilde{m}$ *and any open* d*-dimensional cells* W *of the CW structure induced by* $\mathbf{q}_m$ *on* A_{T} *with* $W \subseteq B(\widetilde{\mathbf{t}}, \varepsilon)$*, the set*

$$T := \{E = \mathbf{0}\} \cap (\mathbb{A}_{\mathbf{q}_m} \times W) = \{(\mathbf{g}, \mathbf{t}) \in \mathbb{A}_{\mathbf{q}_m} \times W \colon \ E(\mathbf{g}, \mathbf{t}, \mathbf{q}_m, \sigma) = \mathbf{0}\}$$

is a finite CW complex of dimension at most $\mathsf{Q}_m + 2$ *(where* $\mathsf{Q}_m = \dim \mathbf{q}_m$*). The higher skeleton of* T *locally is the graph of an analytic function.*

Proof. The restriction of E to $\mathbb{A}_{\mathbf{q}} \times W$ is an analytic function then its zero set $T = \{E = \mathbf{0}\} \cap (\mathbb{A}_{\mathbf{q}} \times W)$ is a semi-analytic set contained in $\mathbb{A}_{\mathbf{q}} \times W$, hence T has a CW structure by Theorem 8.5.
Choose ε and $\widetilde{m}$ as in Lemma 5.3, denote Q_m and $\mathbf{q}_m$ shortly by Q and $\mathbf{q} = (q_i)_{i=1}^{\mathsf{Q}}$ and denote by $\{l_r\}_{r=1}^{d}$ the free coordinates of the d-dimensional cell W.
Even without assuming $E = 0$, by (5.1) we have to consider the intersection of sets $\{E_l = 0\}$ only over indexes l_r related to free coordinates of W: since

$$\left(\bigcap_{l \notin \{l_r\}_{r=1}^{d}} \{E_l = 0\} \right) \cap (\mathbb{A}_{\mathbf{q}} \ \times W) = \mathbb{A}_{\mathbf{q}} \times W$$

we have to study only $\left(\bigcap_{r=1}^{d} \{E_{l_r} = 0\} \right) \cap (\mathbb{A}_{\mathbf{q}} \times W)$.
Hence we are left to study the analytic function $J : \mathbb{R}^{\mathsf{Q}+2} \times W \to \mathbb{R}^d$ defined by $J(\mathbf{g}, \mathbf{t}) = (E_{l_r}(g, \mathbf{t}, \mathbf{q}, \sigma))_{r=1}^{d}$ through the identification (3.16) and (3.17) between g and $\mathbf{g}$.
By Lemma 5.3 there are points $q_{l_r - 1}, q_{l_r} \in (t_{l_r}, t_{l_r + 1})$, $r \in \{1, ..., d\}$ such that the maps ${}_rh(x) = (x - q_{l_r - 1})\chi_{[q_{l_r - 1}, q_{l_r}]}(x) + (q_{l_r} - q_{l_r - 1})\chi_{(q_{l_r}, 1]}(x)$ fulfills (5.6), (5.7), hence ${}_rh$ are $\mathsf{Q} + 2$ linearly independent functions in $\mathbb{A}_{\mathbf{q}}$.
If ${}_r\mathbf{h} \in \mathbb{R}^{\mathsf{Q}+2}$ $(r = 1, ..., d)$ are the vectors related to ${}_rh$ through (3.16) and

(3.17), then $\{{}_r\mathbf{h}\}$ is a set of $\mathsf{Q}+2$ independent vectors.
Moreover the matrix

$$\left(\frac{\partial J_{l_{r'}}}{\partial({}_r\mathbf{h})}\right)_{r,r'=1}^{d} \tag{5.10}$$

is an invertible $d \times d$ matrix for any $(\mathbf{g}, \mathbf{t}) \in \mathbb{R}^{\mathsf{Q}+2} \times W \subseteq \mathbb{R}^{\mathsf{Q}+2} \times \mathbb{R}^d$: in fact by Definition 5.1, (5.5), (5.8) we have $\frac{\partial J_{l_{r'}}}{\partial({}_r\mathbf{h})} = \frac{\partial E_{l_{r'}}}{\partial({}_r h)} = \gamma_{l_{r'}}[{}_r h, \mathbf{t}, \mathbf{q}, \sigma](t_{l_{r'}})$, hence the matrix (5.10) is a lower triangular matrix with diagonal given by the vector

$$\left(\gamma_{l_r}[{}_r h, \mathbf{t}, \mathbf{q}, \sigma](t_{l_r})\right)_{r=1}^{d}$$

whose entries are all non zero by (5.7).
So the matrix $\frac{\partial J}{\partial \mathbf{h}}$ has always maximal rank and, by the Implicit Function Theorem, $\{J = \mathbf{0}\} = \left(\bigcap_{r=1}^{d} \{E_{l_r} = 0\}\right) \cap \mathbb{A}_{\mathbf{q}} \times W$ has dimension $\mathsf{Q}+2$ and locally is the graph of an analytic function. $\square$

6 CW structure of the set $\{\mathfrak{E} = 0\} \cap \{\mathbf{E} = \mathbf{0}\}$ of all data exhibiting non uniqueness of minimizer with same cardinality of singular sets and different arrangement

The main result of this section is Theorem 6.4 which measures how many triplets $(\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$ exist where $\mathbf{g}$ is associated by (3.16) and (3.17) to a continuous piecewise affine function g with no more than m creases and $\mathbf{t}$, $\boldsymbol{\tau}$ are the ordered singular sets of two different (when possible) solutions of Problem 4.2 with same cardinality n of singular set but different arrangement[1] : we prove that the projection on the first component (in $\mathbb{R}^m$) of the whole set of such triplets has zero m dimensional Lebesgue measure.
We introduce two additional excess functionals $\mathfrak{E}$ and $\mathbf{E}$ to represent the deviation of suitable weights evaluated on the solution of Problem 4.4 from (expected for minimizers) vanishing values. The definition is built in such a way that $\{\mathfrak{E} = 0\} \cap \{\mathbf{E} = \mathbf{0}\}$ is the set of all data exhibiting non uniqueness of minimizer with different arrangement and same cardinality of singular sets.

Definition 6.1 *For any* $\mathsf{Q}, \mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\mathbf{q} \in A_{\mathsf{Q}}$, $\sigma, \widetilde{\sigma} \in \Omega_{\mathsf{j},\mathsf{c}}$, *open cell* $W \subseteq A_{\mathsf{T}} \times A_{\mathsf{T}}$ *s.t.* $W = W_0 \times W_1$ *with* W_0, W_1 *open cells of*

[1] together with the same arrangement we would have uniqueness by Remark 4.3, Theorem 2.1 and Theorem 3.8

the CW structure induced by $\mathbf{q}$ *on* A_{T} *and any* $(g,\mathbf{t},\boldsymbol{\tau})\in\mathbb{A}_{\mathbf{q}}\times W$*, referring to Definitions 4.8(2) and 5.1, we define:*

- $\mathfrak{E}:\mathbb{A}_{\mathbf{q}}\times W\times A_{\mathsf{Q}}\times\Omega_{\mathsf{j,c}}\times\Omega_{\mathsf{j,c}}\to\mathbb{R}$,
  such that $\mathfrak{E}(g,\mathbf{t},\boldsymbol{\tau},\mathbf{q},\sigma,\widetilde{\sigma})=\mathfrak{F}(g,\mathbf{t},\mathbf{q},\sigma)-\mathfrak{F}(g,\boldsymbol{\tau},\mathbf{q},\widetilde{\sigma})$,
- $\mathbf{E}:\mathbb{A}_{\mathbf{q}}\times W\times A_{\mathsf{Q}}\times\Omega_{\mathsf{j,c}}\times\Omega_{\mathsf{j,c}}\to\mathbb{R}^{2\mathsf{T}}$,
  such that $\mathbf{E}(g,\mathbf{t},\boldsymbol{\tau},\mathbf{q},\sigma,\widetilde{\sigma})=\big(E(g,\mathbf{t},\mathbf{q},\sigma),E(g,\boldsymbol{\tau},\mathbf{q},\widetilde{\sigma})\big)$.

$\mathfrak{E}(g,\mathbf{t},\boldsymbol{\tau},\mathbf{q},\sigma,\widetilde{\sigma})=0$ means that both $\gamma=\gamma(g,\mathbf{t},\mathbf{q},\sigma)$ and $\widetilde{\gamma}=\gamma(g,\boldsymbol{\tau},\mathbf{q},\widetilde{\sigma})$ have the same energy $\mathcal{F}^g$.
$\mathbf{E}(g,\mathbf{t},\boldsymbol{\tau},\mathbf{q},\sigma,\widetilde{\sigma})=0$ entails that both $\gamma=\gamma(g,\mathbf{t},\mathbf{q},\sigma)$ and $\widetilde{\gamma}=\gamma(g,\boldsymbol{\tau},\mathbf{q},\widetilde{\sigma})$ solve not only Problem 4.2 but also Problem 3.3:

$$b(g,\mathbf{t},\sigma)=\gamma,\ \ b(g,\mathbf{t},\widetilde{\sigma})=\widetilde{\gamma},$$
$$\mathcal{F}^g(\gamma)=\mathfrak{F}(g,\mathbf{t},\mathbf{q},\sigma)=\mathbb{F}(g,\mathbf{t},\sigma),\ \ \mathcal{F}^g(\widetilde{\gamma})=\mathfrak{F}(g,\boldsymbol{\tau},\mathbf{q},\widetilde{\sigma})=\mathbb{F}(g,\mathbf{t},\widetilde{\sigma}).$$

Notice that the existence of two different u_1, u_2 minimizing $F^g_{\alpha,\beta}$ with $(\mathbf{t},\sigma)$ arrangement of u_1 and $(\boldsymbol{\tau},\widetilde{\sigma})$ arrangement of u_2 would entail $\mathfrak{E}(g,\mathbf{t},\boldsymbol{\tau},\mathbf{q},\sigma,\widetilde{\sigma})=0$ and $\mathbf{E}(g,\mathbf{t},\boldsymbol{\tau},\mathbf{q},\sigma,\widetilde{\sigma})=\mathbf{0}$.
In Lemma 6.2 we evaluate the difference $\mathbb{F}(g,\mathbf{t},\omega)-\mathbb{F}(g,\boldsymbol{\tau},\widetilde{\omega})$ when two different minimizers of $F^g_{\alpha,\beta}$ exhibit l_2-l_1 consecutive crease points with the same location between two jumps with the same location: by approximating these crease points and the two jump points with suitable ramps we prove that the contribution of such interval to the above energy difference is different from zero almost everywhere in a non empty neighborhood of the diagonal $\mathbf{t}=\boldsymbol{\tau}$ (recall that such energy difference must vanish on the diagonal).
In Lemma 6.3 and Theorem 6.4, for any cell W and any pair of qualities σ, $\widetilde{\sigma}$, we study the CW structure (induced on $A_{\mathsf{T}}\times A_{\mathsf{T}}$ by $\mathbf{q}$) of the set

$$\begin{aligned}&\{\mathfrak{E}=0\}\cap\{\mathbf{E}=\mathbf{0}\}:=\\&\{(\mathbf{g},\mathbf{t},\boldsymbol{\tau})\in\mathbb{A}_{\mathbf{q}}\times W:\quad \mathfrak{E}(g,\mathbf{t},\boldsymbol{\tau},\mathbf{q},\sigma,\widetilde{\sigma})=0,\ \mathbf{E}(g,\mathbf{t},\boldsymbol{\tau},\mathbf{q},\sigma,\widetilde{\sigma})=\mathbf{0}\}\end{aligned}\tag{6.1}$$

in a small neighborhood of a fixed point $(\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\tau}})\in A_{\mathsf{T}}\times A_{\mathsf{T}}$ when the partition $\mathbf{q}$ appearing in Definition 6.1 is suitably fine.

Lemma 6.2 *Fix* $\mathsf{T},\mathsf{m},\mathsf{n},\widetilde{\mathsf{m}},\widetilde{\mathsf{n}},l_1,l_2,\lambda_1,\lambda_2\in\{0,1,2,...\}$, $\mathsf{T}=\mathsf{m}+\mathsf{n}=\widetilde{\mathsf{m}}+\widetilde{\mathsf{n}}>0$, $0\le l_1,l_2,\lambda_1,\lambda_2\le\mathsf{T}+1$, $(\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\tau}})\in A_{\mathsf{T}}\times A_{\mathsf{T}}$, $\omega\in\Omega_{\mathsf{m,n}}$, $\widetilde{\omega}\in\Omega_{\widetilde{\mathsf{m}},\widetilde{\mathsf{n}}}$. *Assume that*

$$l_2-l_1=\lambda_2-\lambda_1>0,\tag{6.2}$$

$$\widetilde{\mathbf{t}}_{l_1+i-1} = \widetilde{\boldsymbol{\tau}}_{\lambda_1+i-1} \qquad i = 1, ..., l_2 - l_1 + 1, \tag{6.3}$$

$$\omega_{l_1} = \omega_{l_2} = \widetilde{\omega}_{\lambda_1} = \widetilde{\omega}_{\lambda_2} = \mathsf{J}, \qquad \omega_{l_1+i} = \widetilde{\omega}_{\lambda_1+i} = \mathsf{C} \quad i = 1, ..., l_2 - l_1 - 1. \tag{6.4}$$

We insert suitable points $\{\mathbf{x}_k\}$ *between common locations; define an estimate* φ *of* $\{\mathbf{x}_k\}$ *proximity to the given partition* $\widetilde{\mathbf{t}}$*; then define a distance* ψ *from coincidence of* $\{\mathbf{x}_k\}$ *and* $\{t_l\}$ *and from collapse of consecutive pairs in* $\{\mathbf{x}_k\}$*:*

$$d = l_2 - l_1 + 1,$$

$$X = \{\mathbf{x} = (\mathbf{x}_k)_{k=1}^{2d} \in (0,1)^{2d} \colon \; \widetilde{t}_{l_1+i-1} < \mathbf{x}_{2i-1} < \mathbf{x}_{2i} < \widetilde{t}_{l_1+i} \quad i = 1, ..., d-2, \\ \widetilde{t}_{l_2-1} < \mathbf{x}_{2d-3} < \mathbf{x}_{2d-2} < \mathbf{x}_{2d-1} < \mathbf{x}_{2d} < \widetilde{t}_{l_2}\},$$

$$\varphi(\mathbf{x}) = \max\left(\{\mathbf{x}_{2i-1} - \widetilde{t}_{l_1+i-1},\ \widetilde{t}_{l_1+i} - \mathbf{x}_{2i}\}_{i=2}^{d-1} \cup \{\mathbf{x}_2 - \widetilde{t}_{l_1}, \widetilde{t}_{l_2} - \mathbf{x}_{2d-1}\}\right),$$

$$\psi(\mathbf{x}) = \min\left(\{\operatorname{dist}(\{\mathbf{x}_k\}_{k=1}^{2d}, \{\widetilde{t}_l\}_{l=0}^{\mathsf{T}+1})\} \cup \{\mathbf{x}_{2i} - \mathbf{x}_{2i-1}\}_{i=1}^{d}\right).$$

Then

$$\left.\begin{array}{l} \exists \delta = \delta(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}) > 0 \text{ and a closed set } P \subseteq X \text{ with empty interior in } \mathbb{R}^{2d}\colon \\ \forall \overline{\mathbf{x}} \in X \setminus P \text{ with } \psi(\overline{\mathbf{x}}) < \min\{\operatorname{dist}(\widetilde{\mathbf{t}}, \partial A_{\mathsf{T}}), \operatorname{dist}(\widetilde{\boldsymbol{\tau}}, \partial A_{\mathsf{T}}))\} \text{ and } \varphi(\overline{\mathbf{x}}) < \delta \\ \exists \varepsilon = \varepsilon(\overline{\mathbf{x}}, \widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}) \in (0, \psi(\overline{\mathbf{x}})/2) \text{ s.t.} \\ \forall \mathbf{x} \in (X \setminus P) \cap B(\overline{\mathbf{x}}, \varepsilon) \\ \forall (\mathbf{t}, \boldsymbol{\tau}) \in B(\widetilde{\mathbf{t}}, \varepsilon) \times B(\widetilde{\boldsymbol{\tau}}, \varepsilon) \text{ with } (t_{l_1}, t_{l_1+1}, ..., t_{l_2}) \neq (\tau_{\lambda_1}, \tau_{\lambda_1+1}, ..., \tau_{\lambda_2}) \\ \exists i \in \{1, ..., d\} \text{ s.t. } \; \mathbb{F}({}_i h, \mathbf{t}, \omega) - \mathbb{F}({}_i h, \boldsymbol{\tau}, \widetilde{\omega}) \neq 0, \end{array}\right\} \tag{6.5}$$

where we refer to Definition 3.10 of $\mathbb{F}$ *and* ${}_i h$ *is the ramp defined for by*

$$ {}_i h(x) = {}_i h[\mathbf{x}_{2i-1}, \mathbf{x}_{2i}](x) = \frac{x - \mathbf{x}_{2i-1}}{\mathbf{x}_{2i} - \mathbf{x}_{2i-1}} \chi_{[\mathbf{x}_{2i-1}, \mathbf{x}_{2i}]}(x) + \chi_{(\mathbf{x}_{2i}, 1]}(x) \quad x \in [0,1]. \tag{6.6}$$

Proof. There are four possible types of choices for l_1, l_2 fulfilling (6.2):

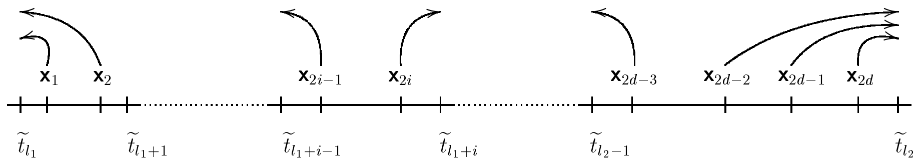
$$1 \leq l_1 < l_2 \leq \mathsf{T}, \quad \text{then set} \begin{cases} r_1 = \rho_1 = l_1, \\ r_2 = \rho_2 = l_2, \\ s = d, \\ \mu = 1,\ \nu = d, \end{cases} \tag{6.7}$$

$$0 = l_1 < l_2 \leq \mathsf{T}, \quad \text{then set} \begin{cases} r_1 = \rho_1 = 1, \\ r_2 = \rho_2 = l_2, \\ s = d - 1, \\ \mu = 2,\ \nu = d, \end{cases} \tag{6.8}$$

$$1 \leq l_1 < l_2 = \mathsf{T} + 1, \quad \text{then set} \begin{cases} r_1 = \rho_1 = l_1, \\ r_2 = \rho_2 = \mathsf{T}, \\ s = d - 1, \\ \mu = 1,\ \nu = d - 1, \end{cases} \tag{6.9}$$

$$l_1 = 0,\, l_2 = \mathsf{T} + 1, \quad \text{then set } \begin{cases} r_1 = \rho_1 = 1, \\ r_2 = \rho_2 = \mathsf{T}, \\ s = d - 2, \\ \mu = 2,\, \nu = d - 1. \end{cases} \tag{6.10}$$

According to Definition 3.10 ${}_ih$ approaches either a ramp or a jump when $\varphi(\mathbf{x}) \to 0^+$ as sketched below, $i = 1, ..., d$:



Define the vector function $\mathbb{L} : X \times A_{\mathsf{T}} \times A_{\mathsf{T}} \to \mathbb{R}^s$ by

$$\mathbb{L}(\mathbf{x}, \mathbf{t}, \boldsymbol{\tau}) = \big(\mathbb{F}({}_ih, \mathbf{t}, \omega) - \mathbb{F}({}_ih, \boldsymbol{\tau}, \widetilde{\omega})\big)_{i=\mu}^{\nu}. \tag{6.11}$$

Actually the dependence of $\mathbb{L}$ on t and τ is restricted to components $(t_{r_1}, ..., t_{r_2})$ and $(\tau_{\rho_1}, ..., \tau_{\rho_2})$ alone whenever $\left|\mathbf{t} - \widetilde{\mathbf{t}}\right| < \psi(\mathbf{x})$ and $|\boldsymbol{\tau} - \widetilde{\boldsymbol{\tau}}| < \psi(\mathbf{x})$ in fact by (6.4) and Remark 3.4, we know that: system (3.2) with data $\mathbf{t}$, ω, ${}_ih$ splits into three uncoupled systems related to intervals $[0, t_{l_1}]$, $[t_{l_1}, t_{l_2}]$, $[t_{l_2}, 1]$; system (3.2) with data $\boldsymbol{\tau}$, $\widetilde{\omega}$, ${}_ih$ splits into three uncoupled systems on $[0, \tau_{\lambda_1}]$, on $[\tau_{\lambda_1}, \tau_{\lambda_2}]$ and on $[\tau_{\lambda_2}, 1]$; $b[{}_ih, \mathbf{t}, \omega] = b[{}_ih, \boldsymbol{\tau}, \widetilde{\omega}]$ on $[0, t_{l_1}] \cup [t_{l_2}, 1]$.
By denoting with $|\cdot|$ the Euclidean norm we define:

- the set $Z \subseteq \mathbb{R}^{2\mathsf{T}}$ by

$$Z = \big\{(t_1, ..., t_{\mathsf{T}}, \tau_1, ..., \tau_{\mathsf{T}}) \colon \quad t_l = 0 \text{ for } l \neq r_1, ..., r_2, \\ \tau_l = 0 \text{ for } l \neq \rho_1, ..., \rho_2\big\}$$

  and the orthogonal projection onto Z, $pr : \mathbb{R}^{2\mathsf{T}} \to Z$;

- the set $\Lambda_{\mathbf{x}} \subseteq Z \subseteq \overline{A_{\mathsf{T}}} \times \overline{A_{\mathsf{T}}}$ by

$$\Lambda_{\mathbf{x}} = \big\{(pr(\mathbf{t}, \boldsymbol{\tau})) \colon \quad \left|\mathbf{t} - \widetilde{\mathbf{t}}\right| < \psi(\mathbf{x}), \ |\boldsymbol{\tau} - \widetilde{\boldsymbol{\tau}}| < \psi(\mathbf{x}), \\ t_{r_1+i} = \tau_{\rho_1+i} \text{ for } i = 0, ..., s-1\big\} \quad \forall \mathbf{x} \in X;$$

- the open set $Y_{\mathbf{x}} \subseteq Z \subseteq \overline{A_{\mathsf{T}}} \times \overline{A_{\mathsf{T}}}$ by

$$Y_{\mathbf{x}} = \big\{(pr(\mathbf{t}, \boldsymbol{\tau})) \colon \quad \left|\mathbf{t} - \widetilde{\mathbf{t}}\right| < \psi(\mathbf{x}) \text{ and } |\boldsymbol{\tau} - \widetilde{\boldsymbol{\tau}}| < \psi(\mathbf{x})\big\} \quad \forall \mathbf{x} \in X;$$

- the open set $Y \subseteq X \times Z \subseteq X \times \overline{A_{\mathsf{T}}} \times \overline{A_{\mathsf{T}}}$ by

$$Y = \left\{(\mathbf{x}, pr(\mathbf{t}, \boldsymbol{\tau})) : \quad \mathbf{x} \in X, \quad \left|\mathbf{t} - \widetilde{\mathbf{t}}\right| < \psi(\mathbf{x}), \quad |\boldsymbol{\tau} - \widetilde{\boldsymbol{\tau}}| < \psi(\mathbf{x})\right\} \subseteq X \times Y_{\mathbf{x}} \subseteq X \times Z.$$

We study the restriction of $\mathbb{L}$ to Y.
By Theorems 3.8 and 3.14(2)

$$(\mathbf{t}, \boldsymbol{\tau}) \longmapsto \mathbb{L}(\mathbf{x}, \mathbf{t}, \boldsymbol{\tau}) \text{ is a real analytic function of free coordinates } Y_{\mathbf{x}} \text{ for any } \mathbf{x} \in X.$$

We state a claim about Jacobian matrix $D\mathbb{L} = \partial\mathbb{L}/\partial(\mathbf{t}, \boldsymbol{\tau}) = (\partial\mathbb{L}/\partial\mathbf{t}, \partial\mathbb{L}/\partial\boldsymbol{\tau})$, where abusing notation $\partial(\mathbf{t}, \boldsymbol{\tau})$ stands for ∂z with $z \in Z$, say we take into account only the derivatives with respect to t_l with $l = r_1, ..., r_2$ and to τ_l with $l = \rho_1, ..., \rho_2$:

$$\left.\begin{array}{l} \exists\delta = \delta(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}) > 0 \text{ and a closed semi-analytic set } P \subseteq \mathbb{R}^{2d} \text{ s.t.} \\ \dim(P) \leq 2d - 1, \text{ hence with empty interior in } \mathbb{R}^{2d}, \\ \forall \mathbf{x} \in X \setminus P \text{ and } \varphi(\mathbf{x}) < \delta, \ \operatorname{rank}(D\mathbb{L}(\mathbf{x}, pr(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}))) = s. \end{array}\right\} \tag{6.12}$$

The matrix $D\mathbb{L}(\mathbf{x}, pr(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}))$ has s row and $2s$ columns such that the first s columns do not depend on $\boldsymbol{\tau}$ and the second s columns do not depend on $\mathbf{t}$, we denote by $M = M(\mathbf{x}, \widetilde{\mathbf{t}}) = \partial\mathbb{L}/\partial\mathbf{t}$ the square matrix given by the first s columns of $D\mathbb{L}(\mathbf{x}, pr(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}))$. Recall that $s = l_2 - l_1 + 1$ in case (6.7), $s = l_2 - l_1$ in cases (6.8), (6.9), $s = l_2 - l_1 - 1$ in case (6.10). We study in detail the behaviour of the entries of M in cases (6.7)-(6.10) when $\varphi(\mathbf{x}) \to 0^+$. By exploiting identity (3.22), (3.2.(iii)), (3.2.(iv)) and (3.2.(vi)) we analyze M.

Entries of type $M_{i,i}$ (diagonal entries).

We study $M_{i,i} = \dfrac{\partial\mathbb{F}}{\partial t_{r_1+i-1}}({}_ih, \widetilde{\mathbf{t}}, \omega)$.

If $i = 1$ and (6.7) or (6.9) occur ($\nexists M_{1,1}$ in cases (6.8), (6.10)), then

$$M_{1,1} = -b_{r_1}^2[{}_1h, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1}) \quad \text{and} \quad M_{1,1} \to -1 \quad \text{when} \quad \varphi(\mathbf{x}) \to 0^+. \tag{6.13}$$

In fact ${}_1h(\widetilde{t}_{r_1}) = 0$, $b_{r_1-1}[{}_1h, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1}) = 0$ by Remark 3.4 and $b_{r_1}'''[{}_1h, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1}) = 0$ since $\widetilde{t}_{r_1}$ has quality J. Moreover data $h = {}_1h[\mathbf{x}_1, \mathbf{x}_2]$ and $g = \chi_{[\widetilde{t}_{r_1}, 1]}$ in Lemma 3.19, estimate (3.31) and $g(\widetilde{t}_{r_1}) = 1$ entail $M_{1,1}(\mathbf{x}, \widetilde{\mathbf{t}}) \to -1$ when $\varphi(\mathbf{x}) \to 0^+$.

If $i = \{2, ..., d-1\}$ then

$$\begin{array}{l} M_{i,i} = -2b_{r_1+i-1}'''[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+i-1}) \times \\ \qquad (b_{r_1+i-1}'[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+i-1}) - b_{r_1+i-2}'[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+i-1})) \end{array} \tag{6.14}$$

and $M_{i,i} \to 0$ of order 1 when $\varphi(\mathbf{x}) \to 0^+$.

In fact $b_{r_1+i-2}[{}_ih,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_1+i-1}) = b_{r_1+i-1}[{}_ih,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_1+i-1})$ since $\widetilde{t}_{r_1+i-1}$ has quality C. Moreover if we chose $\mathbf{t}=\widetilde{\mathbf{t}}$, $g={}_ih[\widetilde{t}_{r_1+i-1},\widetilde{t}_{r_1+i}]$, $g(\widetilde{t}_{r_1+i})=1$, $h={}_ih[\mathbf{x}_{2i-1},\mathbf{x}_{2i}]$ and $(r,s)=(\mathbf{x}_{2i-1},\mathbf{x}_{2i})$ in Lemma 3.19, then (3.30), estimate (3.27) and vanishing set properties of non constant analytic functions allow us to define the following sets for $i=2,...,d-1$,

$$P_i=\left\{\text{pairs } (\mathbf{x}_{2i-1},\mathbf{x}_{2i}):\ \widetilde{t}_{r_1+i-1}<\mathbf{x}_{2i-1}<\mathbf{x}_{2i}<\widetilde{t}_{r_1+i}\ ,\ M_{i,i}(\mathbf{x},\widetilde{\mathbf{t}})=0\right\}. \tag{6.15}$$

Sets P_i fulfill the following properties:

$$P_i \text{ is a closed semi-analytic set contained in } \mathbb{R}^2 \text{ and } \dim(P_i)\le 1. \tag{6.16}$$

Then $M_{i,i}(\mathbf{x},\widetilde{\mathbf{t}})\to 0$ of order 1 when $\varphi(\mathbf{x})\to 0^+$.

If $i=d$ and (6.7) or (6.8) occur ($\nexists M_{d,d}$ in cases (6.9), (6.10)), then

$$M_{d,d}=(b_{r_2-1}[{}_dh,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_2})-1)^2 \quad\text{and}\quad M_{d,d}\to 1 \quad\text{when}\quad \varphi(\mathbf{x})\to 0^+. \tag{6.17}$$

In fact ${}_dh(\widetilde{t}_{r_2})=1$, $b_{r_2}[{}_dh,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_2})=1$ by Remark 3.4 and $b'''_{r_2}[{}_dh,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_2})=0$ since $\widetilde{t}_{r_2}$ has quality J. Moreover data $h={}_dh[\mathbf{x}_{2d-1},\mathbf{x}_{2d}]$ and $g=\chi_{[\widetilde{t}_{r_2},1]}$ in Lemma 3.19 and estimate (3.31) entail $M_{d,d}(\mathbf{x},\widetilde{\mathbf{t}})\to 1$ when $\varphi(\mathbf{x})\to 0^+$.
So far we have all the estimates which are needed about main diagonal, since index i runs respectively from 1 to d in case (6.7), from 2 to d in case (6.8), from 1 to $d-1$ in case (6.9), from 2 to $d-1$ in case (6.10).

Entries of type $M_{i,i+1}$ (entries just above the diagonal).

We study $M_{i,i+1}=\dfrac{\partial\mathbb{F}}{\partial t_{r_1+i}}({}_ih,\widetilde{\mathbf{t}},\omega)$.

If $i=\{\mu,...,d-1\}$ then

$$\begin{aligned}M_{i,i+1}=-2b'''_{r_1+i}[{}_ih,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_1+i})\times\\ (b'_{r_1+i}[{}_ih,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_1+i})-b'_{r_1+i-1}[{}_ih,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_1+i}))\end{aligned} \tag{6.18}$$

and $M_{i,i+1}\to 0$ of order 1 when $\varphi(\mathbf{x})\to 0^+$.

In fact $b_{r_1+i-1}[{}_ih,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_1+i}) = b_{r_1+i}[{}_ih,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_1+i})$ since $\widetilde{t}_{r_1+i}$ has quality C. Moreover data $h={}_ih[\mathbf{x}_{2i-1},\mathbf{x}_{2i}]$ and $g={}_ih[\widetilde{t}_{r_1+i-1},\widetilde{t}_{r_1+i}]$ in Lemma 3.19, (3.30) and estimate (3.27) entail $M_{i,i+1}(\mathbf{x},\widetilde{\mathbf{t}})\to 0$ of order 1 when $\varphi(\mathbf{x})\to 0^+$.

If $i=d-1$ and (6.7) or (6.8) occur, then

$$\begin{aligned}&M_{d-1,d}=(b_{r_2-1}[{}_{d-1}h,\widetilde{\mathbf{t}},\omega](\widetilde{t}_{r_2})-1)^2 \text{ and}\\ &M_{d-1,d}\to 0 \text{ of order at least 2 when } \varphi(\mathbf{x})\to 0^+.\end{aligned} \tag{6.19}$$

In fact we have ${}_{d-1}h(\widetilde{t}_{r_2}) = 1$, $b_{r_2}[{}_{d-1}h, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_2}) = 1$ by Remark 3.4 and $b'''_{r_2}[{}_{d-1}h, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_2}) = 0$ since $\widetilde{t}_{r_2}$ has quality J. Moreover data $h = {}_ih[\mathbf{x}_{2i-1}, \mathbf{x}_{2i}]$ and $g = {}_ih[\widetilde{t}_{r_2-1}, \widetilde{t}_{r_2}]$ in Lemma 3.19 and estimate (3.31) entails $M_{d-1,d}(\mathbf{x}, \widetilde{\mathbf{t}}) \to 0$ of order at least 2 when $\varphi(\mathbf{x}) \to 0^+$.

Entries of type $M_{i,j}$ with $(i,j) \neq (i,i), (i,i+1)$.

We study $M_{i,j} = \dfrac{\partial \mathbb{F}}{\partial t_{r_1+j-1}}({}_ih, \widetilde{\mathbf{t}}, \omega)$.

If $\widetilde{t}_{r_1+j-1}$ has quality C then

$$\begin{aligned} M_{i,j} = &-2b'''_{r_1+j-1}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+j-1}) \times \\ &(b'_{r_1+j-1}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+j-1}) - b'_{r_1+j-2}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+j-1})) \end{aligned} \tag{6.20}$$

and $M_{i,j} \to 0$ of order at least 2 when $\varphi(\mathbf{x}) \to 0^+$.

In fact $b_{r_1+j-1}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+i-1}) = b_{r_1+j-2}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+j-1}) = 0$ since $\widetilde{t}_{r_1+j-1}$ has quality C. Moreover data $h = {}_ih[\mathbf{x}_{2i-1}, \mathbf{x}_{2i}]$ and $g = {}_ih[\widetilde{t}_{r_1+i-1}, \widetilde{t}_{r_1+i}]$ in Lemma 3.19 and estimates (3.27) and (3.31) entail $M_{i,j}(\mathbf{x}, \widetilde{\mathbf{t}}) \to 0$ of order at least 2 when $\varphi(\mathbf{x}) \to 0^+$.

If $\widetilde{t}_{r_1+j-1}$ has quality J then

$$\begin{aligned} M_{i,j} = &(b_{r_1+j-2}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+j-1}) - b_{r_1+j-1}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+j-1})) \times \\ &(b_{r_1+j-2}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+j-1}) + b_{r_1+j-1}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+j-1}) - 2{}_ih(\widetilde{t}_{r_1+j-1})) \end{aligned} \tag{6.21}$$

and $M_{i,j}(\mathbf{x}, \widetilde{\mathbf{t}}) \to 0$ of order at least 2 when $\varphi(\mathbf{x}) \to 0^+$.

In fact $b'''_{r_1+j-1}[{}_ih, \widetilde{\mathbf{t}}, \omega](\widetilde{t}_{r_1+j-1}) = 0$ since $\widetilde{t}_{r_1+j-1}$ has quality J. Moreover data $h = {}_ih[\mathbf{x}_{2i-1}, \mathbf{x}_{2i}]$ and $g = {}_ih[\widetilde{t}_{r_1+i-1}, \widetilde{t}_{r_1+i}]$ in Lemma 3.19 and estimates (3.27) and (3.31) entail $M_{i,j}(\mathbf{x}, \widetilde{\mathbf{t}}) \to 0$ of order at least 2 when $\varphi(\mathbf{x}) \to 0^+$.

Referring to (6.15), and setting by convention $P_1 = P_d = \emptyset$ in all cases (6.7)-(6.10) we define $P \subseteq \mathbb{R}^{2d}$ as follows

$$P = \bigcup_{i=1}^{d} \mathbb{R}^2 \times \ldots \times \underset{\substack{\uparrow \\ i\text{-th position}}}{\mathbb{P}_i} \times \ldots \times \mathbb{R}^2. \tag{6.22}$$

The set P is contained in $\mathbb{R}^{2d}$: actually P is the union of $d-2$ semi-analytic sets since the first and the last one are empty. By denoting $\mathcal{S}$ the group of permutations of s elements and referring to (6.7)-(6.10), we exploit the standard formula

$$\det(M(\mathbf{x}, \widetilde{\mathbf{t}})) = \sum_{p \in \mathcal{S}} sgn(p) \prod_{i=\mu}^{\nu} M_{i,p(i)}, \tag{6.23}$$

where $\nu - \mu = s - 1$ is equal to respectively $d-1$, $d-2$, $d-3$ in cases (6.7), (6.8) and (6.9), (6.10).
We summarize (6.13)-(6.21) as follows: product $\prod_{i=\mu}^{\nu} M_{i,i}$ is an infinitesimal as $\varphi(\mathbf{x}) \to 0^+$ of order respectively $(s-2) \vee 0$, $(s-1) \vee 0$, $s \vee 0$ in cases (6.7), (6.8) and (6.9), (6.10); all other products are of order at least $s+1$. Then $\det(M(\mathbf{x}, \widetilde{\mathbf{t}}))$ tends to 0 of the same order than $\prod_{i=\mu}^{\nu} M_{i,i}$. The claim (6.12) follows by (6.16), (6.22).
For fixed $\overline{\mathbf{x}} \in X \setminus P$ consider the following choices in Definition 8.9: $\mathcal{M} = Y_{\overline{\mathbf{x}}}$, $\mathcal{V} = \mathbb{R}^s$, $\mathcal{N} = \{\mathbf{0}\}$, f defined by $f(\mathbf{t}, \boldsymbol{\tau}) = \mathbb{L}(\overline{\mathbf{x}}, \mathbf{t}, \boldsymbol{\tau})$ with $(\mathbf{t}, \boldsymbol{\tau}) \in Y_{\overline{\mathbf{x}}}$. Then $\dim(\mathcal{M}) = 2s$, $\dim(\mathcal{N}) = 0$, $\operatorname{rank}(D\mathbb{L}) = s$ by (6.12), hence projection $pr(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}})$ is a regular point of f for $\overline{\mathbf{x}} \in X \setminus P$ with $\varphi(\overline{\mathbf{x}}) < \delta$. By Theorem 8.10, $f^{-1}(\mathbf{0})$ is an analytic manifold containing the diagonal set $\Lambda_{\overline{\mathbf{x}}}$ and contained in the open set $Y_{\overline{\mathbf{x}}}$. Since $\partial\,(Y_{\overline{\mathbf{x}}}) \cap \Big(B(\widetilde{\mathbf{t}}, \psi(\overline{\mathbf{x}})/2) \times B(\widetilde{\boldsymbol{\tau}}, \psi(\overline{\mathbf{x}})/2)\Big)$ is the empty set we conclude that $f^{-1}(\mathcal{N}) \cap \Big(B(\widetilde{\mathbf{t}}, \varepsilon) \times B(\widetilde{\boldsymbol{\tau}}, \varepsilon)\Big) = \Lambda_{\overline{\mathbf{x}}} \cap \Big(B(\widetilde{\mathbf{t}}, \varepsilon) \times B(\widetilde{\boldsymbol{\tau}}, \varepsilon)\Big)$ for suitable $\varepsilon \in (0, \psi(\overline{\mathbf{x}})/2)$. $\square$

Lemma 6.3 *Fix* $\mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c} > 0$, $(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}) \in A_{\mathsf{T}} \times A_{\mathsf{T}}$ *and* $\sigma, \widetilde{\sigma} \in \Omega_{\mathsf{j},\mathsf{c}}$.
$\forall \varepsilon$ *s.t.* $0 < \varepsilon < \frac{1}{2} \min \Big\{ \operatorname{dist}(\widetilde{\mathbf{t}}, \partial A_{\mathsf{T}}), \operatorname{dist}(\widetilde{\boldsymbol{\tau}}, \partial A_{\mathsf{T}}) \Big\}$ *and any exhaustive family of partitions* $\{\mathbf{q}_m\}_{m \geq 0}$, $\mathbf{q}_m$ *of cardinality* Q_m, *fix:*

$$\text{a } d\text{-dimensional cell } W = W_0 \times W_1 \subseteq B(\widetilde{\mathbf{t}}, \varepsilon) \times B(\widetilde{\boldsymbol{\tau}}, \varepsilon) \text{ of the CW structure induced on } A_{\mathsf{T}} \times A_{\mathsf{T}} \text{ by } \mathbf{q}_m; \tag{6.24}$$

$$\{t_{l_\nu}\}_{\nu=1}^{L_0},\ \{\tau_{\lambda_\nu}\}_{\nu=1}^{L_1} \text{ respectively denote free coordinates of } W_0,\ W_1; \tag{6.25}$$

$$(\mathbf{t}, \boldsymbol{\tau}) \in W \text{ with } \mathbf{t} \neq \boldsymbol{\tau} \text{ if } \sigma = \widetilde{\sigma}; \tag{6.26}$$

$$L = \sharp\ \left(\{t_{l_\nu}\}_{\nu=1}^{L_0} \cup \{\tau_{\lambda_\nu}\}_{\nu=1}^{L_1}\right) \leq L_0 + L_1; \tag{6.27}$$

$$\mathfrak{E}(g, \mathbf{t}, \boldsymbol{\tau}, \mathbf{q}_m, \sigma, \widetilde{\sigma}) = \mathfrak{F}(g, \mathbf{t}, \mathbf{q}_m, \sigma) - \mathfrak{F}(g, \boldsymbol{\tau}, \mathbf{q}_m, \widetilde{\sigma}) \qquad \forall g \in \mathbb{A}_{\mathbf{q}_m}. \tag{6.28}$$

Then there is $\overline{m} \geq \widetilde{m}$ *(where* $\widetilde{m}$ *is the integer defined in Theorem 5.4) s.t. for any* $\mathbf{q}_m$ *with* $m > \overline{m}$ *there are at least* $L+1$ *independent vectors* $\{{}_\nu\mathbf{h}\}_{\nu=1}^{L+1} \subseteq \mathbb{R}^{\mathsf{Q}_m+2}$, *identified with* $L+1$ *functions in* $\mathbb{A}_{\mathbf{q}_m}$ *by* (3.16) *and* (3.17), *such that*

$$\left\{ \frac{\partial \mathfrak{E}}{\partial({}_\nu\mathbf{h})} \right\}_{\nu=1}^{L+1} \subseteq \mathbb{R}^{\mathsf{Q}_m+2} \text{ is a set of } L+1 \text{ independent vectors} \tag{6.29}$$

or, equivalently,

$$\dim\left(\left(\text{span}\left(\left\{\frac{\partial\mathfrak{E}}{\partial({}_\nu\mathbf{h})}\right\}_{\nu=1}^{L+1}\right)\right)^{\perp}\right) = Q_m+2-(L+1) = Q_m-L+1. \quad (6.30)$$

Here $\mathfrak{E}$ *is defined by* (6.28) *and, analogously to Definition 3.15 of derivative* $\dfrac{\partial\mathbb{F}}{\partial\mathbf{g}}$, *we set*

$$\frac{\partial\mathfrak{F}}{\partial\mathbf{g}} = \lim_{\varepsilon\to0}\frac{\mathfrak{F}(\mathbf{f}+\varepsilon\mathbf{g},\mathbf{t},\sigma)-\mathfrak{F}(\mathbf{f},\mathbf{t},\sigma)}{\varepsilon}. \quad (6.31)$$

Proof. First we introduce some notation. Let $\ll\cdot,\cdot\gg: L^2(0,1)\times L^2(0,1)\to\mathbb{R}$ be the positive definite bilinear map given by

$$\ll u,v\gg = \int_0^1 \ddot{u}(x)\ddot{v}(x)+u(x)v(x)\,dx. \quad (6.32)$$

Let $\Omega_{\mathsf{T}} = \bigcup_{\mathsf{m}+\mathsf{n}=\mathsf{T}} \Omega_{\mathsf{m},\mathsf{n}}$ and denote by $\boldsymbol{\omega}$ the elements of $\Omega_{\mathsf{T}}\times\Omega_{\mathsf{T}}$.
Fix $\boldsymbol{\omega}=(\omega,\widetilde{\omega})\in\Omega_{\mathsf{T}}\times\Omega_{\mathsf{T}}$, $\omega\in\Omega_{\mathsf{m},\mathsf{n}}$ and $\widetilde{\omega}\in\Omega_{\widetilde{\mathsf{m}},\widetilde{\mathsf{n}}}$, and set

$$\mathbb{F}_{\boldsymbol{\omega}}(g,\mathbf{t},\boldsymbol{\tau}) = \mathbb{F}(g,\mathbf{t},\omega)-\mathbb{F}(g,\boldsymbol{\tau},\widetilde{\omega}) \quad \forall(g,\mathbf{t},\boldsymbol{\tau})\in L^2(0,1)\times A_{\mathsf{T}}\times A_{\mathsf{T}}, \quad (6.33)$$

$$T=\{\widetilde{t}_l\,:\,\omega_l=\mathsf{J}\},\quad \mathcal{T}=\{\widetilde{\tau}_l\,:\,\widetilde{\omega}_l=\mathsf{J}\}, \quad (6.34)$$

$$T\cup\mathcal{T}\cup\{0,1\}=(\zeta_r)_{r=0}^{\rho+1},\quad 0=\zeta_0<...<\zeta_r<\zeta_{r+1}<...<\zeta_{\rho+1}=1, \quad (6.35)$$

$$R=\{\,r\in\{0,...,\rho+1\}:\quad \zeta_r\in(T\cap\mathcal{T})\cup\{0,1\}\,\}. \quad (6.36)$$

Any ζ_r with $r\in R$ is called *double point*, we will study intervals $[\zeta_r,\zeta_s]$ where ζ_r and ζ_s are two consecutive double points (notice that there are at least two double points in any case: 0 and 1). Now the proof splits into two steps.
Step 1 - As a first step we prove the following claim.

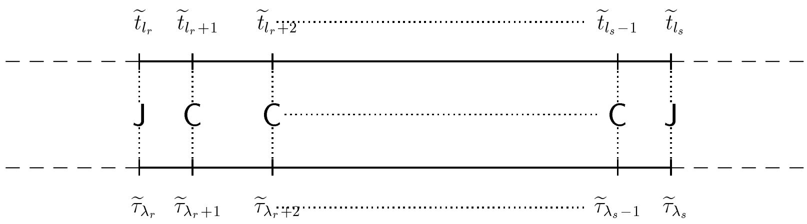
$$\left.\begin{array}{l}\textit{For any interval } [\zeta_r,\zeta_s], \textit{ with } \zeta_r,\ \zeta_s \textit{ consecutive double points,}\\ \textit{there are continuous piecewise affine maps } \{{}_ig\}_{i=1}^{s-r} \textit{ in } [0,1]\\ \textit{such that, by setting (only in this step)}\\ \qquad b^i=b[{}_ig,\mathbf{t},\omega] \textit{ and } \mathfrak{b}^i=b[{}_ig,\boldsymbol{\tau},\widetilde{\omega}],\\ \textit{the following square matrix } M \textit{ is invertible}\\ \qquad \mathbb{M}=\big(\ll b^i-{}_ig,b^k-{}_kg\gg-\ll\mathfrak{b}^i-{}_ig,\mathfrak{b}^k-{}_kg\gg\big)_{i,k=1}^{s-r}.\end{array}\right\} \quad (6.37)$$

Proof of statement (6.37) depends on the nature of the interval $[\zeta_r,\zeta_s]$. Assume $\zeta_r=\widetilde{t}_{l_r}=\widetilde{\tau}_{\lambda_r}$ and $\zeta_s=\widetilde{t}_{l_s}=\widetilde{\tau}_{\lambda_s}$ then we distinguish between three different types of intervals, describing all possible configurations.

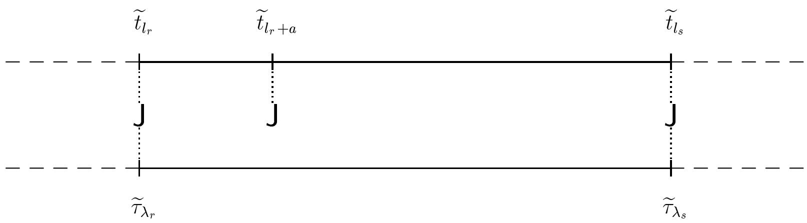
Type 1 intervals: Intervals $[\zeta_r, \zeta_s]$ fulfilling all the following three conditions

$$\left\{ \begin{array}{ll} l_s - l_r = \lambda_s - \lambda_r & \\ \widetilde{t}_{l_r+i-1} = \widetilde{\tau}_{\lambda_r+i-1} & \forall i \in \{1, ...l_s - l_r + 1\} \\ \omega_{l_r+i} = \widetilde{\omega}_{\lambda_r+i} = \mathsf{C} & \forall i \in \{1, ...l_s - l_r - 1\} \end{array} \right\}. \tag{6.38}$$

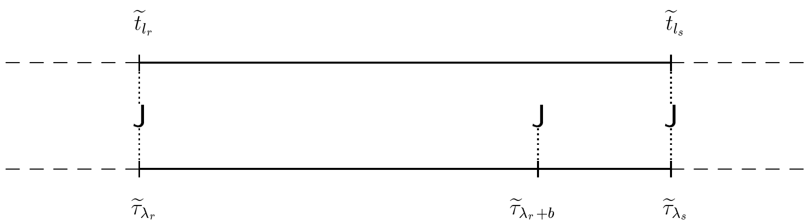
Type 1 intervals $[\zeta_r, \zeta_s]$ look as follows



Type 2 intervals: Intervals $[\zeta_r, \zeta_s]$ containing at least one jump point in $\mathbf{t}$ or $\boldsymbol{\tau}$, hence fulfilling $s - r > 1$. Each type 2 interval belongs to at least one of the two following kinds: either

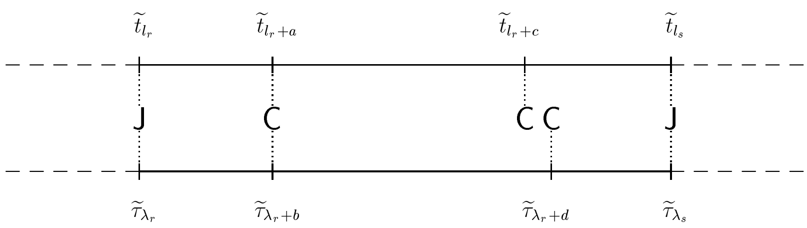


or



Crease points are not drawn in the two figures above, however they could be present possibly not coupled or in different number for ω and $\widetilde{\omega}$.

Type 3 intervals: Intervals $[\zeta_r, \zeta_s]$ fulfilling $s - r = 1$, say without jumps in $[\zeta_r, \zeta_s]$, and not fulfilling all conditions (6.38). Type 3 intervals $[\zeta_r, \zeta_s]$ look as follows



Proof of (6.37) **in case of type 1 intervals.** In this case $r-s=1$ hence the matrix $\mathbb{M}$ is a scalar.
Lemma 6.2 applied with data T, m, n, $\widetilde{\mathsf{m}}$, $\widetilde{\mathsf{n}}$, ω, $\widetilde{\omega}$ given by the pair $\boldsymbol{\omega}$, $l_1=l_r$, $l_2=l_s$, $\lambda_1=\lambda_r$, $\lambda_2=\lambda_s$ entails the existence of

$$X=X_{\boldsymbol{\omega},r},\quad P=P_{\boldsymbol{\omega},r},\quad \overline{\mathbf{x}}={}_{\boldsymbol{\omega},r}\overline{\mathbf{x}},\quad \delta=\delta_{\boldsymbol{\omega},r},\quad \varepsilon=\varepsilon_{\boldsymbol{\omega},r} \tag{6.39}$$

fulfilling

$$\begin{aligned}
&\exists\delta_{\boldsymbol{\omega},r}>0 \text{ and a closed set } P_{\boldsymbol{\omega},r}\subseteq X_{\boldsymbol{\omega},r} \text{ with empty interior in } \mathbb{R}^{2d} \text{ s.t.}\\
&\forall\ {}_{\boldsymbol{\omega},r}\overline{\mathbf{x}}\in X_{\boldsymbol{\omega},r}\setminus P_{\boldsymbol{\omega},r} \text{ with } \begin{cases}\psi(\ {}_{\boldsymbol{\omega},r}\overline{\mathbf{x}}\,)<\min\{\operatorname{dist}(\widetilde{\mathbf{t}},\partial A_{\mathsf{T}}),\operatorname{dist}(\widetilde{\boldsymbol{\tau}},\partial A_{\mathsf{T}}))\},\\ \varphi(\ {}_{\boldsymbol{\omega},r}\overline{\mathbf{x}}\,)<\delta\end{cases}\\
&\exists\varepsilon_{\boldsymbol{\omega},r}\in(0,\psi(\overline{\mathbf{x}})/2) \text{ s.t.}\\
&\forall\ {}_{\boldsymbol{\omega},r}\overline{\mathbf{x}}\in(X_{\boldsymbol{\omega},r}\setminus P_{\boldsymbol{\omega},r})\cap B({}_{\boldsymbol{\omega},r}\overline{\mathbf{x}},\varepsilon_{\boldsymbol{\omega},r})\\
&\forall(\mathbf{t},\boldsymbol{\tau})\in B(\widetilde{\mathbf{t}},\varepsilon_{\boldsymbol{\omega},r})\times B(\widetilde{\boldsymbol{\tau}},\varepsilon_{\boldsymbol{\omega},r}) \text{ with } (t_{l_1},t_{l_1+1},...,t_{l_2})\neq(\tau_{\lambda_1},\tau_{\lambda_1+1},...,\tau_{\lambda_2})\\
&\exists i\in\{1,...,d\} \text{ s.t.}\quad \mathbb{F}({}_ih,\mathbf{t},\omega)-\mathbb{F}({}_ih,\boldsymbol{\tau},\widetilde{\omega})\neq 0,
\end{aligned}$$

We plug

$${}_ig(x)=\frac{x-\mathbf{x}_{2i-1}}{\mathbf{x}_{2i}-\mathbf{x}_{2i-1}}\chi_{[\mathbf{x}_{2i-2},\mathbf{x}_{2i}]}(x)+\chi_{(\mathbf{x}_{2i},1]}(x)$$

in (6.37). Due to (6.32), ${}_i\ddot{g}\equiv 0$ in $[0,1]$, with the choice ${}_ig={}_ih$ in (6.6), (6.5) of Lemma 6.2 entails

$$\mathbb{M}=\big(\mathbb{F}({}_1g,\mathbf{t},\omega)-\mathbb{F}({}_1g,\boldsymbol{\tau},\widetilde{\omega})\big)\neq 0,$$

say $\mathbb{M}$ is a matrix of order $1=s-r$ with non zero determinant.
Proof of (6.37) **in case of type 2 intervals.** In this case $s-r\geq 2$, we set

$$Y=\{\mathbf{y}=(\mathbf{y}_i)_{i=1}^{s-r}\in[0,1]^{s-r}:\ \ \mathbf{y}_i\in(\zeta_{r+i-1},\zeta_{r+i}]\qquad \forall i\in\{1,...,s-r\},$$

$$\widetilde{\psi}(\mathbf{y})=\min\ \big(\{\operatorname{dist}(\mathbf{y}_i,\{\widetilde{t}_l,\widetilde{\tau}_l\}_{l=0}^{\mathsf{T}+1})\}_{i=1}^{s-r}\cup\{\operatorname{dist}(\mathbf{y}_i,\mathbf{y}_k)\}_{i\neq k=1}^{s-r}\big).$$

We denote by $\mathbb{M}(\mathbf{y},\mathbf{t},\boldsymbol{\tau})$ the symmetric matrix $\mathbb{M}$ defined in (6.37) with the choices

$${}_ig=\chi_{[\mathbf{y}_i,1]}\qquad i\in\{1,...,s-r\}.$$

By Theorems 3.11(2), 3.14(2) matrix $\mathbb{M}(\mathbf{y},\mathbf{t},\boldsymbol{\tau})$ is a continuous function on $Y\times A_{\mathsf{T}}\times A_{\mathsf{T}}$ where the topology of Y is induced by $[0,1]^{s-r}$.
We denote by $\mathbb{M}_\zeta$ the matrix $\mathbb{M}(\mathbf{y},\mathbf{t},\boldsymbol{\tau})$ evaluated at $\mathbf{y}=(\zeta_{r+1},...,\zeta_{s-r-1},\mathbf{y}_{s-r})$, $\mathbf{t}=\widetilde{\mathbf{t}}$, $\boldsymbol{\tau}=\widetilde{\boldsymbol{\tau}}$. We claim that

$$\mathbb{M}_\zeta \text{ is non singular.} \tag{6.40}$$

Since $\mathbb{M}(\mathbf{y}, \widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}})$ is continuous on Y, if (6.40) holds true, then we get

$$\begin{aligned}&\exists c>0,\ \ \exists\, {}_{\boldsymbol{\omega},r}\mathbf{y} = ({}_{\boldsymbol{\omega},r}\mathbf{y}_1, \dots, {}_{\boldsymbol{\omega},r}\mathbf{y}_{s-r}) \in Y \ \text{ with } \{{}_{\boldsymbol{\omega},r}\mathbf{y}_i\}_{i=1}^{s-r} \cap \{\widetilde{t}_l, \widetilde{\tau}_l\}_{l=0}^{\mathsf{T}+1} = \emptyset \\ &\text{such that } \quad \left|\det(\mathbb{M}({}_{\boldsymbol{\omega},r}\mathbf{y}, \widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}))\right| > c,\end{aligned}$$

then, referring to Definition (6.37) of $\mathbb{M}$, $\widetilde{\psi}({}_{\boldsymbol{\omega},r}\mathbf{y}) > 0$, Theorem 3.11(2) and Theorem 3.19 entail

$$\left.\begin{aligned}&\exists \varepsilon_{\boldsymbol{\omega},r} \in (0, \tfrac{\widetilde{\psi}({}_{\boldsymbol{\omega},r}\mathbf{y})}{2}) \text{ such that the matrix } \mathbb{M}(\mathbf{y}, \mathbf{t}, \boldsymbol{\tau}) \text{ is invertible} \\ &\forall (\mathbf{t}, \boldsymbol{\tau}) \in B(\widetilde{\mathbf{t}}, \varepsilon_{\boldsymbol{\omega},r}) \times B(\widetilde{\boldsymbol{\tau}}, \varepsilon_{\boldsymbol{\omega},r}), \\ &\forall \text{ partition } \mathbf{q} \text{ and set } \{{}_i g\}_{i=1}^{s-r} \subseteq \mathbb{A}_{\mathbf{q}} \text{ of ramp functions with} \\ &\qquad\qquad {}_i g(x) = \begin{cases} 0 & \text{if } x \le {}_{\boldsymbol{\omega},r}\mathbf{y}_i - \varepsilon_{\boldsymbol{\omega},r}, \\ 1 & \text{if } x \ge {}_{\boldsymbol{\omega},r}\mathbf{y}_i + \varepsilon_{\boldsymbol{\omega},r}. \end{cases}\end{aligned}\right\} \quad (6.41)$$

Eventually we prove claim (6.40) by showing that $\mathbb{M}_\zeta$ is a block diagonal matrix and that each block has non zero determinant.
The matrix $\mathbb{M} = \mathbb{M}(\mathbf{y}, \mathbf{t}, \boldsymbol{\tau})$ defined in (6.37) is symmetric and for any $i, k \in \{1, \dots, s-r\}$ with $i < k$ we have

$$\begin{cases} \sharp\,(\operatorname{spt}(b^i - {}_i g) \cap \operatorname{spt}(b^k - {}_k g)) \le 1 & \text{if } [\mathbf{y}_i, \mathbf{y}_k] \cap T \ne \emptyset, \\ \sharp\,(\operatorname{spt}(\mathfrak{b}^i - {}_i g) \cap \operatorname{spt}(\mathfrak{b}^k - {}_k g)) \le 1 & \text{if } [\mathbf{y}_i, \mathbf{y}_k] \cap \mathcal{T} \ne \emptyset, \end{cases} \tag{6.42}$$

then

$$\begin{cases} (i) & \ll b^i - {}_i g, b^k - {}_k g \gg = 0 \quad \text{if } [\mathbf{y}_i, \mathbf{y}_k] \cap T \ne \emptyset, \\ (ii) & \ll \mathfrak{b}^i - {}_i g, \mathfrak{b}^k - {}_k g \gg = 0 \quad \text{if } [\mathbf{y}_i, \mathbf{y}_k] \cap \mathcal{T} \ne \emptyset, \end{cases} \tag{6.43}$$

hence

$$\mathbb{M}_{i,k} = 0 \ \text{ if both } \ [\mathbf{y}_i, \mathbf{y}_k] \cap T \ne \emptyset \ \text{ and } \ [\mathbf{y}_i, \mathbf{y}_k] \cap \mathcal{T} \ne \emptyset \ \text{ hold true.} \tag{6.44}$$

By (6.44) we have

$$\mathbb{M}_{i,k} = 0 \ \text{ entails } \ M_{a,b} = 0 \ \text{ for } \ a \le i \ \text{ and } \ b \ge k \tag{6.45}$$

and

$$\begin{aligned}&(\mathbb{M}_\zeta)_{i,i+1} = 0 \ \text{ for all } \ i \text{ such that} \\ &\qquad (\zeta_{r+i} \in T \ \textit{and} \ \zeta_{r+i+1} \in \mathcal{T}) \ \textit{or} \ (\zeta_{r+i} \in \mathcal{T} \ \textit{and} \ \zeta_{r+i+1} \in T),\end{aligned} \tag{6.46}$$

then $\mathbb{M}_\zeta$ is a square block diagonal matrix where each block $\mathbb{M}_e^{e'}$ belongs to exactly one kind among the following four ones:

B.1 $e \le e' < s-r$, $\zeta_{r+e-1}, \zeta_{r+e'+1} \in T$ and $\zeta_{r+i} \in \mathcal{T} \setminus T$ for any $i \in \{e, ..., e'\}$;

B.2 $e \le e' < s-r$, $\zeta_{r+e-1}, \zeta_{r+e'+1} \in \mathcal{T}$ and $\zeta_{r+i} \in T \setminus \mathcal{T}$ for any $i \in \{e, ..., e'\}$;

B.3 $e \le e' = s-r$, $\zeta_{r+e-1} \in T$ and $\zeta_{r+i} \in \mathcal{T} \setminus T$ for any $i \in \{e, ..., s-r-1\}$;

B.4 $e \le e' = s-r$, $\zeta_{r+e-1} \in \mathcal{T}$ and $\zeta_{r+i} \in T \setminus \mathcal{T}$ for any $i \in \{e, ..., s-r-1\}$.

By (6.43.(ii)), B.1 blocks have the form $\mathbb{M}_e^{e'} = \big(\ll b^i - {}_ig, b^k - {}_kg \gg \big)_{i,k=e}^{e'}$.
The bilinear map (6.32) is positively defined and $\{b^i - {}_ig\}_{i=e}^{e'} \subseteq L^2(0,1)$ are independent vectors since $\{{}_ig\}_{i=e}^{e'}$ are, then $\det(\mathbb{M}_e^{e'}) \neq 0$.
By (6.43.(i)), B.2 blocks have the form $\mathbb{M}_e^{e'} = \big(- \ll \mathfrak{b}^i - {}_ig, \mathfrak{b}^k - {}_kg \gg \big)_{i,k=e}^{e'}$.
The bilinear map (6.32) is positively defined and $\{\mathfrak{b}^i - {}_ig\}_{i=e}^{e'} \subseteq L^2(0,1)$ are independent vectors since $\{{}_ig\}_{i=e}^{e'}$ are, then $\det(\mathbb{M}_e^{e'}) \neq 0$.
Type B.3 blocks have the form

$$\mathbb{M}_e^{s-r} = \big(\ll b^i - {}_ig, b^k - {}_kg \gg - \ll \mathfrak{b}^i - {}_ig, \mathfrak{b}^k - {}_kg \gg \big)_{i,k=e}^{s-r}$$

where $\ll \mathfrak{b}^i - {}_ig, \mathfrak{b}^k - {}_kg \gg = 0$ whenever $(i,k) \neq (s-r, s-r)$.
Let $\mathbb{M}_e^{s-r} = N_e^{s-r} - \mathcal{E}$, where

$$N_e^{s-r} = \begin{bmatrix} \ll b^e - {}_eg, b^e - {}_eg \gg & \cdots & \ll b^e - {}_eg, b^{s-r} - {}_{s-r}g \gg \\ \vdots & & \vdots \\ \ll b^{s-r} - {}_{s-r}g, b^e - {}_eg \gg & \cdots & \ll b^{s-r} - {}_{s-r}g, b^{s-r} - {}_{s-r}g \gg \end{bmatrix}$$

$$\mathcal{E} = [\mathcal{E}_{i,k}]_{i,k=e}^{s-r}, \quad \mathcal{E}_{i,k} = \begin{cases} 0 & \text{if } (i,k) \neq (s-r, s-r), \\ \ll \mathfrak{b}^{s-r} - {}_{s-r}g, \mathfrak{b}^{s-r} - {}_{s-r}g \gg & \text{if } (i,k) = (s-r, s-r). \end{cases}$$

The same argument used for B.1 blocks proves that N_e^{s-r} is an invertible matrix. Moreover $\mathcal{E} \to 0$ as $\mathbf{y}_{s-r} \to \zeta_{s-1}$, since ${}_{s-r}g \to \chi_{[\zeta_{s-r},1]}$ in $L^2(0,1)$, hence Theorem 3.11(2) entails $\mathfrak{b} \to \chi_{[\zeta_{s-r},1]}$ in $H^2(\tau_l, \tau_{l+1}) \quad \forall l$.
Then any type B.3 block $\mathbb{M}_e^{s-r} = N_e^{s-r} - \mathcal{E}$ is an invertible matrix.
Type B.4 blocks have the form

$$\mathbb{M}_e^{s-r} = \big(\ll b^i - {}_ig, b^k - {}_kg \gg - \ll \mathfrak{b}^i - {}_ig, \mathfrak{b}^k - {}_kg \gg \big)_{i,k=e}^{s-r}$$

where $\ll b^i - {}_ig, b^k - {}_kg \gg = 0$ whenever $(i,k) \neq (s-r, s-r)$.
Let $\mathbb{M}_e^{s-r} = -\mathfrak{N}_e^{s-r} + \widehat{\mathcal{E}}$ where $\mathfrak{N}_e^{s-r}$ is defined like N_e^{s-r} with $\ll b^i - {}_ig, b^k - {}_kg \gg$ replaced by $\ll \mathfrak{b}^i - {}_ig, \mathfrak{b}^k - {}_kg \gg$ and $\widehat{\mathcal{E}}$ is defined like $\mathcal{E}$ with $\ll \mathfrak{b}^{s-r} - {}_{s-r}g, \mathfrak{b}^{s-r} - {}_{s-r}g \gg$ replaced by $\ll b^{s-r} - {}_{s-r}g, b^{s-r} - {}_{s-r}g \gg$.
So we can repeat the same analysis we performed on type B.3 blocks.
Proof of (6.37) **in case of type 3 intervals.** In this case set

$$Z = \{\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2) \in [\zeta_r, \zeta_s]^2,\ \mathbf{z}_1 < \mathbf{z}_2\},$$

$$\widehat{\psi}(\mathbf{z}) = \min\ \big(\{\mathrm{dist}(\{\mathbf{z}_1,\mathbf{z}_1\},\{\widetilde{t}_l,\widetilde{\tau}_l\}_{l=0}^{\mathsf{T}+1})\}\cup\{\mathrm{dist}(\mathbf{z}_1,\mathbf{z}_2)\}\big).$$

We label by $\mathbb{M}(\mathbf{z},\mathbf{t},\boldsymbol{\tau})$ the matrix $\mathbb{M}$ defined in (6.37) with the choice

$$g(x) = \frac{x-\mathbf{z}_1}{\mathbf{z}_2-\mathbf{z}_1}\chi_{[\mathbf{z}_1,\mathbf{z}_2]}(x) + \chi_{[\mathbf{z}_2,1]}(x) \qquad \mathbf{z} = (\mathbf{z}_1,\mathbf{z}_2)\in Z.$$

Actually the matrix $\mathbb{M}(\mathbf{z},\mathbf{t},\boldsymbol{\tau})$ is a scalar whose value is a continuous function on $Z\times A_{\mathsf{T}}\times A_{\mathsf{T}}$, where the topology of Z is induced by $[0,1]^2$, due to Theorems 3.11(2), 3.14(2).
Since $[\zeta_r,\zeta_s]$ is a type 3 interval and $s-r=1$, then at least one among the following two possibilities holds true:

$$\begin{aligned}
&(1)\quad \exists l\in\{1,...,\mathsf{T}\}:\quad \widetilde{\tau}_l\in[\zeta_r,\zeta_s]\setminus(\{\widetilde{t}_l\}_{l=0}^{\mathsf{T}+1}\cup\mathcal{T}),\\
&(2)\quad \exists l\in\{1,...,\mathsf{T}\}:\quad \widetilde{t}_l\in[\zeta_r,\zeta_s]\setminus(\{\widetilde{\tau}_l\}_{l=0}^{\mathsf{T}+1}\cup T).
\end{aligned}$$

We examine only possibility (1) since the other one is analogous.
We evaluate $\mathbb{M}(\mathbf{z},\mathbf{t},\boldsymbol{\tau})$ at: $\mathbf{z}=(\widetilde{\tau}_l,\mathbf{z}_2)$, $\mathbf{t}=\widetilde{\mathbf{t}}$, $\boldsymbol{\tau}=\widetilde{\boldsymbol{\tau}}$, with $\mathbf{z}_2\in(\widetilde{\tau}_l,\widetilde{\tau}_{l+1})$
Since

$$\begin{cases}\displaystyle\lim_{\mathbf{z}_2\to\widetilde{\tau}_{l+1}} \ll \mathfrak{b}-g,\mathfrak{b}-g\gg\, = 0 \quad \text{(by Theorem 3.9)}\\ \displaystyle\lim_{\mathbf{z}_2\to\widetilde{\tau}_{l+1}} \ll b-g,b-g\gg\ >\ 0,\end{cases}$$

Theorem 3.11(2) entails $\mathbb{M}((\widetilde{\tau}_l,\mathbf{z}_2),\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\tau}})\neq 0$ for any $\mathbf{z}_2$ close enough to $\widetilde{\tau}_{l+1}$.
Since $\mathbb{M}(\mathbf{z},\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\tau}})$ is continuous on Z we get

$$\begin{aligned}
&\exists\, c>0,\ \exists\ {}_{\boldsymbol{\omega},r}\mathbf{z} = (\,{}_{\boldsymbol{\omega},r}\mathbf{z}_1,\ {}_{\boldsymbol{\omega},r}\mathbf{z}_2\,)\\
&\qquad\qquad\text{with}\quad \{\,{}_{\boldsymbol{\omega},r}\mathbf{z}_1,\ {}_{\boldsymbol{\omega},r}\mathbf{z}_2\,\}\cap\{\widetilde{t}_l,\widetilde{\tau}_l\}_{l=0}^{\mathsf{T}+1}=\emptyset\\
&\qquad\qquad\qquad\qquad\text{such that}\quad \Big|\mathbb{M}(\,{}_{\boldsymbol{\omega},r}\mathbf{z},\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\tau}}\,)\Big| > c.
\end{aligned}$$

Then, by definition (6.37) of $\mathbb{M}$, $\widehat{\psi}(\,{}_{\boldsymbol{\omega},r}\mathbf{z}\,)>0$, Theorem 3.11(2) and Theorem 3.19 we get

$$\left.\begin{aligned}
&\exists\varepsilon_{\boldsymbol{\omega},r}\in(0,\tfrac{\widehat{\psi}(\,{}_{\boldsymbol{\omega},r}\mathbf{z}\,)}{2})\text{ such that }|\mathbb{M}(\mathbf{z},\mathbf{t},\boldsymbol{\tau})|>\tfrac{c}{2}\\
&\forall(\mathbf{t},\boldsymbol{\tau})\in B(\widetilde{\mathbf{t}},\varepsilon_{\boldsymbol{\omega},r})\times B(\widetilde{\boldsymbol{\tau}},\varepsilon_{\boldsymbol{\omega},r}),\ \forall\mathbf{z}\in B(\,{}_{\boldsymbol{\omega},r}\mathbf{z},\varepsilon_{\boldsymbol{\omega},r}\,).
\end{aligned}\right\}\tag{6.47}$$

So far the claim (6.37) is proven.
Step 2 - To achieve the conclusion we exploit Step 1. First we choose

$$\varepsilon = \min\{\varepsilon_{\boldsymbol{\omega},r}:\ \boldsymbol{\omega}\in\Omega_{\mathsf{T}}\times\Omega_{\mathsf{T}},\ r\in R\}>0.\tag{6.48}$$

We consider $\overline{m}$ such that $\mathbf{q}_{\overline{m}}$ has at least two distinct points in each one of the following intervals

$$\begin{aligned}
&\left({}_{\boldsymbol{\omega},r}\mathsf{x}_{2i-1} - \varepsilon,\ {}_{\boldsymbol{\omega},r}\mathsf{x}_{2i-1} + \varepsilon \right),\ \left({}_{\boldsymbol{\omega},r}\mathsf{x}_{2i} - \varepsilon,\ {}_{\boldsymbol{\omega},r}\mathsf{x}_{2i} + \varepsilon \right) \quad \forall\, {}_{\boldsymbol{\omega},r}\mathsf{x} \qquad \text{(Type 1)},\\
&\left({}_{\boldsymbol{\omega},r}\mathsf{y}_{i} - \varepsilon,\ {}_{\boldsymbol{\omega},r}\mathsf{y}_{i} + \varepsilon \right) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \forall\, {}_{\boldsymbol{\omega},r}\mathsf{y} \qquad \text{(Type 2)},\\
&\left({}_{\boldsymbol{\omega},r}\mathsf{z}_{2i-1} - \varepsilon,\ {}_{\boldsymbol{\omega},r}\mathsf{z}_{2i-1} + \varepsilon \right),\ \left({}_{\boldsymbol{\omega},r}\mathsf{z}_{2i} - \varepsilon,\ {}_{\boldsymbol{\omega},r}\mathsf{z}_{2i} + \varepsilon \right) \quad \forall\, {}_{\boldsymbol{\omega},r}\mathsf{z} \qquad \text{(Type 3)},
\end{aligned}$$

where we refer respectively to (6.39), (6.41), (6.47) for different interval types (only for types which are present, according to arrangements $\mathbf{t}$ and $\boldsymbol{\tau}$).
Obviously the same property holds true for any $\mathbf{q}_m$ with $m \geq \overline{m}$.
To any cell $W = W_0 \times W_1$ fulfilling (6.24) with ε given by (6.48), we associate the following qualities ω and $\widetilde{\omega}$:

$$\omega_l = \begin{cases} \sigma_l & \text{if } t_l \text{ is not a free coordinate of } W_0 \\ \mathsf{J} & \text{otherwise} \end{cases}$$

$$\widetilde{\omega}_l = \begin{cases} \widetilde{\sigma}_l & \text{if } t_l \text{ is not a free coordinate of } W_1 \\ \mathsf{J} & \text{otherwise,} \end{cases}$$

and denote respectively by m, n the number of J, C in quality ω and respectively by $\widetilde{\mathsf{m}}$, $\widetilde{\mathsf{n}}$ the number of J, C in quality $\widetilde{\omega}$.
Then the ordered sequence $(\zeta_r)_{r=1}^{L}$ of jump points (related to ω, $\widetilde{\omega}$) is fixed and for any $(\mathbf{t}, \boldsymbol{\tau}) \in W$: by referring to Definition 6.1 and (6.33), we get $\mathfrak{E}(\cdot,\cdot,\cdot,\mathbf{q}_m,\sigma,\widetilde{\sigma}) = \mathbb{F}_{\boldsymbol{\omega}}(\cdot,\cdot,\cdot)$ where $\boldsymbol{\omega} = (\omega, \widetilde{\omega})$.
By the same procedure used in Step 1 we choose $\{{}_{\nu}g\}_{\nu=1}^{L+1}$ associated to ω, $\widetilde{\omega}$: we notice that the vectors $\{{}_{\nu}g\}_{\nu=1}^{L+1}$ are linearly independent by construction.
We exploit $\gamma[\cdot,\cdot,\mathbf{q}_m,\sigma] = b[\cdot,\cdot,\omega]$, $\gamma[\cdot,\cdot,\mathbf{q}_m,\widetilde{\sigma}] = b[\cdot,\cdot,\widetilde{\omega}]$ and apply Lemma 3.16 to obtain the following identity between square matrices

$$\mathbb{M} = \frac{1}{2}\left(\frac{\partial^2 \mathbb{F}_{\boldsymbol{\omega}}(g,\mathbf{t},\boldsymbol{\tau})}{\partial({}_{\nu}\mathbf{g})\,\partial({}_{\nu}\mathbf{g}')} \right)_{\nu,\nu'=1}^{L+1} .$$

There is an uniform estimate in the choices of $\varepsilon_{\boldsymbol{\omega},r}$ (in case of type 1 intervals by Lemma 6.2; in case of type 2 interval by (6.41); in case of type 3 interval by (6.47)). By summarizing:

$$\varepsilon_{\boldsymbol{\omega},r} < \frac{1}{2}\min\left\{ \psi({}_{\boldsymbol{\omega},r}\overline{\mathsf{x}}), \widetilde{\psi}({}_{\boldsymbol{\omega},r}\mathsf{y}), \widehat{\psi}({}_{\boldsymbol{\omega},r}\mathsf{z}) \right\}.$$

Hence $\mathbb{M}$ is a block diagonal matrix where each block is related to an interval of type either 1 or 2 or 3. Moreover $\mathbb{M}$ turns out to be a constant matrix

once $\mathbf{t}$ and $\boldsymbol{\tau}$ are fixed, due to (6.33) and Theorem 3.14.
By Step 1 each block is an invertible matrix so that the whole matrix $\mathbb{M}$ is invertible. This implies that the normal vectors to the $L+1$ hyperplanes determined by $\{\frac{\partial \mathbb{F}_{\boldsymbol{\omega}}(g,\mathbf{t},\boldsymbol{\tau})}{\partial({}_\nu \mathbf{g})} = 0\}_{\nu=1}^{L+1}$ are independent. $\square$

Theorem 6.4 *Fix* $\mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, 1, 2, ...\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c} > 0$, $(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}) \in A_{\mathsf{T}} \times A_{\mathsf{T}}$ *and* $\sigma, \widetilde{\sigma} \in \Omega_{\mathsf{j},\mathsf{c}}$.
For any ε *s.t.* $0 < \varepsilon < \frac{1}{2} \min \left\{ \mathrm{dist}(\widetilde{\mathbf{t}}, \partial A_{\mathsf{T}}), \mathrm{dist}(\widetilde{\boldsymbol{\tau}}, \partial A_{\mathsf{T}}) \right\}$ *and any exhaustive family of partitions* $\{\mathbf{q}_m\}_{m \geq 0}$, $\mathbf{q}_m$ *of cardinality* Q_m, *there is* $\overline{m}$:
for any $\mathbf{q}_m$ *with* $m > \overline{m}$, *any* $d = 0, ..., 2\mathsf{T}$ *and any open* d*-dimensional cell* $W \subseteq B(\widetilde{\mathbf{t}}, \varepsilon) \times B(\widetilde{\boldsymbol{\tau}}, \varepsilon)$ *s.t.* $W = W_0 \times W_1$ *where* $W_0 \subseteq B(\widetilde{\mathbf{t}}, \varepsilon)$ *and* $W_1 \subseteq B(\widetilde{\boldsymbol{\tau}}, \varepsilon)$ *open cells of the CW structure induced by* $\mathbf{q}_m$ *on* A_{T}, *if* $\mathfrak{E}$ *and* $\mathbf{E}$ *are the maps of Definition 6.1, we have*

$$\begin{cases} \mathcal{L}^{\mathsf{Q}_m+2}\Big(pr\,[\mathbb{A}_{\mathbf{q}_m}]\left(\{\mathfrak{E} = 0\} \cap \{\mathbf{E} = \mathbf{0}\}\right)\Big) = 0 & \text{if } \sigma \neq \widetilde{\sigma} \\ \mathcal{L}^{\mathsf{Q}_m+2}\Big(\,pr\,[\mathbb{A}_{\mathbf{q}_m}]\left((\{\mathfrak{E}=0\}\cap\{\mathbf{E}=\mathbf{0}\}) \ \setminus \ (\mathbb{A}_{\mathbf{q}_m} \times \Lambda\,[A_{\mathsf{T}}])\right)\Big) = 0 & \text{if } \sigma = \widetilde{\sigma}, \end{cases}$$

where $\Lambda\,[A_{\mathsf{T}}] = \{(\mathbf{t}, \boldsymbol{\tau}) \in A_{\mathsf{T}} \times A_{\mathsf{T}} : \ \mathbf{t} = \boldsymbol{\tau}\}$ *and*
$pr\,[\mathbb{A}_{\mathbf{q}_m}] : \mathbb{A}_{\mathbf{q}_m} \times A_{\mathsf{T}} \times A_{\mathsf{T}} \to \mathbb{A}_{\mathbf{q}_m}$ *is the projection on the component* $\mathbb{A}_{\mathbf{q}_m}$:

$$pr\,[\mathbb{A}_{\mathbf{q}_m}]\,(\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) = \mathbf{g}, \quad \forall \mathbf{g} \in \mathbb{R}^{\mathsf{Q}_m+2}. \tag{6.49}$$

Proof. Choose ε and $\overline{m}$ as in Lemma 6.3. Fix $m > \overline{m}$.

Parameters $\mathbf{q}_m$, σ, $\widetilde{\sigma}$ are now fixed: for this reason they are omitted when writing the variables of $\mathfrak{E}$ and $\mathbf{E}$ in the following. As usual we set $\mathsf{Q}_m = \dim \mathbf{q}_m$ and the identification between $\mathbb{A}_{\mathbf{q}_m}$ and $\mathbb{R}^{\mathsf{Q}_m+2}$ through (3.16) and (3.17) will be always understood. We denote by $(\mathbf{t}_{l_1}, ..., \mathbf{t}_{l_{d_0}}, \boldsymbol{\tau}_{\lambda_1}, ..., \boldsymbol{\tau}_{\lambda_{d_1}})$ the free coordinates of W. We set

$$\mathbf{J} = (\mathbf{E}_{l_1}, ..., \mathbf{E}_{l_{d_0}}, \mathbf{E}_{\mathsf{T}+\lambda_1}, ..., \mathbf{E}_{\mathsf{T}+\lambda_{d_1}})$$
$$T = \{(\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) \in \mathbb{A}_{\mathbf{q}_m} \times A_{\mathsf{T}} \times A_{\mathsf{T}} \ : \ \mathbf{E}(\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) = \mathbf{0}\}.$$

We emphasize that

$$S \subseteq \mathbb{A}_{\mathbf{q}_m} \times W, \qquad \dim(\mathbb{A}_{\mathbf{q}_m} \times W) = \mathsf{Q}_m + 2 + d, \qquad \mathbf{H}^{\dim(S)}(S) > 0,$$

since S is at most countable union of analytic graphs; here $\mathbf{H}^d$ denotes d-dimensional Hausdorff measure and $\dim(S)$ denotes the geometric dimension of S which is coincident with the Hausdorff measure of S.
By applying first (5.1) and Theorem 5.4 to W_0 and to W_1 we have:
$T \cap (\mathbb{A}_{\mathbf{q}_m} \times W) = \{(\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) \in \mathbb{A}_{\mathbf{q}_m} \times W \ : \ \mathbf{J}(\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) = \mathbf{0}\}$;

$T \cap (\mathbb{A}_{\mathbf{q}_m} \times W)$ is a semi-analytic set contained in $\mathbb{A}_{\mathbf{q}_m} \times W$ and the higher order skeleton (Definition 5.3 of [23] or Definition 8.1 in the Appendix) S of $T \cap (\mathbb{A}_{\mathbf{q}_m} \times W)$ has dimension at most $\mathsf{Q}_m + 2$.
If $\dim(S) < \mathsf{Q}_m + 2$ then the theorem follows.
If $\dim(S) = \mathsf{Q}_m + 2$ then we can show a contradiction by a three steps proof.

Step 1 - We prove the following statement.

If we set $Z = \{(\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) \in S: \quad \det(D_W \mathbf{J})(\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) = 0\}$
where $D_W \mathbf{J}$ *is the differential of* $\mathbf{J}$ *with respect to free coordinates of d-cell* W,
then $\mathcal{L}^{\mathsf{Q}_m+2}(pr\,[\mathbb{A}_{\mathbf{q}_m}]\,(Z)) = 0.$

Theorem 5.4 entails that the higher order skeleton S is a countable union of graphs of analytic functions $\mathbf{F} : A \to B$ where A and B are connected open sets, $A \subseteq \mathcal{U} \times W$, $B \subseteq \mathcal{V}$ and $\mathcal{U}, \mathcal{V} \subseteq \mathbb{A}_{\mathbf{q}_m}$ are independent linear subspaces of dimension $\mathsf{Q}_m + 2 - d$ and d respectively, we also choose A and B so that $S \cap (A \times B)$ is connected.
For any choice of $\mathbf{F}$, A, B as above we prove:

$$\mathcal{L}^{\mathsf{Q}_m+2}(pr\,[\mathbb{A}_{\mathbf{q}_m}]\,(Z \cap (A \times B))) = 0. \tag{6.50}$$

By denoting $pr[\mathcal{U}] : \mathcal{U} \times W \to \mathcal{U}$ the projection on $\mathcal{U}$ we can say

$$(\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) \in Z \cap (A \times B) \iff \left\{ \begin{array}{l} (\mathbf{g}, \mathbf{t}, \boldsymbol{\tau}) \in (A \times B), \\ \mathbf{g} = (pr[\mathcal{U}](\mathbf{g}), \mathbf{F}(pr[\mathcal{U}](\mathbf{g}), \mathbf{t}, \boldsymbol{\tau})), \\ \det(D_W \mathbf{F})(pr[\mathcal{U}](\mathbf{g}), \mathbf{t}, \boldsymbol{\tau}) = 0. \end{array} \right\} \tag{6.51}$$

We examine two possibilities according to the fact that $\det(D_W \mathbf{J})$ is identically zero or not on $S \cap (A \times B)$.
If $\det(D_W \mathbf{J}) \equiv 0$ on $S \cap (A \times B)$ then $\det(D_W \mathbf{F}) \equiv 0$ on A since, by Dini's Theorem, $(D_W \mathbf{F})(\mathbf{h}, w) = ((D_\mathcal{V} \mathbf{J})(\mathbf{h}, \mathbf{F}(\mathbf{h}, w), w))^{-1}((D_W \mathbf{J})(\mathbf{h}, \mathbf{F}(\mathbf{h}, w), w))$.
By (6.51) $pr\,[\mathbb{A}_{\mathbf{q}_m}]\,(Z \cap (A \times B))$ is the image of the function $\mathbf{G} : A \to \mathbb{A}_{\mathbf{q}_m}$ defined by $\mathbf{G}(\mathbf{h}, w) = (\mathbf{h}, \mathbf{F}(\mathbf{h}, w))$. Hence Theorem 2.71 in [1] together with $D\mathbf{G}$ lower block triangular matrix entail

$$\mathcal{L}^{\mathsf{Q}_m+2}(pr\,[\mathbb{A}_{\mathbf{q}_m}]\,(Z \cap (A \times B))) = \int_A |\det(D\,\mathbf{G})|\, d\mathbf{h}\, dw = \int_A |\det(D_\mathcal{U}\, pr[\mathcal{U}])|\, |\det(D_W\, \mathbf{F})|\, d\mathbf{h}\, dw = 0, \tag{6.52}$$

hence (6.50) holds true.
If $\det(D_W \mathbf{J}) \not\equiv 0$ on $S \cap (A \times B)$ then the semi analytic set $\{\det(D_W \mathbf{F}) = 0\}$ is

a closed subset of A with higher order skeleton of dimension at most Q_m+1: this follows by Dini's Theorem and (6.51) since $S\cap(A\times B)$ is connected. By $A\subseteq\mathcal{U}\times W$ and $\dim W=d$, we get

$$Z\cap(A\times B)=\{(\mathbf{h},\mathbf{F}(\mathbf{h},\mathbf{t},\boldsymbol{\tau}),\mathbf{t},\boldsymbol{\tau})\quad(\mathbf{h},\mathbf{t},\boldsymbol{\tau})\in\{\det(D_W\mathbf{F})=0\}\subseteq\mathcal{U}\times W_0\times W_1\}$$

is a semi-analytic subset of $S\cap(A\times B)$ of dimension at most Q_m+1, hence (6.50) holds true.

Step 2 - We prove the following statement.

> *Referring to* (3.16), (3.17), (6.31), *we denote the differential of* $\mathfrak{E}$ *with respect to* $\mathbf{g}\in\mathbb{R}^{\mathsf{Q}_m+2}$ *by* $D_{\mathbb{A}_{\mathbf{q}_m}}\mathfrak{E}$ *and set*
> $$\begin{cases} Y=\{(\mathbf{g},\mathbf{t},\boldsymbol{\tau})\in S\colon\quad(D_{\mathbb{A}_{\mathbf{q}_m}}\mathfrak{E})(\mathbf{g},\mathbf{t},\boldsymbol{\tau})=\mathbf{0}\} & \text{if } \sigma\neq\widetilde{\sigma},\\ Y=\{(\mathbf{g},\mathbf{t},\boldsymbol{\tau})\in S\setminus\mathcal{D}\colon\quad(D_{\mathbb{A}_{\mathbf{q}_m}}\mathfrak{E})(\mathbf{g},\mathbf{t},\boldsymbol{\tau})=\mathbf{0}\} & \text{if } \sigma=\widetilde{\sigma}.\end{cases}\tag{6.53}$$
> *Then* Y *is contained in a semi-analytic set whose higher order skeleton has dimension strictly less than* Q_m+2.

We introduce V_r as the intersection with the cell W of all $(2\mathsf{T}-r)$-dimensional diagonal hyperplanes, say:

$$R_r=\{\mathbf{r}\subseteq\{1,...,\mathsf{T}\}\times\{1,...,\mathsf{T}\}\colon\quad\sharp(\mathbf{r})=r\}\quad\forall r\in\{0,...,\mathsf{T}\},$$

$$V_{r,\mathbf{r}}=\{(\mathbf{t},\boldsymbol{\tau})\in A_{\mathsf{T}}\times A_{\mathsf{T}}\colon\quad t_i=\tau_k\quad\forall(i,k)\in\mathbf{r}\}\quad\forall r\in\{0,...,\mathsf{T}\}\;\forall\mathbf{r}\in R_r,$$

$$V_r=\Bigg(\Big(\bigcup_{\mathbf{r}\in R_r}V_{r,\mathbf{r}}\Big)\setminus\Big(\bigcup_{s>r,\mathbf{s}\in R_s}V_{s,\mathbf{s}}\Big)\Bigg)\cap W\qquad\forall r\in\{0,...,\mathsf{T}\}.$$

Notice that

$$R_0=\{\emptyset\},\qquad V_0=W\setminus\Big(\bigcup_{s>r,\mathbf{s}\in R_s}V_{s,\mathbf{s}}\Big),$$

$V_{r,\mathbf{r}}$ is a semi-analytic set contained in $A_{\mathsf{T}}\times A_{\mathsf{T}}$,

V_r is a real analytic manifold contained in $W\quad\forall r\in\{1,...,\mathsf{T}\}$,

$$V_{\mathsf{T}}=\Lambda\,[A_{\mathsf{T}}]\,,\qquad V_r\cap V_s=\emptyset\quad\text{if } r\neq s,\qquad W=\bigcup_{r=0}^{\mathsf{T}}V_r.$$

Now fix any $r\in\{0,...,\mathsf{T}\}$, with restriction $r\neq\mathsf{T}$ if $\sigma=\widetilde{\sigma}$, and denote by L the dimension of V_r: $L\leq\min\{2\mathsf{T}-r,d\}$. Lemma 6.3 entails the existence (for any $(\mathbf{t},\boldsymbol{\tau})\in V_r$) of at least $L+1$ vectors $\{{}_\nu h\}_{\nu=1}^{L+1}\subseteq\mathbb{A}_{\mathbf{q}_m}$ s.t. $K_{(\mathbf{t},\boldsymbol{\tau})}=\bigcap_{\nu=1}^{L+1}\{\frac{\partial\mathfrak{E}(g,\mathbf{t},\boldsymbol{\tau})}{\partial({}_\nu\mathbf{h})}=0\}$ is a (Q_m+1-L)-dimensional subspace of $\mathbb{A}_{\mathbf{q}_m}$. Then the set $K_r = \bigcup_{(\mathbf{t},\boldsymbol{\tau})\in V_r} K_{(\mathbf{t},\boldsymbol{\tau})}$ is a semi-analytic set with higher order skeleton of dimension strictly less than $\mathsf{Q}_m + 2$, moreover

$$Y \cap (\mathbb{A}_{\mathbf{q}_m} \times V_r) \;\subseteq\; K_r \;\subseteq\; S \cap (\mathbb{A}_{\mathbf{q}_m} \times V_r).$$

Eventually (6.53) follows by

$$\begin{cases} Y \;\subseteq\; \bigcup_{r=0}^{\mathsf{T}} \big(Y \cap (\mathbb{A}_{\mathbf{q}_m} \times V_r)\big) & \text{if } \sigma \neq \widetilde{\sigma} \\ Y \;\subseteq\; \bigcup_{r=0}^{\mathsf{T}-1} \big(Y \cap (\mathbb{A}_{\mathbf{q}_m} \times V_r)\big) & \text{if } \sigma = \widetilde{\sigma}. \end{cases}$$

Step 3 - By Step 1 and Step 2 we are left to prove the following statement.

$$\textit{We set } \widehat{S} = \begin{cases} S \setminus (Z \cup Y) & \text{if } \sigma \neq \widetilde{\sigma} \\ S \setminus ((\Lambda[\mathbb{A}_{\mathbf{q}_m}] \times W) \cup Z \cup Y) & \text{if } \sigma = \widetilde{\sigma} \end{cases}, \textit{ then}$$

$$\begin{cases} \mathcal{L}^{\mathsf{Q}_m+2}\Big(pr\,[\mathbb{A}_{\mathbf{q}_m}]\,\big(\{\mathfrak{E} = 0\} \cap \widehat{S}\big)\Big) = 0 & \textit{if } \sigma \neq \widetilde{\sigma} \\ \mathcal{L}^{\mathsf{Q}_m+2}\Big(pr\,[\mathbb{A}_{\mathbf{q}_m}]\,\big((\{\mathfrak{E} = 0\} \cap \widehat{S})\big)\Big) = 0 & \textit{if } \sigma = \widetilde{\sigma}. \end{cases}$$

Since S is semi-analytic, there is a covering $\mathcal{C}$ of $\widehat{S}$ defined as follows

- Any element of $\mathcal{C}$ is the product of a connected open subset N of $\mathbb{A}_{\mathbf{q}_m}$ by a connected open subset U of W,
- For any $N \times U \in \mathcal{C}$ there is an analytic function $\Phi : N \times U \to \mathbb{R}^d$ with $(N\times U)\cap\widehat{S} = \{\Phi = \mathbf{0}\}$, moreover $\det(D_W\Phi)(g, \mathbf{t}, \boldsymbol{\tau}) \neq 0$ on $\widehat{S}$ by Step 1.

The differential of map $(\mathfrak{E}, \Phi) : N \times U \to \mathbb{R}^{d+1}$ is a $(\mathsf{Q}_m + 2 + d) \times (d + 1)$ tensor with the following structure:

$$D(\mathfrak{E}, \Phi) = \begin{bmatrix} D_{\mathbb{A}_{\mathbf{q}_m}} \mathfrak{E} & D_W \mathfrak{E} \\ D_{\mathbb{A}_{\mathbf{q}_m}} \Phi & D_W \Phi \end{bmatrix}.$$

We claim that the rank of the matrix $D(\mathfrak{E}, \Phi)$ is $2\mathsf{T} + 1$ on $\widehat{S}$. In fact:

- $\det(D_W\Phi)(g, \mathbf{t}, \boldsymbol{\tau}) \neq 0$ on $\widehat{S}$;
- $(D_W\mathfrak{E})(g, \mathbf{t}, \boldsymbol{\tau}) = 0$ for any $(g, \mathbf{t}, \boldsymbol{\tau}) \in \widehat{S}$, since Lemma 3.17 holds true and $\mathbf{E}(g, \mathbf{t}, \boldsymbol{\tau}) = \mathbf{0}$ entails that the right hand side of (3.22) vanishes on $\widehat{S}$;

- there is at least a coordinate $\mathbf{h}_j$ with $\frac{\partial \mathfrak{E}(g,\mathbf{t},\boldsymbol{\tau})}{\partial ({}_j\mathbf{h})} \neq 0$ for any $(g,\mathbf{t},\boldsymbol{\tau}) \in \widehat{S}$, thanks to Step 2.

Hence, by Theorem 12.17 in [23], (recalled in the Appendix A: Theorem 8.10) the set $\{(\mathfrak{E},\Phi)=0\}$ has co-dimension at least $d+1$ in $N\times U$, that is dimension strictly less than Q_m+2. □

7 Proof of the main theorem

This section is devoted to prove Theorem 1.1.
In Lemma 7.1 we show a compactness property about locations of minimizers valid when data α, β, g, fulfill the assumption that all related minimizers have the same cardinality of both jumps and creases (with possibly different quality of singular set). In Theorem 7.2 we prove the existence of a dense set of continuous piecewise affine data leading to uniqueness. Eventually we deduce Theorem 1.1.

Lemma 7.1 *Fix* $\mathsf{T},\mathsf{j},\mathsf{c} \in \{0,1,2,...\}$, $\mathsf{T}=\mathsf{j}+\mathsf{c}$, $\sigma\in\Omega_{\mathsf{j},\mathsf{c}}$, $g\in L^2$ *and set*

$$T_g^\sigma = \{\mathbf{t}(u)\in A_{\mathsf{T}}: \quad \exists u \in \operatorname{argmin} F^g_{\alpha,\beta} \text{ with } \sigma(u)=\sigma\},$$

$$R_{\mathsf{j},\mathsf{c}} = \left\{h\in L^2: \quad \forall w \in \operatorname{argmin} F^h_{\alpha,\beta} \quad \sharp(S_w)=\mathsf{j}, \quad \sharp(S_{\dot w}\setminus S_w)=\mathsf{c}\right\}.$$

$$\text{Assume} \quad T_g^\sigma \neq \emptyset, \quad \sharp(S_u)=\mathsf{j} \text{ and } \sharp(S_{\dot u}\backslash S_u)=\mathsf{c} \quad \forall u\in \operatorname{argmin} F^g_{\alpha,\beta}. \qquad (7.1)$$

Then:

1. $g\in R_{\mathsf{j},\mathsf{c}}$;
2. *the set* T_g^σ *of locations of* $F^g_{\alpha,\beta}$ *minimizers with quality* σ *is a compact subset of the open set* A_{T}, *hence* $\operatorname{dist}(T_g^\sigma,\partial A_{\mathsf{T}})>0$;
3. *for any neighborhood* A *of* T_g^σ *contained in* A_{T} *there is an* L^2*-neighborhood* V *of* g *such that*
   $\mathbf{t}(u)\in A$ *for any* $u\in\operatorname{argmin} F^h_{\alpha,\beta}$ *with* $\sigma(u)=\sigma$ *and* $h\in V\cap R_{\mathsf{j},\mathsf{c}}$.

Proof. The first point is a restatement of (7.1). Now we prove 2 and 3.
For any fixed choice of sequences $\{\mathbf{t}_n\}\subseteq T_g^\sigma$, $\{u_n\}\subseteq \operatorname{argmin} F^g_{\alpha,\beta}$ such that $\mathbf{t}_n=\mathbf{t}(u_n)$, we have $F^g_{\alpha,\beta}(u_n)=m^g(\alpha,\beta)$ and $\{u_n\}$ satisfies the hypotheses of Theorem 2.5(1) in [4]. Then there is $u_\infty\in\mathcal{H}^2$ such that, up to subsequences, $u_n\to u_\infty$ strongly in L^1, $u_\infty\in\operatorname{argmin} F^g_{\alpha,\beta}$, and $\{\mathbf{t}_n=\mathbf{t}(u_n)\}$ tends to $\mathbf{t}_\infty=\mathbf{t}(u_\infty)$. Actually $\mathbf{t}_\infty\in A_{\mathsf{T}}$ in fact if $i\neq l$ then sequences $\{t_{n,i}\}$, $\{t_{n,l}\}$ cannot have the same limit point without contradiction with (7.1) and

Theorem 2.5(3) in [4]. The number of creases is preserved. Obviously the ordering (say the quality) is preserved too, then the second statement is proven.
The third statement holds true whenever g is an isolated point of $R_{\mathsf{j,c}}$ since $g \in R_{\mathsf{j,c}}$. If g is not an isolated point of $R_{\mathsf{j,c}}$ we argue by contradiction by assuming the existence of a neighborhood U of T_g^σ such that for any n there is $g_n \in R_{\mathsf{j,c}}$ with $\|g - g_n\|_{L^2} < \frac{1}{n}$ and $u_n \in \operatorname{argmin} F_{\alpha,\beta}^{g_n}$ with $\sigma(u_n) = \sigma$ and $\mathbf{t}(u_n) \notin U$.
The sequence $\{u_n\}$ satisfies the hypotheses of Theorem 2.5(1) in [4] then up to subsequences there is u_∞ with $u_n \to u_\infty$ strongly in L^1, and by Theorem 2.5(3) in [4] the sequence $\{\mathbf{t}_n = \mathbf{t}(u_n)\}$ tends to $\mathbf{t}_\infty = \mathbf{t}(u_\infty) \notin U$.
We have that $F_{\alpha,\beta}^g(u_n) \to m_{\alpha,\beta}^g$ since:

$$\left|F_{\alpha,\beta}^g(u_n) - m^g(\alpha,\beta)\right| \leq \left|F_{\alpha,\beta}^g(u_n) - F_{\alpha,\beta}^{g_n}(u_n)\right| + \left|m^{g_n}(\alpha,\beta) - m^g(\alpha,\beta)\right|$$

and the first term in the right-hand side goes to zero by plugging (2.12) of [4] in

$$\begin{aligned}\left|F_{\alpha,\beta}^g(u_n) - F_{\alpha,\beta}^{g_n}(u_n)\right| &= \left|\|u_n - g\|_{L^2}^2 - \|u_n - g_n\|_{L^2}^2\right| = \\ &= \langle g - g_n, g + g_n - 2u_n\rangle_{L^2} \leq \|g + g_n - 2u_n\|_{L^2}\, \|g - g_n\|_{L^2},\end{aligned}$$

while the second term in the right-hand side goes to zero by (2.14) of [4].
Moreover $\sigma(u_\infty) = \sigma$ (since otherwise we get a contradiction with (7.1) and Theorem 2.5(1) in [4]); by lower semi-continuity (Theorem 2.5(2) in [4]) we have $F_{\alpha,\beta}^g(u_\infty) = m^g(\alpha,\beta)$. Then $\mathbf{t}(u_\infty) \in T_g^\sigma \subseteq U$ contradicting $\mathbf{t}(u_\infty) \notin U$. □

Theorem 7.2 *Assume* (1.2) *and* $\alpha/\beta \notin \mathbb{Q}$.
Then there is $A_{\alpha,\beta}$ *dense in* $L^2(0,1)$ *such that*

$$\sharp\,(\operatorname{argmin} F_{\alpha,\beta}^h) = 1 \qquad \forall h \in A_{\alpha,\beta}\,, \tag{7.2}$$

$$A_{\alpha,\beta} \subseteq \{\textit{continuous piecewise affine functions in } [0,1]\}. \tag{7.3}$$

Proof. It is enough proving:

$$\left.\begin{array}{l}\text{for any continuous piecewise linear function } g \in L^2(0,1) \text{ and } \varepsilon > 0 \\ \text{there is a continuous piecewise linear function } f \in L^2(0,1) \text{ s.t.} \\ \qquad\qquad \|f - g\|_{L^2} < \varepsilon, \qquad \sharp\,(\operatorname{argmin} F_{\alpha,\beta}^f) = 1.\end{array}\right\} \tag{7.4}$$

We fix $g \in L^2(0,1)$ continuous piecewise linear. By (2.15) of [4] we know:

$$\begin{array}{l}\exists \mathsf{K} \in \mathbb{N} \quad \exists U = \{f \in L^2(0,1) \colon \|f - g\|_{L^2} < \varepsilon\} \quad \text{s.t.} \\ \sharp\,(S_u \cup S_{\dot u}) \leq \mathsf{K} \;\; \forall u \in \operatorname{argmin} F_{\alpha,\beta}^h,\, \forall h \in U.\end{array} \tag{7.5}$$

So the number of possible pairs $(\sharp\,(S_u), \sharp\,(S_{\dot u}\setminus S_u))$ with $u\in \operatorname{argmin} F^h_{\alpha,\beta}$ and $h\in U$ is finite, say less then $\mathsf{K}(\mathsf{K}+1)/2$. Proof of (7.4) splits in five steps.
Step 1 - We exploit $\alpha/\beta\notin\mathbb{Q}$ to show the following claim.
Let

$$H=H(\mathsf{j},\widetilde{\mathsf{j}},\mathsf{c},\widetilde{\mathsf{c}})=\Big\{h\in U:\ \ \exists u,v\in \operatorname{argmin} F^h_{\alpha,\beta}\ \ \textit{with}\ \ \sharp\,(S_u)=\mathsf{j},\,\sharp\,(S_v)=\widetilde{\mathsf{j}}$$
$$\sharp\,(S_{\dot u}\setminus S_u)=\mathsf{c},\,\sharp\,(S_{\dot v}\setminus S_v)=\widetilde{\mathsf{c}},\quad (\mathsf{j},\mathsf{c})\neq(\widetilde{\mathsf{j}},\widetilde{\mathsf{c}}),\quad \mathsf{j}+\mathsf{c}\le\mathsf{K},\quad \widetilde{\mathsf{j}}+\widetilde{\mathsf{c}}\le\mathsf{K}\Big\}.$$

Then $\;\mathcal{L}^{\mathsf{Q}+2}(H\cap\mathbb{A}_{\mathbf{q}})=0\quad \forall \mathsf{Q}\in\mathbb{N}\quad \forall\mathbf{q}\in A_{\mathsf{Q}}\quad$ *with* $\quad\mathsf{Q}=\dim\mathbf{q}$.
Set $\mathsf{T}=\mathsf{j}+\mathsf{c}$, $\widetilde{\mathsf{T}}=\widetilde{\mathsf{j}}+\widetilde{\mathsf{c}}$. Choose $\sigma\in\Omega_{\mathsf{j},\mathsf{c}}$, $\widetilde\sigma\in\Omega_{\widetilde{\mathsf{j}},\widetilde{\mathsf{c}}}$ and, referring to Definition 3.10, consider the function $\mathcal{E}(\cdot,\sigma,\widetilde\sigma):\mathbb{A}_{\mathbf{q}}\to\mathbb{R}$ defined by

$$\mathcal{E}(h,\sigma,\widetilde\sigma)=\inf_{\mathbf{t}\in A_{\mathsf{T}}}\mathbb{F}(h,\mathbf{t},\sigma)-\inf_{\boldsymbol{\tau}\in A_{\widetilde{\mathsf{T}}}}\mathbb{F}(h,\boldsymbol{\tau},\widetilde\sigma)\qquad\forall h\in A_{\mathbf{q}}.$$

By Theorem 3.11(3) $\mathbb{F}(h,\mathbf{t},\sigma)$ and $\mathbb{F}(h,\boldsymbol{\tau},\widetilde\sigma)$ are non negative continuous functions with respect to h, $\mathbf{t}$ and $\boldsymbol{\tau}$. Then maps $h\mapsto\inf_{\mathbf{t}\in A_{\mathsf{T}}}\mathbb{F}(h,\mathbf{t},\sigma)$ and $h\mapsto\inf_{\boldsymbol{\tau}\in A_{\widetilde{\mathsf{T}}}}\mathbb{F}(h,\boldsymbol{\tau},\widetilde\sigma)$ are Borel functions from $\mathbb{A}_{\mathbf{q}}$ to $\mathbb{R}$, since they are infimum of continuous functions, hence $h\mapsto\mathcal{E}(h,\sigma,\widetilde\sigma)$ is a Borel function of $h\in\mathbb{A}_{\mathbf{q}}\cong\mathbb{R}^{\mathsf{Q}+2}$.
Then

$$\widetilde H\overset{\text{def}}{=}\bigcup_{\sigma,\widetilde\sigma}\Big\{h\in\mathbb{A}_{\mathbf{q}}:\ \mathcal{E}(h,\sigma,\widetilde\sigma)=\alpha\,(\widetilde{\mathsf{j}}-\mathsf{j})+\beta\,(\widetilde{\mathsf{c}}-\mathsf{c})\Big\}\ \text{ is a Borel subset of }\mathbb{A}_{\mathbf{q}}.$$

Since

$$\begin{cases}\mathcal{E}(th,\sigma,\widetilde\sigma)=t^2\mathcal{E}(h,\sigma,\widetilde\sigma)\quad\forall t\in\mathbb{R},\quad\text{by Theorem 3.11(1)},\\ \alpha\,(\widetilde{\mathsf{j}}-\mathsf{j})+\beta\,(\widetilde{\mathsf{c}}-\mathsf{c})\neq0\quad\forall\mathsf{j},\widetilde{\mathsf{j}},\mathsf{c},\widetilde{\mathsf{c}}\in\mathbb{N},\quad(\mathsf{j},\mathsf{c})\neq(\widetilde{\mathsf{j}},\widetilde{\mathsf{c}}),\quad\text{since }\alpha/\beta\notin\mathbb{Q},\end{cases}$$

we deduce

$$\Big\{t\in\mathbb{R}:\ th\in\widetilde H\Big\}=\{-1,1\}\qquad\forall h\in\widetilde H\setminus\{0\}.\tag{7.6}$$

Since $\widetilde H$ is a Borel subset of $\mathbb{A}_{\mathbf{q}}\cong\mathbb{R}^{\mathsf{Q}+2}$ and (7.6) holds true then

$$\mathcal{L}^{\mathsf{Q}+2}(\widetilde H)=\int_{\widetilde H}d\mathbf{x}=$$
$$=\int_{S^{\,\mathsf{Q}+1}}\left(\int_{(0,+\infty)}\chi_{\widetilde H}\,(\rho,\boldsymbol{\vartheta})\,\rho^{\mathsf{Q}+1}\,d\rho\right)\,d\sigma(\boldsymbol{\vartheta})=\int_{S^{\,\mathsf{Q}+1}}0\,d\sigma(\boldsymbol{\vartheta})=0.$$

Since $H\cap\mathbb{A}_{\mathbf{q}}\subseteq\widetilde H$ we have $\mathcal{L}^{\mathsf{Q}+2}(H\cap\mathbb{A}_{\mathbf{q}})=0$.

Step 2 - Referring to (7.5) we introduce the set $\widehat{H}$ (of data g admitting at least two minimizers with different arrangements) and its complement in U:

$$\widehat{H} = \bigcup_{(\mathsf{j},\mathsf{c})\neq(\widetilde{\mathsf{j}},\widetilde{\mathsf{c}})} H(\mathsf{j},\widetilde{\mathsf{j}},\mathsf{c},\widetilde{\mathsf{c}}), \qquad V = U \setminus \widehat{H}. \tag{7.7}$$

By (7.5) $\widehat{H}$ is the union of a finite number of sets, then we deduce by Step 1

$$\mathcal{L}^{\mathsf{Q}+2}(\widehat{H}\cap \mathbb{A}_{\mathbf{q}}) = 0, \quad \mathcal{L}^{\mathsf{Q}+2}(V\cap \mathbb{A}_{\mathbf{q}}) = \mathcal{L}^{\mathsf{Q}+2}(U\cap \mathbb{A}_{\mathbf{q}}) \quad \forall \mathbf{q}\in A_{\mathsf{Q}}. \tag{7.8}$$

By (7.5) and (7.8) there are only the two following possibilities:

$$\text{either} \qquad \exists h\in V\colon\ \sharp\,(\operatorname{argmin} F^h_{\alpha,\beta}) = 1, \tag{7.9}$$

$$\text{or} \quad \begin{cases} \sharp(\operatorname{argmin} F^h_{\alpha,\beta}) > 1 & \forall h\in V \text{ and} \\ \sharp(S_u)=\sharp(S_v),\ \sharp(S_{\dot u}\setminus S_u)=\sharp(S_{\dot v}\setminus S_v) & \forall h\in V,\ \forall u,v\in \operatorname{argmin} F^h_{\alpha,\beta}, \\ 0<\sharp\,((S_u\cup S_{\dot u}) = \sharp\,((S_v\cup S_{\dot v}) \le \mathsf{K} & \forall h\in V,\ \forall u,v\in \operatorname{argmin} F^h_{\alpha,\beta}. \end{cases} \tag{7.10}$$

If (7.9) occurs then claim (7.4) trivially follows.
We show by steps 3,4,5 that (7.10) entails a contradiction.
Step 3 - We prove the following claim.
If (7.10) *occurs then there are*

$\mathsf{j},\mathsf{c},\mathsf{T}\in\{0,...,\mathsf{K}\},\ \ \mathsf{T}=\mathsf{j}+\mathsf{c}\le\mathsf{K},\ \ \sigma,\widetilde{\sigma}\in\Omega_{\mathsf{j},\mathsf{c}}$,
a compact subset $K_0\subseteq A_{\mathsf{T}}\times A_{\mathsf{T}}$,
a subset $\Gamma_0\subseteq V$,
an exhaustive family of partitions $\{\mathbf{q}^0_m=(q_1,q_2,...,q_{\mathsf{Q}^0_m})\}_m$,

such that

$$\begin{cases} S_{\dot g}\subseteq \mathbf{q}^0_0,\ \ S_g=\emptyset, \\ \mathcal{L}^{\mathsf{Q}^0_m+2}(\Gamma_0\cap \mathbb{A}_{\mathbf{q}^0_m})>0 \qquad \forall m\in\{0,1,...\}, \\ (\mathbf{t}(u),\mathbf{t}(v))\in K_0 \qquad \forall h\in\Gamma_0,\ \forall u,v\in \operatorname{argmin} F^h_{\alpha,\beta}\colon\ \sigma(u)=\sigma,\ \sigma(v)=\widetilde{\sigma}. \end{cases}$$

In order to prove the claim, we introduce the following notation:

$$\delta(\mathbf{q}) = \max\{q_{l+1}-q_l\colon\ l\in\{0,...,\mathsf{Q}\}\} \quad \forall \mathbf{q}=(q_l)_{l=1}^{\mathsf{Q}}\in A_{\mathsf{Q}}\ \ \forall \mathsf{Q}\in\{0,1,...\},$$

$$\mathbf{P}=\{\mathbf{q}\colon \quad S_{\dot g}\subseteq \mathbf{q}\},$$

and, for any $\mathsf{m},\mathsf{n}\in\{0,1,...\}$ with $\mathsf{m}+\mathsf{n}\le\mathsf{K}$ and any $\omega,\widetilde{\omega}\in\Omega_{\mathsf{m},\mathsf{n}}$, we set:

$$V(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega}) = \{h\in V\colon\ \ \exists u,v\in \operatorname{argmin} F^h_{\alpha,\beta} \text{ s.t. } u\neq v,\ \sigma(u)=\omega,\ \sigma(v)=\widetilde{\omega}\},$$

$$\mathbf{P}(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega}) = \{\mathbf{q}\in\mathbf{P}\colon\ \ \mathcal{L}^{\mathsf{Q}+2}(V(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})\cap\mathbb{A}_{\mathbf{q}})>0\},$$

$$Z = \{(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})\colon\quad \forall\delta>0\quad \exists\mathbf{q}\in\mathbf{P}(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})\quad\text{and}\quad \delta(\mathbf{q})<\delta\}.$$

We have that Z is a finite non empty set since: the number of quadruples $(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})$ with $\mathsf{m}+\mathsf{n}\le\mathsf{K}$ and $\omega,\widetilde{\omega}\in\Omega_{\mathsf{m},\mathsf{n}}$ is finite; $\bigcup_{\mathsf{m},\mathsf{n},\omega,\widetilde{\omega}} V(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega}) = V$; for any $\delta>0$ the subset of the elements $\mathbf{q}\in\mathbf{P}$ with $\delta(\mathbf{q})<\delta$ is infinite; $\mathcal{L}^{\mathsf{Q}+2}(V\cap\mathbb{A}_{\mathbf{q}}) = \mathcal{L}^{\mathsf{Q}+2}(U\cap\mathbb{A}_{\mathbf{q}})>0$ for any $\mathbf{q}\in\mathbf{P}$ by (7.8).
We label the elements of the finite set Z i.e. $Z=\{z_r=(\mathsf{m}_r,\mathsf{n}_r,\omega_r,\widetilde{\omega}_r)\}_{r=1}^N$.
We set $V_0=U$, for $r\ge1$, if $V_{r-1}\cap V(z_r)=\emptyset$ then we set $({}_rh,\mathbb{T}_r,V_r) = (\emptyset,\emptyset,V_{r-1})$, otherwise we choose ${}_rh\in V_{r-1}\cap V(z_r)$ and observe that Lemma 7.1 entails the existence of a compact neighborhood $\mathbb{T}_r$ of $T^{\omega}_{{}_rh}\times T^{\widetilde{\omega}}_{{}_rh}$ and a neighborhood $V_r\subseteq V_{r-1}$ of ${}_rh$ such that $(\mathbf{t}(u),\mathbf{t}(v))\in\mathbb{T}_r\quad\forall h\in V_r\cap V(z_r)$, $\forall u,v\in\operatorname{argmin}F^h_{\alpha,\beta}$ with $\sigma(u)=\omega_r$, $\sigma(v)=\widetilde{\omega}_r$.
Among all triplets constructed above we consider the collection of the ones whose first two entries are not empty and relabel such triplets $\{({}_sh,\mathbb{T}_s,V_s)\}_{s=1}^M$ with $M\le N$. Summarizing we have

$$\begin{cases} (i) & {}_sh\in V(z_s)\subseteq L^2(0,1),\ \ \mathbb{T}_s \text{ is a pair of locations,}\\ (ii) & {}_sh\in V_s \text{ open set in } L^2,\ \ V_s\subseteq V_{s-1}\subseteq U,\\ (iii) & \mathbb{T}_s\subset A_{\mathsf{T}}\times A_{\mathsf{T}} \text{ is a compact neighborhood of } T^{\omega_r}_{{}_sh}\times T^{\widetilde{\omega}_s}_{{}_sh}\\ & \text{where } T^{\omega}_h \text{ is defined in Lemma 7.1,}\\ (iv) & (\mathbf{t}(u),\mathbf{t}(v))\in\mathbb{T}_s\quad \forall h\in V_s\cap V(z_s),\,\forall u,v\in\operatorname{argmin}F^h_{\alpha,\beta}\text{ s.t.}\\ & \sigma(u)=\omega_s,\,\sigma(v)=\widetilde{\omega}_s. \end{cases} \tag{7.11}$$

For any $(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})\notin Z$ there is $\overline{\delta}=\delta(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})>0$ such that

$$\{\mathbf{q}\in\mathbf{P}(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})\colon\quad \delta(\mathbf{q})<\overline{\delta}\}=\emptyset.$$

Let $\delta_0=\min\{\delta(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega}):\ (\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})\notin Z,\ \mathsf{m}+\mathsf{n}\le\mathsf{K},\ \omega,\widetilde{\omega}\in\Omega_{\mathsf{m},\mathsf{n}}\}>0$.
For any fixed exhaustive family $\{\mathbf{q}_j\}_{j\ge0}\subseteq\mathbf{P}$ with $\delta(\mathbf{q}_0)<\delta_0$, by $V\subseteq U$, (7.8) and definition of $\mathbf{P}$ and Z, we have

$$\mathcal{L}^{\mathsf{Q}_j+2}\Bigg(\Big(\widehat{H}\cup\bigcup_{(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})\notin Z} V(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})\Big)\cap\mathbb{A}_{\mathbf{q}_j}\Bigg)=0\qquad\forall j,$$

hence by (7.8)

$$\mathcal{L}^{\mathsf{Q}_j+2}\Big(\bigcup_{z=(\mathsf{m},\mathsf{n},\omega,\widetilde{\omega})\in Z} V(z)\cap\mathbb{A}_{\mathbf{q}_j}\Big)=\mathcal{L}^{\mathsf{Q}_j+2}(U\cap\mathbb{A}_{\mathbf{q}_j})>0\qquad\forall j\,,$$

then, since V_N is an open set of L^2 and $V_N \subseteq U$,

$$\mathcal{L}^{\mathsf{Q}_j+2}\Big(V_N \cap \bigcup_{z\in Z} V(z) \cap \mathbb{A}_{\mathbf{q}_j}\Big) = \mathcal{L}^{\mathsf{Q}_j+2}(V_N \cap \mathbb{A}_{\mathbf{q}_j}) > 0 \qquad \forall j\,.$$

Since Z is a non empty and finite set there is $z_{\overline{r}} = (\mathsf{m}_{\overline{r}}, \mathsf{n}_{\overline{r}}, \omega_{\overline{r}}, \widetilde{\omega}_{\overline{r}}) \in Z$ and a subsequence $\{\mathbf{q}^0_m\}_m \subseteq \{\mathbf{q}_j\}_j$ such that $\mathbb{T}_{\overline{r}} \neq \emptyset$ and $\mathcal{L}^{\mathsf{Q}^0_m+2}(V_N \cap V(z_{\overline{r}}) \cap \mathbb{A}_{\mathbf{q}^0_m}) > 0$ for any m. We select $\mathsf{j} = \mathsf{m}_{\overline{r}}$, $\mathsf{c} = \mathsf{n}_{\overline{r}}$, $\sigma = \omega_{\overline{r}}$, $\widetilde{\sigma} = \widetilde{\omega}_{\overline{r}}$, $K_0 = \mathbb{T}_{\overline{r}}$ and $\Gamma_0 = V_N \cap V(z_{\overline{r}})$.

Step 4 - We prove the following claim (which is an iteration of Step 3).
If (7.10) *occurs then there are* $\mathsf{T}, \mathsf{j}, \mathsf{c} \in \{0, ..., \mathsf{K}\}$, $\mathsf{T} = \mathsf{j} + \mathsf{c}$, $\sigma, \widetilde{\sigma} \in \Omega_{\mathsf{j},\mathsf{c}}$ *and a family* $F = \{\varphi_i = (K_i, \Gamma_i, \{\mathbf{q}^i_m\}_m)\}_{i\in\mathbb{N}}$ *of triplets where*

> K_i *is a non empty compact subset of* $A_{\mathsf{T}} \times A_{\mathsf{T}} \subseteq \mathbb{R}^{2\mathsf{T}}$,
> Γ_i *is a subset of* V,
> $\{\mathbf{q}^i_m\}_m$*is an exhaustive sequence of partitions* ($\mathsf{Q}^i_m = \sharp\,(\mathbf{q}^i_m)$),

such that, for any $i \in \mathbb{N}$,

$$\begin{cases} S_g = \emptyset, \quad S_{\dot g} \subseteq \mathbf{q}^i_0 \quad \textit{and} \quad \{\mathbf{q}^i_m\}_m \textit{ is a subsequence of } \{\mathbf{q}^{i-1}_m\}_m, \\ K_i \subseteq K_{i-1}, \ \textit{diam}(K_i) \le \frac{1}{2}\textit{diam}(K_{i-1}) \ \textit{ and } \ \Gamma_i \subseteq \Gamma_{i-1}, \\ \mathcal{L}^{\mathsf{Q}^i_m+2}(\Gamma_i \cap \mathbb{A}_{\mathbf{q}^i_m}) > 0 \qquad \forall m \in \{0, 1, ...\}, \\ (\mathbf{t}(u), \mathbf{t}(v)) \in K_i \qquad \forall h \in \Gamma_i, \ \forall u, v \in \operatorname{argmin} F^h_{\alpha,\beta}\colon\ \sigma(u) = \sigma,\ \sigma(v) = \widetilde{\sigma}. \end{cases}$$

We argue by induction. Step 3 is the starting point: we set $\varphi_0 = (K_0, \Gamma_0, \{\mathbf{q}^0_m\}_m)$. Then, by assuming that the family F is defined up to index i, we show how to define $\varphi_{i+1} = (K_{i+1}, \Gamma_{i+1}, \{\mathbf{q}^{i+1}_m\}_m)$.
Choose a finite covering $\{K_{i,k}\}_{k=1}^N$ of K_i by compact subsets with $diam(K_{i,k}) \le diam(K_i)/2$, the choice is possible since K_i is compact by induction.
For any $k \in \{1, ..., N\}$ set

$$\Gamma_{i,k} = \{h \in \Gamma_i\colon \quad \exists u, v \in \operatorname{argmin} F^h_{\alpha,\beta} \text{ with } u \neq v,\ (\mathbf{t}(u), \mathbf{t}(v)) \in K_{i,k}\}.$$

Since $\mathcal{L}^{\mathsf{Q}^i_m+2}(\Gamma_i \cap \mathbb{A}_{\mathbf{q}^i_m}) > 0 \quad \forall m$ by induction,

$$\exists \text{ a sequence } \{k_m\}_m \text{ with values in } \{1, ..., N\}\ : \qquad \mathcal{L}^{\mathsf{Q}^i_m+2}(\Gamma_{i,k_m} \cap \mathbb{A}_{\mathbf{q}^i_m}) > 0 \quad \forall m. \tag{7.12}$$

Hence there is $\overline{k} \in \{1, ..., N\}$ and a subsequence $\{m_n\}_n$ such that $k_{m_n} = \overline{k}\ \forall n$.
We define φ_{i+1} as follows: $K_{i+1} = K_{i,\overline{k}}$, $\Gamma_{i+1} = \Gamma_{i,\overline{k}}$, $\{\mathbf{q}^{i+1}_m\}_m = \{\mathbf{q}^i_{m_n}\}_n$.

Step 5 - We exploit Step 4 and Theorem 6.4 to show that (7.10) cannot hold true (as was claimed at the end of Step 2).
By the construction in Step 4, $\bigcap_i K_i \neq \emptyset$: precisely $\bigcap_i K_i$ is a single point. So we can set $(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\tau}}) = \bigcap_i K_i$. Then we choose: T, j, c, σ, $\widetilde{\sigma}$ as in Step 4; ε as in Theorem 6.4; j such that $K_j \subseteq B(\widetilde{\mathbf{t}}, \varepsilon/2) \times B(\widetilde{\boldsymbol{\tau}}, \varepsilon/2)$; m (large enough) such that the CW structure induced by $\mathbf{q}^j_m$ on $A_\mathsf{T} \times A_\mathsf{T}$ (is so fine that) provides a compact neighborhood K of K_j where $K \subseteq B(\widetilde{\mathbf{t}}, \varepsilon) \times B(\widetilde{\boldsymbol{\tau}}, \varepsilon)$ and K is a union of cells of the CW structure induced by $\mathbf{q}^j_m$.
For the sake of simplicity we drop the indexes j and m and we write $\mathbf{q}$, Q instead of $\mathbf{q}^j_m$, Q_j in the following.
For any $d \in \{0, ..., \min\{2\mathsf{T}, 2\mathsf{K}\}\}$ we set

$$\{C_{d,l}\}_{l=1}^L \text{ the finite set of all } d\text{-dimensional open cells of } K, \quad L = L(d),$$

$$\Phi_{d,l} = \{h \in \Gamma_j \cap \mathbb{A}_{\mathbf{q}^j_m} : \exists u, v \in \operatorname{argmin} F^h_{\alpha,\beta} \text{ s.t. } \sigma(u) = \sigma,\ \sigma(v) = \widetilde{\sigma},\ (\mathbf{t}(u), \mathbf{t}(v)) \in C_{d,l}\}.$$

Form
$\{C_{d,l}\}_{d,l}$ is a finite set of cells, $\mathcal{L}^{\mathsf{Q}+2}(\Gamma_j \cap \mathbb{A}_\mathbf{q}) > 0$ and $\Gamma_j \cap \mathbb{A}_{\mathbf{q}^j_m} \subseteq \bigcup_{d,l} \Phi_{d,l}$,

we deduce: there is a pair $(\overline{d}, \overline{l})$ such that $\mathcal{L}^{\mathsf{Q}+2}(\Phi_{\overline{d},\overline{l}}) > 0$.
On the other hand we prove that $\mathcal{L}^{\mathsf{Q}+2}(\Phi_{\overline{d},\overline{l}}) = 0$ obtaining the contradiction.
By referring to Definition 6.1, (6.1) and (6.49) we set

$$\widetilde{\Gamma} = \begin{cases} pr[\mathbb{A}_\mathbf{q}](\{\mathfrak{E} = \mathbf{0}\} \cap \{\mathbf{E} = \mathbf{0}\}) & \text{if } \sigma \neq \widetilde{\sigma}, \\ pr[\mathbb{A}_\mathbf{q}]\big((\{\mathfrak{E} = 0\} \cap \{\mathbf{E} = \mathbf{0}\}) \setminus (\mathbb{A}_\mathbf{q} \times \Lambda[A_\mathsf{T}])\big) & \text{if } \sigma = \widetilde{\sigma}, \end{cases}$$

where $\Lambda[A_\mathsf{T}] = \{(\mathbf{t}, \boldsymbol{\tau}) \in A_\mathsf{T} \times A_\mathsf{T} : \ \mathbf{t} = \boldsymbol{\tau}\}$.
The choice $W = C_{\overline{d},\overline{l}}$ in Theorem 6.4 entails $\mathcal{L}^{\mathsf{Q}+2}(\widetilde{\Gamma}) = 0$. We claim:

$$\Phi_{\overline{d},\overline{l}} \subseteq \widetilde{\Gamma}. \tag{7.13}$$

To prove (7.13) we choose $h \in \Phi_{\overline{d},\overline{l}}$ and $u, v \in \operatorname{argmin} F^h_{\alpha,\beta}$ with $u \neq v$, $\sigma(u) = \sigma$ and $\sigma(v) = \widetilde{\sigma}$; then, referring to identification (3.16) and (3.17), we have

$$(\mathbf{h}, \mathbf{t}(u), \mathbf{t}(v)) \in \begin{cases} \{\mathfrak{E} = \mathbf{0}\} \cap \{\mathbf{E} = \mathbf{0}\} & \text{if } \sigma \neq \widetilde{\sigma} \\ \{\mathfrak{E} = 0\} \cap \{\mathbf{E} = \mathbf{0}\}) \setminus (\mathbb{A}_\mathbf{q} \times \Lambda[A_\mathsf{T}]) & \text{if } \sigma = \widetilde{\sigma}, \end{cases}$$

since,
by referring to Definition 4.8(1): $u = \gamma[h, \mathbf{t}(u), \mathbf{q}, \sigma]$ and $v = \gamma[h, \mathbf{t}(v), \mathbf{q}, \widetilde{\sigma}]$;

by (7.10): $\sharp(S_u) = \sharp(S_v)$, $\sharp(S_{\dot u}\backslash S_u) = \sharp(S_{\dot v} \setminus S_v)$, $\mathfrak{E}(h, \mathbf{t}(u), \mathbf{t}(v), \mathbf{q}, \sigma, \widetilde{\sigma}) = 0$;
by $u,v \in \operatorname{argmin} F^h_{\alpha,\beta}$ and Theorems 2.1, 4.5: $\mathbf{E}(h,\mathbf{t}(u),\mathbf{t}(v),\mathbf{q},\sigma,\widetilde{\sigma}) = \mathbf{0}$;
by Theorem 3.8 since we have chosen $u \neq v$: $\mathbf{t}(u) \neq \mathbf{t}(v)$ if $\sigma = \widetilde{\sigma}$.
Then (7.13) is proven and, since $\mathcal{L}^{\mathsf{Q}+2}(\widetilde{\Gamma}) = 0$ we deduce $\mathcal{L}^{\mathsf{Q}+2}(\Phi_{\overline{d},\overline{l}}) = 0$, e.g. a contradiction with (7.10). $\square$

Proof of Theorem 1.1. We fix α, β fulfilling (1.2) and $\alpha/\beta \notin \mathbb{Q}$. Then we choose $A_{\alpha,\beta}$ as in Theorem 7.2. We define the function $H : L^2 \to [0, +\infty)$ by

$$H(g) = \sup\{\|u - v\|_{L^1} \quad u, v \in \operatorname{argmin} F^g_{\alpha,\beta}\} \qquad \forall g \in L^2(0,1).$$

Since $\sharp\left(\operatorname{argmin} F^g_{\alpha,\beta}\right) = 1$ for any $g \in A_{\alpha,\beta}$, we get

$$A_{\alpha,\beta} \subseteq \left\{g \in L^2(0,1) \colon \sharp\left(\operatorname{argmin} F^g_{\alpha,\beta}\right) = 1\right\} = H^{-1}(0) = \\ = \left\{g \in L^2(0,1) \colon H(g) = 0\right\} = \bigcap_{n \in \mathbb{N}} \left\{g \in L^2(0,1) \colon H(g) < 1/n\right\}.$$

We claim

$$\forall n \;\; V_n = H^{-1}([0, 1/n)) \text{ is an } L^2\text{-neighborhood of dense set } H^{-1}(0) \supseteq A_{\alpha,\beta}, \tag{7.14}$$

i.e.: $\forall n \;\; \exists\, U_n$ open sets in $L^2(0,1)$, $U_n \overset{\text{ds}}{\subseteq} L^2(0,1)$, $A_{\alpha,\beta} \subseteq U_n \subseteq V_n$.
Then Theorem 1.1 is a consequence of (7.14) by setting

$$E_{\alpha,\beta} \; = \; H^{-1}(0) \; = \; \bigcap_n V_n \; \supseteq \; \bigcap_n U_n.$$

We prove (7.14) by showing that H is continuous at any $g \in H^{-1}(0)$. Arguing by contradiction assume that there are $\varepsilon > 0$, f s.t. $H(f) = 0$ and a family $\{f_n\}_n \subseteq L^2$ with $f_n \to_{L^2} f$ and $H(f_n) > \varepsilon$. Then for any n we can choose

$$u_n, v_n \in \operatorname{argmin} F^{f_n}_{\alpha,\beta} \text{ such that } \|u_n - v_n\|_{L^1} > \varepsilon. \tag{7.15}$$

By Young inequality and (2.12),(2.13) of [4]

$$F^f_{\alpha,\beta}(u_n) = F^{f_n}_{\alpha,\beta}(u_n) + \|u_n - f\|^2_{L^2} - \|u_n - f_n\|^2_{L^2} \leq \\ \leq m^{f_n}(\alpha,\beta) + 2\,\|f_n\|^2_{L^2} + 2\,\|f\|^2_{L^2} \leq 4\,\|f_n\|^2_{L^2} + 2\,\|f\|^2_{L^2} \leq C,$$

in the same way we get $F^f_{\alpha,\beta}(v_n) \leq C$.

By $$m^{f_n}(\alpha,\beta) \overset{\text{def}}{=} \min F^{f_n}_{\alpha,\beta} \; = \; F^{f_n}_{\alpha,\beta}(u_n) \; = \; F^{f_n}_{\alpha,\beta}(v_n),$$

and property (2.14) in [4] we get $F^{f_n}_{\alpha,\beta}(u_n) \to m^f(\alpha,\beta)$, $F^{f_n}_{\alpha,\beta}(v_n) \to m^f(\alpha,\beta)$ then by Theorem 2.5(1) in [4], up to subsequences, we have $u_n \to^{L^1} u \in \mathcal{H}^2$ and $v_n \to^{L^1} v \in \mathcal{H}^2$ with $u, v \in \operatorname{argmin} F^f_{\alpha,\beta}$. Since $\sharp\left(\operatorname{argmin} F^f_{\alpha,\beta}\right) = 1$ we have $u = v$, then $\|u_n - v_n\|_{L^1} \to 0$ which is in contradiction with (7.15). $\square$

8 Appendix A: CW complexes and Transversality

Here follows a short summary of the notions about CW complexes which are needed in this paper.
Let $I = [0,1]$. For any $n \in \{0,1,2,...\}$ we define I^n to be the closed unit n-cube if $n > 0$ and the origin if $n = 0$, we also define $(0,1)^n$ to be the open unit n-cube if $n > 0$ and the origin if $n = 0$, we denote by ∂I^n the topological boundary of I^n if $n > 0$ and set $\partial I^0 = \varnothing$.

Definition 8.1 *A CW complex X is the direct limit of a sequence $\{X_n\}_{n=-1}^{\infty}$ of topological spaces defined inductively as follows:*

- $X_{-1} = \emptyset$,
- *a family of continuous maps $\{f_\lambda^n\colon \ \partial I_\lambda^n \to X_{n-1}\}_{\lambda\in\Lambda_n}$ called gluing maps,*
- *X_n obtained from the following diagram:*

$$\bigsqcup_{\lambda\in\Lambda_n} I_\lambda^n \longleftarrow \bigsqcup_{\lambda\in\Lambda_n} \partial I_\lambda^n \xrightarrow{\quad f^n \quad} X_{n-1}$$

by push-out

$$\begin{array}{ccccc} \bigsqcup_{\lambda\in\Lambda_n} I_\lambda^n & \longleftarrow & \bigsqcup_{\lambda\in\Lambda_n} \partial I_\lambda^n & \xrightarrow{\quad f^n \quad} & X_{n-1} \\ & \searrow & & \swarrow & \\ & & X_n & & \end{array}$$

where $\bigsqcup$ denotes the disjoint union of spaces and the left arrow represents the injective embedding and $f^n = \bigsqcup_{\lambda\in\Lambda_n} f_\lambda^n$.

The subspaces X_n are called n-skeleta, for $n = -1, 0, 1, ...$
A CW complex X is finite of dimension $\overline{n}$ if Λ_n is a finite set for any n, $\Lambda_{\overline{n}} \neq \emptyset$ and $\Lambda_n = \emptyset$ whenever $n > \overline{n}$. In such case $X_{\overline{n}}$ is called higher skeleton and $X_n = X_{\overline{n}}$ for any $n \geq \overline{n}$.

Notice that, by push-out, each gluing function $f_\lambda^n : \partial I_\lambda^n \to X_{n-1}$ extends to a continuous function $g_\lambda^n : I_\lambda^n \to X_n$ which is an homeomorphism on the open n-cube $I_\lambda^n \setminus \partial I_\lambda^n$.

Definition 8.2 *A n-cell of a CW complex X is the image of a n-cube I_λ^n through g_λ^n, an open n-cell of X is the image of an open n-cube $I_\lambda^n \setminus \partial I_\lambda^n$ through g_λ^n.*

We emphasize that the Definitions 8.1 and 8.2 above refers to cubes instead of balls, nevertheless they are equivalent to Definition 5.3 in [23].
The following result due to Lojasiewicz (see [19]) describes a very large class of spaces which are CW complexes.

Definition 8.3 *Consider a real analytic manifold M and a subset $S \subseteq M$. We call S a semi-analytic set if and only if for any $x_0 \in S$ there is a neighborhood V of x_0 and a finite set $\{f_j : V \to \mathbb{R}\}$ of analytic functions such that $S \cap V$ is a finite union of finite intersections of sets of type*

$$\{x\colon\ f_j(x) > 0\}\,, \qquad \{x\colon\ f_j(x) = 0\}\,.$$

Definition 8.4 *Consider an affine space space X, a real analytic manifold Y and a subset $S \subseteq X \times Y$. We call S a partially semi algebraic set with respect to X if and only if for any $y_0 \in Y$ there are:*

- *a neighborhood U of y_0*
- *a finite set $\{f_j : X \times U \to \mathbb{R}\}$ of analytic functions which are polynomials in x for any fixed $y \in U$*

s.t. $S \cap (X \times U)$ is a finite union of finite intersections of sets of type

$$\{x\colon\ f_j(x) > 0\}\,, \qquad \{x\colon\ f_j(x) = 0\}\,.$$

Theorem 8.5 (Lojasiewicz [19]) *Consider a real analytic manifold Y and a locally finite collection $\{B_l\}_{l\in\Lambda}$ of semi-analytic sets of Y s.t. $B_l \subseteq Y$ for any $l \in \Lambda$.*
Then there exist an affine space X, a locally finite symplicial complex K and a homeomorphism $f : |K| \to Y$ such that:

1. *$|K|$ is a subspace of X;*
2. *the set $\{(x, f(x)) \quad x \in |K| \subseteq X \times Y\}$ is partially semi-algebraic with respect to X (see Definition 8.4);*
3. *$f(|\Sigma|)$ is an analytic sub-manifold of M and the restriction of f to $|\Sigma| \subseteq |K|$ is an analytic isomorphism for any simplex $\Sigma \in K$;*

4. $f(|\Sigma|) \subseteq B_l$ *or* $f(|\Sigma|) \subseteq M \setminus B_l$ *for any simplex* $\Sigma \in K$, $l \in \Lambda$.

Here $|K|$ *denotes the geometric realization of the symplicial complex* K *and* $|\Sigma|$ *denotes the geometric realization of the simplex* $\Sigma \in K$.

Example 8.6 *Theorem 8.5 provides many examples of CW complexes: the semi analytic sets and the semi-algebraic sets.*
In this paper we are also interested in a very particular type of CW complexes where cells are cubes and gluing maps are identities on boundaries, as in the case of the whole collection of $(\mathsf{T}+2)$ *dimensional rectangles lying in* A_{T} *of the CW structure induced on* $[0,1]^{\mathsf{T}}$ *by vectors of type* $\mathbf{q}$.

Example 8.7 *Consider the space*

$$X = \left\{(x,y) \in \mathbb{R}^2 : \quad (0 \leq x \leq 1 \; and \, y = 0) \;\; or \;\; (x = 0 \; and \, 0 \leq y \leq 1)\right\}.$$

This space is a 1*-dimensional CW complex with* $X_0 = \{(0,0),\ (0,1),\ (1,0)\}$, $X_1 = X$; $\Lambda_0 = \{1,2,3\}$, $f_1^0, f_2^0, f_3^0 : \partial I^0 \to \emptyset = X_{-1}$; $\Lambda_1 = \{1,2\}$, $f_\lambda^1 : \{0,1\} \to X_0$ *given by* $f_1^1(0) = (0,0)$, $f_1^1(1) = (1,0)$, $f_2^1(0) = (0,0)$, $f_2^1(1) = (0,1)$.
Cells of X *are the following:* $(0,0)$, $(0,1)$, $(1,0)$ *the three* 0*-cells;* $[0,1]\times\{0\}$, $\{0\}\times[0,1]$ *the two* 1*-cells.*
Geometric realization of X *is the collection* $X_0 \cup X_1 \subseteq \mathbb{R}^2$.

Example 8.8 *Consider the space*

$$X = \left\{(x,y) \in \mathbb{R}^2 : \quad \left(0 \leq x \leq \frac{1}{3} \; and \; \frac{1}{3} \leq y \leq 1\right) \; or \; \left(0 \leq x \leq \frac{2}{3} \; and \; \frac{2}{3} \leq y \leq 1\right)\right\}.$$

This space is a 2*-dimensional CW complex with skeleta:*

$$X_0 = \left\{(0,\tfrac{1}{3}),\ (0,\tfrac{2}{3}),\ (0,1),\ (\tfrac{1}{3},\tfrac{1}{3}),\ (\tfrac{1}{3},\tfrac{2}{3}),\ (\tfrac{1}{3},1),\ (\tfrac{2}{3},\tfrac{2}{3}),\ (\tfrac{2}{3},1)\right\},$$
$$X_1 = \left(\{0,\tfrac{1}{3}\}\times[\tfrac{1}{3},1]\right) \cup \left(\{\tfrac{2}{3}\}\times[\tfrac{2}{3},1]\right) \cup \left([0,\tfrac{2}{3}]\times\{\tfrac{2}{3},1\}\right) \cup \left([0,\tfrac{1}{3}]\times\{\tfrac{1}{3}\}\right),$$
$$X_2 = X;$$

and gluing maps:

$\Lambda_0 = \{1,...,8\}$, $f_\lambda^0 : \partial I^0 \to \emptyset = X_{-1}$;

$\Lambda_1 = \{1,...,10\}$, $f_\lambda^1 : \partial I^1 \to X_0$ *whose images are points given by*

$f_1^1(0) = (0,\tfrac{1}{3})$, $f_1^1(1) = (\tfrac{1}{3},\tfrac{1}{3})$, $f_2^1(0) = (0,\tfrac{2}{3})$, $f_2^1(1) = (\tfrac{1}{3},\tfrac{2}{3})$,
$f_3^1(0) = (0,1)$, $f_3^1(1) = (\tfrac{1}{3},1)$, $f_4^1(0) = (\tfrac{1}{3},\tfrac{2}{3})$, $f_4^1(1) = (\tfrac{2}{3},\tfrac{2}{3})$,
$f_5^1(0) = (\tfrac{1}{3},1)$, $f_5^1(1) = (\tfrac{2}{3},1)$, $f_6^1(0) = (0,\tfrac{1}{3})$, $f_6^1(1) = (0,\tfrac{2}{3})$,
$f_7^1(0) = (0,\tfrac{2}{3})$, $f_7^1(1) = (\tfrac{2}{3},1)$, $f_8^1(0) = (\tfrac{1}{3},\tfrac{1}{3})$, $f_8^1(1) = (\tfrac{1}{3},\tfrac{2}{3})$,
$f_9^1(0) = (\tfrac{1}{3},\tfrac{2}{3})$, $f_9^1(1) = (\tfrac{2}{3},1)$, $f_{10}^1(0) = (\tfrac{2}{3},\tfrac{2}{3})$, $f_{10}^1(1) = (\tfrac{2}{3},1)$;

$\Lambda_2 = \{1,2,3\}$, $f^2_\lambda : \partial I^2 \to X_1$ *given by*

$$f^2_\lambda(x,y) = (\tfrac{1}{3}x, \tfrac{1}{3}y) + \begin{cases} (0,\tfrac{2}{3}) & \text{if } \lambda = 1, \\ (0,\tfrac{1}{3}) & \text{if } \lambda = 2, \\ (\tfrac{1}{3},\tfrac{2}{3}), & \text{if } \lambda = 3. \end{cases} \quad (x,y) \in \partial I^2$$

We recall from [23] the definition of transversality and Theorem 12.17.

Definition 8.9 *Let $\mathcal{M}$, $\mathcal{V}$ be C^∞ manifolds with $\partial\mathcal{M} = \emptyset$, $f \in C^\infty(\mathcal{M},\mathcal{V})$, $\mathcal{N}$ be a C^∞ sub-manifold of $\mathcal{V}$.*
We say that f is transverse regular to $\mathcal{N}$ at $x \in f^{-1}(\mathcal{N})$ if

$$Df(x)\,(\tau(\mathcal{M})_x) + \tau(\mathcal{N})_y = \tau(\mathcal{V})_y$$

where $Df(x)$ is the differential of f at x and $\tau(Z)_z$ is the tangent space to Z at point z.
In this case we say that x is a regular point for f.

Theorem 8.10 (Switzer [23]) *Let $\mathcal{M}$, $\mathcal{V}$ be C^∞ manifolds, $f \in C^\infty(\mathcal{M},\mathcal{V})$, $\mathcal{N}$ be a C^∞ sub-manifold of $\mathcal{V}$.*
Suppose $\partial\mathcal{M} = \emptyset$, $\dim \mathcal{N} + \dim \mathcal{M} - \dim \mathcal{V} \geq 0$ and f transverse regular to N at any $x \in f^{-1}(\mathcal{N})$, then $f^{-1}(\mathcal{N})$ is a sub-manifold of $\mathcal{M}$ and

$$\operatorname{codim} f^{-1}(\mathcal{N}) = \operatorname{codim} \mathcal{N}$$

that is

$$\dim \mathcal{M} - \dim f^{-1}(\mathcal{N}) = \dim \mathcal{V} - \dim \mathcal{N}$$

Theorem 8.10 is applied when $\mathcal{N}$ is a single point (hence $\dim f^{-1}(\mathcal{N}) = \dim \mathcal{M} - \dim \mathcal{V}$) in the proof of Lemma 6.2 and, in its general form, of Theorem 5.4 and of Theorem 6.4.

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