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UNIFORM DENSITY ESTIMATES FOR BLAKE & ZISSERMAN FUNCTIONAL

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Abstract. We prove density estimates and elimination properties for minimizing triplets of functionals which are related to contour detection in image segmentation and depend on free discontinuities, free gradient discontinuities and second order derivatives. All the estimates concern optimal segmentation under Dirichlet boundary conditions and are uniform in the image domain up to the boundary.

Dedicated to our teachers Ennio De Giorgi and Guido Stampacchia

1. Introduction.

Image segmentation is a relevant problem both in digital image processing and in the understanding of biological vision.

There exist many different ways to define the tasks of segmentation (template matching, component labelling, thresholding, boundary detection, quad-trees, texture matching, texture segmentation) and there is no universally accepted notion (optimality criteria for segmentation, analogies and differences between biological and automata perspective in segmentation): here the exposition is confined to a model for decomposing an image field, where is given a function describing the signal intensity associate to each point (typically the light intensity on a screen image). Such purpose has a clear connection with the problem of optimal partitions of a domain minimizing the length of the boundaries.

In simple words the segmentation we look for provides a cartoon of the given image satisfying some requirements: the decomposition of the image is performed by choosing a pattern of lines of steepest discontinuity for light intensity, and this pattern will be called segmentation of the image.

The variational formalizations of contour detection in segmentation models provided deeper understanding of image analysis, produced intriguing mathematical questions (some of them still open) and entailed global estimates for geometric quantities in visual and automatic perception at both low and high level vision (see [5], [14], [19], [30]).

Variational models have been extremely successful in a wide variety of restoration problems and are one of the most active areas of research in mathematical image processing and computer vision. Now they are applied not only to the problem of image denoising, but also to other restoration

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tasks such as deblurring, blind deconvolution, and inpainting; in particular, imposing a Dirichlet boundary condition fits the special case of inpainting (see [20], [21], [25]).
These models exhibit the solution of the previous problems as minimizers of appropriately chosen functionals. The minimization technique for such models involves the solution of partial differential equations derived as necessary optimality conditions. Perhaps the most basic (fundamental) image restoration problem is denoising: it forms a significant preliminary step in many machine vision tasks, such as object detection and recognition; it is also one of the mathematically most intriguing problems in vision. A key point in designing image denoising models is to preserve relevant image features, such as those most easily detected by the human visual system, while removing noise. One such relevant image feature are the edges, say locations of the image where there is a sharp change in image properties, as like as image points related to object boundaries. Many research activities aims to models which filter noise but preserve edges; recently there has also been a lot of effort in preserving other fine scale image features like textures. Often variational and PDE based models impose geometric properties on the solutions obtained as denoised images: such as smoothness, or rectifiability or finite length of boundaries.

Here we present some recent results based on the notion of free discontinuity problem which was introduced by Ennio De Giorgi in [22]. Our framework balances carefully signal smoothing and segmentation length and allows the study of problems coupling bulk and surface terms, where discontinuous solutions are admissible and their discontinuity set is the most significant part of the solution.

Precisely we focus our analysis on the strong Blake & Zisserman functional (see [5], [9], [10], [12], [13]) with Dirichlet boundary condition, say

$$\begin{aligned} F(K_0, K_1, v) \;=\; & \int_{\widetilde{\Omega}\setminus(K_0\cup K_1)} \left| D^2 v \right|^2 \, d\mathbf{x} + \mu \int_{\widetilde{\Omega}} |v-g|^q \, d\mathbf{x} \\ & + \alpha\, \mathcal{H}^1\left(K_0 \cap \widetilde{\Omega}\right) + \beta\, \mathcal{H}^1\left((K_1 \setminus K_0)\cap \widetilde{\Omega}\right), \end{aligned} \tag{1}$$

to be minimized over triplets (K_0, K_1, v), where K_0, K_1 are Borel sets, $K_0 \cup K_1$ is a closed set, function v belongs to $C^2\left(\widetilde{\Omega}\setminus(K_0\cup K_1)\right)$, v is approximately continuous in $\left(\widetilde{\Omega}\setminus K_0\right)$, equality $v = w$ holds true a.e. in $\widetilde{\Omega}\setminus\Omega$ and $\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2$ are open sets.

If (K_0, K_1, u) is a minimizing triplet of F, then $K_0 \cup K_1$ can be interpreted as an optimal segmentation of the monochromatic image of brightness intensity g, while the three elements of a minimizing triplet (K_0, K_1, u) play respectively the role of edges, creases and smoothly varying intensity in the region $\widetilde{\Omega}\setminus(K_0\cup K_1)$ for the segmented image. The second-order functional (1) was introduced to overcome the over-segmentation of steep gradients (ramp effect) and other inconvenient which occur in lower order models as in case of Mumford & Shah functional ([5], [30], [31]).

In Definition 2.6 we recall the notion of *essential (locally) minimizing triplet* (see [17]); in Theorem 2.7 and Remark 3 we clarify properties of essential minimizing triplets and essential part of admissible triplets, and provide the essential formulation (23) of Blake & Zisserman functional.

In this paper we prove estimates on minimizers under Dirichlet boundary conditions (see Section 4): *upper and lower energy density* of the *essential locally minimizing triplets*; an *elimination property* and a sharp estimate of the *Minkowski content* for the optimal segmentation set $K_0 \cup K_1$; the main results are stated in Theorems 3.1, 4.1, 4.2, 4.3, 4.4 in case of smooth Dirichlet datum. This kind of estimates are useful in numerical approximation (see [1], [3], [6], [29]). All the estimates on optimal segmentation in the image domain where noise filtering acts (i.e. in the set $\overline{\Omega}$) are uniform up to the boundary $\partial\Omega$.
Discontinuous Dirichlet datum is studied in Section 5: density estimates which are uniform up to the boundary $\partial\Omega$ are proved in this case too (Theorems 5.2, 5.6, 5.7, 5.8, 5.9).

More explicitly the elimination property (Theorems 4.3 and 5.8) states that, when the intersection of optimal segmentation with a small ball has less length than an absolute constant times the radius of the ball, then such segmentation does not intersect the concentric half-radius ball. This information is useful in the numerical analysis of the problem, that is algorithms can be tuned in order to eliminate inessential isolated parts of $K_0 \cup K_1$, because they are "needless energy" for the segmentation.

The results about the Minkowski content (Theorems 4.4, 5.9) expresse the agreement between the Hausdorff one dimensional measure and the Minkowski content of the segmentation $K_0 \cup K_1$. Roughly speaking, the theorem says that a uniform fattening of an optimal segmentation is a reasonable approximation of the segmentation itself.

All the above mentioned results are based on the existence of strong solutions under Dirichlet boundary conditions ([17]) recalled here in Theorem 2.3 (case of smooth Dirichlet datum) and Theorem 5.1 (case of Dirichlet datum with non empty jump and crease sets), moreover they rely on several decay estimates: hessian decay estimate at interior points, at boundary points and close to the boundary for bi-harmonic functions with homogeneous Dirichlet boundary datum (Theorems 3.2, 3.3, 3.4); decay of Blake & Zisserman functional evaluated on minimizing triplets at interior points, at boundary points and close to the boundary (3.6, 3.7, 3.8). Theorem 3.2 is classic, Theorems 3.3, 3.6, 3.7 are proved in [10],[17], while Theorems 3.4 and 3.8 are related to L^2 hessian decay of bi-harmonic functions at points close to the boundary, were announced in [17] (Remark 7.5) and are proved in detail here.

We emphasize that the analysis at points close to the boundary (decay Theorems 3.4 and 3.8) is necessary to avoid counterexamples analogous to Counterexamples 1.1-1.3 in [28] about Mumford & Shah functional, and to achieve regularity $C^2(\widetilde{\Omega} \setminus (K_0 \cup K_1))$ of intensity level for minimizers together with closedness of optimal segmentation $K_0 \cup K_1$ (and not only the property that $K_0 \cup K_1$ is the union of relatively closed subsets of Ω and $\partial\Omega$).

About the quite technical hypotheses ((66)-(72)) assumed in case of nonsmooth Dirichlet datum we remark that actually they are very natural, in particular: (69) means that each datum discontinuity component cannot live on boundary $\partial\Omega$ nor can reach this boundary from interior without crossing it; (72) simply says that the datum is expressed as an essential triplet; the whole set of assumptions on datum says that w represents a Dirichlet datum which is noise-free in the region $\widetilde{\Omega} \setminus \Omega$ (see (69)-(71)), as it is very natural when facing inpainting problem if noise, blotches and all artifact to be removed are contained in $\overline{\Omega}$.

2. **Background and notation.**

We recall the strong formulation F of Blake & Zisserman functional (see [10]) for 2-dimensional images.

We refer to [4], [5], [8]–[11], [15], [16], [29]–[31] for motivation and background analysis of variational approach to image segmentation and digital image processing.

Precisely we study the functional

$$
\begin{aligned}
F(K_0, K_1, v) \;=\; & \int_{\widetilde{\Omega} \setminus (K_0 \cup K_1)} \left| D^2 v \right|^2 \, d\mathbf{x} + \mu \int_{\widetilde{\Omega}} |v - g|^q \, d\mathbf{x} \\
& + \alpha\, \mathcal{H}^1\left(K_0 \cap \widetilde{\Omega}\right) + \beta\, \mathcal{H}^1\left((K_1 \setminus K_0) \cap \widetilde{\Omega}\right),
\end{aligned}
\tag{2}
$$

with the aim of minimizing it among admissible triplets (K_0, K_1, v), say triplets fulfilling a boundary Dirichlet condition in the sense of the following Definition.

Definition 2.1. **(Admissible triplets)** (K_0, K_1, v) is an admissible triplet if

$$\begin{cases} K_0\,,\, K_1 \text{ Borel subsets of } \mathbb{R}^2, \qquad K_0 \cup K_1 \text{ closed}, \\ v \in C^2\left(\widetilde{\Omega} \setminus (K_0 \cup K_1)\right),\ v \text{ approximately continuous in } \left(\widetilde{\Omega} \setminus K_0\right), \\ v = w \text{ a.e. in } \widetilde{\Omega} \setminus \Omega\,. \end{cases} \tag{3}$$

Here and in the following $\Omega\,,\ \widetilde{\Omega}$ are open sets such that

$$\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2\,. \tag{4}$$

Minimizing (2) over (3) corresponds to minimize Blake & Zisserman functional [5] with Dirichlet boundary datum in 2-dimensional image segmentation: the boundary datum is prescribed by penalization as usual when competing functions belong to a non reflexive space.

Definition 2.2. **(Approximate limit)**
For any L^1_{loc} function $v : \Omega \to \mathbb{R}$ and $\mathbf{x} \in \Omega, z \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we set $z = \text{ap} \lim\limits_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ (*approximate limit* of v at $\mathbf{x}$) if, for every $g \in C^0(\overline{\mathbb{R}})$,

$$g(z) = \lim_{\varrho \downarrow 0} \fint_{B_\varrho(0)} g(v(\mathbf{x}+\mathbf{y}))d\mathbf{y}\,; \tag{5}$$

the function $\tilde{v}(\mathbf{x}) = \text{ap} \lim\limits_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ is called *a representative* of v; the *singular set* of v is $S_v = \{\mathbf{x} \in \Omega : \nexists z \ s.t.\ \text{ap} \lim\limits_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y}) = z\}$ (see [2], [22], [23]).

By referring to [2] and [16]: Dv denotes the distributional gradient of v, ∇v denotes the approximate gradient of v, $SBV(\Omega)$ denotes the De Giorgi class of functions $v \in BV(\Omega)$ such that the total variation of Dv is given by:

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\, d\mathbf{x} + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^1.$$

$$\begin{aligned} SBV_{loc}(\Omega) &:= \{v \in SBV(\Omega') :\ \forall\, \Omega' \subset\subset \Omega\}\,, \\ GSBV(\Omega) &:= \left\{v : \Omega \to \mathbb{R} \text{ Borel function}; -k \vee v \wedge k \in SBV_{loc}(\Omega)\ \forall k \in \mathbb{N}\right\}. \\ GSBV^2(\Omega) &:= \left\{v \in GSBV(\Omega),\ \nabla v \in \left(GSBV(\Omega)\right)^2\right\}. \end{aligned}$$

Remark 1. We notice that in the book [2] there is a slightly different definition of *GSBV* (say Definition 4.26 p.235 in [2]): that space strictly contains the space *GSBV* introduced by our definition above; nevertheless they coincide in case of scalar-valued functions. In that space ([2]), our Definition (5) of aplim is equivalent $\mathcal{H}^1$ a.e. in $\widetilde{\Omega}$ to the following one

$$\lim_{\varrho \downarrow 0} \fint_{B_\varrho(0)} |g(v(\mathbf{x}+\mathbf{y})) - g(z)|\, d\mathbf{y} \ = \ 0 \qquad \forall g \in C^0(\overline{\mathbb{R}})\,. \tag{6}$$

This difference does not create any inconvenient since we will apply (5) only component-wise to the gradient of admissible functions.
The same remark apply to compactness Theorem 4.36 page 240 ([2]), a useful tool which is used component-wise on vector-valued gradients.

We recall the following result about existence of strong minimizing triplets for Blake & Zisserman functional under Dirichlet boundary conditions in the simplified case of smooth Dirichlet datum (about general discontinuous Dirichlet datum we refer to the last Section).

Theorem 2.3. *Let $\alpha\,,\,\beta\,,\mu\,,\ q\ ,g\,,\Omega,\ \widetilde\Omega$ and w be s.t.*

$$0<\beta\le\alpha\le 2\beta,\ \ \mu>0,\ \ q>1,\ \ g\in L^q(\widetilde\Omega)\cap L^{2q}_{loc}(\widetilde\Omega)\,,\ \ w\in L^q(\widetilde\Omega)\,, \tag{7}$$

$$\Omega\subset\subset\widetilde\Omega\subset\subset\mathbb{R}^2\,, \tag{8}$$

$$\Omega \text{ is an open set with } C^2 \text{ boundary } \partial\Omega\,,\ \widetilde\Omega \text{ is an open set}\,, \tag{9}$$

$$w\in C^2(\widetilde\Omega)\,, \tag{10}$$

$$D^2w\in L^\infty(\widetilde\Omega\,)\,. \tag{11}$$

Then there exists a triplet (C_0,C_1,u) which minimizes the functional F defined by (2) with finite energy, among admissible triplets (K_0,K_1,v) fulfilling (3).
Moreover any minimizing triplet (K_0,K_1,v) fulfils:

$$K_0\cap\widetilde\Omega \text{ and } K_1\cap\widetilde\Omega \text{ are } (\mathcal{H}^1,1) \text{ rectifiable sets,} \tag{12}$$

$$\mathcal{H}^1(K_0\cap\widetilde\Omega)\ =\ \mathcal{H}^1(\overline{S_v})\,,\quad \mathcal{H}^1(K_1\cap\widetilde\Omega)\ =\ \mathcal{H}^1(\overline{S_{\nabla v}}\setminus S_v)\,, \tag{13}$$

$$\begin{cases} v\in GSBV^2(\widetilde\Omega),\ \text{hence}\\ v \text{ and } \nabla v \text{ have well defined two-sided traces, finite } \mathcal{H}^1\ \text{ a.e. on } K_0\cup K_1,\end{cases} \tag{14}$$

the function v is also a minimizer of the weak functional $\mathcal{F}$ (see [9], [16])

$$\mathcal{F}(z)\ =\ \int_{\widetilde\Omega}(|\nabla^2 z|^2+\mu|z-g|^q)\,d\mathbf{y}+\alpha\mathcal{H}^1(S_z)+\beta\mathcal{H}^1(S_{\nabla z}\setminus S_z) \tag{15}$$

over

$$z\in L^q(\widetilde\Omega)\cap GSBV(\widetilde\Omega)\ \text{s.t.}\ \nabla z\in\Big(GSBV(\widetilde\Omega)\Big)^2\ \text{ and } z=w \text{ a.e. in } \widetilde\Omega\setminus\Omega. \tag{16}$$

Eventually, for any third element of a minimizing triplet v we have

$$\mathcal{F}(v) =\ F(K_0,K_1,v)\,. \tag{17}$$

We emphasize that in (17), by referring to Definition 2.2, $v=\widetilde v$ on $\widetilde\Omega\setminus(K_0\cup K_1)$.

Proof. It is a particular case of Theorem 2.2 in [17]. □

Definition 2.4. (Localization) We will use the symbol F_U to denote the functional (2) when $\widetilde\Omega$ is substituted by a Borel set $U\subset\widetilde\Omega$

Definition 2.5. (**Locally minimizing triplet of** F)
An admissible triplet (K_0,K_1,u), is a locally minimizing triplet of the functional (2) if

$$F_A(K_0,K_1,u)<+\infty \tag{18}$$

$$F_A(K_0,K_1,u)\le F_A(U_0,U_1,v) \tag{19}$$

for every open subset $A\subset\subset\Omega$ and for every admissible triplet (U_0,U_1,v) such that

$$\mathrm{spt}(v-u)\quad\text{and}\quad (U_0\cup U_1)\triangle(K_0\cup K_1)\quad\text{ are subsets of } A.$$

Remark 2. We emphasize that Definition 2.5 is equivalent to say that v (the third element of the triplet) fulfills Definition 3.1 in [16]. This is a consequence of Theorem 2.2 in [17] (on this subject see also Definition 3.6 and Remark 3.8 in [17]) which is stated precisely in Theorem 2.10 in [18].

Definition 2.6. **(Essential locally minimizing triplet of F)** Given a locally minimizing triplet (U_0, U_1, v) of the functional (2), there is another triplet (K_0, K_1, u) , called *essential locally minimizing triplet*, uniquely defined by

$$u = \tilde{v}$$

$$K_0 = \overline{U_0 \cap K} \setminus (U_1 \setminus U_0)$$

$$K_1 = \overline{U_1 \cap K} \setminus U_0$$

where K is the smallest closed subset of $U_0 \cup U_1$ such that $\widetilde{v} \in C^2(\widetilde{\Omega} \setminus K)$.

We emphasize that the above construction entails that

$$F(K_0, K_1, u) = F(U_0, U_1, v)$$

as clarified by the following theorem.

Theorem 2.7. *Assume* (K_0, K_1, u) *is an essential locally minimizing triplet of* F *. Then* u *is a minimizer of functional* $\mathcal{F}$ *,* $K_0 \cup K_1$ *is a closed set and*

$$K_0 \cap K_1 = \emptyset\,, \quad K_0 = \overline{K_0} \setminus K_1 = K_0 \setminus K_1\,, \tag{20}$$

$$K_1 = \overline{K_1} \setminus K_0\,, \quad \overline{K_1} \setminus K_1 \subset K_0\,, \quad K_1 \setminus K_0 = K_1\,, \tag{21}$$

$$\mathcal{H}^1(S_u \triangle K_0) = 0\,, \qquad \mathcal{H}^1\big((S_{\nabla u} \setminus S_u) \triangle K_1\big) = 0\,, \tag{22}$$

$$\begin{aligned} F(K_0, K_1, u) \;=\; & \int_{\widetilde{\Omega} \setminus (K_0 \cup K_1)} \left| D^2 u \right|^2 d\mathbf{x} + \int_{\widetilde{\Omega}} |u - g|^q \, d\mathbf{x} \\ & + \alpha\, \mathcal{H}^1\left(K_0 \cap \widetilde{\Omega}\right) + \beta\, \mathcal{H}^1\left(K_1 \cap \widetilde{\Omega}\right) . \end{aligned} \tag{23}$$

Assume (U_0, U_1, v) *is already an essential locally minimizing triplet of* F*, then, if the construction of Definition 2.6 were repeated it would produce*

$$(K_0, K_1, u) = (U_0, U_1, v).$$

Proof. Let (U_0, U_1, v) be a locally minimizing triplet which produces (K_0, K_1, u) by the construction in Definition 2.6.
Since $u = \widetilde{v}$, Theorem 2.3 entails that u minimizes $\mathcal{F}$, and

$$S_u \subset K_0\,, \qquad (S_{\nabla u} \setminus S_u) \subset (K_1 \setminus K_0)\,, \tag{24}$$

$x \in K_1 \Rightarrow x \notin U_0 \Rightarrow x \notin K_0$; $x \in K_0 \Rightarrow x \notin U_1 \setminus U_0 \Rightarrow x \notin K_1$; hence $K_0 \cap K_1 = \emptyset$. The other relationships in (20) and (21) follow by substitution.
Properties (20), (21) and (24) entail (22), since $\mathcal{H}^1(K_0 \setminus S_u) > 0$ or $\mathcal{H}^1(K_1 \setminus (S_{\nabla u} \setminus S_u)) > 0$ would imply $\min \mathcal{F} < \min F$.
$K_1 \setminus K_0 = K_1$ entails (23), say the essential representation of Blake & Zisserman functional. □

Remark 3. To any admissible triplet we can associate an essential admissible triplet, defined by exactly the same construction given by Definition 2.6 for minimizing triplets. Functional F , when evaluated on essential admissible triplets (K_0, K_1, u), takes the simpler essential formulation given by (23).

Remark 4. Obviously any essential globally minimizing triplet for F (see Definition 2.11 in [16]) is an essential locally minimizing triplet for F.

3. Hessian decay estimates for bi-harmonic functions with null boundary data and decay of weak functional evaluated at local minimizers.

First we show a uniform upper density estimate for the functional F. Then we state hessian decay estimates and decay estimates for weak functional $\mathcal{F}$ in three different geometrical cases: at interior points, at boundary points and at points close to the boundary.

We introduce a suitable constant in order to handle boundary conditions:

$$\begin{cases} L = \big(\mathrm{C}(\partial\Omega)\big)^2 + \big(\mathrm{Lip}(Dw)\big)^2 \\ \text{where } \big(\mathrm{C}(\partial\Omega)\big) \text{ is an uniform estimate of} \\ \text{second derivatives of arc-length parametrization of } \partial\Omega. \end{cases} \tag{25}$$

Theorem 3.1. *(***Density upper bound for functional** F*)* *Let (K_0, K_1, u) be an essential locally minimizing triplet for the functional (2) with (7)-(11) and (25). Then there exist $C > 0$ and $\bar{\varrho} = \bar{\varrho}(\alpha, \beta, L, \|w\|_{L^q}, \|g\|_{L^q}) > 0$ such that*

$$\mathcal{H}^1\big(\partial\Omega \cap B_\varrho(\mathbf{x})\big) < C\varrho \quad \forall \mathbf{x} \in \overline{\Omega}\,,\ \forall \varrho \le \bar{\varrho}\,, \tag{26}$$

and

$$\begin{array}{c} F_{\overline{B}_\varrho(\mathbf{x})\cap\overline{\Omega}}(K_0, K_1, u) \le c_0\varrho \\ \forall \varrho \text{ s.t. } 0 < \varrho \le (\bar{\varrho} \wedge 1) \quad \forall \mathbf{x} \in \overline{\Omega} \text{ s.t. } \overline{B}_\varrho(\mathbf{x}) \subset \widetilde{\Omega}, \end{array} \tag{27}$$

where $c_0 = L\pi + 2^{q-1}\pi^{\frac{1}{2}}\mu\big(\|w\|^q_{L^{2q}(B_\varrho(\mathbf{x}))} + \|g\|^q_{L^{2q}(B_\varrho(\mathbf{x}))}\big) + (2\pi + C)\alpha$.
If $q = 2$ and $g\,, w \in L^\infty(\widetilde{\Omega})$, then we can choose

$$c_0 = L\pi + 2\pi\mu\big(\|w\|^2_{L^\infty} + \|g\|^2_{L^\infty}\big) + (2\pi + C)\alpha\,.$$

Proof. Estimate (26) follows by (25) and Lipschitz property of $\partial\Omega$.
By minimality of (K_0, K_1, u) for F we get

$$F(K_0, K_1, u) \le F(Q_0, Q_1, v)\,,$$

where

$$v = u\chi_{\widetilde{\Omega}\setminus(B_\varrho(\mathbf{x})\cap\Omega)}, \quad Q_0 = \big(K_0 \setminus B_\varrho(\mathbf{x})\big) \cup (\partial B_\varrho(\mathbf{x}) \cap \Omega) \cup (\partial\Omega \cap B_\varrho(\mathbf{x})), \ Q_1 = K_1.$$

Taking into account $\beta \le \alpha$, since $F_{\widetilde{\Omega}\setminus\overline{B_\varrho}(\mathbf{x})}(K_0, K_1, u) = F_{\widetilde{\Omega}\setminus\overline{B_\varrho}(\mathbf{x})}(Q_0, Q_1, v)$ then

$$\begin{aligned} &\int_{B_\varrho(\mathbf{x})\setminus(K_0\cup K_1)} \big(|D^2u|^2 + \mu|u-g|^q\big)\, d\mathbf{y} \\ &\qquad + \alpha\mathcal{H}^1(K_0 \cap \overline{B}_\varrho(\mathbf{x})) + \beta\mathcal{H}^1\big((K_1 \setminus K_0) \cap \overline{B}_\varrho(\mathbf{x})\big) \\ &\le \int_{B_\varrho(\mathbf{x})\setminus\overline{\Omega}} \big(|D^2w|^2 + \mu|w-g|^q\big)\, d\mathbf{y} \\ &\qquad + \mu\int_{B_\varrho(\mathbf{x})\cap\Omega} |g|^q\, d\mathbf{y} + \alpha\mathcal{H}^1\big((\partial B_\varrho(\mathbf{x}) \cap \Omega) \cup (\partial\Omega \cap B_\varrho(\mathbf{x}))\big) \\ &\le L\pi\varrho^2 + 2^{q-1}\mu\int_{B_\varrho(\mathbf{x})\setminus\overline{\Omega}} (|w|^q + |g|^q)\, d\mathbf{y} \\ &\qquad + \mu\int_{B_\varrho(\mathbf{x})\cap\Omega} |g|^q\, d\mathbf{y} + 2\pi\alpha\varrho + \alpha\mathcal{H}^1\big(\partial\Omega \cap B_\varrho(\mathbf{x})\big) \\ &\le L\,\pi\,\varrho^2 + 2^{q-1}\mu\big(\|w\|^q_{L^{2q}(B_\varrho(\mathbf{x}))} + \|g\|^q_{L^{2q}(B_\varrho(\mathbf{x}))}\big)(\pi\rho^2)^{\frac{1}{2}} \\ &\qquad + 2\pi\alpha\varrho + C\alpha\varrho\,, \end{aligned}$$

hence we achieve the proof. □

In order to obtain density lower bounds for the functional F, when evaluated at an essential minimizing triplet, we need some estimate about bi-harmonic functions and some Decay Theorems for the functional $\mathcal{F}$ when evaluated at an essential minimizing triplet, explicitly: L^2 hessian decay for bi-harmonic functions at interior points (Theorem 3.2), at boundary points (Theorem 3.3) and at points close to the boundary (Theorem 3.4).

Theorem 3.2. *(L^2 **hessian decay for bi-harmonic function**) Let $B_1 \subset \mathbb{R}^2$ and let $z \in H^2(B_1)$ be a solution of*

$$\Delta^2 z = 0 \quad on\ B_1.$$

Then for any $q > 1$ and for every affine function ξ the following inequality holds:

$$\int_{B_\varrho} \left(|D^2 z|^2 + |\xi|^q\right)\, d\mathbf{x} \le c_{2,q}\, \varrho^2 \int_{B_1} \left(|D^2 z|^2 + |\xi|^q\right)\, d\mathbf{x} \qquad \forall \varrho < 1, \tag{28}$$

where $c_{2,q}$ is an absolute constant.

Proof. See [27], Chap.III, Sect.2. □

In [17] we proved that any function which is bi-harmonic in a half-disk and vanishes together with its normal derivative on the diameter has a suitable decay of hessian L^2 norm.

Theorem 3.3. *(L^2 **hessian decay for bi-harmonic functions in half-disk which vanish together with normal derivative along diameter**)*
Set $B_1^+ = B_1(\mathbf{0}) \cap \{(x,y) \in \mathbb{R}^2 : y > 0\} \subset \mathbb{R}^2$ and $\Gamma = B_1(\mathbf{0}) \cap \{(x,y) \in \mathbb{R}^2 : y = 0\}$.
Assume $z \in H^2(B_1^+)$, $\Delta^2 z = 0$ on B_1^+, $z = \partial z/\partial y = 0$ on Γ.
Then

$$\|D^2 z\|^2_{L^2(B_\varrho^+)} \ \le\ \varrho^2\ \|D^2 z\|^2_{L^2(B_1^+)} \qquad \forall \varrho \le 1. \tag{29}$$

Proof. See [17], Theorem 6.1. □

An analogous estimate can be proved in disks whose center is close to the boundary, as was announced in Remark 7.5 in [17] and proved in detail here by the following estimate.

Theorem 3.4. *(L^2 **hessian decay for bi-harmonic functions in B_1^τ which vanish together with normal derivative along Γ_τ**)*
Set

$$B_\varrho^\tau = B_\varrho(\mathbf{0}) \cap \{(x,y) \in \mathbb{R}^2 : y > \tau\}, \quad \Gamma_\tau = B_1(\mathbf{0}) \cap \{(x,y) \in \mathbb{R}^2 : y = \tau\}. \tag{30}$$

Assume $-1 < \tau < 0$, $z \in H^2(B_1^\tau)$, $\Delta^2 z = 0$ on B_1^τ, $z = \partial z/\partial y = 0$ on Γ_τ.
Then there is a constant $c_3 = \left(4 + 2\sqrt{3}\right) \vee (16\, c_{2,q}/3)$ such that

$$\|D^2 z\|^2_{L^2(B_\varrho^\tau)} \ \le\ c_3\, \varrho^2\ \|D^2 z\|^2_{L^2(B_1^\tau)} \qquad \forall \varrho < 1. \tag{31}$$

Proof. If $(\sqrt{3} - 1)/2 \le \varrho < 1$ the inequality is trivial since by $B_\varrho^\tau \subset B_1^\tau$ we have

$$\|D^2 z\|^2_{L^2(B_\varrho^\tau)} \ \le\ (4 + 2\sqrt{3})\, \varrho^2\ \|D^2 z\|^2_{L^2(B_1^\tau)}\,. \tag{32}$$

From now on we assume

$$\varrho < \frac{\sqrt{3} - 1}{2}. \tag{33}$$

First we examine the case $-1/2 < \tau < 0$:
set $r = \sqrt{1 - \tau^2} \ge \sqrt{3}/2$, $t = \varrho - \tau$ (so that $t < r$ by condition (33)) and

$$S_r = \left\{\mathbf{x} \in \mathbb{R}^2 ;\ |\mathbf{x} - (0,\tau)| \le r\,,\ y \ge \tau\right\} \subset B_1^\tau\,;$$

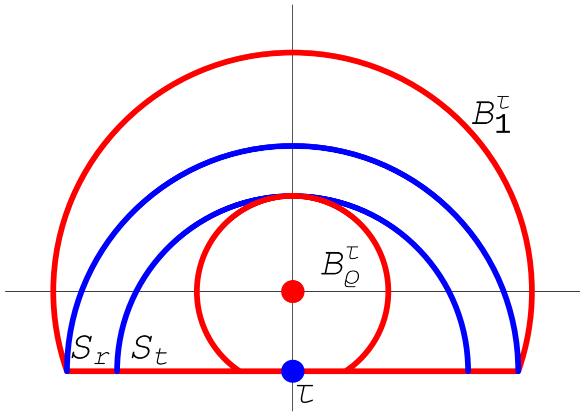

FIGURE 1. $B_\varrho^\tau \subset S_t \subset S_r \subset B_1^\tau$ (case: $0 < \varrho < (\sqrt{3}-1)/2$, $-1/2 < \tau < 0$).

then, by scaling in Theorem 3.3 we have

$$\begin{aligned}\int_{B_\varrho^\tau} |D^2 z|^2\, d\mathbf{x} &\le \int_{S_t} |D^2 z|^2\, d\mathbf{x} \le \left(\frac{t}{r}\right)^2 \int_{S_r} |D^2 z|^2\, d\mathbf{x} \\ &= \frac{(\varrho-\tau)^2}{1-\tau^2} \int_{S_r} |D^2 z|^2\, d\mathbf{x} \le \frac{4}{3}(\varrho-\tau)^2 \int_{S_r} |D^2 z|^2\, d\mathbf{x}\,;\end{aligned} \tag{34}$$

if $\varrho \ge -\tau$ then by (34)

$$\int_{B_\varrho^\tau} |D^2 z|^2\, d\mathbf{x} \le \frac{16}{3}\varrho^2 \int_{S_r} |D^2 z|^2\, d\mathbf{x} \le \frac{16}{3}\varrho^2 \int_{B_1^\tau} |D^2 z|^2\, d\mathbf{x}\,;$$

if $\varrho < -\tau$ then $B_\varrho^\tau = B_\varrho$ and z is a bi-harmonic function in $B_{-\tau}$, hence by exploiting (34) and scaling in Theorem 3.2 with the choice $\xi = 0$ we obtain

$$\begin{aligned}\int_{B_\varrho} |D^2 z|^2\, d\mathbf{x} &\le c_{2,q} \left(\frac{\varrho}{-\tau}\right)^2 \int_{B_{-\tau}} |D^2 z|^2\, d\mathbf{x} = c_{2,q} \left(\frac{\varrho}{-\tau}\right)^2 \int_{B_{-\tau}^\tau} |D^2 z|^2\, d\mathbf{x} \\ &\le c_{2,q} \left(\frac{\varrho}{-\tau}\right)^2 \frac{16}{3}\tau^2 \int_{B_1^\tau} |D^2 z|^2\, d\mathbf{x} = \frac{16}{3} c_{2,q}\, \varrho^2 \int_{B_1^\tau} |D^2 z|^2\, d\mathbf{x}\,,\end{aligned}$$

hence by choosing $c_3 = \big(4 + 2\sqrt{3}\big) \vee (16\, c_{2,q}/3)$ we achieve the thesis for any τ and ϱ such that $-1/2 < \tau < 0$ and $\varrho < (\sqrt{3}-1)/2$.
If $-1 < \tau \le -1/2$ and (33) then $\varrho < 1/2 \le -\tau$, hence $B_\varrho^\tau = B_\varrho$ and $B_{1/2} \subset B_1^\tau$. Then by scaling Theorem 3.2 and choosing $\xi = 0$ we have

$$\begin{aligned}\int_{B_\varrho^\tau} |D^2 z|^2\, d\mathbf{x} &= \int_{B_\varrho} |D^2 z|^2\, d\mathbf{x} \le 4\, c_{2,q}\, \varrho^2 \int_{B_{1/2}} |D^2 z|^2\, d\mathbf{x} \\ &\le 4\, c_{2,q}\, \varrho^2 \int_{B_1^\tau} |D^2 z|^2\, d\mathbf{x},\end{aligned}$$

and the proof is complete since the case $(\sqrt{3}-1)/2 \le \varrho < 1$ is estimated in (32). □

By the previous estimates and a blow-up argument we obtain, as in [10] and [17], the following decay results for functional $\mathcal{F}$: explicitly the decay of functional $\mathcal{F}$ at interior points (Theorem 3.6), at the boundary points (Theorem 3.7) and close to the boundary (Theorem 3.8).
We need a localization of the functional $\mathcal{F}$, in the form provided by the following definition.

Definition 3.5. (Localization of $\mathcal{F}$) We will use the symbol $\mathcal{F}(u,A)$ to denote the localization of the functional (15). For any Borel set $A\subset\widetilde{\Omega}$ we set:

$$\mathcal{F}(u,A) \;=\; \int_A (|\nabla^2 u|^2+\mu|u-g|^q)\,d\mathbf{y}+\alpha\mathcal{H}^1(S_u\cap A)+\beta\mathcal{H}^1\big((S_{\nabla u}\setminus S_u)\cap A\big)\,. \tag{35}$$

Theorem 3.6. ***(Decay of functional $\mathcal{F}$ at interior points of Ω)*** *([10], Theorem 5.4) Assume (7). Then, by referring to (28) and to (27) about the meaning of $c_{2,q}$ and c_0,*

$$\forall k>2,\ \forall\eta,\sigma\in(0,1)\ \textit{with}\ \eta^\sigma<\frac{1}{c_{2,q}},\quad \exists\varepsilon_0>0\ \textit{such that} \tag{36}$$

for all $\varepsilon\in(0,\varepsilon_0]$ and $\overline{B}_\varrho(\mathbf{x})\subset\Omega$, if $u\in GSBV^2(\Omega)$ is a local minimizer of $\mathcal{F}(\cdot,\Omega)$ with

$$\rho\le\varepsilon^k,\qquad \int_{B_\varrho(\mathbf{x})}|g|^{2q}dy\le\varepsilon^k$$

and

$$\alpha\mathcal{H}^1(S_u\cap B_\varrho(\mathbf{x}))+\beta\mathcal{H}^1\big((S_{\nabla u}\setminus S_u)\cap B_\varrho(\mathbf{x})\big)\le\varepsilon\rho, \tag{37}$$

we have

$$\mathcal{F}(u,B_{\eta\rho}(\mathbf{x}))\le\eta^{2-\sigma}\mathcal{F}(u,B_\rho(\mathbf{x})). \tag{38}$$

Theorem 3.7. ***(Decay of functional $\mathcal{F}$ at boundary points)*** *([17], Theorem 7.3) Assume (7)-(11) and (25). Then, by referring to (26) and to (27) about the meaning of $\bar\varrho$ and c_0,*

$$\forall k>2,\ \forall\eta,\sigma\in(0,1),\quad \exists\varepsilon_1>0,\ \exists\vartheta_1>0\quad \textit{such that} \tag{39}$$

for all $\varepsilon\in(0,\varepsilon_1]$, for any $\mathbf{x}\in\partial\Omega$, for any $u\in GSBV^2(\Omega)$ which is an $\overline{\Omega\cap B_\varrho(\mathbf{x})}$ local minimizer of $\mathcal{F}(\cdot,\overline{\Omega\cap B_\varrho(\mathbf{x})})$, for any ϱ s.t. $B_\varrho(\mathbf{x})\subset\widetilde{\Omega}$,

$$0<\varrho\le\widetilde{\varrho}:=\left(\varepsilon^k\wedge\bar\varrho\wedge(c_0\vee 1)^{-1}\right),\qquad \int_{B_\varrho(\mathbf{x})}|g|^{2q}d\mathbf{y}\le\varepsilon^k \tag{40}$$

and

$$\alpha\,\mathcal{H}^1\left(S_u\cap\overline{\Omega\cap B_\varrho(\mathbf{x})}\right)\;+\;\beta\,\mathcal{H}^1\left((S_{\nabla u}\setminus S_u)\cap\overline{\Omega\cap B_\varrho(\mathbf{x})}\right)\;<\;\varepsilon\,\varrho\,, \tag{41}$$

we have

$$\mathcal{F}(u,B_{\eta\varrho}(\mathbf{x}))\le\;\eta^{2-\sigma}\,\max\left\{\mathcal{F}(u,B_\varrho(\mathbf{x}))\;,\;\varrho^2\,\vartheta_1\,L\right\}. \tag{42}$$

The following decay near the boundary also holds.

Theorem 3.8. ***(Decay of functional $\mathcal{F}$ at points close to the boundary $\partial\Omega$)*** *Assume (7)-(11) and (25). Then, by referring to (26), (27) and to (31) about the meaning of $\bar\varrho$, c_0 and c_3,*

$$\forall k>2,\ \forall\eta,\sigma\in(0,1),\ \textit{with}\ \eta^\sigma<\frac{1}{c_3},\quad \exists\varepsilon_2>0,\ \exists\vartheta_2>0\quad \textit{such that} \tag{43}$$

for all $\varepsilon\in(0,\varepsilon_2]$, for any $\varrho\le\widetilde{\varrho}$, for any $\mathbf{x}\in\Omega$ s.t. $\mathrm{dist}(\mathbf{x},\partial\Omega)<\frac{\varrho}{2}$, where $\widetilde{\varrho}$ is defined in (40), for any $u\in GSBV^2(\Omega)$ which is an $\overline{\Omega\cap B_\varrho(\mathbf{x})}$ local minimizer of $\mathcal{F}(\cdot,\overline{\Omega\cap B_\varrho(\mathbf{x})})$, and

$$\alpha\,\mathcal{H}^1\left(S_u\cap\overline{\Omega\cap B_\varrho(\mathbf{x})}\right)\;+\;\beta\,\mathcal{H}^1\left((S_{\nabla u}\setminus S_u)\cap\overline{\Omega\cap B_\varrho(\mathbf{x})}\right)\;<\;\varepsilon\,\varrho\,, \tag{44}$$

we have

$$\mathcal{F}(u,B_{\eta\varrho}(\mathbf{x}))\;\le\;\eta^{2-\sigma}\,\max\left\{\mathcal{F}(u,B_\varrho(\mathbf{x}))\;,\;\varrho^2\,\vartheta_2\,L\right\}. \tag{45}$$

Proof. The proof is identical to the one of Theorem 3.7, except exploiting Theorem 3.4 in place of Theorem 3.3. □

Now we define a set which plays an important role in the analysis of regularity for minimizing triplets.

Definition 3.9. We define the set

$$\Omega_0(K_0, K_1, v) \;=\; \left\{ \mathbf{x} \in \widetilde{\Omega} \,:\, \lim_{r \downarrow 0} \, r^{-1} \, F_{B_r(\mathbf{x})}(K_0, K_1, v) \;=\; 0 \right\} \tag{46}$$

shortly denoted by Ω_0 when there is no risk of confusion.

Theorem 3.10. *Assume (7)-(11) and (K_0, K_1, v) is an essential minimizing triplet of F. Then $\Omega_0(K_0, K_1, v)$ is an open subset of $\widetilde{\Omega}$, precisely:*

$$\left\{ \mathbf{x} \in \widetilde{\Omega} \,:\, \lim_{r \downarrow 0} \, r^{-1} \, \mathcal{F}(v, B_r(\mathbf{x})) \;=\; 0 \right\} \quad \textit{is an open set,} \tag{47}$$

$$\Omega_0 \cap \left(\overline{S_v \cup S_{\nabla v}} \right) = \emptyset \,, \tag{48}$$

$$\mathcal{H}^1 \left(\widetilde{\Omega} \cap \left(\overline{(S_v \cup S_{\nabla v})} \setminus (S_v \cup S_{\nabla v}) \right) \right) = 0 \,, \tag{49}$$

$$\Omega_0 \cap (K_0 \cup K_1) = \emptyset \,, \tag{50}$$

$$\mathcal{H}^1 \left(\widetilde{\Omega} \cap ((K_0 \cup K_1) \setminus (S_v \cup S_{\nabla v})) \right) = 0 \,. \tag{51}$$

Proof. Since (K_0, K_1, v) is an essential minimizing triplet, by (17) we get

$$F_{B_r(\mathbf{x})}(K_0, K_1, v) \;=\; \mathcal{F}(v, B_r(\mathbf{x})) \,. \tag{52}$$

Since v is a minimizer of $\mathcal{F}$ (by Theorem 2.3), the theses follow by (1.13) of Theorem 2.2 in [17] and by Theorem 2.7, Remark 3 in the present paper. □

Remark 5. About the estimate of hessian decay at the boundary (Theorem 3.3) we emphasize a key difference. On the one hand, Schwarz reflection of harmonic functions in upper half disk vanishing on the diameter is bounded by 1 as a linear operator from $H^1(B_1^+)$ to $H^1(B_1^-)$, say L^2 norm of the hessian is the same in upper and lower half disk (see [7]). On the other hand, Duffin extension map for bi-harmonic functions vanishing on the diameter together with normal derivative provides a poor control of $H^2(B_1^-)$ norm in term of $H^2(B_1^+)$ norm as shown by the following example (see [17], [24]). By setting

$$\begin{cases} v_k \;=\; r^{k+1} \left(\sin((k-1)\vartheta)) - \frac{k-1}{k+1} \sin((k+1)\vartheta) \right) \,, & k = 2, 3, 4, \ldots \\ \omega_k \;=\; r^{k+1} \left(\cos((k-1)\vartheta)) - \cos((k+1)\vartheta) \right) \,, & k = -1 \text{ and } 1, 2, 3, \ldots \end{cases}$$

if we choose $z = \omega_2 - v_3 + \omega_4 - v_5$ then $\|D^2 z\|_{L^2(B_1^-)} \approx 12.5761 \, \|D^2 z\|_{L^2(B_1^+)}$.
This depends on the fact that bi-harmonic extension of z may be either even in y (e.g. $z = y^2$) or odd in y (e.g. $z = r^3(3\sin\vartheta - \sin(3\vartheta)) = 4y^3$) or a mixing of the two (e.g. $z = \omega_2 - v_3$).

4. **Uniform density estimates for essential minimizing triplets with smooth Dirichlet datum.**

In this section we state and prove the main results in the case when Dirichlet datum w has neither jump nor crease set.

In all the statements of this section it is understood that the open set Ω is contained in $\mathbb{R}^2$ and the structural assumptions (7)-(11) hold true.

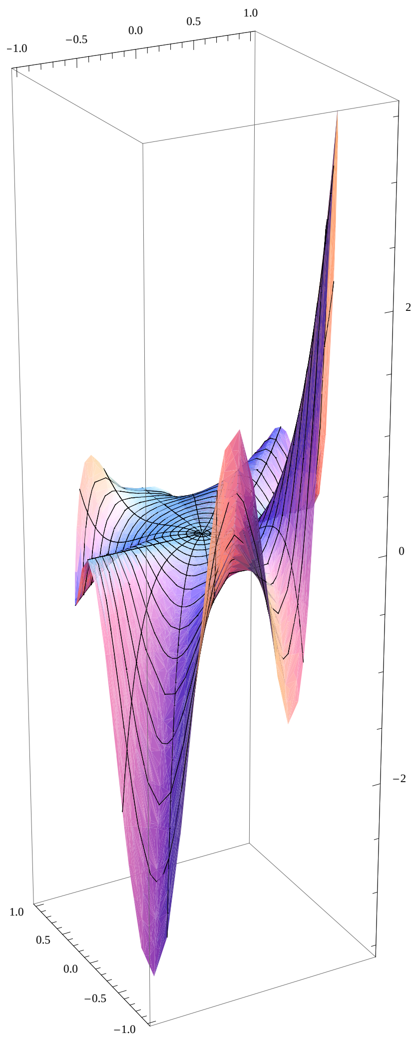

FIGURE 2. Poor L^2 hessian estimate for bi-harmonic extension (see Remark 5).

Theorem 4.1. (**Density lower bound for the functional** F)
Let (K_0, K_1, u) *be an essential minimizing triplet for the functional* F *with (7)-(11). Then there exist* $\varepsilon_3 > 0\,, \varrho_3 > 0$ *such that*

$$F_{B_\varrho(\mathbf{x})}(K_0, K_1, u) \quad \geq \quad \varepsilon_3 \varrho \qquad \forall \mathbf{x} \in (K_0 \cup K_1) \cap \overline{\Omega}, \quad \forall \varrho \leq \varrho_3\,. \tag{53}$$

Proof. Referring to Theorem 3.6, Theorems 3.7 and 3.8, for fixed $k > 2\,, \eta\,, \sigma \in (0,1)$ with $\eta^\sigma < \frac{1}{c_{2,q}} \wedge \frac{1}{c_3}$, choose $\eta' \in (0,1)$ such that $(\eta')^{1-\sigma} c_0 < \varepsilon_0 \wedge \varepsilon_1$, and let $\varepsilon'\,, \vartheta'$ the constants associated to $\eta'\,, \sigma$ by Theorem 3.7.
Set $\overline{\varepsilon} = \varepsilon_0 \wedge \varepsilon_1 \wedge \varepsilon'$ and choose ε_3 and ϱ_3 such that

$$0 < \varepsilon_3 < \frac{1}{2}(\varepsilon_0 \eta \wedge \overline{\varepsilon} \eta' \wedge \varepsilon_2), \quad 0 < \varrho_3 < (\varepsilon_0^k \wedge \varepsilon_1^k \wedge \varepsilon_2^k \wedge \overline{\varrho} \wedge (c_0 \vee 1)^{-1}), \tag{54}$$

$$\int_{B_{\varrho_3}(\mathbf{x})} |g|^{2q} d\mathbf{y} \leq \frac{1}{2}\varepsilon_3^k, \qquad \varrho_3(\vartheta_1 \vee \vartheta' \vee \vartheta_2)\, L < \frac{1}{2}(c_0 \wedge \overline{\varepsilon})\,. \tag{55}$$

We conclude the proof by separately analyzing the three admissible cases.

1^{st} *case.* If $\mathbf{x} \in (K_0 \cup K_1) \cap \Omega$ with $B_{\varrho_3}(\mathbf{x}) \subset \Omega$, then the thesis is proved by Theorem 27.6 in [11] with the constant ε_0 in place of ε_3, hence the thesis holds true since $\varepsilon_3 < \varepsilon_0$.

2^{nd} *case.* If $\mathbf{x} \in (K_0 \cup K_1) \cap \partial\Omega$ then, by Theorem 2.3, u minimizes $\mathcal{F}(z, B_\varrho(\mathbf{x}))$ among z satisfying (16) and $z = u$ in $\widetilde{\Omega} \setminus B_\varrho(\mathbf{x})$ and $F_{B_\varrho(\mathbf{x})}(K_0, K_1, u) = \mathcal{F}(u, B_\varrho(\mathbf{x}))$ for every $B_\varrho(\mathbf{x}) \subset \widetilde{\Omega}$.
By contradiction, assume that the thesis is false. Then there exist $\mathbf{x} \in (K_0 \cup K_1) \cap \partial\Omega$ and $0 < \varrho \leq \varrho_3$ such that

$$F_{B_\varrho(\mathbf{x})}(K_0, K_1, u) < \varepsilon_3 \varrho\,.$$

By exploiting Theorem 3.7 as was done in the last section of [17] with $\eta^{h-1}\varrho$ in place of ϱ, with $h \in \mathbb{N}$, $h \geq 2$, we get

$$F_{B_{\eta^h \varrho}(\mathbf{x})}(K_0, K_1, u) < \eta^{h(2-\sigma)} \max\left\{\varepsilon_3 \varrho\,,\ (\eta^{h-1}\varrho)^2 \vartheta_1 L\right\} \text{ with } h \in \mathbb{N},\ h \geq 2\,. \tag{56}$$

Since

$$r^{-1} F_{B_r(\mathbf{x})}(K_0, K_1, u) \leq \eta^{-1}(\eta^h \varrho)^{-1} F_{B_{\eta^h \varrho}(\mathbf{x})}(K_0, K_1, u) \quad \forall r \text{ s.t. } \eta^{h+1}\varrho \leq r < \eta^h \varrho\,,$$

by letting $h \to +\infty$ in the last inequality and exploiting (56), we obtain

$$\lim_{r \downarrow 0} r^{-1} F_{B_r(\mathbf{x})}(K_0, K_1, u) = 0\,,$$

hence, by referring to Definition 3.9, $\mathbf{x} \in \Omega_0$.
Function u is C^2 in a neighborhood of $\mathbf{x}$ since Ω_0 is open. This fact, together with (K_0, K_1, u) minimizing triplet, $\mathbf{x} \in (K_0 \cup K_1)$ and (50) leads to a contradiction.

3^{rd} *case.* Now assume by contradiction that $\mathbf{x} \in (K_0 \cup K_1) \cap \Omega$ and, referring to (54) and (40) for the definition of ϱ_3 and $\widetilde{\varrho}$, that

$$0 < \mathrm{dist}(\mathbf{x}, \partial\Omega) < \varrho \leq \varrho_3 < \widetilde{\varrho},$$

$$F_{B_\varrho(\mathbf{x})}(K_0, K_1, u) < \varepsilon_3 \varrho.$$

If $B_{\varrho/2}(\mathbf{x}) \subset \Omega$ then

$$F_{B_{\varrho/2}(\mathbf{x})}(K_0, K_1, u) < 2\varepsilon_3 \frac{\varrho}{2} < \varepsilon_0 \frac{\varrho}{2},$$

so we are in the situation of the 1^{st} *case.*
Otherwise $\mathrm{dist}(\mathbf{x}, \partial\Omega) < \frac{\varrho}{2}$, then we can repeat the same discussion of 2^{nd} *case* by using Theorem 3.8 instead of Theorem 3.7: in this way we get $\mathbf{x} \in \Omega_0$. This property leads to a contradiction as was stated at the end of 2^{nd} *case.* □

Theorem 4.2. *(***Density lower bound for the segmentation length***)*
Let (K_0, K_1, u) be an essential minimizing triplet for the functional (2) with (7)-(11). Then there exist $\varepsilon_4 > 0, \varrho_4 > 0$ such that

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(\mathbf{x})\right) \ \geq\ \varepsilon_4 \varrho \qquad \forall \mathbf{x} \in (K_0 \cup K_1) \cap \overline{\Omega},\ \ \forall \varrho \leq \varrho_4. \tag{57}$$

Proof. There are the same three cases to examine as in Theorem 4.1.
We detail only the second one ($\mathbf{x} \in (K_0 \cup K_1) \cap \partial\Omega$), since the other ones require no change with respect to the proof of Theorem 4.1.
Let $k\,, \eta\,, \sigma$ and $\varepsilon_3\,, \varrho_3$ be as in Theorem 4.1. We can fix $h_0 \in \mathbb{N}$ such that $\eta^{h_0(1-\sigma)}(c_0 \vee \varepsilon_1) < \varepsilon_3$, where c_0 is given in Theorem 3.1.

If we define $\varepsilon_4 = \dfrac{\varepsilon_3}{\alpha}\eta^{h_0}$ and $\varrho_4 = \varrho_3 \wedge 1$ and we assume, by contradiction, that there exist $\mathbf{x} \in (K_0 \cup K_1) \cap \partial\Omega$ and $\varrho \le \varrho_4$ such that

$$\mathcal{H}^1((K_0 \cup K_1) \cap B_\varrho(\mathbf{x})) < \varepsilon_4 \varrho,$$

then we get

$$\alpha\mathcal{H}^1(K_0 \cap B_\varrho(\mathbf{x})) + \beta\mathcal{H}^1((K_1 \setminus K_0) \cap B_\varrho(\mathbf{x})) < \varepsilon_3 \eta^{h_0} \varrho \tag{58}$$

and also for all $h < h_0$

$$\alpha\mathcal{H}^1(K_0 \cap B_{\eta^{h+1}\varrho}(\mathbf{x})) + \beta\mathcal{H}^1((K_1 \setminus K_0) \cap B_{\eta^{h+1}\varrho}(\mathbf{x})) < \varepsilon_3 \eta^{h+1} \varrho\,, \tag{59}$$

hence by (52) and Theorem 3.7, and referring to (25) we have

$$\begin{aligned} F_{B_{\eta\varrho}(\mathbf{x})}(K_0, K_1, u) &\le \eta^{2-\sigma} \max\left\{F_{B_\varrho(\mathbf{x})}(K_0, K_1, u)\,, \varrho^2\,\vartheta_1\, L\right\} \\ &\le \eta^{2-\sigma} \max\left\{\, c_0\, \varrho\,, \varepsilon_1\, \varrho\,\right\} \\ &= \eta^{1-\sigma} \max\left\{\, c_0\,(\eta\, \varrho)\,, \varepsilon_1\,(\eta\, \varrho)\,\right\}\,. \end{aligned} \tag{60}$$

Since by (59)

$$\alpha\mathcal{H}^1(K_0 \cap B_{\eta\,\varrho}(\mathbf{x})) + \beta\mathcal{H}^1((K_1 \setminus K_0) \cap B_{\eta\,\varrho}(\mathbf{x})) < \varepsilon_3(\eta\,\varrho)\,,$$

by (60) and Theorem 3.7

$$\begin{aligned} F_{B_{\eta^2\varrho}(\mathbf{x})}(K_0, K_1, u) &\le \eta^{2-\sigma} \max\left\{F_{B_{\eta\varrho}(\mathbf{x})}(K_0, K_1, u)\,, (\eta\varrho)^2\,\vartheta_1\, L\right\} \\ &\le \eta^{1-\sigma}\eta^{2-\sigma} \max\left\{\, c_0\, \eta\varrho\,, \varepsilon_1\, \eta\varrho\,\right\} \\ &= \eta^{2(1-\sigma)} \max\left\{\, c_0\,(\eta^2\, \varrho)\,, \varepsilon_1\,(\eta^2\, \varrho)\,\right\}\,. \end{aligned} \tag{61}$$

By (59) we get

$$\alpha\mathcal{H}^1(K_0 \cap B_{\eta^k\,\varrho}(\mathbf{x})) + \beta\mathcal{H}^1((K_1 \setminus K_0) \cap B_{\eta^k\,\varrho}(\mathbf{x})) < \varepsilon_3(\eta^k\,\varrho)\,, \quad \forall k \le h_0$$

so we can iterate Theorem 3.7 h_0 times until we get

$$\begin{aligned} F_{B_{\eta^{h_0}\varrho}(\mathbf{x})}(K_0, K_1, u) &\le \eta^{h_0(2-\sigma)} \max\left\{F_{B_\varrho(\mathbf{x})}(K_0, K_1, u)\,, \varrho^2\,\vartheta_1\, L\right\} \\ &\le \eta^{h_0(1-\sigma)} \max\left\{c_0(\eta^{h_0}\,\varrho)\,,\ \varepsilon_1(\eta^{h_0}\,\varrho)\,\right\} < \varepsilon_3(\eta^{h_0}\varrho)\,, \end{aligned}$$

which contradicts Theorem 4.1. □

Theorem 4.3. *(***Elimination Property***)*
Let (K_0, K_1, u) be an essential minimizing triplet for the functional (2) with (7)-(11) and let $\varepsilon_4 > 0$, $\varrho_4 > 0$ as in Theorem 4.2 and $\varrho \le \varrho_4$. If $\mathbf{x} \in \overline{\Omega}$ and

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(\mathbf{x})\right) \quad < \quad \frac{\varepsilon_4}{2}\varrho$$

then

$$(K_0 \cup K_1) \cap B_{\varrho/2}(\mathbf{x}) = \emptyset.$$

Proof. Assume, by contradiction, that there exists $\mathbf{y} \in (K_0 \cup K_1) \cap B_{\varrho/2}(\mathbf{x})$. Then $B_{\varrho/2}(\mathbf{y}) \subset B_\varrho(\mathbf{x})$, hence

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_{\varrho/2}(\mathbf{y})\right) \quad \le \quad \mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(\mathbf{x})\right) \quad < \quad \varepsilon_4\left(\frac{\varrho}{2}\right),$$

therefore $\mathbf{y} \notin K_0 \cup K_1$ by Theorem 4.2. □

The Minkowski content $\mathcal{M}^k(E)$ of a set $E \subset \mathbb{R}^n$ is the limit (if it exists and is finite) as $\varrho \downarrow 0$ of the n-dimensional Lebesgue measure of the ϱ-neighborhood of E divided by $\omega_{n-k}\varrho^{n-k}$, where ω_h is the volume of the h-dimensional unit ball. The following result expresses the agreement between one dimensional Hausdorff measure of optimal segmentation $K_0 \cup K_1$ and its Minkowski content. In simple words the theorem says that a uniform fattening of an optimal segmentation is a reasonable approximation of the segmentation itself. This property is useful for a variational approximation of the functional F and for the implementation of a suitable numerical algorithm to find a minimizing triplet for F (see [1]).

Theorem 4.4. *(***Minkowski content of the segmentation***)*
Let (K_0, K_1, u) be an essential minimizing triplet for the functional F.
Assume (7)-(11) and

$$g \in L^{2q}(\widetilde{\Omega})\,. \tag{62}$$

Then

(i) $K_0 \cup K_1$ is $(\mathcal{H}^1, 1)$ rectifiable;
(ii) the following equality holds

$$\lim_{\varrho \downarrow 0} \frac{|\{\mathbf{x} \in \widetilde{\Omega};\ \mathrm{dist}(\mathbf{x}, (K_0 \cup K_1) \cap \overline{\Omega}) < \varrho\ \}|}{2\varrho} = \mathcal{H}^1\left((K_0 \cup K_1) \cap \overline{\Omega}\right)\ .$$

Proof. It can be shown (see [26], Section 3.2.37 and 3.2.39) that

$$\liminf_{\varrho \downarrow 0} \frac{|\{\mathbf{x} \in \widetilde{\Omega};\ \mathrm{dist}(\mathbf{x}, (K_0 \cup K_1) \cap \overline{\Omega}) < \varrho\ \}|}{2\varrho} \geq \mathcal{H}^1\left((K_0 \cup K_1) \cap \overline{\Omega}\right)$$

since $K_0 \cup K_1$ is closed and countably $(\mathcal{H}^1, 1)$ rectifiable.
By Theorem 2.3, Theorem 2.7, (22) and (51) the function u is a minimizer of functional $\mathcal{F}$, $S_u \cup S_{\nabla u}$ is countably $(\mathcal{H}^1, 1)$ rectifiable, $\mathcal{H}^1\left(S_u \cup S_{\nabla u}\right) < +\infty$ and $\mathcal{H}^1\big(\overline{\Omega} \cap ((K_0 \cup K_1) \setminus (S_u \cup S_{\nabla u}))\big) = 0$, then $K_0 \cup K_1$ is $(\mathcal{H}^1, 1)$ rectifiable and (i) follows.

Thanks to (i), $(K_0 \cup K_1) \cap \overline{\Omega}$ compact and density lower bound (57) of Theorem 4.2 we can apply Theorem 2.104 in [2] and get the thesis (ii). □

Remark 6. We emphasize that all the constants $c_0, \varepsilon_3, \varepsilon_4, \varrho_3, \varrho_4$ appearing in Theorems 4.1–4.2 depend on the data α, β, μ, g, w.

5. Uniform density estimates for essential minimizing triplets with discontinuous Dirichlet datum.

In this section we deal with the case of Dirichlet datum that can be discontinuous at the boundary. To this aim we recall the appropriate general theorem about existence of minimizers: a stronger result than Theorem 2.3, allowing discontinuous Dirichlet datum. Then we prove density estimates which are uniform up to the boundary $\partial\Omega$.

Theorem 5.1. *Let α, β, μ, q, g, Ω, $\widetilde{\Omega}$, M, T_0, T_1 and w be s.t.*

$$0 < \beta \leq \alpha \leq 2\beta,\ \ \mu > 0,\ \ q > 1,\ \ g \in L^q(\widetilde{\Omega}) \cap L^{2q}_{loc}(\widetilde{\Omega})\,,\ \ w \in L^q(\widetilde{\Omega}) \tag{63}$$

hold true.

$$\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2\,, \tag{64}$$

$$\Omega \textit{ is an open set with Lipschitz boundary, } \widetilde{\Omega} \textit{ is an open set}\,, \tag{65}$$

$$\exists \text{ a finite set } M \text{ s.t. each connected component of } (\partial\Omega\setminus M) \text{ is uniformly } C^2, \tag{66}$$

$$(T_0 \cup T_1) \cap \partial\Omega \quad \text{is a finite set}, \tag{67}$$

$$T_0,\, T_1 \text{ Borel sets, } T_0 \cup T_1 \text{ closed subset of } \mathbb{R}^2,\ \mathcal{H}^1\left((T_0\cup T_1)\cap\widetilde{\Omega}\right) < +\infty\,, \tag{68}$$

$$\begin{array}{c}\exists\ \varepsilon_5>0,\ \varrho_5>0 \text{ s.t. } \mathcal{H}^1\left((T_0\cup T_1)\cap B_\varrho(\mathbf{x})\cap(\widetilde{\Omega}\setminus\overline{\Omega})\right)\geq \varepsilon_5\varrho \\ \forall\ \mathbf{x}\in(T_0\cup T_1)\cap\partial\Omega, \forall\ \varrho\leq\varrho_5,\end{array} \tag{69}$$

$$w\in C^2\left(\widetilde{\Omega}\setminus(T_0\cup T_1)\right), \quad w \text{ approximately continuous in } \left(\widetilde{\Omega}\setminus T_0\right), \tag{70}$$

$$\begin{cases} D^2w\in L^2(\widetilde{\Omega}\setminus(T_0\cup T_1)\,),\ D^2w\in L^\infty\big(A\setminus(T_0\cup T_1)\big) \\ \text{ with } A \text{ open set s.t. } \partial\Omega\subset A\subset\widetilde{\Omega}\,, \\ \exists C>0\ :\ \|w\|_{L^\infty}, \|\nabla w\|_{L^\infty}, \|\nabla^2 w\|_{L^\infty}\leq C \quad \text{in } A\,, \\ \mathrm{Lip}(\gamma')\leq C \quad \text{with } \gamma \text{ arc-length parametrization of } \partial\Omega\,, \\ \exists\bar\varrho>0\ :\ \mathcal{H}^1\big(\partial\Omega\cap B_\varrho(\mathbf{x})\big)<C\varrho \quad \forall\mathbf{x}\in\partial\Omega\,,\ \forall\varrho\leq\bar\varrho\,, \end{cases} \tag{71}$$

$$\begin{array}{l}\nexists\,(\mathfrak{T}_0,\mathfrak{T}_1,\omega) \text{ fulfilling (68),(70),} \\ \omega = \mathrm{aplim}\, w \text{ in } \widetilde{\Omega}\setminus T_0,\ (\mathfrak{T}_0\cup\mathfrak{T}_1)\subsetneq(T_0\cup T_1)\,.\end{array} \tag{72}$$

Then there exists a triplet (C_0, C_1, u) which minimizes the functional F defined by (2) with finite energy, among admissible triplets (K_0, K_1, v) fulfilling (3).
Moreover any minimizing triplet (K_0, K_1, v) fulfils:

$$K_0\cap\widetilde{\Omega} \text{ and } K_1\cap\widetilde{\Omega} \text{ are } (\mathcal{H}^1,1) \text{ rectifiable sets}, \tag{73}$$

$$\mathcal{H}^1(K_0\cap\widetilde{\Omega}) \;=\; \mathcal{H}^1(\overline{S_v})\,, \quad \mathcal{H}^1(K_1\cap\widetilde{\Omega}) \;=\; \mathcal{H}^1(\overline{S_{\nabla v}}\setminus S_v)\,, \tag{74}$$

$$\left\{\begin{array}{c} v\in GSBV^2(\widetilde{\Omega}),\ \text{hence } v \text{ and } \nabla v \\ \text{have well defined two-sided traces, finite } \mathcal{H}^1 \text{ a.e. on } K_0\cup K_1, \end{array}\right. \tag{75}$$

the function v is also a minimizer of the weak functional $\mathcal{F}$ (see [9],[16])

$$\mathcal{F}(z) \;=\; \int_{\widetilde{\Omega}}(|\nabla^2 z|^2+\mu|z-g|^q)\,d\mathbf{y}+\alpha\mathcal{H}^1(S_z)+\beta\mathcal{H}^1(S_{\nabla z}\setminus S_z) \tag{76}$$

over $z\in L^q(\widetilde{\Omega})\cap GSBV(\widetilde{\Omega})$ s.t. $\nabla z\in\left(GSBV(\widetilde{\Omega})\right)^2$ and $z=w$ a.e. in $\widetilde{\Omega}\setminus\Omega$.
Eventually, for any third element of minimizing triplet v we have

$$\mathcal{F}(v) = \; F(K_0,K_1,v)\,. \tag{77}$$

Proof. The theorem is a restatement of Theorem 2.2 in [17]. □

Remark 7. About all the hypotheses on Dirichlet datum (T_0, T_1, w) we emphasize that, though they are quite technical, actually they are very weak assumptions. Moreover their role in the Theorem is the following: a priori density estimates (69) on the Dirichlet datum reproduce the same density estimates on the optimal segmentation up to the boundary points; actually (69) means that each component of datum discontinuity set cannot live on boundary $\partial\Omega$ nor can reach this boundary from interior without crossing it; (72) simply says that the datum is expressed as an essential triplet (see Remark 3); the whole set of assumptions on data tells that w represents a Dirichlet datum which is noise-free in the region $\widetilde{\Omega}\setminus\Omega$, as it is very natural when facing inpainting problem, if noise, blotches and all artifact to be removed are contained in Ω.

Remark 8. Assumption (67) in Theorem 5.1 can be substituted by the following weaker assumption:

$$\mathcal{H}^1\left((T_0\cup T_1)\cap\partial\Omega\right)=0\,.$$

Remark 9. Due to (74) of Theorem 5.1 the theses (47)-(51) of Theorem 3.10 hold true for essential minimizing triplets of F also under the assumptions of this section: (63)-(72).

We introduce a suitable constant in order to handle boundary conditions:

$$\begin{cases} L = \big(\mathrm{C}(\partial\Omega)\big)^2 + \big(\mathrm{Lip}(\nabla w)\big)^2, \text{ where } \mathrm{C}(\partial\Omega) \text{ is an uniform estimate of} \\ \text{second derivatives of piecewise arc-length parametrization of } \partial\Omega \text{ and} \\ \mathrm{Lip}(\nabla w) \text{ is the Lipschitz constant of } \nabla w \text{ in the neighborhood } A \text{ of } \partial\Omega. \end{cases} \tag{78}$$

Theorem 5.2. *(**Density upper bound for functional** F) Let (K_0, K_1, u) be an essential locally minimizing triplet for the functional (2) with (63)-(72) and (78). Then there exist $C > 0$ and $\bar{\varrho} = \bar{\varrho}(\alpha, \beta, L, \|w\|_{L^q}, \|g\|_{L^q}) > 0$ such that*

$$\mathcal{H}^1\big(\partial\Omega \cap B_\varrho(\mathbf{x})\big) < C\varrho \quad \forall \mathbf{x} \in \overline{\Omega}\,,\ \forall \varrho \leq \bar{\varrho}\,, \tag{79}$$

and

$$\begin{gathered} F_{\overline{B_\varrho(\mathbf{x})}\cap\overline{\Omega}}(K_0, K_1, u) \leq c_0\varrho \\ \forall \varrho \ s.t.\ 0 < \varrho \leq (\bar{\varrho} \wedge 1) \quad \forall \mathbf{x} \in \overline{\Omega} \ s.t.\ \overline{B}_\varrho(\mathbf{x}) \subset \widetilde{\Omega}, \end{gathered} \tag{80}$$

where $c_0 = L\pi + 2^{q-1}\pi^{\frac{1}{2}}\mu\big(\|w\|^q_{L^{2q}(B_\varrho(\mathbf{x}))} + \|g\|^q_{L^{2q}(B_\varrho(\mathbf{x}))}\big) + (2\pi + C)\alpha$.
If $q = 2$ *and* $g\,, w \in L^\infty(\widetilde{\Omega})$, *then we can choose*

$$c_0 = L\pi + 2\pi\mu\big(\|w\|^2_{L^\infty} + \|g\|^2_{L^\infty}\big) + (2\pi + C)\alpha\,.$$

Proof. It is identical to the one of Theorem 3.1 . □

Theorem 5.3. *(**Decay of functional** $\mathcal{F}$ **at interior points of** Ω) Assume (63). Then, by referring to (28) and to (80) about the meaning of $c_{2,q}$ and c_0,*

$$\forall k > 2,\ \forall \eta, \sigma \in (0,1) \ \text{with}\ \eta^\sigma < \frac{1}{c_{2,q}}, \quad \exists \varepsilon_0 > 0 \ \text{such that} \tag{81}$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and* $\overline{B}_\varrho(\mathbf{x}) \subset \Omega$, *if* $u \in GSBV^2(\Omega)$ *is a local minimizer of* $\mathcal{F}(\cdot, \Omega)$ *with*

$$\rho \leq \varepsilon^k, \qquad \int_{B_\varrho(\mathbf{x})} |g|^{2q} dy \leq \varepsilon^k$$

and

$$\alpha\mathcal{H}^1(S_u \cap B_\varrho(\mathbf{x})) + \beta\mathcal{H}^1\big((S_{\nabla u} \setminus S_u) \cap B_\varrho(\mathbf{x})\big) \leq \varepsilon\rho, \tag{82}$$

we have

$$\mathcal{F}(u, B_{\eta\rho}(\mathbf{x})) \leq \eta^{2-\sigma}\mathcal{F}(u, B_\rho(\mathbf{x})). \tag{83}$$

Proof. It is a restatement of Theorem 5.4 in [10] . □

Theorem 5.4. *(**Decay of functional** $\mathcal{F}$ **at boundary points**) ([17], Theorem 7.3) Assume (63)-(72), (78). Then, by referring to (79) and to (80) about the meaning of $\bar{\varrho}$ and c_0,*

$$\forall k > 2,\ \forall \eta, \sigma \in (0,1), \quad \exists \varepsilon_6 > 0,\ \exists \vartheta_6 > 0 \quad \text{such that} \tag{84}$$

for all $\varepsilon \in (0, \varepsilon_6]$, *for any* $\mathbf{x} \in (\partial\Omega) \setminus (T_0 \cup T_1 \cup M)$, *for any* $u \in GSBV^2(\Omega)$ *which is an* $\overline{\Omega \cap B_\varrho(\mathbf{x})}$ *local minimizer of* $\mathcal{F}(\cdot, \overline{\Omega \cap B_\varrho(\mathbf{x})})$, *for any* ϱ *s.t.* $B_\varrho(\mathbf{x}) \subset \widetilde{\Omega}$,

$$0 < \varrho \leq \widetilde{\varrho} := \big(\varepsilon^k \wedge \bar{\varrho} \wedge (c_0 \vee 1)^{-1}\big)\,, \qquad \int_{B_\varrho(\mathbf{x})} |g|^{2q} d\mathbf{y} \leq \varepsilon^k \tag{85}$$

and

$$\alpha\,\mathcal{H}^1\left(S_u \cap \overline{\Omega \cap B_\varrho(\mathbf{x})}\right) \ + \ \beta\,\mathcal{H}^1\left((S_{\nabla u} \setminus S_u) \cap \overline{\Omega \cap B_\varrho(\mathbf{x})}\right) \ < \ \varepsilon\,\varrho\,, \tag{86}$$

we have

$$\mathcal{F}(u, B_{\eta\varrho}(\mathbf{x})) \leq \ \eta^{2-\sigma} \max\big\{\,\mathcal{F}(u, B_\varrho(\mathbf{x}))\ ,\ \varrho^2\,\vartheta_6\,L\,\big\}\ . \tag{87}$$

Proof. The proof is identical to the one of Theorem 3.7 . □

The following decay near the boundary also holds.

Theorem 5.5. *(Decay of functional $\mathcal{F}$ at points close to the boundary $\partial\Omega$)* *Assume (63)-(72), (78). Then, by referring to (79), (80) and to (31) about the meaning of $\bar{\varrho}$, c_0 and c_3,*

$$\forall k > 2,\ \forall \eta, \sigma \in (0,1),\ \textit{with}\ \eta^\sigma < \frac{1}{c_3},\quad \exists \varepsilon_7 > 0,\ \exists \vartheta_7 > 0 \quad \textit{such that} \tag{88}$$

for all $\varepsilon \in (0, \varepsilon_7]$, for any $\varrho \le \widetilde{\varrho}$, for any $\mathbf{x} \in \Omega \setminus (T_0 \cup T_1)$ s.t. $\mathrm{dist}(\mathbf{x}, \partial\Omega) < \frac{\varrho}{2}$, where $\widetilde{\varrho}$ is defined in (85), for any $u \in GSBV^2(\Omega)$ which is an $\overline{\Omega \cap B_\varrho(\mathbf{x})}$ local minimizer of $\mathcal{F}(\cdot, \overline{\Omega \cap B_\varrho(\mathbf{x})})$, and

$$\alpha\, \mathcal{H}^1 \left(S_u \cap \overline{\Omega \cap B_\varrho(\mathbf{x})} \right) \ +\ \beta\, \mathcal{H}^1 \left((S_{\nabla u} \setminus S_u) \cap \overline{\Omega \cap B_\varrho(\mathbf{x})} \right) \ <\ \varepsilon\, \varrho\,, \tag{89}$$

we have

$$\mathcal{F}(u, B_{\eta\varrho}(\mathbf{x})) \ \le\ \eta^{2-\sigma} \max \left\{ \mathcal{F}(u, B_\varrho(\mathbf{x}))\ ,\ \varrho^2\, \vartheta_7\, L \right\} . \tag{90}$$

Proof. The proof is identical to the one of Theorem 3.8 . □

Theorem 5.6. *(***Density lower bound for the functional** *F)* *Let (K_0, K_1, u) be an essential minimizing triplet for the functional (2) with (63)-(72), (78). Then there exist $\varepsilon_8 > 0$, $\varrho_8 > 0$ such that, referring to (69), $\varepsilon_8 \le \varepsilon_5$, $\varrho_8 \le \varrho_5$ and*

$$F_{B_\varrho(\mathbf{x})}(K_0, K_1, u) \ \ge\ \varepsilon_8 \varrho \quad \forall \mathbf{x} \in \left((K_0 \cup K_1) \cap \overline{\Omega} \right) \setminus M\,, \quad \forall \varrho \le \varrho_8\,. \tag{91}$$

Proof. We can repeat the same proof of Theorem 4.1 at every point

$$\mathbf{x} \in \left((K_0 \cup K_1) \cap \overline{\Omega} \right) \setminus (\partial\Omega \cap (T_0 \cup T_1 \cup M)).$$

Due to (69), the points in $\left((K_0 \cup K_1) \cap \partial\Omega \right) \setminus M$ fulfill (91). □

Theorem 5.7. *(***Density lower bound for the segmentation length***)* *Let (K_0, K_1, u) be an essential minimizing triplet for the functional (2) with (63)-(72). Then there exist $\varepsilon_9 > 0, \varrho_9 > 0$ such that, referring to (69), $\varepsilon_9 \le \varepsilon_5$, $\varrho_9 \le \varrho_5$ and*

$$\mathcal{H}^1 \left((K_0 \cup K_1) \cap B_\varrho(\mathbf{x}) \right) \ \ge\ \varepsilon_9 \varrho \quad \forall \mathbf{x} \in \left((K_0 \cup K_1) \cap \overline{\Omega} \right) \setminus M\,,\ \forall \varrho \le \varrho_9. \tag{92}$$

Proof. We can repeat the same proof of Theorem 4.2 at every point

$$\mathbf{x} \in \left((K_0 \cup K_1) \cap \overline{\Omega} \right) \setminus (\partial\Omega \cap (T_0 \cup T_1 \cup M)).$$

Due to (69), the points in $\left((K_0 \cup K_1) \cap \partial\Omega \right) \setminus M$ fulfill (92). □

Theorem 5.8. *(***Elimination Property***)* *Let (K_0, K_1, u) be an essential minimizing triplet for the functional (2) with (63)-(72) and let $\varepsilon_9 > 0, \varrho_9 > 0$ as in Theorem 5.7 and $\varrho \le \varrho_9$. If $\mathbf{x} \in \overline{\Omega} \setminus M$ and*

$$\mathcal{H}^1 \left((K_0 \cup K_1) \cap B_\varrho(\mathbf{x}) \right) \ <\ \frac{\varepsilon_9}{2} \varrho$$

then

$$(K_0 \cup K_1) \cap B_{\varrho/2}(\mathbf{x}) = \emptyset.$$

Proof. Assume, by contradiction, that there exists $\mathbf{y} \in (K_0 \cup K_1) \cap B_{\varrho/2}(\mathbf{x})$. Then $B_{\varrho/2}(\mathbf{y}) \subset B_\varrho(\mathbf{x})$, hence

$$\mathcal{H}^1 \left((K_0 \cup K_1) \cap B_{\varrho/2}(\mathbf{y}) \right) \ \le\ \mathcal{H}^1 \left((K_0 \cup K_1) \cap B_\varrho(\mathbf{x}) \right) \ <\ \varepsilon_9 \left(\frac{\varrho}{2} \right),$$

therefore $\mathbf{y} \notin (K_0 \cup K_1) \setminus M$ by Theorem 5.7. □

Theorem 5.9. *(***Minkowski content of the segmentation***)*
Let (K_0, K_1, u) be an essential minimizing triplet for the functional F.
Assume (63)-(72) and

$$g \in L^{2q}(\widetilde{\Omega}) . \tag{93}$$

Then

(i) $K_0 \cup K_1$ *is* $(\mathcal{H}^1, 1)$ *rectifiable;*
(ii) the following equality holds

$$\lim_{\varrho \downarrow 0} \frac{|\{\mathbf{x} \in \widetilde{\Omega};\ \mathrm{dist}(\mathbf{x}, (K_0 \cup K_1) \cap \overline{\Omega}) < \varrho\ \}|}{2\varrho} = \mathcal{H}^1\left((K_0 \cup K_1) \cap \overline{\Omega}\right) .$$

Proof. By Theorem 5.1, Theorem 2.7, (22), (51) and Remark 9, the function u is a minimizer of functional $\mathcal{F}$, $S_u \cup S_{\nabla u}$ is countably $(\mathcal{H}^1, 1)$ rectifiable, $\mathcal{H}^1(S_u \cup S_{\nabla u}) < +\infty$ and $\mathcal{H}^1\big(\overline{\Omega} \cap ((K_0 \cup K_1) \setminus (S_u \cup S_{\nabla u}))\big) = 0$, then $K_0 \cup K_1$ is $(\mathcal{H}^1, 1)$ rectifiable and (i) follows.

Thanks to (i), $(K_0 \cup K_1) \cap \overline{\Omega}$ compact and density lower bound (92) of Theorem 5.7 we can apply Theorem 2.104 in [2] to the set $(K_0 \cup K_1 \setminus M) \cap \overline{\Omega}$ and get the thesis (ii), since M is finite. ☐

Remark 10. We emphasize that all the constants $c_0, \varepsilon_8, \varepsilon_9, \varrho_8, \varrho_9$ appearing in Theorems 5.6–5.7 depend on the data α, β, μ, g, w.

Remark 11. All the results in this section hold true also for the essential minimizing triplets of the main part E of the functional F:

$$E(K_0, K_1, v) = \int_{\widetilde{\Omega} \setminus (K_0 \cup K_1)} \left|D^2 v\right|^2 \, d\mathbf{x} + \alpha\, \mathcal{H}^1\left(K_0 \cap \widetilde{\Omega}\right) + \beta\, \mathcal{H}^1\left((K_1 \setminus K_0) \cap \widetilde{\Omega}\right), \tag{94}$$

where E is minimized over admissible triplets (Definition 2.1) under assumption:

$$0 < \beta \leq \alpha \leq 2\beta , \tag{95}$$

and (64)–(72).

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