DIPARTIMENTO DI MATEMATICA
"Francesco Brioschi"
POLITECNICO DI MILANO

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C.D.Pagani, D.Pierotti

Collezione dei *Quaderni di Dipartimento*, numero **QDD 45**
Inserito negli *Archivi Digitali di Dipartimento* in data 10-7-2009

Piazza Leonardo da Vinci, 32 - 20133 Milano (Italy)

Variational methods for nonlinear Steklov eigenvalue problems with an indefinite weight function

Carlo Domenico PAGANI – Dario PIEROTTI
Dipartimento di Matematica del Politecnico, Piazza L. da Vinci 32 - 20133 Milano (Italy)

Abstract

We consider the problem of finding a harmonic function u in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, satisfying a nonlinear boundary condition of the form $\partial_\nu u(x) = \lambda\,\mu(x) h(u(x))$, $x \in \partial\Omega$ where μ changes sign and h is an increasing function with superlinear, subcritical growth at infinity. We study the solvability of the problem depending on the parameter λ by using min-max methods.

1 Statement of the problem and main result

We discuss the solvability of the boundary value problem

$$\begin{aligned} \Delta u(x) &= 0 \quad \text{in } \Omega \\ \partial_\nu u(x) &= \lambda\,\mu(x) h(u(x)) \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a bounded smooth domain and $\lambda \in \mathbb{R}\backslash\{0\}$. We suppose that h is an increasing $\mathcal{C}^1$ function on $\mathbb{R}$, satisfying $h(0) = 0$ and with superlinear, subcritical growth at infinity. Finally, we assume $\mu(x) \in L^\infty(\partial\Omega)$ and consider the case that $\mu(x)$ changes its sign along $\partial\Omega$.
Problems of the above type have been recently discussed in quite different contexts, e.g. in [1] motivated by the study of conformal metrics with prescribed (sign-changing) mean curvature on the boundary of a Riemannian manifold (see also [2], [3]) and in [4], [5] in corrosion modeling.
In this latter framework, we mention in particular the model described in [4], where one has problem (1.1) in a two dimensional domain with

$$h(u) = [e^{\alpha u} - e^{-(1-\alpha)u}], \tag{1.2}$$

(α is a known parameter ranging in $[0, 1]$) and where μ is equal to 1 or to the characteristic function of a subset $\Gamma \subset \partial\Omega$; note that the condition of subcritical growth of the exponential boundary term is satisfied in two dimensions. The above boundary condition (Butler-Volmer formula) expresses the relation, on a conducting surface, between a suitably defined internal voltage potential of an electrolyte (overpotential) and the current density; this exponential formula is an accurate model of the physical process in two cases: when the overpotential is "small" (in the so-called "active region") by taking in (1.1) positive values of the parameter λ, or when the overpotential ranges in some interval of intermediate values (the "transition region") and for *negative* values of the parameter.
Actually, in the model discussed in [4], condition (1.2) is justified by assuming *a priori* that the overpotential at the surface (or at some portion of the surface) lies entirely in the active or in the transition region. However, especially when the surface is made of several conducting pieces Γ_1, Γ_2,..., separated by insulating parts, it seems natural to consider the more general situation of a potential in the active region on Γ_i and in the transition region on Γ_j for $i \neq j$. This means that we will consider the boundary term (1.2) with a function μ which changes its sign on the contour $\partial\Omega$.

The case of indefinite μ is also treated for $\lambda = 1$ in [1], where existence of positive solutions was proved (see theorems 1.2 and 1.3) under the additional assumption that $h'(0) = 0$ (valid in the case of zero mean curvature at a given manifold's boundary) which implies that the linearized problem at the origin is positive semidefinite.

Motivated by the previous discussion, we study the problem (1.1) in the case of indefinite μ and assuming $h'(0) > 0$. In this case, we can not treat the problem as an indefinite perturbation of a non negative linear operator as the indefinite weight is already involved in the linearized boundary condition; this requires a particularly careful analysis of the linear eigenvalue problem. Note that, although linear and semi-linear eigenvalue problems with indefinite weight have been widely studied for elliptic operators [7]-[10], less is known about the corresponding problems involving boundary operators.

In this work we begin to address the problem of existence of solutions by assuming that the nonlinear term h is a *strictly increasing* function and that the indefinite weight satisfies $\int_{\partial\Omega} \mu \neq 0$; then, provided the boundary Ω and the function μ satisfy some regularity conditions (depending on the dimension $n \geq 2$) we find solutions to problem (1.1) for λ ranging in some intervals determined by the eigenvalues λ_k, $k \in \mathbb{Z}$, of the first order approximation to the problem at $u = 0$ (see section 2 below).

In particular, if the derivative h' assumes the global minimum at zero, we get a solution for $\lambda_k < \lambda < \lambda_{k+1}$ (with $k \neq 0, -1$ if $\int_{\partial\Omega} \mu < 0$); otherwise, there is at least one solution for small enough λ provided $\int_{\partial\Omega} \mu > 0$; in this case, the solution is necessarily sign-changing.

We prove the above results by a variational approach, by looking at the solutions of (1.1) as critical points of a functional E_λ on $H^1(\Omega)$. As we remarked before, the first step of the proof is the study of a linear Steklov eigenvalue problem with indefinite weight on the boundary (section 2); then, by the properties of the eigenfunctions, we define suitable linking manifolds in $H^1(\Omega)$ and obtain crucial estimates for E_λ on these manifolds in both the cases $\int_{\partial\Omega} \mu > 0$ and $\int_{\partial\Omega} \mu < 0$ (section 3); finally, we prove that the functional E_λ satisfies the Palais-Smale condition with two different kind of assumptions on the functions μ and h (section 4). The main theorem is stated in the last section, together with a discussion of possible developments (positivity, multiplicity, etc.) and open problems .

2 The linear eigenvalue problem

In this section, we first consider the linear eigenvalue problem related to (1.1) in a variational setting; then, we will discuss the regularity of solutions. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain; we investigate the existence of non trivial solutions in $H^1(\Omega)$ to the problem:

$$\begin{aligned} \Delta u(x) &= 0 \quad \text{in } \Omega \\ \gamma(\partial_\nu u)(x) &= \lambda\, \mu(x)\gamma(u)(x) \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

where $\lambda \in \mathbb{R}\backslash\{0\}$, $\mu(x) \in L^\infty(\partial\Omega)$ and γ denotes the trace operator on $\partial\Omega$.

We recall that, for a Lipschitz domain Ω, the trace on $\partial\Omega$ of the normal derivative of a $H^1(\Omega)$ function satisfying $\Delta u \in L^2(\Omega)$ (in the weak sense) is well defined as an element of the Sobolev space $H^{-1/2}(\partial\Omega)$.

The main result of this section is:

Theorem 2.1. *Assume that*

$$\int_{\partial\Omega} \mu \neq 0. \tag{2.2}$$

Then, problem (2.1) has infinitely many eigenvalues λ_n with $|\lambda_n| \to +\infty$.

2.1 Preliminary results

We define the following subspace

$$H^1_\mu \equiv \Big\{ u \in H^1(\Omega), \quad \int_{\partial\Omega} \mu\,\gamma(u) = 0 \Big\}. \tag{2.3}$$

By the continuity of the linear functional

$$u \mapsto \int_{\partial\Omega} \mu\,\gamma(u)$$

on $H^1(\Omega)$, it follows that H^1_μ is a closed subspace of $H^1(\Omega)$.

Lemma 2.2. *Problem* (2.1) *is equivalent to the variational problem : find* $u \in H^1_\mu$ *such that*

$$\int_\Omega \nabla u\, \nabla v = \lambda \int_\Omega \mu\gamma(u)\gamma(v) \tag{2.4}$$

holds for every $v \in H^1_\mu$.

Proof. We first note that every solution u of problem (2.1) must satisfy $\int_{\partial\Omega} \mu\gamma(u) = 0$, that is $u \in H^1_\mu$; moreover, by applying the first Green's formula (which holds in a Lipschitz domain Ω for u, v in $H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$) to the relation

$$0 = \int_\Omega -\Delta u\, v,$$

and using the boundary condition, we readily get (2.4).
Conversely, let (2.4) holds; by taking v in the subspace of the smooth functions with support in Ω we first obtain that $\Delta u = 0$. Moreover, by applying again Green's formula to the left hand side of (2.4), we get

$$\int_{\partial\Omega} \gamma(\partial_\nu u)\, \gamma(v) = \int_{\partial\Omega} \lambda\, \mu\, \gamma(u)\, \gamma(v)$$

for every $v \in H^1_\mu$. By the definition of H^1_μ, this implies

$$\gamma(\partial_\nu u) - \lambda\, \mu\, \gamma(u) = c\mu$$

for some $c \in \mathbb{R}$. But the integral on $\partial\Omega$ of the left side vanishes, so that, by condition (2.2), $c = 0$. □

Lemma 2.3. *The Dirichlet norm*

$$\|u\|^2_D = \int_\Omega |\nabla u|^2 \tag{2.5}$$

is a norm in H^1_μ *equivalent to the* H^1 *norm* $\|u\|_{H^1(\Omega)}$.

Proof.
By a classical *reductio ad absurdum* argument, we will show that there is a constant C such that

$$\|u\|^2_{L^2(\Omega)} \le C\|u\|^2_D, \tag{2.6}$$

for every $u \in H^1_\mu$.
If not, one can find a sequence $u_n \in H^1_\mu$ such that

$$\|u_n\|^2_{L^2(\Omega)} \geq n\,\|u_n\|^2_D \tag{2.7}$$

By omogeneity, we may normalize $\|u_n\|_{L^2(\Omega)} = 1$. Then, $\|u_n\|_{H^1(\Omega)}$ is bounded and we may assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$; by the compactness of the immersions, we also have $u_n \to u$ in $L^2(\Omega)$ and $\gamma(u_n) \to \gamma(u)$ in $L^2(\partial\Omega)$. But from (2.7) we also have $|\nabla u_n| \to 0$ in $L^2(\Omega)$; hence u satisfies $\nabla u = 0$ and therefore $u = c$ costant in Ω.
However, we also have

$$0 = \int_{\partial\Omega} \mu\gamma(u_n) \to c\int_{\partial\Omega} \mu,$$

so that, by (2.2) $c = 0$. But this contradicts $\|u\|_{L^2(\Omega)} = 1$ that follows from $u_n \to u$ in $L^2(\Omega)$. Hence, (2.6) holds. □

Remark 2.4. *By the previous lemma, it follows that the expression*

$$\|u\|_1^2 = \int_\Omega |\nabla u|^2 + \Big(\int_{\partial\Omega} \mu\gamma(u)\Big)^2, \tag{2.8}$$

defines an equivalent norm in $H^1(\Omega)$.

Finally we define the Hilbert space

$$H_\mu^\perp \equiv \Big\{u \in H_\mu^1, \quad \int_\Omega \nabla u\,\nabla v = 0 \quad \forall\, v \in H_0^1(\Omega)\Big\} \tag{2.9}$$

Lemma 2.5. *A function $u \in H_\mu^1$ is a variational solution of the Steklov problem if and only if $u \in H_\mu^\perp$ and* (2.4) *holds for every $v \in H_\mu^\perp$. Moreover, by denoting with $H_\mu^{1/2}(\partial\Omega)$ the subspace of the functions $w \in H^{1/2}(\partial\Omega)$ satisfying $\int_{\partial\Omega} \mu\, w = 0$, the trace operator γ restricted to $H_\mu^\perp$ is an isomorphism between $H_\mu^\perp$ and $H_\mu^{1/2}(\partial\Omega)$.*

Proof. By definition, the functions in $H_\mu^\perp$ are weakly harmonic, hence they satisfy (2.4) for every $v \in H_0^1(\Omega)$; conversely, every solution of (2.4) belongs to $H_\mu^\perp$. Since any $v \in H_\mu^1$ has a unique decomposition $v = v_0 + v_1$ with $v_0 \in H_0^1(\Omega)$ and $v_1 \in H_\mu^\perp$, we are reduced to solve the variational problem in the latter space.
Finally, it is well known that the trace operator γ is continuous from $H^1(\Omega)$ *onto* $H^{1/2}(\partial\Omega)$. Then, by definition (2.3), γ is also continuous from $H_\mu^1(\Omega)$ onto $H_\mu^{1/2}(\partial\Omega)$. But $\operatorname{Ker}\gamma = H_0^1(\Omega)$, so that γ is one-to-one from $H_\mu^\perp$ onto $H_\mu^{1/2}(\partial\Omega)$. Then, the Lemma follows by the bounded inverse theorem.

□

2.2 Proof of theorem 2.1

By the results of the previous section, problem (2.1) is equivalent to the variational problem:
find $u \in H_\mu^\perp$ satisfying

$$\int_\Omega \nabla u\,\nabla v = \lambda \int_{\partial\Omega} \mu\gamma(u)\gamma(v) \tag{2.10}$$

for every $v \in H_\mu^\perp$.
Let us denote by $L_\mu^2(\partial\Omega)$ the space of the functions in $L^2(\partial\Omega)$ orthogonal to μ and introduce the following operators:

$$I: \quad H_\mu^{1/2}(\partial\Omega) \to L_\mu^2(\partial\Omega), \qquad I\,w = w,$$

which is compact, by Sobolev imbeddings.

$$M: \quad L^2_\mu(\partial\Omega) \to L^2(\partial\Omega), \qquad (M\,w)(x) = \mu(x)\,w(x),$$

that is the bounded, self-adjoint multiplication operator by the bounded function μ.

$$J: \quad L^2(\partial\Omega) \to H^{-1/2}_\mu(\partial\Omega), \qquad {}_{H^{-1/2}}< J\,w, z >_{H^{1/2}} = \int_{\partial\Omega} w\,z,$$

for every $w \in L^2(\partial\Omega)$ and $z \in H^{1/2}_\mu(\partial\Omega)$; J is the immersion of L^2 in the dual space of $H^{1/2}_\mu$ and it is also bounded.
Finally, let

$$L: \quad H^\perp_\mu \to (H^\perp_\mu)^*$$

be the Riesz isomorphism defined by

$${}_{(H^1)^*}< L\,u, v >_{H^1} = \int_\Omega \nabla u\,\nabla v,$$

for every u, $v \in H^\perp_\mu$.
Then, the operator

$$K: \quad H^\perp_\mu \to H^\perp_\mu, \qquad K = L^{-1}\gamma^*\,J\,M\,I\,\gamma \tag{2.11}$$

is compact, being the product of bounded operators with the compact imbedding I; furthermore, K is self-adjoint since, by the previous definitions, we have:

$$\int_\Omega \nabla(Ku)\,\nabla v = {}_{(H^1)^*}< \gamma^*\,J\,M\,I\,\gamma(u), v >_{H^1} = {}_{H^{-1/2}}< J\,M\,I\,\gamma(u), \gamma(v) >_{H^{1/2}}$$

$$= \int_{\partial\Omega} \mu\,\gamma(u)\gamma(v) = \int_{\partial\Omega} \mu\,\gamma(v)\gamma(u) = \int_\Omega \nabla(Kv)\,\nabla u.$$

Then K has a complete set of eigenfunctions corresponding to real eigenvalues μ_n, with $\mu_n \to 0$. Note that K has the zero eigenvalue if and only if there exists a non trivial $u \in H^\perp_\mu$ such that

$$\int_{\partial\Omega} \mu\,\gamma(u)\gamma(v) = 0$$

for every $v \in H^\perp_\mu$; by definition (2.3) and again by condition (2.2), this may happen only if $\mu\gamma(u) = c\mu = 0$, that is if the function $\mu(x)$ vanishes on a set of positive Hausdorff measure in $\partial\Omega$. Otherwise, all the eigenvalues are not vanishing. In any case, since by condition (2.2) $\mu \neq 0$ on a set of positive measure, the range of K has infinite dimension and therefore, by compactness, there are infinite non zero eigenvalues (of finite multiplicity). Let u_n, $n \neq 0$, be an eigenfunction corresponding to an eigenvalue $\mu_n \neq 0$; from $K\,u_n = \mu_n\,u_n$ and by the previous relations we have

$$\mu_n \int_\Omega \nabla u_n\,\nabla v = \int_{\partial\Omega} \mu\,\gamma(u_n)\gamma(v),$$

for every $v \in H^\perp_\mu$. Hence u_n solves (2.10) with $\lambda_n = 1/\mu_n$. Finally, as $\mu_n \to 0$, we have $|\lambda_n| \to +\infty$.
In the sequel, we will list all the eigenvalues to problem (2.1) as follows

$$...\lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2...$$

The eigenvalue $\lambda_0 = 0$ corresponds to the constant solutions of the homogeneous Neumann problem. Note that from the relations

$$\int_\Omega |\nabla u_n|^2 = \lambda_n \int_{\partial\Omega} \mu\,\gamma(u_n)^2,$$

we get the inequalities

$$\int_{\partial\Omega} \mu\,\gamma(u_n)^2 > 0, \quad \text{for } n > 0; \qquad \int_{\partial\Omega} \mu\,\gamma(u_n)^2 < 0, \quad \text{for } n < 0. \tag{2.12}$$

Moreover, we can take all the u_n orthogonal and normalized with respect to the scalar product associated to the Dirichlet norm (2.5) and even to the equivalent norm (2.8) by defining $u_0 = \big(\int_{\partial\Omega} \mu\big)^{-1}$; then, we have

$$\int_\Omega \nabla u_n \,\nabla u_m = \int_{\partial\Omega} \mu\,\gamma(u_n)\gamma(u_m) = 0, \tag{2.13}$$

for $n \neq m$.

Now, if we denote by V_μ the subspace spanned by the u_n, $n \in \mathbb{Z}\backslash\{0\}$, and by V_0 the one spanned by the *null eigenfunctions* w_m, $m = 1, 2, \ldots$ *of* K (recall that V_0 is non trivial only if there exists $w \neq 0$ such that $\mu\gamma(w) = 0$ a.e. on $\partial\Omega$) we have

$$H_\mu^\perp = V_\mu \oplus V_0.$$

Finally, it is not difficult to show that

$$H^1 = H_0^1 \oplus c \oplus H_\mu^\perp = H_0^1 \oplus c \oplus V_\mu \oplus V_0. \tag{2.14}$$

where the orthogonal decomposition refers to the scalar product associated to (2.8).

It follows in particular that the variational equation

$$\int_\Omega \nabla u_n \,\nabla v = \lambda_n \int_{\partial\Omega} \mu\,\gamma(u_n)\gamma(v) \tag{2.15}$$

is satisfied for any $n \in \mathbb{Z}$, *and for every* $v \in H^1(\Omega)$.

Remark 2.6. *We stress that the condition* (2.2) *is not necessary for the existence of non zero eigenvalues to problem* (2.1)*, as it is shown by the following example:*
Let Ω *be the rectangle* $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [a, b]$*,* $a < b$*. The harmonic functions* $u_m(x, y) = e^{\lambda_m y}\big(\sin(\lambda_m x) + \cos(\lambda_m x)\big)$*, where* $\lambda_m = 2m + 1$*,* $m \in \mathbb{Z}$*, solve problem* (2.1) *with* $\lambda = \lambda_m$*,* $\mu = 1$ *on the left and upper sides of* Ω*,* $\mu = -1$ *on the other sides.*
Note however that by (2.2) *we got the orthogonal decomposition* (2.14)*, which will be crucial for the estimates of the next section.*

Remark on the regularity of eigenfunctions.

Global regularity of the eigenfunctions of (2.1) depends on the indefinite weight μ and on the regularity of the boundary $\partial\Omega$. For the subsequent discussion of the nonlinear problem, it is important to guarantee that the eigenfunctions u_m are bounded. Recall that any solution of (2.15) belongs to $H^1(\Omega)$ and is harmonic in Ω; hence the trace of its normal derivative, being proportional to $\mu\gamma(u)$, belongs to $L^2(\partial\Omega)$, so that we have $u_m \in H^{3/2}(\Omega)$ even in a Lipschitz domain [11]. Then, in case of dimension $n = 2$ we get $u_m \in \mathcal{C}(\overline{\Omega})$ (by Sobolev imbedding) without any additional assumption. For $n \geq 3$, more regularity of μ and of the boundary $\partial\Omega$ will be required in order to achieve $u_m \in H^s(\Omega)$ with $s > n/2$, which implies the continuity of u_m up to the boundary. For the sake of brevity, we assume in the following that μ and $\partial\Omega$ are smooth enough to satisfy such conditions without entering into further details.

3 Linking manifolds

By the results of the previous sections, we will discuss the solvability of the non linear problem 1.1 by assuming the condition (2.2); since the function μ is indefinite, we may restrict to the case $\lambda > 0$ and consider both the cases $\int_{\partial\Omega} \mu > 0$ and $\int_{\partial\Omega} \mu < 0$. We point out that for positive λ the latter case is a necessary condition for the existence of positive solutions; in fact, if $u > 0$ solves (1.1) and if $1/h(u)$ is integrable at infinity, the function

$$v = \int_u^{+\infty} \frac{ds}{h(s)}$$

satisfies $\Delta v = \frac{h'(u)}{h(u)^2}|\nabla u|^2 > 0$ in Ω and $\partial_\nu v = -\lambda\mu$ on $\partial\Omega$. By the divergence theorem, it follows that the two conditions on v imply

$$\int_{\partial\Omega} \mu < 0, \tag{3.1}$$

In any case, we will assume that the function h in (1.1) is *strictly increasing*, so that the derivative h' has a positive global minimum h'_m. For the sake of simplicity in the estimates below, we also assume that $h'_m = h'(0)$ (this holds, e.g., for $h(u) = \sinh u$); at the end of the section, we will show how the discussion changes in the general case.
By rescaling the parameter λ we can now take

$$h'(u) \geq h'(0) = 1. \tag{3.2}$$

Let us define $H(u) = \int_0^u h(t)dt$ (note that $H \geq 0$) and consider the functional

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} \mu H(u) \tag{3.3}$$

where $u \in H^1_\mu$. We will prove that problem (1.1) has a solution by applying a *linking argument* to the functional E_λ. To begin with, we assume the condition (3.1) and suppose further that

$$\lambda_k < \lambda < \lambda_{k+1} \tag{3.4}$$

where λ_k, $k > 0$ are positive eigenvalues of (2.1).
We now set $H^1(\Omega) = V_1 \oplus V_2$, where

$$V_2 = \text{span}_{1\leq j\leq k} \{u_j\} \tag{3.5}$$

Then, comparing with the decomposition (2.14), we have

$$V_1 = H^1_0 \oplus V_0 \oplus \text{span}_{n\leq 0} \{u_n\} \oplus \text{span}_{n\geq k+1} \{u_n\} \tag{3.6}$$

Let us fix positive R, R_1, R_2 and define the following subsets

$$S = \{v \in V_1 : \ \|v\|_1 = R\} \tag{3.7}$$

$$Q_n = \{su_n + u, \ n > k, \ u \in V_2, \ 0 \leq s \leq R_1, \ \|u\|_1 \leq R_2\} \tag{3.8}$$

It is known that S and ∂Q_n link (see [6]); we need some estimates for the functional E_λ on these subsets. Recalling that $h(0) = 0$ and $h'(0) = 1$, we make the following assumption on the nonquadratic term of the functional:

Let $0 < \epsilon < \frac{2}{n-2}$ *(*$\epsilon > 0$ *for* $n = 2$*) and* $q \geq \frac{2(n-1)}{2-\epsilon(n-2)}$ *(any* $q > 1$ *for* $n = 2$*) and suppose that*

$$\Big|H(u) - \frac{1}{2}\,u^2\,\Big| \leq |u|^{2+\epsilon}\tilde{H}(u), \tag{3.9}$$

where $\tilde{H} : H^1(\Omega) \to L^q(\partial\Omega)$ *is bounded.* For $n \geq 3$, it is readily verified that the above condition holds if there is $C > 0$ such that $h'(u) \leq 1 + C|u|^\epsilon$; in the case $n = 2$, it can be shown ([5], lemma 2.1) that the function $H(u) = \cosh u - 1$ also satisfies the assumption; the same is true for the primitive function of the more general nonlinear term (1.2). Then, we have

Lemma 3.1. *Assuming conditions* (3.1), (3.2), (3.4), *and* (3.9), *there exists* $R > 0$ *such that* $E_\lambda(v) \geq a > 0$, *for every* $v \in S$.

Proof. By (3.9) we have

$$E_\lambda(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega}\mu u^2 - \lambda\|\mu\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}|u|^{2+\epsilon}\tilde{H}(u), \tag{3.10}$$

and the integral in the last term can be bounded as follows

$$\Big|\int_{\partial\Omega} u^{2+\epsilon}\tilde{H}(u)\Big| \leq \|\tilde{H}(u)\|_{L^q(\partial\Omega)}\Big(\int_{\partial\Omega}|u|^{(2+\epsilon)p}\Big)^{1/p} \leq C\,\|u\|_1^{2+\epsilon} = C\,R^{2+\epsilon}, \tag{3.11}$$

where the last estimate follows by $(2+\epsilon)p = (2+\epsilon)\frac{q}{q-1} \leq \frac{2(n-1)}{n-2}$.
Let us decompose an element $u \in S$ as $u = c \oplus \tilde{u}$; by the definition (2.8) of the equivalent norm we have

$$\int_\Omega |\nabla\tilde{u}|^2 + c^2\Big(\int_{\partial\Omega}\mu\Big)^2 = R^2.$$

Now, by the decomposition (3.6) and recalling the second of inequalities (2.12) and the definition of the subspace V_0, we get the following estimate of the quadratic part of the functional

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega}\mu u^2 = \frac{1}{2}\int_\Omega|\nabla\tilde{u}|^2 - \frac{\lambda}{2}\int_{\partial\Omega}\mu\tilde{u}^2 - \frac{\lambda}{2}c^2\int_{\partial\Omega}\mu$$

$$\geq \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_{k+1}}\Big)\int_\Omega|\nabla\tilde{u}|^2 - \frac{\lambda}{2}c^2\int_{\partial\Omega}\mu \geq \frac{1}{2}\min\Big[(1-\lambda/\lambda_{k+1}),\,\lambda\big(-\int_{\partial\Omega}\mu\big)^{-1}\Big]R^2 \tag{3.12}$$

for every $u \in S$. Then, the lemma follows by the assumption (3.1) and taking R small enough.

□

Next we prove that the functional E_λ is non positive on the subspace V_2 given by (3.5).

Proposition 3.2. *With the same assumptions of lemma 3.1, we have* $E_\lambda(u) < 0$ *for every non zero* $u \in V_2$*; moreover,* $E_\lambda(u) \to -\infty$ *for* $\|u\|_1 \to \infty$.

Proof. As in the proof of the previous lemma, one can show that $E_\lambda(u) < 0$ for every non vanishing $u \in V_2$ with small enough norm; for any $u \in V_2$ we can write

$$u = \sum_{i=1}^k t_i u_i$$

and consider the function

$$f(t_1, ..., t_k) \equiv E_\lambda(u) = \sum_{i=1}^k \frac{t_i^2}{2}\int_\Omega|\nabla u_i|^2 - \lambda\int_{\partial\Omega}\mu H(u) = \frac{1}{2}\sum_{i=1}^k t_i^2 - \lambda\int_{\partial\Omega}\mu H(u), \tag{3.13}$$

where the last equality follows from

$$\int_\Omega |\nabla u_i|^2 = \|u_i\|_1^2 = 1.$$

We already know that $f < 0$ in a neighborhood of the origin; we will prove that the inequality holds in the whole space $\mathbb{R}^k \backslash \{\mathbf{0}\}$.
Let $1 \leq j \leq k$ and consider the variational equation

$$\int_\Omega \nabla u_j \, \nabla h(u) = \lambda_j \int_{\partial\Omega} \mu \, u_j \, h(u) \tag{3.14}$$

where $h(u) = h(\sum_{i=1}^k t_i u_i)$; we stress that, by the regularity results of the previous section, u is a bounded continuous function on Ω, so that $h(u) \in H^1(\Omega)$ and (3.14) holds by the discussion at the end of section 2. Then, we get

$$\sum_{i=1}^k t_i \int_\Omega \nabla u_j \, \nabla u_i h'(u) = \lambda_j \int_{\partial\Omega} \mu \, u_j \, h(u).$$

Multiplying by t_j and summing up from $j = 1$ to k, we find

$$\sum_{i,j=1}^k t_i t_j \int_\Omega \nabla u_j \, \nabla u_i h'(u) = \sum_{j=1}^k \lambda_j t_j \int_{\partial\Omega} \mu \, u_j \, h(u),$$

that is

$$\int_\Omega |\nabla u|^2 h'(u) = \sum_{j=1}^k \lambda_j t_j \int_{\partial\Omega} \mu \, u_j \, h(u). \tag{3.15}$$

Let us now calculate

$$\Big(\sum_{j=1}^k \lambda_j t_j \partial_{t_j}\Big) f(t_0, t_1, ..., t_k) = \sum_{j=1}^k \lambda_j t_j^2 - \lambda \sum_{j=1}^k \lambda_j t_j \int_{\partial\Omega} \mu \, u_j \, h(u) =$$

(by (3.15))

$$= \sum_{j=1}^k \lambda_j t_j^2 - \lambda \int_\Omega |\nabla u|^2 h'(u) \leq \lambda_k \sum_{j=1}^k t_j^2 - \lambda \int_\Omega |\nabla u|^2 h'(u) = \int_\Omega |\nabla u|^2 \Big[\lambda_k - \lambda \, h'(u)\Big] \tag{3.16}$$

Then, recalling that $\lambda > \lambda_k$ and $h' \geq 1$, we obtain

$$\Big(\sum_{j=1}^k \lambda_j t_j \partial_{t_j}\Big) f(t_1, ..., t_k) < -(\lambda - \lambda_k)(t_1^2 + ... + t_k^2) < 0, \tag{3.17}$$

for $(t_1, ..., t_k) \neq (0, ..., 0)$.
Now, the right hand side of (3.17) is (proportional to) the derivative of the function f in the direction of the vector $(\lambda_1 t_1, ..., \lambda_k t_k)$; this vector is normal to the hypersurface in $\mathbb{R}^k$ of equation $\lambda_1 x_1^2 + ... + \lambda_k x_k^2 = c$ at the point $(t_1, ..., t_k)$. We conclude that the function is strictly decreasing along the orthogonal curve $x_1 = t_1 e^{\lambda_1 s}, ..., x_k = t_k e^{\lambda_k s}$, $s \in \mathbb{R}$, passing through the point $(t_1, ..., t_k)$; since the origin is an unstable node for all the orbits, we conclude that $f < 0$ in $\mathbb{R}^k \backslash \{\mathbf{0}\}$. Moreover, by (3.17) we also have $\lim_{\|u\|_1 \to \infty} E_\lambda(u) = -\infty$ in V_2. □

By the above proposition, we may prove a crucial estimate on the boundary ∂Q_n of the set (3.8).

Lemma 3.3. *With the same assumptions of lemma 3.1, one can choose* R_1*,* R_2 *in* (3.8) *such that*

$$\sup_{u\in\partial Q_n} E_\lambda(u) = 0. \tag{3.18}$$

Proof. We first evaluate the derivative of the functional E_λ along the ray tu_n, $t \geq 0$:

$$\partial_t E_\lambda(tu_n) = t\int_\Omega |\nabla u_n|^2 - \lambda\int_{\partial\Omega} \mu u_n h(tu_n)$$

From the variational equation

$$\int_\Omega \nabla u_n \nabla h(tu_n) = \lambda_n \int_{\partial\Omega} \mu u_n h(tu_n), \tag{3.19}$$

(see the proof of proposition 3.2) we get

$$\partial_t E_\lambda(tu_n) = t\Big(\int_\Omega |\nabla u_n|^2\big[1 - \frac{\lambda}{\lambda_n} h'(tu_n)\big]\Big) \tag{3.20}$$

Since $h'(s) \to +\infty$ for $|s| \to +\infty$, the expression in the square brackets is positive only for $t|u_n| \leq K$, where K only depends on h' and λ/λ_n; hence, the integrand in (3.20) may be non negative only on the sublevel set $\Omega_t = \{x \in \Omega,\, |u(x)| \leq K/t\}$, where it is bounded by $|\nabla u_n|^2$. Since u_n is harmonic in Ω, we have $\lim_{t\to+\infty} |\Omega_t| = 0$; moreover, the integrand is negative and not identically zero on $\Omega\backslash\Omega_t$. It follows that $\partial_t E_\lambda(tu_n) \to -\infty$ for $t \to +\infty$. Then, $E_\lambda(tu_n)$ is non negative only on a *bounded* interval, $0 \leq t \leq b_n$ and therefore it is bounded from above by some positive constant.
Finally, consider $u = tu_n + \sum_{i=1}^k t_i u_i \in Q_n$ and set $f(t, t_1, ..., t_k) = E_\lambda(u)$. As in the proof of proposition 3.2, one can show that the function $(t_1, ..., t_k) \mapsto f(t, t_1, ..., t_k)$ decreases to $-\infty$ for $t_1^2 + ...t_k^2 \to +\infty$. Hence, by taking large enough R_1 and R_2 we find that $E_\lambda(u) \leq 0$ for $t = 0$, $t = R_1$ *or* for $\sqrt{t_1^2 + ...t_k^2} = R_2$, that is on the boundary ∂Q_n. □

By lemmas 3.1 and 3.3 we now get

$$\alpha = \inf_{u\in S} E_\lambda(u) > \sup_{u\in\partial Q_n} E_\lambda(u) = 0, \tag{3.21}$$

for suitably chosen R, R_1 and R_2 in (3.7) and (3.8).
We stress that the above estimate follows assuming (3.1), (3.4) and by the definitions (3.6), (3.5) of the subspaces V_1 and V_2. We claim that an analogous result holds when

$$\int_{\partial\Omega} \mu > 0, \tag{3.22}$$

and

$$\lambda_k < \lambda < \lambda_{k+1}, \qquad k \geq 0; \tag{3.23}$$

note that we now allow the case $0 < \lambda < \lambda_1$ which was excluded in (3.4).
We only have to change the decomposition of $H^1(\Omega)$ by shifting the constants from V_1 to V_2 and to define the subsets S and Q_n as before; thus, we now set $H^1(\Omega) = \tilde{V}_1 \oplus \tilde{V}_2$, where

$$\tilde{V}_2 = c \oplus V_2 = \mathrm{span}_{0\leq j\leq k}\{u_j\}, \tag{3.24}$$

$$\tilde{V}_1 = H_0^1 \oplus V_0 \oplus \mathrm{span}_{n<0}\{u_n\} \oplus \mathrm{span}_{n\geq k+1}\{u_n\} \tag{3.25}$$

Furthermore, for positive R, R_1, R_2 we set

$$\tilde{S} = \{v \in \tilde{V}_1 : \|v\|_1 = R\} \tag{3.26}$$

$$\tilde{Q}_n = \{su_n + u, \ n > k, \ u \in \tilde{V}_2, \ 0 \le s \le R_1, \ \|u\|_1 \le R_2\} \tag{3.27}$$

We first note that lemma 3.1 still holds for $\tilde{S}$ (the proof is even simpler, since the norm on $\tilde{V}_1$ is the Dirichlet norm). Furthermore, we can show that $E_\lambda(u) < 0$ for every non zero $u \in \tilde{V}_2$, with $E_\lambda(u) \to -\infty$ for $\|u\|_1 \to \infty$. In fact, by (3.22) we get

$$E_\lambda(c) = -\lambda H(c) \int_{\partial\Omega} \mu < 0 \tag{3.28}$$

for every $c \neq 0$. By continuity, we conclude that $E_\lambda(u) < 0$ for every $u \in \tilde{V}_2$ having projection on V_2 sufficiently close to zero. Then by writing any $u \in V_2$ in the form $u = \sum_{i=0}^{k} t_i u_i$ $(t_0 u_0 = c)$ and defining as in (3.13)

$$f(t_0, t_1, ..., t_k) \equiv E_\lambda(u) = \sum_{i=1}^{k} \frac{t_i^2}{2} \int_\Omega |\nabla u_i|^2 - \lambda \int_{\partial\Omega} \mu H(u) = \frac{1}{2} \sum_{i=1}^{k} t_i^2 - \lambda \int_{\partial\Omega} \mu H(u), \tag{3.29}$$

we obtain, by the same arguments as in proposition 3.2, that the function f is strictly decreasing along the curves $(t_0, t_1 e^{\lambda_1 s}, ..., t_k e^{\lambda_k s})$, $s \in \mathbb{R}$ for every $(t_0, t_1, ..., t_n) \in \mathbb{R}^{k+1}$; thus, $f < 0$ in $\mathbb{R}^{k+1} \backslash \{0\}$. Moreover, we also have $\lim_{\|u\|_1 \to \infty} E_\lambda(u) = -\infty$ in $\tilde{V}_2$. Thus, in order to prove our claim, we are left with the analogous of lemma 3.3:

Lemma 3.4. *If now* (3.22) *and* (3.23) *hold (instead of* (3.1) *and* (3.4)*) one can choose* R_1, R_2 *in* (3.27) *such that*

$$\sup_{u \in \partial\tilde{Q}_n} E_\lambda(u) = 0. \tag{3.30}$$

Proof. We first estimate the functional E_λ on the two-dimensional subset (of $\tilde{Q}_n$) of the vectors $c + tu_n$, where $c = t_0 u_0 \in \mathbb{R}$ and $t \ge 0$. Then we can write

$$E_\lambda(c + tu_n) = \frac{t^2}{2} \int_\Omega |u_n|^2 - \lambda \int_{\partial\Omega} \mu H(c + tu_n) = \frac{t^2}{2} - \lambda H(c) \int_{\partial\Omega} \mu - \lambda t \int_0^1 d\tau \int_{\partial\Omega} \mu u_n h(c + \tau t u_n).$$

Now, from the variational equation

$$\int_\Omega \nabla u_n \nabla h(c + \tau t u_n) = \lambda_n \int_{\partial\Omega} \mu u_n h(c + \tau t u_n), \tag{3.31}$$

(see the proof of proposition 3.2) we get

$$\int_{\partial\Omega} \mu u_n h(c + \tau t u_n) = \frac{\tau t}{\lambda_n} \int_\Omega |\nabla u_n|^2 h'(c + \tau t u_n) \ge \frac{\tau t}{\lambda_n},$$

where the last bound follows from $h' \ge 1$. Thus, we have the estimate

$$E_\lambda(c + tu_n) \le \frac{t^2}{2}\Big(1 - \frac{\lambda}{\lambda_n}\Big) - \lambda H(c) \int_{\partial\Omega} \mu \tag{3.32}$$

From this estimate we conclude that $E_\lambda(c + tu_n) \le 0$ for

$$0 \le t \le C_n(\lambda, \mu) H(c)^{1/2}, \tag{3.33}$$

where $C_n(\lambda,\mu) = \left(\frac{2\lambda\lambda_n \int_{\partial\Omega}\mu}{\lambda_n-\lambda}\right)^{1/2}$. Note that in the domain where $t \geq C_n H(c)^{1/2}$ the ratio $H(c)^{1/2}/t$ is bounded and therefore, by the growth assumption on H at infinity, $c/t \to 0$ for $t \to +\infty$ in that region. Let us now calculate the derivative

$$\partial_t E_\lambda(c+tu_n) = t\int_\Omega |u_n|^2 - \lambda\int_{\partial\Omega} \mu u_n h(c+tu_n)$$

By using again the variational equation (3.31) (with $\tau = 1$) we now get

$$\partial_t E_\lambda(c+tu_n) = t\Big(\int_\Omega |u_n|^2\big[1 - \frac{\lambda}{\lambda_n}h'(c+tu_n)\big]\Big) \tag{3.34}$$

Since $h'(s) \to +\infty$ for $|s| \to +\infty$, the expression in the square brackets is positive for $|c+tu_n| \leq K$, where K only depends on h' and λ/λ_n. Summing up the previous discussion, for (c,t) outside the region (3.33) the integrand in (3.34) may be non negative only if $|u_n(x)| = \mathcal{O}(1/t)$. As in the proof of lemma 3.3, we conclude that $E_\lambda(c+tu_n) \geq 0$ only on a *bounded* domain of the plane (c,t), contained in the region (3.33); hence, $E_\lambda(c+tu_n) \leq M$ in the half-plane $t \geq 0$ for some positive constant M.
Finally, by considering $u = tu_n + \sum_{i=0}^k t_i u_i \in \tilde{Q}_n$ and setting $f(t,t_0,t_1,...,t_k) = E_\lambda(u)$ we find as in lemma 3.3 that $f \leq 0$ for $t = R_1$ *or* for $\sqrt{t_0^2+t_1^2+...t_k^2} = R_2$ (that is on the boundary $\partial\tilde{Q}_n$) provided R_1 and R_2 are large enough. □

Thus, we conclude

$$\tilde{\alpha} = \inf_{u\in\tilde{S}} E_\lambda(u) > \sup_{u\in\partial\tilde{Q}_n} E_\lambda(u) = 0, \tag{3.35}$$

for suitably chosen R, R_1 and R_2 in (3.26) and (3.27).
We point out that the crucial estimates (3.21) and (3.35) have been obtained (for λ respectively in the intervals (3.4) and (3.23)) under the assumption (3.2) that the derivative of the non linear term assumes the global minimum ($=1$) precisely at $u=0$. Note that this is not true for the Butler-Volmer condition (1.2) whenever $\alpha \neq 1/2$.
Thus, by defining $h'_m \equiv \min_{u\in\mathbb{R}} h'(u)$, we now consider the case $0 < h'_m < 1$, still normalizing $h'(0) = 1$.

Proposition 3.5. *Define the following sets*

$$I_k = \Big(\frac{\lambda_k}{h'_m}, \lambda_{k+1}\Big), \quad \text{for} \quad \frac{\lambda_k}{\lambda_{k+1}} < h'_m, \quad I_k = \emptyset \quad \text{otherwise}, \tag{3.36}$$

where λ_k, $k = 0,1,...$ *are the (non negative) eigenvalues of problem* (2.1).
We now have:
If $\int_{\partial\Omega}\mu < 0$, *then estimate* (3.21) *holds for* $\lambda \in I_k$, *with* $k \geq 1$.
If $\int_{\partial\Omega}\mu > 0$, *then estimate* (3.35) *holds for* $\lambda \in I_k$, *with* $k \geq 0$.

Proof. The proof closely follows the one given by assuming (3.2). The main difference concerns the estimates (3.16)-(3.17); actually, one can check that the conclusion of proposition 3.2 remains valid if $\lambda_k - \lambda h'_m < 0$. Then, the conclusions in the two cases follow as before. Note that $I_0 = (0,\lambda_1)$ is never the empty set. □

4 The Palais-Smale condition

By (3.21), (3.35), the existence of critical values at levels greater than α or $\tilde{\alpha}$ is assured if E_λ satisfies the Palais-Smale condition (see [6] thm. 8.4). The main difficulty in this task is to prove the boundedness

of a Palais-Smale sequence, since the usual inequality assumptions relating E_λ and E'_λ could not be helpful in presence of an indefinite weight; in addition, we would like to include the case of exponential growth of the nonlinear term for $n = 2$. We will obtain below the desired result with two different kind of hypotheses: in the first case, we assume that the functional H and its derivative h satisfy a kind of 'standard' inequality (see [6], theorems 6.2, 8.5) allowing an exponential growth at infinity of the non linear term, at the cost of introducing an additional assumption on the indefinite weight μ (which is however quite reasonable in the framework of corrosion modeling, see the remark below). In the second case, we make more specific requirements on the functional, which are similar to those previously considered for other problems with indefinite nonlinearities [8], [9].
Let us introduce the following decomposition of the boundary $\partial\Omega$:

$$\partial\Omega = \Gamma_+ \cup \Gamma_- \cup \Gamma_0, \tag{4.1}$$

where $\mu > 0$ on Γ_+, $\mu < 0$ on Γ_- and $\mu = 0$ on Γ_0; recall that $|\Gamma_\pm| > 0$, while Γ_0 may have vanishing measure. We further define $\mu_\pm = \max\{\pm\mu, 0\}$. Finally, we recall that the functional E_λ is given by $E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} \mu H(u)$, with $H' = h$ strictly increasing and superlinear at infinity.
Let us now state our first set of assumptions:
Condition PS1.

1. $$\overline{\Gamma}_+ \cap \overline{\Gamma}_- = \emptyset;$$

2. $$qH(u) \leq uh(u) + Au^2 + B, \tag{4.2}$$

   for some constant $q > 2$, $A \geq 0$, $B \geq 0$.

We can now state:

Proposition 4.1. *Let $u_m \in H^1(\Omega)$ be a sequence such that $E_\lambda(u_m) \to c$ and $E'_\lambda(u_m) \to 0$ in $H^1(\Omega)'$. Assume that (PS1) holds.*
Then, the sequence $\|u_m\|$ is bounded.
Suppose further that h has subcritical growth at infinity; more precisely, $|h(u)| \leq Ce^{\alpha|u|}$ for some $\alpha \in \mathbb{R}$ if $n = 2$ (see [5], lemmas 2.1 and 2.2) and $|h(u)| \leq C(1 + |u|^\beta)$ (with $\beta < \frac{n}{n-2}$) if $n \geq 3$.
Then, the functional (3.3) *satisfies the Palais-Smale condition.*

Remark 4.2. *We stress that condition* 2 *is satisfied by the functions $H(u) = \frac{1}{\beta+1}|u|^{\beta+1} + G(u)$ where $|G(u)| \leq |u|^\gamma$, $1 < \gamma \leq 2$. In fact, in this case the estimate* (4.2) *holds with $q = \beta + 1 > 2$ and suitably chosen A, B. Moreover, in the case $n = 2$, it is readily verified that $H(u) = \cosh u - 1$ also satisfies* (4.2) *with $q = 4$, $A = 1$ and $B = 0$. We point out that in the framework of corrosion modeling, condition* 1 *corresponds to the requirement that the boundary is made of conducting pieces separated by insulating parts (see the introduction).*

Proof of Proposition 4.1. Assume by contradiction (considering a subsequence if necessary) that $\|u_m\|_1 \to +\infty$ and define $v_m = t_m^{-1}u_m$, where $t_m = \|u_m\|_1$. Substituting in the condition $E'_\lambda(u_m)v = o(1)\|v\|_1$, $v \in H^1(\Omega)$, we get

$$\int_\Omega \nabla v_m \nabla v - \lambda \int_{\partial\Omega} \mu \frac{h(t_m v_m)}{t_m} v = o(1)\|v\|_1/t_m \tag{4.3}$$

Since v_m is bounded in $H^1(\Omega)$, there is a subsequence (still denoted by v_m) such that v_m converges strongly in $L^2(\Omega)$ and $v_m|_{\partial\Omega}$ converges strongly in $L^2(\partial\Omega)$. We claim that $v_m|_{\Gamma_\pm} \to 0$ a.e. If not, there exists $\delta > 0$ such that $|v_m| \geq \delta$ on a set of positive measure $\Gamma_\delta \subset \Gamma_\pm$; take now $v = \varphi_+ v_m$, where $0 \leq \varphi_+ \leq 1$ is a smooth function defined in $\overline{\Omega}$, vanishing on $\overline{\Gamma}_-$ and such that $\varphi_+|_{\Gamma_+} = 1$ (such a function exists by assumption 1). From (4.3) we have

$$\int_\Omega [|\nabla v_m|^2 \varphi_+ + \nabla v_m \nabla \varphi_+ v_m] - \lambda \int_{\Gamma_+} \mu_+ \frac{h(t_m v_m)}{t_m} v_m = o(1)\|\varphi_+ v_m\|_1 / t_m, \tag{4.4}$$

where the first term at the left hand side is uniformly bounded. On the other hand

$$\int_{\Gamma_+} \mu_+ \frac{h(t_m v_m)}{t_m} v_m \geq \int_{\Gamma_\delta} \mu_+ \frac{h(t_m v_m)}{t_m} v_m \geq \delta \frac{h(t_m \delta)}{t_m} \int_{\Gamma_\delta} \mu \to +\infty$$

for $t_m \to \infty$, thus contradicting (4.4). We may repeat the previous considerations by choosing a function φ_- vanishing on $\overline{\Gamma}_+$ and equal to 1 on Γ_-; then, the claim is proved.
Thus, we have $v_m \rightharpoonup w$ weakly in $H^1(\Omega)$, with w *harmonic function* in Ω satisfying $w|_{\Gamma_\pm} = 0$. Moreover, by choosing v in (4.3) with supp $v|_{\partial\Omega} \subset \Gamma_0$ and taking again the limit for $m \to \infty$ we also find (in a weak sense) $\partial_\nu w|_{\Gamma_0} = 0$. We conclude that $w = 0$, so that $v_m \to 0$ in $L^2(\Omega)$.
Then, we obtain from (4.4) (and from the corresponding formula with φ_-)

$$\int_\Omega |\nabla v_m|^2 \varphi_+ - \lambda \int_{\Gamma_+} \mu_+ \frac{h(t_m v_m)}{t_m} v_m = o(1), \tag{4.5}$$

$$\int_\Omega |\nabla v_m|^2 \varphi_- + \lambda \int_{\Gamma_-} \mu_- \frac{h(t_m v_m)}{t_m} v_m = o(1). \tag{4.6}$$

We readily get that both the terms at the left side of the last equation goes to zero for $m \to \infty$. On the other hand, from $E_\lambda(u_m) \to c$ we also get

$$\frac{1}{2} \int_\Omega |\nabla v_m|^2 - \lambda \int_{\Gamma_+} \mu_+ \frac{H(t_m v_m)}{t_m^2} + \lambda \int_{\Gamma_-} \mu_- \frac{H(t_m v_m)}{t_m^2} = \mathcal{O}(1/t_m^2). \tag{4.7}$$

Note that, by assumption 2 and by $v_m \to 0$ in $L^2(\Gamma_-)$, also the last integral at the left side is infinitesimal. Then, by comparison of (4.5) and (4.7) we find

$$0 \leq \int_\Omega |\nabla v_m|^2 (1 - \varphi_+) = \lambda \int_{\Gamma_+} \mu_+ \Big[\frac{2H(t_m v_m)}{t_m^2} - \frac{h(t_m v_m)}{t_m} v_m \Big] + o(1) \leq$$

(again by 2 and $v_m \to 0$ in $L^2(\Gamma_+)$)

$$\leq -(q-2) \int_{\Gamma_+} \mu_+ \frac{H(t_m v_m)}{t_m^2} + o(1). \tag{4.8}$$

By the above relation and again by (4.7) we finally have $\|\nabla v_m\|_{L^2(\Omega)} \to 0$ and therefore $v_m \to 0$ in $H^1(\Omega)$, thus contradicting $\|v_m\|_1 = 1$.
Thus, the sequence $\|u_m\|_1$ is bounded and in particular we have $u_m = c_m + \tilde{u}_m$ with $|c_m|$ bounded sequence and $\tilde{u}_m \in H^1_\mu(\Omega)$ (see definition (2.3)) such that $\|\tilde{u}_m\|_1$ $(= \|\nabla \tilde{u}_m\|_{L^2(\Omega)})$ is also bounded; by lemma 2.3 and the Lax-Milgram theorem, the linear map $L : H^1_\mu(\Omega) \to H^1(\Omega)'$ defined by $L(u)v = \int_\Omega \nabla u \nabla v$ is boundedly invertible.
Finally, by the growth assumptions on h, the operator defined through the bilinear form $\int_{\partial\Omega} \mu h(u) v$ maps bounded sets in $H^1(\Omega)$ to relatively compact sets in $H^1(\Omega)'$. By standard results [6], Prop.2.2, it follows that $\tilde{u}_m$ is relatively compact in $H^1_\mu(\Omega)$; then, the same holds for u_m in $H^1(\Omega)$. □

We now describe in detail the second kind of assumptions discussed above; recall that λ_1 and λ_{-1} denote respectively the lowest positive eigenvalue and the highest negative eigenvalue of the linear problem.

Condition PS2.

1. Assume $n \geq 3$ and

$$(\beta+1)H(u) - uh(u) = Au^2 + g(u), \quad 1 < \beta < n/(n-2) \quad \textit{and} \quad A \in \mathbb{R}, \tag{4.9}$$

   where $g(u) = o(u^2)$ for large u and $\int_{|u|\geq R} \frac{|g(u)|}{|u|^3} \leq \infty$ for every $R > 0$.

2. Either one of the following holds:

$$i) \qquad A\int_{\partial\Omega} \mu < 0 \qquad \text{and} \quad \lambda_{-1}\frac{\beta-1}{2} < \lambda A < \lambda_1 \frac{\beta-1}{2} \quad (\lambda > 0); \tag{4.10}$$

$$ii) \qquad A = 0 \qquad \text{and} \quad h'(u) \leq C(1+|u|^{\beta-1}) \tag{4.11}$$

   for some $C > 0$.

We now have:

Proposition 4.3. *Let $u_m \in H^1(\Omega)$ ($\Omega \subset \mathbb{R}^n$, $n \geq 3$) be a sequence such that $E_\lambda(u_m) \to c$ and $E'_\lambda(u_m) \to 0$ in $H^1(\Omega)'$. Assume that (PS2) holds.*
Then, the sequence $\|u_m\|_1$ is bounded and the functional (3.3) *satisfies the Palais-Smale condition.*

Proof. We first show that, assuming h superlinear, (4.9) implies that h is also subcritical. By defining

$$K(u) = H(u) - \frac{A}{\beta-1}u^2,$$

one readily finds that K solves the linear equation

$$K'(u) = \frac{\beta+1}{u}K(u) - \frac{g(u)}{u}.$$

Since h is superlinear, we have $K(u) \geq u^2$ for $|u| \geq R$ large enough; then, dividing the above equation by K, integrating for $u \geq R$ and taking the exponential of both members, we get

$$K_-(R)u^{\beta+1} \leq K(u) \leq K_+(R)u^{\beta+1}, \tag{4.12}$$

for $u \geq R$, where

$$K_\pm(R) = \frac{K(R)}{R^{\beta+1}} e^{\pm\int_R^{+\infty} \frac{|g(u)|}{u^3}}$$

An analogous estimate holds for $u \leq -R$.
Now let $A \neq 0$ and assume $2\,i)$ in $PS2$; by (4.9) we have

$$C + o(1)\|u_m\|_1 = (\beta+1)E_\lambda(u_m) - < u_m, E'_\lambda(u_m) >$$

$$= \frac{\beta-1}{2}\int_\Omega |\nabla u_m|^2 - \lambda\int_{\partial\Omega} \mu[(\beta+1)H(u_m) - u_m h(u_m)]$$

$$\geq \frac{\beta-1}{2}\int_\Omega |\nabla u_m|^2 - \lambda A\int_{\partial\Omega} \mu u_m^2 - C(\epsilon) - \epsilon\|u_m\|_1^2,$$

for any positive ϵ and suitably chosen $C(\epsilon) > 0$. Then, by writing again $u_m = \tilde{u}_m + c_m$, we conclude that for any $\epsilon > 0$ there exists a constant $K(\epsilon)$ such that

$$K(\epsilon) + o(1)\|u_m\|_{H^1} \geq \frac{\beta - 1}{2}\int_\Omega |\nabla \tilde{u}_m|^2 - \lambda A \int_{\partial\Omega} \mu \tilde{u}_m^2 - \lambda c_m^2 A \int_{\partial\Omega} \mu - \epsilon \|u_m\|_1^2 \tag{4.13}$$

Thus, by recalling (2.8), we get

$$K(\epsilon) + o(1)\|u_m\|_1 \geq \min\Big\{\frac{\beta - 1}{2} - \frac{\lambda}{\lambda_1} A - \epsilon; \lambda A\big(-\int_{\partial\Omega} \mu\big)^{-1} - \epsilon\Big\}\|u_m\|_1^2$$

if $A > 0$ and

$$K(\epsilon) + o(1)\|u_m\|_1 \geq \min\Big\{\frac{\beta - 1}{2} - \frac{\lambda}{\lambda_{-1}} A - \epsilon; \lambda A\big(-\int_{\partial\Omega} \mu\big)^{-1} - \epsilon\Big\}\|u_m\|_1^2$$

if $A < 0$. From the previous estimates the boundedness of a Palais-Smale sequence follows.
Let us now consider the case $2\,ii)$. By putting $A = 0$ in (4.13) and again by (2.8) we get

$$K(\epsilon) + o(1)\big[\|\nabla \tilde{u}_m\|_{L^2(\Omega)} + M|c_m|\big] + \epsilon M^2 c_m^2 \geq \Big(\frac{\beta - 1}{2} - \epsilon\Big)\|\nabla \tilde{u}_m\|_{L^2(\Omega)}^2,$$

where $M = \big|\int_{\partial\Omega} \mu\big|$. Note that if c_m is bounded, we readily get that the same is true for $\|\nabla \tilde{u}_m\|_{L^2(\Omega)}$ and therefore for $\|u_m\|_1$. Thus, we may assume that $c_m \to +\infty$; hence, for large enough m we may write

$$K(\epsilon) + o(1)\|\nabla \tilde{u}_m\|_{L^2(\Omega)} + 2\epsilon M^2 c_m^2 \geq \Big(\frac{\beta - 1}{2} - \epsilon\Big)\|\nabla \tilde{u}_m\|_{L^2(\Omega)}^2,$$

which also implies

$$K(\epsilon) + 3\epsilon M^2 c_m^2 \geq \Big(\frac{\beta - 1}{2} - 2\epsilon\Big)\|\nabla \tilde{u}_m\|_{L^2(\Omega)}^2,$$

From the last estimate it follows that for small positive ϵ there exist constants $B_1(\epsilon)$ and B_2 such that

$$\|\nabla \tilde{u}_m\|_{L^2(\Omega)} \leq B_1(\epsilon) + \epsilon^{1/2} B_2 |c_m|$$

that is,

$$\|\tilde{u}_m\|_1 \leq B_1(\epsilon) + \epsilon^{1/2} B_2 |c_m| \tag{4.14}$$

Consider again the condition $E'_\lambda(u_m)v \to 0$; by choosing $v = 1$ we obtain

$$\int_{\partial\Omega} \mu h(u_m) \to 0,$$

that is

$$h(c_m)\int_{\partial\Omega} \mu + \int_{\partial\Omega}\int_0^1 \mu h'(c_m + t\tilde{u}_m)\, \tilde{u}_m \, dt \to 0. \tag{4.15}$$

By the second assumption $ii)$, we have

$$|h'(c_m + t\tilde{u}_m)\, \tilde{u}_m| \leq \tilde{C}\big[(1 + |c_m|^{\beta-1})|\tilde{u}_m| + |\tilde{u}_m|^\beta\big]$$

for some positive constant $\tilde{C}$; hence, by (4.14) and by Sobolev immersion, it can be shown that for $|c_m|$ large enough

$$\Big|\int_{\partial\Omega}\int_0^1 \mu h'(c_m + t\tilde{u}_m)\, \tilde{u}_m \, dt\Big| \leq \tilde{B}_1(\epsilon)|c_m|^{\beta-1} + \sqrt{\epsilon}\tilde{B}_2 |c_m|^\beta$$

where $\tilde{B}_1(\epsilon)$, $\tilde{B}_2$ are suitable constants. We now observe that for $A = 0$ the bound (4.12) holds with $K(u) = H(u)$; then, by assumption (4.9) we obtain that $h(c_m)/c_m^\beta$ is bounded away from zero for $c_m \to +\infty$; hence, by taking ϵ small enough in the previous estimate, we contradict (4.15). Then, the sequence c_m must be bounded and the same holds for u_m.
Finally, by the previously established subcritical growth of h, the operator defined through the bilinear form $\int_{\partial\Omega} \mu h(u)v$ maps bounded sets in $H^1(\Omega)$ to relatively compact sets in $H^1(\Omega)'$, so that Palais-Smale condition again follows by [6], Prop. 2.2. □

5 Main theorem and final comments

In order to clearly state our existence results, we recall here the general assumptions about problem (1.1).

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open set with sufficiently regular boundary $\partial\Omega$ (see below);
- h is a strictly increasing $\mathcal{C}^1$ function, $h(0) = 0$; in addition h is superlinear and subcritical at infinity;
- μ is a bounded function satisfying $\int_{\partial\Omega} \mu \neq 0$ and with some additional regularity depending on dimension (see below);

A more specific assumption is needed in connection with the eigenfunctions of the linear problem (2.1):

- for $n \geq 3$ we suppose that $\partial\Omega$ and μ are regular enough to guarantee that the eigenfunctions u_k belong to $H^s(\Omega)$ with $s > n/2$, so that $u_k \in \mathcal{C}(\overline{\Omega})$.

We can now state our main result; recall that we may assume $\lambda > 0$ since μ is indefinite.

Theorem 5.1. *Let condition* (3.9) *hold and assume* $PS1$ *or* $PS2$*. Let* I_k *be defined by* (3.36).
Then, if $\int_{\partial\Omega} \mu \neq 0$ *there exists a solution* u *of problem* (1.1) *for every* $\lambda \in \cup_{k=1}^\infty I_k$*.*
Furthermore if $\int_{\partial\Omega} \mu > 0$ *there exists a solution* u *of problem* (1.1) *for* $0 < \lambda < \lambda_1$*.*

Proof. By the results of sections 3 and 4, we get existence of critical points for the functional E_λ defined by (3.3) when the hypotheses of the theorem are satisfied. Then, problem (1.1) has a solution in $H^1(\Omega)$. By the regularity results in [2], if Ω is smooth and $\mu \in \mathcal{C}^\infty(\partial\Omega)$, we have $u \in \mathcal{C}^\infty(\overline{\Omega})$ □

Concluding remarks.
By the definition (3.36), we see that some I_k with $k \geq 1$ could be empty if $h'_m = \min_{u\in\mathbb{R}} h'(u) < 1$ (recall the normalization $h'(0) = 1$). Consider, e.g., $h(u) = [e^{\alpha u} - e^{-(1-\alpha)u}]$; a straightforward computation gives $h'(0) = 1$ and $h'_m = \left(\frac{1-\alpha}{\alpha}\right)^{2\alpha-1} < 1$ if $\alpha \neq 1/2$ (the minimum is achieved at $u = 2\log\left(\frac{1-\alpha}{\alpha}\right) \neq 0$ if $\alpha \neq 1/2$). Now take $\Omega = B_1$, the unit ball in $\mathbb{R}^2$, and $\mu = 1$. The eigenvalues of the linearized problem (Steklov eigenvalues) are $\lambda_k = k$, $k = 0, 1, ...$ For $\alpha \neq 1/2$ in the interval $[0, 1]$, we have only a finite number of intervals $I_k = \left(\frac{k}{h'_m}, k+1\right)$; in the limit cases $\alpha = 0$, $\alpha = 1$, we are left with the interval $I_0 = (0, 1)$. Thus, for $\alpha \neq 1/2$ theorem 5.1 asserts that the nonlinear problem admits a solution for every λ belonging to a finite set of bounded intervals. We may compare this result with the one given in [4] (for $g = 0$) where the authors prove that a solution does exist for λ positive and less than some value λ^*. Here, we characterize λ^* as the first positive eigenvalue of the linear problem, and furthermore we obtain a number of other intervals of existence.
The case $\alpha = 1/2$ is quite special since $h(u) = 2\sinh(u/2)$ and E_λ is a symmetric functional. In this case it was proved [5] that, still with $\mu = 1$, problem (1.1) has infinitely many solutions for every

positive λ. Multiplicity of solutions in the general case (no symmetry and μ sign changing) seems to be an open problem.
Another interesting question is the existence of positive solutions; we notice, for instance, that a positive solutions of (1.1) with $h(u) = u + c|u|^{\beta-1}u$, and with $c \in \mathbb{R}_+$, $1 < \beta < \frac{n}{n-2}$, solves a special case of a problem studied by Escobar ([3] problem (2.1)) in connection with the problem of finding conformal metrics already quoted in the introduction.

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