ESTIMATES FOR KOTTMAN'S SEPARATION CONSTANT IN REFLEXIVE BANACH SPACES

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Abstract. In reflexive Banach spaces which possess some degree of uniform convexity, we obtain estimates for Kottman's separation constant in terms of the corresponding modulus.

Introduction

For X an infinite-dimensional Banach spaces, *Kottman's constant* $K(X)$, which measures how big the separation of an infinite subset of the unit ball can be, was introduced in the seventies ([11], [12]). Its exact value is known in quite a few classical spaces: moreover, Elton and Odell ([5], 1981) proved that $K(X) > 1$ in every infinite dimensional space.

A new interest on this constant arose recently; what is relevant is the fact that, as it has been shown in [10], such constant gives exact estimates concerning extensions of Lipschitz maps in some Banach spaces. Estimates from below have been obtained in the last years in nonreflexive spaces ([13]) as well as in uniformly convex spaces ([17]).

In this paper, working mainly in reflexive spaces, we provide for Kottman's constant some estimates from below and from above in terms of the modulus of convexity δ or of the modulus of smoothness. More precisely, we obtain estimates from below for all spaces with $\delta(\sqrt{2}) > 0$ and from above for all spaces with $\delta(1) > 0$. Our estimates (part of which are sharp) apply to classes of spaces much wider than the class of uniformly convex spaces and, in uniformly convex spaces, are more accurate than the ones already known in literature.

The paper is organized in the following way: in Section 1 we recall the relevant definitions and some known results. The whole Section 2 is devoted to estimates which rely on the modulus of convexity. In Section 3 we discuss the extreme values concerning Kottman's constant and renormings. Finally, in Section 4 we discuss conditions under which $K(X)$ can be defined using only basic sequences.

1. Definitions and known results

Let X be a real infinite-dimensional Banach space; denote by S_X its unit sphere; by B_X its unit ball; by $B(x, r)$, for $r > 0$, the ball centered at x with radius r.

We recall the definitions of the moduli of convexity and of smoothness.

For $\varepsilon \in [0, 2]$ we call *modulus of convexity* of X the function

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$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \; x, y \in S_X; \; \|x-y\| \geq \varepsilon\right\}.$$

We simply write $\delta(\varepsilon)$ instead of $\delta_X(\varepsilon)$ when no misunderstanding can arise.

A space X is *uniformly convex*, (UC) for short, if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$, and *uniformly non square*, (UNS), if $lim_{\varepsilon \to 2}\, \delta_X(\varepsilon) = \delta_X(2^-) > 0$.

We recall that:

$$\|x\| \leq r; \; \|y\| \leq r; \; \|x-y\| \geq \varepsilon \quad \text{imply} \quad \left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right) \quad (\varepsilon \leq 2r). \tag{1.1}$$

Given a space X, its *characteristic of convexity* is defined as

$$\varepsilon_0 = \sup\{\varepsilon \geq 0 : \; \delta_X(\varepsilon) = 0\}.$$

The following equalities hold (see for example [8] p.56):

$$1 - \frac{\varepsilon}{2} = \delta(2 - 2\delta(\varepsilon)) \quad \text{for} \quad \text{all} \quad \varepsilon \in [\varepsilon_0, 2] \tag{1.2}$$

and

$$\delta(2^-) = 1 - \frac{\varepsilon_0}{2}. \tag{1.3}$$

We call *modulus of smoothness* of X, for $\tau \in R^+$, the function

$$\rho_X(\tau) = \sup\left\{\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} - 1 : \; \|x\| = 1, \|y\| = \tau\right\}.$$

The space X is *uniformly smooth (US)* if $\; lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} = 0$.

We call *separation* of a sequence $\{x_n\}$ in X the number

$$\text{sep}(\{x_n\}) = \inf\{\|x_i - x_j\| : \; i \neq j\}.$$

The following constant was defined in [11]:

$$K(X) = \sup\{\text{sep}\,(\{x_n\}) : \{x_n\} \subset S_X\}.$$

$K(X)$ is the *separation measure of noncompactness* of S_X and it is called *separation constant* or *Kottman's constant* of X.

We recall some properties of $K(X)$ (see [18] for references):

i_1) in the definition of $K(X)$, we can substitute S_X with B_X;

i_2) for any infinite dimensional space we have $K(X) > 1$ (this is a deep result proved in [5]); the range of $K(X)$, even if we restrict ourselves to the class of reflexive spaces, is $(1, 2]$ (see [12], p.21);

i_3) the constant $K(X)$ is functionally related to a *packing constant*, concerning the size of infinite sets of balls which can be packed in B_X (see [18]);

i_4) an easy application of Ramsey's theorem implies that for every $\varepsilon > 0$ there exists an infinite sequence $\{x_n\}$ in B_X such that

$$| \, \|x_i - x_j\| - K(X) \, | < \varepsilon \quad \text{for} \quad i \neq j; \tag{1.4}$$

i_5) If X is a (UC) space or a (US) space, then $K(X) < 2$ (see [11], Theorems 3.6 and 3.7) while (UNS) spaces do not satisfy in general the condition $K(X) < 2$ (see [16], Example 3.2.);

i_6) K(X)= 2 if X contains l_1 or c_0 isomorphically; but the condition $K(X) < 2$ does not imply reflexivity (see [11], Example 3.3);

i_7) if X is nonreflexive, then $K(X)$ is larger than $4^{\frac{1}{5}}$ (see [13]).

We recall also the following results, related to i_1); the first part seems to be a well known fact; nevertheless we provide a proof, since we cannot quote any reference for it. The second part is due to Lyusternik and Šnirel'man (see, e.g. [3]).

Lemma 1.1. *Let $dim(X) = \infty$, and let F be a finite family of balls covering S_X. Then:*
- F covers also B_X;
- at least one of the balls must contain an antipodal pair.

Proof. Let $B_i = B(x_i, r_i)$, $i = 1, 2, ..., n$, such that $S_X \subset \cup_{i=1}^n B_i$. Assume there exists $x \in B_X$, $x \notin \cup_{i=1}^n B_i$ and let Y an n-dimensional subspace of X such that $x \in Y$. Of course $\cup_{i=1}^n (B_i \cap Y)$ covers $S_X \cap Y$ but does not contain x.

Since $x \notin B_1 \cap Y$ we can find in Y a hyperplane H_{n-1} through x which does not intersect $B_1 \cap Y$: therefore $S_X \cap H_{n-1} \subset \cup_{i=2}^n (B_i \cap H_{n-1})$; now, since $x \notin B_2 \cap H_{n-1}$ we can find in the affine space H_{n-1} a hyperplane H_{n-2} through x which does not intersect $B_2 \cap H_{n-1}$: therefore $S \cap H_{n-2} \subset \cup_{i=3}^n (B_i \cap H_{n-2})$. Iterating the process $n-1$ times, we obtain an affine 1-dimensional space H_1 through x, that separates, in H_2, x from $B_{n-1} \cap H_2$. Therefore $S \cap H_1 \subset B_n \cap H_1$. By convexity, $B_n \cap H_1$ must contain $\overline{\text{conv}}\,(S \cap H_1)$, i.e. $B \cap H_1$, hence x, a contradiction. □

To obtain estimates for $K(X)$, we consider also two other constants from the literature.

The first one, $T(X)$, called *thickness* of X, was introduced by Whitney in [22] (see [15] for sharper results on it).

To define it, recall that a set A is an $\varepsilon - net$ for a set E if for every $x \in E$ there exists $a \in A$ such that $\|x - a\| \leq \varepsilon$; we set

$$T(X) = \inf\{\varepsilon : \text{there exists a finite}\, \varepsilon - net \text{ for } S_X \text{ in } S_X\}.$$

$T(X)$ has the following properties (the first one being a consequence of Lemma 1.1):

t_1) $T(X) = \inf\{\varepsilon : \text{there exists a finite}\, \varepsilon - net \text{ for } B_X \text{ in } S_X\}$;

t_2) If X is (UNS), then $T(X) > 1$ (see [15], Corollary 5.5)

t_3) If X is (UNS), then $T(X) < 2$ (see [15], Theorem 5.10).

It is not difficult to prove (see [18], (6.3)) that, for any X,

$$T(X) \leq K(X). \tag{1.5}$$

Equality holds in some classical spaces (for example, in Hilbert spaces the value of both constants is $\sqrt{2}$): but in general their values are different; in particular, we have $T(X) = 1$ in some classical Banach spaces, while $K(X) > 1$ always.

The other constant that we consider appeared in literature under two different aspects and names. It can be defined as

$$J(X) = \sup\{\min\{\|x-y\|, \|x+y\|\} : \ x, y \in S_X\}.$$

Though not explicitly introduced there, $J(X) < 2$ is exactly the condition used by James in [9] when defining uniformly non square spaces. It is usually called James' constant. It is immediate to see that

$\sqrt{2} \leq J(X) \leq 2.$

Later Gao [6] introduced the constant

$$g(X) = \inf\{\max\{\|x-y\|, \|x+y\|\} : \ x, y \in S_X\}.$$

This constant has been studied in several papers (see [2], [7], [18], [20]).

It follows from [20], Proposition 2, that $J(X)$ and $g(X)$ can be defined equivalently considering only $x, y \in S_X$ such that $\|x-y\| = \|x+y\|$.

Actually, Casini [2] first proved that

Lemma 1.2. *In any Banach space X*

$$g(X)J(X) = 2. \tag{1.6}$$

We obtain, as a consequence of Lemma 1.2, that

g_1) $1 \leq g(X) \leq \sqrt{2}$;

g_2) $g(X) > 1$ if and only if X is (UNS).

Though second to appear, we use Gao's formulation of the constant, because its comparison with the separation constant is easier. Moreover it has a clear geometrical meaning: it gives the lower bound for numbers g's such that for some point $x \in S$, the ball $B(x, g)$ contains an antipodal pair $(y, -y)$.

The next Lemma, which can easily be proved directly, is also an immediate consequence of Theorem 5.4 in [7] and Lemma 1.2.

Lemma 1.3. *In every (UNS) space X ($g(X) > 1$) we have:*

$$g(X) = \frac{1}{1 - \delta\left(\frac{2}{g(X)}\right)}\,; \tag{1.7}$$

equivalently

$$\delta\left(\frac{2}{g(X)}\right) = 1 - \frac{1}{g(X)}\,. \tag{1.8}$$

As a consequence, in every space X

$$g(X) \geq \frac{1}{1-\delta_X\left(\sqrt{2}\right)}\,; \tag{1.9}$$

In fact, (1.9) follows immediately from the previous Lemma when $g(X) > 1$ while it is trivially true when $g(X) = 1$, i.e. X is not (UNS), because $\delta_X(\sqrt{2}) = 0$.

The next Lemma summarizes relationships among $g(X)$, $T(X)$ and $K(X)$: the first inequality follows from Lemma 1.1, while the second one is (1.5).

Lemma 1.4. *For any X,*

$$g(X) \leq T(X) \leq K(X); \tag{1.10}$$

2. Estimates with the modulus of convexity

In this section we obtain several inequalities concerning our constants, based on the modulus of convexity of the space. Theorem 2.3 and the following ones provide our main results on estimates of $K(X)$ from below and from above; the best estimate from below for $K(X)$ will be given by Corollary 2.15.

We recall that the well known Day-Nordlander' s inequality (see for example [14], p.63) says that

$$\delta(\varepsilon) \leq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \tag{2.1}$$

and equality characterizes Hilbert spaces.

The following estimate was given in [17], Theorem 1.2.

Theorem 2.1. *(Van Neerven) Let X be (UC); then:*

$$K(X) \geq 1 + \frac{1}{2}\,\delta\left(\frac{2}{3}\right). \tag{2.2}$$

The above estimate appears to be rather weak: for example, in Hilbert spaces the value of the right hand side of the inequality is around 1,0286, and this is the best lower bound we can obtain by (2.2). Better estimates are known in the literature; in fact, as it was already noticed in [19], we have:

$$K(X) \geq \frac{1}{1-\delta(1)}\,. \tag{2.3}$$

In Hilbert spaces, this gives

$$K(X) \geq \frac{2}{\sqrt{3}} \sim 1,155. \tag{2.4}$$

Another estimate by the modulus of convexity was obtained in [18], Theorem 5.4, which, in Hilbert spaces, gives

$$K(X) \geq \beta \sim 1,215. \tag{2.5}$$

Such estimates are drastically improved by the following results.

Remark 2.2. From (1.9) and (1.10), in any space X we obtain immediately an estimate sharper than those already quoted:

$$K(X) \geq \frac{1}{1-\delta(\sqrt{2})}\,. \tag{2.6}$$

An even sharper result is provided by the next theorem.

Theorem 2.3. *In every space X we have*

$$K(X) \geq \frac{1}{1-\delta\left(\frac{2}{K(X)}\right)}\,. \tag{2.7}$$

Proof. If X is not (UNS), then, since $2/(K(X)) < 2$, $\delta\left(\frac{2}{K(X)}\right) = 0$ and (2.7) is trivially true. Otherwise ($g(X) > 1$), use (1.7), (1.10) and the fact that $\frac{1}{1-\delta(2/t)}$ is a decreasing function of t to obtain

$$K(X) \geq g(X) = \frac{1}{1-\delta\left(\frac{2}{g(X)}\right)} \geq \frac{1}{1-\delta\left(\frac{2}{K(X)}\right)}\,,$$

which is the thesis. □

Remark 2.4. Since $g(X) \leq T(X) \leq K(X)$, when $T(X) > 1$ inequality (2.7) holds also with K(X) replaced by $T(X)$; this result is contained in Theorem 5.3 in [15].

Remark 2.5. The estimate given by (2.7) is better than (2.6) as far as $K(X) < \sqrt{2}$. Also, both estimates are sharp, in the sense that they become equalities in Hilbert spaces; (2.7) becomes an equality also for l_p spaces, $2 < p < \infty$ (where $K(X) = 2^{1/p}$).

It is known (see [11], Theorem 3.6) that $K(X) < 2$ whenever X is uniformly convex (in fact, the condition $\delta(2/3) > 0$ is sufficient), but no estimates are provided there. Now, using the modulus of convexity of X, we shall give a sharper result.

Theorem 2.6. *For every Banach space X we have:*

$$K(X) \leq 2 - 2\delta(1). \tag{2.8}$$

Also: if $K(X) < 2$, then we have

$$\delta(K(X)) \leq \frac{1}{2}\,. \tag{2.9}$$

Proof. Given $\varepsilon > 0$, we choose in S - according to (1.4) in Section 1 - an infinite sequence $\{x_i\}, i = 0, 1, 2, ..., n, ...$ such that

$$|\,\|x_i - x_j\| - K(X)\,| < \varepsilon \quad \text{for} \quad i \neq j;$$

in particular

$$K(X) - \varepsilon < \|x_0 - x_i\| < K(X) + \varepsilon \quad \text{for all } \ i \in N = \{1, 2, \dots\}\,. \tag{2.10}$$

Now set $y_i = \frac{x_0 + x_i}{2}$ $(i \in N)$. From (2.10) we have

$$\|y_i\| \leq 1 - \delta(K(X) - \varepsilon)$$

and moreover

$$\|y_i - y_j\| = \left\| \frac{x_0 + x_i}{2} - \frac{x_0 + x_j}{2} \right\| = \frac{1}{2}\|x_i - x_j\|,$$

hence

$$\frac{1}{2}(K(X) - \varepsilon) \leq \|y_i - y_j\| \leq \frac{1}{2}(K(X) + \varepsilon) \ \text{ for } i \neq j\,.$$

Thus $\{y_i\}, i \in N$, is a $\left(\frac{K(X)-\varepsilon}{2}\right)$-separated sequence.

Clearly, the largest *separation* for a sequence in $B(0, 1 - \delta(K(X) - \varepsilon))$ is $K(X)\,(1 - \delta(K(X) - \varepsilon))$. Therefore

$$\frac{K(X) - \varepsilon}{2} \leq K(X)\,(1 - \delta(K(X) - \varepsilon))\,.$$

But $\varepsilon > 0$ is arbitrary. So, if $K(X) = 2$, we obtain $2\delta(2^-) \leq 1$, i.e. (according to (1.3)) $\varepsilon_0 \geq 1$ and (2.8) is true for $K(X) = 2$.

Otherwise, let $K(X) < 2$; then, by continuity of δ in $[0, 2)$, we obtain:

$$\frac{K(X)}{2} \leq K(X)\,(1 - \delta(K(X)))$$

i.e. (2.9).

If $K(X) \geq \varepsilon_0$, according to (1.2) we have $\frac{1}{2} = \delta(2 - 2\delta(1)$, hence (2.9) is equivalent to (2.8) because δ is strictly increasing in $[\varepsilon_0, 2]$. On the other hand, if $K(X) < \varepsilon_0$, then $\delta(K(X)) = \delta(1) = 0$, and then (2.8) and (2.9) are trivially true. □

Remark 2.7. If H is a Hilbert space, then the estimate (2.8) gives

$$K(H) \leq \sqrt{3}$$

which is not sharp. In any case, due to (2.1) which gives

$$\min_X(2 - 2\delta_X(1)) = \sqrt{3}$$

the best estimate we can obtain from Theorem 2.6 is $K(X) \leq k$ for some $k \geq \sqrt{3}$.

Remark 2.8. Theorem 2.6 contains Theorem 17 of [23], which states that $\delta(1) > 0$ implies $K(X) < 2$.

It is easy to prove that $\delta_X(1)$ can be estimated from below using $\rho_X(1)$; precisely

Lemma 2.9. *In any space X*

$$\delta_X(1) + \rho_X(1) \geq \frac{1}{2}. \tag{2.11}$$

Proof. From the definitions of δ and ρ it follows

$$\rho_X(1) \geq \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 \,:\, x, y \in S_X,\ \|x-y\| = 1\right\} \tag{2.12}$$

$$= \sup\left\{\frac{\|x+y\|}{2} - \frac{1}{2} \,:\, x, y \in S_X,\ \|x-y\| = 1\right\} = \frac{1}{2} - \delta_X(1).$$

□

Moreover, in [1], Proposition 2.2, it was proved that

$$\rho_X(1) = \rho^*_X(1). \tag{2.13}$$

Therefore, from Theorem 2.6 and Lemma 2.9 we obtain the following

Corollary 2.10. *For any space X,*

$$K(X) \leq 1 + 2\rho_X(1) \qquad K(X^*) \leq 1 + 2\rho_X(1). \tag{2.14}$$

Remark 2.11. The estimate (2.14) is meaningful only when $\rho_X(1) < \frac{1}{2}$ (this implies $\delta_X(1) > 0$); this happens for instance if H is a Hilbert space, where

$$\rho_H(1) = \sqrt{2} - 1. \tag{2.15}$$

In fact it is known (see, e.g. [14]) that

$$\rho_H(\tau) = \sqrt{1+\tau^2} - 1 \leq \rho_X(\tau) \tag{2.16}$$

and equality holds on the right hand side only if X is a Hilbert space.

In particular:

$$\rho_X(1) \geq \sqrt{2} - 1. \tag{2.17}$$

Remark 2.12. (2.14) is of course not sharp, and it is strictly weaker than (2.8): actually, in a Hilbert space H

$$\sqrt{2} = K(H) < 2 - 2\delta_H(1) = \sqrt{3} < 2\sqrt{2} - 1 = 1 + 2\rho_H(1). \tag{2.18}$$

On the other hand the inequality $K(X) \leq 1 + \rho_X(1)$, that would give the right value for Hilbert spaces, is not true; in fact, $\rho_X(1) < 1$ clearly characterizes (UNS) spaces, while there exist (UNS) spaces with $K(X) = 2$ (see [16]).

Now we consider the following known result (see for example [14], p.66).

Proposition 2.13. *The function $\frac{\delta(t)}{t}$ is non decreasing on $(0, 2]$.*

By this result, we can obtain some other nice estimates.

Theorem 2.14. *In every space X we have:*

$$g(X) \geq 1 + \sqrt{2}\,\delta(\sqrt{2}); \tag{2.19}$$

$$g(X) \leq 1 + \lim_{\varepsilon \to 2^-} \delta(\varepsilon). \tag{2.20}$$

Proof. If $g(X) = 1$, the result is trivial.

Assume that $g(X) > 1$: since $1 < g(X) \leq \sqrt{2}$, for every $a \leq \sqrt{2}$ and $b \in [\frac{2}{g}, 2)$ we obtain from Proposition 2.13:

$$\frac{\delta(a)}{a} \leq \frac{\delta(\sqrt{2})}{\sqrt{2}} \leq \frac{\delta(\frac{2}{g(X)})}{\frac{2}{g(X)}} \leq \frac{\delta(b)}{b}. \tag{2.21}$$

Now Lemma 1.3 in Section 1 implies $\frac{\delta(\frac{2}{g(X)})}{\frac{2}{g(X)}} = \frac{g(X)-1}{2}$; then:

$$\frac{\delta(a)}{a} \leq \frac{\delta(\sqrt{2})}{\sqrt{2}} \leq \frac{g(X)-1}{2} \leq \frac{\delta(b)}{b}\ . \tag{2.22}$$

The middle inequality is (2.19) while we obtain (2.20) letting $b \to 2^-$. □

By (2.19) and (1.10) we obtain

Corollary 2.15. *In every space X we have:*

$$K(X) \geq 1 + \sqrt{2}\,\delta(\sqrt{2}). \tag{2.23}$$

From (2.1), it is possible to see that (2.19) and (2.23) (which are sharp in Hilbert spaces) always give better estimates than (1.9) and (2.6).

Remark 2.16. Both (2.8) and (2.22) with $a = 1$ can also be seen as formulas to estimate $\delta(1)$, once $g(X)$ or $K(X)$ is known.

(2.8) can be written as:

$$\delta(1) \leq 1 - \frac{K(X)}{2}\ . \tag{2.24}$$

From (2.22), setting $a = 1$ we obtain

$$\delta(1) \leq \frac{K(X)-1}{2} \tag{2.25}$$

which is stronger than (2.24) if $K(X) < \frac{3}{2}$.

Due to (2.1), the estimate:

$$\delta(1) \leq \min\left\{1 - \frac{K(X)}{2},\ \frac{K(X)-1}{2}\right\} \tag{2.26}$$

is not trivial for $K(X) \notin [3-\sqrt{3}, \sqrt{3}]$. Also: (2.8) together with (2.22) (with $a = 1$) gives

$$1 + 2\delta(1) \leq K(X) \leq 2 - 2\delta(1). \tag{2.27}$$

We add some more estimates connecting $K(X)$ and the modulus of convexity of X: these estimates, in Hilbert spaces, give again the bound $K(X) \leq \sqrt{3}$.

Theorem 2.17. *Let $K(X) < 2$; then*
(2.28)
$$\max\left\{\frac{\delta(K(X))}{2}, 1 - \frac{K(X)}{2}(1-\delta(1))\right\} \leq (1-\delta(K(X))\left(1-\delta\left(\frac{K(X)}{2(1-\delta(K(X)))}\right)\right).$$

Proof. For $\varepsilon > 0$, we consider a sequence $\{x_n\} \subset S_X$ satisfying (1.4) and, as in Theorem 2.6, the sequence $\{y_n\}$ defined by $y_n = \frac{x_0+x_n}{2}$. We have

$$\|y_n\| = \left\|\frac{x_0 + x_n}{2}\right\| \leq 1 - \delta(K(X) - \varepsilon)$$

and

$$\operatorname{sep}(\{y_n\}) \geq \frac{K(X) - \varepsilon}{2}.$$

For any $i, j,\ i \neq j$ set $z_{ij} = \frac{y_i+y_j}{2} = \frac{1}{2}(x_0 + \frac{x_i+x_j}{2})$.

As a first estimate from below we obtain

$$\|z_{ij}\| \geq \frac{1}{2}\left(\|x_0\| - \left\|\frac{x_i + x_j}{2}\right\|\right) \geq \frac{\delta(K(X) - \varepsilon)}{2}. \tag{2.29}$$

Then, using (1.1), we obtain a second estimate from below (an easy computation proves that this second one is better when $K(X) \leq \sqrt{3}$); precisely, taking into account that, from (1.4), it follows that $\|\frac{x_i-x_0}{2}\| \leq \frac{K(X)+\varepsilon}{2}$ we have (use (1.1))

$$\|z_{ij}\| \geq \big|\|x_0\| - \|z_{ij} - x_0\|\big| = 1 - \left\|\frac{\frac{x_i-x_0}{2} + \frac{x_j-x_0}{2}}{2}\right\| \geq \tag{2.30}$$
$$\geq 1 - \frac{K(X)+\varepsilon}{2}\left(1 - \delta\left(\frac{K(X)-\varepsilon}{K(X)+\varepsilon}\right)\right).$$

To get an estimate from above we remark that, according to (2.9), the assumption $K(X) < 2$ guarantees that $\frac{K(X)}{2} < 2(1 - \delta(K(X)))$. So, for ε small

$$\frac{K(X) - \varepsilon}{2} < 2(1 - \delta(K(X) - \varepsilon));$$

hence we can apply (1.1) to obtain

$$\|z_{ij}\| \leq (1 - \delta(K(X) - \varepsilon))\left(1 - \delta\left(\frac{\frac{K(X)-\varepsilon}{2}}{1 - \delta(K(X) - \varepsilon)}\right)\right). \tag{2.31}$$

Letting $\varepsilon \to 0$ in (2.29), (2.30) and (2.31), since δ is a continuous function on $[0, 2)$, we obtain (2.28). □

3. Near the extremes, subspaces and renormings: general discussion

The range of $K(X)$ is $(1,2]$ (see Section 1); when $K(X)$ approaches the extremes of its range, $\delta_X(1)$ must be small and $\rho_X(1)$ must be not too small. Precisely, when $K(X)$ is close to 1, according to our estimates, $\delta_X(\sqrt{2})$ (hence $\delta_X(1)$) must be near 0 (see (2.6) or (2.23)).

Moreover, according to (1.10), also $g(X)$ is near 1, therefore $\rho_X(1)$ is near 1; in fact this follows from

$$J(X) = \frac{2}{g(X)} \leq \rho_X(1) + 1. \tag{3.1}$$

On the other hand, when $K(X) = 2$ we have, from inequality (2.8), $\delta_X(1) = 0$, and, from inequality (2.14), $\rho_X(1) \geq 1/2$.

It can be remarked that for the modulus of convexity to be small it is enough that X admits a 2-dimensional subspace whose unit sphere has *almost flat* sides: we can produce spaces X with any admissible value of $K(X)$ and containing such a 2-dimensional space. Therefore we cannot expect to obtain sharp estimates for $K(X)$ using the modulus of convexity except for spaces in which finite dimensional subspaces combine in a very regular way.

As about renorming, it is known that all spaces can be renormed so as to have $K(X) = 2$ (see [12], Theorem 7). Clearly each renorming X of a space which contains isomorphically l_1 or c_0 has $K(X) = 2$ while all superreflexive Banach spaces admit renormings such that $K(X) < 2$. We do not know whether every space which does not contain an isomorphic copy of l_1 or c_0 or at least every reflexive space admits a renorming with $K(X) < 2$.

4. Reflexive spaces: a related constant

In [4], J. Dronka, L. Olszowy and L. Rybarska-Rusinek asked whether, in reflexive spaces, it is possible to obtain $K(X)$ considering, in the unit ball, only sequences w-converging to 0 or, equivalently, considering only basic sequences. Precisely, they defined a constant of the space X, that they called $\gamma_0(X)$, as

$$\gamma_0(X) = \sup\{\text{sep}\left(\{x_n\}_{n=1}^{+\infty}\right) : \|x_n\| = 1 \ \wedge w - lim_{n\to+\infty} x_n = 0\}$$

and they proved that $\gamma_0(X) = K(X)$ in reflexive spaces admitting a Schauder basis $\{e_n\}$ with the property

$$\Big\| \sum_{i=n}^{+\infty} a_i e_i \Big\| \leq \Big\| \sum_{i=1}^{+\infty} a_i e_i \Big\| \tag{4.1}$$

for every $n \in N$ and every choice of the $a_i' s$ such that $\sum_{i=1}^{+\infty} a_i e_i \in X$; moreover they showed that, for the space c of convergent sequences, $1 = \gamma_0(c) \neq K(c) = 2$. (For bases satisfying (4.1), see Chapter I-19 in [21]; norms of spaces with such bases are usually called K-norms or comonotone norms).

We prove that equality holds in the larger class of spaces satisfying the non-strict Opial's property.

We recall that a space X satisfies the *non-strict Opial's property* if, for any sequence $\{x_n\} \subset X$, if $w - \lim_{n\to+\infty} x_n = x$ then, for every $y \in X$

$$\liminf \|x_n - x\| \le \liminf \|x_n - y\|. \tag{4.2}$$

Theorem 4.1. *Let X be a reflexive Banach space which satisfies the non-strict Opial's property; then*

$$K(X) = \gamma_0(X).$$

Proof. Obviously, in reflexive spaces,

$$K(X) = \sup\{\text{sep}(\{x_n\}_{n=1}^{+\infty}) : \{x_n\}_{n=1}^{+\infty} \subset B_X \wedge \{x_n\}\ w - \text{convergent}\}.$$

Clearly, $\gamma_0(X) \le K(X)$. Now, for any $\varepsilon > 0$, choose $\{x_n\} \subset B_X$ such that $\text{sep}(\{x_n\}) > K(X) - \varepsilon$ and $w - lim_{n\to+\infty} x_n = x$ and set $y_n = x_n - x$. Then $w - lim_{n\to+\infty} y_n = 0$, $sep(\{y_n\}) = \text{sep}(\{x_n\}) \ge K(X) - \varepsilon$ and, by Opial's property, after passing to suitable subsequences

$$lim_{k\to+\infty}\|y_{n_k}\| = lim_{k\to+\infty}\|x_{n_k} - x\| \le lim_{k\to+\infty}\|x_{n_k}\| \le 1,$$

hence

$$\gamma_0(X) \ge K(X)$$

which proves the thesis. ☐

The next proposition shows that the above theorem really improves the result proved in [4].

Proposition 4.2. *Let X a Banach space with a Schauder basis $\{e_n\}$ satisfying condition* (4.1)*: then X has the non-strict Opial's property.*

Proof. Let $\{x_n\}$ a sequence in X such that $w - lim_{n\to+\infty} x_n = x$ and y any element of X. Set

$$x_n = \sum_{i=1}^{+\infty} a_i^n e_i \qquad x = \sum_{i=1}^{+\infty} a_i e_i \qquad y = \sum_{i=1}^{+\infty} b_i e_i.$$

For any $\varepsilon > 0$ take k such that

$$\Big\| \sum_{i=k+1}^{+\infty} (a_i - b_i) e_i \Big\| < \varepsilon;$$

for such k,

$$\Big\| \sum_{i=1}^{k} (a_i^n - a_i) e_i \Big\| = \varepsilon_n \to 0 \ \text{ for } n \to +\infty$$

therefore

$$\|x_n - x\| = \Big\| \sum_{i=1}^{+\infty} a_i^n e_i - \sum_{i=1}^{+\infty} a_i e_i \Big\| \le \Big\| \sum_{i=1}^{k} (a_i^n - a_i) e_i \Big\| + \Big\| \sum_{i=k+1}^{+\infty} (a_i^n - a_i) e_i \Big\| \le$$

$$\varepsilon_n + \Big\| \sum_{i=k+1}^{+\infty} (a_i^n - b_i) e_i \Big\| + \Big\| \sum_{i=k+1}^{+\infty} (a_i - b_i) e_i \Big\| \le$$

$$\varepsilon_n + \Big\| \sum_{i=k+1}^{+\infty} (a_i^n - b_i)e_i \Big\| + \varepsilon \le \varepsilon_n + \Big\| \sum_{i=1}^{+\infty} (a_i^n - b_i)e_i \Big\| + \varepsilon = \varepsilon_n + \|x_n - y\| + \varepsilon\,;$$

then

$$\liminf \|x_n - x\| \le \liminf \|x_n - y\| + \varepsilon$$

and, since ε is arbitrary

$$\liminf \|x_n - x\| \le \liminf \|x_n - y\|.$$

□

Recently, S. Prus (private communication) constructed an example of a superreflexive space X with $\gamma_0(X) \neq K(X)$ thus confirming that, to obtain equality, it is necessary to require some additional property for the norm.

Added in proof: S.Prus allowed us to add here his example, which has not been published elsewhere.

Example 4.3. Let $x = \{x_i\} \in l_2$. We set

$$\|x\| = \sup_{k>1} \left\{ (x_1 + x_k)^2 + \frac{1}{3} \sum_{i=k+1}^{\infty} x_i^2 \right\}^{\frac{1}{2}}.$$

This formula gives a norm on l_2 which is equivalent to the standard one. Indeed

$$\|x\| \le 2 \left(\sum_{i=1}^{\infty} x_i^2 \right)^{\frac{1}{2}}.$$

Moreover, $\|x\| \ge |x_1|$ and

$$\|x\| \ge \left((x_1 + x_2)^2 + \frac{1}{3} \sum_{i=3}^{\infty} x_i^2 \right)^{\frac{1}{2}}.$$

Hence

$$2\|x\| \ge \left(x_2^2 + \frac{1}{3} \sum_{i=3}^{\infty} x_i^2 \right)^{\frac{1}{2}}$$

and

$$5\|x\|^2 \ge x_1^2 + x_2^2 + \frac{1}{3} \sum_{i=3}^{\infty} x_i^2 \ge \frac{1}{3} \sum_{i=1}^{\infty} x_i^2.$$

Let $x_n = \left(-\frac{1}{2}, 0, \ldots, 0, \frac{3}{2}, 0 \ldots\right)$ where $\frac{3}{2}$ is the n-th coordinate of x_n. Then $\|x_n\| = 1$ and $\|x_n - x_m\| = \sqrt{3}$ if $n \neq m$. This shows that $K(X) \ge \sqrt{3}$.

Let now (u_n) be a weakly null sequence in B_X and $\varepsilon > 0$. There exist a subsequence (u_{n_k}) and a block basic sequence (v_k) such that $\|u_{n_k} - v_k\| < \varepsilon$ for every k and all vectors v_k have the first coordinate 0. We have $\|u_{n_k} - u_{n_m}\| \le \|v_k - v_m\| + 2\varepsilon$ and

$$\|v_k - v_m\|^2 \le \|v_k\|^2 + \|v_m\|^2 \le 2(1+\varepsilon)^2$$

for all k, m. Therefore, $\mathrm{sep}(u_n) \le \sqrt{2}$ which shows that $\gamma_0 \le \sqrt{2}$.

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References

[1] M. Baronti, E. Casini and P.L.Papini, *Triangles inscribed in a semicircle, in Minkowski planes, and in normed spaces*, J. Math. Anal. Appl. **252** (2000), 124–146.

[2] E. Casini, *About some parameters of normed linear spaces*, Atti Accad. Naz. Lincei fis. **80** (1986), 11–15.

[3] K. Deimling *Nonlinear functional analysis*, Springer-Verlag, Berlin (1979).

[4] J. Dronka, L. Olszowy and L. Rybarska-Rusinek, *Separability of weakly convergent sequences in Banach spaces*, Panamer. Math. J. **16** (2006), 67-82.

[5] J. Elton and E. Odell, *The unit ball of every infinite-dimensional normed linear space contains a (1+ε)-separated sequence,* Colloq. Math. **44** (1981), 105–109.

[6] J. Gao, *The uniform degree of the unit ball of Banach space (I)* (Chinese, english summary), Nanjing Daxue Xuebao **1** (1982), 14–28.

[7] J. Gao and K.-S. Lau, *On the geometry of spheres in normed linear spaces*, J.Austral. Math. Soc. **48-A** (1990), 101–112.

[8] K. Goebel and W.A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge (1990).

[9] R.C. James, *Uniformly Non-square Banach Spaces*, Ann. Math. **80** (1964), 542–550.

[10] N.J. Kalton, *Extending Lipschitz maps into $C(K)$ spaces*, Israel J. Math. **162** (2007), 275–315.

[11] C.A. Kottman, *Packing and reflexivity in Banach spaces*, Trans. Amer. Math. Soc. **150** (1970), 565–576.

[12] C.A. Kottman, *Subsets of the unit ball that are separated by more than one*, Studia Math. **53** (1975), 15–27.

[13] A. Kryczka and S. Prus, *Separated sequences in nonreflexive Banach spaces*, Proc. Amer. Math. Soc. **129** (2001), 155–163.

[14] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, Springer-Verlag, Berlin (1979).

[15] E. Maluta and P.L. Papini, *Relative centers and finite nets for the unit ball and its subsets*, Boll. U.M.I. **7-B** (1993), 451–472.

[16] S.V.R. Naidu and K.P.R. Sastry, *Convexity conditions in normed linear spaces*, J. Reine Angew. Math. **297** (1978), 35–53.

[17] J.M.A.M. van Neerven, *Separated sequences in nonreflexive Banach spaces*, Colloq. Math. **102** (2005), 147–153.

[18] P.L. Papini, *Some parameters of Banach spaces*, Rend. Sem. Mat. Fis. Milano **53** (1983), 131–148.

[19] P.L. Papini, *Jung's constant and around*, Seminaire mathématique Univ. de Louvain, (1989), Rapport n.160.

[20] P.L. Papini, *Constants and symmetries in Banach spaces*, Ann. Univ. M. Curie-Skłodowska, Sectio A **56** (2002), 65–76.

[21] I. Singer, *Bases in Banach spaces I*, Springer-Verlag, Berlin (1970) .

[22] R. Whitley, *The size of the unit sphere*, Canadian J. Math. **20** (1968), 450–455.

[23] W. Zhao, *Normal structure coefficients and seminorms of linear operators in Banach spaces*, Nonlin. Anal. **20** (1993), 249–259.

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