About well-posedness of optimal segmentation for Blake & Zisserman functional

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SOMMARIO: Si discute la buona posizione di un problema di minimo con discontinuità libera nel gradiente: sono provate varie condizioni di estremalità e sono esibite varie tipologie di non unicità dei minimi.

ABSTRACT: We focus well-posedness in the minimization of a second order free discontinuity problem. Several extremality conditions are proven. Various examples of multiplicity for minimizers are shown.

Contents

1 Introduction

The interest in image segmentation arises in image analysis and computer vision theory. The first variational model for image segmentation was suggested by Mumford and Shah in [18]: they introduced the following functional

$$\int_{\Omega\setminus K} \left(|Du(x)|^2 + |u(x)-g(x)|^2\right)\, dx \;+\; \gamma\, \mathbf{H}^{n-1}(K\cap\Omega) \qquad (1.1)$$

where Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$, u is a scalar function, $K \subset \mathbb{R}^n$, D denotes the distributional gradient, $g \in L^2(\Omega)$ is a function representing the grey levels of the image, γ is a given positive real number related to scale and contrast threshold and $\mathbf{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure. According to this model ([18], [17], [8]) the segmentation of the given image is achieved through the minimization of (1.1) over u and K where K is a closed subset of $\mathbb{R}^n$ and $u \in C^1(\Omega \setminus K)$.

The existence of minimizers for the functional (1.1) was proven in [16] starting from the functional framework introduced in [15]. The existence of minimizers was proven also in [14] by a different approach in the case $n = 2$. The uniqueness of such these minimizers may fail ([2]).

Blake and Zisserman showed some limitations of the Mumford-Shah functional and introduced an alternative way to translate the image segmentation problem into a variational problem in [3]. The strong formulation of the Blake-Zisserman functional is the following functional ([6]) to be minimized among triplets u, K_0 and K_1 where K_0 and K_1 are closed sets in $\mathbb{R}^n$ and $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ and is approximately continuous on $\Omega \setminus K_0$:

$$\int_{\Omega \setminus (K_0 \cup K_1)} \left(\left| D^2 u(x) \right|^2 + |u(x) - g(x)|^2 \right) dx + \\ + \alpha\, \mathbf{H}^{n-1}(K_0) + \beta\, \mathbf{H}^{n-1}(K_1 \setminus K_0) . \quad (1.2)$$

In (1.2) Ω is an open set of $\mathbb{R}^n$, $n \geq 1$, $g \in L^2(\Omega)$ is a function representing the grey levels of the given image, α and β are given positive real numbers related to scale and contrast threshold, D^2 denotes the distributional hessian and $\mathbf{H}^{n-1}$ is the $(n-1)$ dimensional Hausdorff measure.

According to this model ([3], [8]) an optimal segmentation of the given image is achieved through the minimization of functional (1.2) over u, K_0 and K_1. The existence of minimizers for (1.2) was proven in [13] for $n = 1$ and then in [6] for $n = 2$ starting from the weak formulation framework introduced in [5] for any dimension $n \geq 2$.

The non convex functionals (1.1) and (1.2) depend on functions and sets: in fact Mumford-Shah functional involves the two unknowns u and K, while Blake-Zisserman functional involves the three unknowns u, K_0 and K_1.

De Giorgi introduced the basic idea to deal with problems with free discontinuity: formulate and study a relaxed version in the unknown u alone, then prove regularity results for optimal u and eventually recover the discontinuity as the singular set of such optimal u.

This program was achieved for Mumford-Shah functional in [15] by introducing a weak formulation of (1.1) where u belongs to $SBV(\Omega)$, S_u replaces K and $\int_\Omega |\nabla u(x)|^2\, dx$ replaces $\int_{\Omega \setminus K} |Du(x)|^2\, dx$ where ∇u is the absolutely continuous part of Du.

This program was achieved for the Blake-Zisserman functional in [5] by introducing a weak formulation of (1.2) where u belongs to $GSBV(\Omega)$ with ∇u in $GSBV(\Omega)^n$, S_u replaces K_0, $S_{\nabla u} \setminus S_u$ replaces K_1 and $\int_\Omega |\nabla^2 u(x)|^2 \, dx$ replaces $\int_{\Omega\setminus(K_0\cup K_1)} |D^2 u(x)|^2 \, dx$.
Concerning free discontinuity problems in image segmentation the only available result about uniqueness of the minimizer is given in [2] for the 1 dimensional Mumford-Shah functional (1.1).
In this paper we face the question of uniqueness for minimizer of 1-dimensional Blake-Zisserman functional $F^g_{\alpha,\beta}$ below. Given $g \in L^2(0,1)$, $\alpha, \beta \in \mathbb{R}$ and $u \in \mathcal{H}^2$ we define $F^g_{\alpha,\beta} : \mathcal{H}^2 \to [0, +\infty)$ as follows

$$F^g_{\alpha,\beta}(u) = \int_0^1 |\ddot{u}(x)|^2 \, dx + \int_0^1 |u(x) - g(x)|^2 \, dx + \alpha \sharp (S_u) + \beta \sharp (S_{\dot{u}} \setminus S_u). \quad (1.3)$$

Here and in the sequel for all $u \in L^2(0,1)$, $\dot{u}$ denotes the absolutely continuous part of the distributional derivative u' of u, $\ddot{u}$ denotes the absolutely continuous part of $(\dot{u})'$, $S_u \subseteq (0,1)$ denotes the approximate discontinuity set ([1]) of u and $S_{\dot{u}} \subseteq (0,1)$ the approximate discontinuity set of $\dot{u}$, $\sharp$ denotes the counting measure and

$$H^2(I) = \left\{ v \in L^2(I) : \, v', v'' \in L^2(I) \right\} \qquad \text{for any interval } I \subseteq \mathbb{R}$$

$$\mathcal{H}^2 = \Big\{ v \in L^2(0,1) : \\ \sharp (S_v \cup S_{\dot{v}}) < +\infty, \; v \in H^2(I) \; \forall \text{interval } I \subseteq (0,1) \setminus (S_v \cup S_{\dot{v}}) \Big\} .$$

We will call *singular set* of u the set $S_u \cup S_{\dot{u}}$ and we denote

$$m^g(\alpha, \beta) = \inf\{ F^g_{\alpha,\beta}(u) \quad \forall u \in \mathcal{H}^2 \},$$

$$\operatorname{argmin} F^g_{\alpha,\beta} = \{ u \in \mathcal{H}^2 : \quad F^g_{\alpha,\beta}(u) = m^g(\alpha,\beta) \},$$

the absolutely continuous part of functional $F^g_{\alpha,\beta}$ is denoted by

$$\mathcal{F}^g(u) = \int_0^1 |\ddot{u}(x)|^2 \, dx + \int_0^1 |u(x) - g(x)|^2 \, dx. \quad (1.4)$$

We emphasize that, in the 1-d case the strong and the weak version of Blake-Zisserman functional coincide: in fact if $u \in L^2(0,1)$ with $F^g_{\alpha,\beta}(u) < +\infty$ then $\sharp (S_{\dot{u}} \cup S_u) < +\infty$, hence $u \in C^1((0,1) \setminus (S_u \cup S_{\dot{u}})) \cap C^0((0,1) \setminus S_u) \cap \mathcal{H}^2$ and

$$\int_{(0,1)} |\ddot{u}|^2 \, dx = \int_{(0,1)\setminus(S_u \cup S_{\dot{u}})} |u''|^2 \, dx$$

so that minimizers of $F^g_{\alpha,\beta}$ authomatically belong to $C^2\left((0,1)\setminus(S_u\cup S_{\dot u})\right)$. The complete set of Euler equations for minimizers, a compliance identity formula for functional $F^g_{\alpha,\beta}$, a priori estimates and continuous dependence of $m^g(\alpha,\beta)$ with respect to g, α, β are proven in Section 2: Theorems 2.1, 2.2, 2.3 (and 2.4 about n-d case).
It is known that $F^g_{\alpha,\beta}$ achieves a finite minimum (say argmin $F^g_{\alpha,\beta}\neq\emptyset$) whenever the two following conditions are satisfied ([13]):

$$0<\beta\leq\alpha\leq 2\beta<+\infty \tag{1.5}$$

$$g\in L^2(0,1). \tag{1.6}$$

Nevertheless minimizers are not unique in general, due to non convexity of functional (1.3). In Section 3 we show some examples of multiplicity: we exhibit $\alpha>0$ such that $F^g_{\alpha,\alpha}$ has exactly two minimizers if $g=\chi_{[\frac{1}{2},1]}$ (see Counterexample 3.1); there are $\alpha>0$ and $g\in L^2(0,1)$ such that uniqueness fails for any β belonging to a non empty interval $(\alpha-\varepsilon,\alpha]$ (see Counterexample 3.2); for any α and β satisfying $0<\beta\leq\alpha<2\beta$ there is $g\in L^2(0,1)$ with $\sharp\,(\text{argmin}\,F^g_{\alpha,\beta})>1$ (see Counterexample 3.3). Moreover we give an example of a non empty open subset $\mathcal{N}\subseteq L^2(0,1)$ such that for any $g\in\mathcal{N}$ there are α and β satisfying (1.5) and $\sharp\,(\text{argmin}\,F^g_{\alpha,\beta})\geq 2$ (see Counterexample 3.4). The resulting picture is coherent with the appearance of instable patterns and bifurcation of optimal segmentation upon variation of parameters α and β related contrast threshold and luminance sensitivity.
In a forthcoming paper (see [4]) we will show generic uniqueness of minimizers starting from the properties shown in the present paper. We emphasize that, even for continuous piecewise affine functions g, jump and crease points of minimizers are not necessarily localized among those of g (see Section 4): hence the techniques used in [2] to prove generic uniqueness for Mumford-Shah functional cannot be directly applied here. For this reason we will follow a different strategy in [4], by carefully exploiting some intersection properties between real analytic varieties.

2 Euler equations

In this section we deduce Euler equations for minimizers of the functional $F^g_{\alpha,\beta}$. For the multidimensional situation ($n\geq 2$) we refer to [7], [10] and [12].

Theorem 2.1 (Euler equations) *If (1.5) and (1.6) hold true then every u which minimizes (1.3) in $\mathcal{H}^2$ is also a solution of the following system:*

$$\begin{cases} (i) & u'''' + u = g & \text{on } (0,1) \setminus (S_{\dot u} \cup S_u) \\ (ii) & \ddot u_+ = \ddot u_- = 0 & \text{on } S_{\dot u} \cup S_u \cup \{0,1\} \\ (iii) & \dddot u_+ = \dddot u_- = 0 & \text{on } S_u \cup \{0,1\} \\ (iv) & \dddot u_+ = \dddot u_- & \text{on } S_{\dot u} \\ (v) & \frac{1}{2}(u_+ + u_-) = g & \text{on } S_u \cap \{\text{continuity points of } g\} \end{cases}$$

In (ii) and (iii) we conventionally set $\ddot u_-(0) = \ddot u_+(1) = 0 = \dddot u_+(1) = \dddot u_-(0)$. If, in addition to (1.5) and (1.6), $\alpha = \beta$ then (iii),(iv) improve as follows

$$\dddot u_+ = \dddot u_- = 0 \qquad \text{on } S_u \cup S_{\dot u} \cup \{0,1\}\,. \tag{2.1}$$

By summarizing:

$$\ddot u \in H^2(0,1) \quad \text{and} \quad (\ddot u)'' + u = g \;\; \text{in} \;\; \mathcal{D}'(0,1). \tag{2.2}$$

Proof. Let u be a minimizer in $\mathcal{H}^2$ of $F^g_{\alpha,\beta}$. For any $v \in BV$ we set $\big[v\big] = v_+ - v_-$ where v_-, v_+ denote respectively the left and right values of v on S_v.
We introduce the localized version of functional $F^g_{\alpha,\beta}$: once fixed g, α, β, we set, for any v in $\mathcal{H}^2(0,1)$ and any Borel set $A \subset [0,1]$,

$$F(v,A) = \int_A \left(|\ddot v^2| + |v-g|^2\right) dx + \alpha\, \mathcal{H}^{n-1}(S_v \cap A) + \beta\, \mathcal{H}^{n-1}\left((S_{\dot v} \setminus S_v) \cap A\right) \tag{2.3}$$

Step 1 - (Green formula) Assume $u \in \mathcal{H}^2 \cap H^4((0,1) \setminus \{S_u \cup S_{\dot u}\})$ then, by labelling t_l, $l = 1, ..., \mathsf{T}$, the ordered finite set $S_u \cup S_{\dot u}$, and $t_0 = 0$, $t_{\mathsf{T}+1} = 1$, for any $\varphi \in \mathcal{H}^2$ the following identity holds true

$$\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} u''\varphi''\, dx = \sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} u''''\varphi\, dx + \sum_{l=0}^{\mathsf{T}} \Big(\big(-u'''_-(t_{l+1})\varphi_-(t_{l+1}) + u'''_+(t_l)\varphi_+(t_l)\big) + \big(u''_-(t_{l+1})\varphi'_-(t_{l+1}) - u''_+(t_l)\varphi'_+(t_l)\big)\Big) \tag{2.4}$$

Step 2 - At first we show that each minimizer u solves the fourth order elliptic equation (i) in the interior of $(0,1) \setminus (S_u \cup S_{\dot u})$, by performing smooth variations. For every open set $A \subset\subset I \setminus (S_u \cup S_{\dot u})$, for every $\varepsilon \in \mathbb{R}$ and for every $\varphi \in C_0^\infty(A)$ we have

$$0 \le F(u+\varepsilon\varphi, A) - F(u, A) = 2\varepsilon\biggl(\int_A u''\varphi''\,dx + \int_A (u-g)\varphi\,dx\biggr) + o(\varepsilon)$$

where $o(\varepsilon)$ is an infinitesimal of order greater than ε. Hence

$$\int_A u''\varphi''\,dx = -\int_A (u-g)\varphi\,dx$$

for every $\varphi \in C_0^\infty(A)$. Then (i) follows integrating by parts with (2.4).
Now we seek the Euler conditions on the discontinuity set.
Step 3 - We prove the necessary conditions for extremality on S_u :

$$\ddot u_\pm = 0 \quad \text{on } S_u \cup \{0,1\} \tag{2.5}$$

$$\dddot u_\pm = 0 \quad \text{on } S_u \cup \{0,1\} \tag{2.6}$$

In fact, let $\varphi \in \mathcal{H}^2(0,1) \cap C^2([t_l, t_{l+1}])$, $l = 0, ..., \mathsf{T}$, $\mathrm{spt}(\varphi) \subset A$, where A is a Borel set with $(S_{\dot u} \setminus S_u) \cap A = \emptyset$. Then for every $\varepsilon \in \mathbb{R}$ we have

$$\bigl(S_{u+\varepsilon\varphi} \cup S_{\dot u+\varepsilon\dot\varphi}\bigr) \cap A \ \subset\ S_u \cap A$$

By (2.4) we have:

$$\begin{aligned}
0 \le\ & F(u+\varepsilon\varphi, A) - F(u, A) \\
&= \alpha\left(\sharp(S_{u+\varepsilon\varphi} \cap A) - \sharp(S_u \cap A)\right) + \beta\sharp\left((S_{\dot\varphi} \setminus S_{u+\varepsilon\varphi}) \cap A\right) + \\
&\quad 2\varepsilon\left(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} (u''\varphi'' + (u-g)\varphi)\,dx\right) + o(\varepsilon) \\
&= \alpha\left(\sharp(S_{u+\varepsilon\varphi} \cap A) - \sharp(S_u \cap A)\right) + \beta\sharp\left((S_{\dot\varphi} \setminus S_{u+\varepsilon\varphi}) \cap A\right) + \\
&\quad 2\varepsilon\biggl(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} (u''''\varphi + (u-g)\varphi)\,dx \\
&\quad + \dddot u_+(0)\varphi_+(0) - \ddot u_+(0)\dot\varphi_+(0) - \dddot u_-(1)\varphi_-(1) + \ddot u_-(1)\dot\varphi_-(1) \\
&\quad \sum_{S_u \cap A}\left(\left[+\dddot u\varphi\right] - \left[\ddot u\dot\varphi\right]\right)\biggr) + o(\varepsilon)
\end{aligned}$$

Up to a finite set of values of ε, we have $S_{u+\varepsilon\varphi} \cap A = S_u \cap A$ so that we can choose arbitrarily small ε satisfying

$$\sharp\left((S_{\dot\varphi} \setminus S_{u+\varepsilon\varphi}) \cap A\right) = \sharp\left((S_{\dot\varphi} \setminus S_u) \cap A\right) = 0$$

By taking into account (i) and the arbitrariness of the two traces of φ and $\dot\varphi$ on the two sides of points in S_u, for small ε, we can choose φ with $\varphi_\pm = 0$, $\dot\varphi_+ = 0$ and $\dot\varphi_-$ arbitrary or viceversa to get (2.5). Similarly, we obtain (2.6) by choosing $\dot\varphi_\pm = 0$, $\varphi_+ = 0$ and φ_- arbitrary or vice-versa.

Step 4 - We prove the necessary conditions for extremality on $S_{\dot u}$:

$$\ddot u_\pm = 0 \quad \text{on } S_{\dot u} \tag{2.7}$$

$$\left[\dddot u\right] = 0 \quad \text{on } S_{\dot u} \setminus S_u \tag{2.8}$$

Let $\varphi \in \mathcal{H}^2(0,1) \cap C^2([t_l, t_{l+1}])$, $l = 0, ..., \mathsf{T}$, $\mathrm{spt}(\varphi) \subset A$, and $S_\varphi = \emptyset = (S_u \setminus S_{\dot u}) \cap A$. Up to a finite set of values of ε, so that we can choose ε arbitrarily small, we have:

$$(S_{u+\varepsilon\varphi} \cup S_{\dot u + \varepsilon\dot\varphi}) \cap A = S_{\dot u+\varepsilon\dot\varphi} \cap A = S_{\dot u}$$

Moreover, by (2.4):

$$\begin{aligned}
0 \le F(u+\varepsilon\varphi, A) - F(u, A)& \\
&\le \beta\big(\sharp\,(S_{\dot u+\varepsilon\dot\varphi} \cap A) - \sharp\,(S_{\dot u} \cap A)\big) + \\
&2\varepsilon\left(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} (u''\varphi'' + (u-g)\varphi)\,dx\right) + o(\varepsilon) \\
&= 2\varepsilon\Bigg(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} u''''\varphi\,dx + (u-g)\varphi\,dx + \\
&+ \dddot u_+(0)\varphi_+(0) - \ddot u_+(0)\dot\varphi_+(0) - \dddot u_-(1)\varphi_-(1) + \ddot u_-(1)\dot\varphi_-(1) \\
&\sum_{S_{\dot u}\cap A}\Big(\left[+\dddot u\varphi\right] - \left[\ddot u\dot\varphi\right]\Big)\Bigg) + o(\varepsilon)
\end{aligned}$$

By taking into account (i), for small ε and by the arbitrariness of φ and of the two traces of $\dot\varphi$ on the two sides of $S_{\dot u}$, we can choose φ with $\varphi_\pm = 0$, and arbitrary $\dot\varphi_+ = \dot\varphi_-$, to get (2.7). Analogous by choosing $\dot\varphi_\pm = 0$ and $\left[\dot\varphi\right] = 0$ or viceversa, we obtain (2.8).

Then (ii),(iii) and (iv) follows from (2.5)-(2.8) of steps 3 and 4.

Step 5 - We prove (v).

Assume $t \in S_u$ and g continuous at t. If $s = \frac{1}{2}(u_+(t) + u_-(t)) \neq g(t)$ then only one of the following eight cases occurs:

1)	$u_-(t) > u_+(t) \ge g(t)$	5)	$u_+(t) > u_-(t) \ge g(t)$
2)	$u_-(t) > s > g(t) > u_+(t)$	6)	$u_+(t) > s > g(t) > u_-(t)$
3)	$u_-(t) > g(t) > s > u_+(t)$	7)	$u_+(t) > g(t) > s > u_-(t)$
4)	$g(t) \ge u_-(t) > u_+(t)$	8)	$g(t) \ge u_+(t) > u_-(t)$

To deal with 1), 2), 6), 7) choose $0 < \varepsilon << \mathrm{dist}(t, (S_u \cup S_{\dot u} \cup \{0,1\}) \setminus \{t\})$ and explicit the minimality of u by comparison with a variation v in a small interval:

$$v(x) = \begin{cases} u(x) & \text{if } x \in [0, t-\varepsilon) \cup (t, 1] \\ \gamma(x) = u_+(t) + \dot u_+(t)(x-t) & \text{if } x \in [t-\varepsilon, t] \end{cases}$$

Since $\ddot v \equiv 0$ in $(t-\varepsilon, t)$ and g is continuous at t then

$$\begin{aligned} F^g_{\alpha,\beta}(v) - F^g_{\alpha,\beta}(u) &= \int_0^1 |\ddot v(x)|^2\,dx + \int_0^1 |v(x)-g(x)|^2\,dx \\ &\quad - \int_0^1 |\ddot u(x)|^2\,dx - \int_0^1 |u(x)-g(x)|^2\,dx \\ &\le \int_{t-\varepsilon}^t |v(x)-g(x)|^2\,dx - \int_{t-\varepsilon}^t |u(x)-g(x)|^2\,dx \\ &= \int_{t-\varepsilon}^t \left((\gamma(x)-g(x))^2 - (u(x)-g(x))^2\right)dx \\ &= \int_{t-\varepsilon}^t (\gamma(x)-u(x))(\gamma(x)+u(x)-2g(x))\,dx \\ &\sim \int_{t-\varepsilon}^t (u_+(t)-u_-(t))(u_+(t)+u_-(t)-2g(t))\,dx < 0 \end{aligned}$$

This contradicts the minimality of u.
To deal with 3), 4), 5), 6) choose $0 < \varepsilon << \mathrm{dist}(t, (S_u \cup S_{\dot u} \cup \{0,1\}) \setminus \{t\})$ and explicit the minimality of u by comparison with a variation w in a small interval:

$$w(x) = \begin{cases} u(x) & \text{if } x \in [0, t) \cup (t+\varepsilon, 1] \\ \delta(x) = u_-(t) + \dot u_-(t)(x-t) & \text{if } x \in [t, t+\varepsilon] \end{cases}$$

which leads to the contradiction:

$$F^g_{\alpha,\beta}(w) - F^g_{\alpha,\beta}(u) \sim \int_t^{t+\varepsilon} (u_-(t)-u_+(t))(u_+(t)+u_-(t)-2g(t))\,dx < 0.$$

Step 6 - Eventually we prove (2.1): due to (iii) we have only to show

$$\dddot u_\pm = 0 \quad \text{on } (S_{\dot u} \setminus S_u) \text{ if } \alpha = \beta$$

Fix a Borel set A s.t. $A \subset\subset (0,1)$, $S_u \cap A = \emptyset \neq S_{\dot u} \cap A$.
Let $\varphi \in \mathcal{H}^2(0,1) \cap C^2([t_l, t_{l+1}])$, $l = 0, ..., \mathsf{T}$ and

$$S_{\dot\varphi} \cap A = S_u \cap A = \emptyset \neq S_\varphi \cap A = S_{\dot u} \cap A$$

Then for every $\varepsilon \in \mathbb{R}$ we have $S_{u+\varepsilon\varphi} \cap A = S_\varphi \cap A$ and

$$\left(S_{u+\varepsilon\varphi} \cup S_{(\dot{u}+\varepsilon\dot{\varphi})}\right) \cap A = S_{\dot{u}} \cap A$$

By (2.4), (i) and (ii) we have

$$\begin{aligned}
0 &\leq F(u+\varepsilon\varphi, A) - F(u,A) \\
&= \alpha\,\sharp\left(S_{u+\varepsilon\varphi} \cap A\right) + \beta\left(\sharp\left(\left(S_{(\dot{u}+\varepsilon\dot{\varphi})} \setminus S_{u+\varepsilon\varphi}\right) \cap A\right) - \beta\sharp\left(S_{\dot{u}} \cap A\right)\right) \\
&\quad + 2\varepsilon\left(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_l+1} \left(u''\varphi'' + (u-g)\varphi\right) dx\right) + o(\varepsilon) \\
&= \alpha\,\sharp(S_\varphi \cap A) + \beta\,\sharp\left((S_{\dot{u}} \setminus S_\varphi) \cap A\right) - \beta\,\sharp(S_{\dot{u}} \cap A) \\
&\quad + 2\varepsilon\left(\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_l+1} \left(u''''\varphi + (u-g)\varphi\right) dx + \sum_{S_{\dot{u}}}\left(\left[\dddot{u}\varphi\right] - \left[\ddot{u}\dot{\varphi}\right]\right)\right) + o(\varepsilon) \\
&= \alpha\,\sharp(S_\varphi \cap A) - \beta\,\sharp(S_{\dot{u}} \cap A) + 2\varepsilon \sum_{S_{\dot{u}} \cap A} \left[\dddot{u}\varphi\right] + o(\varepsilon)
\end{aligned}$$

Since $S_\varphi \cap A = S_{\dot{u}} \cap A$, when $\alpha > \beta$ then the inequality is fulfilled for ε small enough, hence we do not obtain further information. On the other hand, when $\alpha = \beta$, we get

$$0 \leq F(u+\varepsilon\varphi, A) - F(u,A) = 2\varepsilon \sum_{S_{\dot{u}} \cap A} \left[\dddot{u}\varphi\right] + o(\varepsilon)$$

Then the coefficient of 2ε must vanish, hence by the arbitrariness of the two traces of φ we get (2.1).
Step 7 - The proof of (2.2) is a straightforward consequence of (i)-(iv). □

Theorem 2.2 (Compliance identity) *Assume (1.5) and (1.6). Then any $u \in \mathcal{H}^2$ fulfilling the Euler equations (i)-(iv) of Theorem 2.1 verifies also*

$$\mathcal{F}^g(u) = \int_0^1 (gu - u^2)\, dx \tag{2.9}$$

and

$$F^g_{\alpha,\beta}(u) \;=\; \int_0^1 (gu - u^2)\, dx \;+\; \alpha\,\sharp\,(S_u) \;+\; \beta\,\sharp\,(S_{\dot{u}} \setminus S_u). \tag{2.10}$$

In particular any u minimizing $F^g_{\alpha,\beta}$ over $\mathcal{H}^2$ fulfills (2.9) and (2.10).

Proof. Label t_l, $l = 1, ..., \mathsf{T}$, the ordered finite set $S_u \cup S_{\dot u}$ and $t_0 = 0$, $t_{\mathsf{T}+1} = 1$. Integration by parts in $\int_0^1 \mid \ddot u \mid^2 \, dx$ and (i)-(iv) of Theorem 2.1 entail

$$\int_0^1 \mid \ddot u \mid^2 \, dx =$$

$$\sum_{l=0}^{\mathsf{T}} \int_{t_l}^{t_{l+1}} (\ddot u)'' u \, dx + \sum_{l=0}^{\mathsf{T}} \Big(\big(- u'''_-(t_{l+1}) u_-(t_{l+1}) + u'''_+(t_l) u_+(t_l) \big) +$$

$$\big(u''_-(t_{l+1}) u'_-(t_{l+1}) - u''_+(t_l) u'_+(t_l) \big) \Big) = \int_0^1 (\ddot u)'' u \, dx$$

$$= \int_0^1 (g - u) u \, dx = \int_0^1 (gu - u^2) \, dx$$

and the theorem follows. □

We show a priori estimates for minima, minimizers, singular set of minimizers of $F^g_{\alpha,\beta}$ and continuous dependence of minimum value $m^g(\alpha, \beta)$ with respect to α, β in $\{(\alpha, \beta) \in \mathbb{R}^2 : \; 0 < \beta \le \alpha \le 2\beta\}$ and g in $L^2(0, 1)$.

Theorem 2.3 *Assume* $f, g \in L^2(0, 1)$ *and*

$$0 < \beta \le \alpha \le 2\beta < +\infty \, , \qquad 0 < b \le a \le 2b < +\infty \, . \tag{2.11}$$

Then

$$\|u\|_{L^2} \le 2 \, \|g\|_{L^2} \qquad \forall u \in \operatorname{argmin} F^g_{\alpha,\beta}, \tag{2.12}$$

$$0 \le m^g(\alpha, \beta) \le \|g\|^2_{L^2} \, , \tag{2.13}$$

$$\begin{aligned} \left| m^g(\alpha, \beta) - m^h(a, b) \right| \; &\le \; 5(\|g\|_{L^2} + \|h\|_{L^2}) \, \|g - h\|_{L^2} + \\ &+ \frac{\min\left\{ \|g\|^2_{L^2}, \|h\|^2_{L^2} \right\}}{\min\{\alpha, a\}} \, |\alpha - a| + \frac{\min\left\{ \|g\|^2_{L^2}, \|h\|^2_{L^2} \right\}}{\min\{\beta, b\}} \, |\beta - b| \; , \end{aligned} \tag{2.14}$$

$$\left. \begin{array}{c} \sharp \, (S_u) \le \max\{ j \in \mathbb{N} : \; j \le 2(\|g\|^2_{L^2} + \eta^2)/\alpha \}, \\ \sharp \, (S_{\dot u} \setminus S_u) \le \max\{ j \in \mathbb{N} : \; j \le 2(\|g\|^2_{L^2} + \eta^2)/\beta \}, \\ \forall u \in \operatorname{argmin} F^h_{\alpha,\beta} \; \textit{with} \; \|h - g\|_{L^2} < \eta. \end{array} \right\} \tag{2.15}$$

Proof. Estimate (2.13) follows from $0 \le m^g(\alpha, \beta) \le F^g_{\alpha,\beta}(0) = \|g\|^2_{L^2}$.
By (2.13) we get the following inequality equivalent to (2.12)

$$\|u\|^2_{L^2} \le 2(\|u - g\|^2_{L^2} + \|g\|^2_{L^2}) \le 2(m^g(\alpha, \beta) + \|g\|^2_{L^2}) \le 4 \, \|g\|^2_{L^2} \, .$$

Fix $u_g \in \operatorname{argmin} F^g_{\alpha,\beta}$, $u_h \in \operatorname{argmin} F^h_{\alpha,\beta}$; then by Schwarz inequality and (2.12)

$$\begin{aligned} m^g(\alpha,\beta) &= F^g_{\alpha,\beta}(u_g) \le F^g_{\alpha,\beta}(u_h) = F^h_{\alpha,\beta}(u_h) - \|u_h - h\|^2_{L^2} + \|u_h - g\|^2_{L^2} = \\ &= m^h(\alpha,\beta) - \|u_h - h\|^2_{L^2} + \|u_h - g\|^2_{L^2} \le \\ &\le m^h(\alpha,\beta) + \langle g-h, g+h-2u_h\rangle_{L^2} \le \\ &\le m^h(\alpha,\beta) + (\|g\|_{L^2} + 5\,\|h\|_{L^2})\,\|g-h\|_{L^2}\,, \end{aligned}$$

similarly $m^h(\alpha,\beta) \le m^g(\alpha,\beta) + (\|h\|_{L^2} + 5\,\|g\|_{L^2})\,\|g-h\|_{L^2}$. Then

$$\left|m^g(\alpha,\beta) - m^h(\alpha,\beta)\right| \le 5(\|g\|_{L^2} + \|h\|_{L^2})\,\|g-h\|_{L^2}\,. \tag{2.16}$$

Fix $u_{\alpha,\beta} \in \operatorname{argmin} F^g_{\alpha,\beta}$, $u_{a,b} \in \operatorname{argmin} F^g_{a,b}$; then by (1.5) and (2.13)

$$\begin{aligned} m^g(a,b) &\le F^g_{a,b}(u_{\alpha,\beta}) = F^g_{\alpha,\beta}(u_{\alpha,\beta}) + (a-\alpha)\,\sharp\,(S_{u_{\alpha,\beta}}) + (b-\beta)\,\sharp\,(S_{\dot u_{\alpha,\beta}} \setminus S_{u_{\alpha,\beta}}) \\ &= m^g(\alpha,\beta) + \tfrac{a-\alpha}{\alpha}\;\alpha\,\sharp\,(S_{u_{\alpha,\beta}}) + \tfrac{b-\beta}{\beta}\;\beta\,\sharp\,(S_{\dot u_{\alpha,\beta}} \setminus S_{u_{\alpha,\beta}}) \le \\ &\le m^g(\alpha,\beta) + \tfrac{|\alpha-a|}{\alpha}\; m^g(\alpha,\beta) + \tfrac{|\beta-b|}{\beta}\; m^g(\alpha,\beta) \le \\ &\le m^g(\alpha,\beta) + \tfrac{\|g\|^2}{\alpha}\,|\alpha-a| + \tfrac{\|g\|^2}{\beta}\,|\beta-b|\,, \end{aligned}$$

similarly $m^g(\alpha,\beta) \le m^g(a,b) + \tfrac{\|g\|^2}{a}\,|\alpha-a| + \tfrac{\|g\|^2}{b}\,|\beta-b|$. Then

$$|m^g(\alpha,\beta) - m^g(a,b)| \le \frac{\|g\|^2_{L^2}}{\min\{\alpha,a\}}\,|\alpha-a| + \frac{\|g\|^2_{L^2}}{\min\{\beta,b\}}\,|\beta-b|\,. \tag{2.17}$$

Eventually inequality (2.14) follows by (2.16), (2.17) and

$$\begin{aligned} \left|m^g(\alpha,\beta) - m^h(a,b)\right| &\le \min\Big\{ \left|m^g(\alpha,\beta) - m^h(\alpha,\beta)\right| + \left|m^h(\alpha,\beta) - m^h(a,b)\right|, \\ &\qquad |m^g(\alpha,\beta) - m^g(a,b)| + \left|m^g(a,b) - m^h(a,b)\right| \Big\}. \end{aligned}$$

To prove (2.15) choose $h \in L^2(0,1)$ with $\|h-g\|_{L^2} < \eta$ and $u \in \operatorname{argmin} F^h_{\alpha,\beta}$, then (2.13) entails

$$\alpha\,\sharp\,(S_u) + \beta\,\sharp\,(S_{\dot u} \setminus S_u) \le m^h(\alpha,\beta) \le \|h\|^2_{L^2} \le 2\,\|g\|^2_{L^2} + 2\eta^2. \quad \square$$

Analogous properties hold true for n-dimensional Blake-Zisserman functional.

Theorem 2.4 *Fix an open set $\Omega \subseteq \mathbb{R}^n$, denote by $\mathbf{F}^g_{\alpha,\beta}$ the functional (1.2), by* argmin $\mathbf{F}^g_{\alpha,\beta}$ *the set of minimizers of $\mathbf{F}^g_{\alpha,\beta}$, by $m^g(\alpha,\beta)$ the minimum value of $\mathbf{F}^g_{\alpha,\beta}$. Assume $f, g \in L^2(\Omega)$ and α, β and a, b fulfill (2.11). Then*

$$\|u\|_{L^2} \le 2\,\|g\|_{L^2} \qquad \forall u \in \operatorname{argmin} \mathbf{F}^g_{\alpha,\beta}, \tag{2.18}$$

$$0 \leq m^g(\alpha,\beta) \leq \|g\|^2_{L^2}\,, \tag{2.19}$$

$$\begin{aligned} \left|m^g(\alpha,\beta) - m^h(a,b)\right| \;\leq\; & 5(\|g\|_{L^2} + \|h\|_{L^2})\,\|g-h\|_{L^2} + \\ & \tfrac{\min\left\{\|g\|^2_{L^2},\|h\|^2_{L^2}\right\}}{\min\{\alpha,a\}}\,|\alpha - a| + \tfrac{\min\left\{\|g\|^2_{L^2},\|h\|^2_{L^2}\right\}}{\min\{\beta,b\}}\,|\beta - b|\,, \end{aligned} \tag{2.20}$$

$$\left.\begin{array}{c} \mathbf{H}^{n-1}(S_u) \leq \dfrac{2}{\alpha}\left(\|g\|^2 + \eta^2\right), \quad \mathbf{H}^{n-1}(S_{\nabla u}\setminus S_u) \leq \dfrac{2}{\beta}\left(\|g\|^2 + \eta^2\right) \\ \forall u \in \operatorname{argmin} \mathbf{F}^h_{\alpha,\beta} \textit{ with } \|h-g\|_{L^2} < \eta. \end{array}\right\} \tag{2.21}$$

Proof. Repeat the proof of the 1-d case (Theorem 2.3) by substituting $\mathbf{H}^{n-1}$ to $\sharp$. $\square$
In the following Lemma we summarize and restate in a form suitable for our purposes Theorems 2.1, 3.1 and Lemma 3.6 of [13].

Theorem 2.5 *Assume $g \in L^2(0,1)$, α, β fulfilling (1.2), $(u_l)_{l\in\mathbb{N}} \subseteq \mathcal{H}^2(0,1)$ and $\left\{F^g_{\alpha,\beta}(u_l)\right\}_{l\in\mathbb{N}}$ is bounded.*
1. **Compactness**
Then there are $u \in \mathcal{H}^2(0,1)$ and a subsequence $(u_{l_n})_{n\in\mathbb{N}}$ such that

$$\begin{cases} (u_{l_n})_{n\in\mathbb{N}} \textit{ converges to } u \textit{ in the strong topology of } L^1(0,1), \\ (\dot u_{l_n})_{n\in\mathbb{N}} \textit{ converges almost everywhere to } \dot u, \\ (\ddot u_{l_n})_{n\in\mathbb{N}} \textit{ converges to } \ddot u \textit{ in the weak topology of } L^2(0,1). \end{cases}$$

2. **Lower semicontinuity**
If $(u_l)_{l\in\mathbb{N}}$ converges strongly in L^1 to $u \in \mathcal{H}^2(0,1)$, then

$$F^g_{\alpha,\beta}(u) \;\leq\; \liminf_{l\to+\infty} F^g_{\alpha,\beta}(u_l).$$

3. **A confined single crease sequence (of a minimizing sequence) cannot converge to a jump**
If $(u_l)_{l\in\mathbb{N}}$ converges strongly in L^1 to $u \in \mathcal{H}^2(0,1)$, $(a,b) \subseteq (0,1)$ and

$$x_l \in S_{\dot u_l}\setminus S_{u_l}, \qquad (S_{u_l}\cup S_{\dot u_l})\cap(a,b) = \{x_l\}, \qquad x_l \to \overline{x} \in (a,b)$$

then $\overline{x} \notin S_u$, more precisely

$$S_u \cap (a,b) = \emptyset, \qquad S_{\dot u}\cap(a,b) \subseteq \{\overline{x}\}. \;\square$$

We show that $\mathcal{F}^g$ has strictly positive infimum over suitable subsets of $\mathcal{H}^2$.

Theorem 2.6 *For any possibly discontinuous piecewise affine function g with $S_g \cup S_{\dot g} \neq \varnothing$ we introduce the subset $\mathcal{S}[g]$ of $\mathcal{H}^2$ as follows:*
$v \in \mathcal{S}[g]$ if and only if, either

$$(i) \begin{cases} \sharp\,(S_{\dot v} \setminus S_v) \;<\; \sharp\,(S_{\dot g} \setminus S_g) \\ \sharp\,(S_v) \;<\; \sharp\,(S_g) + \sharp\,(S_{\dot g} \setminus S_g) - \sharp\,(S_{\dot v} \setminus S_v)\,, \end{cases}$$

or

$$(ii) \begin{cases} \sharp\,(S_v) \;<\; \sharp\,(S_g) \\ \sharp\,(S_{\dot v} \setminus S_v) \;<\; \sharp\,(S_{\dot g} \setminus S_g) + 2(\sharp\,(S_g) - \sharp\,(S_v)). \end{cases}$$

Then $\mathcal{S}[g] \neq \varnothing$ and

$$\inf_{v \in \mathcal{S}[g]} \mathcal{F}^g(v) \;>\; 0. \tag{2.22}$$

Proof. $\mathcal{S}[g]$ is not empty since $H^2(0,1) \subseteq \mathcal{S}[g]$.
In order to show (2.22) we argue by contradiction: suppose that there is a sequence $\{v_n\}_n$ in $\mathcal{S}[g]$ with $\lim_{n\to+\infty} \mathcal{F}^g(v_n) = 0$.
Condition (i), (ii) and $\mathcal{S}[g] \subseteq \mathcal{H}^2$ entail

$$\mathcal{F}^g(v_n) + \alpha\,\sharp\,(S_{v_n}) + \beta\,\sharp\,(S_{\dot v_n} \setminus S_{v_n}) \le C < +\infty \qquad \forall n.$$

By Theorem 2.5(1), up to subsequences, v_n converges strongly in $L^1(0,1)$ to a function $w \in \mathcal{H}^2$, $\dot v_n \to \dot v$ a.e, and $\ddot v_n \to \ddot v$ weakly in $L^2(0,1)$. Lower semicontinuity of $\mathcal{F}^g$ (Theorem 2.5(2)) implies $\mathcal{F}^g(w) = 0$ then $w = g$ a.e. in $(0,1)$ and, by $g, w \in \mathcal{H}^2$, we have $g = w$.
Let $\mathsf{s}_n = \sharp\,(S_{v_n})$ and $\mathsf{p}_n = \sharp\,(S_{\dot v_n} \setminus S_{v_n})$. Up to subsequences we can assume the existence of non negative integers s, p such that, for any n, $\mathsf{s}_n = \mathsf{s}$, $\mathsf{p}_n = \mathsf{p}$ and the ordering of jumps and creases is independent of n. By introducing the sets $\{y_n^a\}_{a=1}^{\mathsf{s}} = S_{v_n}$ and $\{y_n^b\}_{b=1}^{\mathsf{p}} = S_{\dot v_n} \setminus S_{v_n}$ with $y_n^a < y_{n+1}^a$ and $y_n^b < y_{n+1}^b$ we can also assume

$$\lim_{n\to+\infty} y_n^a = y^a \quad \text{and} \quad \lim_{n\to+\infty} y_n^b = y^b.$$

The assumptions read, either $(i) \begin{cases} \mathsf{p} < \mathsf{c} \\ \mathsf{s} < \mathsf{j} + \mathsf{c} - \mathsf{p} \end{cases}$, or $(ii) \begin{cases} \mathsf{s} < \mathsf{j} \\ \mathsf{p} < \mathsf{c} + 2(\mathsf{j} - \mathsf{s}). \end{cases}$

In case (i) there is $\overline{x} \in S_{\dot g} \cup S_g$ such that $\overline{x} \notin \{y^a\}_{a=1}^{\mathsf{s}} \cup \{y^b\}_{b=1}^{\mathsf{p}}$, then the term $\int |\ddot v_n|^2$ blows up around $\overline{x}$, hence the contradiction $\lim_{n\to+\infty} \mathcal{F}^g(v_n) = +\infty$.
In case (ii) there is $\overline{x} \in S_g$ such that $\overline{x} \notin \{y^a\}_{a=1}^{\mathsf{s}}$ by the first condition in (ii) and, at the same time, by the second condition in (ii)

$$\lim_{n\to+\infty} y_n^{b_1} \neq \overline{x} \quad \text{or} \quad \lim_{n\to+\infty} y_n^{b_2} \neq \overline{x} \qquad \forall b_1, b_2 \in \{1, ..., \mathsf{p}\}, \quad b_1 \neq b_2;$$

then by Theorem 2.5(3) we get the contradiction $\lim_{n\to+\infty} \mathcal{F}^g(v_n) = +\infty$ as in the previous case. $\square$

We introduce and study the family Φ_λ of affine transformations of $L^2(0,1)$ which are useful in exhibiting examples without uniqueness of minimizers.

Lemma 2.7 *Given $\alpha, \beta, \lambda \in \mathbb{R}$ with (1.5), we set*

$$\Phi_\lambda[v](x) = \lambda - v(1-x), \qquad \forall v \in L^2(0,1). \tag{2.23}$$

Then for any $g \in L^2(0,1)$ and $u \in \operatorname{argmin} F^g_{\alpha,\beta}$ we have $\Phi_\lambda[u] \in \operatorname{argmin} F^{\Phi_\lambda[g]}_{\alpha,\beta}$.
In particular if $\Phi_\lambda[g] = g$ then also $\Phi_\lambda[u] \in \operatorname{argmin} F^g_{\alpha,\beta}$.
Hence there is no uniqueness of minimizers for $F^g_{\alpha,\beta}$ whenever $\Phi_\lambda[g] = g$ and one can prove that a minimizer u fulfills $\Phi_\lambda[u] \neq u$.
If $g \in L^2(0,1)$ fulfills $\sharp\,(\operatorname{argmin} F^g_{\alpha,\beta}) = 1$ then $\sharp\,(\operatorname{argmin} F^{\Phi_\lambda[g]}_{\alpha,\beta}) = 1$.
If $g \in L^2(0,1)$ fulfills $\Phi_\lambda[g] = g$ and $\sharp\,(\operatorname{argmin} F^g_{\alpha,\beta}) = 1$, then $\Phi_\lambda[u] = u$.
The set $E^n_{\alpha,\beta} = \{g \in L^2(0,1) : \quad \sharp\,(\operatorname{argmin} F^g_{\alpha,\beta}) = n\}$ fulfills

$$\Phi_\lambda[E^n_{\alpha,\beta}] = E^n_{\alpha,\beta} \qquad \forall \lambda \in \mathbb{R}.$$

Proof. For any $v, w \in \mathcal{H}^2(0,1)$ we have

$$\|\Phi_\lambda[v] - \Phi_\lambda[w]\|^2_{L^2} = \|v-w\|^2_{L^2}\,, \qquad \|(\Phi_\lambda[v])^{\cdot\cdot}\|^2_{L^2} = \|\ddot v\|^2_{L^2}$$

$$\sharp\,(S_v) = \sharp\,(S_{\Phi_\lambda[v]}), \qquad \sharp\,(S_{\dot v} \setminus S_v) = \sharp\,\left(S_{(\Phi_\lambda[v])^{\cdot}} \setminus S_{\Phi_\lambda[v]}\right),$$

then

$$F^g_{\alpha,\beta}(v) = F^{\Phi_\lambda[g]}_{\alpha,\beta}(\Phi_\lambda[v]) \qquad \forall v \in \mathcal{H}^2(0,1)$$

hence

$$m^{\Phi_\lambda[g]}(\alpha,\beta) \le m^g(\alpha,\beta).$$

Since $\Phi_\lambda[\Phi_\lambda[v]] = v$ the above argument is symmetric hence

$$m^{\Phi_\lambda[g]}(\alpha,\beta) = m^g(\alpha,\beta) = F^g_{\alpha,\beta}(u) = F^{\Phi_\lambda[g]}_{\alpha,\beta}(\Phi_\lambda[u])$$

where u belongs to $\operatorname{argmin} F^g_{\alpha,\beta}$. $\square$

3 Counterexamples to uniqueness

In this section we show that uniqueness for the minimizer of $F^g_{\alpha,\beta}$ cannot be proven for generic data α, β and g.
The first example is given in the case $\alpha = \beta$.

Counterexample 3.1 *Set* $\chi = \chi_{[\frac{1}{2},1]}$. *Let* w *be the unique minimizer of* $\mathcal{F}^\chi$ *in* $H^2(0,1)$. *Observe that* $F^\chi_{\alpha,\alpha}(\chi) = \alpha$ *and, since* $\chi \notin H^2(0,1)$, *there is* $\mu = \mu(\chi) > 0$ *with* $\mu := \mathcal{F}^\chi(w) = F^\chi_{\alpha,\alpha}(w)$. *Such* μ *is independent of* α. *Then the functional* $F^\chi_{\mu,\mu}$ *has at least two minimizers:* χ *and* w, *with* $\chi \neq w$ *since* $\chi \notin H^2(0,1)$.
Actually $F^\chi_{\mu,\mu}$ *has exactly two minimizers.*

To prove the last claim observe first that $F^\chi_{\mu,\mu}(u) > \mu$ if $\sharp\,(S_u \cup S_{\dot u}) \geq 2$.
Set $\mathcal{B} = \{u \in \mathcal{H}^2 : \quad \sharp\,(S_u) = 0,\ \sharp\,(S_{\dot u}) \leq 1\}$ and $\rho = \rho(\chi) = \inf\limits_{u\in\mathcal{B}} \mathcal{F}^\chi(z)$.
Referring to Theorem 2.6 case (i), $\mathcal{B} \subseteq \mathcal{S}[\chi]$ hence $\rho > 0$, in any case $\rho(\chi) \leq \mu(\chi)$ since $H^2(0,1) \subseteq \mathcal{B}$.
If $u \in \mathcal{B}$, we have either $S_u = S_{\dot u} = \emptyset$ then $F^\chi_{\mu,\mu}(u) \geq \mu$ with equality if and only if $u = w$; or $S_u = \emptyset$ and $\sharp\,(S_{\dot u}) = 1$, hence $F^\chi_{\mu,\mu}(u) \geq \rho + \mu > \mu$.
Eventually if $S_{\dot u} = \emptyset$ and $\sharp\,(S_u) = 1$ then either $u = \chi$ or $F^\chi_{\mu,\mu}(u) > \mu$. □

The previous example proves that there are α and g such that $F^g_{\alpha,\alpha}$ has exactly two minimizers. Now we show that $F^\chi_{\alpha,\beta}$ may have more than one minimizer for suitable α and a continuum of choices of β, say even if (1.5) holds true and $\frac{\alpha}{\beta} \notin \mathbb{Q}$. About irrational quotient of data α, β we refer to generic uniqueness statement in Theorem 1.1 of [4].

Counterexample 3.2 *Define* $\chi = \chi_{[\frac{1}{2},1]}$, w, $\mu = \mu(\chi)$, ρ *and* $\mathcal{B}$ *as in Counterexample 3.1: say* $F^\chi_{\alpha,\beta}(\chi) = \alpha$ *and* $F^\chi_{\alpha,\beta}(w) = \mu \geq \rho > 0$ *with* μ *and* ρ *independent of* α *and* β, *so that* $F^\chi_{\mu,\mu}$ *has exactly two minimizers (*χ, w*).*
We claim that for any $\beta \in (\mu - \varepsilon, \mu]$, $\varepsilon = \min\{\frac{\mu}{2}, \rho\} > 0$, *the functional* $F^\chi_{\mu,\beta}$ *has the same two minimizers* χ *and* w *and none more.*
In fact $\beta > \mu/2$, $\beta > \mu - \rho$ and

$$\beta \in (\mu - \varepsilon, \mu] \subseteq (\frac{\mu}{2}, \mu] \Rightarrow \begin{cases} F^\chi_{\mu,\beta}(\chi) = \mu \\ F^\chi_{\mu,\beta}(u) > \mu & \text{if } \sharp\,(S_u \cup S_{\dot u}) \geq 2. \end{cases}$$

Moreover $0 < \beta \leq \mu < 2\beta$ hence inequality (1.5) is fulfilled by the pair μ, β.
If $u \in \mathcal{B}$ we have: either $S_u = S_{\dot u} = \emptyset$ hence $F^\chi_{\mu,\beta}(u) \geq \mu$ with equality if and only if $u = w$, or $S_u = \emptyset$ and $\sharp\,(S_{\dot u}) = 1$ hence $F^\chi_{\mu,\beta}(u) \geq \rho + \beta > \rho + (\mu - \rho) = \mu$.
Eventually if $S_{\dot u} = \emptyset$ and $\sharp\,(S_u) = 1$ then either $u = \chi$ or $F^\chi_{\mu,\beta}(u) > \mu$. □

Counterexample 3.3 *Here we show that for any* α, β *satisfying the inequality* $0 < \beta \leq \alpha < 2\beta$ *(say a stronger constraint than (1.5)), there is* $g \in L^2(0,1)$, *for instance a multiple of* χ, *such that* $\sharp\,(\mathrm{argmin}\, F^g_{\alpha,\beta}) \geq 2$.

To prove the claim we exploit the homogeneity of $F^g_{\alpha,\beta}$:

$$F^{\lambda g}_{\lambda^2\alpha,\lambda^2\beta}(\lambda v) = \lambda^2\, F^g_{\alpha,\beta}(v) \qquad \forall \lambda \in \mathbb{R},\ \ \forall v \in \mathcal{H}^2,\ \ \forall \alpha, \beta \ \text{ s.t. } (1.5).$$

Then $F^{\lambda g}_{\lambda^2\alpha,\lambda^2\beta}$ has the same qualitative behaviour (with respect to uniqueness or non uniqueness of minimizers) of $F^{g}_{\alpha,\beta}$ for any $g \in L^2(0,1)$ and α, β satisfying (1.5).
Minimizers and minima of $F^{\lambda g}_{\lambda^2\alpha,\lambda^2\beta}$ are respectively λ and λ^2 times the minimizers and minima of $F^{g}_{\alpha,\beta}$.
We set $\lambda = \sqrt{\frac{\alpha}{\mu(\chi)}}$ where $\mu(\chi) = \min_{H^2}\{\mathcal{F}^\chi\} = \mathcal{F}^\chi(w)$.
If $u \in H^2(0,1)$, then either $F^{\lambda\chi}_{\alpha,\beta}(u) > F^{\lambda\chi}_{\alpha,\beta}(\lambda w) = \lambda^2\mu = \alpha$, or $u = \lambda w$ and $F^{\lambda\chi}_{\alpha,\beta}(u) = \alpha$.
If $\sharp\,(S_u) = 1$ and $\sharp\,(S_{\dot u}) = 0$, then either $F^{\lambda\chi}_{\alpha,\beta}(u) > F^{\lambda\chi}_{\alpha,\beta}(\lambda\chi) = \alpha$, or $u = \lambda\chi$.
If $\sharp\,(S_u \cup S_{\dot u}) \geq 2$, then $F^{\lambda\chi}_{\alpha,\beta}(u) > 2\beta \geq \alpha$, since $\int_0^1 |u - \lambda\chi|^2\, dx > 0$.

We are left to analyze the behaviour of functional $F^{\lambda\chi}_{\alpha,\beta}$ only in the set $\{u \in \mathcal{H}^2 : \quad \sharp\,(S_u) = 0,\ \sharp\,(S_{\dot u}) = 1\} \subset \mathcal{B}$.
Suppose first that $\rho(\chi) \geq \mu(\chi)/2$.
Since $1/2 < \beta/\alpha \leq 1$, Counterexample 3.2 implies that $F^{\chi}_{\mu,\mu\frac{\beta}{\alpha}} = F^{\chi}_{\lambda^{-2}\alpha,\lambda^{-2}\beta}$ admits exactly χ and w as minimizers. By scaling $F^{\chi}_{\lambda^{-2}\alpha,\lambda^{-2}\beta}$ behaves as $F^{\lambda\chi}_{\alpha,\beta}$.
Then $F^{\lambda\chi}_{\alpha,\beta}$ admits exactly $\lambda\chi$ and λw as minimizers and no more.
On the other hand suppose $\rho(\chi) < \mu(\chi)/2$.
Then, either we have the two minimizers $\lambda\chi$ and λw *of* $F^{\lambda\chi}_{\alpha,\beta}$, or there is a minimizer u of $F^{\lambda\chi}_{\alpha,\beta}$ with $S_u = \emptyset$ and $\sharp\,(S_{\dot u}) = 1$. In this last case consider the transformation Φ_λ defined by (2.23): since $\Phi_\lambda(\lambda\chi) = \lambda\chi$ Proposition 2.7 entails that $F^{\lambda\chi}_{\alpha,\beta}$ has at least two minimizers u and $\Phi_\lambda(u)$ which must be different since they have exactly one crease point. $\square$

Counterexample 3.4 *Here we show the existence of* $\mathcal{N} \subseteq L^2(0,1)$ *with non empty interior in the strong topology of* $L^2(0,1)$ *and such that for any* $g \in \mathcal{N}$ *there is* $\beta = \beta(g)$ *with* $0 < \beta \leq \min_{H^2(0,1)} \mathcal{F}^g < 2\beta$ *and* $\sharp\,(\operatorname{argmin} F^g_{\alpha,\beta}) \geq 2$ *for any* α *satisfying*

$$\beta \leq \min_{H^2(0,1)} \mathcal{F}^g < \alpha < 2\beta. \tag{3.1}$$

Notice that (3.1) entails (1.5).
To prove the above claim we choose $\mathcal{N}$ as a suitable L^2 neighborhood of a fixed function. Precisely we set

$$h(x) \overset{\text{def}}{=} \left| x - \frac{1}{2} \right|,\ \mu(g) \overset{\text{def}}{=} \min_{H^2(0,1)} \mathcal{F}^g,\ \mathcal{B} \overset{\text{def}}{=} \{u \in \mathcal{H}^2 : \quad \sharp\,(S_u) = 0,\ \sharp\,(S_{\dot u}) \leq 1\}.$$

We claim that

$$\exists\ L^2(0,1) \text{ open neighborhood } \mathcal{N} \text{ of } h: \quad \inf_{\mathcal{B}} \mathcal{F}^g < \frac{1}{2}\mu(g) \quad \forall g \in \mathcal{N}, \tag{3.2}$$

and this will be the choice of $\mathcal{N}$ leading to the counterexample.
To prove (3.2) we argue as follows. Consider $b = b[g](\cdot) \in \mathcal{H}^2(0,1)$ fulfilling

$$\left.\begin{array}{ll} b''''(x) + b(x) = g(x) & \textit{on } (0,1) \setminus \{1/2\}\,, \\ b''_+(1/2) = b''_-(1/2) = 0, & \\ b'''_+(0) = b'''_-(1) = 0, & \\ b'''_+(1/2) = b'''_-(1/2), & \\ b_+(1/2) = b_-(1/2). & \end{array}\right\} \qquad (3.3)$$

By direct inspection problem (3.3) has a unique solution. Moreover $\mathcal{F}^h(b[g])$ depends continuously in L^2 with respect to g. Also $\mu(g)$ has continuous dependence on g by elliptic regularity and Theorem 2.2.
Since $h \notin H^2$, we have

$$\mathcal{F}^h(b[h]) = \mathcal{F}^h(h) = 0 < \mu(h). \qquad (3.4)$$

Then (3.4) entails $\exists \mathcal{N} : 0 \le \mathcal{F}^g(b[g]) < \frac{1}{3}\mu(h) < \frac{2}{3}\mu(h) < \mu(g) \quad \forall g \in \mathcal{N}$, say

$\exists\, L^2(0,1)$ open neighborhood $\mathcal{N}$ of h :

$$0 \le \mathcal{F}^g(b[g]) < \frac{1}{2}\mu(g) \quad \forall g \in \mathcal{N}. \qquad (3.5)$$

For any $g \in \mathcal{N}$, $\mathcal{F}^g$ admits a minimizer over $\mathcal{B}$. In fact given $g \in \mathcal{N}$ and a minimizing sequence of $\mathcal{F}^g$ over $\mathcal{B}$, by Theorem 2.5(1,3) we can extract a subsequence w_n strongly convergent in L^1 to a function $w \in \mathcal{B}$ with $\dot{w}_n \to \dot{w}$ a.e. and $\ddot{w}_n \to \ddot{w}$ weakly in $L^2(0,1)$. By lower semicontinuity of $\mathcal{F}^g$ we have that w minimizes $\mathcal{F}^g$ over $\mathcal{B}$. By (3.5) w cannot belong to H^2, hence $S_{\dot{w}} \neq \emptyset$. By the same argument used in the proof of Theorem 2.1, w fulfills (i)-(iii) of Theorem 2.1. Then

$$\min_{\mathcal{B}} \mathcal{F}^g(u) = \mathcal{F}^g(w) \qquad \forall g \in \mathcal{N}.$$

Then claim (3.2) follows by (3.5) since

$$\min_{\mathcal{B}} \mathcal{F}^g \le \mathcal{F}^g(b[g]) < \frac{1}{2}\mu(g) \qquad \forall g \in \mathcal{N}. \qquad (3.6)$$

For any $g \in \mathcal{N}$ we set

$$\beta = \beta(g) \stackrel{\text{def}}{=} \mu(g) - \min_{\mathcal{B}} \mathcal{F}^g > \frac{1}{2}\mu(g) > 0. \qquad (3.7)$$

Then $\beta < \mu(g) < 2\beta$ and we can choose any α such that

$$0 < \beta \le \mu(g) < \alpha < 2\beta. \qquad (3.8)$$

With the above choices for α, β and $\mathcal{N}$ by (3.2)-(3.8) we get:

- $F^g_{\alpha,\beta}(u) \geq 2\beta > \mu(g)$ for any $u \in \mathcal{H}^2$ with $\sharp\,(S_{\dot u} \setminus S_u) > 1$,
- $F^g_{\alpha,\beta}(u) \geq \alpha > \mu(g)$ for any $u \in \mathcal{H}^2$ with $\sharp\,(S_u) > 0$,
- $\min_{\mathcal{H}^2} F^g_{\alpha,\beta} = \min_{\mathcal{B}} F^g_{\alpha,\beta} = F^g_{\alpha,\beta}(w) = \min_{\mathcal{B}} \mathcal{F}^g + \beta = \mu = \min_{H^2} F^g_{\alpha,\beta}$.

Since $w \notin H^2$, the minimizers of $F^g_{\alpha,\beta}$ over $\mathcal{H}^2$ are at least two: the minimizers of $F^g_{\alpha,\beta}$ over $\mathcal{B}$ and the unique minimizer of $F^g_{\alpha,\beta}$ over H^2. $\square$

4 Free discontinuity set of a minimizer may live outside $S_g \cup S_{\dot g}$

Besides the non convexity of $F^g_{\alpha,\beta}$ the following issue is among the main difficulties in the proof of generic uniqueness of minimizers: jump and crease points of a minimizer are not necessarily contained in $S_g \cup S_{\dot g}$. Moreover a minimizer u with $S_u \cup S_{\dot u} \not\subseteq S_g \cup S_{\dot g}$ may occur even with continuous piecewise affine datum g. This issue and the presence of the two parameters α and β instead of one prevents straightforward adaptation of methods used in [2], therefore we will employ different technical arguments in the proof of generic uniqueness of minimizers (see [4]). In this section we give an example of piecewise affine continuous functions exhibiting such phenomenon.

Theorem 4.1 *Define the following family of functions $g \in L^2(0,1)$ dependent on the parameter $a \in \mathbb{R}$*

$$g[a](x) = \left(\left| x - \frac{1}{2} \right| - a \right) \vee 0, \qquad x \in [0,1]. \tag{4.1}$$

Then:

$$\begin{gathered} S_{g[a]} = \emptyset \textit{ and } S_{\dot g[a]} = \left\{ \frac{1}{2} - a, \frac{1}{2} + a \right\} \qquad \forall a \in [0, \frac{1}{2}), \\ \exists \alpha, \beta \textit{ fulfilling with (1.5)}, \widetilde{a} > 0 \textit{ s.t.} \\ S_u = \emptyset, \quad S_{\dot u} \neq \emptyset, \quad S_{\dot u} \cap S_{\dot g[a]} = \emptyset \qquad \forall u \in \operatorname{argmin} F^{g[a]}_{\alpha,\beta} \quad \forall a \in (0, \widetilde{a}), \end{gathered} \tag{4.2}$$

so that $\emptyset \neq S_{\dot u} \not\subseteq S_g \cup S_{\dot g}$ for any $a \in (0, \widetilde{a})$.
Moreover either $S_{\dot u} = \left\{\frac{1}{2}\right\}$ or there is non uniqueness of minimizers for $F^{g[a]}_{\alpha,\beta}$.

Proof. Define $\mathcal{H}^{2,\mathsf{j},\mathsf{c}} = \{u \in \mathcal{H}^2$ such that $\sharp\,(S_u) = \mathsf{j}$ and $\sharp\,(S_{\dot u} \setminus S_u) = \mathsf{c}\}$.
Step 1 - We claim

$$\exists \overline{a} > 0,\ \alpha,\ \beta \text{ with (1.5) s.t. } \min_{\mathcal{H}^2} F^{g[a]}_{\alpha,\beta} = \min_{\mathcal{H}^{2,0,1}} F^{g[a]}_{\alpha,\beta} \quad \forall a \in (0, \overline{a}). \tag{4.3}$$

To prove (4.3), we set

$$\mu_1 = \mu_1(a) = \min_{u\in H^2(0,1)} \mathcal{F}^{g[a]}(u),$$

$$\mu_2 = \mu_2(a) = \inf_{u\in \mathcal{H}^{2,0,1}} \mathcal{F}^{g[a]}(u), \qquad \mu_3 = \mu_3(a) = \inf_{u\in \mathcal{H}^{2,1,0}} \mathcal{F}^{g[a]}(u),$$

then μ_1 depends continuously on a since the map $a \mapsto g[a]$ is continuous from $\mathbb{R}$ to $L^2([0,1])$, $m = \mu_1(0) > 0$ since $g[0] = \left|x - \frac{1}{2}\right| \in \mathcal{H}^{2,0,1} \setminus H^2(0,1)$. Moreover

$$0 < \mu_3(a) \le \mu_2(a) \le \mu_1(a)\,, \tag{4.4}$$

in fact the first inequality in (4.4) holds true since $g[a]$does not belong to $H^2(0,1) \cup \mathcal{H}^{2,0,1} \cup \mathcal{H}^{2,1,0}$, the second inequality holds true by semicontinuity and the fact that for any $u \in \mathcal{H}^{2,0,1}$ there is a sequence $\{u_n\} \subseteq \mathcal{H}^{2,1,0}$ with $S_{u_n} = S_{\dot u}$ for any n such that $u_n \to u$ strongly in $H^2((0,1)\setminus S_{\dot u})$, and the last inequality follows from the embedding $H^2 \subseteq \mathcal{H}^{2,0,1}$. Then

$$\lim_{a\to 0^+} \mu_2(a) = \mu_2(0) = 0 \qquad \lim_{a\to 0^+} \mu_3(a) = \mu_3(0) = 0 \tag{4.5}$$

For any $\eta \in [1,2)$ we choose $\delta = \delta(a) > \left(\mu_3 - \frac{\mu_1}{2}\right) \vee \frac{\mu_3}{\eta} > 0$ and define

$$\alpha = \alpha(a,\delta) = \mu_1 - \mu_3 + \delta\,, \qquad \beta = \beta(a,\eta,\delta) = \tfrac{\mu_1 - \mu_3 + \eta\delta}{2} \tag{4.6}$$

which will be briefly denoted α and β whenever there is no risk of confusion. Then

$$0 < \beta < \alpha \le 2\beta, \tag{4.7}$$

$$F^{g[a]}_{\alpha,\beta}(u) \ge \mu_3 + \alpha > \mu_1 \quad \text{for any} \quad u \in \mathcal{H}^{2,1,0}, \tag{4.8}$$

$$F^{g[a]}_{\alpha,\beta}(u) \ge 2\alpha > \mu_1 \quad \text{for any} \quad u \in \mathcal{H}^{2,\mathsf{j},\mathsf{c}} \quad \text{with} \quad \mathsf{j} > 1, \tag{4.9}$$

$$F^{g[a]}_{\alpha,\beta}(u) \ge 2\beta > \mu_1 \text{ for any } u \in \mathcal{H}^{2,\mathsf{j},\mathsf{c}} \text{ with } \mathsf{c} > 1 \text{ or } (\mathsf{j},\mathsf{c}) = (1,1). \tag{4.10}$$

By summarizing (4.7)-(4.10)

$$\left\{\operatorname{argmin} F^{g[a]}_{\alpha,\beta}\right\} \subseteq H^2(0,1) \cup \mathcal{H}^{2,0,1}. \tag{4.11}$$

Since $\mu_1 \to m$ and $\mu_2, \mu_3 \to 0$ as $a \to 0$ we can fix η and δ as before and such that $m > \eta\delta$ and choose $\varepsilon \in (0, \frac{1}{6}(m - \eta\delta))$ and $\overline{a}$ such that

$$0 < \mu_3 \le \mu_2 < \varepsilon, \qquad \left|\beta - \frac{1}{2}(m + \eta\delta)\right| < \varepsilon, \quad |m - \mu_1| < \varepsilon \qquad \forall a \in (0, \overline{a}). \tag{4.12}$$

Hence inequalities (4.12) entail

$$\mu_2 + \beta - \mu_1 \leq \varepsilon + \frac{1}{2}(m + \eta\delta) + \varepsilon - m + \varepsilon = 3\varepsilon - \frac{1}{2}(m - \eta\delta) < 0,$$

say $\mu_2 + \beta < \mu_1$ for any $a \in (0, \overline{a})$, hence (4.3) follows by (4.11).

Step 2 - We deduce the thesis starting by (4.3) and solving the Euler system of Theorem 2.1 related to one crease point at $x = t \in (0,1)$ and no jump point.

Consider $b = b[a,t](\cdot) \in \mathcal{H}^2(0,1)$ fulfilling

$$\left.\begin{array}{l} b''''(x) + b(x) = g[a](x) \quad \text{on } (0,1) \setminus \{t\}\,, \\ b''_+(t) = b''_-(t) = 0, \\ b'''_+(0) = b'''_-(1) = 0, \\ b'''_+(t) = b'''_-(t), \\ b_+(t) = b_-(t). \end{array}\right\} \tag{4.13}$$

By direct inspection problem (4.13) has a unique solution. We emphasize that problem (4.13) is a particular case of a general differential problem related to multiple jump points and crease points which will be discussed in [4], Theorem 2.8. Then we can define

$$\psi(a,t) = \mathcal{F}^{g[a]}(b[a,t]).$$

Symmetry of $g[a]$ with respect to $\frac{1}{2}$ (say $g[a](x) = g[a](1-x)$) entails analogous symmetry for the solution of differential problem (4.13):

$$b[a,t](x) = b[a,1-t](1-x) \quad \forall a \in (0,\overline{a}), \tag{4.14}$$

$$\psi(a,t) = \psi(a,1-t) \quad \forall a \in (0,\overline{a}), \tag{4.15}$$

$$\psi(a,\frac{1}{2}-a) = \psi(a,\frac{1}{2}+a) \quad \forall a \in (0,\overline{a}). \tag{4.16}$$

Eventually we set $\varphi(a) = \psi(a,\frac{1}{2}-a) - \psi(a,\frac{1}{2})$. Since $\varphi(0) = 0$, if we prove $\varphi'_+(0) > 0$ then for suitable $\widetilde{a} \in (0,\overline{a})$ the thesis (4.2) follows.

To establish inequality $\varphi'_+(0) > 0$ we exploit Euler equations and compliance identity and we employ the software Maple© as follows (the coded instruction is contained in the appendix): first we use the symbolic computation to find the exact formula for $\psi(a,\frac{1}{2}-a)$, $\psi(a,\frac{1}{2})$ and $\varphi(a)$, then we compute exactly the right total derivarive $\varphi'_+(0)$ of φ at $a = 0$, eventually we numerically compute the value of $\varphi'_+(0)$ with error estimates and get $\varphi'_+(0) > 0$.

The above proof shows only that

$$F^{g[a]}_{\alpha,\beta}(b[a,1/2]) < F^{g[a]}_{\alpha,\beta}(b[a,1/2 \pm a]) \quad \forall a \in (0,\widetilde{a})$$

but does not entail $b[a, 1/2] \in \operatorname{argmin} F^{g[a]}_{\alpha,\beta}$. Nevertheless, if $b[a, 1/2] \notin \operatorname{argmin} F^{g[a]}_{\alpha,\beta}$, then $u(x)$ and $u(1-x)$ are both minimizers and they do not coincide, since any minimizer must have exactly one crease point. □

5 Appendix: Symbolic and numeric computations

In this section we provide the Maple© procedure used to show that $\varphi'_+(0) > 0$ in the proof of Theorem 4.1.

1. Canonical base of $\ker\left(\frac{d^4}{dt^4} + I\right)$**.**

```
> w_1(x) := exp(-1/2*sqrt(2)*x)*cos(1/2*sqrt(2)*x);
> w_2(x) := exp(1/2*sqrt(2)*x)*cos(1/2*sqrt(2)*x);
> w_3(x) := exp(-1/2*sqrt(2)*x)*sin(1/2*sqrt(2)*x);
> w_4(x) := exp(1/2*sqrt(2)*x)*sin(1/2*sqrt(2)*x);
```

2. A solution of the homogeneous equation in $[1/2 - a, 1/2 + a]$**.**

```
> dsolve({diff(d(x),x,x,x,x)+d(x)=0,
> d(1/2-a)=0,D(d)(1/2-a)=-1,
> D(D(d))(1/2-a)=0,D(D(D(d)))(1/2-a)=0});
```

3. Solution of differential system (4.13) with $t = 1/2$**, and compliance identity.**

```
> w(C_1,C_2,C_3,C_4,x) :=
> C_1*w_1(x)+C_2*w_2(x)+C_3*w_3(x)+C_4*w_4(x);
> dsolve(
> {diff(d(x),x,x,x,x)+d(x)=0,
> d(1/2-a)=0,
> D(d)(1/2-a)=-1,
> D(D(d))(1/2-a)=0,
> D(D(D(d)))(1/2-a)=0});
> solve(
> {eval(diff(w(C_1,C_2,C_3,C_4,x),x,x),x=0)=0,
> eval(diff(w(C_1,C_2,C_3,C_4,x),x,x,x),x=0)=0,
> eval(diff(d(x),x,x),x=1/2)+
> eval(diff(w(C_1,C_2,C_3,C_4,x),x,x),x=1/2)=0,
> eval(diff(d(x),x,x,x),x=1/2)+
> eval(diff(w(C_1,C_2,C_3,C_4,x),x,x,x),x=1/2)=0},
> {C_1,C_2,C_3,C_4});
```

```
> v(x) :=
> C_1*w_1(x)+C_2*w_2(x)+C_3*w_3(x)+C_4*w_4(x);
```

```
> ComplianceInTheMiddle(a) :=
> 2*(int((-x+1/2-a)^2,x=0..1/2-a)-
> int((-x+1/2-a)*(-x+1/2-a+v(x)),x=0..1/2-a));
```

```
> FirstDerivativeComplianceInTheMiddle :=
> simplify(coeftayl(ComplianceInTheMiddle(a),a=0,1));
```

4. Solution of differential system (4.13) with t $= 1/2 + a$, and compliance identity.

```
> w_0(C_01,C_02,C_03,C_04,x) :=
> C_01*w_1(x)+C_02*w_2(x)+C_03*w_3(x)+C_04*w_4(x);
```

```
> w_1(C_11,C_12,C_13,C_14,x) :=
> C_11*w_1(x)+C_12*w_2(x)+C_13*w_3(3)+C_14*w_4(x);
```

```
> solve(
> {eval(diff(w_0(C_01,C_02,C_03,C_04,x),x,x),x=0)=0,
> eval(diff(w_0(C_01,C_02,C_03,C_04,x),x,x,x),x=0)=0,
> eval(diff(d(x),x,x),x=1/2+a)+
> eval(diff(w_0(C_01,C_02,C_03,C_04,x),x,x),x=1/2+a)=0,
> eval(diff(d(x),x,x,x),x=1/2+a)+
> eval(diff(w_0(C_01,C_02,C_03,C_04,x),x,x,x),x=1/2+a)=
> eval(diff(w_1(C_11,C_12,C_13,C_14,x),x,x,x),x=1/2+a),
> eval(d(x),x=1/2+a)+eval(w_0(C_01,C_02,C_03,C_04,x),x=1/2+a)=
> eval(w_1(C_11,C_12,C_13,C_14,x),x=1/2+a),
> eval(diff(w_1(C_11,C_12,C_13,C_14,x),x,x),x=1)=0,
> eval(diff(w_1(C_11,C_12,C_13,C_14,x),x,x),x=1/2+a)=0,
> eval(diff(w_1(C_11,C_12,C_13,C_14,x),x,x,x),x=1)=0},
> {C_01,C_02,C_03,C_04,C_11,C_12,C_13,C_14});
```

```
> u_0(x) :=
> C_01*w_1(x)+C_02*w_2(x)+C_03*w_3(x)+C_04*w_4(x);
> u_1(x) :=
> C_11*w_1(x)+C_12*w_2(x)+C_13*w_3(x)+C_14*w_4(x);
> ComplianceRight(a) :=
> 2*int((-x+1/2-a)^2,x=0..1/2-a)-
> int((-x+1/2-a)*(-x+1/2-a+u_0(x)),x=0..1/2-a)-
```

```
> int((x-1/2-a)*(x-1/2-a+u_1(x)),x=1/2+a..1);
> FirstDerivativeComplianceRight :=
> coeftayl(ComplianceRight(a),a=0,1);
```

5. Evaluation of the first derivative for $a = 0$.

```
> FinalEvaluation :=
> evalf(FirstDerivativeComplianceRight-
> FirstDerivativeComplianceInTheMiddle);
```

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