Optimal design of thin plates by a dimension reduction for linear constrained problems.

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Abstract

Our main goal is to give a rigorous justification for the Hessian-constrained problems introduced in [13], and to show how they are linked to the optimal design of thin plates. To that aim, we study the asymptotic behaviour of a sequence of optimal elastic compliance problems, in the double limit when both the maximal height of the design region and the total volume of the material tend to zero. In the vanishing volume limit, a sequence of linear constrained first order vector problems is obtained, which in turn - in the vanishing thickness limit - produces a new linear constrained problem where both first and second order gradients appear. When the load is suitably chosen, only the Hessian constraint is active, and we recover exactly the plate optimization problem studied in [13]. Some attention is also paid to the possible different approaches to the afore mentioned double limit process, in both the cases of real and ficticious materials, which might favour some debate on the modelling of thin plates.

Keywords: thin plates, optimization, compliance, linear constrained problems, positive measures, Γ-convergence.

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1 Introduction

Let Ω be an open bounded connected subset of $\mathbb{R}^2$ with a smooth boundary. In [13] we considered the following *mass optimization problem*, which consists in finding the optimal distribution of a given amount of plate-like material in the design region $\overline{\Omega}$ in order to minimize the work made on it by a given system of forces:

$$\mathcal{I} = \inf \left\{ \mathcal{C}^{\rm pl}(\mu, j, f) \ : \ \mu \in \mathcal{P}(\overline{\Omega}) \right\} . \tag{1.1}$$

Here measures μ in the space $\mathcal{P}(\overline{\Omega})$ of probabilities on $\overline{\Omega}$ represent the admissible designs, which are allowed to be diffused as well as concentrated on low-dimensional sets. The cost $\mathcal{C}^{\rm pl}(\mu, j, f)$

that we want to minimize is the *plate compliance*: for any $\mu \in \mathcal{P}(\overline{\Omega})$, for a given stored energy density $j : \mathbb{R}^{2\times 2}_{\rm sym} \to \mathbb{R}$, and for a given real measure $f \in \mathcal{M}(\overline{\Omega};\mathbb{R})$, it is obtained as

$$\mathcal{C}^{\rm pl}(\mu, j, f) := -\inf \left\{ \int j(\nabla^2 u)\, d\mu - \langle f, u\rangle_{\mathbb{R}^2} \ : \ u \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}) \right\} . \tag{1.2}$$

In particular, in [13] we established the equality

$$\mathcal{I} = \mathcal{S}^2/2 , \tag{1.3}$$

where $\mathcal{S}$ is computed through the following *linear constrained problem*:

$$\mathcal{S} = \sup \left\{ \langle f, u\rangle_{\mathbb{R}^2} \ : \ u \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}) \text{ such that } \rho(\nabla^2 u) \le 1 \text{ on } \Omega \right\} \tag{1.4}$$

(being ρ related to j by $j(z) = (1/2)\rho^2(z)$). Moreover, we proved that problems (1.1) and (1.4) share the same optimality conditions, which can be explicitly determined.
The goal of this paper is to give a rigorous justification for problems of kind (1.1) or (1.4), and show how they are linked to the optimal design of thin plates. In fact in [13] these problems were introduced just *formally*, as the second order analogous of their corresponding first order problems. When the design region is a subset of $\mathbb{R}^3$ of the form $Q = \overline{\Omega} \times [-h, h] \subset \mathbb{R}^2 \times \mathbb{R}$, the *elastic compliance* of a mass distribution $\mu \in \mathcal{P}(Q)$, for a given density $j : \mathbb{R}^{3\times 3}_{\rm sym} \to \mathbb{R}$ and a given measure load $F \in \mathcal{M}(Q;\mathbb{R}^3)$ is given by

$$\mathcal{C}^{\rm el}(\mu, j, F) := -\inf \left\{ \int j(e(U))\, d\mu - \langle F, U\rangle_{\mathbb{R}^3} \ : \ U \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \right\} , \tag{1.5}$$

where $e(U)$ denotes the symmetric gradient of U. Then the first order $3D$-versions of (1.1) and (1.4) read respectively:

$$\inf \left\{ \mathcal{C}^{\rm el}(\mu, j, F) \ : \ \mu \in \mathcal{P}(Q) \right\} \tag{1.6}$$

$$\sup \left\{ \langle F, U\rangle_{\mathbb{R}^3} \ : \ U \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \text{ such that } \rho(e(U)) \le 1 \text{ on } Q \right\} . \tag{1.7}$$

These problems were studied in detail in [10]; in particular it turns out that they are related to each other by the condition analogous to (1.3). From a mechanical point of view, they are perfectly justified: when one tries to optimize the compliance of an elastic material under a given load, in the limit of *vanishing volume* microstructures appear - meaning that the material tends to occupy low-dimensional networks - and the limit problem is of type (1.6). This is true both in the case of real materials, due to a common-use result in shape optimization, and in the case of so-called "ficticious" materials, see Section 3.1 for more details.
The question is now: *do problems of type* (1.1) *admit any mechanical justification?* In particular: *are they somehow linked to problems* (1.6) *and to the extensive existing literature on thin plates?* This paper is an attempt to answer these questions. The approach we adopt is new, and consists in performing a $3D-2D$ reduction dimension analysis for problems of type (1.6). More precisely, we investigate the asymptotics of problems (1.6) in the *vanishing thickness* limit, namely when

the maximal height h is multiplied by an infinitesimal parameter δ and the design region is taken of the form $Q_\delta = \overline{\Omega} \times [-h\delta, h\delta] \subset \mathbb{R}^2 \times \mathbb{R}$. To that purpose, a quite natural idea - formerly unexplored to our knowledge - is to exploit the first order analogous of (1.3). Indeed by this way one is led to study the asymptotics as $\delta \to 0$ of the simpler problems (1.7), when Q is replaced by Q_δ. Now, one might expect that such suprema remain finite as $\delta \to 0$, and that the convex set of constraint appearing in the limit problem, which is none else than the Kuratowski limit of the sets

$$K_\delta := \Big\{ U \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \text{ such that } \rho(e(U)) \leq 1 \text{ on } \Omega \times (-\delta h, \delta h) \Big\} ,$$

is given by functions whose first order gradient satisfies some suitable relation. Actually, facts come up to these expectations only in the scalar case, namely when functions U in K_δ take real values (see Remark 3.7). In spite, in the vector case when functions U in K_δ take values in $\mathbb{R}^3$, the situation is dramatically different. Firstly, if the vertical component of the force is of order 1, the suprema in (1.7) blow up to infinity (like δ^{-1}). Then, we need to rescale the third component of the force by a factor δ. After such scaling, another crucial difference with respect to the scalar case shows up when studying the Kuratowski limit of K_δ: indeed, due to the role played by a specific strain-displacement relation (of Kirchoff-Love type), two independent constraints appear, each one involving both first and second order derivatives. This analytical fact has an immediate mechanical counterpart: when the load is suitably scaled, a bending effect coupled with membrane energy appears in the limit problem, which can be written as

$$(\mathcal{P}) \qquad \sup \Big\{ \langle \overline{F}, v \rangle_{\mathbb{R}^2} \ : \ v \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}^3) \text{ such that } \ \overline{\rho}(e(v_1, v_2) \pm h \nabla^2 v_3) \leq 1 \text{ on } \Omega \Big\} ,$$

for a suitably averaged system of forces $\overline{F}$ and a suitably modified function $\overline{\rho}$ (see Theorem 3.3). Problem $(\mathcal{P})$ reduces to a problem of type (1.4) in the particular case when the unique nonzero component of the load is the vertical one, because in such case the double constraint imposed on fields v simplifies into one inequality for the Hessian matrix of their third component v_3.
This amounts to say - see Corollary 3.5 - that problems of type (1.1) are recovered as $3D - 2D$ limits of problems of type (1.6) when the load is a vertical one. In particular, for such kind of loads, the optimality conditions found in [13] can be fruitfully employed in order to determine explicit solutions to problem $(\mathcal{P})$. For arbitrary loads, the optimality system has to be suitably generalized in order to cover the case of mixed regimes, see Proposition 3.9 and the examples in Section 5.
Further possible justifications for problems of type (1.1) are discussed in this paper, and will be studied more in detail in a forthcoming one. The background is still a sequence of classical $3D$-elasticity problems, where both the maximal height of the design and the total volume of the material are multiplied by infinitesimal parameters, say δ and ε respectively. Actually, the strategy described above consists in passing to the limit first in ε - which yields problems of type (1.7) - and then in δ, ending up with problems of type $(\mathcal{P})$ (or (1.1)). However, it is tempting to look at different ways of performing the double limit in δ and ε. More precisely, we believe that problems of type (1.1) can be recast by passing to the limit *contemporarily* in ε and δ, keeping

the quotient $\eta := \delta/\varepsilon$ fixed, and eventually letting η tend to $+\infty$. Following this alternative strategy, we are led to propose a limit compliance model of the kind (1.1) which fits together the original shape optimization problem for real materials and the case of ficticious materials, see Proposition 4.1.
Finally, we can also link our approach with the classical thin plates model widely studied in the literature, where a cubic dependance on the profile of the plate appears (without any attempt of being complete, let us refer the reader to [4, 5, 6, 7, 8, 16, 18, 20, 22, 23, 24, 25]). In our setting, this corresponds to enclose a topologicalconstraint on the admissible sets. After a suitable scaling, the limit problem is conjectured to be once again of type (1.1), for a different stored energy, see Proposition 4.3.
The paper is organized as follows. In Section 2 we fix some notation and the setting of the problem, then we state our main results in Section 3. In Section 4 we discuss the above mentioned alternative genesis for problems of type (1.1). Section 5 is entirely devoted to exemplify the application of the results obtained in Section 3. Proofs are collected in Section 6. Finally in the Appendix we compute the possible different effective energy densities when one starts from a classical elastic potential.

2 Preliminaries and setting of the problem.

Let us take a design region in $\mathbb{R}^3$ of the form $Q = \overline{\Omega} \times [-h, h]$, where Ω is an open bounded connected subset of $\mathbb{R}^2$ and h is fixed in $\mathbb{R}^+$; the spatial variable in Q will be denoted by (x', x_3). Consider a given amount m of elastic material placed in a subset A of the design region: thus A is subject to the constraints

$$A \subseteq Q = \overline{\Omega} \times [-h, h] \ , \qquad \mathrm{vol}(A) = m \ .$$

If the stored energy density is represented by a given integrand $j : \mathbb{R}^{3\times3}_{\rm sym} \to \mathbb{R}$ and the material is subject to a given system of forces $F = (F_1, F_2, F_3) \in \mathcal{M}(Q; \mathbb{R}^3)$, the resulting elastic compliance is given by

$$\mathcal{C}^{\rm el}\left(A, j, F\right) := -\inf\Big\{ \int_A j(e(U))\, dx - \langle F, U\rangle_{\mathbb{R}^3} \ : \ U \in \mathcal{C}^\infty(\mathbb{R}^3; \mathbb{R}^3) \Big\}$$

(here and in the following, $e(U)$ denotes the symmetric part of the gradient of U).
We assume that j is convex, 2-homogeneous, and coercive, so that it can be written as

$$j(z) = \frac{1}{2}\rho^2(z) \ , \qquad \text{with } \inf_{z\neq 0} \frac{\rho(z)}{|z|} > 0 \ . \tag{2.1}$$

The typical choice of j is the usual quadratic elastic potential of the kind

$$j(z) = \frac{\lambda}{2}(\mathrm{tr}(z))^2 + \mu|z|^2 \ . \tag{2.2}$$

Moreover, for the compliance to be finite, we ask that the system of forces is *balanced*, namely

$$\langle F, U\rangle_{\mathbb{R}^3} = 0 \quad \text{whenever } e(U) = 0 \tag{2.3}$$

and also that it belongs to the Sobolev space $H^{-1}(Q;\mathbb{R}^3)$.
We want to consider now the problem of optimizing the compliance when both the maximal height of the design and the total volume of the material become very small. In this situation the maximal height and the total volume will be multiplied by two positive vanishing parameters, say δ and ε respectively:

$$A \subseteq Q_\delta = \overline{\Omega} \times [-\delta h, \delta h] \;, \qquad \mathrm{vol}(A) = \varepsilon m \;. \tag{2.4}$$

The same optimization problem can be considered also for "fictitious materials", that is when the set A is replaced by a density θ satisfying

$$\theta \in L^\infty(\mathbb{R}^3;[0,1]) \;, \qquad \mathrm{spt}(\theta) \subseteq Q_\delta \;, \qquad \int \theta \, dx = \varepsilon m \;, \tag{2.5}$$

and the definition of compliance is extended by setting

$$\mathcal{C}^{\mathrm{el}}\left(\theta, j, F\right) := -\inf \Big\{ \int j(e(U))\,\theta\, dx - \langle F, U\rangle_{\mathbb{R}^3} \;:\; U \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \Big\} \;. \tag{2.6}$$

So we focus attention on the two variational problems

$$\inf \Big\{ \mathcal{C}^{\mathrm{el}}\left(A, j, F\right) \;:\; A \text{ satisfying } (2.4) \Big\} \tag{2.7}$$

$$\inf \Big\{ \mathcal{C}^{\mathrm{el}}\left(\theta, j, F\right) \;:\; \theta \text{ satisfying } (2.5) \Big\} \;. \tag{2.8}$$

The asymptotics of the above infima for $\varepsilon, \delta \to 0$ can be investigated by adopting one of the two following strategies (A) or (B) (notice indeed that δ cannot go to zero for fixed ε):

(A) *Step 1.* Keeping δ fixed, let ε tend to zero (so that the quotient $\eta := \delta/\varepsilon$ tends to $+\infty$).
Step 2. Let δ tend to zero.

(B) *Step 1.* Keeping the quotient $\eta = \delta/\varepsilon$ fixed, let ε and δ tend to zero contemporarily.
(*Step 2.* Possibly let η tend to $+\infty$.)

In this paper we are mainly concerned with strategy (A), which seems to be the simpler way leading from the infima in (2.7) or (2.8) to a problem of kind (1.1) (see Section 3). However, we also discuss briefly strategy (B), showing that it may lead to a limit problem of the same kind, see Section 4.
The first crucial remark when approaching the problem through strategy (A) is that the infima in (2.7) or (2.8) blow up at each of the two steps.

More precisely, if δ is fixed and ε tend to zero, the infima are of order ε^{-1}. Indeed, via the change of variables $V = U/\varepsilon$, it is easy to obtain the identity

$$\mathcal{C}^{\rm el}\left(\frac{\theta}{\varepsilon}, j, F\right) = \varepsilon\,\mathcal{C}^{\rm el}\left(\theta, j, F\right) \ ,$$

whose left hand side has a finite infimum for θ satisfying (2.5). Therefore, we are led to rescale the system of forces into $\sqrt{\varepsilon}F$; this will ensure that the infimum of the compliance remains finite as ε tends to zero in view of the identity

$$\mathcal{C}^{\rm el}\left(\theta, j, \sqrt{\varepsilon}F\right) = \varepsilon\,\mathcal{C}^{\rm el}\left(\theta, j, F\right) \ .$$

In turn, the infima obtained through the first step of strategy (A) blow up again when performing the second step, that is when also δ tends to zero. Thus, we need to rescale the system of forces also with respect to δ. It will be more clear later on (see the proof of Theorem 3.3), that the right scaling of the load in order to keep finite the suprema in (3.3) as $\delta \to 0$ is the following one: set $Q_\delta := \overline{\Omega} \times [-\delta h, \delta h]$, and change F into the element $F^\delta \in H^{-1}(Q_\delta; \mathbb{R}^3)$ which acts on any test function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^3; \mathbb{R}^3)$ as

$$\langle F^\delta, \varphi \rangle_{\mathbb{R}^3} := \sum_{i=1}^{2} \langle F_i(x), \varphi_i(x', \delta x_3) \rangle_{\mathbb{R}^3} + \delta \langle F_3(x), \varphi_3(x', \delta x_3) \rangle_{\mathbb{R}^3} \ .$$

We stress that, in the above definition, the vertical component F_3 is multiplied by δ, as it is usual when dealing with plates in flexion regime.

Summarizing, our rescaled optimization problems read

$$\mathcal{I}_{\varepsilon,\delta} := \inf \Big\{ \mathcal{C}^{\rm el}\left(A, j, \sqrt{\varepsilon}F^\delta\right) \ : \ A \text{ satisfying } (2.4) \Big\} \ , \tag{2.9}$$

$$\widetilde{\mathcal{I}}_{\varepsilon,\delta} := \inf \Big\{ \mathcal{C}^{\rm el}\left(\theta, j, \sqrt{\varepsilon}F^\delta\right) \ : \ \theta \text{ satisfying } (2.5) \Big\} \ . \tag{2.10}$$

Notice that, for each fixed (ε, δ), $\mathcal{I}_{\varepsilon,\delta}$ and $\widetilde{\mathcal{I}}_{\varepsilon,\delta}$ should remain finite because $\sqrt{\varepsilon}F^\delta$ is still balanced, that is it fulfills (2.3). Further, in view of the heuristic considerations above, we expect that $\mathcal{I}_{\varepsilon,\delta}$ and $\widetilde{\mathcal{I}}_{\varepsilon,\delta}$ admit finite limits as ε and δ tend to zero. In the remaining of the paper our goal is to identify such limits.

For simplicity of notation, in the sequel we take the volume parameter m appearing in (2.4) and (2.5) equal to 1 (this is not restrictive up to a multiplicative factor).

3 Strategy (A): main results

Subsections 3.1 and 3.2 below are devoted respectively to steps 1 and 2 of strategy (A). All the statements (but the one of Proposition 3.2) will be proved in Section 6.

3.1 Step 1: $\varepsilon \to 0$ with δ fixed.

The main advantage of strategy (A) is actually that its first step yields a pretty tractable limit problem. Indeed, when one performs the limit as $\varepsilon \to 0$ of $\mathcal{I}_{\varepsilon,\delta}$ or of $\widetilde{\mathcal{I}}_{\varepsilon,\delta}$, in both cases one falls upon an infimum problem over the space $\mathcal{P}(Q_\delta)$ of probabilities on Q_δ. Moreover, in both cases the functional to be minimized in the limit problem is of the kind $\mu \mapsto \mathcal{C}^{\mathrm{el}}\left(\mu, \mathcal{J}, F^\delta\right)$, where for a given integrand $\mathcal{J}$ we have set

$$\mathcal{C}^{\mathrm{el}}\left(\mu, \mathcal{J}, F^\delta\right) := -\inf\left\{ \int \mathcal{J}(e(U))\, d\mu - \langle F^\delta, U\rangle_{\mathbb{R}^3} \ :\ U \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)\right\} .$$

The only difference between the real and the ficticious case lies in the determination of the integrand $\mathcal{J}$: in the ficticious case one can take simply $\mathcal{J} = j$, while in the real case one has to take $\mathcal{J} = j_0$, being j_0 obtained from j through a suitable formula. This is stated more precisely in the next two propositions.

Proposition 3.1 (Fictitious materials.) *There holds:*

$$\lim_{\varepsilon\to 0} \widetilde{\mathcal{I}}_{\varepsilon,\delta} = \widetilde{\mathcal{I}}_\delta := \inf\left\{ \mathcal{C}^{\mathrm{el}}\left(\mu, j, F^\delta\right) \ :\ \mu \in \mathcal{P}(Q_\delta)\right\} .$$

Proposition 3.2 (Real materials.) *Assume that j is taken of the form* (2.2). *Then there holds:*

$$\lim_{\varepsilon\to 0} \mathcal{I}_{\varepsilon,\delta} = \mathcal{I}_\delta := \inf\left\{ \mathcal{C}^{\mathrm{el}}\left(\mu, j_0, F^\delta\right) \ :\ \mu \in \mathcal{P}(Q_\delta)\right\} ,$$

where $j_0 : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$ denotes the following modified integrand :

$$j_0(z) = \frac{1}{2}\rho_0(z)^2 := \sup\left\{ z\cdot z^* - j^*(z^*) \ : z \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \ ,\ \det(z^*) = 0\right\} . \tag{3.1}$$

Proposition 3.2 is actually a reformulation of the results in [2, 3] (to which we refer for a proof), where the effective stress potential - the Fenchel conjugate $j_0^*(z^*)$ of $j_0(z)$ - is characterized explicitly in terms of the eigenvalues of the symmetric tensor z^*. Formula (3.1) is a concise way to recover directly the related effective strain potential j_0; we refer to the Appendix for some explicit computations. We believe that Proposition 3.2 remains true even for non-quadratic strain potentials, see [9].

3.2 Step 2: $\delta \to 0$.

The kind of mass optimization problem given by Proposition 3.1 and Proposition 3.2 has been widely studied in [10], where it is proved in particular that

$$\widetilde{\mathcal{I}}_\delta = \widetilde{\mathcal{S}}_\delta^2/2 , \tag{3.2}$$

being

$$\widetilde{\mathcal{S}}_\delta := \sup\left\{ \langle F^\delta, U\rangle_{\mathbb{R}^3} \ :\ U \in \mathcal{C}^\infty(\mathbb{R}^3; \mathbb{R}^3) \ \text{ such that } \ \rho(e(U)) \le 1 \text{ on } Q_\delta\right\} \tag{3.3}$$

(or equivalently $\mathcal{I}_\delta = \mathcal{S}_\delta^2/2$, where $\mathcal{S}_\delta$ is defined as in (3.3) just replacing ρ by ρ_0).
Thanks to the crucial equality (3.2), this second step in strategy (A) is reduced to determining the limit of $\widetilde{\mathcal{S}}_\delta$ (resp. $\mathcal{S}_\delta$) as $\delta \to 0$. The main contribution of this paper is actually performing the $3D-2D$ reduction dimension analysis for such a sequence of linear constrained problems. We write down the results for $\widetilde{\mathcal{S}}_\delta$, being those for $\mathcal{S}_\delta$ identical up to replacing ρ by ρ_0.
In order to state our main theorem, we need to introduce an effective system of forces $\overline{F} \in \mathcal{M}(\overline{\Omega};\mathbb{R}^3)$ and an effective integrand $\overline{j} : \mathbb{R}^{2\times 2}_{\text{sym}} \to \mathbb{R}$.
For any $\lambda \in \mathcal{M}(Q;\mathbb{R})$, we denote by $[\lambda] \in \mathcal{M}(\overline{\Omega},\mathbb{R})$ the marginal measure defined by the equality

$$\langle [\lambda], \varphi\rangle_{\mathbb{R}^2} := \langle \lambda, \varphi\rangle_{\mathbb{R}^3} \qquad \forall \varphi \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}) \ ; \tag{3.4}$$

then we define the effective system of forces $\overline{F} = (\overline{F}_1, \overline{F}_2, \overline{F}_3) \in \mathcal{M}(\overline{\Omega};\mathbb{R}^3)$ componentwise by:

$$\overline{F}_i := [F_i] \quad i = 1,2 \qquad \text{and} \qquad \overline{F}_3 := \big[F_3 + x_3 \sum_{i=1}^{2} \frac{\partial F_i}{\partial x_i}\big] \ . \tag{3.5}$$

The effective density $\overline{j} : \mathbb{R}^{2\times 2}_{\text{sym}} \to \mathbb{R}$ is obtained from j through the following formula:

$$\overline{j}(z) = \frac{1}{2}\overline{\rho}(z)^2 := \inf \Big\{ j\big(z + \sum_{i=1}^{3} \xi_i (e_i \otimes e_3)^*\big) \ : \ \xi_i \in \mathbb{R} \Big\} \ . \tag{3.6}$$

Theorem 3.3 *The limit as $\delta \to 0$ of the sequence $\{\widetilde{\mathcal{S}}_\delta\}$ defined by* (3.3) *is given by*

$$\mathcal{S}_0 := \sup \Big\{ \langle \overline{F}, v\rangle_{\mathbb{R}^2} \ : \ v \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}^3) \text{ such that } \ \overline{\rho}(e(v_1,v_2) \pm h\nabla^2 v_3) \le 1 \text{ on } \Omega \Big\} \ , \tag{3.7}$$

where $\overline{F}$ and $\overline{\rho}$ are given by (3.5) *and* (3.6) *respectively.*

It is clear that, in general, the limit problem given by Theorem 3.3 cannot be "decoupled" into two separate problems respectively of first order in (v_1, v_2) and of second order in v_3. Nevertheless, there are special cases when it simplifies into one of them:

Corollary 3.4 *(i) If $\overline{F}_1 = \overline{F}_2 = 0$, then*

$$\mathcal{S}_0 = \sup \Big\{ \langle \overline{F}_3, v_3\rangle_{\mathbb{R}^2} \ : \ v_3 \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}) \text{ such that } \ \overline{\rho}(\nabla^2 v_3) \le 1/h \text{ on } \Omega \Big\} \ .$$

(ii) If $\overline{F}_3 = 0$, then

$$\mathcal{S}_0 = \sup \Big\{ \sum_{i=1}^{2} \langle \overline{F}_i, v_i\rangle_{\mathbb{R}^2} \ : \ (v_1,v_2) \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}^2) \text{ such that } \overline{\rho}(e(v_1,v_2)) \le 1 \text{ on } \Omega \Big\} \ .$$

When $\overline{F}_1 = \overline{F}_2 = 0$, combining case (i) of the above corollary with our results in [13], we are finally able to prove that the infima in (2.10) converge to a limit problem of type (1.1).

Corollary 3.5 *Let* $\widetilde{\mathcal{I}}_{\varepsilon,\delta}$ *be defined by* (2.10). *If* $\overline{F}_1 = \overline{F}_2 = 0$, *there holds*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\widetilde{\mathcal{I}}_{\varepsilon,\delta} = h^{-2}\inf\left\{\,\mathcal{C}^{\rm pl}\left(\mu,\bar{j},\overline{F}_3\right)\ :\ \mu\in\mathcal{P}(\overline{\Omega})\right\}\ , \tag{3.8}$$

where the plate compliance $\mathcal{C}^{\rm pl}\left(\mu,\bar{j},\overline{F}_3\right)$ *is defined according to* (1.2).

Remark 3.6 Let us emphasize that the assumption $F\in H^{-1}(Q;\mathbb{R}^3)$ stated in Section 2 is not needed for the well-posedness of the variational problems in (3.7) or (3.8). For instance, it is enough to ask that F is a measure with finite variation. In particular, pointwise applied forces are allowed in our limit problem.

Remark 3.7 The scalar analogue of Theorem 3.3 is simpler, and it can be easily obtained with the same proof. For any $f\in\mathcal{M}(Q;\mathbb{R})$ (with $f\in H^{-1}(Q;\mathbb{R})$ and $\int_Q f=0$), and any convex, 1-homogeneous, coercive function $\rho:\mathbb{R}^3\to\mathbb{R}$, it can be stated as follows: the limit as $\delta\to 0$ of

$$s_\delta := \sup\Big\{\langle f^\delta, u\rangle_{\mathbb{R}^3}\ :\ u\in\mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R})\ \text{ such that }\ \rho(\nabla u)\le 1 \text{ on } Q_\delta\Big\}\ .$$

is given by

$$s_0 := \sup\Big\{\langle [f], v\rangle_{\mathbb{R}^2}\ :\ v\in\mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}) \text{ such that }\ \overline{\rho}(\nabla v)\le 1 \text{ on } \Omega\Big\}\ .$$

Here $f^\delta\in\mathcal{M}(Q;\mathbb{R})$ is the measure which acts on any test function $\varphi\in\mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R})$ as $\langle f^\delta,\varphi\rangle_{\mathbb{R}^3} := \langle f,\varphi(x',\delta x_3)\rangle_{\mathbb{R}^3}$, while $[f]\in\mathcal{M}(\overline{\Omega};\mathbb{R})$ is defined according to (3.4), and $\overline{\rho}:\mathbb{R}^2\to\mathbb{R}$ is given by $\overline{\rho}(z) := \inf\big\{\rho(z+\xi e_3)\ :\ \xi\in\mathbb{R}\big\}$.

Let us turn to the practice computation of $\mathcal{S}_0$. To that purpose, one needs to determine optimality conditions for the infimum problem $(\mathcal{P})$ which defines $\mathcal{S}_0$. Such optimality conditions are obtained in [13], by exploiting the results of [14], in the special situation of Corollary 3.4 (i). Let us see how they look like in the more general situation of Theorem 3.3. As a preliminary step, we begin by writing the dual problem of $(\mathcal{P})$ (intended in the usual sense of Convex Analysis, see *e.g.* [19]). We denote by $\rho^o:\mathbb{R}^{2\times 2}\to\mathbb{R}$ the *polar function of* ρ, that is,

$$\rho^o(\xi) := \sup\{\xi\cdot z\ :\ \rho(z)\le 1\}\ ,$$

where $\xi\cdot z$ indicates the Euclidean scalar product. Then, for λ in the space $\mathcal{M}=\mathcal{M}(\overline{\Omega};\mathbb{R}^{2\times 2}_{\rm sym})$ of $\mathbb{R}^{2\times 2}_{\rm sym}$-valued measures supported on $\overline{\Omega}$ with finite total variation, we use the notation $\int\rho^o(\lambda)$ in the usual sense of convex 1-homogeneous functionals on measures (see for instance [21]).

Lemma 3.8 *The dual problem* $(\mathcal{P}^*)$ *of* $(\mathcal{P})$ *is given by*

$$\min\Big\{\int\overline{\rho}^o(\lambda^+)+\int\overline{\rho}^o(\lambda^-)\ :\ \lambda^\pm\in\mathcal{M}\ ,\ -\operatorname{div}(\lambda^+ +\lambda^-) = (\overline{F}_1,\overline{F}_2)\ ,\ h\operatorname{div}^2(\lambda^+ -\lambda^-) = \overline{F}_3\Big\}\ ,$$

where the operators div *and* div^2 *are intended in distributional sense.*

Proposition 3.9 *Let v be admissible for $(\mathcal{P})$ and $\lambda^{\pm}$ be admissible for $(\mathcal{P})^*$. They are optimal for the respective problems if and only if the following two equations are satisfied:*

$$\overline{\rho}^o(\lambda^+) = \left\langle \lambda^+ ,\, e(v_1, v_2) + h\nabla^2 v_3 \right\rangle_{\mathbb{R}^2} \quad , \quad \overline{\rho}^o(\lambda^-) = \left\langle \lambda^- ,\, e(v_1, v_2) - h\nabla^2 v_3 \right\rangle_{\mathbb{R}^2} \, . \tag{3.9}$$

The application of Proposition 3.9 is exemplified in two concrete cases in Section 5.

4 Strategy (B): some insights.

When strategy (B) is adopted we believe that, for both real and fictious materials, the limit problem is the same as the one obtained through strategy (A) in the ficticious case. More precisely, in parallel with Corollary 3.5, we announce the following result:

Proposition 4.1 *Let $\mathcal{I}_{\varepsilon,\delta}$ and $\widetilde{\mathcal{I}}_{\varepsilon,\delta}$ be defined by* (2.9) *and* (2.10) *respectively, and assume that $\overline{F}_1 = \overline{F}_2 = 0$. Then there holds :*

$$\lim_{\eta\to+\infty} \lim_{\varepsilon\to 0} \widetilde{\mathcal{I}}_{\varepsilon,\eta\varepsilon} = h^{-2} \inf \left\{ \mathcal{C}^{\rm pl} \left(\mu, \bar{j}, \overline{F}_3 \right) \ : \ \mu \in \mathcal{P}(\overline{\Omega}) \right\} = \lim_{\eta\to+\infty} \lim_{\varepsilon\to 0} \mathcal{I}_{\varepsilon,\eta\varepsilon} \, . \tag{4.1}$$

The first equality in (4.1) can be obtained by the same duality methods used in the proof of Proposition 3.1. The proof of the second equality is more delicate: it involves some homogenization process occurring around the middle of the design, and it will be detailed in a forthcoming paper.
Another reason of interest in strategy (B) is that it allows to relate our approach with the standard method used in the literature to describe the compliance of a thin plate. This relation comes out when the following *topological constraint* is added in the model: take the set A appearing in (2.9) of the form

$$A(g) := \{|x_3| < g(x')\} \, ,$$

for some profile function g which must satisfy

$$0 < g(x') \le \delta h \, , \qquad \int_\Omega g(x')\, dx' = \varepsilon \, .$$

In this framework, if we set $\delta = \eta\varepsilon$ and we let ε go to zero, the result of the limit process is well-known (see *e.g.* [1, 18]), and it can be expressed according to the next proposition. Therein for convenience the function g is written as $g = \varepsilon\psi$, where ψ satisfies

$$0 < \psi(x') < \eta h \qquad \text{and} \qquad \int_\Omega \psi(x')\, dx' = 1 \, .$$

Lemma 4.2 *If the integrand j is quadratic and if the function ψ is bounded from below by a positive constant, then*

$$\lim_{\varepsilon\to 0} \mathcal{C}^{\rm el} \left(A(\varepsilon\psi), j, \sqrt{\varepsilon} F^{\eta\varepsilon} \right) = \mathcal{C}^{\rm el} \left(\psi \mathcal{L}^2, \bar{j}, (\overline{F}_1, \overline{F}_2) \right) + \mathcal{C}^{\rm pl} \left(\frac{2\psi^3}{3} \mathcal{L}^2, \bar{j}, \eta \overline{F}_3 \right) \, .$$

Here the effective forces $\overline{F}_i$ are given by (3.5)*, the plate compliance $\mathcal{C}^{\rm pl}\left(\frac{2\psi^3}{3}\mathcal{L}^2, \overline{j}, \eta\overline{F}_3\right)$ is defined according to* (1.2)*, and the elastic compliance $\mathcal{C}^{\rm el}\left(\psi\mathcal{L}^2, \overline{j}, (\overline{F}_1, \overline{F}_2)\right)$ according to (the* 2D *analogous of)* (1.5)*.*

Let us mention how the cubic dependence in ψ appearing in the above statement comes out (see [1, 18] for more details). Roughly, it arises when evaluating the integral of the bulk density $j(e_{\alpha\beta}(u))$. Indeed, in terms of the Kirchoff-Love strain displacement v, $e_{\alpha\beta}(u)$ is given by $e_{\alpha\beta}(u) = e(v_1, v_2) - x_3\nabla^2 v_3$ (see (6.7) below). Thus, since j is 2-homogeneous and quadratic, one has to compute the one-dimensional integral $\int_{-\psi}^{\psi} x_3^2\, dx_3$, which yields the cubic dependence in ψ.
Differently from the limit problem found in Theorem 3.3, the limit compliance given by Lemma 4.2 is always "decoupled" into the sum of an elastic compliance plus a plate compliance. On the other hand, it is well-known since [17] that the problem of minimizing it over the class of admissible profiles ψ in general has no solution, and there is a large literature investigating its possible relaxations (see [4, 6, 7, 8, 23, 24]). Hence it can be interesting to investigate what happens as $\eta \to +\infty$. If we assume for simplicity that $F_1 = F_2 = 0$, the counterpart of the double limit appearing in the last term of (4.1) becomes:

$$\lim_{\eta\to+\infty} \inf\left\{ \mathcal{C}^{\rm pl}\left(\frac{2}{3}\psi^3\mathcal{L}^2, \overline{j}, \eta\overline{F}_3\right) \ :\ 0 < \psi \le \eta h\ ,\ \int_\Omega \psi = 1\right\} . \tag{4.2}$$

Notice that, if we choose $\psi_\eta := \eta h\, \chi_{E_\eta}$ as a competitor, with $|E_\eta| = (\eta h)^{-1}$, then, by using a suitable scaling factor on the strain displacement, we obtain the equality

$$\mathcal{C}^{\rm pl}\left(\frac{2}{3}\psi_\eta^3\mathcal{L}^2, \overline{j}, \eta\overline{F}_3\right) \ =\ \frac{3}{2h^2}\,\mathcal{C}^{\rm pl}\left(\theta_\eta\mathcal{L}^2, \overline{j}, \overline{F}_3\right)\ ,$$

where $\theta_\eta = \eta h \chi_{E_\eta}$ is a probability density. This suggests that limit in (4.2) will remain finite. Actually, as already mentioned in [13], we believe that, as $\eta \to +\infty$, optimal sequences $\{\psi_n\}$ for the infimum problem in (4.2) will saturate the upper bound constraint $\psi_n = \eta h$ on a subset E_n which tends to concentrate on one dimensional structures, whereas ψ_n will be very close to zero away from E_n. The optimization of such one-dimensional microstructures through homogenization techniques (as the one developed in [15]) brings us to formulate the following conjecture, whose full proof seems by now out of reach.

Proposition 4.3 (conjectured) *The limit in (4.2) exists and is given by*

$$\frac{3}{2h^2}\inf\left\{ \mathcal{C}^{\rm pl}\left(\mu, \overline{j}_\ominus, \overline{F}_3\right) \ :\ \mu \in \mathcal{P}(\overline{\Omega})\right\}\ , \tag{4.3}$$

where $(\overline{j})_\ominus : \mathbb{R}^{2\times2}_{\rm sym} \to \mathbb{R}$ is the following modified integrand:

$$\overline{j}_\ominus(z) = \frac{1}{2}\overline{\rho}_\ominus(z)^2 := \sup\left\{ z\cdot z^* - (\overline{j})^*(z^*) \ : z \in \mathbb{R}^{2\times2}_{\rm sym}\ ,\ \det(z^*) \le 0\right\} .$$

Remark 4.4 Similarly as in Corollary 3.5, the limit problem in (4.3) is of type (1.1). Notice also that (4.3) can be rewritten as

$$h^{-2} \inf \left\{ \mathcal{C}^{\mathrm{pl}} \left(\mu, \sqrt{2/3}\, \overline{j}_{\ominus}, \overline{F}_3 \right) \ : \ \mu \in \mathcal{P}(\overline{\Omega}) \right\} . \tag{4.4}$$

In particular, it is natural to compare the effective potentials $\overline{j_0}$, $\overline{j}$ and $\sqrt{2/3}\, \overline{j}_{\ominus}$ obtained for real materials respectively through strategy A (*cf.* Corollary 3.5), strategy B without topological constraint (*cf.* Proposition 4.1), and strategy B under topological constraint (*cf.* Proposition 4.3). In general, such potentials will not coincide with each other. For instance, if we take $j(z) = (1/2)|z|^2$ (on $\mathbb{R}^{3\times3}_{\mathrm{sym}}$), it is immediate to get $\overline{j}(z) = (1/2)|z|^2$ (on $\mathbb{R}^{2\times2}_{\mathrm{sym}}$), while the explicit computation of $\overline{j_0}$ and $\overline{j}_{\ominus}$ is more delicate and can be found in the Appendix.

5 Examples

In the examples we are going to discuss, the systems of loads are discrete (see Remark 3.6). Moreover, they lie into a plane, so that the corresponding optimal structures are supported into that plane. As a consequence, we take a planar design region Q of the form $\overline{\Omega} \times [-h, h]$, being Ω an open bounded interval of the real line. Thus throughout this section the spatial variable $x' \in \overline{\Omega}$ will become x_1, and the role of the "vertical variable" x_3 will be played by x_2. Clearly, the limit problem will reduce simply to a $1D$-problem.
We take as a function ρ in (3.3) the Euclidean norm on $\mathbb{R}^{2\times2}_{\mathrm{sym}}$, so that the corresponding function $\overline{\rho_0}$ will be simply the Euclidean norm on $\mathbb{R}$ (see the Appendix).

Example 5.1 *(pure flexion regime).*
For fixed nonnegative parameters l and h_0, let the points O, A, B have coordinates

$$O := \big(0, 0\big) \ , \qquad A := \big(l, 0\big) \ , \qquad B := \big(0, h_0\big)$$

and let us consider the following system of forces:

$$F_1 := \delta_O - \delta_B \ , \qquad F_2 = \frac{h_0}{l}(\delta_B - \delta_A) \ .$$

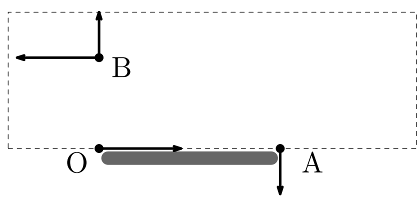

Figure 1: *loads yielding a pure flexion regime in the whole of* $\overline{OA}$

This system of forces is supported on the design region $Q = \overline{\Omega} \times [-h, h]$ provided Ω is an interval containing both O and A, and $h \geq h_0$. Moreover, it is immediate to check that this system is balanced. Then we can apply Theorem 3.3 to compute $\mathcal{S}_0$, namely the limit as $\delta \to 0$ of the suprema $\widetilde{\mathcal{S}}_\delta$ in (3.3). The effective system of forces on the x_1-axis is easily obtained:

$$\overline{F}_1 := 0 \ , \qquad \overline{F}_2 = \frac{h_0}{l}(\delta_O - \delta_A) - h_0 \delta_O' \ .$$

Then according to (3.7) $\mathcal{S}_0$ can be expressed as:

$$\sup\left\{ \frac{h_0}{l}\big(v_2(O) - v_2(A)\big) + h_0 v_2'(O) \ : \ v_2 \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}) \text{ such that } \ |(v_2)''| \le \frac{1}{h} \text{ on } \Omega \right\} .$$

In order to compute the explicit value of $\mathcal{S}_0$, we apply Proposition 3.9. Given $v = (v_1, v_2) \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^2)$ and $\lambda^\pm \in \mathcal{M}(\overline{\Omega};\mathbb{R})$, they are solutions to problem $(\mathcal{P})$ and its dual $(\mathcal{P}^*)$ if the following system is satisfied:

$$\begin{cases} (\lambda^+ + \lambda^-)' = 0 \\ h(\lambda^+ - \lambda^-)'' = \dfrac{h_0}{l}(\delta_O - \delta_A) - h_0 \delta_O' \\ |(v_2)''| \le \dfrac{1}{h} \\ |\lambda^\pm| = \langle \lambda^\pm, \pm h (v_2)'' \rangle_{\mathbb{R}} \ , \end{cases}$$

where the first two equations select admissible $\lambda^\pm$ in problem $(\mathcal{P})^*$ (see Lemma 3.8), the third equation selects admissible v inproblem $(\mathcal{P})$, and the last couple of equations corresponds to the optimality conditions (3.9).
Solutions $\lambda^\pm$ to the first two equations are determined by

$$\lambda^+ = -\lambda^- = \frac{1}{2}\frac{h_0}{hl}(x_1 - l)\chi_{\overline{OA}}(x_1)\mathcal{L}^1 \llcorner \overline{OA} \ ,$$

and the remaining conditions are satisfied if we take

$$v_2(x_1) = -\frac{x_1^2}{2h} \ .$$

Thus we find for the value of the energy

$$\mathcal{S}_0 = \frac{l h_0}{2h} \ .$$

Remark 5.2 (i) Exactly the same result above holds if, in the system of forces, the point $A$is replaced by any other point of the type (l, h_1), with $|h_1| \le h$ (or even more generally if δ_A is replaced by any probability on the segment $l \times [-h, h]$).
(ii) Exactly the same result above holds if, with the same system of forces, the design region is changed into $\overline{\Omega} \times [0, h]$.
(iii) Note that $\mathcal{S}_0$ is infinitesimal as $h \to +\infty$, as it always happens is a pure flexion regime (see Corollary 3.4 (i)).

(iv) The role of $\lambda^\pm$ in the reconstruction of $3D$-optimal structures will be investigated more into deep in a subsequent work. In the above example we guess that, for any $\delta > 0$, optimal structures are given by two horizontal bars at heights 0 and h, connected by some diagonal bars of vanishing mass.

Example 5.3 *(mixed regime).*
For fixed nonnegative parameters l, h_0, α, let the points O, A, B, C have coordinates

$$O := (0,0) \ , \qquad A := \Big(-\frac{l}{2}, 0\Big) \ , \qquad B := \Big(\frac{l}{2}, 0\Big) \ , \qquad C := (0, h_0) \ ,$$

and let us consider the axially symmetric system of forces:

$$F_1 := \alpha(\delta_B - \delta_A) \ , \qquad F_2 = \delta_C - \frac{1}{2}(\delta_A + \delta_B) \ .$$

Figure 2: *loads yielding a membrane/flexion regime in the clear/dark part of $\overline{AB}$*

This system of forces is balanced and it is supported on the design region $Q = \overline{\Omega} \times [-h, h]$ provided the interval Ω contains both A and B, and $h \geq h_0$. The effective system of forces is given on the x_1-axis by:

$$\overline{F}_1 := \alpha\big(\delta_B - \delta_A\big) \ , \qquad \overline{F}_2 = \delta_O - \frac{1}{2}\big(\delta_A + \delta_B\big) \ .$$

Then according to (3.7) the limit $\mathcal{S}_0$ of the suprema $\widetilde{\mathcal{S}}_\delta$ in (3.3) can be expressed as:

$$\sup\Big\{\alpha\big[v_1(B) - v_1(A)\big] + v_2(O) - \frac{1}{2}\big[v_2(A) + v_2(B)\big] \ : \ v \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^2) \text{ such that } \ |(v_1)' \pm h(v_2)''| \leq 1 \text{ on } \Omega\Big\} \ .$$

Let us compute the explicit value of $\mathcal{S}_0$ in terms of the involved parameters.
By Proposition 3.9, given $v = (v_1, v_2) \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and $\lambda^\pm \in \mathcal{M}(\overline{\Omega};\mathbb{R})$, they are solutions to problem $(\mathcal{P})$ and its dual $(\mathcal{P}^*)$ if the following system is satisfied:

$$\begin{cases} -(\lambda^+ + \lambda^-)' = \alpha\big(\delta_A - \delta_B\big) \\ h(\lambda^+ - \lambda^-)'' = \delta_O - \dfrac{1}{2}\big(\delta_A + \delta_B\big) \\ |(v_1)' \pm h(v_2)''| \leq 1 \\ |\lambda^\pm| = \langle \lambda^\pm, (v_1)' \pm h(v_2)'' \rangle_{\mathbb{R}} \ . \end{cases}$$

Solutions $\lambda^{\pm}$ to the first two equations are determined by

$$\lambda^+ + \lambda^- = \alpha \mathcal{L}^1 \llcorner \overline{AB} \ , \qquad \lambda^+ - \lambda^- = \frac{1}{2h}\Big(|x_1| - \frac{l}{2}\Big) \mathcal{L}^1 \llcorner \overline{AB} \ , \tag{5.1}$$

and the remaining conditions are satisfied provided

$$(v_1)' \pm h (v_2)'' = \mathrm{sign}\,(\lambda^{\pm}) \ , \tag{5.2}$$

where $\mathrm{sign}\,(\lambda^{\pm})$ denotes the sign of (the densities of) $\lambda^{\pm}$.
¿From (5.1), we see in particular that λ^- remains always nonnegative, whereas for λ^+ two cases may occur:

case 1): if $h \geq l/(4\alpha)$, then λ^+ remains nonnegative;

case 2): if $h < l/(4\alpha)$, then

$$\begin{cases} \lambda^+ \geq 0 & \text{if } |x_1| \geq (l/2) - 2h\alpha \\ \lambda^+ < 0 & \text{if } |x_1| < (l/2) - 2h\alpha \ . \end{cases}$$

Accordingly, solutions to (5.2) and the value of $\mathcal{S}_0$ can be easily computed:

case 1): we have $(v_1)' = 1$, $(v_2)'' = 0$, and

$$\mathcal{S}_0 = \int \lambda^+ + \int \lambda^- = \alpha l \ ;$$

case 2): we have

$$\begin{cases} (v_1)' = 1 \text{ and } (v_2)'' = 0 & \text{if } |x_1| \geq (l/2) - 2h\alpha \\ (v_1)' = 0 \text{ and } (v_2)'' = 1/h & \text{if } |x_1| < (l/2) - 2h\alpha \ , \end{cases}$$

and

$$\mathcal{S}_0 = \int |\lambda^+| + \int \lambda^- = 2h\big[\alpha^2 + l^2/(16h^2)\big] \ .$$

Summing up, we have obtained

$$\mathcal{S}_0 = \begin{cases} \alpha l & \text{if } h \geq l/(4\alpha) \\ 2h\big[\alpha^2 + l^2/(16h^2)\big] & \text{if } h < l/(4\alpha) \ . \end{cases}$$

Remark 5.4 (i) The value found above for $\mathcal{S}_0$ is always independent of the parameter h_0.
(ii) The critical height $h_c := l/(4\alpha)$ is the second coordinate of the intersection point between the straight lines $A + t(-\alpha, -1/2)$ and $B + t(\alpha, -1/2)$ (namely the point where the two forces $(-\alpha, -1/2)\delta_A$ and $(\alpha, -1/2)\delta_B$ concur). If $h \geq h_c$, then the value of $\mathcal{S}_0$ is independent of h. In spite, if $h < h_c$, then the dependance of $\mathcal{S}_0$ on h tells that optimal structures for $\widetilde{\mathcal{S}}_\delta$ do "touch" the bottom of the design region (independently of the choice of h_0).

6 Proofs of the results in Section 3

Proof of Proposition 3.1. Let δ be fixed. We introduce, for every ε, the functional J_ε and the function φ_ε defined respectively on $\mathcal{M}(Q_\delta;\mathbb{R}^+)$ and on $\mathbb{R}$ by:

$$J_\varepsilon(\mu) \;:=\; \begin{cases} \mathcal{C}^{\mathrm{el}}\big(\mu, j, F^\delta\big) & \text{if } \mu = \theta\,dx\ ,\ \ \theta \in L^\infty(\mathbb{R}^3;[0,\varepsilon^{-1}])\ ,\ \operatorname{spt}(\theta) \subseteq Q_\delta \\ +\infty & \text{otherwise} \end{cases}$$

$$\varphi_\varepsilon(t) \;:=\; \begin{cases} \inf\Big\{ J_\varepsilon(\mu)\ :\ \mu \in \mathcal{M}(Q_\delta;\mathbb{R}^+)\ ,\ \displaystyle\int d\mu = t \Big\} & \text{if } 0 < t \le \varepsilon^{-1}|Q_\delta| \\ +\infty & \text{otherwise}\ . \end{cases}$$

It is easy to check that J_ε and φ_ε are convex and decrease as ε goes down to zero. In particular the limit $\varphi_0(t) = \lim_{\varepsilon\to 0}\varphi_\varepsilon(t)$ exists and is convex as a function of t. We claim that, for every $t > 0$, there holds

$$\varphi_0(t) = \frac{(\widetilde{\mathcal{S}}_\delta)^2}{2\,t} \tag{6.1}$$

Recalling (3.2), the proposition will follow by taking $t = 1$, since by (2.10) and (2.6):

$$\widetilde{\mathcal{I}}_{\varepsilon,\delta} = \inf\Big\{ \mathcal{C}^{\mathrm{el}}\big(\frac{\theta}{\varepsilon}, j, F^\delta\big)\ :\ \theta \text{ satisfying (2.5)} \Big\} \;=\; \varphi_\varepsilon(1)\ .$$

For proving (6.1), we are going to identify the Fenchel conjugate of φ_0 through the formula

$$\varphi_0^* = \big(\inf_\varepsilon \varphi_\varepsilon\big)^* = \sup_\varepsilon \varphi_\varepsilon^*\ . \tag{6.2}$$

To compute φ_ε^*, we begin by noticing that $\varphi_\varepsilon^*(t) = +\infty$ for any $t \le 0$ and that, for every $k > 0$, φ_ε^* computed at $-k$ coincides with the Fenchel conjugate of J_ε computed at the constant function identically equal to $-k$. Indeed:

$$\varphi_\varepsilon^*(-k) = \sup\Big\{ -\int k\,d\mu - J_\varepsilon(\mu)\ :\ \mu \in \mathcal{M}(Q_\delta;\mathbb{R}^+)\ ,\ \int d\mu = t \Big\} = J_\varepsilon^*(-k)\ . \tag{6.3}$$

Let us compute $J_\varepsilon^*(-k)$. By definition we have

$$J_\varepsilon^*(-k) = \sup_\mu \inf_U \Big\{ \int j\big(e(U) - k\big)\ d\mu - \langle F^\delta, U\rangle_{\mathbb{R}^3} \Big\}\ ,$$

where the infimum in U is taken over $\mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3)$, while the supremum in μ is taken over the class of measures of the form $\mu = \theta\,dx$ with $\theta \in L^\infty(\mathbb{R}^3;[0,\varepsilon^{-1}])$ and $\operatorname{spt}(\theta) \subseteq Q_\delta$. Since the latter class is compact and since the dependence with respect to (μ, U) is convex-concave, we may exchange the supremum and the infimum (see*e.g.* [13, Proposition 2.2]) so that

$$\begin{aligned} J_\varepsilon^*(-k) &= \inf_U \Big\{ -\langle F^\delta, U\rangle_{\mathbb{R}^3} + \sup_\mu \int \big(j(e(U)) - k\big)\ d\mu \Big\} \\ &= \inf_U \Big\{ -\langle F^\delta, U\rangle_{\mathbb{R}^3} + \varepsilon^{-1}\int_{Q_\delta} \big(j(e(U)) - k\big)^+\ dx \Big\}\ . \end{aligned}$$

Then, in order to compute the limit as $\varepsilon \to 0$ of $J^*_\varepsilon(-k)$ (which is also their supremum), we are led to consider the functionals G_ε defined on $H^1(\mathbb{R}^3;\mathbb{R}^3)$ by

$$G_\varepsilon(U) := \begin{cases} -\langle F^\delta, U\rangle_{\mathbb{R}^3} + \varepsilon^{-1} \displaystyle\int_{Q_\delta} \big(j(e(U)) - k\big)^+ \, dx & \text{if } U \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \\ +\infty & \text{otherwise .} \end{cases}$$

Since G_ε are increasing in ε (and since by assumption $F^\delta \in H^{-1}(Q_\delta;\mathbb{R}^3)$), their Γ-limit with respect to the weak convergence on $H^1(\mathbb{R}^3;\mathbb{R}^3)$ coincide with the functional G_0 defined by

$$G_0(U) := \begin{cases} -\langle F^\delta, U\rangle_{\mathbb{R}^3} & \text{if } U \in H^1(\mathbb{R}^3;\mathbb{R}^3) \text{ such that } j(e(U)) \leq k \text{ a.e. on } Q_\delta \\ +\infty & \text{otherwise .} \end{cases}$$

Moreover, by using the coercivity of j and the Korn inequality, one can easily check that any sequence $\{U^\varepsilon\}$ with $\sup_\varepsilon G_\varepsilon(U^\varepsilon) < +\infty$ is weakly precompact in $H^1(\mathbb{R}^3;\mathbb{R}^3)$ (up to subtracting a rigid displacement, which is not restrictive thanks to (2.3)). This compactness property, combined with the Γ-convergence of G_ε to G_0, ensures that the infima of G_ε converge to the infimum of G_0. Therefore

$$\begin{aligned} -\lim_\varepsilon J^*_\varepsilon(-k) &= -\inf\Big\{ -\langle F^\delta, U\rangle_{\mathbb{R}^3} \ : \ U \in H^1(\mathbb{R}^3;\mathbb{R}^3) \text{ such that } j(e(U)) \leq k \text{ a.e. on } Q_\delta \Big\} \\ &= \sup\Big\{ \langle F^\delta, U\rangle_{\mathbb{R}^3} \ : \ U \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \text{ such that } j(e(U)) \leq k \text{ on } Q_\delta \Big\} \ . \end{aligned}$$

Recalling the definition of $\widetilde{\mathcal{S}}_\delta$ in (3.3) and by the 2-homogeneity of j (see(2.1)), we deduce after an easy computation that

$$-\lim_\varepsilon J^*_\varepsilon(-k) = \sqrt{2k}\, \widetilde{\mathcal{S}}_\delta \ .$$

By (6.2) and (6.3), we arrive then to $\varphi_0^*(-k) = \sqrt{2k}\,\widetilde{\mathcal{S}}_\delta$. Passing to the biconjugate, we infer that

$$\varphi_0^{**}(t) = \sup_{k\geq 0} \Big\{ -kt - \varphi_0^*(-k) \Big\} = \sup_{k\geq 0} \Big\{ -kt + \sqrt{2k}\,\widetilde{\mathcal{S}}_\delta \Big\} = \frac{1}{2}\frac{(\widetilde{\mathcal{S}}_\delta)^2}{t} \qquad \text{if } t > 0 \qquad (+\infty \quad \text{otherwise}) \ .$$

Finally, to deduce (6.1), it remains to check that φ_0^{**} coincides with φ_0. This is a consequence of the fact that φ_0 is convex continuous on $\mathbb{R}^+$. Indeed, let μ_0 be the uniform probability density on Q_δ. As F^δ belongs to $H^{-1}(Q_\delta;\mathbb{R}^3)$, we have that $k(\delta) := \mathcal{C}^{\mathrm{el}}(\mu_0, j, F^\delta) < +\infty$. Then, for every $t > 0$, the measure $t\mu_0$ is admissible for $\varphi_\varepsilon(t)$ whenever $\varepsilon \leq t^{-1}|Q_\delta|$. Thus

$$\varphi_0(t) \ \leq \ \varphi_\varepsilon(t) \ \leq \ \mathcal{C}^{\mathrm{el}}(t\mu_0, j, F^\delta) \ = \ \frac{k(\delta)}{t} \ ,$$

where the last equality is obtained performing the rescaling $V = tU$ on the competing strain displacements. The continuity of the convex function φ_0 on $(0, +\infty)$ follows from the latter upperbound, and the proof of Proposition 3.1 is concluded.

□

Proof of Theorem 3.3. Let us begin by writing $\widetilde{\mathcal{S}}_\delta$ in a more convenient way. We set

$$U(x) = \Big(u_1\big(x', \delta^{-1}x_3\big),\ u_2\big(x', \delta^{-1}x_3\big),\ \delta^{-1}u_3\big(x', \delta^{-1}x_3\big)\Big)\ ,$$

so that

$$e(U)(x) = e_\delta(u)\big(x', \delta^{-1}x_3\big) := \begin{bmatrix} e_{\alpha\beta}(u) & \delta^{-1}e_{\alpha 3}(u) \\ \delta^{-1}e_{\alpha 3}(u) & \delta^{-2}e_{33}(u) \end{bmatrix} (x', \delta^{-1}x_3)\ , \tag{6.4}$$

where the indices α and β take values into $\{1, 2\}$. Hence

$$\begin{aligned} \widetilde{\mathcal{S}}_\delta &= \sup\Big\{\langle F, u\rangle_{\mathbb{R}^3}\ :\ u \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \text{ such that } \rho(e_\delta(u)(x', \delta^{-1}x_3)) \le 1 \text{ on } Q_\delta\Big\} \\ &= \sup\Big\{\langle F, u\rangle_{\mathbb{R}^3}\ :\ u \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \text{ such that } \rho(e_\delta(u)) \le 1 \text{ on } Q\Big\} \\ &= \sup\Big\{\langle F, u\rangle_{\mathbb{R}^3}\ :\ u \in K_\delta\Big\}\ , \end{aligned}$$

where K_δ denotes the the convex set

$$K_\delta := \Big\{u \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3)\ :\ \rho(e_\delta(u)) \le 1 \text{ on } Q\Big\}\ .$$

As a preliminary remark, we notice that the following compactness property holds: if we take a sequence $\{u^\delta\}$ such that $u^\delta \in K_\delta$, then up to subsequences and up to arigid motion, it converges uniformly on Q. Indeed by (2.1) we have that $e_\delta(u^\delta)$ is uniformly bounded in $L^\infty(Q)$; hence, up to subtracting a rigid displacement (which isnot restrictive thanks to (2.3)), by the Korn inequality $\{u^\delta\}$ is equibounded in $W^{1,p}(Q;\mathbb{R}^3)$ for every $p \in (1, +\infty)$.
In view of this remark, we are reduced to identify the Kuratowski limit (if any) of the sequence $\{K_\delta\}$ with respect to the uniform convergence on the compact Q. Indeed if $\overline{K}$ denotes such a Kuratowski limit, since the linear form $u \mapsto \langle F, u\rangle$ is continuous with respect to the uniform convergence, we will have that

$$\lim_{\delta\to 0} \widetilde{\mathcal{S}}_\delta\ =\ \sup\Big\{\langle F, u\rangle_{\mathbb{R}^3}\ :\ u \in \overline{K}\Big\}\ . \tag{6.5}$$

We claim that the set $\overline{K}$ can be characterized as follows:

$$\overline{K} = \Big\{u \in L^\infty(Q;\mathbb{R}^3)\ :\ e(u) \in L^\infty(Q;\mathbb{R}^{3\times 3}_{\text{sym}})\ ,\ \overline{\rho}(e_{\alpha\beta}(u)) \le 1\ ,\ e_{i3}(u) = 0 \text{ a.e. on } Q\Big\}\ . \tag{6.6}$$

Let us first show how Theorem 3.3 follows from (6.6), and then give theproof of (6.6).
As a slight variant of Theorem 3.1 in [12], it is easy to check that the r.h.s. of (6.6) is the closure in the uniform norm of the set of Kirchoff-Love displacements

$$K = \Big\{u \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3)\ :\ \overline{\rho}(e_{\alpha\beta}(u)) \le 1\ ,\ e_{i3}(u) = 0 \text{ on } Q\Big\}\ .$$

As it is well known, any function $u \in K$ may be written under the form

$$u_i(x) = v_i(x') - \frac{\partial v_3}{\partial x_i}(x') x_3 \quad \text{for } i = 1, 2 \ , \qquad u_3(x) = v_3(x') \ .$$

In terms of the function v, the matrix $e_{\alpha\beta}(u)$ is given by

$$e_{\alpha\beta}(u) = e(v_1, v_2) - x_3 \nabla^2 v_3 \ , \tag{6.7}$$

hence v must satisfy the inequality

$$\overline{\rho}\big(e(v_1, v_2) - x_3 \nabla^2 v_3\big) \leq 1 \qquad \forall \, (x', x_3) \in \Omega \times \big(-h, h\big) \ ,$$

which by convexity is equivalent to

$$\overline{\rho}\big(e(v_1, v_2) \pm h \nabla^2 v_3\big) \leq 1 \qquad \text{on } \Omega \ . \tag{6.8}$$

On the other hand we have

$$\langle F, u \rangle_{\mathbb{R}^3} = \sum_{i=1}^{3} \langle F_i, v_i \rangle_{\mathbb{R}^3} + \sum_{i=1}^{2} \langle x_3 \frac{\partial F_i}{\partial x_i}, v_3 \rangle_{\mathbb{R}^3} = \langle \overline{F}, v \rangle_{\mathbb{R}^2} \ . \tag{6.9}$$

By (6.5), (6.8), (6.9), and recalling the definition of $\mathcal{S}_0$ in (3.7), we conclde that

$$\lim_{\delta \to 0} \widetilde{\mathcal{S}}_\delta \ = \ \sup \Big\{ \langle F, u \rangle_{\mathbb{R}^3} \ : \ u \in K \Big\} = \mathcal{S}_0 \ .$$

It remains to establish (6.6). Such equality holds provided one has:

(i) $u^\delta \in K_\delta$, $u^\delta \to u$ uniformly on $Q \implies u \in \overline{K}$;

(ii) $u \in K \implies \exists u^\delta \in K_\delta$ such that $u^\delta \to u$ uniformly on Q.

Proof of (i). Let $u^\delta \in K_\delta$ such that $u^\delta \to u$ uniformly on Q. As already noticed above in this proof, such a sequence $\{u^\delta\}$ is weakly precompact in $W^{1,p}(Q; \mathbb{R}^3)$ for every $p \in (1, +\infty)$, which ensures that u belongs $W^{1,p}(Q; \mathbb{R}^3)$ for every such p. Possibly passing to a subsequence, we may assume that $\{e_\delta(u^\delta)\}$converges weakly in $L^p(Q; \mathbb{R}^{3\times 3}_{\text{sym}})$ to some matrix valued function $M(x)$ which is of the form

$$M = \begin{bmatrix} e_{\alpha\beta}(u) & \xi_{\alpha 3} \\ \xi_{\alpha 3} & \xi_{33} \end{bmatrix} \ .$$

By the convexity of ρ, one has

$$\| \rho(M) \|_{L^\infty(Q)} \ \leq \liminf_{\delta} \| \rho(e_\delta(u^\delta)) \|_{L^\infty(Q)} \leq 1 \ .$$

Thus, by the definition (3.6) of $\overline{\rho}$, it follows that

$$\| \overline{\rho}(e_{\alpha\beta}(u)) \|_{L^\infty(Q)} \leq \| \rho(M) \|_{L^\infty(Q)} \leq 1 \ .$$

On the other hand, it is clear that, for $i = 1, 2, 3$, $e_{i3}(u^\delta)$ does converge strongly to 0 in $L^p(Q)$ and therefore $e_{i3}(u) = 0$. Summarizing we have proved that u belongs to $\overline{K}$.

Proof of (ii). Let $u \in K$. We search for $u^\delta \in K_\delta$ such that $u^\delta \to u$ uniformly on Q. To this end, it not restrictive to assume that the *strict* inequality $\overline{\rho}(e_{\alpha\beta}(u)) < 1$ holds on Q (indeed, for any $u \in K$ the function $\tilde{u} := (1-\delta)u$ satisfies $e_{i3}\tilde{u} = 0$ and $\overline{\rho}(e_{\alpha\beta}\tilde{u}) < 1$). Let $\xi^i = \xi^i(x', x_3)$ be arbitrary smooth functions, and let Φ_i denote their primitives with respect to the x_3 variable:

$$\Phi_i(x', x_3) := \int_0^{x_3} \xi_i(x', s)\, ds \ .$$

We define the sequence $\{u^\delta\}$ componentwise by:

$$u_1^\delta = u_1 + \delta\Phi_1 \ , \qquad u_2^\delta = u_2 + \delta\Phi_2 \ , \qquad u_3^\delta = u_3 + \delta^2\Phi_3 \ .$$

Clearly $\{u^\delta\}$ converges uniformly to u and, according to definition (6.4), an immediate calculation gives

$$e_\delta(u^\delta) = e_{\alpha\beta}(u) + \sum_{i=1}^{2} \Big(\xi_i + \delta \frac{\partial \Phi_3}{\partial x_i}\Big)(e_i \otimes e_3)^* + \xi_3 (e_3 \otimes e_3) \ ,$$

so that

$$\rho(e_\delta(u^\delta)) \leq \rho\big(e_{\alpha\beta}(u) + \sum_{i=1}^{3} \xi^i (e_i \otimes e_3)^*\big) + o(1) \ .$$

The proof of (ii) is concluded by the arbitrariness of the functions ξ_i. □

Proof of Corollary 3.4. In case (i) it is immediate that

$$\mathcal{S}_0 \geq \sup \Big\{ \langle \overline{F}_3, v_3 \rangle_{\mathbb{R}^2} \ : \ v_3 \in \mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R}) \text{ such that } \ \overline{\rho}(\nabla^2 v_3) \leq 1/h \text{ on } \Omega \Big\} \ .$$

The converse inequality is obtained by noticing that, since $\overline{\rho}$ is even and subadditive, the constraint $\overline{\rho}\big(e(v_1, v_2) \pm h\nabla^2 v_3\big) \leq 1$ implies $\overline{\rho}\big(\nabla^2 v_3\big) \leq 1/h$. The proof in case (ii) is analogous. □

Proof of Corollary 3.5. First we recall that there holds $\lim_\varepsilon \widetilde{\mathcal{I}}_{\varepsilon,\delta} = \widetilde{\mathcal{I}}_\delta$ (see Proposition 3.1) and that $\widetilde{\mathcal{I}}_\delta = \widetilde{\mathcal{S}}_\delta^2/2$ [10, Theorem 2.3]. Then Theorem 3.3 gives $\lim_\delta \widetilde{\mathcal{S}}_\delta = \mathcal{S}_0$. Finally we apply Corollary 3.4 (i) and [13, Theorem 2.4] to conclude that $\mathcal{S}_0^2/2 = h^{-2} \inf \big\{ \mathcal{C}^{\rm pl}\left(\mu, \overline{j}, \overline{F}_3\right) \ : \ \mu \in \mathcal{P}(\overline{\Omega}) \big\}$. □

Proof of Lemma 3.8. Let us rewrite $(\mathcal{P})$ as

$$(\mathcal{P}) \qquad\qquad -\inf \Big\{ -\langle \overline{F}, v \rangle_{\mathbb{R}^2} + \chi_{\mathcal{K}}(A^+ v) + \chi_{\mathcal{K}}(A^- v) \ : \ v \in \mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R}^3) \Big\} \ ,$$

where $\chi_{\mathcal{K}}$ is the characteristic function of the set

$$\mathcal{K} = \left\{ M \in \mathcal{C}_0(\Omega; \mathbb{R}^{2\times 2}_{\rm sym}) \ : \ \overline{\rho}(M) \le 1 \right\} ,$$

and $A : \mathcal{C}_0(\Omega; \mathbb{R}^3) \ni v \mapsto (A^+v, A^-v) \in [\mathcal{C}_0(\Omega; \mathbb{R}^{2\times 2}_{\rm sym})]^2$ is the linear operator densely defined by $A^{\pm}v := e(v_1, v_2) \pm h\nabla^2 v_3$ for all smooth functions v.
By standard duality theory (see for instance [19]), there holds

$$(\mathcal{P}^*) \qquad \min\left\{ \int \overline{\rho}^o(\lambda^+) + \int \overline{\rho}^o(\lambda^-) \ : \ (\lambda^+, \lambda^-) \in [\mathcal{M}(\overline{\Omega}; \mathbb{R}^{2\times 2}_{\rm sym})]^2 \ , \ A^*(\lambda^+, \lambda^-) = \overline{F} \right\} ,$$

where $A^* : [\mathcal{M}(\overline{\Omega}; \mathbb{R}^{2\times 2}_{\rm sym})]^2 \to \mathcal{M}(\overline{\Omega}; \mathbb{R}^3)$ is the adjoint operator of A. It is determined by the following identity (valid for every smooth v):

$$\begin{aligned} \langle A^*(\lambda^+, \lambda^-), v\rangle_{\mathbb{R}^2} &= \langle (\lambda^+, \lambda^-), (A^+v, A^-v)\rangle_{\mathbb{R}^2} \\ &= \langle \lambda^+, e(v_1, v_2) + h\nabla^2 v_3\rangle_{\mathbb{R}^2} + \langle \lambda^-, e(v_1, v_2) - h\nabla^2 v_3\rangle_{\mathbb{R}^2} \\ &= -\langle {\rm div}(\lambda^+ + \lambda^-), (v_1, v_2)\rangle_{\mathbb{R}^2} + \langle h\, {\rm div}^2(\lambda^+ - \lambda^-), v_3\rangle_{\mathbb{R}^2} \ . \end{aligned}$$

Therefore, when rewritten componentwise, the constraint $A^*(\lambda^+, \lambda^-) = \overline{F}$ is equivalent to the system of two conditions: $-\,{\rm div}(\lambda^+ + \lambda^-) = (\overline{F}_1, \overline{F}_2)$ and $h\, {\rm div}^2(\lambda^+ - \lambda^-) = \overline{F}_3$. □

Proof of Proposition 3.9. Let v and $\lambda^{\pm}$ be optimal respectively for problems $(\mathcal{P})$ and $(\mathcal{P})^*$. By Lemma 3.8 there holds:

$$\int \overline{\rho}^o(\lambda^+) + \int \overline{\rho}^o(\lambda^-) = \langle \overline{F}, v\rangle \ . \tag{6.10}$$

On the other hand, if the operator $Av = (A^+v, A^-v)$ is defined as in the proof of Lemma 3.8, we have

$$\overline{\rho}^o(\lambda^{\pm}) \ge \overline{\rho}^o(\lambda^{\pm})\, \overline{\rho}(A^{\pm}v) \ge \langle \lambda^{\pm}, A^{\pm}v\rangle_{\mathbb{R}^2} \ , \tag{6.11}$$

which implies

$$\int \overline{\rho}^o(\lambda^+) + \int \overline{\rho}^o(\lambda^-) \ge \langle (\lambda^+, \lambda^-), Av\rangle_{\mathbb{R}^2} = \langle A^*(\lambda^+, \lambda^-), v\rangle_{\mathbb{R}^2} = \langle \overline{F}, v\rangle_{\mathbb{R}^2} \ . \tag{6.12}$$

Combining (6.10) and (6.12), we deduce that the inequalities in (6.11) must turn into equalities, so that the optimality conditions (3.9) hold.
Conversely, any v and $\lambda^{\pm}$ which are admissible respectively for problems $(\mathcal{P})$ and $(\mathcal{P})^*$ satisfy

$$\langle \overline{F}, v\rangle_{\mathbb{R}^2} \le \mathcal{S}_0 \le \int \overline{\rho}^o(\lambda^+) + \int \overline{\rho}^o(\lambda^-) \ . \tag{6.13}$$

If equations (3.9) hold, we have

$$\langle \overline{F}, v\rangle_{\mathbb{R}^2} = \langle A^*(\lambda^+, \lambda^-), v\rangle_{\mathbb{R}^2} = \langle (\lambda^+, \lambda^-), Av\rangle_{\mathbb{R}^2} = \int \overline{\rho}^o(\lambda^+) + \int \overline{\rho}^o(\lambda^-) \ ,$$

hence the inequalities in (6.13) must turn into equalities, which means that v and $\lambda^{\pm}$ are optimal.
□

7 Appendix: computation of effective densities

The following result shows that in general the effective energies $\overline{j_0}$ and $\overline{j}_\ominus$ do not coincide (*cf.* Remark 4.4).

Proposition 7.1 *For $z \in \mathbb{R}^{3\times 3}_{\rm sym}$, take $j(z) = (1/2)|z|^2$. Then, for $z \in \mathbb{R}^{2\times 2}_{\rm sym}$, denoting by $\lambda_1(z)$ the eigenvalue of z which is largest in modulus, there holds*

$$\overline{j_0}(z) = \frac{1}{2}(\lambda_1(z))^2 \qquad \text{and} \qquad (\overline{j})_\ominus(z) = \begin{cases} \frac{1}{2}(\lambda_1(z))^2 & \text{if } \det(z) > 0 \\ \frac{1}{2}|z|^2 & \text{if } \det(z) \le 0 \ . \end{cases}$$

Proof.

Computation of $\overline{j_0}$. For $z \in \mathbb{R}^{3\times 3}_{\rm sym}$, the modified integrand $j_0(z)$ is given by (see [9])

$$j_0(z) = \frac{1}{2}\big(\lambda_1(z)^2 + \lambda_2(z)^2\big) \ ,$$

being $\lambda_i(z)$ the eigenvalues of z with $|\lambda_1(z)| \ge |\lambda_2(z)| \ge |\lambda_3(z)|$.
The Fenchel conjugate is given by [2]

$$j_0^*(z^*) = \begin{cases} \frac{1}{2}\big(|\tau_1| + |\tau_2| + |\tau_3|\big)^2 & \text{if } |\tau_3| \le |\tau_1| + |\tau_2| \\ \frac{1}{2}\big((|\tau_1| + |\tau_2|)^2 + |\tau_3|^2\big) & \text{otherwise} \ , \end{cases} \tag{7.1}$$

where $\tau_i = \tau_i(z^*)$ are the eigenvalues of the matrix $z^* \in \mathbb{R}^{3\times 3}_{\rm sym}$.
Take now $z^* \in \mathbb{R}^{2\times 2}_{\rm sym}$, and denote by $(z^*|0) \in \mathbb{R}^{3\times 3}_{\rm sym}$ the matrix obtained by adding to z^* a line and a column of zeros. It is easy to check directly from the definition of $\overline{j_0}$ that the Fenchel conjugates of $\overline{j_0}$ and j_0 are related by the identity

$$\big(\overline{j_0}\big)^*(z^*) = (j_0)^*((z^*|0)) \ . \tag{7.2}$$

By using (7.1) and (7.2), for any $z \in \mathbb{R}^{2\times 2}_{\rm sym}$, we obtain

$$\begin{aligned} \overline{j_0}(z) &= \sup\Big\{ z \cdot z^* - \big(\overline{j_0}\big)^*(z^*) \ : \ z^* \in \mathbb{R}^{2\times 2}_{\rm sym} \Big\} \\ &= \sup\Big\{ (ze\cdot e)\tau_1 + (ze^\perp \cdot e^\perp)\tau_2 - \tfrac{1}{2}\big(|\tau_1| + |\tau_2|\big)^2 \ : \ \tau_i \in \mathbb{R} \ , e \in S^1 \Big\} \ . \\ &= \sup_{e\in S^1}\Big\{ \max\Big\{ \sup\big\{ (ze\cdot e)\tau_1 + (ze^\perp \cdot e^\perp)\tau_2 - \tfrac{1}{2}\big(|\tau_1| + |\tau_2|\big)^2 \ : \ \tau_1 \cdot \tau_2 \in \mathbb{R}^\pm \big\} \Big\}. \end{aligned}$$

For a fixed $e \in S^1$, let us consider the supremum over $\tau_1 \cdot \tau_2 \in \mathbb{R}^+$: the optimality conditions are

$$ze \cdot e = \tau_1 + \tau_2 \ , \qquad ze^\perp \cdot e^\perp = \tau_1 + \tau_2 \ .$$

Therefore, the value of the supremum is $(1/2)\max\{(ze\cdot e)^2,(ze^\perp\cdot e^\perp)^2\}$. The supremum over $\tau_1\cdot\tau_2\in\mathbb{R}^-$ has the same value as it can be obtained from the supremum over $\tau_1\cdot\tau_2\in\mathbb{R}^+$ up to changing the sign of $ze^\perp\cdot e^\perp$. Finally, optimizing with respect to e gives

$$\overline{j_0}(z)=\sup_{e\in S^1}\left\{\frac{1}{2}\max\left\{(ze\cdot e)^2,(ze^\perp\cdot e^\perp)^2\right\}\right\}=\frac{1}{2}(\lambda_1(z))^2\ .$$

Computation of $(\bar{j})_\ominus$. For $z\in\mathbb{R}^{2\times2}_{\rm sym}$, we have $\bar{j}(z)=(1/2)|z|^2$ (see [11]). Then

$$\begin{aligned}(\bar{j})_\ominus(z)\ &=\sup\left\{z\cdot z^*-(1/2)|z^*|^2\ :\ z^*\in\mathbb{R}^{2\times2}_{\rm sym}\ ,\ \det(z^*)\le 0\right\}\\ &=\sup\left\{(ze\cdot e)\tau_1+(ze^\perp\cdot e^\perp)\tau_2-\tfrac{1}{2}\left(\tau_1^2+\tau_2^2\right)\ :\ \tau_1\cdot\tau_2\in\mathbb{R}^-,e\in S^1\right\}\ .\end{aligned}$$

For a fixed $e\in S^1$, let us consider the supremum over $\tau_1\cdot\tau_2\in\mathbb{R}^-$: the optimaliy conditions are

$$ze\cdot e=\tau_1\ ,\qquad ze^\perp\cdot e^\perp=\tau_2\ .$$

Therefore, the value of the supremum is

$$\psi(e):=\begin{cases}\psi_1(e):=\dfrac{1}{2}\left[(ze\cdot e)^2+(ze^\perp\cdot e^\perp)^2\right] & \text{if } (ze\cdot e)\cdot(ze^\perp\cdot e^\perp)\in\mathbb{R}^-\\ \psi_2(e):=\dfrac{1}{2}\max\left\{(ze\cdot e)^2,(ze^\perp\cdot e^\perp)^2\right\} & \text{otherwise}\ .\end{cases}$$

If we write $e=(\cos\theta,\sin\theta)$ in a basis made by eigenvectors of z, and we denote by λ_i the eigenvalues of z (with $|\lambda_1|\ge|\lambda_2|$), we have

$$(ze\cdot e)\cdot(ze^\perp\cdot e^\perp)\in\mathbb{R}^-\iff\frac{\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}\le-\frac{\lambda_1\lambda_2}{\lambda_1^2+\lambda_2^2}\ . \tag{7.3}$$

Then two cases may occur.
Case 1: $\det(z)>0$. In this case (7.3) cannot hold, so that for every $e\in S^1$ we have $\psi(e)=\psi_2(e)$. In terms of θ the function ψ_2 is written as

$$\psi_2(\theta)=\frac{1}{2}\max\left\{(\lambda_1\cos^2\theta+\lambda_2\sin^2\theta)^2\ ,\ (\lambda_1\sin^2\theta+\lambda_2\cos^2\theta)^2\right\};$$

then the supremum of $\psi_2(\theta)$ over S^1 equals $(1/2)\lambda_1^2$.
Case 2: $\det(z)\le 0$. In terms of θ the function $\psi_1(e)$ is written as

$$\psi_1(\theta)=\frac{1}{2}(\lambda_1^2+\lambda_2^2)(\cos^4\theta+\sin^4\theta)+2\lambda_1\lambda_2\cos^2\theta\sin^2\theta;$$

then the supremum of $\psi_1(\theta)$ for θ satisfying (7.3) is easily computed to be $(1/2)(\lambda_1^2+\lambda_2^2)$. Since such value is larger than (or equal to) the supremum of ψ_2 on S^1, it is also the supremum of ψ on S^1.

□

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