Nodal solutions to critical growth elliptic problems under Steklov boundary conditions

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April 30, 2008

Abstract

We study two elliptic problems, respectively in the second and in the fourth order case, both under Steklov-type boundary conditions and critical growth. In the second order case, by standard tools of critical point theory, we give existence and nonexistence regions for nontrivial nodal solutions. The basic ideas here are to concentrate the Sobolev minimizers on the boundary and to perform a suitable orthogonal decomposition of the functional set of the solutions. In the fourth order, in spite of the similarity between the variational structures of the two problems, concentration doesn't work and we only have partial results.

1 Introduction

In a celebrated paper, Pohozaev [25] proved that the semilinear elliptic equation

$$-\Delta u = |u|^{2^*-2}u \qquad \text{in } \Omega \tag{1}$$

admits no positive solutions in a bounded smooth starshaped domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) under homogeneous Dirichlet boundary conditions. In fact, in these domains, Pohozaev's identity combined with the unique continuation property rules out also the existence of nodal solutions (see [19]) so that (1) admits only the trivial solution $u \equiv 0$. Here $2^* = \frac{2n}{n-2}$ denotes the critical exponent for the embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$. Since then, in order to obtain existence results for the *Dirichlet* problem associated to (1), many attempts were made to modify the geometry (topology) of the domain Ω or to perturb the critical nonlinearity $|u|^{2^*-2}u$ in (1). It appears an impossible task to exhaust all the literature. In these papers, existence of nontrivial solutions was obtained.

Much less is known when different boundary value problems are considered. Brezis [10, Section 6.4] suggested to study (1) under *Neumann* boundary conditions:

$$u_\nu = 0 \qquad \text{on } \partial\Omega \tag{2}$$

where u_ν denotes the outer normal derivative of u on $\partial\Omega$. Problem (1)-(2) was studied by Compte-Knaap [15]: it is shown there that if $n \geq 4$ then it admits nontrivial solutions in any domain Ω.

One of the purposes of the present paper is to study existence of nodal solutions for a different boundary value problem. For $\delta \in \mathbb{R}$, we consider the following (second order) elliptic problem with purely critical growth and *Steklov* boundary conditions:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega \\ u_\nu = \delta u & \text{on } \partial\Omega. \end{cases} \tag{3}$$

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Clearly, (3) becomes the Neumann problem when $\delta = 0$ and tends to the Dirichlet problem as $\delta \to -\infty$. Hence, one expects nonexistence results in the spirit of [25] for δ sufficiently negative: when Ω is the ball, this was proved independently in [1, 29]. When $\delta < 0$, existence of positive solutions to (3) in general domains was studied in [1, 29], see also [18] for the case $n = 3$ in the ball. In these papers, the authors take advantage of the mountain-pass variational structure (constrained minimization over the whole space).
We are here interested in the case where $\delta > 0$ and we obtain existence results for (3) by using variational methods. Since the variational structure of the problem is no longer of mountain-pass type, linking arguments are required. In this case, it is well-known that in order to lower the energy level of Palais-Smale sequences one needs to estimate "mixed terms" which are difficult to estimate, see [14, 17]. The basic idea is to concentrate Sobolev minimizers on the boundary as in [2, 3] but before concentrating we need to subtract their mean value on the boundary.
A further goal of this paper is to highlight the nonstandard variational structure of (3). The space spanned by the eigenfunctions of the linear boundary value problem does not exhaust all the functional space under consideration. Therefore, the linking argument used for its study has somehow a more complicated behaviour. We collect the main properties describing the variational structure in the Appendix.
Finally, we emphasize that a quite similar structure may also be observed for the corresponding fourth order critical growth problem

$$\begin{cases} \Delta^2 u = |u|^{2_*-2}u & \text{in } \Omega \\ u = 0, \quad \Delta u = du_\nu & \text{on } \partial\Omega \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 5$) is a smooth bounded domain, $d \in \mathbb{R}$ and $2_* = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embedding $H^2(\Omega) \subset L^{2_*}(\Omega)$. Also the boundary conditions in (4) are named after *Steklov*. They were first studied for the eigenvalue problem in the two dimensional case [20, 23] and more recently for the same problem in any dimension [16]. For some nonlinear problems and for the positivity preserving property we refer to [7, 8]. In particular, in [8] the existence of positive solutions of (4) was studied. Here, we are again concerned with the existence of nodal solutions. Although (4) has the same variational structure as (3), it exhibits several different features. In particular, we cannot expect concentration phenomena on the boundary since $u = 0$ on $\partial\Omega$. Moreover, since (4) requires several hard computations, we obtain existence results only when Ω is the unit ball in dimensions $n = 5, 6, 8$.

2 Main results

We say that a function $u \in H^1(\Omega)$ is a weak solution of (3) if

$$\int_\Omega \nabla u \nabla v - \delta \int_{\partial\Omega} uv = \int_\Omega |u|^{2^*-2}uv \qquad \text{for all } v \in H^1(\Omega) .$$

We say that a function $u \in H^2 \cap H^1_0(\Omega)$ is a weak solution of (4) if

$$\int_\Omega \Delta u \Delta v - d \int_{\partial\Omega} u_\nu v_\nu = \int_\Omega |u|^{2_*-2}uv \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega) .$$

It can be shown that weak solutions in these senses are in fact strong (classical) solutions, see [11] for the second order equation and [7, Proposition 23] for the fourth order equation.

Here and in the following, we denote by $\|\cdot\|_p$ the $L^p(\Omega)$-norm ($1 \le p \le \infty$), and we put

$$\|u\|_\partial^2 = \int_{\partial\Omega} u^2 \qquad \text{for } u \in H^1(\Omega)\ , \qquad\qquad \|u\|_{\partial_\nu}^2 = \int_{\partial\Omega} u_\nu^2 \qquad \text{for } u \in H^2 \cap H_0^1(\Omega).$$

Set

$$H_{\max} := \max_{x\in\partial\Omega} H(x), \tag{5}$$

where $H(x)$ is the mean curvature of $\partial\Omega$. Let us recall the statement concerning positive solutions:

Theorem 1. [1, 29]
Let $\Omega \subset \mathbb{R}^n$ ($n \ge 4$) be a smooth bounded domain.
(i) If $\delta \ge 0$, then (3) admits no positive solutions.
(ii) If $\delta \in \left(\frac{2-n}{2} H_{\max}, 0\right)$, then (3) admits a positive solution.
Moreover, if $\Omega = B$ (the unit ball of $\mathbb{R}^n$, $n \ge 3$), then:
(iii) If $\delta \le 2-n$, (3) admits no positive radial solutions.
(iv) If $\delta \in (2-n, 0)$, then problem (3) admits a unique positive radial solution u_δ which is explicitly given by

$$u_\delta(x) = \frac{[n(n-2)C_{\delta,n}]^{\frac{n-2}{4}}}{(C_{\delta,n} + |x|^2)^{\frac{n-2}{2}}},$$

where $C_{\delta,n} := \frac{2-n}{\delta} - 1$.

In order to state our result about nodal solutions, we introduce the set

$$\mathcal{X}(\Omega) := \left\{u \in H^1(\Omega) : \int_{\partial\Omega} u = 0\right\} \setminus H_0^1(\Omega)$$

and define

$$\delta_1 := \inf_{u\in\mathcal{X}(\Omega)} \frac{\|\nabla u\|_2^2}{\|u\|_\partial^2} \tag{6}$$

so that δ_1 is the largest constant satisfying

$$\|\nabla u\|_2^2 \ge \delta_1 \|u\|_\partial^2 \qquad \text{for all } u \in \mathcal{X}(\Omega).$$

Moreover, δ_1 is the first nontrivial eigenvalue of $-\Delta$ under the Steklov boundary conditions, see the Appendix. Then, we have

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ ($n \ge 4$) be a smooth bounded domain. If $\delta \in (0, \delta_1)$, then (3) admits a pair of nontrivial nodal solutions.*

In the case where Ω is the unit ball, Theorem 2 combined with Theorem 13 in the Appendix, states that (3) has nontrivial nonradial solutions for all $\delta \in (0,1)$.

For the fourth order problem (4) we only consider the case where $\Omega = B$ so that the first boundary eigenvalue is $d_1 = n$, see [7] and Theorem 16 in the Appendix. Let us also recall results from [8] about positive solutions. For $n \ge 5$, let

$$\sigma_n = \begin{cases} n - (n-4)(n^2-4)\frac{\Gamma(\frac{n}{2})}{2^{\frac{8}{n}+1}}\left(\frac{n\Gamma(\frac{n}{2})}{\Gamma(n)}\right)^{\frac{4}{n}}\left(\frac{\Gamma(\frac{2n}{n-4})}{\Gamma(\frac{n^2}{2(n-4)})}\right)^{1-\frac{4}{n}} & \text{if } n = 5 \text{ or } n = 6 \\ \frac{4(n-3)}{n-4} & \text{if } n \ge 7\ . \end{cases}$$

In particular, $\sigma_5 \approx 4.5$ and $\sigma_6 \approx 5.2$, see [4]. Then, we have

Theorem 3. [8]
Assume that $\Omega = B$ (the unit ball of $\mathbb{R}^n$, $n \geq 5$).
(i) If $d \leq 4$ or $d \geq n$, then (4) admits no positive solution.
(ii) If $d \in (\sigma_n, n)$ problem (4) admits a radial positive solution.
(iii) For every $d \in \mathbb{R}$, problem (4) admits no radial nodal solutions.

For $n \geq 5$, put

$$g(n) := \frac{n^2(n-2)\Gamma\left(\frac{n}{2}\right)}{4}\left[\frac{(n-4)(n+2)\Gamma(\frac{n}{2})}{2\Gamma(n)}\right]^{4/n}\left[\frac{(n+4)\Gamma(\frac{2n}{n-4})\Gamma(\frac{n+4}{2(n-4)})}{\sqrt{\pi}\,\Gamma(\frac{n^2+2n}{2(n-4)})}\right]^{1-4/n}. \tag{7}$$

Then, in some dimensions, we can prove existence and multiplicity results for $d \geq n$:

Theorem 4. *Assume that $\Omega = B$ (the unit ball of $\mathbb{R}^n$) and let $n = 5, 6, 8$.*
If $d \in (n+2-g(n), n+2)$ problem (4) admits at least n pairs of nontrivial solutions.

Remark 5. As we explain in Section 5, even if we do not have a complete proof, we believe that Theorem 4 holds for every $n \geq 5$. If this is true, since $g(n) \geq 2$ for $n \geq 16$, this means that the existence result, for n large, covers the whole range between n and $n+2$.

3 The Palais-Smale condition

Let

$$S_2 = \min_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$

By [21] we know that there exists $K = K(\Omega) > 0$ such that

$$\frac{S_2}{2^{2/n}}\|u\|_{2^*}^2 \leq \|\nabla u\|_2^2 + K\|u\|_\partial^2 \qquad \text{for all } u \in H^1(\Omega). \tag{8}$$

Consider the space $H^1(\Omega)$ endowed with the scalar product

$$(u, v)_1 := \int_\Omega \nabla u \nabla v + \int_{\partial\Omega} uv \qquad \text{for all } u, v \in H^1(\Omega) \tag{9}$$

and the induced norm

$$\|u\|^2 := \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} |u|^2 \qquad \text{for all } u \in H^1(\Omega). \tag{10}$$

Consider the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\delta}{2}\int_{\partial\Omega} u^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} \tag{11}$$

whose critical points are weak solutions of (3). We prove

Lemma 6. *The functional I satisfies the Palais-Smale condition at levels $c \in (-\infty, \frac{S_2^{n/2}}{2n})$, that is, if $\{u_m\}_{m\geq 0} \subset H^1(\Omega)$ is such that*

$$I(u_m) \to c < \frac{S_2^{n/2}}{2n}, \quad dI(u_m) \to 0 \quad \text{in} \quad (H^1(\Omega))', \tag{12}$$

then there exists $u \in H^1(\Omega)$ such that $u_m \to u$ in $H^1(\Omega)$, up to a subsequence.

Proof. To deduce that $\{u_m\}_{m\geq 0}$ is bounded in $H^1(\Omega)$ we follow [26, Theorem 4.12]. Let $\{\delta_j\}_{j\geq 0}$ be the set of the eigenvalues of $-\Delta$ under the Steklov boundary condition and denote with M_j the eigenspace associated to δ_j. If $\delta = \delta_k$, for some $k \geq 0$, we define:

$$H_+ := \overline{\bigoplus_{j\geq k+1} M_j} \bigoplus H_0^1(\Omega), \quad H_0 := M_k \quad \text{and} \quad H_- := \bigoplus_{j\leq k-1} M_j$$

and, in view of Theorem 13 in the Appendix, we have

$$H^1(\Omega) = H_+ \oplus H_0 \oplus H_-.$$

Thus we may decompose $u_m = u_m^+ + u_m^0 + u_m^-$, where $u_m^+ \in H_+$, $u_m^0 \in H_0$ and $u_m^- \in H_-$. If $\delta \neq \delta_k$, for every $k \geq 0$, and $\delta_k < \delta < \delta_{k+1}$, we just have the two spaces H_+ and H_- but the decomposition works similarly. By (12) and arguing as in [26], one can prove that each of the components of u_m, and in turn u_m, is bounded in $H^1(\Omega)$. By this we conclude that (up to a subsequence) there exists $u \in H^1(\Omega)$ such that

$$u_m \rightharpoonup u \quad \text{in } H^1(\Omega) \quad \text{and} \quad u_m \to u \quad \text{a.e. in } \Omega. \tag{13}$$

Hence, by compactness of the map $H^1(\Omega) \to L^2(\partial\Omega)$ defined by $u \mapsto u|_{\partial\Omega}$, we have:

$$u_m\,|_{\partial\Omega} \to u\,|_{\partial\Omega} \quad \text{in} \quad L^2(\partial\Omega). \tag{14}$$

We apply (8) to the function $u_m - u$ and, in view of (14), we get

$$\frac{S_2}{2^{2/n}}\,\|u_m - u\|_{2^*}^2 \leq \|\nabla(u_m - u)\|_2^2 + o(1). \tag{15}$$

On the other hand, by the Brezis-Lieb Lemma [12], we know that

$$\|u_m\|_{2^*}^{2^*} - \|u\|_{2^*}^{2^*} = \|u_m - u\|_{2^*}^{2^*} + o(1). \tag{16}$$

Exploiting (12), (13), (14) and (16) we have

$$\begin{aligned} o(1) &= \langle dI(u_m), u_m - u\rangle \\ &= \int_\Omega |\nabla u_m|^2 - \int_\Omega \nabla u_m \cdot \nabla u - \delta \int_{\partial\Omega} u_m(u_m - u) - \int_\Omega |u_m|^{2^*-2} u_m(u_m - u) \\ &= \int_\Omega (|\nabla u_m|^2 - 2\nabla u_m \cdot \nabla u + |\nabla u|^2) - \int_\Omega |u_m|^{2^*} + \int_\Omega |u|^{2^*} + o(1) \\ &= \int_\Omega |\nabla(u_m - u)|^2 - \int_\Omega |u_m - u|^{2^*} + o(1), \end{aligned}$$

so that

$$\|\nabla(u_m - u)\|_2^2 = \|u_m - u\|_{2^*}^{2^*} + o(1). \tag{17}$$

By (12) we also get that

$$o(1) = \langle dI(u_m), u_m\rangle = \|\nabla u_m\|_2^2 - \delta\|u_m\|_\partial^2 - \|u_m\|_{2^*}^{2^*},$$

that is,

$$\|u_m\|_{2^*}^{2^*} = \|\nabla u_m\|_2^2 - \delta\|u_m\|_\partial^2 + o(1). \tag{18}$$

Inserting (18) into (12) we obtain

$$\frac{1}{n}\|\nabla u_m\|_2^2 - \frac{\delta}{n}\|u_m\|_\partial^2 = c + o(1)$$

and therefore

$$\|\nabla u\|_2^2 - \delta\|u\|_\partial^2 + \|\nabla(u_m - u)\|_2^2 = nc + o(1). \tag{19}$$

On the other hand, exploiting the convergence $\langle dI(u_m), v\rangle \to \langle dI(u), v\rangle$ for any fixed $v \in H^1(\Omega)$, we deduce that u solves (3) (that is, $dI(u) = 0$) so that

$$\|\nabla u\|_2^2 - \delta\|u\|_\partial^2 = \|u\|_{2^*}^{2^*} \geq 0.$$

The last inequality combined with (19) gives

$$\|\nabla(u_m - u)\|_2^2 \leq nc + o(1) < \frac{S_2^{n/2}}{2} + o(1). \tag{20}$$

Furthermore (15) and (17) give

$$\|\nabla(u_m - u)\|_2^{2-\frac{4}{n}} \left(\frac{S_2}{2^{2/n}} - \|\nabla(u_m - u)\|_2^{\frac{4}{n}}\right) \leq o(1).$$

This, combined with (20), shows that $\|\nabla(u_m - u)\|_2 = o(1)$. And this, together with (14), proves that $u_m \to u$ in $H^1(\Omega)$. □

We now turn to the fourth order problem. Let

$$S_4 = \min_{u \in \mathcal{D}^{2,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\Delta u\|_2^2}{\|u\|_{2_*}^2}.$$

Consider the space $H^2 \cap H_0^1(\Omega)$ endowed with the scalar product

$$(u, v)_2 := \int_\Omega \Delta u \Delta v \qquad \text{for all } u, v \in H^2 \cap H_0^1(\Omega) \tag{21}$$

and the induced norm

$$|||u|||^2 := \int_\Omega |\Delta u|^2 \qquad \text{for all } u \in H^2 \cap H_0^1(\Omega). \tag{22}$$

Consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{d}{2}\int_{\partial\Omega} u_\nu^2 - \frac{1}{2_*}\int_\Omega |u|^{2_*} \tag{23}$$

whose critical points are weak solutions of (4). We have

Lemma 7. *The functional J satisfies the Palais-Smale condition at levels $c \in (-\infty, \frac{2S_4^{n/4}}{n})$, that is, if $\{u_m\}_{m\geq 0} \subset H^2 \cap H_0^1(\Omega)$ is such that*

$$J(u_m) \to c < \frac{2}{n} S_4^{n/4}, \quad dJ(u_m) \to 0 \quad \textit{in} \quad (H^2 \cap H_0^1(\Omega))', \tag{24}$$

then there exists $u \in H^2 \cap H_0^1(\Omega)$ such that $u_m \to u$ in $H^2 \cap H_0^1(\Omega)$, up to a subsequence.

Proof. The first step consists in showing that $\{u_m\}_{m\geq 0}$ is bounded in $H^2 \cap H_0^1(\Omega)$. As in Lemma 6, this follows by arguing as in Theorem 4.12 in [26], suitably adapted to this case. For the rest of the proof one can follow the same lines as the proof of Lemma 6 except that, now, one has to exploit the compactness of the linear map $H^2 \cap H_0^1(\Omega) \ni u \mapsto u_\nu|_{\partial\Omega} \in L^2(\partial\Omega)$ and the inequality (8) must be replaced by the Sobolev inequality: $S_4\|u\|_{2_*}^2 \leq \|\Delta u\|_2^2$, for all $u \in H^2 \cap H_0^1(\Omega)$. □

4 Proof of Theorem 2

The nonexistence result for $\delta \geq 0$ is a consequence of the divergence Theorem combined with the boundary condition. Indeed, if $u > 0$ is a solution of (3) and $\delta \geq 0$, we have:

$$0 < \int_\Omega u^{2^*-1} = -\int_\Omega \Delta u = -\int_{\partial\Omega} u_\nu = -\delta \int_{\partial\Omega} u \leq 0,$$

which is impossible.

Concerning the existence result, we prove it by showing that there exists a critical level for the functional (11) below the compactness threshold found in Lemma 6. In order to do this, we need some estimates that we collect in the following subsection.

4.1 Estimates

For our convenience, we introduce the notation $\overline{x} \equiv (x_1, ..., x_{n-1})$, $\overline{\nabla} \equiv (\partial_{x_1}, ..., \partial_{x_{n-1}})$. We choose a point $x_0 \in \partial\Omega$ such that $H(x_0) = H_{\max}$ (see (5)), a neighborhood N of x_0 and a coordinate system with origin in x_0 such that the domain $\Omega \cap N$ is described by the relation

$$\Omega \cap N = \{x \in N : \, x_n \geq f(\overline{x})\}, \tag{25}$$

where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a smooth function satisfying $f(0) = 0$, $\overline{\nabla} f(0) = 0$. Define the transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi := \begin{cases} \overline{y} = \overline{x}\,, \\ y_n = x_n - f(\overline{x})\,. \end{cases} \tag{26}$$

It is easily checked that Φ transforms the region $x_n \geq f(\overline{x})$ into the half-space $y_n \geq 0$ and that its Jacobian is 1. Moreover, we have the relation between the surface elements $d\sigma = \sqrt{1 + |\overline{\nabla} f|^2} d\overline{y}$. We may also assume that N contains $\Phi^{-1}(B_r \times [0,1])$, where B_r is the closed ball of radius r centered at the origin in $\mathbb{R}^{n-1}$. Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a fixed cut-off function such that $\eta \circ \Phi$ has support contained in N and is equal to one in $\Phi^{-1}(B_r \times [0,1])$. Then, we define

$$u_\epsilon^*(y) = \eta(y) u_\epsilon(y), \tag{27}$$

where

$$u_\epsilon(y) = \frac{[n(n-2)]^{\frac{n-2}{4}} \epsilon^{\frac{n-2}{2}}}{(\epsilon^2 + |y|^2)^{\frac{n-2}{2}}}. \tag{28}$$

Finally, we set

$$v_\epsilon^*(x) = u_\epsilon^*(\Phi(x)). \tag{29}$$

When $\epsilon \to 0$ the functions v_ϵ^* "concentrate" at the origin which, by construction, is a point of $\partial\Omega$ where the mean curvature attains its maximum.

We now prove some estimates when $\epsilon \to 0$. We first observe that from (27)-(29) the following identities are easily verified

$$\int_\Omega |v_\epsilon^*(x)|^{2^*} dx = \int_{\mathbb{R}^n_+} |u_\epsilon^*(y)|^{2^*} dy, \tag{30}$$

$$\int_{\partial\Omega} |v_\epsilon^*(x)|^2 d\sigma = \int_{\mathbb{R}^{n-1}} |u_\epsilon^*(\overline{y}, 0)|^2 \sqrt{1 + |\overline{\nabla} f(\overline{y})|^2} d\overline{y}. \tag{31}$$

From [13] we have

$$\int_{\mathbb{R}^n_+} |u^*_\epsilon(y)|^{2^*} dy = \int_{\mathbb{R}^n_+} |u_\epsilon(y)|^{2^*} dy + O(\epsilon^n)$$

which, combined with (30), yields

$$\int_\Omega |v^*_\epsilon(x)|^{2^*} dx = \int_{\mathbb{R}^n_+} |u_\epsilon(y)|^{2^*} dy + O(\epsilon^n). \tag{32}$$

The last term in (31) can be estimated by bounding $|\overline{\nabla} f|$ and scaling, see [24]; then we obtain

$$\int_{\mathbb{R}^{n-1}} |u^*_\epsilon(\overline{y},0)|^2 \sqrt{1+|\overline{\nabla} f(\overline{y})|^2} d\overline{y} = \int_{B_r} |u_\epsilon(\overline{y},0)|^2 d\overline{y} + \begin{cases} O(\epsilon^2) & \text{if } n=4 \\ O(\epsilon^3|\log\epsilon|) & \text{if } n=5 \\ O(\epsilon^3) & \text{if } n\geq 6. \end{cases}$$

By scaling we also get

$$\int_{B_r} |u_\epsilon(\overline{y},0)|^2 d\overline{y} = \epsilon \int_{B_{r/\epsilon}} |u_1(\overline{y},0)|^2 d\overline{y} = \epsilon \int_{\mathbb{R}^{n-1}} |u_1(\overline{y},0)|^2 d\overline{y} + O(\epsilon^n) \equiv K\epsilon + O(\epsilon^n), \tag{33}$$

so that, for any $n \geq 4$,

$$\int_{\partial\Omega} |v^*_\epsilon(x)|^2 d\sigma = K\epsilon + o(\epsilon)\ . \tag{34}$$

Next, from $\nabla v^*_\epsilon(x) = D\Phi(x)\nabla u^*_\epsilon(\Phi(x))$ we obtain after some calculations

$$\int_\Omega |\nabla v^*_\epsilon(x)|^2 dx = \int_{\mathbb{R}^n_+} \left[|\nabla u^*_\epsilon(y)|^2 - 2\overline{\nabla} f(\overline{y})\overline{\nabla} u^*_\epsilon(y)\partial_\nu u^*_\epsilon(y) + |\overline{\nabla} f(\overline{y})\overline{\nabla} u^*_\epsilon(y)|^2\right] dy.$$

Hence, assuming that Δf is bounded in B_r, by [24, Lemmas 5.2 and 5.3], we have for $\epsilon \to 0$:

$$\int_\Omega |\nabla v^*_\epsilon(x)|^2 dx = \int_{\mathbb{R}^n_+} |\nabla u_\epsilon(y)|^2 dy - \frac{n-2}{2(n-1)} \int_{B_r} \Delta f(\overline{y}) u^2_\epsilon(\overline{y},0) d\overline{y} + R(\epsilon), \tag{35}$$

where, for some positive constant c,

$$R(\epsilon) = \begin{cases} c\epsilon^2|\log\epsilon| + O(\epsilon^2) & \text{if } n=4 \\ c\epsilon^2 + O(\epsilon^{n-2}) & \text{if } n\geq 5. \end{cases}$$

Set $h(\overline{y}) = \Delta f(\overline{y})/(n-1)$ so that $h(0) = H_{\max}$ is the mean curvature of the boundary at the origin. Therefore, for every $\gamma < H_{\max}$, we have $h(\overline{y}) \geq \gamma$ for $\overline{y} \in B_r$ with small enough r. This combined with (33) gives

$$\int_\Omega |\nabla v^*_\epsilon(x)|^2 dx \leq \int_{\mathbb{R}^n_+} |\nabla u_\epsilon(y)|^2 dy - \gamma\frac{n-2}{2} K\epsilon + R(\epsilon). \tag{36}$$

We conclude with two further estimates. Let $B_R \subset \mathbb{R}^n$ be a ball containing the support of u^*_ϵ; then, for any $\alpha > 0$ we have

$$I_\alpha \equiv \int_\Omega |v^*_\epsilon(x)|^\alpha dx = \int_{\mathbb{R}^n_+} |u^*_\epsilon(y)|^\alpha dy = \int_{\mathbb{R}^n_+\cap B_R} |u_\epsilon(y)|^\alpha dy = \quad \text{(by (28))}$$

$$= C\epsilon^{\alpha\frac{n-2}{2}} \int_{\mathbb{R}^n_+\cap B_R} \frac{dy}{(\epsilon^2+|y|^2)^{\alpha\frac{n-2}{2}}} = \quad (y=\epsilon z,\ |z|=\rho)$$

$$= C\epsilon^{n-\alpha\frac{n-2}{2}}\int_0^{R/\epsilon}\frac{\rho^{n-1}}{(1+\rho^2)^{\alpha\frac{n-2}{2}}}d\rho \le C\epsilon^{n-\alpha\frac{n-2}{2}}\Big(C_0+\int_1^{R/\epsilon}\rho^{n-1-\alpha(n-2)}d\rho\Big)$$

$$\le \begin{cases} C_1\epsilon^{n-\alpha\frac{n-2}{2}}+C_2\epsilon^{\alpha\frac{n-2}{2}} & \text{for} \quad \alpha\neq\frac{n}{n-2} \\ \epsilon^{n/2}(C_1+C_2|\ln\epsilon|) & \text{for} \quad \alpha=\frac{n}{n-2}. \end{cases}$$

In particular, we get (for $n \ge 4$)

$$I_{(2^*-1)} = I_{\frac{n+2}{n-2}} = O(\epsilon^{(n-2)/2}), \qquad I_1 = O(\epsilon^{(n-2)/2}); \tag{37}$$

$$I_{(2^*-2)} = \begin{cases} I_2 = O(\epsilon^2\ln\epsilon) & \text{if} \quad n=4 \\ I_{\frac{4}{n-2}} = O(\epsilon^2) & \text{if} \quad n\ge 5. \end{cases} \tag{38}$$

4.2 Linking argument

For any $u \in H^1(\Omega)\setminus\{0\}$ define the functional

$$F(v) = \int_\Omega |\nabla v|^2dx - \delta\int_{\partial\Omega}|v|^2d\sigma. \tag{39}$$

We consider $H^1(\Omega)$ equipped with the norm (10). Let M_0 be the closed subspace of $H^1(\Omega)$ of the functions with zero mean value on $\partial\Omega$. From Theorem 13 in the Appendix, we know that $M_0 = H_0^1(\Omega)\oplus V^+$, where V^+ is the subspace spanned by the eigenfunctions e_n of problem (53) with positive eigenvalues $0<\delta_1<\delta_2<...$

Let F be the functional defined in (39). We want to minimize the ratio

$$\frac{F(v)}{\|v\|_{2^*}^2}$$

over M_0. Note that if $\delta<\delta_1$, then $F(v)>0$ for all $v\in M_0$.

We consider the functions v_ϵ^* in (29) and we define $\overline{v}_\epsilon^* = v_\epsilon^* - m_\epsilon$, where

$$m_\epsilon = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} v_\epsilon^* d\sigma, \tag{40}$$

so that $\overline{v}_\epsilon^*\in M_0$. We have

$$\int_{\partial\Omega} v_\epsilon^*d\sigma = \int_{\mathbb{R}^{n-1}} u_\epsilon^*(\overline{y},0)\sqrt{1+|\overline{\nabla}f(\overline{y})|^2}d\overline{y} \le C\int_{B_R} u_\epsilon(\overline{y},0)d\overline{y},$$

where $B_R\subset\mathbb{R}^{n-1}$ is a ball containing the support of $\eta(\overline{y},0)$ and $C=\max_{B_R}\sqrt{1+|\overline{\nabla}f|^2}$; hence, by scaling as before we get

$$m_\epsilon = O(\epsilon^{(n-2)/2}).$$

Then we obtain:

$$F(\overline{v}_\epsilon^*) = \int_\Omega|\nabla\overline{v}_\epsilon^*|^2dx - \delta\int_{\partial\Omega}|\overline{v}_\epsilon^*|^2d\sigma = \int_\Omega|\nabla v_\epsilon^*|^2dx - \delta\int_{\partial\Omega}|v_\epsilon^*|^2d\sigma + \delta m_\epsilon^2|\Omega| = F(v_\epsilon^*)+O(\epsilon^{n-2}). \tag{41}$$

Furthermore, we have

$$\int_\Omega|\overline{v}_\epsilon^*|^{2^*}dx = \int_\Omega|v_\epsilon^*|^{2^*}dx - 2^*m_\epsilon\int_0^1dt\int_\Omega|v_\epsilon^*-tm_\epsilon|^{2^*-2}(v_\epsilon^*-tm_\epsilon)dx.$$

The estimate of the last term (see (37) above) gives

$$\int_\Omega |\bar v_\epsilon^*|^{2^*}dx = \int_\Omega |v_\epsilon^*|^{2^*}dx + O(\epsilon^{n-2}). \tag{42}$$

Finally, by the last identities and by (34), (32), (36), we get

$$\begin{aligned}\frac{F(\bar v_\epsilon^*)}{\left(\int_\Omega |\bar v_\epsilon^*|^{2^*}dx\right)^{2/2^*}} &= \frac{F(v_\epsilon^*) + O(\epsilon^{n-2})}{\left(\int_\Omega |v_\epsilon^*|^{2^*}dx + O(\epsilon^{n-2})\right)^{2/2^*}}\\ &\le \frac{\int_{\mathbb{R}^n_+}|\nabla u_\epsilon(y)|^2 - \epsilon K\big(\delta+\gamma\frac{n-2}{2}\big) + R(\epsilon)}{\left(\int_{\mathbb{R}^n_+}|u_\epsilon(y)|^{2^*}dy + O(\epsilon^{n-2})\right)^{2/2^*}}\\ = \frac{\frac12 S_2^{n/2} - \epsilon K\big(\delta+\gamma\frac{n-2}{2}\big)+R(\epsilon)}{\left(\frac12 S_2^{n/2} + O(\epsilon^{n-2})\right)^{2/2^*}} &= \frac{S_2}{2^{2/n}} - \epsilon K'\big(\delta+\gamma\frac{n-2}{2}\big) + R(\epsilon),\end{aligned} \tag{43}$$

where $K' > 0$. Conclusion: since $R(\epsilon)$ comes from (35), we go below the critical level for ϵ sufficiently small.

Let us consider the direct sum

$$H^1(\Omega) = M_0 \oplus \mathbb{R};$$

furthermore, suppose $0 < \rho < R_1$, $0 < R_2$ and let

$$S = \Big\{u \in V\ :\ \|u\| = \rho\Big\}$$

$$Q = \Big\{s\bar v_\epsilon^* + c, \quad 0 \le s \le R_1, \quad |c| \le R_2\Big\}. \tag{44}$$

Assume that $\|\bar v_\epsilon^*\| = 1$ in (44), then S and ∂Q link (see [27] Example 8.3).
We are now ready to prove the existence of a critical level below the compactness threshold for the functional (11). We first remark that for $\delta < \delta_1$ one has $\inf_{v\in S} I(v) = \alpha > 0$ for small enough ρ. Let us now evaluate the functional I on the manifold Q:

$$\begin{aligned}I(s\bar v_\epsilon^* + c) &= \frac{s^2}{2}\Big[\int_\Omega |\nabla \bar v_\epsilon^*|^2dx - \delta\int_{\partial\Omega}|\bar v_\epsilon^*|^2 d\sigma\Big] - \delta|\partial\Omega|c^2 - \frac{1}{2^*}\int_\Omega |s\bar v_\epsilon^* + c|^{2^*}dx\\ &= \frac{s^2}{2}\Big[\int_\Omega |\nabla \bar v_\epsilon^*|^2dx - \delta\int_{\partial\Omega}|\bar v_\epsilon^*|^2 d\sigma\Big] - \delta|\partial\Omega|c^2 - \frac{s^{2^*}}{2^*}\int_\Omega |\bar v_\epsilon^*|^{2^*}dx\\ &\qquad -c\int_0^1 dt\int_\Omega |s\bar v_\epsilon^* + tc|^{2^*-2}(s\bar v_\epsilon^* + tc)dx.\end{aligned} \tag{45}$$

By using the inequality $(a+b+c)^{2^*-2} \le K(a^{2^*-2} + b^{2^*-2} + c^{2^*-2})$ we estimate :

$$\begin{aligned}\Big|\int_\Omega \bar v_\epsilon^*|s\bar v_\epsilon^* + tc|^{2^*-2}dx\Big| &\le \int_\Omega v_\epsilon^*|sv_\epsilon^* - sm_\epsilon + tc|^{2^*-2}dx + m_\epsilon\int_\Omega |sv_\epsilon^* - sm_\epsilon + tc|^{2^*-2}dx\\ &\le K\Big\{s^{2^*-2}\Big[\int_\Omega |v_\epsilon^*|^{2^*-1}dx + m_\epsilon\int_\Omega |v_\epsilon^*|^{2^*-2}dx + m_\epsilon^{2^*-2}\int_\Omega v_\epsilon^*\,dx + m_\epsilon^{2^*-1}|\Omega|\Big]\\ &\qquad +(tc)^{2^*-2}\Big[\int_\Omega v_\epsilon^*\,dx + m_\epsilon|\Omega|\Big]\Big\}\end{aligned}$$

Then, by (37), (38) and (40) and recalling that s is bounded in Q, we can estimate the non negative term in the last line of (45) as follows :

$$\Big|c\int_0^1 dt\int_\Omega |s\bar{v}_\epsilon^* + tc|^{2^*-2} s\bar{v}_\epsilon^*\, dx\Big| \le K(\epsilon)\big(|c| + |c|^{2^*-1}\big),$$

where $K(\epsilon) = O(\epsilon^{(n-2)/2})$. Therefore, we can write :

$$I(s\bar{v}_\epsilon^* + c) \le \frac{s^2}{2}\Big[\int_\Omega |\nabla\bar{v}_\epsilon^*|^2 dx - \delta\int_{\partial\Omega} |\bar{v}_\epsilon^*|^2 d\sigma\Big] - \frac{s^{2^*}}{2^*}\int_\Omega |\bar{v}_\epsilon^*|^{2^*} dx - p_\epsilon(|c|), \tag{46}$$

where

$$p_\epsilon(\tau) = \delta|\partial\Omega|\tau^2 - K(\epsilon)(\tau + \tau^{2^*-1}).$$

Since $2^*-1 = \frac{n+2}{n-2} \in (1,3]$ (for $n \ge 4$) we see that (for small enough ϵ) $p_\epsilon(\tau) \ge 0$ for every $\tau \ge \frac{2}{\delta|\partial\Omega|}K(\epsilon)$ if $n \ge 6$ and for τ in the interval $[\frac{2}{\delta|\partial\Omega|}K(\epsilon), R(\epsilon)]$ if $n = 4$ or $n = 5$, where $R(\epsilon) \approx K(\epsilon)^{\frac{n-2}{n-6}}$; note that the latter quantity is $O(1/\epsilon)$ for $n = 4$ and $O((1/\epsilon)^{9/2})$ for $n = 5$. In these two cases, the function p_ϵ takes a maximum value of order $1/\epsilon^2$ and $1/\epsilon^9$ respectively.
By the above discussion, it follows in particular that the term $-p_\epsilon(|c|)$ in the right hand side of (46) is positive of order $\epsilon^{(n-2)}$ for $|c| \le O(\epsilon^{(n-2)/2})$ and assumes arbitrarily large negative values for large $|c|$ (and small enough ϵ if $n = 4$, 5).
We can now verify the assumptions of [27, Theorem 8.4] : by the definition of I we have $I(c) \le 0$ for every c. Moreover, by taking $|c| = R_2$ large enough in (46), one easily get $I(s\bar{v}_\epsilon^* \pm R_2) \le 0$ for all $s \ge 0$. Finally, let R_1 be chosen to satisfy $I(R_1\bar{v}_\epsilon^*) < 0$; then, again by (46) and recalling that the term $-p_\epsilon(|c|)$ is either negative or arbitrarily small for $\epsilon \to 0$, we obtain $I(R_1\bar{v}_\epsilon^* + c) \le 0\ \forall\ |c| \le R_2$. Then, we have proved that

$$\alpha = \inf_{v\in S} I(v) > \sup_{v\in\partial Q} I(v) = 0.$$

Now, by defining

$$\Gamma = \{h \in C^0(H^1, H^1)\,;\ h|_{\partial Q} = I\},$$

it follows that the number

$$\beta = \inf_{h\in\Gamma}\sup_{v\in Q} I(h(v))$$

is a critical value of I, whenever $\beta < S_2^{n/2}/2n$. Since $\beta \le \sup_{u\in Q} I(v) \equiv \beta_0$, it is sufficient to prove that $\beta_0 < S_2^{n/2}/2n$. Actually, by the estimate (43) and by standard arguments we have

$$I(s\bar{v}_\epsilon^* + c) \le \frac{1}{n}\Big[\frac{S_2}{2^{2/n}} - \epsilon k\big(\delta + \gamma\frac{n-2}{2}\big) + R(\epsilon)\Big]^{n/2} - p_\epsilon(|c|),$$

where $k > 0$. As previously remarked, for $|c| \le R_2$ the last term is either negative or $O(\epsilon^{(n-2)})$, so that our claim follows.

5 Proof of Theorem 4

As shown in Lemma 7, the compactness threshold for the corresponding functional J (see (23)) is $2\,S_4^{n/4}/n$. Since (4) does not admit nodal radial solutions (see Theorem 3 (iii)), to go below the compactness threshold one cannot exploit the functions $u_\epsilon(x) := (\epsilon^2 + |x|^2)^{-\frac{n-4}{2}}$ $(\epsilon > 0)$, which attain the constant S_4. Moreover, in view of the first boundary condition ($u = 0$ on ∂B), we cannot bypass this difficulty by concentrating the functions u_ϵ on the boundary as done in the second order case. This makes necessary to introduce a different procedure. For $j \geq 1$, we denote by M_j the eigenspace associated to d_j, where the d_j's are the positive eigenvalues of Δ^2 under Steklov boundary conditions in the ball and we define

$$M_+ := \overline{\bigoplus_{j\geq 2} M_j} \quad \text{and} \quad M_- := M_1 \bigoplus M_2.$$

By Theorem 16 in the Appendix we have

$$M_1 = span\{\phi_1\} \quad \text{and} \quad M_2 = span\{\phi_2^i\}_{1\leq i\leq n},$$

where $\phi_1(x) = (1 - |x|^2)$ and $\phi_2^i = x_i(1 - |x|^2)$, for $i = 1, ..., n$. We set

$$Q(u) := \frac{\|\Delta u\|_2^2}{\|u\|_{2_*}^2}\ , \quad K := \sup_{M_-} Q(u)\ , \tag{47}$$

and we prove

Lemma 8. *If $n = 5, 6, 8$, then $K = Q(\phi_2^1)$.*

Proof. Let $\omega_n := |\partial B|$. First we note that

$$\|\Delta\phi_2^i\|_2^2 = 4\frac{n+2}{n}\omega_n, \quad \|\Delta\phi_1\|_2^2 = 4n\omega_n. \tag{48}$$

Next, let $u \in M_2$ so that $u(x) = \sum_1^n \alpha_i\phi_2^i(x) = (1 - |x|^2)\sum_1^n \alpha_i x_i$, where the α_i are the components of a real vector $\alpha \in \mathbb{R}^n$. We denote by $\{y_i\}_{1\leq i\leq n}$ a complete orthonormal system of coordinates in $\mathbb{R}^n$, obtained by rotating $\{x_i\}_{1\leq i\leq n}$ and such that $y_1 := \frac{1}{|\alpha|}\sum_1^n \alpha_i x_i$. Then, we get

$$Q(u) = \frac{\sum_1^n \alpha_i^2\|\Delta\phi_2^i\|_2^2}{\left(\int_B |\sum_1^n \alpha_i x_i|^{2_*}(1 - |x|^2)^{2_*}\,dx\right)^{2/2_*}} = \frac{4\frac{n+2}{n}\omega_n|\alpha|^2}{\left(\int_B |\alpha|^{2_*}|y_1|^{2_*}(1 - |y|^2)^{2_*}\,dy\right)^{2/2_*}} = Q(\phi_2^1),$$

for all $u \in M_2$. Similarly, one can prove that $\|u + t\phi_1\|_{2_*}^{2_*} = \|\phi_2^1 + t\phi_1\|_{2_*}^{2_*}$, for all $t \geq 0$ and all $u \in M_2$ such that $|\alpha| = 1$. This, combined with (48), shows that it suffices to study the real function

$$f(t) = Q(\phi_2^1 + t\phi_1) = \frac{\|\Delta\phi_2^1\|_2^2 + t^2\|\Delta\phi_1\|_2^2}{\|\phi_2^1 + t\phi_1\|_{2_*}^2}, \quad t \geq 0$$

and prove that

$$\max_{t\geq 0} f(t) = f(0). \tag{49}$$

Let us simplify (49). Writing $x = (x_1, x')$, where $x' \in \mathbb{R}^{n-1}$, and denoting with B_r the ball in $\mathbb{R}^{n-1}$ of radius r and center 0, we deduce:

$$\|\phi_2^1 + t\phi_1\|_{2_*}^{2_*} = \int_B (1 - |x|^2)^{2_*}|x_1 + t|^{2_*}\,dx = \int_{-1}^1 \int_{B_{(1-x_1^2)^{1/2}}} (1 - x_1^2 - |x'|^2)^{2_*}|x_1 + t|^{2_*}\,dx'\,dx_1$$

$$= \omega_{n-1}\left(\int_{-1}^1 |x_1 + t|^{2_*}\int_0^{(1-x_1^2)^{1/2}} (1 - x_1^2 - \rho^2)^{2_*}\rho^{n-2}\,d\rho\,dx_1\right)$$

$$[\rho=(1-x_1^2)^{1/2}r]=\omega_{n-1}\left(\int_{-1}^{1}|x_1+t|^{2_*}(1-x_1^2)^{2_*+(n-1)/2}\,dx_1\right)\left(\int_0^1(1-r^2)^{2_*}r^{n-2}\,dr\right)$$

$$=\frac{\omega_{n-1}}{2}\,\beta\left(\frac{n-1}{2},\frac{3n-4}{n-4}\right)\left(\int_{-1}^{1}|s+t|^{2_*}(1-s^2)^{\frac{n^2-n+4}{2(n-4)}}\,ds\right)=:\frac{\omega_{n-1}}{2}\,\beta\left(\frac{n-1}{2},\frac{3n-4}{n-4}\right)\varphi(t).$$

We have so found that $f(t)=C_nF(t)$, where $C_n=\frac{8\omega_n}{n\left(\omega_{n-1}\beta\left(\frac{n-1}{2},\frac{3n-4}{n-4}\right)\right)^{2/2_*}}$ and

$$F(t)=\frac{n+2+n^2t^2}{(\varphi(t))^{2/2_*}}.$$

The claim (49) becomes

$$\max_{t\geq 0}F(t)=F(0). \tag{50}$$

When $n=5,6,8$, the number 2_* is an even integer so we may expand the term $|s+t|^{2_*}$ and write φ explicitly.

Case $n=5$. Here, $2_*=10$ and

$$\varphi(t)=\int_{-1}^{1}(s+t)^{10}(1-s^2)^{12}\,ds=\sum_{k=0}^{10}\left(\begin{array}{c}10\\k\end{array}\right)t^k\int_{-1}^{1}s^{10-k}(1-s^2)^{12}\,ds$$

$$=\frac{\beta(\frac{1}{2},13)}{29667}\left(1+175t^2+3850t^4+23870t^6+49445t^8+29667t^{10}\right)$$

so that

$$F(t)=C_5\frac{7+25t^2}{(1+175t^2+3850t^4+23870t^6+49445t^8+29667t^{10})^{\frac{1}{5}}},$$

where $C_5:=\left(\frac{29667}{\beta(\frac{1}{2},13)}\right)^{\frac{1}{5}}$. Let now

$$\widetilde{F}(t):=\frac{F(\sqrt{t})}{C_5}=\frac{7+25t}{(1+175t+3850t^2+23870t^3+49445t^4+29667t^5)^{\frac{1}{5}}},$$

so that by direct computations we get

$$\widetilde{F}'(t)=4\frac{9889t^4-9548t^3-10626t^2-1820t-55}{(1+175t+3850t^2+23870t^3+49445t^4+29667t^5)^{\frac{6}{5}}}.$$

Consider the function

$$g(t):=9889t^4-9548t^3-10626t^2-1820t-55,\quad t\geq 0,$$

we have $g'(t)=4\left(9889t^3-7161t^2-5313t-455\right)$ and $g''(t)=132\left(161t^2-434t-899\right)$. Therefore there exists a unique $\overline{t}>0$ such that

$$g''(t)<0\quad\text{if}\quad t<\overline{t},\quad g''(\overline{t})=0,\quad g''(t)>0\quad\text{if}\quad t>\overline{t}.$$

This, together with $g'(0)<0$ and $\lim_{t\to+\infty}g'(t)=+\infty$, shows that g' has a global minimum at $\overline{t}$ and $g'(\overline{t})<0$. Hence, there exists a unique $\sigma>\overline{t}$ such that

$$g'(t)<0\quad\text{if}\quad t<\sigma,\quad g'(\sigma)=0,\quad g'(t)>0\quad\text{if}\quad t>\sigma.$$

Similarly, since $g(0) < 0$ and $\lim_{t\to+\infty} g(t) = +\infty$, we know that g has a global minimum at σ and $g(\sigma) < 0$. This proves that there exists a unique $\tau > \sigma$ such that

$$g(t) < 0 \quad \text{if} \quad t < \tau, \quad g(\tau) = 0, \quad g(t) > 0 \quad \text{if} \quad t > \tau.$$

Finally, this shows that $\widetilde{F}$ has a global minimum at τ, whereas F has a global minimum at $\sqrt{\tau}$. Since $F(0) = 7C_5 > \lim_{t\to+\infty} F(t) = 25C_5(29667)^{-1/5}$, this proves that (50) holds when $n = 5$.
Case $n = 6$. Here $2_* = 6$,

$$\varphi(t) = \int_{-1}^{1} (s+t)^6 (1-s^2)^{\frac{17}{2}}\, ds = \frac{\beta(\frac{1}{2}, \frac{19}{2})}{704} \left(1 + 72t^2 + 528t^4 + 704t^6\right)$$

and

$$F(t) = C_6 \frac{8 + 36t^2}{(1 + 72t^2 + 528t^4 + 704t^6)^{\frac{1}{3}}},$$

where $C_6 := \left(\frac{704}{\beta(\frac{1}{2}, \frac{19}{2})}\right)^{\frac{1}{3}}$. To simplify further, we set

$$\widetilde{F}(t) := \frac{F(\sqrt{t}/2)}{C_6} = \frac{8 + 9t}{(1 + 18t + 33t^2 + 11t^3)^{\frac{1}{3}}}$$

and we compute

$$\widetilde{F}'(t) = \frac{11t^2 - 68t - 39}{(1 + 18t + 33t^2 + 11t^3)^{\frac{4}{3}}}.$$

This shows that F has a global minimum for $t = \bar{t} > 0$ and no local maximum for $t > 0$. Hence, since $F(0) = 8C_6 > \lim_{t\to+\infty} F(t) = 36C_6(704)^{-1/3}$, we conclude that (50) holds when $n = 6$.
Case $n = 8$. Here $2_* = 4$,

$$\varphi(t) = \int_{-1}^{1} (s+t)^4 (1-s^2)^{\frac{15}{2}}\, ds = \frac{\beta(\frac{1}{2}, \frac{17}{2})}{120} \left(1 + 40t^2 + 120t^4\right)$$

and

$$F(t) = C_8 \frac{10 + 64t^2}{(1 + 40t^2 + 120t^4)^{\frac{1}{2}}},$$

where $C_8 := \left(\frac{120}{\beta(\frac{1}{2}, \frac{17}{2})}\right)^{\frac{1}{2}}$. Consider

$$\widetilde{F}(t) =: \frac{F(\sqrt{t/2})}{2C_8} = \frac{5 + 16t}{(1 + 20t + 30t^2)^{\frac{1}{2}}},$$

we have

$$\widetilde{F}'(t) = 2\frac{5t - 17}{(1 + 20t + 30t^2)^{\frac{3}{2}}}.$$

Coming back to the function F, this means that F has a global minimum for $t = \bar{t} > 0$ and no local maximum for $t > 0$. Thus, since $F(0) = 10C_8 > \lim_{t\to+\infty} F(t) = 64C_8(120)^{-1/2}$, we conclude that (50) holds also when $n = 8$.

$\square$

Lemma 9. *Let K be as in (47). If*

$$d > n+2-\frac{n+2}{K}S_4,$$

then

$$\mu := \sup_{u\in M_-} J(u) < \frac{2}{n}S_4^{n/4}.$$

Moreover, there exist $\rho,\eta>0$ such that

$$J(u)\ge\eta,\quad \textit{for all}\quad u\in M_+\oplus H_0^2(B):\quad \|\Delta u\|_2=\rho.$$

Proof. Let $u\in M_-$. Since $d_2=n+2$ (see Theorem 16), we have

$$J(u)=\frac{1}{2}\left(\|\Delta u\|_2^2-d\|u\|_{\partial_\nu}^2\right)-\frac{1}{2_*}\|u\|_{2_*}^{2_*}\le\frac{1}{2}\left(\frac{n+2-d}{n+2}\right)\|\Delta u\|_2^2-\frac{1}{2_*}\|u\|_{2_*}^{2_*}$$

$$\le\frac{1}{2}\left(\frac{n+2-d}{n+2}\right)K\,\|u\|_{2_*}^2-\frac{1}{2_*}\|u\|_{2_*}^{2_*}\le\frac{2}{n}\left(\frac{n+2-d}{n+2}\,K\right)^{\frac{n}{4}},$$

where the last inequality follows from

$$\max_{s\ge0}\left(as-bs^{\frac{n}{n-4}}\right)=\left(\frac{n-4}{n}\right)^{\frac{n-4}{4}}\frac{4}{n}\frac{a^{n/4}}{b^{(n-4)/4}},\quad\text{for all}\quad a,b>0.$$

Therefore,

$$\mu\le\frac{2}{n}\left(\frac{n+2-d}{n+2}\,K\right)^{\frac{n}{4}}. \tag{51}$$

Let now $u\in M_+\oplus H_0^2(B)$ and $\rho=S_4^{\frac{n}{8}}\left(\frac{n+2-d}{n+2}\right)^{\frac{n-4}{8}}$, for $\|\Delta u\|_2=\rho$ we have

$$J(u)\ge\frac{1}{2}\left(\frac{n+2-d}{n+2}\right)\|\Delta u\|_2^2-\frac{1}{2_*S_4^{n/(n-4)}}\|\Delta u\|_2^{2_*}=\frac{2}{n}\left(\frac{n+2-d}{n+2}S_4\right)^{\frac{n}{4}}=:\eta.$$

To conclude we observe that $\mu<\frac{2}{n}S_4^{\frac{n}{4}}$ for $n+2-d<\dfrac{S_4(n+2)}{K}$. □

Lemma 9 allows us to apply a result of Bartolo-Benci-Fortunato [5, Theorem 2.4] from which we deduce that, if $n+2-d<S_4(n+2)/K$, then J admits at least n (the multiplicity of d_2) pairs of critical points at levels below $(2/n)\,S_4^{n/4}$. Set $g(n):=\frac{S_4(n+2)}{K}$ and compute directly (using Lemma 8) to obtain (7).

6 Remarks on Theorem 4 in general dimensions

As already mentioned in Section 2, we do not have a proof of Theorem 4 in general dimensions $n\ge5$. However, we make the following

Conjecture 10. *Assume that $\Omega=B$ (the unit ball of $\mathbb{R}^n$) and let $n\ge5$.*
If $d\in(n+2-g(n),n+2)$ problem (4) admits at least n pairs of nontrivial solutions.

Let us explain the two main reasons why we believe this conjecture to be true. First, we notice that what is missing for the proof of this conjecture is Lemma 8. In turn, this reduces to show that $F(0) \geq F(t)$, for every $t \geq 0$, or that $G(t) \geq 0$, where

$$G(t) := (n+2)^{\frac{n}{n-4}} \varphi(t) - \varphi(0)(n+2+n^2t^2)^{\frac{n}{n-4}} = (n+2)^{\frac{n}{n-4}} \varphi(t) - b(n+2+n^2t^2)^{\frac{n}{n-4}} \tag{52}$$

and $b := \beta\left(\frac{3n-4}{2(n-4)}, \frac{n^2+n-4}{2(n-4)}\right)$.
We can prove this property only locally:

Lemma 11. *For any $n \geq 5$, we have $G(0) = G'(0) = 0$ and $G''(0) > 0$.*

Proof. Consider first the function φ. We have

$$\varphi'(t) = 2_* \int_{-1}^{1} |s+t|^{2_*-2}(s+t)(1-s^2)^a \, ds > 0 \quad \text{for } t > 0 \quad \text{and } \varphi'(0) = 0,$$

$$\varphi''(t) = 2_*(2_*-1) \int_{-1}^{1} |s+t|^{2_*-2}(1-s^2)^a \, ds > 0 \quad \text{for } t \geq 0,$$

where $a := \frac{n^2-n+4}{2(n-4)}$. Thus φ is an increasing and convex function. Since

$$G'(t) = (n+2)^{\frac{n}{n-4}} \varphi'(t) - b\, 2_* n^2 t (n+2+n^2t^2)^{\frac{4}{n-4}},$$

we have $G(0) = G'(0) = 0$. On the other hand,

$$G''(t) = (n+2)^{\frac{n}{n-4}} \varphi''(t) - b\, 2_* n^2 (n+2+n^2t^2)^{\frac{8-n}{n-4}} (n+2+n^2t^2+4n2_*t^2),$$

so that

$$G''(0) = (n+2)^{\frac{n}{n-4}} \varphi''(0) - b\, 2_* n^2 (n+2)^{\frac{4}{n-4}} = \frac{8n^2(n+2)^{\frac{4}{n-4}}(2n+1)}{(n-4)^2}\, b > 0,$$

where in the last step we exploited the property $\beta(p+1,q) = \frac{p}{p+q}\beta(p,q)$ to deduce that

$$\varphi''(0) = 2_*(2_*-1)\, b\left(\frac{n+4}{2(n-4)}, \frac{n^2+n-4}{2(n-4)}\right) = 2_*(2_*-1)\frac{n(n+2)}{n+4}\, b.$$

□

The second argument which brings some evidence to Conjecture 6 are the numerical plots (obtained with Mathematica) of the functions G defined in (52) when $n = 7, 9, 10, ..., 20$. Not only it seems that $G(t) \geq 0$ for all $t \geq 0$ but also that G is increasing and convex.

Remark 12. The above proofs can be extended to get an existence result for d lying in a suitable left neighborhood of any eigenvalue d_k. Of course the computations become very difficult.

7 Appendix: some results about the eigenvalue problems

In this section we collect some facts about the two boundary eigenvalue problems

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u_\nu = \delta u & \text{on } \partial\Omega \end{cases} \tag{53}$$

and

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega \\ u = \Delta u - du_\nu = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{54}$$

Consider first (53); its smallest eigenvalue is $\delta_0 = 0$. This turns (53) into a Neumann problem which is solved by any constant function in Ω. The smallest (positive) nontrivial eigenvalue δ_1 of (53) is characterized variationally by (6).
Consider the space

$$Z_1 = \{v \in C^\infty(\overline{\Omega}) : \Delta u = 0 \text{ in } \Omega\}$$

and denote by V its completion with respect to the norm (10). Then, we have:

Theorem 13. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open bounded domain with smooth boundary. Then:*
– Problem (53) admits infinitely many (countable) eigenvalues.
– The first eigenvalue $\delta_0 = 0$ is simple, it is associated to constant eigenfunctions and eigenfunctions of one sign necessarily correspond to δ_0.
– The set of eigenfunctions forms a complete orthonormal system in V.
– Any eigenfunction e of (53) corresponding to a positive eigenvalue satisfies $\int_{\partial\Omega} e = 0$.
– The space $H^1(\Omega)$ endowed with (9) admits the following orthogonal decomposition

$$H^1(\Omega) = V \oplus H^1_0(\Omega).$$

– If $v \in H^1(\Omega)$ and if $v = v_1 + v_2$ is the corresponding orthogonal decomposition with $v_1 \in V$ and $v_2 \in H^1_0(\Omega)$, then v_1 and v_2 are weak solutions of

$$\begin{cases} \Delta v_1 = 0 & \textit{in } \Omega \\ v_1 = v & \textit{on } \partial\Omega \end{cases} \qquad \textit{and} \qquad \begin{cases} \Delta v_2 = \Delta v & \textit{in } \Omega \\ v_2 = 0 & \textit{on } \partial\Omega\ . \end{cases}$$

Proof. With the scalar product (9) we decompose the space $H^1(\Omega)$ as

$$H^1(\Omega) = H^1_0(\Omega) \oplus H^1_0(\Omega)^\perp.$$

Thus, every $v \in H^1(\Omega)$ may be written in a unique way as $v = v_1 + v_2$, where $v_2 \in H^1_0(\Omega)$ and v_1 satisfies

$$v_1 = v \quad \text{on } \partial\Omega \qquad \text{and} \qquad \int_\Omega \nabla v_1 \nabla v_0 = 0 \quad \text{for all } v_0 \in H^1_0(\Omega).$$

Hence, v_1 weakly solves the problem

$$\begin{cases} \Delta v_1 = 0 & \text{in } \Omega \\ v_1 = v & \text{on } \partial\Omega \end{cases}$$

and $v_2 = v - v_1$ weakly solves

$$\begin{cases} \Delta v_2 = \Delta v & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega\ . \end{cases}$$

The kernel of the trace operator γ of $H^1(\Omega)$ is $H^1_0(\Omega)$ so that γ is an isomorphism between $H^1_0(\Omega)^\perp$ and $H^{1/2}(\partial\Omega)$. Therefore, the embedding $I_1 : H^1_0(\Omega)^\perp \subset L^2(\partial\Omega)$ is compact and $L^2(\partial\Omega)$ can be identified to a subspace of the dual space $(H^1_0(\Omega)^\perp)'$. In view of this, we have

$$H^1_0(\Omega)^\perp \subset L^2(\partial\Omega) \subset (H^1_0(\Omega)^\perp)'.$$

Next, let $I_2 : L^2(\partial\Omega) \to (H^1_0(\Omega)^\perp)'$ be the continuous linear operator such that

$$\langle I_2 u, v\rangle = \int_{\partial\Omega} uv \quad \text{for all } u \in L^2(\partial\Omega),\ v \in H^1_0(\Omega)^\perp$$

and by $L : H^1_0(\Omega)^\perp \to (H^1_0(\Omega)^\perp)'$ the linear operator defined by:

$$\langle Lu, v\rangle = \int_\Omega \nabla u \nabla v + \int_{\partial\Omega} uv \quad \text{for all } u, v \in H^1_0(\Omega)^\perp.$$

Then, L is an isomorphism and the linear operator $K = L^{-1} I_2 I_1 : H^1_0(\Omega)^\perp \to H^1_0(\Omega)^\perp$ is a compact self-adjoint operator with strictly positive eigenvalues, $H^1_0(\Omega)^\perp$ admits an othonormal basis of eigenfunctions of K and the set of eigenvalues of K can be ordered in a strictly decreasing sequence λ_i which converges to zero. Thus, problem (53) admits infinitely many eigenvalues given by $\delta_i = \frac{1}{\lambda_i}$ and the eigenfunctions coincide with the eigenfunctions of K. Hence, $H^1_0(\Omega)^\perp \equiv V$.
By the divergence Theorem, we see that any solution u of (53) with $\delta > 0$ satisfies $\int_{\partial\Omega} u = 0$. To conclude the proof it remains to show that the unique eigenvalue corresponding to a positive eigenfunction is $\delta_0 = 0$. To see this, let $\delta \geq 0$ be an eigenvalue corresponding to a positive eigenfunction $e > 0$ in Ω. By definition, we know that e satisfies

$$\int_\Omega \nabla e \nabla v = \delta \int_{\partial\Omega} ev \quad \text{for all } v \in H^1(\Omega).$$

Choosing $v \equiv 1$ and recalling that $e \in V$, the above identity shows that necessarily $\delta = 0$. □

When $\Omega = B$ (the unit ball) we may determine explicitly all the eigenvalues of (53). To this end, consider the spaces of harmonic polynomials [4, Sect. 9.3-9.4]:

$$\mathcal{D}_k := \{P \in C^\infty(\mathbb{R}^n);\ \Delta P = 0 \text{ in } \mathbb{R}^n,\ P \text{ is an homogeneous polynomial of degree } k\}.$$

Also, denote by μ_k the dimension of $\mathcal{D}_k$ so that [4, p.450]

$$\mu_k = \frac{(2k+n-2)(k+n-3)!}{k!(n-2)!}.$$

Then, from [9, p.160] we easily infer

Theorem 14. [9]
If $n \geq 2$ and $\Omega = B$, then for all $k = 0, 1, 2, \ldots$:
(i) the eigenvalues of (53) are $\delta_k = k$;
(ii) the multiplicity of δ_k equals μ_k;
(iii) any $\psi \in \mathcal{D}_k$ is an eigenfunction corresponding to δ_k.

We now turn to the fourth order problem (54). Let $\mathcal{H}(\Omega) := [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$. The smallest (positive) eigenvalue d_1 of (54) is characterized variationally as

$$d_1 := \inf_{u \in \mathcal{H}(\Omega)} \frac{\|\Delta u\|_2^2}{\|u\|_{\partial_\nu}^2}.$$

Hence, d_1 is the largest constant satisfying

$$\|\Delta u\|_2^2 \geq d_1 \|u\|_{\partial_\nu}^2 \qquad \text{for all } u \in H^2 \cap H^1_0(\Omega)$$

and $d_1^{-1/2}$ is the norm of the compact linear operator $H^2 \cap H^1_0(\Omega) \to L^2(\partial\Omega)$, $u \mapsto u_\nu$.
Consider the space

$$Z_2 = \left\{v \in C^\infty(\overline{\Omega}) : \Delta^2 u = 0,\ u = 0 \text{ on } \partial\Omega\right\}$$

and denote by W its completion with respect to the norm (22). Then, we have

Theorem 15. [16]
Assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is an open bounded domain with smooth boundary. Then:
– Problem (54) admits infinitely many (countable) eigenvalues.
– The first eigenvalue d_1 is simple and eigenfunctions of one sign necessarily correspond to d_1.
– The set of eigenfunctions forms a complete orthonormal system in W.
– The space $H^2 \cap H^1_0(\Omega)$ endowed with (21) admits the following orthogonal decomposition

$$H^2 \cap H^1_0(\Omega) = W \oplus H^2_0(\Omega).$$

– If $v \in H^2 \cap H^1_0(\Omega)$ and if $v = v_1 + v_2$ is the corresponding orthogonal decomposition with $v_1 \in W$ and $v_2 \in H^2_0(\Omega)$, then v_1 and v_2 are weak solutions of

$$\begin{cases} \Delta^2 v_1 = 0 & \text{in } \Omega \\ v_1 = 0 & \text{on } \partial\Omega \\ (v_1)_\nu = v_\nu & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta^2 v_2 = \Delta^2 v & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega \\ (v_2)_\nu = 0 & \text{on } \partial\Omega\ . \end{cases}$$

Again, when $\Omega = B$ (the unit ball) we may determine explicitly all the eigenvalues of (54):

Theorem 16. [16]
If $n \geq 2$ and $\Omega = B$, then for all $k = 1, 2, 3, \ldots$:
(i) the eigenvalues of (54) are $d_k = n + 2(k-1)$;
(ii) the multiplicity of d_k equals μ_{k-1};
(iii) for all $\psi \in \mathcal{D}_{k-1}$, the function $\phi(x) := (1 - |x|^2)\psi(x)$ is an eigenfunction corresponding to d_k.

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