Wiener criterion for relaxed problems related to p-homogeneous Riemannian Dirichlet forms

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Abstract

We state a Wiener criterion for the regularity of points with respect to a relaxed Dirichlet problem for a p-homogeneous Riemannian Dirichlet form.

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1 Introduction

The relaxed Dirichlet problem was introduced in [17] in relation with the Γ-limits of problems relative to a coercive elliptic operator (with bounded measurable coefficients) in open sets with holes and homogeneous Dirichlet condition on the boundaries of the holes. In [17] a notion of regular points is defined; a point is called regular if any local solution of the relaxed Dirichlet problem in a neighborhood of the point takes the value 0 at the point with continuity. In the same paper a Wiener criterion for the regularity of the point is proved using a suitable notion of capacity connected with the positive Borel

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measure appearing in the problem. The result was extended to the framework of Riemannian bilinear Dirichlet forms in [1].
Concerning the nonlinear case we recall that a notion of Kato measure is given in [9] in relation with the subelliptic p-Laplacian and a Wiener criterion for regular points of the corresponding relaxed Dirichlet problem (with a source term, which is Kato measure) was obtained in [10] using for the proof of the necessity part of the result a generalization to the subelliptic framework of an estimate proved by Malỳ in the Euclidean setting, [21], see [9]; for the proof of the sufficient part of the result in the case of a zero source term an adaptation to the subelliptic framework of a method given in [19] in the Euclidean setting is used and for the general case the fundamental tool in the proof is a comparison method founded on local uniform monotonicity properties.
In [11] the notions of p-homogeneous strongly local Dirichlet functionals and forms are introduced and, in [13], the Hölder continuity of harmonic function is proved in the Riemannian case as a consequence of an Harnack inequality for the metric related to the form. Particular p-homogeneous Riemannian Dirichlet forms are related to the subelliptic p-Laplacian (eventually weighted) and to the p-Laplacian in a metric measurable structure, [14][20].
In the present paper we are interested in the Wiener criterion for regular points of a relaxed Dirichlet problem relative to a p-homogeneous Riemannian Dirichlet form (with a source term, which is Kato measure, see [6] for the definition). The interest of relaxed Dirichlet problems is twofold:
(1) From the Wiener criterion for relaxed Dirichlet problems a Wiener criterion for regular point of the boundary follows, see [7] for the direct proof of Wiener criterion for regular point of the boundary. The proof is immediate in the case where the boundary data can have an extension to a function in the domain of the form on all the space; the proof in the general case requires also some approximation methods.
(2) The class of relaxed Dirichlet problems is closed for Γ-convergence and in particular the Γ-limits of Dirichlet problems in open sets with holes and zero Dirichlet condition on the boundary of holes are relaxed Dirichlet problems, see [5] where the result is proved by methods of Γ-convergence, which are a refinement of the methods used in the linear Euclidean case in [18].
In section 2 we introduce the notion of p-homogeneous Riemannian Dirichlet form and the definition of the Kato class of measures relative to the form. In section 3 we give the main result in the paper, i.e. a Wiener criterion for regular points for the relaxed Dirichlet problem. In section 4 we prove some preliminaries results, in section 5 we prove our criterion. We observe that the methods used in section 5 in the proof of the sufficient part of the criterion are essentially different from the ones used in [10] due to the absence of local uniform monotonicity properties for our form; the methods used here are founded on the extension to our general framework of an estimate of [21] (see [6]) and on a finite iteration method of Nash- Moser type (see [19] for the Euclidean framework). For the proof of the necessary part of the criterion we use an adaptation of the proof in [10] for the subelliptic framework.

2 Notations and main result.

2.1 Riemannian p-homogeneous Dirichlet forms

We consider a locally compact separable Hausdorff space X with a metrizable topology and a positive Radon measure m on X such that supp$[m]= X$. We assume that $\Phi(v) = \int_X \alpha(u)(dx)$ is a strongly local, strictly convex p-homogeneous Dirichlet functional, $p > 1$, with domain D_0 and that $\Psi(u,v) = \int_X \mu(u,v)(dx)$ is the related strongly local p-homogeneous Dirichlet form (with domain $D_0 \times D_0$) as defined in [11]. We refer to [11] for the properties of the Radon measures α and μ (in particular the chain rule, the truncation rule, the Leibnitz rule for $\mu(u,v)$ with respect to v, and the Schwartz inequality for $\mu(u,v)$), and to [2] for a Leibnitz type inequality for α. The above notions allow us to define a capacity relative to the functional Φ (and to the measure space(X,m)). The capacity of an open set O is defined as

$$p - cap(O) = inf\{\Phi_1(v);\ v \in D_0,\ v \geq 1 \text{ a.e. on } O\}$$

if the set $\{v \in D_0\ , v \geq 1$ a.e. on $O\}$ is not empty and

$$p - cap(O) = +\infty$$

otherwise. Let E be a subset of X, we define

$$p - cap(E) = inf\{p - cap(O); O \text{ open set with } E \subset O\}.$$

We recall that the above defined capacity is a Choquet capacity [11]. Moreover we can prove that every function in D_0 is defined quasi-everywhere (i.e. up to sets of zero capacity), [11].
We recall that the Radon measures α and μ are assumed to charge no sets of zero capacity.
The strong locality property allows us to define the domain of the form with respect to an open set O, denoted by $D_0[O]$ and the local domain of the form with respect to an open set O, denoted by $D_{loc}[O]$. We recall that, given an open set O in X we can define a Choquet capacity $p - cap(E;O)$ for a set $E \subset \overline{E} \subset O$ with respect to the open set O. Moreover the sets in O of zero capacity are the same for the p-capacities with respect to O and to X. We also observe that using the truncation rule we can prove that $\mu(u,v) = \mu(w,v)$ on the set where $u = w$ (the set is defined up to sets of zero capacity) for every $v \in D_0$.
Assume that the following hold
(i) A distance d could be defined on X, such that $\alpha(d) \leq m$ in the sense of the measures and the metric topology induced by d is equivalent to the original topology of X.

(ii) Denoting by $B(x, r)$ the ball of center x and radius r (for the distance d), for every fixed compact set K there exist positive constants $\nu \geq 1$, c_0 and R_0 such that

$$m(B(x,r)) \leq c_0 m(B(x,s)) \left(\frac{r}{s}\right)^{\nu} \tag{2.1}$$

$\forall\, x \in K$ and for $0 < s < r < r_0$.

We can assume without loss of generality $p < \nu$.

From the properties of d it follows that for any $x \in X$ there exists a function $\phi(.) = \phi(d(x, .))$ such that $\phi \in D_0[B(x, 2r)]$, $0 \leq \phi \leq 1$, $\phi = 1$ on $B(x, r)$ and

$$\alpha(\phi) \leq \frac{2}{r^p} m$$

.

(iii) We assume also that the following scaled *Poincaré inequality* holds: For every fixed compact set K there exist positive constants c_2, R_1 and $k \geq 1$ such that for every $x \in K$ and every $0 < r < R_1$

$$\int_{B(x,r)} |u - av(u, B(x,r))|^p m(dx) \leq c_2 r^p \int_{B(x,kr)} \alpha(u)(dx) \tag{2.2}$$

for every $u \in D_{loc}[B(x, kr)]$, where $av(u, B(x,r)) = \frac{1}{m(B(x,r))} \int_{B(x,r)} u\ m(dx)$ (scaled *Poincaré inequality*).

A strongly local p-homogeneous Dirichlet form, such that the above assumptions hold, is called a *Riemannian Dirichlet form.*

As proved in [22] the Poincaré inequality implies the following *Sobolev inequality*: for every fixed compact set K there exist positive constants c_3, r_2 and $k \geq 1$ such that for every $x \in K$ and every $0 < r < R_2$

$$(av(u^{p^*}, B(x,r)))^{\frac{1}{p^*}} \leq \tag{2.3}$$

$$\leq c_3 \Big(\frac{r^p}{m(B(x,r))} \int_{B(x,kr)} \alpha(u)(dx) + av(|u|^p, B(x,r))\Big)^{\frac{1}{p}}$$

with $p^* = \frac{p\nu}{\nu - p}$ and c_3, R_2 depending only on c_0, c_2, R_0, R_1. We observe that we can assume without loss of generality $R_0 = R_1 = R_2$.

Remark 2.1 From (2.3) we can easily deduce by standard methods that for every fixed compact set K, such that the neighborhood of K of radius R_0 is strictly contained in X, for every $x \in K$ and $0 < 2r < R_0$

$$\int_{B(x,r)} |u|^p m(dx) \leq c_2^{\star} r^p \int_{B(x,kr)} \alpha(u)(dx)$$

for every $u \in D_0[B(x_0, r)]$, where $c_2^{\star}$ depends only on c_2 and c_0.

As a consequence of the assumptions on X and d and of the Poincaré inequality we have the following estimate on the capacity of a ball, [13]:

Proposition 2.1 *For every fixed compact set K there exists positive constants c_4 and c_5 such that*

$$c_4 \frac{m(B(x,r))}{r^p} \leq p - cap(B(x,r), B(x,2r)) \leq c_5 \frac{m(B(x,r))}{r^p}$$

where $x \in K$ and $0 < 2r < R_0$.

2.2 The σ-p-capacity

Let $\int_X \mu(u,v)(dx)$ be the p-homogeneous Riemannian Dirichlet form relative to the Dirichlet functional $\int_X \alpha(u)dx$ and let Ω be a relatively compact open set in X. We denote by $M_0^p(\Omega)$ the set of the nonnegative Borel measures on Ω, which does not charge sets of zero capacity (with respect to the given form).
Let $\sigma \in M_0^p(\Omega)$. We say that a Borel subset E of Ω is σ-admissible if there exists a function $w \in L^p(\Omega, \sigma_E)$ such that $(w-1) \in D_0[\Omega]$, where $\sigma_E = \sigma|_E$ is the restriction of σ to E.
If E is not σ-admissible, then we define $p - cap_\sigma(E, \Omega) = +\infty$.
If E is σ-admissible, then we define

$$p - cap_\sigma(E, \Omega) = \tag{2.4}$$

$$= min \left\{ \int_\Omega \alpha(v)(dx) + \int_\Omega |v|^p \sigma_E(dx) \ | \ (v-1) \in D_0[\Omega] \right\}$$

The function w_E which realizes the minimum in (2.4) is called the σ-potential of E relative to Ω.
We observe that the σ-potential of E relative to Ω is the solution of the problem

$$\int_\Omega \mu(w_E, v)(dx) + \int_\Omega |w_E|^{p-2} w_E v \ \sigma_E(dx) = 0 \tag{2.5}$$

$w_E \in D_0[\Omega] \cap L^p(\Omega, \sigma_E)$, $w_E - 1 \in D_0[\Omega]$, for every $v \in D_0[\Omega] \cap L^p(\Omega, \sigma_E)$.

2.3 The Kato class

The definition of Kato class of measures was initially given by T. Kato in 1972 in the case of Laplacian and extended in [15] to the case of elliptic operators with bounded measurable coefficients. The Kato class relative to the subelliptic Laplacian was defined in [16], and the case of (bilinear) Riemannian Dirichlet forms was considered in [8] and [3].
In [2] the Kato class was defined in the case of the subelliptic p-Laplacian

and in [6] the following definition of Kato class relative to a Riemannian p-homogeneous Dirichlet form has been given:

Definition 2.1 *Let λ be a Radon measure. We say that λ is in the p-Kato space $K_p(X)$ $(p > 1)$ if*

$$lim_{r\to 0}\Lambda(r) = 0$$

where

$$\Lambda(r) = sup_{x\in X} \int_0^{2r} \left(\frac{|\lambda|(B(x,\rho))}{m(B(x,\rho))}\rho^p \right)^{1/(p-1)} \frac{d\rho}{\rho}$$

Let $\Omega \subset X$ be an open set; $K_p(\Omega)$ is defined as the space of Radon measures λ on Ω such that the extension of λ by 0 out of Ω is in $K_p(X)$.

In [6] the properties of the space $K_p(\Omega)$ are investigated. In particular it is proved that if Ω is a relatively compact open set of diameter $\bar{R}$, then

$$||\lambda||_{K_p(\Omega)} = \Lambda(\frac{\bar{R}}{2})^{p-1}$$

is a norm on $K_p(\Omega)$ and that $K_p(\Omega)$ endowed with this norm is a Banach space, [6]. Moreover, [6], $K_p(\Omega)$ is contained in $D'[\Omega]$, where $D'[\Omega]$ denotes the dual of $D_0[\Omega]$, and

$$||\lambda||_{D'[\Omega]} \leq c_4(\lambda(\Omega)\Lambda(\frac{\bar{R}}{2}))^{\frac{p-1}{p}}$$

2.4 The relaxed Dirichlet problem and the related regular points

Let Ω be a relatively compact subset of X, σ a nonnegative measure in $M_0^p(\Omega)$, $g \in C(\Omega) \cap D_{loc}[\Omega]$ and $\lambda \in K_p(\Omega)$.

Definition 2.2 *The function $u \in D_{loc}[\Omega] \cap L^p_{loc}(\Omega, \sigma)$ is a local solution of the relaxed Dirichlet problem relative to μ, Ω, σ, g, λ if $u - g \in L^p_{loc}(\Omega, \sigma)$ and*

$$\int_\Omega \mu(u, v)(dx) + \int_\Omega |u - g|^{p-2}(u - g)v\, \sigma(dx) = \int_\Omega v\ \lambda(dx) \qquad (2.6)$$

for any $v \in D_0[\Omega] \cap L^p(\Omega, \sigma)$ with compact support in Ω. We observe that the condition $u-g \in L^p_{loc}(\Omega, \sigma)$ can be imposed due to the fact that u is q.e defined on every compact subset of Ω, [11].

Definition 2.3 *A point $x_0 \in \Omega$ is a regular point for (2.6) if, for arbitrary g and λ satisfying the conditions in Definition 2.2, every local solution u of (2.6) relative to a neighborhood of x_0 in Ω is continuous at x_0 and $u(x_0) = g(x_0)$.*

Remark 2.2 The regularity of a point x_0 for (2.6) does not depend on Ω, g, λ.

2.5 The main result

We are now in position to state the main result of this paper.

Definition 2.4 *A point x_0 in Ω is called a Wiener point (for the relaxed Dirichlet problem (2.6)) if and only if*

$$\int_0^R \delta(\rho)^{\frac{1}{p-1}} \frac{d\rho}{\rho} = +\infty \tag{2.7}$$

where $\delta(\rho) = \frac{p-cap_\sigma(B(x_0,\rho),B(x_0,2\rho))}{p-cap(B(x_0,\rho),B(x_0,2\rho))}$ *$(\leq 1)$and $B(x_0,R) \subset \Omega$.*

Theorem 2.1 *Let $x_0 \in \Omega$. The point x_0 is regular (for the relaxed Dirichlet problem (2.6)) if and only if it is a Wiener point.*

3 Preliminaries results.

Proposition 3.1 *Let λ be a Radon measure in Ω such that $\lambda \in D'[\Omega]$, and let u be a local solution of (2.6). Then*

$$\int_\Omega \mu((u \mp k)^{\pm}, v)(dx) \leq \int_\Omega v|\lambda|(dx)$$

$\forall v \in D_0[\Omega]$, $v \geq 0$ a.e. in Ω, where $g^{\pm} \leq k$ in Ω.

The proof is similar to the one of Proposition 2.1 in [10] (where the subelliptic case is considered) using the truncation rule for the form, [11].

Definition 3.1 *Let u, $v \in D_{loc}[\Omega]$. We say that $u \leq v$ on $\partial\Omega$ if $(u-v)^+ \in D_0[\Omega]$.*

Definition 3.2 *Let f, $g \in D'[\Omega]$. We say that $f \leq g$ iff $< f - g, v > \leq 0$ $\forall v \in D_0[\Omega]$, $v \geq 0$ a.e. in Ω.*

Proposition 3.2 *Let u be a local weak solution of (2.6) with $g = 0$. If $\lambda \geq 0$ and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ a.e. in Ω.*

The proof is similar to the one of Proposition 2.2 in [10] (where the subelliptic case is considered) using the truncation rule for the form, [11].

Proposition 3.3 *Let u_1 and u_2 be local weak solutions of (2.6) with $g = 0$ relative to the Borel measures σ_1 and σ_2 in $M_0^p(\Omega)$ with $\sigma_1 \leq \sigma_2$ (in Borel measure sense) and to the Radon measures $\lambda_1, \lambda_2 \in D'[\Omega]$ with $0 \leq \lambda_2 \leq \lambda_1$.*

Assume that $0 \le u_2 \le u_1$ *on* $\partial\Omega$ *and that* u_1 *has an extension to a function in* D_0 *. Then* $0 \le u_2 \le u_1$ *a.e. in* Ω.

Proof. By Proposition 3.2 we have $u_1,\, u_2 \ge 0$ a.e. in Ω. Let $v = (u_2 - u_1) \vee 0$. Since $u_2 \le u_1$ on $\partial\Omega$ we have $v \in D_0[\Omega]$. Since $u_2,\, u_1 \ge 0$ q.e. in Ω we have $0 \le v \le u_2$ q.e. in Ω, therefore $v \in L^p_{loc}(\Omega, \sigma_2) \subset L^p_{loc}(\Omega, \sigma_1)$. There exists a sequence of functions $v_h \in D_0[\Omega]$ with compact support in Ω which converges strongly in $D_0[\Omega]$ to v and such that $0 \le v_h \le v$ q.e. in Ω. We can take v_h as test function in the problems (2.6) relative to λ_1 and λ_2. Since $u_2 v_h \ge 0$ a.e. in Ω and $\sigma_1 \le \sigma_2$ we obtain

$$\int_\Omega \left[\mu(u_2, v_h) - \mu(u_1, v_h)\right](dx)$$

$$+ \int_\Omega \left[|u_2|^{p-2} u_2 v_h - |u_1|^{p-2} u_1\right] v_h \sigma_1(dx) \le \int_\Omega v_h \left[\lambda_2 - \lambda_1\right](dx)$$

Since $\left[|u_2|^{p-2} u_2 v_h - |u_1|^{p-2} u_1\right] v_h \ge 0$ and $v_h \left[\lambda_2 - \lambda_1\right] \le 0$ a.e. in Ω, we obtain

$$\int_\Omega \left[\mu(u_2, v_h) - \mu(u_1, v_h)\right](dx) \le 0$$

and the limit $h \to \infty$ gives

$$\int_{\Omega \cap \{u_2 - u_1 > 0\}} \left[\mu(u_2, v) - \mu(u_1, v)\right](dx) = \int \left[\mu(u_1 + v, v) - \mu(u_1, v)\right](dx) \le 0$$

Taking into account the assumption on Φ of strict convexity, and then that Ψ is strictly monotone, we obtain $v = 0$, so $u_2 \le u_1$ a.e. in Ω.

Proposition 3.4 *(Properties of the potential) Let* $E \subseteq \bar{E} \subseteq \Omega$ *be* σ*-admissible and* w_E *be the* σ*-potential of* E *on* Ω*. Then there is a positive measure* $\zeta_E \in D'[\Omega]$ *such that*

$$\int_\Omega \mu(w_E, v)(dx) + \int_\Omega v \zeta_E(dx) = 0$$

$\forall v \in D_0[\Omega]$*. The measure* ζ_E *has support in* $\bar{E}$ *and* $p - cap_\sigma(E, \Omega) = \zeta_E(\Omega)$*.*

The proof is similar to the one of Proposition 2.4 in [11] (where the subelliptic case is considered). We use also the fact that a positive functional in $D'[\Omega]$ is a measure.

4 Proof of Theorem 2.1

Let $x_0 \in \Omega$, we may assume without loss of generality $g(x_0) = 0$. Let u be a local weak solution of (2.6) we may assume without loss of generality

$u \in L^p(\Omega, m)$. Let $r \leq \frac{3R}{4}$, $\overline{B(x_0, 2R)} \subseteq \Omega$, $R \leq R_0$. From Proposition 3.1 the function $u_k = (u-k)^+$, where $k \geq sup_{B(x_0,2R)} g$, is a local weak subsolution of (2.6) in $B(x_0, 2r)$ with $\sigma = 0$, that is it satisfies

$$\int_{B(x_0,2R)} \mu(u_k, \varphi)(dx) \leq \int_{B(x_0,2R)} \varphi |\lambda|(dx) \tag{4.1}$$

$\forall \varphi \in D_0[B(x_0, 2R)]$, $\varphi \geq 0$ a.e. in $B(x_0, 2r)$. Then u_k is locally bounded in $B(x_0, 2R)$ and its supremum on $B(x_0, R)$ depends on R, $||u||_{L^p(\Omega,m)}$. [6]. Let us define $M(r) = sup_{B(x_0,r)} u_k$. Let $\xi(r) \leq 1$ be a positive increasing function such that $\xi(r) \to 0$ when $r \to 0$ and suppose $\xi(r)^{-2}\Lambda(r)$ bounded on $(0, R)$. For example, if $\Lambda(r) \leq \Lambda$, we can choose $\xi(r) = (\frac{\Lambda(r)}{\Lambda})^{\frac{1}{2}}$. Let us observe that we will suppose r so small that $\xi(r) \leq 1$. Let $v = \frac{1}{M - u_k + \xi(r)}$.

Proposition 4.1 *Let $p \in (1, \nu)$ and $\eta \in D_0[B(x_0, \frac{r}{2})] \cap L^\infty(B(x_0, \frac{r}{2}), m)$, $r \leq \frac{3}{48k} R$, with $\alpha(\eta) \leq \frac{c}{r^p} m$ a.e. in Ω, for a positive constant c. Then there exists a constant $C > 0$ dependent on Ω, p, R, $||u||_{L^p(\Omega,m)}$, such that*

$$\frac{r^p}{m(B(x_0,r))}\left[\int_\Omega \alpha(\eta v^{-1})(dx) + \int_\Omega |v^{-1} - (M(r) + \xi(r))|^p \eta^p \sigma(dx)\right] \tag{4.2}$$

$$\leq CM(r)\left\{\left[M(r) - M(\frac{r}{2}) + \xi(r)\right]^{p-1} + \Sigma(r)^{(p-1)}\right\}$$

where $\Sigma(r)^{p-1} := (\xi(r)^{-1}\Lambda(r))^{(p-1)\wedge 1}$

We assume now the Proposition 4.1 and we prove the sufficient part of Theorem 2.1. Let $k = sup_{B(x_0,2r)} g$ and let $\eta = 1$ on $B(x_0, \frac{r}{4})$. Multiplying (4.2) by $(M(r) + \xi(r))^{-1}$, we obtain

$$(M(r) + \xi(r))^{p-1} \frac{r^p}{m(B(x_0,r))}\left[\int_\Omega \alpha(\eta \widetilde{v}^{-1})(dx) + \int_\Omega |\widetilde{v}^{-1} - 1|^p \eta^p \sigma(dx)\right] \tag{4.3}$$

$$\leq C\left[\left(M(r) - M(\frac{r}{2}) + \xi(r)\right)^{p-1} + \Sigma(r)^{(p-1)}\right]$$

where $\widetilde{v} = \frac{v}{(M(r)+\xi(r))}$. From the definition of $p - cap_\sigma$ and we obtain

$$(M(r) + \xi(r))\left[\frac{p - cap_\sigma(B(x_0, \frac{r}{4}, B(x_0, \frac{r}{2}))}{p - cap(B(x_0, \frac{r}{4}), B(x_0, \frac{r}{2}))}\right]^{\frac{1}{p-1}} \leq$$

$$\leq C\left[M(r) - M(\frac{r}{2}) + \xi(r) + \Sigma(r)\right]$$

where here and in the following C denotes a possibly different constants dependent on Ω, p, R, $||u||_{L^p(\Omega,m)}$. Here we assume $C \geq 1$. The above inequality gives

$$M(\frac{r}{2}) \leq \left[1 - C^{-1}\delta\left(\frac{r}{2}\right)^{\frac{1}{p-1}}\right] M(r) + 2\xi(r) + \Sigma(r)$$

where $\delta(r) = \frac{p-cap_\sigma(B(x_0,\frac{r}{2}),B(x_0,r))}{p-cap(B(x_0,\frac{r}{2}),B(x_0,r))}$. It follows

$$sup_{B(x_0,\frac{r}{2})} u^+ \leq \left[1 - C^{-1}\delta\left(\frac{r}{2}\right)^{\frac{1}{p-1}}\right] sup_{B(x_0,r)} u^+ + \Sigma_1(r)$$

where $\Sigma_1(r) = 2sup_{B(x_0,2R)} g + 2\xi(r) + \Sigma(r)$. Taking into account that $-u$ is a local solution of (2.6) relative to $-g$, $-\lambda$, we obtain an analogous inequality for u^-. Then

$$sup_{B(x_0,\frac{r}{2})} |u| \leq \left[1 - C^{-1}\delta\left(\frac{r}{2}\right)^{\frac{1}{p-1}}\right] sup_{B(x_0,r)} |u| + \Sigma_1(r) \qquad (4.4)$$

where $r \leq \frac{3R}{48k}$ and $\overline{B(x_0,2R)} \subseteq \Omega$. From (4.4) by iteration, see [23], we obtain

$$sup_{B(x_0,s)}|u| \leq$$

$$\leq C_1 exp\left[-C_2 \int_s^r \delta(\rho)^{\frac{1}{p-1}} \frac{d\rho}{\rho}\right] sup_{B(x_0,r)}|u| + 2osc_{B(x_0,2R)} g + 2\xi(r) + \Sigma(r)$$

where $0 < s < \frac{r}{2} < r < \frac{3R}{48k}$ and $\overline{B(x_0,2R)} \subseteq \Omega$. The result follows.

We prove now the sufficient part of Proposition 4.1.
The first step is to prove that suitable powers of v are in the A_2 Muckenhoupt (with respect to the form). Let $\eta \in D_0[B(x_0,r)] \cap L^\infty(B(x_0,r),m)$ with $\eta = 1$ in $B(x_0,\frac{3}{4}r)$ and $\alpha(\eta) \leq cr^{-p}m$ for a positive constant c, where $r \leq R$. If $w = v^{-1}$, we have that w is a supersolution of (2.6) relative to $\sigma = 0$ and $-\lambda$. Then

$$\int_{B(x_0,r)} \eta^p \alpha(lgw)(dx) = \int_{B(x_0,r)} \left(\frac{1}{w}\right)^p \eta^p \alpha(u_k)(dx)$$

$$= \frac{p}{1-p} \int_{B(x_0,r)} \mu(w, \eta^p \left(\frac{1}{w}\right)^{p-1})(dx) - \frac{p^2}{1-p} \int_{B(x_0,r)} \left(\frac{\eta}{w}\right)^{p-1} \mu(w,\eta)(dx)$$

$$\leq \frac{p^2}{p-1} \int_{B(x_0,r)} \eta^p \left(\frac{1}{w}\right)^{p-1} |\lambda|(dx) + \frac{1}{2} \int_{B(x_0,r)} \left(\frac{1}{w}\right)^p \eta^p \alpha(w)(dx)$$

$$+ C_1(p) \int_{B(x_0,r)} \alpha(\eta)(dx)$$

As $\xi(r)^{-1}\Lambda(r)$ is bounded, then it follows

$$\int_{B(x_0,\frac{3}{4}r)} \alpha(lg(w))(dx) \leq C_2(p) \left[\frac{|\lambda|(B(x_0,r))}{\xi(r)^{(p-1)}} + \frac{m(B(x_0,r))}{r^p}\right]$$

$$\leq C_3(p)\left[\left(\xi(r)^{-1}\Lambda(r)\right)^{p-1}+1\right]\frac{m(B(x_0,r))}{r^p}\leq C_4(p)\frac{m(B(x_0,r))}{r^p}$$

Taking into account that $\alpha(lg(v))=\alpha(lg(w))$ we have

$$\int_{B(x_0,\frac{3r}{4})}\alpha(lg(v))(dx)\leq C_4(p)\frac{m(B(x_0,r))}{r^p} \tag{4.5}$$

From (4.5) we obtain as in [13] that there are constants C and σ_0 such that for $|\sigma|\leq\sigma_0$, and $0<r<\frac{3}{48k}R$

$$av(v^{\sigma},B(x_0,r))av(v^{-\sigma},B(x_0,r))\leq C_5 \tag{4.6}$$

As a second step we prove a weak Harnack inequality for v.
For any $\varphi\in D_0[B(x_0,r)]$, $\varphi\geq 0$ a.e. in $B(x_0,r)$ we have

$$\int_{B(x_0,r)}\mu(v,\varphi)(dx)=\int_{B(x_0,r)}v^{2(p-1)}\mu(u_k,\varphi)(dx)$$

$$\leq\frac{1}{\xi(r)^{2(p-1)}}\int_{B(x_0,2r)}\varphi|\lambda|(dx)$$

Then v is a subsolution of (2.6) with $\sigma=0$ in $B(x_0,r)$ for the measure $\frac{|\lambda|}{\xi(r)^{2(p-1)}}$.
From [6] we obtain

$$sup_{B(x_0,r/2)}v\leq C_6\left[\left(\frac{1}{m(B(x_0,\frac{3r}{4}))}\int_{B(x_0,\frac{3r}{4})}v^q m(dx)\right)^{\frac{1}{q}}+C\xi(r)^{-2}\Lambda(r)\right]$$

for any $q>0$, and then using (4.6) we obtain for $r\leq\frac{R}{12k}$ and we can

$$\frac{1}{m(B(x_0,3r/4))}\int_{B(x_0,3r/4)}v^{-q}m(dx)\leq C_7\left[M(r)-M(\frac{r}{2})+\xi(r)\right]^q \tag{4.7}$$

where $0<q\leq\sigma_0$. We observe that the constant C_7 depends on R, $||\lambda||_{K_p(\Omega)}$, $sup_{\{0\leq r\leq R\}}\xi(r)^{-2}\Lambda(r)$ and on $||u||_{L^p(\Omega,m)}$.
Now we want to extend (4.7) to an exponent q greater than σ_0. Let $\tau<0$ such that $p(\tau+1)>1$. Let $\beta=\tau p+p-1$. Let us observe that β is positive. Let $\varphi=\eta^p\psi\geq 0$ where $\eta\in D_0[B(x_0,r)]\cap L^{\infty}(B(x_0,r),m)$, $\eta\geq 0$, $\alpha(\eta)$ has a bounded density with respect to m and $\psi=\left(v^{\beta}-\left(\frac{1}{(M(r)+\xi(r))}\right)^{\beta}\right)$. Let us observe that $\psi\geq 0$, since β is positive. Recalling that u_k is a subsolution of the problem (2.6) with $\sigma=0$ and using φ as test function, we obtain

$$\beta\int_{B(x_0,r)}\eta^p v^{\beta+1}\alpha(u_k)(dx)\leq p^2|\int_{B(x_0,r)}\eta^{p-1}\psi\mu(u_k,\eta)(dx)|+p\int_{B(x_0,r)}\varphi|\lambda|(dx)$$

Since $\psi \leq v^\beta$, using the Young's inequality we have

$$|\int_{B(x_0,r)} \eta^{p-1}\psi\mu(u_k,\eta)(dx)| \leq \tag{4.8}$$

$$\leq \theta^{\frac{p}{p-1}}\frac{p-1}{p}\int_{B(x_0,r)} \eta^p v^{\beta+1}\mu(u_k,u_k)(dx) + \theta^{-p}\frac{1}{p}\int_{B(x_0,r)} v^{\beta-p+1}\alpha(\eta)(dx)$$

We have $\xi(r)v \leq 1$ and then from (M. Biroli & S. Marchi , 2006, Theorem 3.1) we have

$$\int_{B(x_0,r)} \varphi|\lambda|(dx) \leq \int_{B(x_0,r)} v^\beta\eta|\lambda|(dx) \leq \tag{4.9}$$

$$\leq \xi(r)^{-\beta+\tau}\int_{B(x_0,r)} v^\tau\eta|\lambda|(dx) \leq \xi(r)^{-\beta+\tau}||\eta v^\tau||_{D_0[B(x_0,r)]}||\lambda||_{D'[B(x_0,r)]} \leq$$

$$\leq \xi(r)^{-(p-1)(\tau+1)}\left[|\lambda|(B(x_0,r))\Lambda(r)\right]^{\frac{p-1}{p}}||\eta v^\tau||_{D_0[B(x_0,r)]} \leq$$

$$\leq \theta^{-p}\frac{1}{p}\overline{\Sigma}(r)\frac{m(B(x_0,r))}{r^p} + \theta^{\frac{p}{p-1}}\frac{p-1}{p}||\eta v^\tau||^p_{D_0[B(x_0,r)]}$$

where $\overline{\Sigma}(r) = \xi(r)^{-p}\Lambda(r)^p$. Choosing suitable values for θ in (4.8) and (4.9) we have

$$\frac{r^p}{m(B(x_0,r))}\int_{B(x_0,r)} \alpha(\eta v^\tau)(dx) \leq \tag{4.10}$$

$$\leq K(\tau)\left[\frac{1}{m(B(x_0,r))}\int_{B(x_0,r)} v^{p\tau}\alpha(\eta)(dx) + \overline{\Sigma}(r)\right]$$

where $K(\tau) \simeq \beta^{-p}$ is an decreasing function of τ.
Let us choose $\eta \in D_0[B(x_0,tr)] \cap L^\infty(B(x_0,tr),m)$, $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x_0,sr)$, $\alpha(\eta) \leq \frac{C}{r^p(t-s)^p}m$, where $0 < s < t \leq 1$. Using the Sobolev inequality in (4.10) we obtain

$$(av(v^{\gamma p\tau}, B(x_0,sr)))^{\frac{1}{\gamma}} \leq CK(\tau)\left[\frac{1}{(t-s)^p}av(v^{p\tau}, B(x_0,tr)) + \overline{\Sigma}(r)\right] \tag{4.11}$$

where $\frac{1-p}{p} < \tau < 0$, $\gamma = \frac{\nu}{\nu-p}$.
Our aim is now to iterate inequality (4.11) a finite number of times.
Let $0 < \overline{\sigma} < (p-1)$ and $\sigma_1 = \overline{\sigma}\gamma^{-n} \leq \sigma_0$ where n is a positive integer such that $(p-1) < \sigma_0\gamma^n$. Let us observe that the choice of $\tau = -\sigma_1\gamma^j p^{-1}$ satisfies $\frac{1-p}{p} < \tau < 0$, $0 \leq j \leq n$. Moreover $K(-\sigma_1\gamma^j p^{-1}) \leq K(-\overline{\sigma}p^{-1})$, $0 \leq j \leq n$. Let $r_j = \frac{r}{4}\left[3 - \frac{j}{n+1}\right]$ for $0 \leq j \leq n+1$. Iterating (4.11)for n times with the choices $p\tau = -\sigma_1\gamma^j$, $0 \leq j \leq n$, we obtain

$$(av(v^{-\sigma_1\gamma^{n+1}}, B(x_0,r/2)))^{\frac{1}{\gamma^{n+1}}} \leq \tag{4.12}$$

$$\leq C_8 \left[K(-\overline{\sigma} p^{-1}) \frac{4(n+1)^p}{3} \right]^{\frac{\gamma}{\gamma-1}} \left[av(v^{-\sigma_1}, B(x_0, \frac{3r}{4})) + (n+1) \overline{\Sigma}(r)^{\frac{1}{\gamma^{n+1}}} \right]$$

Then, since $0 < \sigma_1 = \overline{\sigma}\gamma^{-n} \leq \sigma_0$, by (4.7) we obtain

$$av(v^{-\overline{\sigma}\gamma}, B(x_0, r/2) \leq C_9(\overline{\sigma}) \left[\left(M(r) - M(\frac{r}{2}) + \xi(r) \right)^{\overline{\sigma}\gamma} + \overline{\Sigma}(r) \right] \tag{4.13}$$

where $C_9(\overline{\sigma})$ is a finite valued increasing function of $\overline{\sigma}$ for any $0 < \overline{\sigma} < p-1$. Using (4.10) and (4.13) we are finally able to conclude the proof of Proposition 4.1. Let now τ satisfy $\frac{1-p}{p} < \tau < (\frac{\gamma}{p} - 1) \wedge 0$, then. Let $\eta \in D_0[B(x_0, \frac{r}{2})] \cap L^\infty(B(x_0, \frac{r}{2}), m)$ with $\alpha(\eta) \leq \frac{c}{r^p}$ for a positive constant c and choose as test function in (2.6) the function $\varphi = \eta^p u_k$. We have

$$\int_{B(x_0, \frac{r}{2})} \eta^p \mu(u_k, u_k)(dx) + p \int_{B(x_0, \frac{r}{2})} u_k \eta^{p-1} \mu(u_k, \eta)(dx) +$$

$$+ \int_{B(x_0, \frac{r}{2})} \eta^p u_k^p \sigma(dx) \leq M(r) \int_{B(x_0, \frac{r}{2})} \eta^p |\lambda|(dx)$$

Let us observe that

$$\frac{1}{m(B(x_0, \frac{r}{2}))} \int_{B(x_0, \frac{r}{2})} u_k \eta^{p-1} |\mu(u_k, \eta)| =$$

$$= \frac{|\tau|^{(p-1)}}{m(B(x_0, \frac{r}{2}))} \int_{B(x_0, \frac{r}{2})} u_k \eta^{p-1} v^{-(\tau+1)(p-1)} |\mu(v^\tau, \eta)| \leq$$

$$\leq C_{10} M(r) \left(\frac{1}{m(B(x_0, \frac{r}{2}))} \int_{B(x_0, \frac{r}{2})} \eta^p \alpha(v^\tau)(dx) \right)^{\frac{p-1}{p}} \times$$

$$\times \left(\frac{1}{m(B(x_0, \frac{r}{2}))} \int_{B(x_0, \frac{r}{2})} v^{-(\tau+1)(p-1)p} \alpha(\eta)(dx) \right)^{\frac{1}{p}}$$

$$\leq C_{11} M(r) r^{-p} \left[\left(M(r) - M(\frac{r}{2}) + \xi(r) \right)^{-\tau p} + \overline{\Sigma}(r) \right]^{\frac{p-1}{p}} \times$$

$$\times \left[\left(M(r) - M(\frac{r}{2}) + \xi(r) \right)^{(\tau+1)(p-1)p} + \overline{\Sigma}(r) \right]^{\frac{1}{p}}$$

Then we obtain

$$\int_{B(x_0, \frac{r}{2})} \eta^p \alpha(u_k)(dx) + \int_{B(x_0, \frac{r}{2})} \eta^p |M(r) + \xi(r) - v^{-1}|^p \sigma(dx) \leq \tag{4.14}$$

$$\leq C_{12}M(r)\left[\left(M(r)-M(\frac{r}{2})+\xi(r)\right)^{-\tau p}+\overline{\Sigma}(r)\right]^{\frac{p-1}{p}}\times$$

$$\times\left[\left(M(r)-M(\frac{r}{2})+\xi(r)\right)^{(\tau+1)(p-1)p}+\overline{\Sigma}(r)\right]^{\frac{1}{p}}r^{-p}m(B(x_0,r))$$

$$+C_{12}M(r)|\lambda|(B(x_0,r))$$

We have taken into account that $\frac{(\tau+1)(p-1)p}{\gamma}<p-1$. Hence from (4.14) we obtain

$$\int_{B(x_0,\frac{r}{2})}\alpha(\eta v^{-1})(dx)+\int_{B(x_0,\frac{r}{2})}\eta^p|M(r)+\xi(r)-v^{-1}|^p\sigma(dx)$$

$$\leq CM(r)\left[\left(M(r)-M(\frac{r}{2})+\xi(r)\right)^{p-1}+(\xi(r)^{-1}\Lambda(r))^{(p-1)\wedge 1}\right]r^{-p}m(B(x_0,r))$$

where the constant C depends on on Ω, p, R, $||u||_{L^p(\Omega,m)}$.

The necessary part of Theorem 2.1 can be proved by the same methods of [10] using a proof by contradiction. We can prove that if x_0 is a regular point, which is not a Wiener point there exists a suitable ball $B(x_0,R)$ such that the σ-potential of $B(x_0,R)$ in $B(x_0,2R)$ has a value in x_0 greater than $\frac{3}{4}$, then we have a contradiction. We observe also that a result similar to Lemma 4.1 in [10] can be proved by methods similar to the ones in Proposition 3.3.

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