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About a new approach to the characterization of D-stability

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Dedicated to Russell A. Johnson on the occasion of his 70th birthday

Abstract

The concept of $D-$stability is relevant for stable square matrices of any order, especially when they appear in ordinary differential systems modelling physical problems. Indeed, $D-$stability was treated from different points of view in the last fifty years, but the problem of characterization of a general D-stable matrix was solved for low order matrices only (i.e. up to order 4). Here a new approach is proposed within the context of numerical linear algebra. Starting from a known necessary and sufficient condition, other simpler equivalent necessary and sufficient conditions for $D-$stability are proved and yield to a computational method which reveals easy and efficient, so that matrices of order greater than 4 can be characterized.

1 Introduction

Recall that a $n \times n$ real matrix A is called D- (diagonally-) stable if and only if for every diagonal matrix $D = diag(d_1, ..., d_n)$ with positive diagonal entries, all the eigenvalues of the matrix DA lie in the left-half plane.

In the last fifty years, D-stable matrices were involved in many applications, mainly in economics, but also in control systems theory, neural networks, large scale systems, mathematical ecology, etc. For example, this notion arises naturally in problems exhibiting different time scales. In fact, consider a problem of the form

$$\begin{aligned} \varepsilon_1 x_1' &= f_1(x_1, ..., x_n) \\ \varepsilon_2 x_2' &= f_2(x_1, ..., x_n) \\ &\vdots \\ \varepsilon_n x_n' &= f_n(x_1, ..., x_n) \end{aligned}$$

where $f_i(0, ..0) = 0, i = 1, ..., n$. Let A be the $n \times n$ matrix obtained by linearizing this differential system at the origin $\underline{0} \in \mathbb{R}^n$. Then $\underline{0}$ is a linearly stable equilibrium for all positive values of parameters $\varepsilon_1, ..., \varepsilon_n$ if and if A is D-stable.

A long list of references about this topic can be found in Giorgi and Zuccotti (2015).

Here our purpose is to present a new approach to the characterization of a D- (or diagonal-) stable matrix, which is a long lasting problem. Indeed, this problem was theoretically solved for matrices of orders from 2 to 4 (even if general full 4×4 matrices were in practice not tractable), whereas some partial results concerning sufficient or necessary conditions are available for matrices of larger orders or for particular classes of stable matrices, i.e. P-matrices. Among others,, we recall results by Cain (1976), Johnson and Tesi (1999), Impram et al. (2005), Kanovei and Logofet (2001), Kushel (2016) and references therein. We remark that for general matrices of orders larger than 4, all the proposed methods are unfeasible.

Our new approach exploits the concept of Schur-complement applied to results presented by Johnson and Tesi (1999); this allows to prove general theoretical results to characterize a D-stable matrix. Consequently, a practical method is implemented by a symbolic algorithm, which remarkably does not use the Schur-complement, but characteristic polynomials and their roots only. An analogous approach was introduced in Pavani (2013), but here new theoretical results enforce the method so that in practice it reveals capable to characterize general full matrices of order 4 in a quite easy way; extension to matrices of larger order is straightforward, but depends on the power of the used symbolic software. Here the largest order treated in our examples is 5.

The paper is organized as follows. Section 2 presents main theoretical results. Section 3 presents briefly the computing method. Section 4 is concerned with some significant numerical examples. Section 5 contains some concluding remarks.

2 Main results

In Johnson and Tesi (1999) it was proved that a $n \times n$ matrix A is D-stable *if and only if* A *is stable and*

$$\det \begin{pmatrix} A & -D \\ D & A \end{pmatrix} \neq 0 \tag{1}$$

for all positive diagonal D. Then, in Pavani (2013) a related corollary was shown which we report here for reader's convenience. We recall that matrix A is assumed **real**.

COROLLARY 1 *A $n \times n$ matrix A is D-stable if and only if A is stable and, for all positive diagonal D,*

$$\det \left(AD^{-1} + DA^{-1} \right) \neq 0 \tag{2}$$

Proof. We call M the non-singular partitioned matrix in (1)

$$M = \begin{pmatrix} A & -D \\ D & A \end{pmatrix} \tag{3}$$

Then we call M/A the Schur complement of M with respect to A (see e.g. Trefethen and Bau (1997)); for this particular matrix, M/A is given by

$$M/A = A - DA^{-1}(-D) \tag{4}$$

It is clear that

$$\begin{aligned} A - DA^{-1}(-D) = A + DA^{-1}D = AD^{-1}D + DA^{-1}D = \\ = (AD^{-1} + DA^{-1})D. \end{aligned}$$

Therefore $\det(M/A) = \det(AD^{-1} + DA^{-1})\det(D)$.
Since it is known that $\det(M) = \det(A)\det(M/A)$, then we have

$$\det(M) = \det(A)\det(AD^{-1} + DA^{-1})\det(D)$$

From this, it derives that $\det(M) \neq 0$ if and only if $\det(AD^{-1} + DA^{-1}) \neq 0$, since A and D are non-singular as well as AD^{-1} and DA^{-1}. ■

These results allow to simplify the computation of $\det(M)$ in order to check whether it is different from 0. Actually, instead of a matrix of order $2n$, such as M is, we have to compute the determinant of a matrix of order n, given by $(AD^{-1} + DA^{-1})$.

The main feature of our new approach is changing the problem from the symbolic computation of $\det(AD^{-1} + DA^{-1})$ to the symbolic computation of eigenvalues of $(AD^{-1} + DA^{-1})$. From the point of view of symbolic software, when we compute $\det(AD^{-1} + DA^{-1})$ we obtain a function of n variables which can be hardly treated; instead, if we compute the symbolic characteristic polynomial of $\left(AD^{-1} + DA^{-1}\right)$ we obtain a polynomial of degree n in one variable, say x, with coefficients depending on n parameters. This function of x can be studied in a much easier way and this explains why here we prefer this approach. Indeed, we exploit the propriety of the determinant of a matrix to be the product of its eigenvalues; consequently, $\det(AD^{-1} + DA^{-1}) = 0$ if and only if at least one of its eigenvalues is equal to 0. This happens only when AD^{-1} exhibits the imaginary unit as a root of its characteristic polynomial. About this event, we notice that $AD^{-1} = (DA^{-1})^{-1}$ and consequently matrices AD^{-1} and DA^{-1} commute and share the same set of eigenvectors. Resorting to these facts, we can prove the following results.

COROLLARY 2. *A stable matrix A is $D-$stable if and only if the characteristic polynomial of AD^{-1} is **not** divisible by $x^2 + 1$, for all positive diagonal D.*

Proof.

Since AD^{-1} and DA^{-1} commute, they are diagonalized by the same non-singular matrix, say S, according to the following relations

$$S^{-1}AD^{-1}S = D_1, \qquad S^{-1}DA^{-1}S = D_2$$

where D_1 and D_2 are the diagonal matrices of eigenvalues of matrix AD^{-1} and DA^{-1}, respectively.
Then we have

$$\begin{aligned} \det(AD^{-1} + DA^{-1}) &= \det(SD_1S^{-1} + SD_2S^{-1}) = \\ &= \det(S)\det(D_1 + D_2)\det(S^{-1}) \end{aligned}$$

Therefore $\det(AD^{-1} + DA^{-1}) \neq 0 \Leftrightarrow \det(D_1 + D_2) \neq 0$.
Call d_{1i}, $i = 1, ..., n$ and d_{2i}, $i = 1, ..., n$ the diagonal elements of D_1 and D_2, which are the eigenvalues of AD^{-1} and DA^{-1}, respectively; they are connected by the relation $d_{2i} = 1/d_{1i}, i = 1, ..., n$. Therefore $\det(D_1+D_2) = \Pi_i(d_{1i}+1/d_{1i})$.

Since $\Pi_i(d_{1i} + 1/d_{1i})$ is equal to 0 if and only if at least one of the factors $(d_{1i} + 1/d_{1i})$ is equal to zero, this means that $\det(AD^{-1} + DA^{-1}) = 0$ if and only if, at least for one i, it happens $d_{1i}^2 + 1 = 0$; from this, it derives that for at least one i, d_{1i} must be equal to the imaginary unit.
This means that $\det(AD^{-1} + DA^{-1}) = 0$ if and only if the characteristic polynomial of AD^{-1} has at least one factor equal to (x^2+1), i.e. it is divisible by i and $-i$ (since complex roots are always present in couple of conjugate complex numbers).
Consequently, we have $\det(D_1 + D_2) \neq 0$ if and only if AD^{-1} has **no** eigenvalue equal to the imaginary unit, or equivalently AD^{-1}has the characteristic polynomial **not** divisible by $(x^2 + 1)$. ■

REMARK 1. It is worth noticing that the characteristic polynomial of matrix AD^{-1} cannot have any real root equal to zero because it is nonsingular, instead matrix $(AD^{-1}+DA^{-1})$ can be singular, as well as matrix M; therefore this happens if and only if matrix AD^{-1} has at least two conjugate eigenvalues equal to the imaginary unit and, due to its structure, matrix $(AD^{-1} + DA^{-1})$ has two corresponding real eigenvalues equal to 0 .

Summing up our results, the following scheme can be helpful (where all the conditions are equivalent).

$$\begin{aligned} & \text{Stable matrix } A \text{ is } D-\text{stable} \\ \Leftrightarrow\ & \det(M) \neq 0 \text{ for all positive diagonal } D && (5) \\ \Leftrightarrow\ & \det(AD^{-1} + DA^{-1}) \neq 0, \text{ for all positive diagonal } D && (6) \\ \Leftrightarrow\ & (AD^{-1} + DA^{-1}) \text{ has no null eigenvalue, for all positive diagonal } D && (7) \\ \Leftrightarrow\ & AD^{-1}(\text{or } DA^{-1}) \text{ has no eigenvalue equal to imaginary unit,} \\ & \text{for all positive diagonal } D && (8) \end{aligned}$$

3 Numerical-symbolic method

From the results presented in the previous Section, we can build the following method which we will use to carry out the examples reported in Section 4. We point out that this method requires any symbolic program as long as it is powerful enough; we used both Mathematica® and MuPad®. Here are steps which have to be followed for correct computations:

- Build numerical matrix A and symbolic matrix D
- Build matrix AD^{-1}
- Compute the symbolic characteristic polynomial of matrix $(AD^{-1}+DA^{-1})$ which turns out to be $x^n + K_{n-1}x^{n-1} + ... + K_1 x + K_0$ and check its coefficients; if it is easy to recognize that K_0 cannot annihilate, stop since matrix A is $D-$ stable
- Otherwise, compute the symbolic characteristic polynomial of matrix AD^{-1} and its reminder for division by (x^2+1)
- Evaluate if there exist positive numbers which make the reminder equal to 0.
- If and only if they do not exist, then matrix A is $D-$ stable.

The practical implementation of this method by any symbolic software is worth investigating; therefore we report and discuss some detailed examples in the following Section.

4 Numerical Examples

4.1 4×4 matrices

A) At first we consider the stable matrix $A = \begin{bmatrix} -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$.

This matrix is known to be $D-$ stable by Johnson and Tesi (1999). Assume parameters $\alpha, \beta, \gamma, \delta > 0$ and $D = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{bmatrix}$.

Then we have

$$(AD^{-1} + DA^{-1}) = \begin{bmatrix} -\alpha & 2\alpha & -2\alpha & 0 \\ \beta & -3\beta & 2\beta & 0 \\ 0 & \gamma & -\gamma & 0 \\ 0 & 0 & \delta & -\delta \end{bmatrix},$$

Its characteristic polynomial is given by

$$x^4 + (\alpha + 3\beta + \gamma + \delta)\, x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + 3\beta\delta + \gamma\delta)\, x^2 \\ + (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)\, x + \alpha\beta\gamma\delta$$

Using a symbolic software, the computation of the eigenvalues of $(AD^{-1} + DA^{-1})$ (as well as the roots of its characteristic polynomial, by a different function) cannot be managed. However, it is easy to check that the last coefficient $K_0 = \alpha\beta\gamma\delta$ is always positive and this means that no null root appears, since the last coefficient is given by the product of all the roots of the polynomial; hence the given matrix $(AD^{-1} + DA^{-1})$ has no zero eigenvalue and is nonsingular. Consequently, for condition (7) matrix A is D-stable.

B) Then we consider the following full matrix (negative Hilbert matrix of order 4), which is negative definite, and consequently stable and $D-$stable, and we confirm this known result by our approach

$$A = \begin{bmatrix} -1.0 & -0.5 & -0.333\,33 & -0.25 \\ -0.5 & -0.333\,33 & -0.25 & -0.2 \\ -0.333\,33 & -0.25 & -0.2 & -0.166\,67 \\ -0.25 & -0.2 & -0.166\,67 & -0.142\,86 \end{bmatrix}$$

Then we have

$$(AD^{-1} + DA^{-1}) =$$

$$\begin{bmatrix} -15.966\alpha - \frac{1.0}{\alpha} & 119.59\alpha - \frac{0.5}{\beta} & -238.97\alpha - \frac{0.33333}{\gamma} & 139.32\alpha - \frac{0.25}{\delta} \\ 119.59\beta - \frac{0.5}{\alpha} & -1195.1\beta - \frac{0.33333}{\beta} & 2687.8\beta - \frac{0.25}{\gamma} & -1671.9\beta - \frac{0.2}{\delta} \\ -238.\,7\gamma - \frac{0.33333}{\alpha} & 2687.8\gamma - \frac{0.25}{\beta} & -6449.8\gamma - \frac{0.2}{\gamma} & 4180.1\gamma - \frac{0.16667}{\delta} \\ 139.32\delta - \frac{0.25}{\alpha} & -1671.9\delta - \frac{0.2}{\beta} & 4180.1\delta - \frac{0.16667}{\gamma} & -2786.8\delta - \frac{0.14286}{\delta} \end{bmatrix}$$

Once its characteristic polynomial is simplified and reordered, it turns out to be of the form $x^4 + K_3x^3 + K_2x^2 + K_1x + K_0$ with $K_i > 0$, $i = 0, ..., 3$ (see Appendix for computational details).
Straightforwardly, we can state that this stable matrix A is $D-$stable, analogously to the previous example.
It is clear how simple this method is for any 4×4 matrix.

4.2 5×5 matrices

A) Then consider the stable matrix

$$A = \begin{bmatrix} -1 & -1/2 & -1/3 & -1/4 & -1/5 \\ -1/2 & -1 & -2/3 & -1/2 & -2/5 \\ -1/3 & -2/3 & -1 & -3/4 & -3/5 \\ -1/4 & -1/2 & -3/4 & -1 & -4/5 \\ -1/5 & -2/5 & -3/5 & -4/5 & -1 \end{bmatrix}$$

It is stable symmetric, definite negative and consequently it is $D-$stable. We confirm this result by our approach. We use the previous notation with $\alpha, \beta, \gamma, \delta, \omega > 0$. By any symbolic software, we have

$$(AD^{-1} + DA^{-1}) =$$

$$\begin{bmatrix} -\frac{4}{3}\alpha - \frac{1}{\alpha} & \frac{2}{3}\alpha - \frac{1}{2\beta} & -\frac{1}{3\gamma} & -\frac{1}{4\delta} & -\frac{1}{5\omega} \\ \frac{2}{3}\beta - \frac{1}{2\alpha} & -\frac{32}{15}\beta - \frac{1}{\beta} & \frac{6}{5}\beta - \frac{2}{3\gamma} & -\frac{1}{2\delta} & -\frac{2}{5\omega} \\ -\frac{1}{3\alpha} & \frac{6}{5}\gamma - \frac{2}{3\beta} & -\frac{108}{35}\gamma - \frac{1}{\gamma} & \frac{12}{7}\gamma - \frac{3}{4\delta} & -\frac{3}{5\omega} \\ -\frac{1}{4\alpha} & -\frac{1}{2\beta} & \frac{12}{7}\delta - \frac{3}{4\gamma} & -\frac{256}{63}\delta - \frac{1}{\delta} & \frac{20}{9}\delta - \frac{4}{5\omega} \\ -\frac{1}{5\alpha} & -\frac{2}{5\beta} & -\frac{3}{5\gamma} & \frac{20}{9}\omega - \frac{4}{5\delta} & -\frac{25}{9}\omega - \frac{1}{\omega} \end{bmatrix}$$

By simplifying and reordering, symbolic computation provides its characteristic polynomial in the form $x^5 + K_4x^4 + K_3x^3 + K_2x^2 + K_1x + K_0$ with $K_i > 0$, $i = 0, ..., 4$.
Immediately, it derives that stable matrix A is $D-$stable for condition (7), analogously to the cases already seen above.

B) Consider the following matrix

$$A = \begin{bmatrix} -20 & -20 & -40 & 10 & -5 \\ -20 & -20 & -80 & 10 & 0 \\ 10 & 20 & 0 & 20 & 5 \\ -10 & 0 & -20 & 0 & 4 \\ 20 & -10 & -10 & 0 & -10 \end{bmatrix}$$

This stable matrix is known to be not $D-$ stable (see e.g. Kanovei and Logofet (1998)). Here we want to use our approach to check this result. Using the previous notation, at first, we have

$$(AD^{-1} + DA^{-1}) =$$

$$\begin{bmatrix} -\frac{7}{68}\alpha - \frac{20}{\alpha} & \frac{41}{340}\alpha - \frac{20}{\beta} & -\frac{3}{340}\alpha - \frac{40}{\gamma} & \frac{10}{\delta} - \frac{1}{4}\alpha & -\frac{9}{170}\alpha - \frac{5}{\omega} \\ \frac{113}{680}\beta - \frac{20}{\alpha} & -\frac{179}{680}\beta - \frac{20}{\beta} & \frac{33}{680}\beta - \frac{80}{\gamma} & \frac{5}{8}\beta + \frac{10}{\delta} & \frac{13}{68}\beta \\ \frac{10}{\alpha} - \frac{13}{680}\gamma & \frac{23}{680}\gamma + \frac{20}{\beta} & -\frac{1}{136}\gamma & \frac{20}{\delta} - \frac{1}{8}\gamma & \frac{5}{\omega} - \frac{3}{68}\gamma \\ -\frac{9}{340}\delta - \frac{10}{\alpha} & \frac{29}{340}\delta & \frac{7}{340}\delta - \frac{20}{\gamma} & -\frac{1}{4}\delta & \frac{4}{\omega} - \frac{13}{170}\delta \\ \frac{20}{\alpha} - \frac{6}{17}\omega & \frac{8}{17}\omega - \frac{10}{\beta} & -\frac{1}{17}\omega - \frac{10}{\gamma} & -\omega & -\frac{6}{17}\omega - \frac{10}{\omega} \end{bmatrix}$$

Since its characteristic polynomial exhibits coefficients which cannot be simplified, we have to resort to the last condition (8).
Hence we consider

$$AD^{-1} = \begin{bmatrix} -20/\alpha & -20/\beta & -40/\gamma & 10/\delta & -5/\omega \\ -20/\alpha & -20/\beta & -80/\gamma & 10/\delta & 0 \\ 10/\alpha & 20/\beta & 0 & 20/\delta & 5/\omega \\ -10/\alpha & 0 & -20/\gamma & 0 & 4/\omega \\ 20/\alpha & -10/\beta & -10/\gamma & 0 & -10/\omega \end{bmatrix}$$

Its characteristic polynomial is

$$P(x) = x^5 + \left(\frac{20.0}{\alpha} + \frac{20.0}{\beta} + \frac{10.0}{\omega}\right)x^4$$
$$+ \left(\frac{400.0}{\alpha\gamma} + \frac{100.0}{\alpha\delta} + \frac{1600.0}{\beta\gamma} + \frac{400.0}{\gamma\delta} + \frac{300.0}{\alpha\omega} + \frac{200.0}{\beta\omega} + \frac{50.0}{\gamma\omega}\right)x^3$$
$$+ \left(\frac{8000.0}{\alpha\beta\gamma} + \frac{2000.0}{\alpha\gamma\delta} + \frac{12000.0}{\beta\gamma\delta} + \frac{3000.0}{\alpha\beta\omega} + \frac{8500.0}{\alpha\gamma\omega} + \frac{200.0}{\alpha\delta\omega} + \frac{13000.0}{\beta\gamma\omega} + \frac{400.0}{\beta\delta\omega} + \frac{4800.0}{\gamma\delta\omega}\right)x^2$$

$$+\left(\frac{2.4\times10^5}{\alpha\beta\gamma\delta}+\frac{1.7\times10^5}{\alpha\beta\gamma\omega}+\frac{5000.0}{\alpha\beta\delta\omega}+\frac{1.79\times10^5}{\alpha\gamma\delta\omega}+\frac{70000.0}{\beta\gamma\delta\omega}\right)x+\frac{6.8\times10^5}{\alpha\beta\gamma\delta\omega}$$

and the reminder of division $P(x)/(x^2+1)$ turns out to be

$$x\left(\frac{400.0}{\alpha\gamma}+\frac{100.0}{\alpha\delta}+\frac{1600.0}{\beta\gamma}+\frac{400.0}{\gamma\delta}+\frac{300.0}{\alpha\omega}+\frac{200.0}{\beta\omega}+\frac{50.0}{\gamma\omega}-\frac{2.4\times10^5}{\alpha\beta\gamma\delta}-\frac{1.7\times10^5}{\alpha\beta\gamma\omega}\right.$$
$$\left.-\frac{5000.0}{\alpha\beta\delta\omega}-\frac{1.79\times10^5}{\alpha\gamma\delta\omega}-\frac{70000.0}{\beta\gamma\delta\omega}-1\right)$$
$$-\frac{20.0}{\alpha}-\frac{20.0}{\beta}-\frac{10.0}{\omega}+\frac{8000.0}{\alpha\beta\gamma}+\frac{2000.0}{\alpha\gamma\delta}+\frac{12000.0}{\beta\gamma\delta}+\frac{3000.0}{\alpha\beta\omega}+\frac{8500.0}{\alpha\gamma\omega}+\frac{200.0}{\alpha\delta\omega}$$
$$+\frac{13000.0}{\beta\gamma\omega}+\frac{400.0}{\beta\delta\omega}+\frac{4800.0}{\gamma\delta\omega}-\frac{6.8\times10^5}{\alpha\beta\gamma\delta\omega}.$$

After some numerical trials, by numerical refinement, it is easy to find out that the reminder achieves the numerical zero (i.e. within the numerical precision) for the following values

$$\alpha=0.10038\ ,\ \beta=6.6250\ ,\ \gamma=439.30\ ,$$
$$\delta=0.47818\ ,\ \omega=58.068.$$

Indeed, using these values in the expression of AD^{-1}, given above, we obtain a numerical matrix whose eigenvalues are:

$-i,\ i,\ -3.9210\times10^{-2}, -11.181, -191.21$

We notice that numerical computations are very ill-conditioned, since we add and subtract numbers which are much different in order of magnitude; consequently, we cannot achieve a high accuracy and the computed values of 0 are approximated. Nevertheless, it is clear that there exist values of parameters which make the reminder equal to 0 and this means that matrix A is **not** D-stable, as expected.

5 Conclusion

We have presented a new method to characterize a $D-$ stable matrix. Starting form a well known necessary and sufficient condition, we prove that it is equivalent to other necessary and sufficient conditions which are simpler to be checked by a symbolic software, within the context of numerical linear algebra. The main features of the presented approach are : i) instead of a $2n\times 2n$ matrix as in the original necessary and sufficient condition, the method exploits $n\times n$ matrices; ii) instead of determinants, the method evaluates roots of symbolic polynomials of degree n.

Theoretical results are proved and a related computational method is discussed in details by significant examples (up to matrices of order 5). The computational method reveals simple and capable to treat matrices of order larger than usual, but obviously it depends on the power of the available symbolic software. However this is not a real bound, since software is getting more and more powerful.

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APPENDIX

Here we report some computational details.

Referrimg to Example B) in Subsect. 4.1, we have

$(AD^{-1} + DA^{-1}) =$

$$\begin{bmatrix} -15.966\alpha - \frac{1.0}{\alpha} & 119.59\alpha - \frac{0.5}{\beta} & -238.97\alpha - \frac{0.33333}{\gamma} & 139.32\alpha - \frac{0.25}{\delta} \\ 119.59\beta - \frac{0.5}{\alpha} & -1195.1\beta - \frac{0.33333}{\beta} & 2687.8\beta - \frac{0.25}{\gamma} & -1671.9\beta - \frac{0.2}{\delta} \\ -238.97\gamma - \frac{0.33333}{\alpha} & 2687.8\gamma - \frac{0.25}{\beta} & -6449.8\gamma - \frac{0.2}{\gamma} & 4180.1\gamma - \frac{0.16667}{\delta} \\ 139.32\delta - \frac{0.25}{\alpha} & -1671.9\delta - \frac{02}{\beta} & 4180.1\delta - \frac{0.16667}{\gamma} & -2786.8\delta - \frac{0.14286}{\delta} \end{bmatrix}$$

Characteristic polynomial of $(AD^{-1} + DA^{-1})$ is given by $x^4 + K_3x^3 + ... + K_1x + K_0$, where

$$K_3 = \frac{2.0\times10^{-20}}{\alpha\beta\gamma\delta}\times \begin{pmatrix} 7.983\times10^{20}\alpha^2\beta\gamma\delta + 5.9755\times10^{22}\alpha\beta^2\gamma\delta + 3.2249\times10^{23}\alpha\beta\gamma^2\delta \\ +1.3934\times10^{23}\alpha\beta\gamma\delta^2 + 7.143\times10^{18}\alpha\beta\gamma + 1.0\times10^{19}\alpha\beta\delta \\ +1.6667\times10^{19}\alpha\gamma\delta + 5.0\times10^{19}\beta\gamma\delta \end{pmatrix}$$

$$K_2 = \frac{2.0\times10^{-20}}{\alpha\beta\gamma\delta}\times \begin{pmatrix} 2.3896\times10^{23}\alpha^2\beta^2\gamma\delta + 2.2935\times10^{24}\alpha^2\beta\gamma^2\delta + 1.2542\times10^{24}\alpha^2\beta\gamma\delta^2 \\ +1.1405\times10^{20}\alpha^2\beta\gamma + 1.5966\times10^{20}\alpha^2\beta\delta + 2.661\times10^{20}\alpha^2\gamma\delta \\ +2.4194\times10^{25}\alpha\beta^2\gamma^2\delta + 2.6763\times10^{25}\alpha\beta^2\gamma\delta^2 + 8.5366\times10^{21}\alpha\beta^2\gamma \\ +1.1951\times10^{22}\alpha\beta^2\delta + 2.5053\times10^{25}\alpha\beta\gamma^2\delta^2 \\ +4.6071\times10^{22}\alpha\beta\gamma^2 + 1.0492\times10^{23}\alpha\beta\gamma\delta + 2.7868\times10^{22}\alpha\beta\delta^2 \\ +3.9656\times10^{16}\alpha\beta + 1.075\times10^{23}\alpha\gamma^2\delta + 4.6446\times10^{22}\alpha\gamma\delta^2 \\ +3.8098\times10^{17}\alpha\gamma + 2.083\times10^{17}\alpha\delta + 5.9755\times10^{22}\beta^2\gamma\delta \\ +3.2249\times10^{23}\beta\gamma^2\delta + 1.3934\times10^{23}\beta\gamma\delta^2 + 4.018\times10^{18}\beta\gamma \\ +4.4446\times10^{18}\beta\delta + 4.1665\times10^{18}\gamma\delta \end{pmatrix}$$

$$K_1 = \frac{2.0\times 10^{-20}}{\alpha\beta\gamma\delta}\times$$
$$\left(\begin{array}{c}
4.301\times 10^{25}\alpha^2\beta^2\gamma^2\delta + 6.0237\times 10^{25}\alpha^2\beta^2\gamma\delta^2 + 3.4138\times 10^{22}\alpha^2\beta^2\gamma \\
+4.7792\times 10^{22}\alpha^2\beta^2\delta + 1.0014\times 10^{26}\alpha^2\beta\gamma^2\delta^2 + 3.2766\times 10^{23}\alpha^2\beta\gamma^2 \\
+7.1518\times 10^{23}\alpha^2\beta\gamma\delta + 2.5084\times 10^{23}\alpha^2\beta\delta^2 + 6.3314\times 10^{17}\alpha^2\beta \\
+7.6451\times 10^{23}\alpha^2\gamma^2\delta + 4.1806\times 10^{23}\alpha^2\gamma\delta^2 + 6.0827\times 10^{18}\alpha^2\gamma \\
+3.3257\times 10^{18}\alpha^2\delta + 2.9683\times 10^{26}\alpha\beta^2\gamma^2\delta^2 + 3.4564\times 10^{24}\alpha\beta^2\gamma^2 \\
+8.7239\times 10^{24}\alpha\beta^2\gamma\delta + 5.3526\times 10^{24}\alpha\beta^2\delta^2 + 4.7392\times 10^{19}\alpha\beta^2 \\
+1.298\times 10^{25}\alpha\beta\gamma^2\delta + 1.4772\times 10^{25}\alpha\beta\gamma\delta^2 + 7.5401\times 10^{20}\alpha\beta\gamma \\
+4.0071\times 10^{20}\alpha\beta\delta + 8.351\times 10^{24}\alpha\gamma^2\delta^2 + 2.4572\times 10^{21}\alpha\gamma^2 \\
+2.4223\times 10^{21}\alpha\gamma\delta + 5.8049\times 10^{20}\alpha\delta^2 + 1.3089\times 10^{14}\alpha \\
+2.4194\times 10^{25}\beta^2\gamma^2\delta + 2.6763\times 10^{25}\beta^2\gamma\delta^2 + 4.8019\times 10^{21}\beta^2\gamma \\
+5.3117\times 10^{21}\beta^2\delta + 2.5053\times 10^{25}\beta\gamma^2\delta^2 + 2.5915\times 10^{22}\beta\gamma^2 \\
+4.4695\times 10^{22}\beta\gamma\delta + 1.2386\times 10^{22}\beta\delta^2 + 9.9075\times 10^{15}\beta \\
+2.6873\times 10^{22}\gamma^2\delta + 1.1611\times 10^{22}\gamma\delta^2 + 5.357\times 10^{16}\gamma \\
+2.3129\times 10^{16}\delta
\end{array}\right)$$

$$K_0 = \frac{2.0\times 10^{-20}}{\alpha\beta\gamma\delta}\times$$
$$\left(\begin{array}{c}
2.9144\times 10^{26}\alpha^2\beta^2\gamma^2\delta^2 + 6.1444\times 10^{24}\alpha^2\beta^2\gamma^2 + 1.6737\times 10^{25}\alpha^2\beta^2\gamma\delta \\
+1.2047\times 10^{25}\alpha^2\beta^2\delta^2 + 1.8952\times 10^{20}\alpha^2\beta^2 + 2.4128\times 10^{25}\alpha^2\beta\gamma^2\delta \\
+3.7639\times 10^{25}\alpha^2\beta\gamma\delta^2 + 3.4132\times 10^{21}\alpha^2\beta\gamma + 1.6729\times 10^{21}\alpha^2\beta\delta \\
+3.338\times 10^{25}\alpha^2\gamma^2\delta^2 + 1.7476\times 10^{22}\alpha^2\gamma^2 + 1.8583\times 10^{22}\alpha^2\gamma\delta \\
+5.225\times 10^{21}\alpha^2\delta^2 + 2.0897\times 10^{15}\alpha^2 + 3.78\times 10^{25}\alpha\beta^2\gamma^2\delta \\
+6.6933\times 10^{25}\alpha\beta^2\gamma\delta^2 + 2.1333\times 10^{22}\alpha\beta^2\gamma + 1.858\times 10^{22}\alpha\beta^2\delta \\
+1.4974\times 10^{26}\alpha\beta\gamma^2\delta^2 + 2.7651\times 10^{23}\alpha\beta\gamma^2 + 4.3908\times 10^{23}\alpha\beta\gamma\delta \\
+1.6725\times 10^{23}\alpha\beta\delta^2 + 2.3645\times 10^{17}\alpha\beta + 1.6723\times 10^{23}\alpha\gamma^2\delta \\
+1.1612\times 10^{23}\alpha\gamma\delta^2 + 9.49\times 10^{17}\alpha\gamma + 3.2366\times 10^{17}\alpha\delta \\
+2.9683\times 10^{26}\beta^2\gamma^2\delta^2 + 1.9443\times 10^{24}\beta^2\gamma^2 + 4.1827\times 10^{24}\beta^2\gamma\delta \\
+2.379\times 10^{24}\beta^2\delta^2 + 1.184\times 10^{19}\beta^2 + 3.3889\times 10^{24}\beta\gamma^2\delta \\
+4.1805\times 10^{24}\beta\gamma\delta^2 + 1.2001\times 10^{20}\beta\gamma + 4.6495\times 10^{19}\beta\delta \\
+2.0877\times 10^{24}\gamma^2\delta^2 + 3.4552\times 10^{20}\gamma^2 + 2.9026\times 10^{20}\gamma\delta \\
+6.4455\times 10^{19}\delta^2 + 8.0556\times 10^{12}
\end{array}\right)$$