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OPTIMIZATION OF THE POSITIVE PRINCIPAL EIGENVALUE FOR INDEFINITE FRACTIONAL NEUMANN PROBLEMS

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ABSTRACT. We study the positive principal eigenvalue of a weighted problem associated with the Neumann spectral fractional Laplacian. This analysis is related to the investigation of the survival threshold in population dynamic. Our main result concerns the optimization of such eigenvalue with respect to the fractional order $s \in (0,1]$, the case $s = 1$ corresponding to the standard Neumann Laplacian: when the habitat is not too hostile in average, the principal positive eigenvalue can not have local minima for $0 < s < 1$. As a consequence, the best strategy for survival is either following the diffusion with the lowest possible s, or with $s = 1$, depending on the size of the domain. In addition, we show that analogous results hold for the standard fractional Laplacian in $\mathbb{R}^N$, in periodic environments.

1. INTRODUCTION

Let $u = u(x,t)$ denote the density of a population in position x at time t. The common mathematical model [30] for the evolution of u, in case it undergoes some kind of dispersal, is given by a reaction-diffusion equation

$$u_t + Lu = f(x,u),$$

on some spatial domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with suitable boundary conditions. The internal reaction $f(x,u)$, which takes into account also the heterogeneity of the habitat, can take various forms: for our purposes it is sufficient to consider the simplest case, i.e. that of a logistic nonlinearity

$$f(x,u) = m(x)u - u^2,$$

where the weight m changes sign, distinguishing regions of either favorable or hostile habitat. The diffusion operator is denoted by L, and in principle it can incorporate a number of different features of the model. Here, we consider linear, homogeneous and isotropic, but possibly non-local, operators. If the individuals tend to move within the nearest neighborhoods, then the spatial spread of u is triggered by an underlying random walk of Brownian type, and it is customary to choose $L = -K\Delta$, for some motility coefficient $K > 0$. On the other hand, in case the resources are sparse, it is expected that more elaborate hunting strategies, allowing for long jumps, may favour the population survival. Actually, this guess has been supported also by experimental studies [38, 22]. In this case the underlying random walk is of Levy flight-type,

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rather that Brownian, and one is lead to consider fractional diffusion operators, where $-\Delta$ is replaced by $(-\Delta)^s$ [28, 36].

When $0 < s < 1$ the fractional Laplacian in the entire space $\mathbb{R}^N$ can be defined via different—but equivalent—definitions [7]: for instance via an integral expression

$$(-\Delta_{\mathbb{R}^N})^s u(x) = C_{N,s}\, \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(\xi)}{|x-\xi|^{N+2s}}\, \mathrm{d}\xi,$$

for some dimensional constant $C_{N,s}$, or as a pseudo-differential operator, in terms of its Fourier transform:

$$\widehat{(-\Delta_{\mathbb{R}^N})^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi). \tag{1}$$

When dealing with a Lipschitz bounded domain $\Omega \subset \mathbb{R}^N$, the situation is more variegated and different, non-equivalent operators have been proposed, in dependence of how the boundary conditions are interpreted. This complexity of the model is particularly evident when dealing with Neumann, i.e. no flux boundary conditions, see [2, 14]. In this paper we consider the boundary as a "reflecting barrier", namely a barrier that, in the discrete time counterpart, acts on the long jump by means of an elastic reflection; this corresponds to the so-called "mirror reflection" case considered in [2]. Reasoning in terms of random walk and imposing the presence of a reflecting barrier on $\partial\Omega$, one is lead, at least heuristically, to consider the Neumann spectral fractional Laplacian, i.e.

$$(-\Delta_{\mathrm{N}})^s u = \sum_{k=1}^{\infty} \mu_k^s \left(\int_\Omega u \phi_k \, dx \right) \phi_k, \tag{2}$$

where $0 = \mu_0 < \mu_1 \le \mu_2 \le \dots$ denote the Neumann eigenvalues of $-\Delta$ in $H^1(\Omega)$, and $(\phi_k)_k$ the corresponding normalized eigenfunctions. This operator has been considered in different models and applications, see for instance [8, 35, 15]. The relation between the Neumann spectral fractional Laplacian and random walks with long jumps and reflections has been discussed in [29], in dimension $N = 1$, and those arguments can be easily extended to higher dimensions in case Ω is a rectangle: in fact, the correspondence holds true as far as the reflecting barrier can be treated by the method of images, by introducing reflected domains in which the motion can be continued, and then by quotienting by the symmetries. Several other interpretations of the boundary as a reflecting barrier are available in the literature: for instance, in [1] the barrier acts on the long jump by just stopping the particle at the boundary, without any rebound; in [14], also the action of the boundary is not deterministic.

Incidentally, another established point of view is that of dealing with a periodically fragmented environment in $\mathbb{R}^N$ [3, 4, 32, 6]. Actually, for our purposes, the treatment of the periodic model is very similar to that with mirror reflections, so that our results can be rewritten also in this context, see below.

Another controversial feature of the model we are describing regards the form of the generalized diffusion coefficient: a number of contributions deals with the difficulty of properly defining (and measuring) the motility coefficient K [39, 37, 20]. Motivated by dimensional arguments and modeling ones, in this context K is supposed to be dependent on s, and a commonly accepted expression for it has been introduced in [28, Section 3.5] as

$$K(s) = \frac{\sigma^{2s}}{\tau}$$

(see also [17, 18]), where the scales σ and τ are respectively characteristic length and time associated with the diffusion process. Here, for simplicity, we assume $\tau = 1$ and write $d = \sigma^2 > 0$. Summing up, we consider the equation

$$\partial_t u + d^s (-\Delta_{\mathrm{N}})^s u = m(x) u - u^2, \qquad x \in \Omega, \tag{3}$$

for $0 < s \leq 1$, where $s = 1$ corresponds to the case of standard (local) diffusion.

A main question related to (3) concerns survival of the population, that is, the identification of conditions (on Ω, s, d, m) which imply that solutions to (3) do not vanish asymptotically for $t \to +\infty$. When $s = 1$, it is well known that such conditions are related to the existence of a positive steady state, which attracts every non-negative non-trivial solution. In turn, the existence of such steady state can be expressed in terms of the principal eigenvalue of the associated linearized problem [21, 11, 3]. These results can be extended also to the fractional setting [4, 23, 29].

Taking $m \in L^\infty(\Omega)$, two different situations may occur in dependence on its average: if m has non-negative average (and it is non-trivial) then there is always survival. On the other hand, in the case

$$m \in \mathcal{M} := \left\{ m \in L^\infty(\Omega) : \int_\Omega m < 0,\ m^+ \not\equiv 0 \right\}, \tag{4}$$

the survival is related to the weighted eigenvalue problem

$$d^s(-\Delta_{\mathrm{N}})^s u = \lambda m u, \qquad x \in \Omega. \tag{5}$$

More precisely, in Section 2 we show that, under condition (4), there exists a unique positive principal eigenvalue

$$\lambda_1 = \lambda_1(m, d, s) > 0,$$

with a positive eigenfunction. Moreover, reasoning as in [4, Theorem 1.2], one has that solutions to (3) survive (i.e. they tend to the unique positive steady state, as $t \to +\infty$) if and only if $\lambda_1(m, d, s) < 1$. Then, natural questions concern the dependence of λ_1 on the parameters of the problem, and in particular its optimization. Note that, through a change of variables, rescaling the size of the domain is equivalent to rescaling the diffusion coefficient d while keeping Ω fixed. Here we choose this second point of view, and this is the reason why we do not consider explicitly the dependence of the eigenvalue on the domain.

In the case $s = 1$ of standard diffusion, the dependence of λ_1 on d can be easily scaled out and the eigenvalue actually depends only on m. Accordingly, the problem of minimizing λ_1 has been mainly considered, when m varies within a suitable admissible class, see [9, 10, 11, 19, 27, 33, 24, 13] and references therein. The typical result obtained is that the minimizer m exists and it is of bang-bang type (i.e. it coincides with its maximum value $\overline{m} > 0$ on some $D \subset \Omega$, and with its minimum $-\underline{m} < 0$ on $\Omega \setminus D$). Furthermore, the best environment has a few number of relatively large favorable regions. As observed in [9], this has significant implications for the design of wildlife refugees. Part of these results can be extended also to the case $s < 1$, as discussed in Section 5 below, but our main interest in the present paper is to analyze the properties of the map

$$(m, d, s) \mapsto \lambda_1(m, d, s),$$

aiming at optimizing λ_1, mainly with respect to $0 < s \leq 1$. From a modelistic viewpoint, this amounts to wonder whether, for given population and habitat, the Brownian hunting strategy is more effective than the long jumps one, in order to survive. The good starting point in our analysis is that the map $s \mapsto \lambda_1(m, d, s)$ is smooth in $(0, 1]$ (see Section 2). Up to our knowledge, there are very few contributions concerning the optimization of the order s in fractional diffusion equations; in particular, a related but different problem has been considered in [34].

It is worth noticing that part of the cited above literature does not treat exactly problem (5) (with $s = 1$), but rather the related version

$$-d\Delta u - mu = \tilde{\lambda} u, \qquad x \in \Omega. \tag{6}$$

It is easy to show that $\lambda_1 < 1$ if and only if $\tilde{\lambda}_1 < 0$, therefore both these eigenvalues play an analogous role for survival. One main advantage of the latter problem is that a principal eigenvalue $\tilde{\lambda}_1$ always exists, regardless of the average of m; on the other hand, we prefer to deal

with (5) because, among other properties, the dependence of λ_1 on d can be treated in a simpler way. Indeed, it is easy to see that

$$\lambda_1(m,d,s) = d^s \lambda_1(m,1,s),$$

and this property allows us to prove that, if d is very large or very small, with respect to the size of Ω, then $s \mapsto \lambda_1(m,d,s)$ becomes monotone, and therefore it is minimized either for $s=1$ or for s small. More precisely, recalling that $\mu_1 > \mu_0 = 0$ denotes the first positive Neumann eigenvalue of $-\Delta$ in $H^1(\Omega)$, we have the following.

Proposition 1.1. *Let Ω and m be fixed and satisfying (4). Then:*

- *if $d \geq \dfrac{1}{\mu_1}$ then the map $s \mapsto \lambda_1(m,d,s)$ is monotone increasing in $(0,1]$;*
- *for any $0 < a < 1$ there exists $\underline{d} > 0$, depending only on a, Ω and m, such that if $d \leq \underline{d}$ then the map $s \mapsto \lambda_1(m,d,s)$ is monotone decreasing in $[a,1]$.*

Recall that d is related to a characteristic length associated with the diffusion process. As we already noticed, a small d corresponds to the case of a domain which is large, with respect to the diffusion characteristic length, and vice versa. As a consequence, the biological interpretation of Proposition 1.1 is that in very large environments the local diffusion is more successful, while in very small ones a nonlocal strategy would be preferable. Similar effects in related models were already noticed in [6], Theorem 1.5 (and the subsequent discussion).

From Proposition 1.1 it is clear that, when d increases from $\underline{d}$ to $1/\mu_1$, then the map $s \mapsto \lambda_1(m,d,s)$ has a transition from decreasing to increasing, and therefore it develops internal critical points. The main result we obtain in this paper is that, if the habitat is not too hostile in average, in such transition only internal maxima appear.

Theorem 1.2. *Let $M, \rho, \delta \in \mathbb{R}^+$ and set*

$$\widetilde{\mathcal{M}} := \left\{ m \in \mathcal{M} : \|m\|_{L^\infty} \leq M,\ \exists\, B_\rho(x_0) \subset \Omega \text{ with } m|_{B_\rho(x_0)} \geq \delta \right\}. \tag{7}$$

There exists $A = A(M,\rho,\delta) > 0$ such that, if $m \in \widetilde{\mathcal{M}}$ and

$$-A \leq \int_\Omega m < 0, \tag{8}$$

then, depending on $d > 0$, either the map $s \mapsto \lambda_1(m,d,s)$ is monotone, or it has exactly one maximum in $(0,1)$. In particular, the limit $\lambda_1(m,d,0^+)$ is well defined for every d, and

$$\inf_{0<s\leq 1} \lambda_1(m,d,s) = \begin{cases} \lambda_1(m,d,1) & \text{when } 0 < d \leq d^* \\ \lambda_1(m,d,0^+) & \text{when } d \geq d^*, \end{cases}$$

where $d^* = \lambda_1(m,1,0^+)/\lambda_1(m,1,1)$.

Therefore, if $m \in \widetilde{\mathcal{M}}$ and (8) holds, two main consequences follow: the first is that, in the second case treated by Proposition 1.1, we can remove the assumption $s \geq a$, covering the full interval $0 < s \leq 1$; the second is that the best choice of s is always either the smallest one, or the biggest one. Let us observe that non-degeneracy conditions as the one present in (7) have been already considered in the literature to avoid an excessive fragmentation of the favorable region, see for instance Theorem 3.1 in [9].

One may wonder whether assumption (8) is merely technical, and Theorem 1.2 may hold for more general m. At the end of Section 4 we provide some simple numerical simulations which suggest that this is not the case and that, when the environment is strongly hostile in average, the map $s \mapsto \lambda_1(m,d,s)$ may present interior minima, as well as multiple local extrema (see Figure 2).

As we mentioned, one can also deal with the fractional Laplacian on the whole $\mathbb{R}^N$, instead of the Neumann spectral one, by assuming that the environment is periodic. More precisely, following [4], let us introduce the hyperrectangle $\mathcal{C}_l = (0, l_1) \times \cdots \times (0, l_N) \subset \mathbb{R}^N$, and let us assume that

$$m : \mathbb{R}^N \to \mathbb{R} \text{ is } \mathcal{C}_l\text{-periodic.}$$

In case $m|_{\mathcal{C}_l}$ satisfies (4) (with Ω replaced by $\mathcal{C}_l$), we have the existence of a positive principal eigenvalue $\lambda_{\rm per} = \lambda_{\rm per}(m, d, s)$, with positive periodic eigenfunction, for the problem

$$d^s(-\Delta_{\mathbb{R}^N})^s u = \lambda m u, \qquad x \in \mathbb{T}^N := \mathbb{R}^N/\mathcal{C}_l. \tag{9}$$

Moreover, the solutions to the problem

$$\partial_t u + d^s(-\Delta_{\mathbb{R}^N})^s u = m(x)u - u^2, \qquad x \in \mathbb{R}^N,$$

where no periodicity condition is assumed on u, survive if and only if $\lambda_{\rm per} < 1$. Actually, these results are proved in [4] in terms of the eigenvalue $\tilde\lambda_{\rm per}$ corresponding to (6), but the two conditions can be easily proved to be equivalent.

Now, it is easy to be convinced that in some particular cases the Neumann spectral eigenvalue problem (5) and the periodic one (9) are equivalent. For instance, if m is defined in a hyperrectangle Ω, then one can extend it to 2Ω by even reflection, and then to $\mathbb{R}^N$ by periodicity; hence, using the uniqueness properties of the principal eigenfunctions, one can reduce the Neumann problem in Ω to the periodic one; also the opposite reduction can be done, in case m is $\mathcal{C}_l$-periodic, and even with respect to the directions of its sides (up to translations). In any case, also for general Ω and m, the two problems have the same structure. Indeed, let $\mathcal{C}_l$ be fixed and let $(\nu_k)_k$, $(\varphi_k)_k$ denote the periodic eigenvalues and eigenfunctions of $-\Delta$ in $\mathcal{C}_l$ (which can be explicitly computed). Then we can introduce the periodic spectral fractional Laplacian as

$$(-\Delta_{\rm per})^s u = \sum_{k=1}^{\infty} \nu_k^s \left(\int_{\mathcal{C}_l} u\varphi_k \, dx \right) \varphi_k. \tag{10}$$

The following result allows us to connect spectral operators with periodic ones.

Proposition 1.3. *If u is continuous and $\mathcal{C}_l$-periodic then*

$$(-\Delta_{\mathbb{R}^N})^s u = (-\Delta_{\rm per})^s u, \qquad x \in \mathcal{C}_l.$$

Once we have interpreted the periodic operator as a spectral one, we can extend the analysis of the Neumann problem also to the periodic case. The key observation, with this respect, is that the spectrum of the Neumann problem and that of the periodic one share the same main properties, namely, they both consist in a diverging sequence of eigenvalues, with first, simple element $\mu_0 = \nu_0 = 0$, and they both are associated with a basis of eigenfunctions which are orthogonal in H^1 and orthonormal in L^2. Then all the results for the Neumann case hold true also in the periodic one. In particular, we have the following counterpart of Theorem 1.2 in the periodic setting.

Theorem 1.4. *Let m be $\mathcal{C}_l$-periodic. There exists $A = A(M, \rho, \delta) > 0$ such that, if $m|_{\mathcal{C}_l} \in \widetilde{\mathcal{M}}$ and*

$$-A \le \int_\Omega m < 0,$$

then, for every $d > 0$, the map $s \mapsto \lambda_{\rm per}(m, d, s)$ is either monotone or it has exactly one internal maximum in $(0, 1)$, and

$$\inf_{0<s\le 1} \lambda_{\rm per}(m, d, s) = \begin{cases} \lambda_{\rm per}(m, d, 1) & \text{when } 0 < d \le d^* \\ \lambda_{\rm per}(m, d, 0^+) & \text{when } d \ge d^*, \end{cases}$$

where $d^* = \lambda_{\rm per}(m, 1, 0^+)/\lambda_{\rm per}(m, 1, 1)$.

Let us point out that with our techniques we can also deal with other fractional spectral operators. For instance, the Dirichlet case can be treated in an even easier way, since in such case zero is not an eigenvalue of $-\Delta$.

The paper is structured as follows. In Section 2 we show the existence and regularity properties of eigenvalues and eigenfunctions associated with (5). In Section 3 we collect a number of results about the dependence of $\lambda_1(m,d,s)$ on d and s. We exploit such results in Section 4 to conclude the proof of Theorem 1.2. In Section 5 we briefly discuss the optimization of λ_1 with respect to m. Finally, Section 6 is devoted to the treatment of the periodic case.

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Notation. We write $(\cdot,\cdot)$ for the scalar product in $L^2(\Omega)$. We will denote with ϕ_k the eigenfunctions of the classical Laplace operator in Ω with Neumann homogeneous boundary conditions, normalized in $L^2(\Omega)$. Their associated eigenvalues will be denoted by μ_k. For a function $u\in L^2(\Omega)$, we write

$$u = u_0 + \sum_{k=1}^{\infty} u_k\phi_k, \qquad \text{where } u_0 = \frac{1}{|\Omega|}\int_\Omega u, \quad u_k = (u,\phi_k),\ k\geq 1. \tag{11}$$

Often we will write $u = u_0 + \tilde{u}$. Finally, C denotes every (positive) constant we do not need to specify, whose value may change also within the same formula.

2. Preliminary Results

In all the paper we will assume, up to further restrictions, $d>0$, $0<s\leq 1$, and $m\in\mathcal{M}$, defined in (4). Taking into account the L^2-spectral decomposition (see (11)), we consider the functional space

$$H^s(\Omega) = \left\{ u = u_0 + \sum_{k=1}^{\infty} u_k\phi_k \in L^2(\Omega) \,:\, \sum_{k=1}^{\infty} \mu_k^s u_k^2 < +\infty \right\}. \tag{12}$$

Note that, as shown in [8, Lemma 7.1], this definition of $H^s(\Omega)$ is equivalent to the usual one given in terms of the Gagliardo semi-norm

$$[u]_{2,s}^2 = \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy.$$

In $H^s(\Omega)$ it is well-defined the fractional differential operator

$$(-\Delta_{\mathrm{N}})^s u = \sum_{k=1}^{\infty} \mu_k^s u_k\phi_k,$$

and, taking into account (11), the norm in $H^s(\Omega)$ can be written as

$$\|u\|_{H^s}^2 = u_0^2 + \sum_{k=1}^{\infty}\mu_k^s u_k^2 = \left(\frac{1}{|\Omega|}\int_\Omega u\right)^2 + ((-\Delta_{\mathrm{N}})^s u, u).$$

In the following, we will prove the existence of a double sequence of eigenvalues for problem (5), and some qualitative properties of the eigenfunctions.

Proposition 2.1. *Problem (5) admits two unbounded sequences of eigenvalues:*

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

Furthermore, both the eigenvalues and the (normalized) eigenfunctions depend continuously on m.

In particular,

$$\lambda_1(m,d,s) = d^s \min_{H^s(\Omega)} \left\{ ((-\Delta_{\mathrm{N}})^s u, u) : \int_\Omega m u^2 = 1 \right\}. \tag{13}$$

Proof. The results for $s = 1$ are standard, so we restrict to the case $0 < s < 1$. First of all, the simple eigenvalue $\lambda_0 = 0$, with constant eigenfunction, can be computed directly. The other eigenvalues can be obtained by restricting to the space

$$V := \{u \in H^s(\Omega) \,:\, (u,1) = 0\}.$$

Indeed, in this space we can use the equivalent scalar product

$$(u,v)_V = \sum_{k=1}^{\infty} \mu_k^s u_k v_k$$

and we have that the linear operator $T : V \to V$ defined by

$$(Tu,v)_V = \int_\Omega muv$$

is symmetric and compact, thanks to the compact embedding of $H^s(\Omega)$ in $L^2(\Omega)$ (recall the definition of $H^s(\Omega)$ in (12)). As a consequence, we can apply standard results in spectral theory of self-adjoint compact operators to obtain the existence and the variational characterization of the eigenvalues (see e.g. [12, Propositions 1.3, 1.10]), as well as the continuity property of the spectrum (see the book by Kato [25]). □

Remark 2.2. Alternatively, following [5, 8], the above result can be obtained by means of an extension problem in $\mathcal{C} := \Omega \times (0,\infty)$. Indeed, let $1 - 2s =: a \in (-1,1)$ and

$$\mathcal{H}^{1;a}(\mathcal{C}) := \left\{ v = v_0 + \tilde{v} : v_0 \in \mathbb{R},\ \int_{\mathcal{C}} y^a \left(|\nabla \tilde{v}|^2 + \tilde{v}^2 \right) dxdy < +\infty \right\}.$$

It is known (see [31]) that, for $\partial\Omega$ sufficiently smooth, the elements of $H^s(\Omega)$ coincide with the traces of functions in $\mathcal{H}^{1;a}(\mathcal{C})$. As a consequence, any $u \in H^s(\Omega)$ admits a unique extension $v \in \mathcal{H}^{1;a}(\mathcal{C})$ which achieves

$$\min \left\{ \int_{\mathcal{C}} y^a |\nabla v|^2 \, dxdy : v(x,0) = u(x) \right\}. \tag{14}$$

Then (5) is equivalent to

$$\begin{cases} \mathrm{div}(y^a \nabla v) = 0 & \text{in } \mathcal{C} \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,\infty) \\ D(s) d^s \partial_\nu^a v(x,0) = \lambda m(x) v(x,0) & \text{in } \Omega, \end{cases}$$

where the structural constant $D(s)$ is known to be

$$D(s) = 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)},$$

so that one has the following characterization:

$$\lambda_1(m,d,s) = d^s D(s) \min_{\mathcal{H}^{1;a}(\mathcal{C})} \left\{ \int_{\mathcal{C}} y^a |\nabla v|^2 dxdy : \int_\Omega m v^2(x,0) dx = 1 \right\}.$$

Note that the last formulation can be rewritten in terms of a suitable Rayleigh quotient.

Remark 2.3. In the next sections we will also deal with

$$-\lambda_{-1}(m,d,s) = d^s \min_{H^s(\Omega)} \left\{ ((-\Delta_{\mathrm{N}})^s u, u) : \int_\Omega m u^2 = -1, \int_\Omega mu = 0 \right\}. \tag{15}$$

Notice that λ_{-1} is actually a second eigenvalue and it may not be simple. Indeed, since m has negative mean, the constant eigenfunction ψ_0 associated with $\lambda_0 = 0$ satisfies

$$\int_\Omega m\psi_0^2 < 0.$$

As a consequence, without imposing that

$$\int_\Omega mu = 0$$

the minimization problem (15) has the solution $\lambda_0 = 0$.

Proposition 2.4. *Let ψ be any eigenfunction of problem (5). Then*

$$\psi \in H^{2s}(\Omega).$$

Furthermore, $\psi \in C^{0,\alpha}(\Omega)$ for every $\alpha < 2s$, whenever $s \le 1/2$, and $\psi \in C^{1,\alpha}(\Omega)$ for every $\alpha < 2s-1$, in case $s > 1/2$.

Proof. Since $m \in L^\infty(\Omega)$, equation (5) implies that $(-\Delta_{\mathrm{N}})^s u \in L^2(\Omega)$, that is, recalling (11)

$$\sum_k (\mu_k^s u_k)^2 < +\infty;$$

the Sobolev regularity follows by the definition of $H^{2s}(\Omega)$ given in (12). On the other hand, the Hölder regularity of the eigenfunctions is a consequence of the regularity theory developed by Caffarelli and Stinga [8, Theorem 1.5], and of a standard bootstrap argument. □

Thanks to Proposition 2.1 we have that there exists a solution to the linear problem

$$d^s(-\Delta_{\mathrm{N}})^s \psi = \lambda_1(m,d,s) m\psi \qquad x \in \Omega, \tag{16}$$

and now we turn to the study of the properties of the first eigenvalue λ_1 and the associated eigenfunction ψ. First of all, in order to show that λ_1 is simple, we will exploit the following lemma, which concerns a convexity property of the $H^s(\Omega)$ semi-norm.

Lemma 2.5. *Let $u \in H^s(\Omega)$, $0 < s < 1$. Then $u^\pm \in H^s(\Omega)$ and*

$$((-\Delta_{\mathrm{N}})^s u, u) \ge \left((-\Delta_{\mathrm{N}})^s u^+, u^+\right) + \left((-\Delta_{\mathrm{N}})^s u^-, u^-\right), \tag{17}$$

and the strict inequality holds whenever $u^\pm$ are both nontrivial.

Remark 2.6. The lemma enlightens a substantial difference between the nonlocal and the local case. Indeed, when $s = 1$, the equality sign in (17) always holds for any u. A similar result in the periodic case has been shown in [4, Proposition 3.1].

Proof of Lemma 2.5. Let $v \in \mathcal{H}^{1;a}(\mathcal{C})$ be the extension of u given in (14). Then $v^\pm \in \mathcal{H}^{1;a}(\mathcal{C})$ and, taking into account Remark 2.2, their traces $u^\pm$ belong to $H^s(\Omega)$. Therefore

$$\begin{aligned}((-\Delta_{\mathrm{N}})^s u, u) = \int_{\mathcal{C}} y^a |\nabla v|^2 \, dxdy &= \int_{\mathcal{C}} y^a |\nabla v^+|^2 \, dxdy + \int_{\mathcal{C}} y^a |\nabla v^-|^2 \, dxdy \\ &\ge \int_{\mathcal{C}} y^a |\nabla w_+|^2 \, dxdy + \int_{\mathcal{C}} y^a |\nabla w_-|^2 \, dxdy \\ &= \left((-\Delta_{\mathrm{N}})^s u^+, u^+\right) + \left((-\Delta_{\mathrm{N}})^s u^-, u^-\right),\end{aligned}$$

where $w_\pm$ solve the minimization problem (14) with traces $u^\pm$.

Finally, if $v^{\pm}$ are both nontrivial, the strong maximum principle [5, Remark 4.2.] implies that they cannot solve (14), thus the strict inequality holds. □

Proposition 2.7. *The eigenvalue $\lambda_1(m,d,s)$ is simple, and the associated eigenfunction does not change sign. Moreover the map*

$$\mathcal{M}\times\mathbb{R}^+\times(0,1]\ni(m,d,s)\mapsto\lambda_1$$

is analytic.

Proof. The fact that λ_1 is simple, with one-signed eigenfunction, can be deduced arguing as in [12, Theorem 1.13], taking into account Lemma 2.5.

To prove the second part of the statement, let $(m^*,d^*,s^*)\in\mathcal{M}\times\mathbb{R}^+\times(0,1]$ be fixed and λ_1,ψ_1 denote the first eigenvalue and the non-negative, normalized first eigenfunction for the corresponding problem (16) with weight m^*, coefficient d^* and exponent s^*. If $\sigma>0$ is sufficiently small we have that, by Propositions 2.1 and 2.4, the map

$$\begin{aligned}\mathcal{F}\colon\mathcal{M}\times\mathbb{R}^+\times(s^*-\sigma,s^*+\sigma)\times H^{2(s^*-\sigma)}(\Omega)\times\mathbb{R}^+&\to L^2(\Omega)\times\mathbb{R},\\ \mathcal{F}(m,d,s,u,\lambda)&=\left(d^s(-\Delta_{\mathrm{N}})^s u-\lambda mu,\int_\Omega mu^2-1\right),\end{aligned}$$

is well defined, and

$$\mathcal{F}(m^*,d^*,s^*,\psi_1,\lambda_1)=(0,0).$$

In order to reach the conclusion, we are going to apply the Implicit Function Theorem to $\mathcal{F}$, expressing the pair (u,λ) as function of (m,d,s). To this aim, computing the derivative one obtains

$$\partial_{(u,\lambda)}\mathcal{F}(m^*,d^*,s^*,\psi_1,\lambda_1)[v,l]=\begin{pmatrix}(d^*)^{s^*}(-\Delta_{\mathrm{N}})^{s^*}v-\lambda_1 m^*v-lm^*\psi_1\\ 2\displaystyle\int_\Omega m^*\psi_1 v\end{pmatrix}.$$

By Fredholm's Alternative, it suffices to show that the linear operator above is injective and this is a straightforward consequence of (13) and of the fact that λ_1 is simple. □

As a direct consequence of Proposition 2.7 we have the following result.

Corollary 2.8. *Let us denote with ψ_1 the non-negative, normalized eigenfunction corresponding to λ_1. For every $0<s_1<s_2\le 1$, the map*

$$F\colon\mathcal{M}\times\mathbb{R}^+\times(s_1,s_2]\to\mathbb{R}^+\times H^{2s_1}(\Omega),\qquad F(m,d,s)=(\lambda_1,\psi_1)$$

is smooth.

Remark 2.9. It is natural to wonder whether the eigenfunction corresponding to λ_1 can be chosen to be strictly positive on $\overline{\Omega}$. To obtain this result, one may invoke the strong maximum principle [5, Remark 4.2, Proposition 4.11]. This requires more regularity, and the proof can be completed in case $m\in C^{0,\alpha}(\overline{\Omega})$ (and $\partial\Omega$ is smooth) by using [8, Theorem 1.4]. For a general $m\in L^\infty(\Omega)$, we can only deduce that the eigenfunction can not vanish on a set with non-empty interior, but we can not exclude vanishing points, in particular when s is small.

To end this section, we provide some estimates on $\lambda_1(m,d,s)$.

Proposition 2.10. *It holds*

$$\lambda_1(m,d,s)\le(d\mu_1)^{s-1}\lambda_1(m,d,1).$$

Proof. Let ψ_1 denote the first normalized eigenfunction associated with $\lambda_1(m,d,1)$. Then

$$\lambda_1(m,d,s) \le (d^s(-\Delta_{\mathrm{N}})^s\psi_1,\psi_1) = \sum_k (d\mu_k)^s(\psi_1)_k^2 = \sum_k (d\mu_k)^{s-1}d\mu_k(\psi_1)_k^2$$
$$\le (d\mu_1)^{s-1}\sum_k d\mu_k(\psi_1)_k^2 = (d\mu_1)^{s-1}\lambda_1(m,d,1) \qquad \square$$

Proposition 2.11. *It holds*

$$\lambda_1(m,d,s) \ge \frac{d^s\mu_1^s\left|\int_\Omega m\right|}{\sup_\Omega m\left|\int_\Omega m\right| + \|m\|_{L^2}^2}. \tag{18}$$

Proof. To start with, notice that, using (11), the following Poincaré inequality holds:

$$u\in H^s(\Omega),\ u_0=0 \quad\Longrightarrow\quad \int_\Omega u^2 \le \frac{1}{\mu_1^s}((-\Delta_{\mathrm{N}})^s u,u). \tag{19}$$

Indeed,

$$\int_\Omega u^2 = \sum_{k\ge1} u_k^2 \le \frac{1}{\mu_1^s}\sum_{k\ge1}\mu_k^s u_k^2 = \frac{1}{\mu_1^s}((-\Delta_{\mathrm{N}})^s u,u).$$

Using the decomposition $\psi_1 = h+\tilde\psi_1$, with $h\in\mathbb{R}$ and $\tilde\psi_1$ with zero average, we can exploit the fact that $(-\Delta_{\mathrm{N}})^s\psi_1$ has zero average to get

$$0=\lambda_1(m,d,s)\int_\Omega m(h+\tilde\psi_1) \quad\Longrightarrow\quad h=-\frac{\int_\Omega m\tilde\psi_1}{\int_\Omega m}.$$

Then

$$\begin{aligned} d^s((-\Delta_{\mathrm{N}})^s\psi_1,\psi_1) &= \lambda_1(m,d,s)\int_\Omega m(h+\tilde\psi_1)^2 \\ &= \lambda_1(m,d,s)\left\{\int_\Omega m\tilde\psi_1^2 - \frac{1}{\int_\Omega m}\left[\int_\Omega m\tilde\psi_1\right]^2\right\} \\ &\le \lambda_1(m,d,s)\left\{\sup_\Omega m + \frac{1}{\left|\int_\Omega m\right|}\|m\|_{L^2}^2\right\}\int_\Omega\tilde\psi_1^2 \\ &\le \frac{\lambda_1(m,d,s)}{\mu_1^s}\left\{\sup_\Omega m + \frac{1}{\left|\int_\Omega m\right|}\|m\|_{L^2}^2\right\}((-\Delta_{\mathrm{N}})^s\psi_1,\psi_1), \end{aligned}$$

where we used (19) in the last line. Then (18) holds. $\square$

3. Dependence on d and s

Throughout this section $m\in\mathcal{M}$ is fixed, therefore, for easier notation, we omit the dependence on m and write $\lambda_1=\lambda_1(d,s)$.

We first observe that, according to the characterization (13), the parameter d only affects the first eigenvalue λ_1 and not the corresponding eigenfunction.

Lemma 3.1. *For any $d_1,d_2\in\mathbb{R}^+$ we have that*

$$\lambda_1(d_2,s) = \frac{d_2^s}{d_1^s}\lambda_1(d_1,s),$$

and both eigenvalues share the same eigenfunction.

Proof. The lemma is an immediate consequence Propositions 2.1 and 2.7. Indeed, let ψ_1 be the first positive normalized eigenfunction associated with $\lambda_1(d_1, s)$. Then

$$d_2^s(-\Delta_{\mathrm{N}})^s\psi_1 = \frac{d_2^s}{d_1^s}\,(d_1^s(-\Delta_{\mathrm{N}})^s\psi_1) = \frac{d_2^s}{d_1^s}\lambda_1(d_1,s)m\psi_1.$$

Therefore, ψ_1 is also a positive eigenfunction associated with the eigenvalue $\nu = d_2^s\lambda_1(d_1,s)/d_1^s$. Taking into account that $\lambda_1(d_2,s)$ is simple we obtain that $\lambda_1(d_2,s) = d_2^s\lambda_1(d_1,s)/d_1^s$, with same eigenfunction as $\lambda_1(d_1,s)$. □

In particular, we obtain that, for any $d > 0$,

$$\lambda_1(d,s) = d^s\lambda_1(1,s). \tag{20}$$

Proposition 3.2. *For any fixed $s \in (0,1]$, the map*

$$d \mapsto \lambda_1(d,s)$$

is monotone increasing. On the other hand, for any fixed $d \geq \dfrac{1}{\mu_1}$, the map

$$s \mapsto \lambda_1(d,s)$$

is monotone increasing.

Proof. The first statement is an easy consequence of equation (20), which implies that $\lambda_1(d,s)$ is monotone increasing with respect to d as $\lambda_1(1,s)$ is positive.

On the other hand, let $d\mu_1 \geq 1$ and $s_1 < s_2$. For any fixed u, we have

$$\lambda_1(d,s_1) \leq d^{s_1}((-\Delta_{\mathrm{N}})^{s_1}u,u) = \sum_k (d\mu_k)^{s_1}u_k^2 \leq \sum_k (d\mu_k)^{s_2}u_k^2 = d^{s_2}((-\Delta_{\mathrm{N}})^{s_2}u,u).$$

Recalling (13), we obtain that λ_1 is non-decreasing in s. To conclude, let us assume by contradiction that $\lambda_1(d,s_1) = \lambda_1(d,s_2)$, and let u denote the first eigenfunction associated with $\lambda_1(d,s_2)$. As $\int_\Omega mu^2 = 1$, by the above inequality we deduce that u achieves also $\lambda_1(d,s_1)$; thus $d\mu_1 = 1$ and $u_k = 0$ whenever $k \geq \nu_1 + 1$, where ν_1 is the multiplicity of μ_1 as a Neumann eigenvalue of $-\Delta$. As a consequence,

$$\lambda_1(d,s_1)mu = d^{s_1}(-\Delta_{\mathrm{N}})^{s_1}u = \sum_{k=1}^{\nu_1}(d\mu_1)^{s_1}u_k\phi_k = \sum_{k=1}^{\nu_1}u_k\phi_k = u,$$

yielding a contradiction since m is not constant. □

Proposition 3.3. *For any $0 < a < 1$, let*

$$\underline{d} = \min_{s\in[a,1]} \exp\frac{-\frac{\partial}{\partial s}\lambda_1(1,s)}{\lambda_1(1,s)}.$$

Then

$$\frac{\partial}{\partial s}\lambda_1(d,s) < 0 \qquad \text{for every } s \in [a,1] \text{ and } d < \underline{d}.$$

Proof. Differentiating (20) with respect to s we have

$$\frac{\partial}{\partial s}\lambda_1(d,s) = d^s\left[(\log d)\lambda_1(1,s) + \frac{\partial}{\partial s}\lambda_1(1,s)\right].$$

By Propositions 2.7 and 2.1, for any $a \in (0,1)$ the map $s \mapsto \lambda_1(1,s)$ is $C^1([a,1])$, and $\lambda_1(1,s) > 0$ for $s \in [a,1]$. Then, the conclusion easily follows. □

Remark 3.4. Analogously, one can show that, for any $0 < a < 1$,

$$\frac{\partial}{\partial s}\lambda_1(d,s) > 0 \text{ for every } s \in [a,1]$$

whenever

$$d > \overline{d} = \max_{s\in[a,1]} \exp \frac{-\frac{\partial}{\partial s}\lambda_1(1,s)}{\lambda_1(1,s)}.$$

Note that Proposition 3.2 implies that $\overline{d} \le 1/\mu_1$, for every $a \in (0,1)$, implying

$$\frac{\partial}{\partial s}\lambda_1(1,s) \ge \ln(\mu_1)\lambda_1(1,s), \qquad s \in (0,1].$$

This provides a uniform condition on d in order to have $\lambda_1(d,s)$ increasing for $s \in (0,1)$. On the other hand, we are not able to give an analogous uniform assumption implying the opposite monotonicity.

Proof of Proposition 1.1. Proposition 1.1 is an immediate consequence of Propositions 3.2 and 3.3. □

Next we turn to the study of the intermediate values of d, when the map $s \mapsto \lambda_1(d,s)$ has a transition in its monotonicity properties. In the following, we will denote the normalized first eigenfunction associated with $\lambda_1(d,s)$, which does not depend on d, as ψ_s:

$$d^s(-\Delta_{\mathrm{N}})^s\psi_s = \lambda_1(d,s)m\psi_s, \qquad \int_\Omega m\psi_s^2 = 1. \tag{21}$$

Lemma 3.5. *Let $\bar s \in (0,1)$, $0 < \varepsilon < \bar s$, and $w \in H^{\bar s+\varepsilon}(\Omega)$ be such that*

$$\int_\Omega m\psi_{\bar s} w = 0. \tag{22}$$

Then there exists a C^2 curve $u : (\bar s - \varepsilon, \bar s + \varepsilon) \mapsto H^{\bar s+\varepsilon}(\Omega)$ such that

$$u(\bar s) = \psi_{\bar s}, \qquad \dot u(\bar s) = w, \qquad \int_\Omega m u^2(t) = 1 \ \text{ for every } t. \tag{23}$$

Furthermore

$$\left(d^{\bar s}(-\Delta_{\mathrm{N}})^{\bar s}\psi_{\bar s}, \ddot u(\bar s)\right) = -\lambda_1(d,\bar s)\int_\Omega m w^2. \tag{24}$$

Proof. Let us define $\gamma(t) := \psi_{\bar s} + (t - \bar s)w$, then

$$u(t) := \frac{\gamma(t)}{\sqrt{\int_\Omega m\gamma^2(t)}}$$

satisfies all the requested properties.

Indeed, $u \in H^{\bar s+\varepsilon}(\Omega)$ by Proposition 2.4, and (21) yields the first equality in (23), while the third one holds by the definition of u. Moreover, (22) implies

$$\dot u(\bar s) = \frac{\dot\gamma(\bar s)}{\sqrt{\int_\Omega m\gamma^2(\bar s)}} - \frac{\gamma(\bar s)}{\left[\int_\Omega m\gamma^2(\bar s)\right]^{3/2}}\int_\Omega m\gamma(\bar s)\dot\gamma(\bar s) = w - \psi_{\bar s}\int_\Omega m\psi_{\bar s} w = w.$$

In addition, deriving twice the last equality in (23) we have

$$\int_\Omega m\dot u(t)^2 + \int_\Omega m u(t)\ddot u(t) = 0 \quad \forall t \in (\bar s - \varepsilon, \bar s + \varepsilon),$$

so that

$$\int_\Omega m\psi_{\bar s}\ddot u(\bar s) = -\int_\Omega m w^2.$$

This equality implies (24) when taking as test function $\ddot{u}(\bar{s})$ in (21). □

Taking into account the spectral decomposition of any $w \in H^{s+\varepsilon}(\Omega)$,

$$w = \sum_{k=0}^{\infty} w_k \phi_k,$$

and recalling (2), we can define the following operators, derivatives of $d^s(-\Delta_{\mathrm{N}})^s$ with respect to s:

$$\begin{aligned} L_s(w) &:= \partial_s \left[d^s(-\Delta_{\mathrm{N}})^s\right](w) = \sum_{k=1}^{\infty} (d\mu_k)^s \ln(d\mu_k) w_k \phi_k, \\ T_s(w) &:= \partial_{ss}^2 \left[d^s(-\Delta_{\mathrm{N}})^s\right](w) = \sum_{k=1}^{\infty} (d\mu_k)^s \ln^2(d\mu_k) w_k \phi_k. \end{aligned} \tag{25}$$

Lemma 3.6. *For ψ_s as in (21) there exists a unique $v \in H^s(\Omega)$, solution of the problem*

$$d^s(-\Delta_{\mathrm{N}})^s v = L_s \psi_s, \qquad \int_\Omega m v = 0. \tag{26}$$

Furthermore $v \in H^{s'}(\Omega)$ for every $s' < 2s$, and

$$\int_\Omega m v^2 = \max_{c \in \mathbb{R}} \int_\Omega m(c+v)^2.$$

Proof. Exploiting (2), the equation in (26) rewrites as

$$\sum_{k=1}^{\infty} (d\mu_k)^s v_k \phi_k = \sum_{k=1}^{\infty} (d\mu_k)^s \ln(d\mu_k) a_k \phi_k,$$

where v_k and a_k are the Fourier coefficients of v and ψ_s respectively. Such problem is solved in $L^2(\Omega)$ by

$$v_k = \ln(d\mu_k) a_k, \qquad k \geq 1, \tag{27}$$

and any $v_0 \in \mathbb{R}$. Moreover, Proposition 2.4 yields

$$\sum_{k=1}^{\infty} (d\mu_k)^{s'} v_k^2 = \sum_{k=1}^{\infty} (d\mu_k)^{s'} \ln^2(d\mu_k) a_k^2 \leq C + \sum_{k=1}^{\infty} (d\mu_k)^{2s} a_k^2 < +\infty,$$

so that $v \in H^{s'}(\Omega)$, for every $v_0 \in \mathbb{R}$. Finally, recalling (11) and hypothesis (4), we have that

$$\max_{v_0 \in \mathbb{R}} \int_\Omega m(v_0 + \tilde{v})^2$$

is uniquely achieved by

$$v_0 = -\frac{1}{m_0 |\Omega|} \int_\Omega m \tilde{v},$$

which also satisfies the second condition in (26). □

The construction of the function v will be crucial in proving the following result.

Theorem 3.7. *Let $d > 0$ be fixed and $\bar{s} \in (0,1)$ be such that*

$$\frac{\partial}{\partial s} \lambda_1(d, \bar{s}) = 0. \tag{28}$$

If

$$-\lambda_{-1}(1, \bar{s}) > \lambda_1(1, \bar{s}),$$

then $\bar{s}$ is a point of local maximum of the map $s \mapsto \lambda_1(d, s)$.

Proof. Let us first note that, using (21) and (28), we have

$$0 = \frac{\partial}{\partial s}(d^s(-\Delta_{\mathrm{N}})^s\psi_s,\psi_s)|_{s=\bar s} = 2\Big(d^{\bar s}(-\Delta_{\mathrm{N}})^{\bar s}\psi_{\bar s},\dot\psi_{\bar s}\Big) + (L_{\bar s}\psi_{\bar s},\psi_{\bar s}), \tag{29}$$

where $\dot\psi_s = (d/ds)\psi_s$ and L_s is defined in (25). We infer that

$$\begin{aligned}(L_{\bar s}\psi_{\bar s},\psi_{\bar s}) &= -2\Big(d^{\bar s}(-\Delta_{\mathrm{N}})^{\bar s}\psi_{\bar s},\dot\psi_{\bar s}\Big) = -2\lambda_1(d,\bar s)\int_\Omega m\psi_{\bar s}\dot\psi_{\bar s}\\ &= -\lambda_1(d,\bar s)\frac{d}{ds}\int_\Omega m\psi_s^2\Big|_{s=\bar s} = 0.\end{aligned}$$

For v as in Lemma 3.6, with $s=\bar s$, and $\alpha\in\mathbb{R}$, let $w=\alpha v$. We deduce that $w\in H^{\bar s+\varepsilon}(\Omega)$ for $\varepsilon>0$ small, and that

$$\lambda_1(d,\bar s)\int_\Omega m\psi_{\bar s}w = \alpha\big(d^{\bar s}(-\Delta_{\mathrm{N}})^{\bar s}v,\psi_{\bar s}\big) = \alpha(L_{\bar s}\psi_{\bar s},\psi_{\bar s}) = 0,$$

that is w satisfies (22). Thus Lemma 3.5 applies, and we denote with $u(s)$ the corresponding curve. Let us consider the map

$$f(s) := (d^s(-\Delta_{\mathrm{N}})^s u(s),u(s)),$$

that, thanks to (23), satisfies

$$f(s)\geq\lambda_1(d,s)\quad\text{and}\quad f(\bar s)=\lambda_1(d,\bar s).$$

Then it will be enough to show that $\bar s$ is a maximum point of f. By direct computation we obtain

$$\begin{aligned}f'(s) =&(L_s u(s),u(s)) + 2(d^s(-\Delta_{\mathrm{N}})^s\dot u(s),u(s)),\\ f''(s) =&(T_s u(s),u(s)) + 4(L_s\dot u(s),u(s)) + 2(d^s(-\Delta_{\mathrm{N}})^s\dot u(s),\dot u(s))\\ &+ 2(d^s(-\Delta_{\mathrm{N}})^s\ddot u(s),u(s))\end{aligned}$$

where T_s is defined in (25). Notice that (29) implies that $f'(\bar s)=0$. Recalling (27), we have

$$(T_{\bar s}\psi_{\bar s},\psi_{\bar s}) = \sum_k (d\mu_k)^{\bar s}\ln^2(d\mu_k)a_k^2 = (L_{\bar s}v,\psi_{\bar s}) = \big(d^{\bar s}(-\Delta_{\mathrm{N}})^{\bar s}v,v\big).$$

On the other hand, (24) yields

$$(d^s(-\Delta_{\mathrm{N}})^s\ddot u(s),u(s))|_{s=\bar s} = \big(d^{\bar s}(-\Delta_{\mathrm{N}})^{\bar s}\psi_{\bar s},\ddot u(\bar s)\big) = -\lambda_1(d,\bar s)\int_\Omega mw^2.$$

Recalling that $w=\alpha v$ and using (23), we obtain

$$f''(\bar s) = \big(d^{\bar s}(-\Delta_{\mathrm{N}})^{\bar s}v,v\big)\left[1+4\alpha+2\alpha^2\right] - 2\alpha^2\lambda_1(d,\bar s)\int_\Omega mv^2. \tag{30}$$

Now, in case the last integral in (30) is nonnegative, then choosing $\alpha=-1$ we obtain $f''(\bar s)<0$, namely $\bar s$ is a maximum point for f and the result follows. While, in case $\displaystyle\int_\Omega mv^2<0$, we take into account (26) and we exploit the definition of $\lambda_{-1}(1,s)$ in (15) to obtain

$$\int_\Omega mv^2 \geq \frac{1}{\lambda_{-1}(1,\bar s)}\big((-\Delta_{\mathrm{N}})^{\bar s}v,v\big).$$

Then, recalling Lemma 3.1, equation (30) becomes

$$f''(\bar s)\leq\big(d^{\bar s}(-\Delta_{\mathrm{N}})^{\bar s}v,v\big)\left[1+4\alpha+2\alpha^2\left(1-\frac{\lambda_1(1,\bar s)}{\lambda_{-1}(1,\bar s)}\right)\right]<0$$

when choosing

$$\alpha = -\left[1 - \frac{\lambda_1(1,\bar{s})}{\lambda_{-1}(1,\bar{s})}\right]^{-1}. \qquad \square$$

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we intend to apply Theorem 3.7, and to this aim we need to investigate the validity of the condition

$$-\lambda_{-1}(m,1,s) > \lambda_1(m,1,s) \tag{31}$$

in dependence on m. To do that, we restrict our attention to $m \in \widetilde{\mathcal{M}}$, defined in (7) and we first prove the following result.

Lemma 4.1. *Let M, ρ, δ be positive and fixed. For every $\sigma > 0$ there exists $A = A(M,\rho,\delta,\sigma) > 0$ such that if*

$$m \in \widetilde{\mathcal{M}}, \qquad \text{and} \qquad -A \le \int_\Omega m < 0,$$

then

$$\lambda_1(m,1,1) < \sigma.$$

Proof. Let $m \in \widetilde{\mathcal{M}}$, so that there exists $x_0 \in \Omega$ and $r > 0$ such that $m|_{B_\rho(x_0)} \ge \delta$, and let $\lambda_1^{\mathrm{Dir}}(B_\rho)$ (with $B_\rho = B_\rho(x_0)$) denote the first Dirichlet eigenvalue of $-\Delta$ in B_ρ, with eigenfunction $\omega \in H_0^1(B_\rho) \subset H^1(\Omega)$. Taking into account the equivalent expression of $\lambda(m,1,1)$ in term of the Rayleigh quotient and using (7), we infer

$$\lambda_1(m,1,1) \le \frac{\displaystyle\int_\Omega |\nabla\omega|^2}{\displaystyle\int_\Omega m\omega^2} \le \frac{\displaystyle\int_{B_\rho(x_0)} |\nabla\omega|^2}{\delta \displaystyle\int_{B_\rho(x_0)} \omega^2} = \frac{\lambda_1^{\mathrm{Dir}}(B_\rho)}{\delta}.$$

Now we argue by contradiction and suppose that $(m_n)_n$ is such that

$$(m_n)_n \subset \widetilde{\mathcal{M}}, \qquad \int_\Omega m_n \to 0^-, \qquad \lambda_1(m_n,1,1) =: \lambda_n \to \sigma > 0. \tag{32}$$

Since $\|m_n\|_{L^\infty} \le M$, we deduce that there exists $m \in L^p(\Omega)$, for every $1 \le p < +\infty$, with zero average and such that $m_n \rightharpoonup m_\infty$ in $L^p(\Omega)$ (up to subsequences). Furthermore, letting $m_n|_{B_\rho(x_n)} \ge \delta$, with $\mathrm{dist}(x_n, \partial\Omega) \ge \rho$, we can assume that, up to a subsequence, $x_n \to x_\infty \in \Omega$ so that

$$\delta|B_\rho| \le \int_{B_\rho(x_n)} m_n \to \int_{B_\rho(x_\infty)} m_\infty,$$

and we infer that $m_\infty \not\equiv 0$. For easier notation we denote the eigenfunction corresponding to λ_n by $\psi_n = \psi_{0,n} + \tilde{\psi}_n$, with $\psi_{0,n} \in \mathbb{R}$ and $\tilde{\psi}_n$ with zero average. Since $\|\nabla\tilde{\psi}_n\|_{L^2}^2 \to \sigma$ we obtain that $\tilde{\psi}_n$ is bounded in $H^1(\Omega)$. Without loss of generality we have two possibilities: either $\psi_{0,n} \to +\infty$, or it is bounded. In the first case, we have that

$$v_n = \frac{\psi_n}{\psi_{0,n}} = 1 + \frac{\tilde{\psi}_n}{\psi_{0,n}} \to 1 \qquad \text{strongly in } H^1(\Omega).$$

Passing to the limit in the equation satisfied by v_n we obtain $0 \equiv \sigma m_\infty$, a contradiction. On the other hand, when $\psi_{0,n}$ is bounded, we have that, up to subsequences, $\psi_n \to \psi_\infty$, weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. As a consequence

$$-\Delta\psi_\infty = \sigma m_\infty \psi_\infty, \qquad \int_\Omega m_\infty \psi_\infty^2 = 1. \tag{33}$$

From the second equality and recalling that $m_\infty \not\equiv 0$ we deduce that $\psi_\infty \not\equiv 0$. Since, by almost everywhere convergence, ψ_∞ is non-negative, the maximum principle and Hopf lemma imply that $\psi_\infty > 0$ in $\overline{\Omega}$. From (32) we deduce

$$0 = \sigma \int_\Omega m_\infty = \int_\Omega \nabla\psi_\infty \cdot \nabla \frac{1}{\psi_\infty} = -\int_\Omega \frac{|\nabla\psi_\infty|^2}{\psi_\infty^2},$$

hence ψ_∞ is constant, and finally, (33) yields $\sigma = 0$, a contradiction. □

Proof of Theorem 1.2. Let $m \in \widetilde{\mathcal{M}}$ and let $\psi_{-1,s}$ be an eigenfunction associated with $\lambda_{-1}(m,1,s)$ as in (15). Writing $\psi_{-1,s} = h + \tilde{\psi}_{-1,s}$, with $h \in \mathbb{R}$ and $\tilde{\psi}_{-1,s}$ with zero average, we have that, as usual,

$$\int_\Omega m\psi_{-1,s} = 0 \qquad \Longrightarrow \qquad h = -\frac{\displaystyle\int_\Omega m\tilde{\psi}_{-1,s}}{\displaystyle\int_\Omega m}.$$

Then

$$-1 = \int_\Omega m\psi_{-1,s}^2 = -\frac{\left(\displaystyle\int_\Omega m\tilde{\psi}_{-1,s}\right)^2}{\displaystyle\int_\Omega m} + \int_\Omega m\tilde{\psi}_{-1,s}^2 \geq \int_\Omega m\tilde{\psi}_{-1,s}^2.$$

Recalling (7) and the Poincaré inequality (19) we have

$$1 \leq \|m\|_{L^\infty} \left\|\tilde{\psi}_{-1,s}^2\right\|_{L_2}^2 \leq \frac{M}{\mu_1^s}((-\Delta_{\mathrm{N}})^s\psi_{-1,s}, \psi_{-1,s}) = -\lambda_{-1}(m,1,s)\cdot\frac{M}{\mu_1^s}.$$

As a consequence, we can choose $\sigma = \mu_1/M$ and we apply Proposition 2.10 and Lemma 4.1, to deduce

$$\lambda_1(m,1,s) \leq \mu_1^{s-1}\lambda_1(m,1,1) < \mu_1^{s-1}\sigma \leq -\lambda_{-1}(m,d,s),$$

for every $m \in \widetilde{\mathcal{M}}$ with $m_0 \geq -A$. Then Theorem 3.7 implies that, for any $d > 0$, the map $s \mapsto \lambda_1(m,d,s)$ has at most one critical point in $(0,1)$, hence it is well defined

$$\lambda_1(m,d,0^+) = \lim_{s\to 0^+} \lambda_1(m,d,s).$$

Furthermore, since any interior critical point is of maximum type, it follows that

$$\inf_{0<s\leq 1} \lambda_1(m,d,s) = \min\left\{\lambda_1(m,d,0^+), \lambda_1(m,d,1)\right\}.$$

Moreover, (20) implies

$$\lambda_1(m,d,0^+) = \lambda_1(m,1,0^+), \qquad \lambda_1(m,d,1) = d\lambda_1(m,1,1),$$

yielding the conclusion of the proof. □

We conclude this section by showing some numerical simulations that suggest that the map $s \mapsto \lambda_1(m,d,s)$ may not admit only interior maxima, in case condition (31) does not hold. In the square $\Omega = (0,\pi)\times(0,\pi) \subset \mathbb{R}^2$ we consider the two environments

$$m_1(x_1,x_2) := \begin{cases} 8 & x_1^2+x_2^2 < 1 \\ -1 & x_1^2+x_2^2 > 1, \end{cases} \qquad m_2(x_1,x_2) := \begin{cases} 1 & x_1^2+x_2^2 < 1 \\ -1 & x_1^2+x_2^2 > 1. \end{cases} \tag{34}$$

These particular choices of m are motivated by the discussion in Section 5 ahead, in particular by Remark 5.4. Notice that, with both choices, $m \in \widetilde{\mathcal{M}}$. For the two possibilities, the eigenvalues $\lambda_1(m,1,s)$, $\lambda_{-1}(m,1,s)$ are numerically evaluated, by truncating the Fourier series, for $s \in \{i/100 : i = 1,\dots,100\}$. As shown in Figure 1, condition (31) is satisfied for $m = m_1$, whereas it does not hold when $m = m_2$. However, notice also that in this case $\mu_1 = \mu_2 = 1$ (achieved by

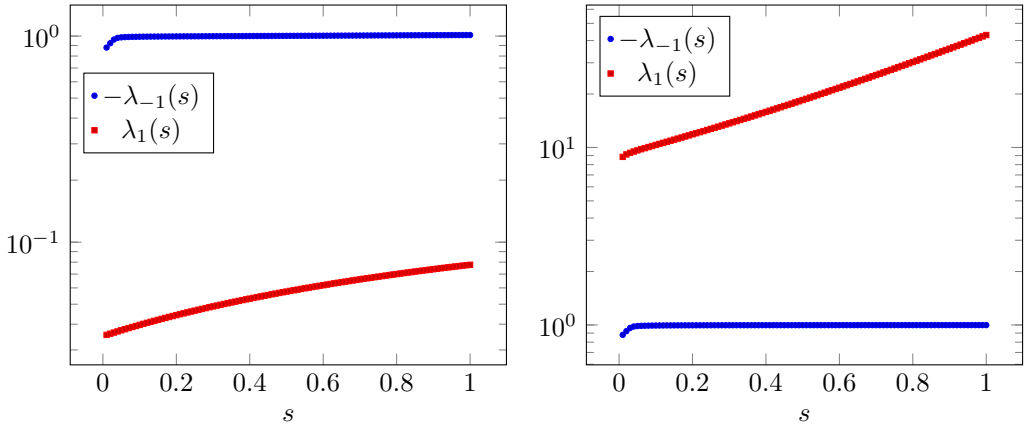

FIGURE 1. Testing condition (31) for the model environment (34) with $m = m_1$ (on the left) and $m = m_2$ (on the right).

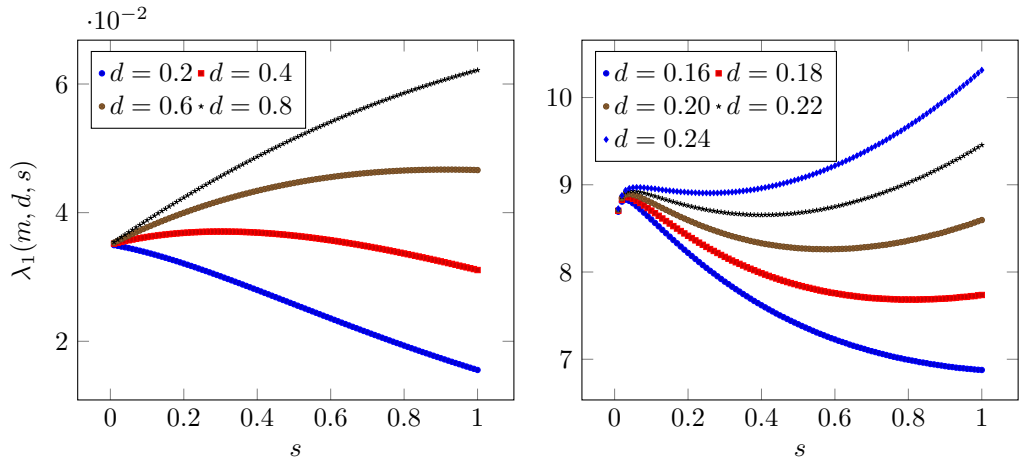

FIGURE 2. The map $s \mapsto \lambda_1(m, d, s)$, for several values of d, with $m = m_1$ (on the left) and $m = m_2$ (on the right) both given in (34).

$\phi_1(x) = \cos x_1$, $\phi_2(x) = \cos x_2$), so that we are in the situation described by the first conclusion of Proposition 1.1, namely, all the graphs are increasing in s. Then in this case the minimum of $\lambda_1(m, 1, s)$ is achieved in $\lambda_1(m, 1, 0^+)$, no matter whether or not condition (31) is verified.

In Figure 2, $\lambda_1(m, 1, s) = d^s \lambda_1(m, 1, s)$ is plotted for different choices of the motility coefficient $d < 1$. When $m = m_1$ (so that condition (31) is satisfied) it is possible to observe the transition of the behaviour of $\lambda_1(m, 1, s)$ from decreasing to increasing (while d increases) developing in the meanwhile a critical point of maximum type. But, when $m = m_2$, in which case condition (31) is violated, $\lambda_1(m, 1, s)$ develops also critical points of minimum type, while moving from decreasing to increasing.

5. Optimization on m

In this section we will briefly analyse the optimization of $\lambda_1(m,d,s)$ with respect to m. In this analysis it is convenient to fix $\underline{m}$, $\overline{m} \in \mathbb{R}^+$, $m_0 \in (-\infty, 0)$ and take m in the following class.

$$m \in \overline{\mathcal{M}} := \left\{ -\underline{m} \le m(x) \le \overline{m},\ \int_\Omega m = m_0|\Omega|,\ m^+ \not\equiv 0 \right\}. \tag{35}$$

Remark 5.1. When m satisfies (35), condition (18) can be rewritten as

$$\lambda_1(m,d,s) \ge \frac{d^s \mu_1^s |m_0|}{\overline{m}|m_0| + [\max(\overline{m},\underline{m})]^2}.$$

The following result is proved in [26, 13, 27].

Lemma 5.2. *Let $f \in L^1(\Omega)$. Then the maximization problem $\sup_{m\in\overline{\mathcal{M}}} \int_\Omega fm$ is solved by*

$$m = \overline{m}\chi_D - \underline{m}\chi_{D^c}$$

for some subsets $D \subset \Omega$, $D^c = \Omega \setminus D$, such that

$$|D| = |\Omega| \frac{\underline{m} + m_0}{\underline{m} + \overline{m}}. \tag{36}$$

Theorem 5.3. *For every $d > 0$ and s fixed, there exists $\underline{\lambda}_1(d,s)$ solution of the minimization problem*

$$\underline{\lambda}_1(d,s) = \inf_{\overline{\mathcal{M}}} \lambda_1(m,d,s)$$

Moreover, $\underline{\lambda}_1(d,s)$ is achieved by $m = \overline{m}\chi_D - \underline{m}\chi_{D^c}$, for some $D \subset \Omega$, independent of d, which satisfies (36).

Proof. Notice that, for every $m \in \overline{\mathcal{M}}$, Lemma 5.2 implies

$$\begin{aligned} \lambda_1(m,d,s) = d^s \frac{((-\Delta)^s\psi_1,\psi_1)}{\int_\Omega m\psi_1^2} &\ge d^s \frac{((-\Delta)^s\psi_1,\psi_1)}{\int_\Omega (\overline{m}\chi_D - \underline{m}\chi_{D^c})\,\psi_1^2} \\ &\ge \lambda_1(\overline{m}\chi_D - \underline{m}\chi_{D^c}, d, s), \end{aligned}$$

for D satisfying (36). Since $\overline{m}\chi_D - \underline{m}\chi_{D^c} \in \overline{\mathcal{M}}$ the conclusion follows. □

Remark 5.4. Once Theorem 5.3 is proved, it is natural to deepen the knowledge of the favorable region D, in particular, to wonder whether or not D is connected, as this is related to the detection of possible fragmentation of the optimal environment. The connectedness of D has been obtained in the local diffusion case, for $N = 1$, in [11, 27, 13]. This line of research has been pursued in higher dimension in [33], where a sharp analysis of the optimal environment is performed in the standard diffusion case $s = 1$. In particular, when Ω is a bi-dimensional rectangle, by combining monotone Steiner rearrangements and numerical simulations it appears that D and $\Omega \setminus D$ can be of two main types: ball-shaped or stripe-shaped. In addition, when the ratio $|D|/|\Omega \setminus D|$ is sufficiently small, it appears that D should be a quarter of circle, centered in one of the corners of Ω. By using symmetrization arguments on the extension problem (Remark 2.2), we expect that part of such analysis may be carried also to the case $s < 1$, even though this falls beyond the scope of the present paper.

6. The periodic problem

Let the spectral periodic fractional Laplacian $(-\Delta_{\rm per})^s$ be defined as in (10). The aim of this section is to put in evidence that our problem and the periodic one enjoy the same underlying structure. To start with we prove that, on periodic functions, it coincides with the fractional Laplacian on the full space $(-\Delta_{\mathbb{R}^N})^s$ (see (1)). For easier notation, let us fix $l_1 = \cdots = l_N = 2\pi$, i.e. $\mathcal{C}_l = (0,2\pi)^N$. Using complex notation, we have that the periodic eigenfunctions of $-\Delta$ in $(0,2\pi)^N$ are the functions $\varphi_k(x) = e^{-ik\cdot x}$, indexed by $k \in \mathbb{Z}^N$ and corresponding to the eigenvalues $\nu_k = |k|^2$. Let u be $(0,2\pi)^N$-periodic; up to normalization factors we obtain

$$u(x) = \sum_{k\in\mathbb{Z}^N} u_k e^{-ik\cdot x}, \qquad \text{where } u_k = \int_{\mathcal{C}_l} u(x) e^{-ik\cdot x}\,dx, \tag{37}$$

and consequently

$$(-\Delta_{\rm per})^s u(x) = \sum_{k\in\mathbb{Z}^N} |k|^{2s} u_k e^{-ik\cdot x}.$$

Proof of Proposition 1.3. Let $f\colon \mathbb{R}\to\mathbb{R}$ be continuous and 2π-periodic; then f belongs to the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions, and it is well known (see e.g. [16, Ch. II]) that its Fourier transform is, up to normalization factors,

$$\hat f(\xi) = \sum_{k\in\mathbb{Z}} f_k \delta_1(\xi - k), \quad \xi\in\mathbb{R}, \qquad \text{where } f_k = \int_0^{2\pi} f(x) e^{-ikx}\,dx,$$

and δ_n denotes the Dirac delta in $\mathbb{R}^n$, $n\geq 1$. Indeed, it suffices to transform both sides of the identity

$$f(x) = \sum_{k\in\mathbb{Z}} f_k e^{-ikx},$$

which holds true in $\mathcal{S}'(\mathbb{R})$. Recalling that

$$\delta_N(x_1,\dots,x_N) = \delta_1(x_1)\otimes\cdots\otimes\delta_1(x_N),$$

it is not difficult to generalize the above formula to the N-dimensional setting, obtaining that, if u is $(0,2\pi)^N$-periodic, then

$$\hat u(\xi) = \sum_{k\in\mathbb{Z}^N} u_k \delta_N(\xi - k), \quad \xi\in\mathbb{R}^N,$$

and u_k are given in (37). Exploiting (1), (10) and recalling that $\nu_k = |k|^2$, we obtain:

$$\widehat{(-\Delta_{\mathbb{R}^N})^s u}(\xi) = |\xi|^{2s}\hat u(\xi) = \sum_{k\in\mathbb{Z}^N} |\xi|^{2s} u_k \delta_N(\xi-k) = \sum_{k\in\mathbb{Z}^N} |k|^{2s} u_k \delta_N(\xi-k) = \widehat{(-\Delta_{\rm per})^s u}(\xi),$$

and the desired result follows. □

Once the equivalence between $(-\Delta_{\rm per})^s$ and $(-\Delta_{\mathbb{R}^N})^s$ is established, one can easily repeat the arguments introduced for the Neumann case, because $(-\Delta_{\rm per})^s$ is a spectral operator as pointed out in (10): since $\nu_0 = 0$ and $\nu_k\to+\infty$ as $|k|\to\infty$, these arguments are exactly the same, except for two minor points, namely:

(1) the use of the regularity results from [8] we did in Section 2: following the arguments in [8], these results can be proved also for $(-\Delta_{\rm per})^s$, even though in this case it is much easier to use the regularity theory for $(-\Delta_{\mathbb{R}^N})^s$, which is well established in [5];
(2) the results in Section 5 about the optimization with respect to m: also in this case only minor changes are needed; we refer the interested reader to [33], where the relation between the Neumann case and the periodic one has been deeply analyzed, for $s = 1$.

On the contrary, Sections 3 and 4 can be rewritten in the periodic case without any change but replacing $(\mu_k, \phi_k)_{k\in\mathbb{Z}}$ with $(\nu_k, \varphi_k)_{k\in\mathbb{Z}^N}$. In particular, the proof of Theorem 1.4 follows as well.

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