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Normalized bound states for the nonlinear Schrödinger equation in bounded domains

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Abstract

Given $\rho > 0$, we study the elliptic problem

$$\text{find } (U, \lambda) \in H_0^1(\Omega) \times \mathbb{R} \text{ such that } \begin{cases} -\Delta U + \lambda U = |U|^{p-1}U \\ \int_\Omega U^2 \, dx = \rho, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $p > 1$ is Sobolev-subcritical, searching for conditions (about ρ, N and p) for the existence of solutions. By the Gagliardo-Nirenberg inequality it follows that, when p is L^2-subcritical, i.e. $1 < p \leq 1 + 4/N$, the problem admits solution for every $\rho > 0$. In the L^2-critical and supercritical case, i.e. when $1 + 4/N \leq p < 2^* - 1$, we show that, for any $k \in \mathbb{N}$, the problem admits solutions having Morse index bounded above by k only if ρ is sufficiently small. Next we provide existence results for certain ranges of ρ, which can be estimated in terms of the Dirichlet eigenvalues of $-\Delta$ in $H_0^1(\Omega)$, extending to general domains and to changing sign solutions some results obtained in [21] for positive solutions in the ball.

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1 Introduction

Given $\rho > 0$, we consider the problem

$$\begin{cases} -\Delta U + \lambda U = |U|^{p-1}U & \text{in } \Omega, \\ \int_\Omega U^2 \, dx = \rho, \quad U = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz, bounded domain, $1 < p < 2^* - 1$, $\rho > 0$ is a fixed parameter, and both $U \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$ are unknown. More precisely, we investigate conditions on p and ρ (and also Ω) for the solvability of the problem.

The main interest in (1.1) relies on the investigation of standing wave solutions for the nonlinear Schrödinger equation

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + |\Phi|^{p-1}\Phi = 0, \qquad (t, x) \in \mathbb{R} \times \Omega$$

with Dirichlet boundary conditions on $\partial\Omega$. This equation appears in several different physical models, both in the case $\Omega = \mathbb{R}^N$ [6], and on bounded domains [16]. In particular, the latter case appears in nonlinear optics and in the theory of Bose-Einstein condensation, also as a limiting case of the equation on $\mathbb{R}^N$ with confining potential. When searching for solutions having the wave function Φ factorized as $\Phi(x,t) = e^{i\lambda t}U(x)$, one obtains that the real valued function U must solve

$$-\Delta U + \lambda U = |U|^{p-1}U, \qquad U \in H_0^1(\Omega), \tag{1.2}$$

and two points of view are available. The first possibility is to assign the chemical potential $\lambda \in \mathbb{R}$, and search for solutions of (1.2) as critical points of the related action functional. The literature concerning this approach is huge and we do not even make an attempt to summarize it here. On the contrary, we focus on the second possibility, which consists in considering λ as part of the unknown and prescribing the mass (or charge) $\|U\|^2_{L^2(\Omega)}$ as a natural additional condition. Up to our knowledge, the only previous paper dealing with this case, in bounded domains, is [21], which we describe below. The problem of searching for normalized solutions in $\mathbb{R}^N$, with non-homogeneous nonlinearities, is more investigated [4, 18], even though the methods used there can not be easily extended to bounded domains, where dilations are not allowed. Very recently, also the case of partial confinement has been considered [5].

Solutions of (1.1) can be identified with critical points of the associated energy functional

$$\mathcal{E}(U) = \frac{1}{2}\int_\Omega |\nabla U|^2\,dx - \frac{1}{p+1}\int_\Omega |U|^{p+1}\,dx$$

restricted to the mass constraint

$$\mathcal{M}_\rho = \{U \in H_0^1(\Omega) : \|U\|_{L^2(\Omega)} = \rho\},$$

with λ playing the role of a Lagrange multiplier.

A cricial role in the discussion of the above problem is played by the Gagliardo-Nirenberg inequality: for any Ω and for any $v \in H_0^1(\Omega)$,

$$\|v\|^{p+1}_{L^{p+1}(\Omega)} \le C_{N,p}\|\nabla v\|^{N(p-1)/2}_{L^2(\Omega)}\|v\|^{(p+1)-N(p-1)/2}_{L^2(\Omega)}, \tag{1.3}$$

the equality holding only when $\Omega = \mathbb{R}^N$ and $v = Z_{N,p}$, the positive solution of $-\Delta Z + Z = Z^p$ (which is unique up to translations [19]). Accordingly, the exponent p can be classified in relation with the so called L^2-*critical exponent* $1 + 4/N$ (throughout all the paper, p will be always Sobolev-subcritical and its criticality will be understood in the L^2 sense). Indeed we have that $\mathcal{E}$ is bounded below and coercive on $\mathcal{M}_\rho$ if and only if either p is subcritical, or it is critical and ρ is sufficiently small.

The recent paper [21] deals with problem (1.1) in the case of the spherical domain $\Omega = B_1$, when searching for positive solutions U. In particular, it is shown that the solvability of (1.1) is strongly influenced by the exponent p, indeed:

- in the subcritical case $1 < p < 1 + 4/N$, (1.1) admits a unique positive solution for every $\rho > 0$;
- if $p = 1 + 4/N$ then (1.1) admits a unique positive solution for

  $$0 < \rho < \rho^* = \left(\frac{p+1}{2C_{N,p}}\right)^{N/2} = \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)},$$

  and no positive solutions for $\rho \ge \rho^*$;
- finally, in the supercritical regime $1 + 4/N < p < 2^* - 1$, (1.1) admits positive solutions if and only if $0 < \rho \le \rho^*$ (the threshold ρ^* depending on p), and such solutions are at least two for $\rho < \rho^*$.

In this paper we carry on such analysis, dealing with a general domain Ω and with solutions which are not necessarily positive. More precisely, let us recall that for any U solving (1.1) for some λ, it is well-defined the Morse index

$$m(U) = \max \left\{ k : \begin{array}{l} \exists V \subset H_0^1(\Omega), \dim(V) = k : \forall v \in V \setminus \{0\} \\ \displaystyle\int_\Omega |\nabla v|^2 + \lambda v^2 - p|U|^{p-1} v^2 \, dx < 0 \end{array} \right\} \in \mathbb{N}.$$

Then, if $\Omega = B_1$, it is well known that a solution U of (1.1) is positive if and only if $m(U) = 1$. Under this perspective, the results in [21] can be read in terms of Morse index one–solutions, rather than positive ones: introducing the sets of admissible masses

$$\mathfrak{A}_k = \mathfrak{A}_k(p, \Omega) := \left\{ \rho > 0 : \begin{array}{l} \text{(1.1) admits a solution } U \text{ (for some } \lambda) \\ \text{having Morse index } m(U) \le k \end{array} \right\},$$

then [21] implies that $\mathfrak{A}_1(p, B_1)$ is a bounded interval if and only if p is critical or supercritical, while $\mathfrak{A}_1(p, B_1) = \mathbb{R}^+$ in the subcritical case. On the contrary, when considering general domains and higher Morse index, the situation may become much more complicated. We collect some examples in the following remark.

Remark 1.1. In the case of a symmetric domain, one can use any solution as a building block to construct other solutions with a more complex behavior, obtaining the so-called necklace solitary waves. Such kind of solutions are constructed in [17], even though in such paper the focus is on stability, rather than on normalization conditions. For instance, by scaling argument, any Dirichlet solution of $-\Delta U + \lambda U = |U|^{p-1}U$ in a rectangle $R = \prod_{i=1}^N (a_i, b_i)$ can be scaled to a solution of $-\Delta U + k^2 \lambda U = |U|^{p-1}U$ in R/k, $k \in \mathbb{N}_+$, and then k^N copies of it can be juxtaposed, with alternating sign. In this way one obtains a new solution on R having $k^{4/(p-1)}$ times the mass of the starting one, and eventually solutions in R with arbitrarily high mass (but with higher Morse index) can be constructed even in the critical and supercritical case. An analogous construction can be performed in the disk, using solutions in circular sectors as building blocks, even though in this case explicit bounds on the mass obtained are more delicate. Also, instead of symmetric domains, singular perturbed ones can be considered, such as dumbbell domains [10]: for instance, using [22, Theorem 3.5], one can show that for any k, there exists a domain Ω, which is close in a suitable sense to the disjoint union of k domains, such that (1.1) has a *positive* solution on Ω with Morse index k and $\rho = \rho_k \to +\infty$ as $k \to +\infty$. This kind of results justifies the choice of classifying the solutions in terms of their Morse index, rather than in terms of their nodal properties.

Motivated by the previous remark, the first question we address in this paper concerns the boundedness of $\mathfrak{A}_k$. We provide the following complete classification.

Theorem 1.2. *For every $\Omega \subset \mathbb{R}^N$ bounded C^1 domain, $k \ge 1$, $1 < p < 2^* - 1$,*

$$\sup \mathfrak{A}_k(p, \Omega) < +\infty \qquad \Longleftrightarrow \qquad p \ge 1 + \frac{4}{N}.$$

The proof of such result, which is outlined in Section 2, is obtained by a detailed blow-up analysis of sequences of solutions with bounded Morse index, via suitable a priori pointwise estimates (see [12]). In this respect, the regularity assumption on $\partial\Omega$ simplifies the treatment of possible concentration phenomena towards the boundary. The argument, which holds for solutions which possibly change sign, is inspired by [13], where the case of positive solutions is treated.

Once Theorem 1.2 is established, in case $p \ge 1 + 4/N$ two questions arise, namely:

1. is it possible to provide lower bounds for $\sup \mathfrak{A}_k$? Is it true that $\sup \mathfrak{A}_k$ is strictly increasing in k, or, at least, that $\sup \mathfrak{A}_k > \sup \mathfrak{A}_1$ for some k?

2. is (1.1) solvable for every $\rho \in (0, \sup \mathfrak{A}_k)$, or at least can we characterize some subinterval of solvability?

It is clear that both issues can be addressed by characterizing values of ρ for which existence (and multiplicity) of solutions with bounded Morse index can be guaranteed. To this aim, it can be useful to restate problem (1.1) as

$$\begin{cases} -\Delta u + \lambda u = \mu |u|^{p-1} u & \text{in } \Omega, \\ \int_\Omega u^2 \, dx = 1, \quad u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{where} \qquad \begin{cases} U = \sqrt{\rho} u \\ \mu = \rho^{(p-1)/2}, \end{cases} \tag{1.4}$$

where now $\mu > 0$ is prescribed. Since

$$\text{both } \mathcal{E}_\mu(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\mu}{p+1} \int_\Omega |u|^{p+1} \qquad \text{and } \mathcal{M} = \mathcal{M}_1 = \{u : \|u\|_{L^2(\Omega)} = 1\} \tag{1.5}$$

are invariant under the $\mathbb{Z}_2$-action of the involution $u \mapsto -u$, solutions of (1.4) can be found via min-max principles in the framework of index theories (see e.g. [24, Ch. II.5]). Notice that in the supercritical case $\mathcal{E}_\mu$ is not bounded from below on $\mathcal{M}$. Following [21], it can be convenient to parameterize solutions to (1.4) with respect to the H^1_0-norm, therefore we introduce the sets

$$\mathcal{B}_\alpha := \left\{ u \in \mathcal{M} : \int_\Omega |\nabla u|^2 \, dx < \alpha \right\}, \qquad \mathcal{U}_\alpha := \left\{ u \in \mathcal{M} : \int_\Omega |\nabla u|^2 \, dx = \alpha \right\}. \tag{1.6}$$

Introducing the first Dirichlet eigenvalue of $-\Delta$ in $H^1_0(\Omega)$, $\lambda_1(\Omega)$, we have that the sets above are non-empty whenever $\alpha > \lambda_1(\Omega)$. Since we are interested in critical points having Morse index bounded from above, following [3, 20, 23] we introduce the following notion of genus.

Definition 1.3. Let $A \subset H^1_0(\Omega)$ be a closed set, symmetric with respect to the origin (i.e. $-A = A$). We define the *genus* γ of a A as

$$\gamma(A) := \sup\{m : \exists h \in C(\mathbb{S}^{m-1}; A),\ h(-u) = -h(u)\}.$$

Furthermore, we define

$$\Sigma_\alpha = \{A \subset \overline{\mathcal{B}}_\alpha : A \text{ is closed and } -A = A\}, \qquad \Sigma_\alpha^{(k)} = \{A \in \Sigma_\alpha : \gamma(A) \geq k\},$$

We remark that this notion of genus is different from the classical one of *Krasnoselskii genus*, which is well suited for estimates of the Morse index from below, rather than above. Nonetheless, γ shares with the Krasnoselskii genus most of the main properties of an index [9, 26]. In particular, by the Borsuk-Ulam Theorem, any set A homeomorphic to the sphere $\mathbb{S}^{m-1} := \partial B_1 \subset \mathbb{R}^m$ has genus $\gamma(A) = m$. Furthermore, we show in Section 3 that $\Sigma_\alpha^{(k)}$ is not empty, provided $\alpha > \lambda_k(\Omega)$ (the k-th Dirichlet eigenvalue of $-\Delta$ in $H^1_0(\Omega)$).

Equipped with this notion of genus we provide two different variational principles for solutions of (1.4) (and thus of (1.1)). The first one is based on a variational problem with *two constraints*, which was exploited as the main tool in proving the results in [21].

Theorem 1.4. *Let $k \geq 1$ and $\alpha > \lambda_k(\Omega)$. Then*

$$M_{\alpha,k} := \sup_{A \in \Sigma_\alpha^{(k)}} \inf_{u \in A} \int_\Omega |u|^{p+1} \tag{1.7}$$

is achieved on $\mathcal{U}_\alpha$, and there exists a critical point $u_\alpha \in \mathcal{M}$ such that, for some $\lambda_\alpha \in \mathbb{R}$ and $\mu_\alpha > 0$,

$$\int_\Omega |\nabla u_\alpha|^2 = \alpha \qquad \textit{and} \qquad -\Delta u_\alpha + \lambda_\alpha u_\alpha = \mu_\alpha |u_\alpha|^{p-1} u_\alpha \quad \textit{in } \Omega. \tag{1.8}$$

As a matter of fact, the results in [21] were obtained by a detailed analysis of the map $\alpha \mapsto \mu_\alpha$ in the case $k = 1$, i.e. when dealing with

$$M_{\alpha,1} = \max\left\{\|u\|_{L^{p+1}}^{p+1} : \|u\|_{L^2}^2 = 1,\ \|\nabla u\|_{L^2}^2 = \alpha\right\}.$$

In the present paper we do not investigate the properties of the map $\alpha \mapsto \mu_\alpha$ for general k, but we rather prefer to exploit the characterization of $M_{\alpha,k}$ in connection with a second variational principle, which deals with only *one constraint*.

Theorem 1.5. *Let $1 + N/4 \le p < 2^* - 1$. There exists a sequence $(\hat\mu_k)_k$ (depending on Ω and p) such that, for every $k \ge 1$ and $0 < \mu < \hat\mu_k$, the value*

$$c_k := \inf_{A \in \Sigma_\alpha^{(k)}} \sup_A \mathcal{E}_\mu, \tag{1.9}$$

is achieved in $\mathcal{B}_\alpha$, for a suitable $\alpha > \lambda_k(\Omega)$. Furthermore there exists a critical point $u_\mu \in \mathcal{M}$ such that, for some $\lambda_\mu \in \mathbb{R}$,

$$-\Delta u_\mu + \lambda_\mu u_\mu = \mu |u_\mu|^{p-1} u_\mu \quad \text{in } \Omega,$$

$\|\nabla u\|_{L^2}^2 < \alpha$, and $m(u_\mu) \le k$.

Remark 1.6. Of course, if $p < 1 + 4/N$, the above theorem holds with $\hat\mu_k = +\infty$ for every k.

Corollary 1.7. *Let $\hat\rho_k := \hat\mu_k^{2/(p-1)}$. Then*

$$(0, \hat\rho_k) \subset \mathfrak{A}_k.$$

The link between Theorem 1.4 and Theorem 1.5 is that we can provide explicit estimates of $\hat\mu_k$ (and hence of $\hat\rho_k$) in terms of the map $\alpha \mapsto M_{\alpha,k}$ (see Section 4).

We stress that the above results hold for any Lipschitz Ω. As a first consequence, this allows to extend the existence result in [21] to non-radial domains.

Theorem 1.8. *For every $0 < \rho < \hat\rho_1 = \hat\rho_1(\Omega, p)$ problem (1.1) admits a solution which is a local minimum of the energy $\mathcal{E}$ on $\mathcal{M}_\rho$. In particular, U is positive, has Morse index one and the associated solitary wave is orbitally stable.*

Furthermore, for every Lipschitz Ω,

- $1 < p < 1 + \dfrac{4}{N} \implies \hat\rho_1(\Omega, p) = +\infty$,
- $p = 1 + \dfrac{4}{N} \implies \hat\rho_1(\Omega, p) \ge \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2$,
- $1 + \dfrac{4}{N} < p < 2^* - 1 \implies \hat\rho_1(\Omega, p) \ge D_{N,p} \lambda_1(\Omega)^{\frac{2}{p-1} - \frac{N}{2}}$,

where the universal constant $D_{N,p}$ is explicitly written in terms of N and p in Section 4.

Remark 1.9. Of course, in the subcritical and critical cases, c_1 is actually a global minimum. Furthermore, the lower bound for the supercritical case agrees with that of the critical one since, as shown in Section 4, $D_{N,1+4/N} = \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2$ (and $\lambda_1(\Omega)$ is raised to the 0^{th}-power). Notice that the estimate for the supercritical case is new also in the case $\Omega = B_1$.

We observe that the exponent of $\lambda_1(\Omega)$ in the supercritical threshold is negative, therefore such threshold decreases with the size of Ω.

Once the first thresholds have been estimated, we turn to the higher ones: by exploiting the relations between $M_{\alpha,k}$ and c_k, we can show that the thresholds obtained for Morse index one–solutions in Theorem 1.8 can be increased, by considering higher Morse index–solutions, at least for some exponent.

Proposition 1.10. *For every Ω and $1 < p < 2^* - 1$,*

$$\hat{\rho}_3\,(\Omega, p) \geq 2 \cdot D_{N,p} \lambda_3(\Omega)^{\frac{2}{p-1} - \frac{N}{2}}.$$

Remark 1.11. In the critical case, the lower bound for $\hat{\rho}_3$ provided by Proposition 1.10 is twice that for $\hat{\rho}_1$ obtained in Theorem 1.8. By continuity, the estimate for $\hat{\rho}_3$ is larger than that for $\hat{\rho}_1$ also when p is supercritical, but not too large. To quantify such assertion, we can use Yang's inequality [2, 8], which implies that for every Ω it holds

$$\lambda_3(\Omega) \leq \left(1 + \frac{N}{4}\right) 2^{2/N} \lambda_1(\Omega).$$

We deduce that $2 \cdot D_{N,p} \lambda_3(\Omega)^{\frac{2}{p-1} - \frac{N}{2}} \geq D_{N,p} \lambda_1(\Omega)^{\frac{2}{p-1} - \frac{N}{2}}$ whenever

$$p \leq 1 + \frac{4}{N} + \frac{8}{N^2 \log_2 \left(1 + \frac{4}{N}\right)}.$$

In particular, the physically relevant case $N = 3$, $p = 3$ is covered. Furthermore, if $N \geq 7$, the above condition holds for every $p < 2^* - 1$.

Beyond existence results for (1.1), also multiplicity results can be achieved. A first general consideration, with this respect, is that Theorem 1.5 holds true also when using the standard Krasnoselskii genus instead of γ; this allows to obtain critical points having Morse index bounded from below (see [3, 20, 23]), and therefore to obtain infinitely many solutions, at least when ρ is less than some threshold. More specifically, we can also prove the existence of a second solution in the supercritical case, thus extending to any Ω the multiplicity result obtained in [21] for the ball. Indeed, on the one hand, in the supercritical case $\mathcal{E}_\mu$ is unbounded from below; on the other hand the solution obtained in Theorem 1.5, for $k = 1$, is a local minimum. Thus the Mountain Pass Theorem [1] applies on $\mathcal{M}$, and a second solution can be found for $\mu < \hat{\mu}_1$, see Proposition 4.4 for further details (and also Remark 4.5 for an analogous construction for $k \geq 2$).

To conclude this introduction, let us mention that the explicit lower bounds obtained in Theorem 1.8 can be easily applied in order to gain much more information also in the case of special domains, as those considered in Remark 1.1. For instance, we can prove then following.

Theorem 1.12. *Let $\Omega = B$ be a ball in $\mathbb{R}^N$. Then*

$$p < 1 + \frac{4}{N-1} \quad \implies \quad (1.1) \textit{ admits a solution for every } \rho > 0.$$

An analogous result holds when $\Omega = R$ is a rectangle, without further restrictions on $p < 2^* - 1$.

Therefore our starting problem in $\Omega = B$ can be solved for any mass value also in the critical and supercritical regime, at least for p smaller than this further critical exponent $1 + 4/(N-1) > 1 + 4/N$. Of course, higher masses require higher Morse index–solutions. In particular, since by [21] we know that $\mathfrak{A}_1(B, 1 + 4/N) = (0, \|Z_{N,p}\|_{L^2})$, we have that for larger masses, even though no positive solution exists, nodal solutions with higher Morse index can be obtained: in such cases (1.1) admits *nodal ground states with higher Morse index*.

The paper is structured as follows: in Section 2 we perform a blow-up analysis of solutions with bounded Morse index, in order to prove Theorem 1.2; Section 3 is devoted to the analysis of the variational problem with two constraints (1.7) and to the proof of Theorem 1.4; that of Theorems 1.5, 1.8 and Proposition 1.10 is developed in Section 4, by means of the variational problem with one constraint (1.9); finally, Section 5 contains the proof of Theorem 1.12.

Notation. We use the standard notation $\{\varphi_k\}_{k\geq 1}$ for a basis of eigenfunctions of the Dirichlet laplacian in Ω, orthogonal in $H^1_0(\Omega)$ and orthonormal in $L^2(\Omega)$. Such functions are ordered in such a way that the corresponding eigenvalues $\lambda_k(\Omega)$ satisfy

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots,$$

and φ_1 is chosen to be positive on Ω. $C_{N,p}$ denotes the universal constant in the Gagliardo-Nirenberg inequality (1.3), which is achieved (uniquely, up to translations and dilations) by the positive, radially symmetric function $Z_{N,p} \in H^1(\mathbb{R}^N)$, with

$$\|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)} = \left(\frac{p+1}{2C_{N,p}}\right)^{N/2}.$$

Finally, C denotes every (positive) constant we need not to specify, whose value may change also within the same formula.

2 Blow-up analysis of solutions with bounded Morse index

Throughout this section we will deal with a sequence $\{(u_n, \mu_n, \lambda_n)\}_n \subset H^1_0(\Omega) \times \mathbb{R}^+ \times \mathbb{R}$ satisfying

$$-\Delta u_n + \lambda_n u_n = \mu_n |u_n|^{p-1} u_n, \qquad \int_\Omega u_n^2\,dx = 1, \qquad \int_\Omega |\nabla u_n|^2\,dx =: \alpha_n. \tag{2.1}$$

To start with, we recall the following result (actually, in [21], the result is stated for positive solution, but the proof does not require such assumption).

Lemma 2.1 ([21, Lemma 2.5])**.** *Take a sequence $\{(u_n, \mu_n, \lambda_n)\}_n$ as in (2.1). Then*

$$\{\alpha_n\}_n \text{ bounded} \qquad \Longrightarrow \qquad \{\lambda_n\}_n,\ \{\mu_n\}_n \text{ bounded}.$$

Next we turn to the study of sequences having arbitrarily large H^1_0-norm. In particular, we will focus on sequences of solutions having a common upper bound on the Morse index

$$m(u_n) = \max\left\{ k : \begin{array}{c} \exists V \subset H^1_0(\Omega),\ \dim(V) = k : \forall v \in V \setminus \{0\} \\ \displaystyle\int_\Omega |\nabla v|^2 + \lambda_n v^2 - p\mu_n |u_n|^{p-1} v^2\,dx < 0 \end{array} \right\}.$$

Throughout this section we will assume that

$$\text{the sequence } \{(u_n, \mu_n, \lambda_n)\}_n \text{ satisfies (2.1), with } \alpha_n \to +\infty \text{ and } m(u_n) \leq \bar{k}, \tag{2.2}$$

for some $\bar{k} \in \mathbb{N}$ not depending on n.

Lemma 2.2. *Let (2.2) hold. Then $\lambda_n \geq -\lambda_{\bar{k}}(\Omega)$.*

Proof. Assume, to the contrary, that for some n it holds $\lambda_n < -\lambda_{\bar{k}}(\Omega)$. For any real $t_1, \dots t_{\bar{k}}$ we define

$$\phi := \sum_{h=1}^{\bar{k}} t_h \varphi_h.$$

By denoting $J_{\lambda,\mu}(u)=\mathcal{E}_\mu(u)+\frac{\lambda}{2}\|u\|_{L^2}^2$, so that Morse index properties can be written in terms of $J''_{\lambda,\mu}$, we have

$$\begin{aligned}
J''_{\lambda_n,\mu_n}(u_n)[u_n,\phi]&=-(p-1)\mu_n\int_\Omega |u_n|^{p-1}u_n\phi,\\
J''_{\lambda_n,\mu_n}(u_n)[\phi,\phi]&=\sum_{h=1}^{\bar{k}} t_h^2\int_\Omega(|\nabla\varphi_h|+\lambda_n\varphi_h^2)\,dx-p\mu_n\int_\Omega |u_n|^{p-1}\phi^2\,dx\\
&\le\sum_{h=1}^{\bar{k}} t_h^2(\lambda_h(\Omega)+\lambda_n)-(p-1)\mu_n\int_\Omega |u_n|^{p-1}\phi^2\,dx\le-(p-1)\mu_n\int_\Omega |u_n|^{p-1}\phi^2\,dx,
\end{aligned}$$

where equality holds if and only if $t_1=\cdots=t_{\bar{k}}=0$. As a consequence

$$\begin{aligned}
J''_{\lambda_n,\mu_n}(u_n)[t_0u_n+\phi,t_0u_n+\phi]\le{}&-t_0^2(p-1)\mu_n\int_\Omega |u_n|^{p-1}u_n^2\\
&-2t_0(p-1)\mu_n\int_\Omega |u_n|^{p-1}u_n\phi\,dx-(p-1)\mu_n\int_\Omega |u_n|^{p-1}\phi^2\,dx.
\end{aligned}$$

We deduce that $J''_{\lambda_n,\mu_n}(u_n)$ is negative definite on $\operatorname{span}\{u_n,\varphi_1,\dots,\varphi_{\bar{k}}\}$, in contradiction with the bound on the Morse index (note that u_n cannot be a linear combination of a finite number of eigenfunctions, otherwise using the equations we would obtain that such eigenfunctions are linearly dependent). □

Lemma 2.3. *Let* (2.2) *hold. Then* $\lambda_n\to+\infty$.

Proof. By Lemma 2.2 we have that λ_n is bounded below. As a consequence, we can use Hölder inequality with $\|u_n\|_{L^2}=1$ and (2.1) to write

$$\mu_n\,\|u_n\|_{L^\infty}^{p-1}\ge\mu_n\,\|u_n\|_{L^{p+1}}^{p+1}=\alpha_n+\lambda_n\to+\infty.$$

Let us define

$$U_n:=\mu_n^{\frac{1}{p-1}}\,u_n,\qquad\text{so that }-\Delta U_n+\lambda_nU_n=|U_n|^{p-1}U_n\quad\text{in }\Omega,\quad U|_{\partial\Omega}=0.\tag{2.3}$$

Pick $P_n\in\Omega$ such that $|U_n(P_n)|=\|U_n\|_{L^\infty(\Omega)}$ and set

$$\tilde{\varepsilon}_n:=|U_n(P_n)|^{-\frac{p-1}{2}}=\frac{1}{\sqrt{\mu_n\,\|u_n\|_{L^\infty}^{p-1}}}\longrightarrow0\tag{2.4}$$

Hence, $|U_n(P_n)|\to+\infty$; moreover, as P_n is a point of positive maximum or of negative minimum, we have

$$0\le\frac{-\Delta U_n(P_n)}{U_n(P_n)}=|U_n(P_n)|^{p-1}-\lambda_n\,.$$

Thus $\lambda_n|U_n(P_n)|^{1-p}\le1$, and since λ_n is bounded from below, we conclude

$$\frac{\lambda_n}{|U_n(P_n)|^{p-1}}\longrightarrow\tilde{\lambda}\in[0,1].\tag{2.5}$$

Now, we are left to prove that $\tilde{\lambda}>0$. Let us define

$$\tilde{V}_n(y)=\tilde{\varepsilon}_n^{\frac{2}{p-1}}\,U_n(\tilde{\varepsilon}_n\,y+P_n),\qquad y\in\tilde{\Omega}_n:=(\Omega-P_n)/\tilde{\varepsilon}_n,\tag{2.6}$$

and let $d_n := d(P_n, \partial\Omega)$; we have, up to subsequences,

$$\frac{\tilde\varepsilon_n}{d_n} \longrightarrow L \in [0, +\infty] \qquad \text{and} \qquad \tilde\Omega_n \to \begin{cases} \mathbb{R}^n, & \text{if } L = 0; \\ H, & \text{if } L > 0, \end{cases}$$

where H is a half-space such that $0 \in \overline{H}$ and $d(0, \partial H) = 1/L$. The function $\tilde V_n$ satisfies

$$\begin{cases} -\Delta \tilde V_n + \lambda_n \tilde\varepsilon_n^2 \tilde V_n = |\tilde V_n|^{p-1}\tilde V_n, & \text{in } \tilde\Omega_n; \\ |\tilde V_n| \le |\tilde V_n(0)| = 1, & \text{in } \tilde\Omega_n; \\ \tilde V_n = 0, & \text{on } \partial\tilde\Omega_n. \end{cases}$$

From (2.4) and (2.5) we get $\tilde\varepsilon_n^2 \lambda_n \to \tilde\lambda$; hence, by elliptic regularity and up to a further subsequence, $\tilde V_n \to \tilde V$ in $C^1_{\mathrm{loc}}(\overline{H})$ where $\tilde V$ solves

$$\begin{cases} -\Delta \tilde V + \tilde\lambda \tilde V = |\tilde V|^{p-1}\tilde V, & \text{in } H; \\ |\tilde V| \le |\tilde V(0)| = 1, & \text{in } H; \\ \tilde V = 0, & \text{on } \partial H. \end{cases} \tag{2.7}$$

Since $\sup_n m(U_n) \le \bar k$ (as a solution to (2.3)), one can show as in Theorem 3.1 of [13] that $m(\tilde V) \le \bar k$. In particular, $\tilde V$ is stable outside a compact set (see Definition 2.1 in [13]) so that, by Theorem 2.3 and Remark 2.4 of [13], we have

$$\tilde V(x) \to 0 \qquad \text{as} \qquad |x| \to +\infty.$$

Moreover, since $\tilde V$ is not trivial, we also have that $\tilde\lambda > 0$. For, if $\tilde\lambda = 0$ the function $\tilde V$ would be a solution of the Lane-Emden equation $-\Delta u = |u|^{p-1}u$ either in $\mathbb{R}^n$ or in H. In both cases, $\tilde V$ would contradict Theorems 2 and 9 of [15], being non trivial and stable outside a compact set. Thus, $\tilde\lambda > 0$ and by (2.5) we conclude $\lambda_n \to +\infty$. □

Remark 2.4. We stress that the scaling argument in Lemma 2.3, leading to the limit problem (2.7) (with $\tilde\lambda > 0$), can be repeated also near points of *local* extremum. More precisely, let Q_n be such that $|U_n(Q_n)| \to +\infty$ and

$$|U_n(Q_n)| = \max_{\Omega \cap B_{R_n \tilde\varepsilon_n}(Q_n)} U_n,$$

for some $R_n \to +\infty$. Then the above procedure can be repeated by replacing P_n with Q_n in definition (2.4).

The local description of the asymptotic behaviour of the solutions U_n to (2.3) with bounded Morse index can be carried out more conveniently by defining the sequence (see [13, Theorem 3.1])

$$V_n(y) = \varepsilon_n^{\frac{2}{p-1}} U_n(\varepsilon_n y + P_n), \qquad y \in \Omega_n := \frac{\Omega - P_n}{\varepsilon_n}, \tag{2.8}$$

where P_n is defined before (2.4), and $\varepsilon_n = \frac{1}{\sqrt{\lambda_n}} \to 0$. Then, V_n satisfies

$$\begin{cases} -\Delta V_n + V_n = |V_n|^{p-1}V_n, & \text{in } \Omega_n; \\ |V_n| \le |V_n(0)| = (\varepsilon_n/\tilde\varepsilon_n)^{\frac{2}{p-1}} \to \tilde\lambda^{-\frac{1}{p-1}}, & \text{in } \Omega_n; \\ V_n = 0, & \text{on } \partial\Omega_n. \end{cases}$$

As before, we have (up to a subsequence) $V_n \to V$ in $C^1_{\mathrm{loc}}(\overline{H})$ where H is either $\mathbb{R}^N$ or a half space and V solves

$$\begin{cases} -\Delta V + V = |V|^{p-1}V, & \text{in } H; \\ |V| \le |V(0)| = \tilde\lambda^{-\frac{1}{p-1}}, & \text{in } H; \\ V = 0, & \text{on } \partial H. \end{cases} \tag{2.9}$$

By recalling the discussion following (2.7) we also have $m(V) < +\infty$. We collect some well known property of such a V in the following result.

Theorem 2.5 ([14, 13, 15, 11]). *Let V be a classical solution to (2.9) such that $m(V) \leq \bar{k}$. Then:*

1. $H = \mathbb{R}^N$*;*
2. $V(x) \to 0$ *as* $|x| \to +\infty$, $V \in H^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$*;*
3. *there exist C only depending on $\bar{k}$ (and not on V) such that*

$$\|V\|_{L^\infty} + \|\nabla V\|_{L^\infty} < C.$$

Proof. Claim 2 follows from Theorem 2.3 and Remark 2.4 of [13], see also [14, Remark 1.4]. As a consequence, Theorem 1.1 of [14, Remark 1.4] readily applies, providing claim 1 (V is not trivial as $V(0) > 0$). On the other hand, the L^∞ estimates in claim 3. are proved in Theorem 1.9 of [11]. □

Corollary 2.6. *If the sequence $\{U_n\}$ of solutions to (2.3) has uniformly bounded Morse index, and if $P_n \in \Omega$ is such that $|U_n(P_n)| = \|U_n\|_{L^\infty(\Omega)} \to +\infty$, then*

$$\sqrt{\lambda_n}\, d(P_n, \partial\Omega) \to +\infty, \qquad \text{where } \frac{\lambda_n}{|U_n(P_n)|^{p-1}} \to \tilde{\lambda} \in (0, 1].$$

Remark 2.7. Recall that $Z_{N,p}$, the unique positive solution to $-\Delta u + u = |u|^{p-1}u$ in $\mathbb{R}^N$, has Morse index 1 [19]; then, if V solves (2.9) in $\mathbb{R}^N$ and $1 < m(V) < +\infty$, then V is necessarily sign-changing.

Following the same pattern as in [13], we now analyze the global behaviour of a sequence $\{U_n\}$ of solutions to (2.3) for $\lambda_n \to +\infty$, assuming that $\lim_{n\to+\infty} m(U_n) \leq \bar{k} < \infty$.

By the previous discussion, if P_n^1 is a sequence of points such that $|U_n(P_n^1)| = \|U_n\|_{L^\infty(\Omega)}$, we have $|U_n(P_n^1)| \to +\infty$ and $\lambda_n\, d(P_n^1, \partial\Omega)^2 \to +\infty$. We now look for other possible sequences of (local) extremum points P_n^i, $i = 2, 3, ..$, along which $|U_n|$ goes to infinity. For any $R > 0$, consider the quantity

$$h_1(R) = \limsup_{n\to+\infty} \Big(\lambda_n^{-\frac{1}{p-1}} \max_{|x - P_n^1| \geq R\, \lambda_n^{-1/2}} |U_n(x)| \Big).$$

We will prove that if $h_1(R)$ is *not vanishing* for large R, then there exists a 'blow-up' sequence P_n^2 for u_n, 'disjoint' from P_n^1. Indeed, let us suppose that

$$\limsup_{R\to+\infty} h_1(R) = 4\delta > 0.$$

Hence, up to a subsequence and for arbitrarily large R, we have

$$\lambda_n^{-\frac{1}{p-1}} \max_{|x - P_n^1| \geq R\, \lambda_n^{-1/2}} |U_n(x)| \geq 2\delta. \tag{2.10}$$

Since U_n vanishes on $\partial\Omega$, there exists $P_n^2 \in \Omega \backslash B_{R\, \lambda_n^{-1/2}}(P_n^1)$ such that

$$|U_n(P_n^2)| = \max_{|x - P_n^1| \geq R\, \lambda_n^{-1/2}} |U_n(x)|. \tag{2.11}$$

Clearly, assumption (2.10) implies that $|U_n(P_n^2)| \to +\infty$. We first prove that the sequences P_n^1 and P_n^2 are far away each other.

Lemma 2.8. *Take R such that (2.10) holds, and let P_n^2 be defined as in (2.11); then*

$$\lambda_n^{1/2}|P_n^2 - P_n^1| \to +\infty \tag{2.12}$$

as $n \to \infty$.

Proof. Assuming the contrary one would get, up to a subsequence

$$\lambda_n^{1/2}|P_n^2 - P_n^1| \to R' \geq R.$$

Let us now recall that by (2.8) and the subsequent discussion, we have:

$$\lambda_n^{-\frac{1}{p-1}} U_n(\lambda_n^{-1/2} y + P_n^1) =: V_n^1(y) \to V(y) \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^N) \tag{2.13}$$

as $n \to +\infty$. Then, up to subsequences,

$$\lambda_n^{-\frac{1}{p-1}} |U_n(P_n^2)| = |V_n^1(\lambda_n^{1/2}(P_n^2 - P_n^1))| \to |V(y')|, \quad |y'| = R' \geq R.$$

Since V is vanishing for $|y| \to +\infty$, one can choose R such that $|V(y)| \leq \delta$ for every $|y| \geq R$. But this contradicts (2.10). □

Furthermore, we also have that the blow-up points stay far away from the boundary.

Lemma 2.9. *Assume (2.10) and let P_n^2 be defined as in (2.11); then*

$$\sqrt{\lambda_n}\, d(P_n^2, \partial\Omega) \to +\infty \tag{2.14}$$

as $n \to \infty$. Moreover,

$$|U_n(P_n^2)| = \max_{\Omega \cap B_{R_n \lambda_n^{-1/2}}(P_n^2)} |U_n| \tag{2.15}$$

for some $R_n \to +\infty$.

Proof. Let us set

$$\tilde{\varepsilon}_n^2 := |U_n(P_n^2)|^{-\frac{p-1}{2}} \quad \text{and} \quad R_n^{(2)} := \frac{1}{2}\, \frac{|P_n^2 - P_n^1|}{\tilde{\varepsilon}_n^2}.$$

Clearly, $\tilde{\varepsilon}_n^2 \to 0$; moreover, by (2.10) and (2.11), $\tilde{\varepsilon}_n^2 \leq (2\delta)^{-\frac{p-1}{2}} \lambda_n^{-1/2}$, so that

$$R_n^{(2)} \geq \frac{(2\delta)^{\frac{p-1}{2}}}{2} \lambda_n^{1/2} |P_n^2 - P_n^1| \to +\infty,$$

as $n \to +\infty$ by Lemma 2.8. We claim that this implies

$$|U_n(P_n^2)| = \max_{\Omega \cap B_{R_n^{(2)} \tilde{\varepsilon}_n^2}(P_n^2)} |U_n|. \tag{2.16}$$

For, if $x \in B_{R_n^{(2)} \tilde{\varepsilon}_n^2}(P_n^2)$, by (2.12) we would have

$$|x - P_n^1| \geq |P_n^2 - P_n^1| - |x - P_n^2| \geq \frac{1}{2}|P_n^2 - P_n^1| \geq R\,\lambda_n^{-1/2},$$

for arbitrarily large R. This means that

$$\Omega \cap B_{R_n^{(2)}\tilde{\varepsilon}_n^2}(P_n^2) \subset \Omega \backslash B_{R\,\lambda_n^{-1/2}}(P_n^1).$$

Then, the claim follows. Now, by recalling Remark 2.4, we can apply to U_n satisfying (2.16) the same scaling arguments as in the proof of Lemma 2.3, so that we conclude

$$0 < \lim_{n\to+\infty} \tilde{\varepsilon}_n^2 \sqrt{\lambda_n}.$$

Hence, (2.15) holds by defining $R_n = R_n^{(2)}\tilde{\varepsilon}_n^2 \sqrt{\lambda_n}$, and (2.14) follows by Corollary 2.6. □

We can now iterate the previous arguments: let us define, for $k \geq 1$,

$$h_k(R) = \limsup_{n\to+\infty}\Big(\lambda_n^{-\frac{1}{p-1}} \max_{d_{n,k}(x)\geq R\,\lambda_n^{-1/2}} |U_n(x)|\Big), \tag{2.17}$$

where

$$d_{n,k}(x) := \min\{|x - P_n^i| \ : \ i = 1, \ldots, k\}$$

and the sequences P_n^i are such that

$$\sqrt{\lambda_n}\, d(P_n^i, \partial\Omega) \to +\infty; \quad \lambda_n^{1/2}|P_n^i - P_n^j| \to +\infty, \qquad i, j = 1, \ldots, k, \quad i \neq j$$

as $n \to +\infty$. Assume that

$$\limsup_{n\to+\infty} h_k(R) = 4\delta > 0.$$

As before, up to a subsequence and for arbitrarily large R, we have

$$\lambda_n^{-\frac{1}{p-1}} \max_{d_{n,k}(x)\geq R\,\lambda_n^{-1/2}} |U_n(x)| \geq 2\delta \tag{2.18}$$

and there exist P_n^{k+1} so that

$$|U_n(P_n^{k+1})| = \max_{d_{n,k}(x)\geq R\,\lambda_n^{-1/2}} |U_n(x)|$$

with $\lim_{n\to+\infty} |U_n(P_n^{k+1})| = +\infty$. Moreover, as in Lemma 2.8 we deduce that, for every $i = 1, \ldots, k$

$$\lambda_n^{-\frac{1}{p-1}}\, U_n(\lambda_n^{-1/2}\, y + P_n^i) := V_n^i(y) \to V^i(y) \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^N) \tag{2.19}$$

as $n \to +\infty$; hence, by (2.18) and again from the vanishing of V at infinity, we conclude that

$$\lambda_n^{1/2}|P_n^{k+1} - P_n^i| \to +\infty \tag{2.20}$$

as $n \to \infty$, for every $i = 1, \ldots, k$. Setting now

$$\tilde{\varepsilon}_n^{k+1} := |U_n(P_n^{k+1})|^{-\frac{p-1}{2}} \quad \text{and} \quad R_n^{(k+1)} := \frac{1}{2}\,\frac{d_{n,k}(P_n^{k+1})}{\tilde{\varepsilon}_n^{k+1}}$$

we still have $\tilde{\varepsilon}_n^{k+1} \to 0$ and, by (2.18), $R_n^{(k+1)} \to +\infty$ as $n \to \infty$ (see Lemma 2.9). Then, by the same arguments as in Lemma 2.9, we get

$$|U_n(P_n^{k+1})| = \max_{\Omega\cap B_{R_n^{(k+1)}\tilde{\varepsilon}_n^{k+1}}(P_n^{k+1})} |u_n|, \tag{2.21}$$

and furthermore

$$\lim_{n\to+\infty} \tilde{\varepsilon}_n^{k+1}\sqrt{\lambda_n} > 0\,,$$

so that by defining $R_n =: R_n^{(k+1)}\tilde{\varepsilon}_n^{k+1}\sqrt{\lambda_n} \to +\infty$ we have

$$|U_n(P_n^{k+1})| = \max_{\Omega\cap B_{R_n\lambda_n^{-1/2}}(P_n^{k+1})} |U_n|. \tag{2.22}$$

Now, by the same arguments as in [13], it turns out that the iterative procedure must stop after *at most* $\bar{k}-1$ steps, where $\bar{k} = \lim_{n\to+\infty} m(u_n)$. Thus, we have proved:

Proposition 2.10. *Let $\{U_n\}_n$ be a solution sequence to (2.3) such that $\lambda_n \to +\infty$ and $m(U_n) \le \bar{k}$. Then, up to a subsequence, there exist $P_n^1, \ldots, P_n^k$, with $k \le \bar{k}$ such that*

$$\sqrt{\lambda_n}\, d(P_n^i, \partial\Omega) \to +\infty; \quad \lambda_n^{1/2}|P_n^i - P_n^j| \to +\infty, \qquad i,j = 1,\ldots,k, \quad i \ne j \tag{2.23}$$

as $n\to+\infty$ and

$$|U_n(P_n^i)| = \max_{\Omega\cap B_{R_n\lambda_n^{-1/2}}(P_n^i)} |U_n|, \quad i = 1,\ldots,k,$$

for some $R_n \to +\infty$ as $n\to+\infty$. Finally,

$$\lim_{R\to+\infty} h_k(R) = 0 \tag{2.24}$$

where $h_k(R)$ is given by (2.17).

We now show that the sequence U_n decays exponentially away from the blow-up points.

Proposition 2.11. *Let $\{U_n\}_n$ satisfy the assumptions of Proposition 2.10. Then, there exist $P_n^1, \ldots, P_n^k$ and positive constants C, γ, such that*

$$|U_n(x)| \le C\lambda_n^{\frac{1}{p-1}} \sum_{i=1}^{k} e^{-\gamma\sqrt{\lambda_n}|x-P_n^i|}, \qquad \forall\, x\in\Omega, \quad n\in\mathbb{N}\,. \tag{2.25}$$

Proof. By (2.24), for large $R>0$ and $n > n_0(R)$ it holds

$$\lambda_n^{-\frac{1}{p-1}} \max_{d_{n,k}(x)\ge R\,\lambda_n^{-1/2}} |U_n(x)| \le \Big(\frac{1}{2p}\Big)^{\frac{1}{p-1}}$$

Then, for $n > n_0(R)$ and for $x \in \{d_{n,k}(x) \ge R\,\lambda_n^{-1/2}\}$, we have

$$a_n(x) := \lambda_n - p|U_n(x)|^{p-1} \ge \lambda_n - \frac{\lambda_n}{2} = \frac{\lambda_n}{2}$$

We stress that the linear operator

$$L_n := -\Delta + a_n(x)$$

comes from the linearization of equation (2.3) at U_n; let us compute this operator on the functions

$$\phi_n^i(x) = e^{-\gamma\sqrt{\lambda_n}\,|x-P_n^i|}, \qquad \gamma > 0, \qquad i = 1,\ldots,k$$

in $\{d_{n,k}(x) \geq R\,\lambda_n^{-1/2}\}$. We obtain:

$$L_n\phi_n^i(x) = \lambda_n\phi_n^i(x)\Big[-\gamma^2 + (N-1)\frac{\gamma}{\sqrt{\lambda_n}\,|x-P_n^i|} + \frac{a_n(x)}{\lambda_n}\Big] \geq \lambda_n\phi_n^i(x)\big[-\gamma^2 + 1/2\big] \geq 0$$

for n large, provided $0 < \gamma \leq 1/\sqrt{2}$. Moreover, for $|x - P_n^i| = R\lambda_n^{-1/2}$, $i = 1, \ldots, k$, and R large we have

$$e^{\gamma R}\phi_n^i(x) - \lambda_n^{-\frac{1}{p-1}}|U_n(x)| = 1 - \lambda_n^{-\frac{1}{p-1}}|U_n(x)| > 0$$

as $n \to +\infty$, by (2.13). Note further that

$$\{x : d_{n,k}(x) = R\,\lambda_n^{-1/2}\} = \bigcup_{i=1}^{k} \partial B_{R\,\lambda_n^{-1/2}}(P_n^i) \subset \Omega$$

for large enough n. Then, by defining

$$\phi_n := e^{\gamma R}\,\lambda_n^{\frac{1}{p-1}} \sum_{i=1}^{k} \phi_n^i$$

we have

$$\phi_n(x) - |U_n(x)| \geq 0 \qquad \text{on} \qquad \{d_{n,k}(x) = R\,\lambda_n^{-1/2}\} \cup \partial\Omega$$

and

$$L_n(\phi_n - |U_n|) \geq -L_n\,|U_n| = \Delta\,|U_n| - \lambda_n\,|U_n| + p|U_n|^p \geq (p-1)\,|U_n|^p \geq 0$$

in $\Omega\backslash\{d_{n,k}(x) \leq R\,\lambda_n^{-1/2}\}$. Then (for R large and $n \geq n_0(R)$) we obtain $|U_n| \leq \phi_n$ in the same set, by the minimum principle. Moreover, since by (2.5)

$$|U_n(x)| \leq \|U_n\|_{L^\infty(\Omega)} = |U_n(P_n^1)| \leq C\lambda_n^{\frac{1}{p-1}}$$

for some $C > 0$, we also have, in $\{d_{n,k}(x) \leq R\,\lambda_n^{-1/2}\}$,

$$|U_n(x)| \leq \|U_n(x)\|_{L^\infty(\Omega)} = |U_n(P_n^1)| \leq Ce^{\gamma R}\,\lambda_n^{\frac{1}{p-1}} \sum_{i=1}^{k} e^{-\gamma\sqrt{\lambda_n}|x-P_n^i|}.$$

Then, possibly by choosing a larger C, estimate (2.25) follows for every n. □

We now exploit the previous results to show that suitable rescalings of the solutions to (2.1) converge (locally) to some bounded solution V of

$$-\Delta V + V = |V|^{p-1}V \tag{2.26}$$

in $\mathbb{R}^N$.

Lemma 2.12. *Let (2.2) hold. Then $|u_n|$ admits $k \leq \bar{k}$ local maxima $P_n^1, \ldots, P_n^k$ in Ω such that, defining*

$$u_{i,n}(x) = \Big(\frac{\mu_n}{\lambda_n}\Big)^{\frac{1}{p-1}} u_n\Big(\frac{x}{\sqrt{\lambda_n}} + P_n^i\Big), \qquad x \in \Omega_{n,i} := \sqrt{\lambda_n}(\Omega - P_n^i), \tag{2.27}$$

it results, up to a subsequence,

$$u_{i,n}(x) \to V_i \qquad \text{in } C^1_{\text{loc}}(\mathbb{R}^n) \quad \text{as } n \to +\infty, \quad \forall\ i = 1, 2, \ldots, k, \tag{2.28}$$

where V_i is a bounded solution of (2.26) *with $m(V_i) \le \bar{k}$.*
As a consequence, for every $q \ge 1$,

$$\Big(\frac{\mu_n}{\lambda_n}\Big)^{\frac{q}{p-1}} \lambda_n^{N/2} \int_\Omega |u_n|^q \, dx \to \sum_{i=1}^{k} \int_{\mathbb{R}^n} |V_i|^q \, dx \qquad \text{as } n \to +\infty. \tag{2.29}$$

Proof. By Lemma 2.3 we have $\lambda_n \to +\infty$; then, the first part of the lemma follows by definition (2.3), by (2.19) and by Proposition 2.10; by the same proposition and by Proposition 2.11 we also have that the local maxima P_n^i satisfies (2.23) and that the pointwise estimate

$$|u_n(x)| \le C\Big(\frac{\lambda_n}{\mu_n}\Big)^{\frac{1}{p-1}} \sum_{i=1}^{k} e^{-\gamma\sqrt{\lambda_n}|x-P_n^i|}, \qquad \forall\, x \in \Omega, \quad n \in \mathbb{N}. \tag{2.30}$$

holds. Let us fix $R > 0$ and set $r_n = R/\sqrt{\lambda_n}$; for large enough n, (2.23) implies

$$B_{r_n}(P_n^i) \subset \Omega, \qquad B_{r_n}(P_n^i) \cap B_{r_n}(P_n^j) = \emptyset, \quad i \ne j.$$

Then we obtain

$$\begin{aligned}
&\Big|\Big(\tfrac{\mu_n}{\lambda_n}\Big)^{\frac{q}{p-1}} \lambda_n^{N/2} \textstyle\int_\Omega |u_n|^q \, dx - \sum_{j=1}^k \int_{B_R(0)} |u_{j,n}|^q \, dx \Big| \\
&= \Big(\tfrac{\mu_n}{\lambda_n}\Big)^{\frac{q}{p-1}} \lambda_n^{N/2} \Big|\textstyle\int_\Omega |u_n|^q \, dx - \sum_{j=1}^k \int_{B_{r_n}(P_n^j)} |u_n|^q \, dx \Big| \\
&= \Big(\tfrac{\mu_n}{\lambda_n}\Big)^{\frac{q}{p-1}} \lambda_n^{N/2} \textstyle\int_{\Omega\setminus \bigcup_{j=1}^k B_{r_n}(P_n^j)} |u_n|^q \, dx \le C^q \lambda_n^{N/2} \int_{\Omega\setminus \bigcup_{j=1}^k B_{r_n}(P_n^j)} \Big|\sum_{i=1}^k e^{-\gamma\sqrt{\lambda_n}|x-P_n^i|}\Big|^q \, dx \\
&\le C^q k^{q-1} \lambda_n^{N/2} \textstyle\sum_{i=1}^k \int_{\Omega\setminus \bigcup_{j=1}^k B_{r_n}(P_n^j)} e^{-q\gamma\sqrt{\lambda_n}|x-P_n^i|} \, dx \\
&\le C^q k^{q-1} \lambda_n^{N/2} \textstyle\sum_{i=1}^k \int_{\mathbb{R}^N \setminus B_{r_n}(P_n^i)} e^{-q\gamma\sqrt{\lambda_n}|x-P_n^i|} \, dx \\
&\le (Ck)^q \textstyle\sum_{i=1}^k \int_{\mathbb{R}^N \setminus B_R(0)} e^{-q\gamma|y|} \, dy \le C_1 e^{-C_2 R},
\end{aligned}$$

for some positive C_1, C_2. Letting $n \to +\infty$ we have, up to subsequences,

$$\begin{aligned}
&\Big| \lim_{n\to+\infty} \Big(\frac{\mu_n}{\lambda_n}\Big)^{\frac{q}{p-1}} \lambda_n^{N/2} \int_\Omega |u_n|^q \, dx - \sum_{i=1}^k \int_{B_R(0)} |V_i|^q \, dx \Big| \\
&\qquad = \lim_{n\to+\infty} \Big| \Big(\frac{\mu_n}{\lambda_n}\Big)^{\frac{q}{p-1}} \lambda_n^{N/2} \int_\Omega |u_n|^q \, dx - \sum_{i=1}^k \int_{B_R(0)} |u_{i,n}|^q \, dx \Big| \le C_1 e^{-C_2 R}.
\end{aligned}$$

Then, (2.29) follows by taking $R \to +\infty$. □

The previous lemma allows us to gain some information on the asymptotic behavior of the sequences λ_n, μ_n and $\|u_n\|_{L^{p+1}(\Omega)}$. We first provide some bounds for the solutions of the limit problem (2.26) which will be useful in the sequel.

Lemma 2.13. *Let V_i, $i = 1, \dots, k$ be as in Lemma 2.12 (so that $m(V_i) \le \bar{k}$). There exists a constant C, only depending on the full sequence $\{u_n\}_n$ and not on V_i (and on the particular associated subsequence), such that*

$$\|V_i\|_{H^1}^2 = \|V_i\|_{L^{p+1}}^{p+1} \le C.$$

Furthermore, if also $m(V_i) \ge 2$ (or, equivalently, if V_i changes sign) the following estimates hold:

$$\|V_i\|_{L^{p+1}}^{p+1} > 2\,\|Z\|_{L^{p+1}}^{p+1}, \qquad \|V_i\|_{L^2}^2 > 2\,\|Z\|_{L^2}^2, \tag{2.31}$$

where $Z \equiv Z_{N,p}$ is the unique positive solution to (2.26).

Proof. To prove the bounds from above we claim that there exists $\bar{R} > 0$, not depending on i, such that V_i is stable outside $\overline{B_{\bar{R}}}$. Then the desired estimate will follow, since

$$\|V_i\|_{L^{p+1}}^{p+1} = \int_{B_{\bar{R}}} |V_i|^{p+1} + \int_{\mathbb{R}^N \setminus B_{\bar{R}}} |V_i|^{p+1},$$

where the first term is uniformly bounded by Theorem 2.5, while the second one can be estimated in an uniform way by reasoning as in the proof of [13, Theorem 2.3]. To prove the claim, recalling (2.17) and (2.24), let $\bar{R}$ be such that

$$h_k(\bar{R}) \leq \left(\frac{1}{p}\right)^{1/(p-1)}.$$

Then $|V_i(x)|^{p-1} \leq 1/p$ on $\mathbb{R}^N \setminus B_{\bar{R}}$ and thus, for any $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \equiv 0$ in $B_{\bar{R}}$, it holds

$$\int_{\mathbb{R}^N} |\nabla\psi|^2 + \psi^2 - p|V_i|^{p-1}\psi^2\, dx \geq \left(1 - p\|V_i\|_{L^\infty(\mathbb{R}^N\setminus B_{\bar{R}})}^{p-1}\right) \int_{\mathbb{R}^N} \psi^2 \geq 0.$$

Hence V_i is stable outside $B_{\bar{R}}$, and the first part of the lemma follows.

On the other hand, if V_i is a sign-changing solution to (2.26), the associated energy functional

$$E(V_i) = \frac{1}{2}\|\nabla V_i\|_{L^2}^2 + \frac{1}{2}\|V_i\|_{L^2}^2 - \frac{1}{p+1}\|V_i\|_{L^{p+1}}^{p+1}$$

satisfies the following *energy doubling property* (see [25]):

$$E(V_i) > 2\,E(Z)$$

On the other hand, by using the equation $E'(V_i)V_i = 0$ and the Pohozaev identity one gets

$$\|V_i\|_{L^{p+1}}^{p+1} = 2\,\frac{p+1}{p-1}\,E(V_i), \qquad \|V_i\|_{L^2}^2 = \frac{N+2-p\,(N-2)}{p-1}\,E(V_i) \tag{2.32}$$

Since the ground state solution Z satisfies the same identities, the bounds (2.31) are readily verified. □

Proposition 2.14. *Let (2.2) hold and the functions V_i be defined as in Lemma 2.12. We have, as $n \to +\infty$,*

$$\mu_n^{\frac{2}{p-1}}\,\lambda_n^{N/2-2/(p-1)} \longrightarrow \sum_{i=1}^k \int_{\mathbb{R}^n} |V_i|^2\,dx \tag{2.33}$$

$$\mu_n^{\frac{p+1}{p-1}}\,\lambda_n^{N/2-(p+1)/(p-1)} \int_\Omega |u_n|^{p+1}\,dx \longrightarrow \sum_{i=1}^k \int_{\mathbb{R}^n} |V_i|^{p+1}\,dx \tag{2.34}$$

$$\alpha_n\,\mu_n^{\frac{2}{p-1}}\,\lambda_n^{N/2-(p+1)/(p-1)} \longrightarrow \sum_{i=1}^k \int_{\mathbb{R}^n} |\nabla V_i|^2\,dx. \tag{2.35}$$

Proof. The limits (2.33) and (2.34) follow respectively by choosing $q = 2$ and $q = p+1$ in (2.29) (recall that $\|u_n\|_{L^2} = 1$). Furthermore, from the equations for u_n and V_k, we have

$$\alpha_n + \lambda_n = \mu_n\|u_n\|_{L^{p+1}}^{p+1}, \qquad \int_{\mathbb{R}^n} |\nabla V_i|^2\,dx + \int_{\mathbb{R}^n} |V_i|^2\,dx = \int_{\mathbb{R}^n} |V_i|^{p+1}\,dx,$$

and also (2.35) follows. □

Corollary 2.15. *With the same assumptions as above, we have that*

1. *if* $1 < p < 1 + \frac{4}{N}$, *then* $\mu_n \to +\infty$
2. *if* $p = 1 + \frac{4}{N}$, *then* $\mu_n \to \left(\sum_{i=1}^k \|V_i\|_{L^2}^2\right)^{2/N} \geq k^{2/N}\|Z\|_{L^2}^{4/N}$

3. *if* $1+\frac{4}{N}<p<2^*-1$, *then* $\mu_n\to 0$.

Furthermore

$$\frac{\alpha_n}{\lambda_n}\longrightarrow\frac{N(p-1)}{N+2-p(N-2)}. \tag{2.36}$$

Proof. The limits of μ_n follow by the previous proposition. To prove the lower bound in 2, recall that either $V_i=Z$ or V_i satisfies (2.31). Finally, taking the quotient between (2.35) and (2.33), we have

$$\frac{\alpha_n}{\lambda_n}\longrightarrow\frac{\sum_{i=1}^k\int_{\mathbb{R}^n}|\nabla V_i|^2\,dx}{\sum_{i=1}^k\int_{\mathbb{R}^n}|V_i|^2\,dx}$$

On the other hand, for every $i=1,2,\dots,k$ it holds

$$\|\nabla V_i\|_{L^2}^2=\left(\frac{\|V_i\|_{L^{p+1}}^{p+1}}{\|V_i\|_{L^2}^2}-1\right)\|V_i\|_{L^2}^2=\frac{N(p-1)}{N+2-p(N-2)}\|V_i\|_{L^2}^2$$

where the last equality follows by (2.32). By inserting this into the above limit, we get (2.36). □

Proof of Theorem 1.2. Let (U_n,λ_n) solve (1.1), with $\rho=\rho_n\to+\infty$ and $m(U_n)\le k$. Changing variables as in (1.4), we have that $u_n=\rho_n^{-1/2}U_n$ satisfies (2.1) with $\mu_n=\rho_n^{(p-1)/2}\to+\infty$. As a consequence, Lemma 2.1 guarantees that $\alpha_n\to+\infty$, and Corollary 2.15 yields $p<1+4/N$.

On the other hand, by direct minimization of the energy one can show that, if $p<1+4/N$, for every $\rho>0$ there exists a solution of (1.1) having Morse index one (see also Section 4). □

Remark 2.16. Reasoning as above we can also show that

$$\frac{\int_\Omega|u_n|^{p+1}\,dx}{\alpha_n^{N(p-1)/4}}\longrightarrow C_{N,p}\frac{\|Z\|_{L^2}^{p-1}}{\left(\sum_{i=1}^k\|V_i\|_{L^2}^2\right)^{(p-1)/2}}. \tag{2.37}$$

3 Max-min principles with two constraints

In this section we deal with the maximization problem with two constraints introduced in [21], aiming at considering more general max-min classes of critical points. Let $\mathcal{M}$ be defined in (1.5) and, for any fixed $\alpha>\lambda_1(\Omega)$, let $\mathcal{B}_\alpha$, $\mathcal{U}_\alpha$ be defined as in (1.6). We will look for critical points of the C^2 functional

$$f(u)=\int_\Omega|u|^{p+1},\qquad u\in\mathcal{M},$$

constrained to $\mathcal{U}_\alpha$. To start with, we notice that the topological properties of such set depend on α.

Lemma 3.1. *Let* $\alpha>\lambda_1(\Omega)$. *Then the set*

$$\mathcal{U}_\alpha\setminus\{\varphi\in\mathcal{U}_\alpha:-\Delta\varphi=\alpha\varphi\}$$

is a smooth submanifold of $H_0^1(\Omega)$ *of codimension 2. In particular, this property holds true for* $\mathcal{U}_\alpha$ *itself, provided* $\alpha\ne\lambda_k(\Omega)$, *for every* k.

Proof. Let us set $F(u) = (\int_\Omega u^2\,dx - 1,\ \int_\Omega |\nabla u|^2\,dx)$. For every $u \in \mathcal{U}_\alpha$, if the range of $F'(u)$ is $\mathbb{R}^2$ then $\mathcal{U}_\alpha$ is a smooth manifold at u. Since

$$F'(u)[v] = 2\left(\int_\Omega uv\,dx,\ \int_\Omega \nabla u\cdot\nabla v\,dx\right), \qquad \text{for every } v \in H_0^1(\Omega),$$

and $F'(u)[u] = 2(1,\alpha)$, we have that $F'(u)$ is not surjective if and only if

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \alpha\int_\Omega uv\,dx \qquad \text{for every } v \in H_0^1(\Omega). \qquad \square$$

Remark 3.2. If φ belongs to the eigenspace corresponding to $\lambda_k(\Omega)$, then $\varphi \in \mathcal{U}_{\lambda_k(\Omega)}$. As a consequence $\mathcal{U}_{\lambda_k(\Omega)}$ may not be smooth near φ. For instance, $\mathcal{U}_{\lambda_1(\Omega)}$ consists of two isolated points, $\pm\varphi_1$.

Of course $\mathcal{U}_\alpha$ is closed and odd, for any α. Recalling Definition 1.3 we deduce that its genus $\gamma(\mathcal{U}_\alpha)$ is well defined.

Lemma 3.3. *If $\alpha < \lambda_{k+1}(\Omega)$, for some k, then $\gamma(\mathcal{U}_\alpha) \le k$.*

Proof. Let $V_k := \operatorname{span}\{\varphi_1,\dots,\varphi_k\}$. Since

$$\min\left\{\int_\Omega |\nabla u|^2\,dx : u \in V_k^\perp,\ \int_\Omega u^2\,dx = 1\right\} = \lambda_{k+1}(\Omega),$$

we have that $\mathcal{U} \cap V_k^\perp = \emptyset$, thus the projection

$$g := \operatorname{proj}_{V_k} : \mathcal{U}_\alpha \to V_k \setminus \{0\}$$

is a continuous odd map of $\mathcal{U}_\alpha$ into $V_k \setminus \{0\}$. Now, let $h\colon \mathbb{S}^m \to \mathcal{U}$ be continuous and odd. Then $g \circ h$ is continuous and odd from $\mathbb{S}^m$ to $V_k \setminus \{0\}$, and Borsuk-Ulam's Theorem forces $m \le k-1$. $\square$

Lemma 3.4. *If $\alpha > \lambda_k(\Omega)$, for some k, then $\gamma(\mathcal{U}_\alpha) \ge k$.*

Proof. To prove the lemma we will construct a continuous map $h\colon \mathbb{S}^{k-1} \to \mathcal{U}$. Let $\ell \in \mathbb{N}$ be such that $\lambda_{\ell+1}(\Omega) > \alpha$. For every $i = 1,\dots,k$ we define the functions

$$u_i := \left(\frac{\lambda_{\ell+i}(\Omega) - \alpha}{\lambda_{\ell+i}(\Omega) - \lambda_i(\Omega)}\right)^{1/2} \varphi_i + \left(\frac{\alpha - \lambda_i(\Omega)}{\lambda_{\ell+i}(\Omega) - \lambda_i(\Omega)}\right)^{1/2} \varphi_{\ell+i}.$$

We obtain the following straightforward consequences:

1. as $\lambda_i(\Omega) < \alpha < \lambda_{\ell+i}(\Omega)$, for every i, u_i is well defined;
2. $\int_\Omega u_i^2\,dx = 1$, $\int_\Omega |\nabla u_i|^2\,dx = \alpha$;
3. for every $j \ne i$ it holds $\int_\Omega u_i u_j\,dx = \int_\Omega \nabla u_i\cdot\nabla u_j\,dx = 0$.

Therefore the map $h\colon \mathbb{S}^{k-1} \to \mathcal{U}$ defined as

$$h\colon x = (x_1,\dots,x_k) \mapsto \sum_{i=1}^k x_i u_i$$

has the required properties. $\square$

Now we turn to the properties of the functional f. To start with, it satisfies the Palais-Smale (P.S. for short) condition on $\overline{\mathcal{B}}_\alpha$; more precisely, the following holds.

Lemma 3.5. *Every P.S. sequence u_n for $f|_{\mathcal{B}_\alpha}$ is a P.S. sequence for $f|_{\mathcal{U}_\alpha}$ and has a strongly convergent subsequence in $\mathcal{U}_\alpha$.*

Proof. We first show that there are no P.S. sequences in $\mathcal{B}_\alpha$. In fact, if u_n is such a sequence, there is a sequence of real numbers k_n such that

$$\int_\Omega |u_n|^{p-1} u_n\, v - k_n \int_\Omega u_n\, v = o(1)\, \|v\|_{H_0^1} \tag{3.1}$$

for every $v \in H_0^1(\Omega)$. Since u_n is bounded in $H_0^1(\Omega)$, there is a subsequence (still denoted by u_n) weakly convergent to $u \in H_0^1(\Omega)$; moreover, u_n converges strongly in $L^{p+1}(\Omega)$ and in $L^2(\Omega)$ to the same limit. By choosing $v = u_n$, we see that k_n is bounded, so that we can also assume that $k_n \to k$. By taking the limit of (3.1) for $n \to \infty$ we get

$$\int_\Omega |u|^{p-1} u\, v = k \int_\Omega u\, v$$

for every $v \in H_0^1(\Omega)$. Hence u is constant, but this contradicts $u \in \mathcal{M}$.

Now, if u_n is a P.S. sequence for f on $\mathcal{U}_\alpha$, there are sequences of real numbers k_n, l_n such that

$$\int_\Omega |u_n|^{p-1} u_n\, v - k_n \int_\Omega u_n\, v - l_n \int_\Omega \nabla u_n\, \nabla v = o(1)\, \|v\|_{H_0^1}. \tag{3.2}$$

It is readily seen that l_n is bounded away from zero, otherwise (3.2) is equivalent to (3.1) (for some subsequence) and we still reach a contradiction. Then, we can divide both sides by l_n and find that there are sequences $\{\lambda_n\}_n$, $\{\mu_n\}_n$, with μ_n bounded, such that

$$\int_\Omega \nabla u_n\, \nabla v + \lambda_n \int_\Omega u_n\, v - \mu_n \int_\Omega |u_n|^{p-1} u_n\, v = o(1)\, \|v\|_{H_0^1}.$$

Now, by reasoning as before one finds that also the sequence $\{\lambda_n\}_n$ is bounded, so that by the relation

$$-\Delta u_n + \lambda_n u_n - \mu_n |u_n|^{p-1} u_n = o(1) \quad \text{in } H^{-1}(\Omega)$$

and by the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, the P.S. condition holds for the functional $f|_{\mathcal{U}_\alpha}$. □

We can combine the previous lemmas to prove one of the main results stated in the introduction.

Proof of Theorem 1.4. Lemma 3.5 allows to apply standard variational methods (see e.g. [24, Thm. II.5.7]). We deduce that $M_{\alpha,k}$ is achieved at some critical point u of $f|_{\mathcal{U}_\alpha}$. This amounts to say that u satisfies (1.8) for some real λ and $\mu \neq 0$. We claim that there exists at least one $u \in f^{-1}(M_{\alpha,k}) \cap \mathcal{U}_\alpha$ such that (1.8) holds with $\mu > 0$. Assume by contradiction that for *every* critical point of $f|_{\mathcal{U}_\alpha}$ at level $M_{\alpha,k}$ it holds $\mu < 0$ in equation (1.8).

Let us define the functional $T : H_0^1(\Omega) \to \mathbb{R}$ as

$$T(u) = \frac{1}{2} \int_\Omega |\nabla u|^2.$$

By denoting with D the Fréchet derivative and by $<, >$ the pairing between H_0^1 and its dual H^{-1}, our assumption can be restated as follows:
if there are $u \in f^{-1}(M_{\alpha,k}) \cap \mathcal{U}_\alpha$ and $\mu \neq 0$ such that

$$\langle DT(u), \phi\rangle = \mu \langle Df(u), \phi\rangle \tag{3.3}$$

for every $\phi \in H_0^1(\Omega)$ satisfying $\int_\Omega \phi u = 0$ (that is for every ϕ tangent to $\mathcal{M}$ at u) then $\mu < 0$.

We stress that both $DT(u)$ and $Df(u)$ in the above equation are bounded away from zero, since there are no Dirichlet eigenfunctions in $\mathcal{U}_\alpha$ nor critical points of f on $\mathcal{M}$. Hence, by denoting with $\nabla_{T\mathcal{M}}$ the gradient of a functional (in H_0^1) in the direction tangent to $\mathcal{M}$, if $u \in f^{-1}(M_{\alpha,k}) \cap \mathcal{U}_\alpha$ then $\nabla_{T\mathcal{M}} T(u)$ and $\nabla_{T\mathcal{M}} f(u)$ *are either opposite or not parallel*. Moreover, the angle between these (non vanishing) vectors is *bounded away from zero*; otherwise, we would find sequences $u_n \in \mathcal{U}_\alpha$, $\mu_n > 0$ such that

$$(\nabla_{T\mathcal{M}} T(u_n), v)_{H_0^1} - \mu_n (\nabla_{T\mathcal{M}} f(u_n), v)_{H_0^1} = o(1)\|v\|_{H_0^1} \tag{3.4}$$

for every $v \in H_0^1(\Omega)$; but since

$$(\nabla_{T\mathcal{M}} T(u_n), v)_{H_0^1} = \int_\Omega \nabla u_n \nabla v - \lambda_n^T \int_\Omega u_n v ,$$

$$(\nabla_{T\mathcal{M}} f(u_n), v)_{H_0^1} = \int_\Omega |u_n|^{p-1} u_n v - \lambda_n^f \int_\Omega u_n v ,$$

for suitable bounded sequences λ_n^T, λ_n^f, this is equivalent to saying that u_n is a P.S. sequence for $f|_{\mathcal{U}_\alpha}$, so that, by Lemma 3.5, we would get a constrained critical point with $\mu > 0$.

Then, by choosing suitable linear combinations of the above tangential components one can define a bounded C^1 map $u \mapsto v(u) \in H_0^1(\Omega)$, with $v(u)$ tangent to $\mathcal{M}$ and satisfying the following property: there is $\delta > 0$ such that

$$\int_\Omega \nabla u \nabla v(u) < -\delta , \qquad \int_\Omega |u|^{p-1} u\, v(u) > \delta , \tag{3.5}$$

for every $u \in f^{-1}(M_{\alpha,k}) \cap \mathcal{U}_\alpha$. By continuity and possibly by decreasing δ, inequalities (3.5) extend to

$$f^{-1}(M_{\alpha,k} - \bar\varepsilon, M_{\alpha,k} + \bar\varepsilon) \cap (\overline{\mathcal{B}}_\alpha \backslash \overline{\mathcal{B}}_{\alpha-\tau}) \tag{3.6}$$

for small enough, positive $\bar\varepsilon$ and τ. Finally, since there are no critical points of f in $\mathcal{B}_\alpha$ we can take that the *second of* (3.5) *holds on*

$$f^{-1}(M_{\alpha,k} - \bar\varepsilon, M_{\alpha,k} + \bar\varepsilon) \cap \overline{\mathcal{B}}_\alpha . \tag{3.7}$$

Let φ be a C^1 function on $\mathbb{R}$ such that:

$$0 \le \varphi \le 1, \quad \varphi \equiv 1 \text{ in } (M_{\alpha,k} - \bar\varepsilon/2, M_{\alpha,k} + \bar\varepsilon/2), \quad \varphi \equiv 0 \text{ in } \mathbb{R}\backslash(M_{\alpha,k} - \bar\varepsilon, M_{\alpha,k} + \bar\varepsilon),$$

and define

$$e(u) = \varphi(f(u))\, v(u). \tag{3.8}$$

Clearly, e is a C^1 vector field on $\mathcal{M}$ and is uniformly bounded, so that there exists a global solution $\Phi(u,t)$ of the initial value problem

$$\partial_t \Phi(u,t) = e(\Phi(u,t)), \qquad \Phi(u,0) = 0.$$

By definition (3.8) and by the first of (3.5) (on (3.6)) we get $\Phi(u,t_0) \in \overline{\mathcal{B}}_\alpha$ for $t_0 > 0$ and for any $u \in \overline{\mathcal{B}}_\alpha$; moreover, by the second inequality of (3.5) (on (3.7)) there exists $\varepsilon \in (0, \bar\varepsilon)$ such that

$$f(\Phi(u,t_0)) > M_{\alpha,k} + \varepsilon$$

for every $u \in f^{-1}(M_{\alpha,k} - \varepsilon, +\infty) \cap \overline{\mathcal{B}}_\alpha$.

Now, by (1.7), there is $A_\varepsilon \subset \overline{\mathcal{B}}_\alpha$ such that $\gamma(A_\varepsilon) \geq k$ and

$$\inf_{u \in A_\varepsilon} f(u) \geq M_{\alpha,\,k} - \varepsilon.$$

Hence, $\gamma(\Phi(A_\varepsilon, t_0)) \geq k$ and

$$\inf_{u \in \Phi(A_\varepsilon, t_0)} f(u) \geq M_{\alpha,\,k} + \varepsilon$$

contradicting the definition of $M_{\alpha,\,k}$. □

Remark 3.6. If $\mu > 0$, by testing (1.8) with u and by integration by parts we readily get $\lambda > -\alpha$. An alternative lower bound, independent of α, could be obtained by adapting arguments from [3, 20, 23] in order to prove that the Morse index of u (as a solution of (1.8)) is less or equal than k. Then Lemma 2.2 would provide $\lambda \geq -\lambda_k$.

Remark 3.7. By the Gagliardo-Nirenberg inequality (1.3) we readily obtain that, for every $k \geq 1$,

$$M_{\alpha,k} \leq C_{N,p}\alpha^{N(p-1)/4}.$$

Taking into account the previous remark, this agrees with Remark 2.16.

We conclude this section with the following estimate.

Lemma 3.8. *Under the assumptions and notation of Theorem 1.4,*

$$M_{\alpha,3} \leq 2^{-(p-1)/2} M_{\alpha,1}.$$

Proof. Let $A \in \Sigma_\alpha^{(3)}$, according to Definition 1.3. Notice that the map

$$A \ni u \mapsto \left(\int_\Omega |u|u, \int_\Omega |u|^p u \right) \in \mathbb{R}^2$$

is continuous and equivariant. By the Borsuk-Ulam Theorem, we deduce the existence of $u_a \in A$ such that

$$\int_\Omega |u_a^+|^2 = \int_\Omega |u_a^-|^2 = \frac{1}{2}, \qquad \int_\Omega |u_a^+|^{p+1} = \int_\Omega |u_a^-|^{p+1} = \frac{1}{2}\int_\Omega |u_a|^{p+1},$$

while

$$\text{either} \int_\Omega |\nabla u_a^+|^2 \leq \frac{\alpha}{2} \qquad \text{or} \int_\Omega |\nabla u_a^-|^2 \leq \frac{\alpha}{2}.$$

For concreteness let us assume that the first alternative holds; as a consequence, we obtain that $v := \sqrt{2}u_a^+$ belongs to $\overline{\mathcal{B}}_\alpha$. This yields

$$M_{\alpha,1} \geq \int_\Omega |v|^{p+1} = 2^{(p+1)/2} \int_\Omega |u_a^+|^{p+1} = \frac{2^{(p+1)/2}}{2} \int_\Omega |u_a|^{p+1} \geq 2^{(p-1)/2} \inf_{u \in A} \int_\Omega |u|^{p+1},$$

and since $A \in \Sigma_\alpha^{(3)}$ is arbitrary the proposition follows. □

4 Min-max principles on the unit sphere in L^2

According to equation (1.5), let $\mathcal{M} \subset H_0^1(\Omega)$ denote the unit sphere with respect to the L^2 norm and $\mathcal{E}_\mu$ the energy functional associated to (1.4). In this section we are concerned with critical points of $\mathcal{E}_\mu$ on $\mathcal{M}$ (which, in turn, correspond to solutions of our starting problem (1.1)).

By the Gagliardo-Nirenberg inequality (1.3), setting $\|\nabla u\|_{L^2}^2 = \alpha$, one obtains

$$\frac{1}{2}\alpha - \mu\frac{C_{N,p}}{p+1}\alpha^{N(p-1)/4} \leq \mathcal{E}_\mu(u) \leq \frac{1}{2}\alpha. \tag{4.1}$$

In particular, $\mathcal{E}_\mu$ is bounded on any bounded subset of $\mathcal{M}$, and it is bounded from below (and coercive) on the entire $\mathcal{M}$ for subcritical $p < 1+4/N$ and for critical $p = 1+4/N$ whenever $\mu < \frac{p+1}{2}C_{N,p}^{-1}$. In these cases, one can easily show that $\mathcal{E}_\mu$ satifies the P.S. condition and apply the classical minimax principle for even functionals on a closed symmetric submanifold (see e.g. [24, Thm. II.5.7]).

In the complementary case, when p is either supercritical, i.e. $p > 1 + 4/N$, or critical and μ is large, then $\mathcal{E}_\mu$ is not bounded from below (see e.g. (4.9) below). In order to provide a minimax principle suitable for this case, we recall the Definition 1.3 of genus and that of $\mathcal{B}_\alpha$ (see equation (1.6)). Furthermore, we denote with K_c the (closed and symmetric) set of critical points of $\mathcal{E}_\mu$ at level c contained in $\mathcal{B}_\alpha$. The following theorem is an adaptation of well known arguments of previous critical point theorems relying on index theory.

Theorem 4.1. *Let $k \geq 1$, $\alpha > \lambda_k(\Omega)$, $\mu > 0$ and $\tau > 0$ be fixed, and let c_k be defined as in Theorem 1.5, equation (1.9). If*

$$c_k < \hat{c}_k := \inf_{\substack{A\in\Sigma_\alpha^{(k)} \\ A\setminus\mathcal{B}_{\alpha-\tau}\neq\emptyset}} \sup_{A\setminus\mathcal{B}_{\alpha-\tau}} \mathcal{E}_\mu, \tag{4.2}$$

then $K_{c_k} \neq \emptyset$, and it contains a critical point of Morse index less or equal to k.

Remark 4.2. In case assumption (4.2) holds for $k, k+1, \dots, k+r$, and $c = c_k = \dots c_{k+r}$, then it is standard to extend Theorem 4.1 to obtain

$$\gamma(K_c) \geq r + 1, \tag{4.3}$$

so that K_c contains infinitely many critical points.

Proof of Theorem 4.1. For any $a \in \mathbb{R}$ we denote by $\mathcal{M}_a$ the sublevel set $\{\mathcal{E}_\mu < a\}$. First of all we notice that both c_k and $\hat{c}_k$ are well defined and finite, by Lemma 3.4 and equation (4.1). Suppose now by contradiction that $K_{c_k} = \emptyset$. By a suitably modified version of the Deformation Lemma (recall that $\mathcal{E}_\mu$ satisfies the P.S. condition on $\mathcal{M}$), there exist $\delta > 0$ and an equivariant homeomorphism η such that $\eta(u) = u$ outside $\mathcal{B}_\alpha \cap \mathcal{M}_{c_k+2\delta}$ and

$$\eta(\mathcal{M}_{c_k+\delta} \cap \mathcal{B}_{\alpha-\tau}) \subset \mathcal{M}_{c_k-\delta} \cap \mathcal{B}_\alpha. \tag{4.4}$$

By definition of c_k there exists $A \in \Sigma_\alpha^{(k)}$ such that $A \subset \mathcal{M}_{c_k+\delta}$; it follows by assumption (4.2) (and by decreasing δ if necessary) that $A \subset \mathcal{M}_{c_k+\delta}\cap\mathcal{B}_{\alpha-\tau}$. Then, since η is an odd homeomorphism, $\eta(A) \in \Sigma_\alpha^{(k)}$ and, by definition, $\sup_{\eta(A)} \mathcal{E}_\mu \geq c_k$, in contradiction with (4.4). Finally, the estimate of the Morse index is a direct consequence of the definition of genus we deal with: see [3], Proposition on page 1030, or the discussion at the end of Section 2 in [23]. ◻

We now provide a sufficient condition to guarantee the validity of assumption (4.2).

Lemma 4.3. *Let $k \geq 1$, $\alpha > \lambda_k(\Omega)$ and $\mu > 0$ satisfy*

$$0 < \mu < \frac{p+1}{2} \frac{\alpha - \lambda_k(\Omega)}{M_{\alpha,k} - |\Omega|^{-\frac{p-1}{2}}} \tag{4.5}$$

where $M_{\alpha,k}$ is defined in Theorem 1.4. Then, for $\tau > 0$ sufficiently small, (4.2) holds true.

Proof. We first estimate c_k from above. To this aim, we construct a subset $\tilde{A} \in \Sigma^{(k)}_{\alpha-\tau}$ (for any τ sufficiently small) as

$$\tilde{A} = \left\{ \sum_{i=1}^{k} x_i \varphi_i : x = (x_1, \ldots, x_k) \in \mathbb{S}^{k-1} \right\}, \tag{4.6}$$

where, as usual φ_i denotes the Dirichlet eigenfunction associated to $\lambda_i(\Omega)$. Indeed $\gamma(\tilde{A}) = k$ (it is homeomorphic to $\mathbb{S}^{k-1}$), and $\max_{u\in\tilde{A}} \|u\|^2_{H^1_0} = \lambda_k(\Omega) < \alpha - \tau$ for τ small. Hence Holder inequality yields

$$c_k \leq \sup_{\tilde{A}} \mathcal{E}_\mu \leq \frac{1}{2}\lambda_k(\Omega) - \frac{\mu}{p+1}|\Omega|^{-\frac{p-1}{2}}. \tag{4.7}$$

On the other hand, let $A \in \Sigma^{(k)}_\alpha$. Theorem 1.4 implies

$$\inf_{u\in A} \int_\Omega |u|^{p+1} \leq M_{\alpha,k}.$$

If moreover $A \setminus \mathcal{B}_{\alpha-\tau} \neq \emptyset$ we infer

$$\sup_{A\setminus\mathcal{B}_{\alpha-\tau}} \mathcal{E}_\mu \geq \frac{1}{2}(\alpha - \tau) - \frac{\mu}{p+1} M_{\alpha,k},$$

and taking the infimum an analogous inequality holds true for $\hat{c}_k$. Comparing with (4.7) the lemma follows. □

Exploiting the results above, we are ready to prove our main existence results.

End of the proof of Theorem 1.5. By Theorem 4.1 and Lemma 4.3 the proof is completed by choosing

$$\hat{\mu}_k := \sup_{\alpha>\lambda_k(\Omega)} \frac{p+1}{2} \frac{\alpha - \lambda_k(\Omega)}{M_{\alpha,k} - |\Omega|^{-\frac{p-1}{2}}}. \qquad \square$$

Proof of Theorem 1.8. We write the proof in terms of $\mathcal{E}_\mu$, the theorem following by the relations in (1.4). Recall that, for every $u \in \overline{\mathcal{B}}_\alpha$, $\gamma(\{u, -u\}) = 1$. We deduce that c_1 is actually a local minimum for $\mathcal{E}_\mu$, achieved by some u which solves (1.4) (for a suitable λ), and it can be chosen positive by symmetry. Since

$$\int_\Omega |\nabla u|^2 + \lambda u^2 - p\mu|u|^{p+1}\,dx = -(p-1)\int_\Omega \mu|u|^{p+1}\,dx < 0,$$

and $H^1_0(\Omega) = \mathrm{span}\{u\} \oplus T_u\mathcal{M}$, we have that u has Morse index 1. In a standard way, the minimality property of u implies also orbital stability of the associated solitary wave (see e.g. [7]). Turning to the estimates for $\hat{\mu}_1 = \hat{\rho}_1^{(p-1)/2}$, we can deduce it using Lemma 4.3 and Remark 3.7, which yield

$$\hat{\mu}_1(\Omega, p) := \sup_{\alpha>\lambda_1(\Omega)} \frac{p+1}{2} \frac{\alpha - \lambda_1(\Omega)}{C_{N,p}\alpha^{\frac{N(p-1)}{4}} - |\Omega|^{-\frac{p-1}{2}}} \geq \frac{p+1}{2C_{N,p}} \sup_{\alpha>\lambda_1(\Omega)} \frac{\alpha - \lambda_1(\Omega)}{\alpha^\beta},$$

where $\beta := N(p-1)/4$. Now, if $\beta \le 1$ we obtain the desired bound for the subcritical and critical cases. On the other hand, when $\beta > 1$, elementary calculations show that

$$\hat{\mu}_1(\Omega, p) \ge \frac{p+1}{2C_{N,p}} \frac{(\beta-1)^{(\beta-1)}}{\beta^\beta} \lambda_1(\Omega)^{-(\beta-1)},$$

and finally

$$\hat{\rho}_1(\Omega, p) \ge \underbrace{\left[\frac{p+1}{2C_{N,p}} \frac{(\beta-1)^{(\beta-1)}}{\beta^\beta}\right]^{\frac{2}{p-1}}}_{D_{N,p}} \lambda_1(\Omega)^{\frac{2}{p-1}-\frac{N}{2}}. \qquad \square$$

Proof of Proposition 1.10. As usual, by (1.4), we have to prove that

$$\hat{\mu}_3(\Omega, p) \ge 2^{(p-1)/2} D_{N,p} \lambda_3(\Omega)^{\frac{2}{p-1}-\frac{N}{2}}.$$

By Lemmas 4.3, 3.8, and Remark 3.7 we obtain

$$\begin{aligned}\hat{\mu}_3 &= \sup_{\alpha > \lambda_3(\Omega)} \frac{p+1}{2} \frac{\alpha - \lambda_3(\Omega)}{M_{\alpha,3} - |\Omega|^{-\frac{p-1}{2}}} \ge \sup_{\alpha > \lambda_3(\Omega)} \frac{p+1}{2} \frac{\alpha - \lambda_3(\Omega)}{2^{-(p-1)/2} M_{\alpha,1} - |\Omega|^{-\frac{p-1}{2}}} \\ &\ge 2^{(p-1)/2} \sup_{\alpha > \lambda_3(\Omega)} \frac{p+1}{2} \frac{\alpha - \lambda_3(\Omega)}{C_{N,p}\alpha^\beta - 2^{(p-1)/2}|\Omega|^{-\frac{p-1}{2}}},\end{aligned}$$

where $\beta := N(p-1)/4$, and the desired result follows by arguing as in the proof of Theorem 1.8. $\square$

To conclude this section we prove that in the supercritical case, if μ is not too large, in addition to $(c_k)_k$ there is a further sequence of critical levels $(\bar{c}_k)$ of $\mathcal{E}_\mu$ constrained to $\mathcal{M}$. For concreteness, let us first consider the case $k = 1$: since in such case c_1 is a local minimum of $\mathcal{E}_\mu$ in $\mathcal{M}$, and $\mathcal{E}_\mu$ is unbounded from below in $\mathcal{M}$, the critical level $\bar{c}_1$ is of mountain pass type.

Proposition 4.4. *Let $p > 1+4/N$, $\mu < \hat{\mu}_1$, and u_1 denote the local minimum point of $\mathcal{E}_\mu$ in $\mathcal{M}$, according to Theorems 4.1 and 1.8. The value*

$$\bar{c}_1 := \inf_{\gamma\in\Gamma} \sup_{[0,1]} \mathcal{E}_\mu(\gamma(s)), \quad \text{where } \Gamma := \{\gamma \in C([0,1]; \mathcal{M}) : \gamma(0) = u_1,\ \gamma(1) < c_1 - 1\},$$

is a critical level for $\mathcal{E}_\mu$ in $\mathcal{M}$.

Proof. Notice that, if $p > 1+4/N$, then $\mathcal{E}_\mu \to -\infty$ along some sequence in $\mathcal{M}$. Indeed, by defining

$$w_n(x) := \eta(x) Z_{N,p}((x-x_0)/a_n) \quad \text{and} \quad \tilde{w}_n := \frac{w_n}{\|w_n\|^2_{L^2(\Omega)}} \in \mathcal{M}, \tag{4.8}$$

where $a_n \to 0^+$, $x_0 \in \Omega$ and $\eta \in C_0^\infty(\Omega)$, $\eta(x_0) = 1$, we obtain

$$\alpha_n := \|\nabla \tilde{w}_n\|^2_{L^2(\Omega)} \to +\infty, \qquad \frac{\int_\Omega |\tilde{w}_n|^{p+1}\,dx}{\alpha_n^{N(p-1)/4}} \to C_{N,p}, \qquad \mathcal{E}_\mu(\tilde{w}_n) \to -\infty \tag{4.9}$$

for $n \to +\infty$. Since u_1 is a local minimum, the functional $\mathcal{E}_\mu$ has a mountain pass structure on $\mathcal{M}$; by recalling that $\mathcal{E}_\mu$ satisfies the P.S. condition the proposition follows. $\square$

Remark 4.5. One can generalize Proposition 4.4 by constructing critical points via a saddle-point theorem in the following way: let us pick k points $x_1, x_2, ..., x_k$ in Ω and consider the corresponding function $\tilde{w}_i$; we may assume that $\operatorname{supp} \tilde{w}_i \cap \operatorname{supp} \tilde{w}_j = \emptyset$ for $i \neq j$, so that these functions are orthogonal. Let us now define the subspace $V_k = \operatorname{span}\{\varphi_1, \ldots, \varphi_k; \tilde{w}_1, ..., \tilde{w}_k\}$; note that $\dim V_k = 2k$. Let R be an operator (in $L^2(\Omega)$) such that $R = I$ on $V_k^\perp$, $Ru_i = \tilde{w}_i$, $i = 1, 2, .., k$. Possibly after permutations, we can choose R such that $R|_{V_k} \in SO(2k)$ (actually, there are infinitely many different choices of R). Now, since $SO(2k)$ is (arcwise) connected, there is a continuous path $\tilde{\gamma} : [0, 1] \to SO(2k)$ such that $\gamma(0) = I$, $\gamma(1) = R|_{V_k}$. Then, we can define the following map

$$\gamma : [0, 1] \times S^{k-1} \to \mathcal{M}, \qquad \gamma(s; t_1, ..., t_k) = \sum_{i=1}^{k} t_i \tilde{\gamma}(s) u_i,$$

where $\sum_{i=1}^k t_i^2 = 1$. It is clear that γ is continuous; moreover,

$$\gamma(0; t_1, ..., t_k) \in \operatorname{span}\{\varphi_1, \ldots, \varphi_k\} \cap \mathcal{M} \quad \text{and} \quad \gamma(1; t_1, ..., t_k) \in \operatorname{span}\{\tilde{w}_1, \ldots, \tilde{w}_k\} \cap \mathcal{M}.$$

Then, by denoting with Γ_k the set of the above paths, if μ is sufficiently small we obtain the critical levels

$$\bar{c}_k := \inf_{\gamma \in \Gamma_k} \sup_{[0,1]\times S^{k-1}} \mathcal{E}_\mu(\gamma(s; t_1, ..., t_k)).$$

5 Results in symmetric domains

This section is devoted to the proof of Theorem 1.12, therefore we assume $1 + 4/N \leq p < 2^* - 1$. We perform the proof in the case of $\Omega = B$, but it will be clear that the main assumption on Ω is the following:

(T) there is a tiling of Ω, made by h copies of a subdomain D, in such a way that from any solution U_D of (1.1) on D one can construct, using reflections, a solution U_Ω of (1.1) on Ω.

Then U_Ω has h times the mass of U_D, and recalling Theorem 1.8 we deduce that (1.1) on Ω is solvable for any $\rho < h \cdot D_{N,p} \lambda_1(D)^{\frac{2}{p-1} - \frac{N}{2}}$. At this point, for a sequence $(D_k, h_k)_k$ of tilings satisfying **(T)**, we obtain the solvability of (1.1) on Ω whenever

$$\rho < h_k \cdot D_{N,p} \lambda_1(D_k)^{\frac{2}{p-1} - \frac{N}{2}},$$

and if we can show that

$$\frac{h_k}{\lambda_1(D_k)^{\frac{N}{2} - \frac{2}{p-1}}} \to +\infty \qquad \text{as } k \to +\infty, \tag{5.1}$$

we deduce the solvability of (1.1) on Ω for every mass. Having this scheme in mind, it is easy to prove analogous results on rectangles and also in other kind of domains.

Then let $B \subset \mathbb{R}^N$ be the ball (w.l.o.g. of radius one), and let

$$D_k := \left\{(r\cos\theta, r\sin\theta, x_3, \ldots, x_N) \in B : -\frac{\pi}{k} < \theta < \frac{\pi}{k}\right\}$$

Then D_k satisfies **(T)**, with $h_k = k$. In order to estimate $\lambda_1(D_k)$ we observe that, by elementary trigonometry,

$$B'_k = B_{\frac{\sin(\pi/k)}{\sin(\pi/k)+1}}\left(\frac{1}{\sin(\pi/k) + 1}, 0, 0, \ldots, 0\right) \subset D_k,$$

and therefore

$$\lambda_1(D_k) \le \lambda_1(B'_k) \le Ck^2,$$

for some dimensional constant $C = C(N)$ and k large. Then

$$\frac{h_k}{\lambda_1(D_k)^{\frac{N}{2}-\frac{2}{p-1}}} \ge C\frac{k}{k^{N-\frac{4}{p-1}}} = Ck^{1-N+\frac{4}{p-1}} = Ck^{\frac{N-1}{p-1}\left[1+\frac{4}{N-1}-p\right]},$$

and finally (5.1) holds true whenever $p < 1 + \frac{4}{N-1}$, thus completing the proof of Theorem 1.12.

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