Orthogonal double covers of complete bipartite graphs by the union of a cycle and a star

by
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Abstract

Let H be a graph on n vertices and $\mathcal{G}$ a collection of n subgraphs of H, one for each vertex. Then $\mathcal{G}$ is an orthogonal double cover (ODC) of H if every edge of H occurs in exactly two members of $\mathcal{G}$ and any two members of $\mathcal{G}$ share exactly an edge whenever the corresponding vertices are adjacent in H. If all subgraphs in $\mathcal{G}$ are isomorphic to a given spanning subgraph G, then $\mathcal{G}$ is said to be an ODC of H by G.

We construct ODCs of $H = K_{n,n}$ by $G = C_m \cup^v S_{n-m}$ (union of a cycle C_m and a star S_{n-m} whose center vertex v belongs to that cycle and $m = 6, 8, 10, 12$ and $m < n$). Furthermore, we construct ODCs of $H = K_{n,n}$ by $G = C_m \cup S_{n-m}$ (disjoint union of a cycle and a star) where $m = 4, 8$ and $m < n$. In all cases, G is a symmetric starter of the cyclic group of order n.

Keywords: Graph decomposition; Symmetric starter; Orthogonal double cover.

AMS Subject Classification: 05C70, 05B30

1. Introduction

Let H be a graph with n vertices and let $\mathcal{G} = \{G_0, \ldots, G_{n-1}\}$ be a collection of n spanning subgraphs of H (called *pages*). $\mathcal{G}$ is called an *orthogonal double cover* (ODC) of H if there exists a bijection $\varphi : V(H) \rightarrow \mathcal{G}$ such that:

(i) every edge of H is contained in exactly two of the graphs $G_0, \ldots, G_{n-1}$.

(ii) for every choice of different vertices a, b of H

$$|E(\varphi(a)) \cap E(\varphi(b))| = \begin{cases} 1 & \text{if } \{a,b\} \in E(H) \\ 0 & \text{otherwise.} \end{cases}$$

If all pages in $\mathcal{G}$ are isomorphic to a given graph G, then $\mathcal{G}$ is said to be an ODC of H by G. Note that in this case H is necessarily a regular graph of degree $|E(G)|$. Moreover, if H is not complete, G must be disconnected.

This concept was originally defined for the case where H is a complete graph. We refer the reader to the survey [3] for more details.

While in principle any regular graph H is worth considering (e.g., the remarkable case of hypercubes has been investigated in [4]), the choice of $H = K_{n,n}$ is quite natural, also in view of a technical motivation: ODCs in such graphs are of help in order to obtain ODCs of K_n (see [2], p. 48).

An algebraic construction of ODCs via "*symmetric starters*" (see Section 2) has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \le 9$: a few exceptions apart, all graphs G are found this way (see [2], Table 1). This method has been applied in [2] to detect some infinite classes of graphs G for which there is an ODC of $K_{n,n}$ by G.

Much of research on this subject focused with the detection of ODCs with pages isomorphic to a given graph G. So in [1] the graph considered was $G = (C_m \cup^v S_{n-m}) \cup nK_1$, where $\cup^v$ denotes the union of a cycle of length m and a $(n-m)$-star, whose center v lies in C_m, together with n isolated vertices (nK_1), as shown in Figure 1. For $m = 4$ and $m < n$ it was established in [1] that there is a symmetric starter of an ODC of $K_{n,n}$ by G as described above.

Here, we will improve this result, by showing that the same is true for $m = 6, 8, 10, 12$ and $m < n$. Namely, we shall prove the following.

Theorem 1.1. *Let n and m be integers such that $6 \le m \le 12$, $m < n$. Then there* is a *symmetric starter of an ODC of $K_{n,n}$ by $G = (C_m \cup^v S_{n-m}) \cup nK_1$.*

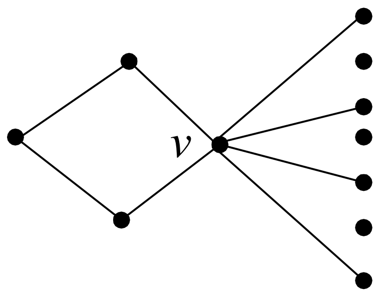

Figure 1: The graph $C_4 \cup^{v} S_4 \cup 3K_1$.

Furthermore, we will construct symmetric starters of an ODC of $K_{n,n}$ by $G = C_m \cup S_{n-m} \cup (n-1)K_1$ (the disjoint union of a cycle and a star and $n-1$ isolated vertices) where $m = 4, 8$ and $m < n$. Namely, we shall prove the following.

Theorem 1.2. *Let n and m be integers such that $m = 4, 8$ and $m < n$. Then there* is a *symmetric starter of an ODC of $K_{n,n}$ by $G = C_m \cup S_{n-m} \cup (n-1)K_1$.*

In addition, we will construct a symmetric starter of an ODC of $K_{n,n}$ by $G = C_6 \cup K_2 \cup S_{n-7} \cup (n-2)K_1$ (see Proposition 4.3).

Preliminaries will be exposed in Section 2. The case where the center of the star lies in the cycle will be discussed in Section 3, leading to the proof of Theorem 1.1. Likewise, the case where the cycle and the star are disjoint will be considered in Section 4, where Theorem 1.2 will be proved.

2. Symmetric starters

All graphs here are finite, simple and undirected.

Let $\Gamma = \{\gamma_0, \ldots, \gamma_{n-1}\}$ be an (additive) abelian group of order *n*. The vertices of $K_{n,n}$ will be labeled by the elements of $\Gamma \times \mathbb{Z}_2$. Namely, for $(v,i) \in \Gamma \times \mathbb{Z}_2$ we will write v_i for the corresponding vertex and define $\{w_i, u_j\} \in E(K_{n,n})$ if and only if $i \neq j$, for all $w, u \in \Gamma$ and $i, j \in \mathbb{Z}_2$.

Let G be a spanning subgraph of $K_{n,n}$ and let $a \in \Gamma$. Then the graph G with $E(G+a) = \{(u+a, v+a) : (u,v) \in E(G)\}$ is called the *a-translate* of G. The length of an edge $e = (u,v) \in E(G)$ is defined by $d(e) = v - u$. As an example, Figure 2 shows the edges of G_{0_0} labeled by their lengths.

G is called a half starter with respect to Γ if $|E(G)|=n$ and the lengths of all edges in G are different, i.e. $\{d(e): e\in E(G)\}=\Gamma$. The following three results were established in [2].

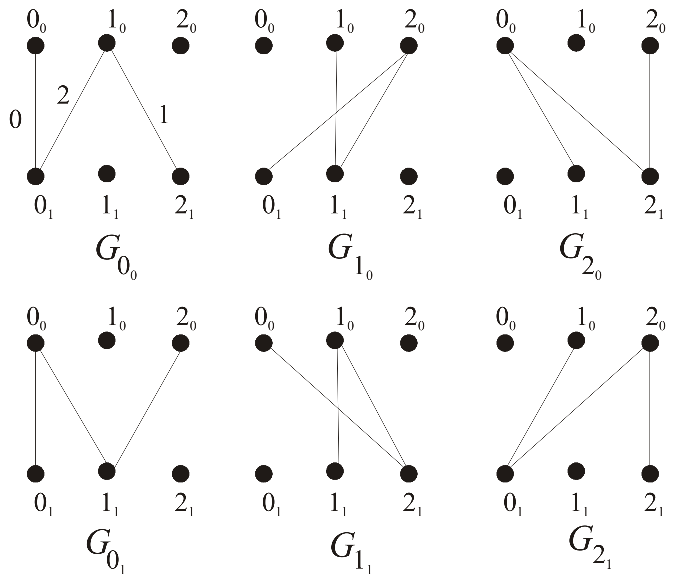

Figure 2: *ODC* of $K_{3,3}$ by $G=P_4$ with $\Gamma=\mathbb{Z}_3$.

Theorem 2.1. *If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$, i.e.* $\bigcup_{a\in\Gamma} E(G+a)=E(K_{n,n})$.

Here, the half starter will be represented by the vector: $v(G)=(v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}})$. Where $v_{\gamma_i}\in\Gamma$ and $(v_{\gamma_i})_0$ is the unique vertex $((v_{\gamma_i},0)\in\Gamma\times\{0\})$ that belongs to the unique edge of length γ_i. For example, in Figure 2 the graph G_{0_0} is a half starter with respect to $\mathbb{Z}_3$ represented by $(0,1,1)$ (e.g. $\{1_0,2_1\}$ is the unique edge of length 1, thus $v_1=1$).

Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be orthogonal if $\{v_\gamma(G_0)-v_\gamma(G_1):\gamma\in\Gamma\}=\Gamma$.

Theorem 2.2. *If two half starters $v(G_0)$ and $v(G_1)$ are orthogonal, then $G=\{G_{a,i} : (a,i)\in\Gamma\times\mathbb{Z}_2\}$ with $G_{a,i}=G_i+a$ is an ODC of $K_{n,n}$.*

The subgraph G_s of $K_{n,n}$ with $E(G_s)=\{\{u_0,v_1\}:\{v_0,u_1\}\in E(G)\}$ is called the symmetric graph of G. Note that if G is a half starter, then G_s is also a half starter .

A half starter G is called a symmetric starter with respect Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 2.3. *Let n be a positive integer and let G* be *a half starter represented by $v(G)=(v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}})$. Then G is symmetric starter if and only if*

$$\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma.$$

3. ODCs of $K_{n,n}$ by $G=(C_m \cup^v S_{n-m}) \cup nK_1$.

In view of Section 2, all we need is to find suitable symmetric starters for all the concerned parameters *n* and *m*. Each of the following lemmas provides a construction for a value of *m*.

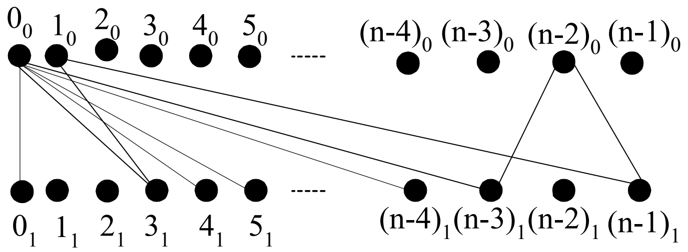

Figure 3: A symmetric starter of an ODC of $K_{n,n}$ by $G=(C_6 \cup^{0_0} S_{n-6}) \cup nK_1$.

Lemma 3.1. *For each integer $n \geq 7$ there is a symmetric starter of an ODC of $K_{n,n}$* by $G=(C_6 \cup^{0_0} S_{n-6}) \cup nK_1$.

Proof. Define the vector $v(G)$ as follows:

$$v_i(G)=\begin{cases} 1 & \text{if } i=2,\ n-2, \text{ or} \\ n-2 & \text{if } i=1,\ n-1, \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $v_i(G)=v_{-i}(G)$, hence $\{v_i(G)-v_{-i}(G)+i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$. The claim now follows from Theorem 2.3.

Note that the *i*-th graph isomorphic to $G=(C_6 \cup^{0_0} S_{n-6}) \cup nK_1$ has the edges:

$$\left\{\left\{(i+1)_0,(i+j)_1\right\} : j=3,n-1\right\} \cup$$
$$\left\{\left\{(i+n-2)_0,(i+j)_1\right\} : j=n-3,n-1\right\} \cup$$
$$\left\{\left\{i_0,(i+j)_1\right\} : j=0,3,4,\cdots,n-3\right\}$$

as shown in Figure 3.

Lemma 3.2. *For each integer* $n \geq 9$ *there is a is a symmetric starter of an ODC of* $K_{n,n}$ *by* $G = (C_8 \cup^{2_1} S_{n-8}) \cup nK_1$.

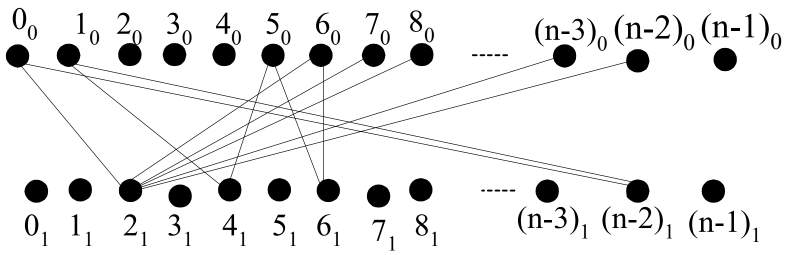

Figure 4: A symmetric starter of an ODC of $K_{n,n}$ by $G = (C_8 \cup^{2_1} S_{n-8}) \cup nK_1$.

Proof. Define the vector $v(G)$ as follows:

$$v_i(G) = \begin{cases} 6 & i = 0,\ n-4, \text{ or} \\ 5 & i = 1,\ n-1, \text{ or} \\ 0 & i = 2,\ n-2, \text{ or} \\ 1 & i = 3,\ n-3, \text{ or} \\ n-i+2 & \text{otherwise.} \end{cases}$$

Hence

$$v_{-i}(G) = \begin{cases} 6 & i = 0,\ n-4, \text{ or} \\ 5 & i = 1,\ n-1, \text{ or} \\ 0 & i = 2,\ n-2, \text{ or} \\ 1 & i = 3,\ n-3, \text{ or} \\ i+2 & \text{otherwise.} \end{cases}$$

So we get

$$v_i(G) - v_{-i}(G) + i = \begin{cases} i & i = 0,\ 1,\ 2,\ 3,\ n-4,\ n-3,\ n-2,\ n-1, \text{ or} \\ -i & \text{otherwise.} \end{cases}$$

The above implies $\{v_i(G) - v_{-i}(G) + i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$, so the claim follows from. Theorem 2.3.

Note that the i-th graph isomorphic to $G = (C_8 \cup^{2_1} S_{n-8}) \cup nK_1$ has the edges:

$$\left\{\{(i+j)_0, (i+2)_1\} : j = 0, 6, 7, \ldots, n-2\right\} \cup$$
$$\left\{\{(i+j)_0, (i+4)_1\} : j = 1, 5\right\} \cup$$
$$\left\{\{(i+j)_0, (i+6)_1\} : j = 5, 6\right\} \cup$$
$$\left\{\{(i+j)_0, (i+n-2)_1\} : j = 0, 1\right\}$$

as shown in Figure 4.

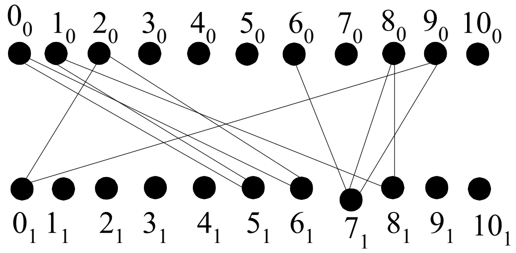

Figure 5: A symmetric starter of an ODC of $K_{11,11}$ by $G = (C_{10} \cup^{7_1} S_1) \cup 11K_1$.

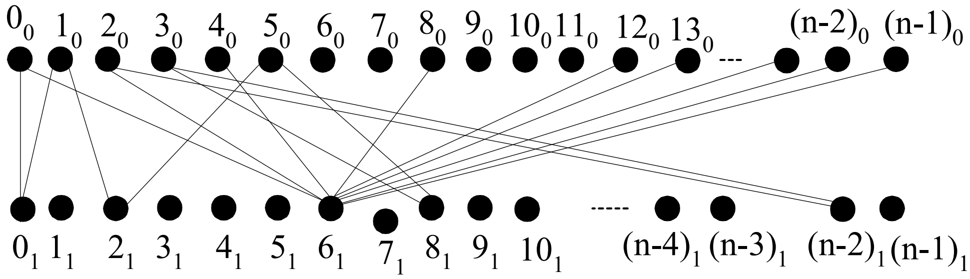

Figure 6: A symmetric starter of an ODC of $K_{n,n}$ by $G = (C_{10} \cup^{6_1} S_{n-10}) \cup nK_1$, $n \geq 12$.

Lemma 3.3. *For each integer $n \geq 11$ there is a symmetric starter of an ODC of $K_{n,n}$ by $G = (C_{10} \cup^{6_1} S_{n-10}) \cup nK_1$.*

Proof. For $n = 11$, the vector $v(G) = (8, 6, 9, 3, 1, 0, 0, 1, 3, 9, 8)$ is a symmetric starter of an ODC of $K_{11,11}$ by $G = (C_{10} \cup^{7_1} S_1) \cup 11K_1$, as shown in Figure 5.

Assume now that $n \geq 12$. Define the vector $v(G)$ as follows:

$$v_i(G) = \begin{cases} 0 & i = 0,\ 6, \text{ or} \\ 1 & i = 1,\ n-1, \text{ or} \\ 4 & i = 2, \text{ or} \\ 5 & i = 3,\ n-3, \text{ or} \\ 2 & i = 4,\ n-4, \text{ or} \\ 3 & i = 5,\ n-5, \text{ or} \\ 8 & i = n-2, \text{ or} \\ n-i+6 & \text{otherwise.} \end{cases}$$

Hence

$$v_{-i}(G)=\begin{cases}0 & i=0,\ 6,\text{ or}\\ 1 & i=1,\ n-1,\text{ or}\\ 4 & i=n-2,\text{ or}\\ 5 & i=3,\ n-3,\text{ or}\\ 2 & i=4,\ n-4,\text{ or}\\ 3 & i=5,\ n-5,\text{ or}\\ 8 & i=2,\text{ or}\\ i+6 & \text{otherwise.}\end{cases}$$

Then we get

$$v_i(G)-v_{-i}(G)+i=\begin{cases}i & i=0,1,3,4,5,n-5,n-4,n-3,n-1,\text{ or}\\ -i & \text{otherwise.}\end{cases}$$

This implies $\{v_i(G)-v_{-i}(G)+i : i\in\mathbb{Z}_n\}=\mathbb{Z}_n$, so by Theorem 2.3 $v(G)$ is a symmetric starter of an ODC of $K_{n,n}$ by $G=(C_{10}\cup^{6_1} S_{n-10})\cup nK_1$.

Note that the *i*-th graph isomorphic to $G=(C_{10}\cup^{6_1} S_{n-10})\cup nK_1$ has the edges:

$$\left\{\left\{(i+j)_0,\ i_1\right\}: j=0,\ 1\right\}\cup\left\{\left\{(i+j)_0,\ (i+2)_1\right\}: j=1,\ 5\right\}\cup$$
$$\left\{\left\{(i+j)_0,\ (i+6)_1\right\}: j=0,\ 2,\ 4,\ 8,\ 12,\ 13,\ 14,\ \ldots,\ n-1\right\}\cup$$
$$\left\{\left\{(i+j)_0,\ (i+8)_1\right\}: j=3,\ 5\right\}\cup\left\{\left\{(i+j)_0,\ (i+n-2)_1\right\}: j=2,\ 3\right\}$$

as shown in Figure 6. ■

Lemma 3.4. *For each integer* $n\ge 13$ *there is a symmetric starter of an ODC of* $K_{n,n}$ *by* $G=(C_{12}\cup^{v} S_{n-12})\cup nK_1$.

Proof. Let us first deal with two special cases.

For $n=14$, the vector $v(G)=(0,\ 2,\ 11,\ 0,\ 9,\ 10,\ 1,\ 0,\ 1,\ 10,\ 9,\ 0,\ 11,\ 2)$ is a symmetric starter of an ODC of $K_{14,14}$ by $G=(C_{12}\cup^{0_0} S_2)\cup 14K_1$ as shown in Figure 7.

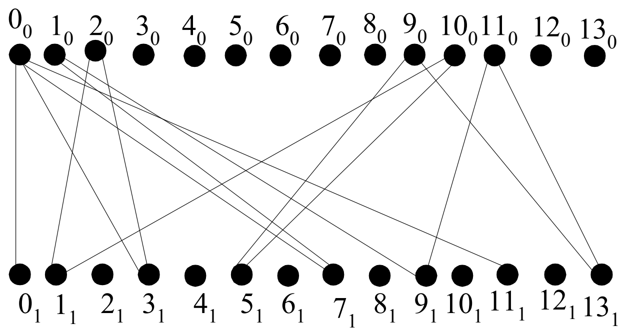

Figure 7: A symmetric starter of an ODC of $K_{14,14}$ by $G=(C_{12}\cup^{0_0} S_2)\cup 14K_1$.

For $n=16$, the vector $v(G)=(0,\ 10,\ 11,\ 0,\ 0,\ 6,\ 9,\ 8,\ 0,\ 8,\ 9,\ 6,\ 0,\ 0,\ 11,\ 10)$

is a symmetric starter of an ODC of $K_{16,16}$ by $G=(C_{12}\cup^{0_0} S_4)\cup 16K_1$ as shown in Figure 8.

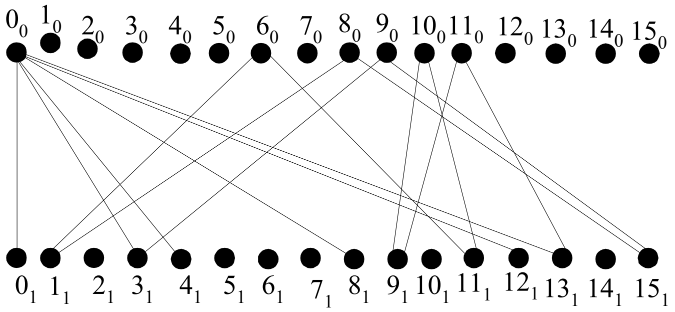

Figure 8: A symmetric starter of an ODC of $K_{16,16}$ by $G=(C_{12}\cup^{0_0} S_4)\cup 16K_1$.

From now on, assume that $n\neq 14,16$. Consider the vector
$v(G)=(4,\ n-1,\ 2,\ 9,\ 0,\ 1,\ n-2,\ 5,\ n-4,\ n-5,\ n-6,\ \ldots,\ 14,\ 13,\ 12,\ 5,\ 10,\ 1,\ 0,\ 9,\ 2,\ n-1)$
that is

$$v_i(G)=\begin{cases}4 & i=0, \text{ or}\\ n-1 & i=1,n-1, \text{ or}\\ 2 & i=2,n-2, \text{ or}\\ 9 & i=3,n-3, \text{ or}\\ 0 & i=4,n-4, \text{ or}\\ 1 & i=5,n-5, \text{ or}\\ 5 & i=7,n-7, \text{ or}\\ n-i+4 & \text{otherwise.}\end{cases}$$

Hence

$$v_{-i}(G)=\begin{cases}4 & i=0, \text{ or}\\ n-1 & i=1,n-1, \text{ or}\\ 2 & i=2,n-2, \text{ or}\\ 9 & i=3,n-3, \text{ or}\\ 0 & i=4,n-4, \text{ or}\\ 1 & i=5,n-5, \text{ or}\\ 5 & i=7,n-7, \text{ or}\\ i+4 & \text{otherwise.}\end{cases}$$

Then we get

$$v_i(G)-v_{-i}(G)+i=\begin{cases}i & i=0,1,2,3,4,5,7,n-1,n-2,n-3,n-4,n-5,n-7, \text{ or}\\ -i & \text{otherwise.}\end{cases}$$

This implies $\{v_i(G)-v_{-i}(G)+i : i\in\mathbb{Z}_n\}=\mathbb{Z}_n$, so by Theorem 2.3 it is a symmetric starter of an ODC of $K_{n,n}$ by $G=(C_{10}\cup^{6_1} S_{n-8})\cup nK_1$. Note that the i-th graph isomorphic to $G=(C_{12}\cup^{4_1} S_{n-12})\cup nK_1$, $n\geq 13$, and $n\neq 14,16$ has the edges:

$$\left\{\left\{(i+j)_0, i_1\right\}: j=2, n-1\right\} \cup$$
$$\left\{\left\{(i+j)_0, (i+4)_1\right\}: j=0,2,4,10,12,13,15,17,18,\ldots,n-2\right\} \cup$$
$$\left\{\left\{(i+j)_0, (i+6)_1\right\}: j=1,9\right\} \cup \left\{\left\{(i+j)_0, (i+12)_1\right\}: j=5,9\right\} \cup$$
$$\left\{\left\{(i+j)_0, (i+13)_1\right\}: j=0,1\right\} \cup \left\{\left\{(i+j)_0, (i+n-2)_1\right\}: j=5,n-1\right\}.$$

For illustration, Figure 9 shows the symmetric starter of an ODC of $K_{17,17}$ by $G=(C_{12} \cup^{4_1} S_5) \cup 17K_1$ and $v(G)=(4, 16, 2, 9, 0, 1, 15, 5, 13, 12, 5, 10, 1, 0, 9, 2, 16)$.

Proof of Theorem 1.1. For $m=4$ the statement was already proved in [1]. For $6 \le m \le 12$, $m<n$, Lemmas 3.1 to 3.4 provide symmetric starters of an ODC of $K_{n,n}$ by $G=(C_m \cup^{v} S_{n-m}) \cup nK_1$ with respect to $\mathbb{Z}_n$.

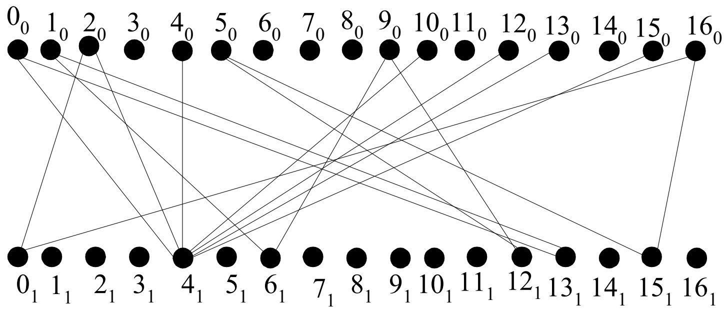

Figure 9: A symmetric starter of an ODC of $K_{17,17}$ by $G=(C_{12} \cup^{4_1} S_5) \cup 17K_1$.

4. ODC of $K_{n,n}$ by $C_m \cup S_{n-m} \cup (n-1)K_1$

In this section, we will construct symmetric starters of an ODC of $K_{n,n}$ by $G=C_m \cup S_{n-m} \cup (n-1)K_1$ (the disjoint union of a cycle and a star and n-1 isolated vertices) where $m=4, 8$ and $m<n$. The following two lemmas take care of cases $m=4$ and $m=8$ separately.

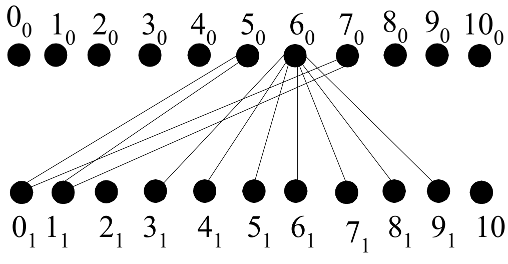

Figure 10: A symmetric starter of an ODC of $K_{11,11}$ by $G=C_4 \cup S_7 \cup 10K_1$.

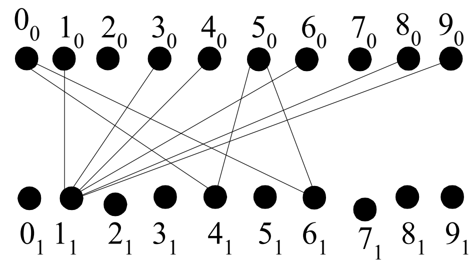

Figure 11: A symmetric starter of an ODC of $K_{10,10}$ by $G = C_4 \cup S_6 \cup 9K_1$.

Lemma 4.1. *For each integer* $n \geq 5$*, there exists a symmetric starter of an ODC of* $K_{n,n}$ *by* $G = C_4 \cup S_{n-4} \cup (n-1)K_1$.

Proof. Assume first that *n* is odd, say $n = 2h+1$. Define $v(G)$ as follows:

$$v_i(G) = \begin{cases} h+2 & i = h-1,\ h, \text{ or} \\ h & i = h+1,\ h+2, \text{ or} \\ h+1 & \text{otherwise.} \end{cases}$$

Hence

$$v_{-i}(G) = \begin{cases} h+2 & i = h+1,\ h+2, \text{ or} \\ h & i = h-1,\ h, \text{ or} \\ h+1 & \text{otherwise.} \end{cases}$$

Therefore

$$v_i(G) - v_{-i}(G) + i = \begin{cases} h+1 & i = h-1, \text{ or} \\ h+2 & i = h, \text{ or} \\ h-1 & i = h+1, \text{ or} \\ h & i = h+2, \text{ or} \\ i & \text{otherwise.} \end{cases}$$

This implies $\{v_i(G) - v_{-i}(G) + i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Thus by Theorem 2.3 $v(G)$ is a symmetric starter of an ODC of $K_{n,n}$ by $G = C_4 \cup S_{n-4} \cup (n-1)K_1$.
Note that the *i*-th *graph* isomorphic to $G = C_4 \cup S_{n-4} \cup (n-1)K_1$ has the following edges: $\{(i+h)_0, i_1\}, \{i_1, (i+h+2)_0\}, \{(i+h+2)_0, i_1\}, \{i_1, (i+h)_0\}$ and $\{(i+h+1)_0, (i+j)_1\}$ where $j = 3, 4, 5, \ldots, h-2, h-1, h, h+1, \ldots, n-2$. Figure 10 shows a symmetric starter of an ODC of $K_{11,11}$ by $G = C_4 \cup S_7 \cup 10K_1$.

Assume now that *n* is even, say $n = 2h$. Define the vector $v(G) = (1, h, n-1, n-2, n-3, \ldots, h+4, h+3, 0, h+1, 0, h-1, h-2, \ldots, 4, 3, h) \in \mathbb{Z}_n^n$ as follows:

$$v_i(G) = \begin{cases} h & i = 1,\ n-1, \text{ or} \\ 0 & i = h-1,\ h+1, \text{ or} \\ 2h-i+1 & \text{otherwise.} \end{cases}$$

Hence

$$v_{-i}(G)=\begin{cases} h & i=1,\ n-1, \text{ or} \\ 0 & i=h-1,\ h+1, \text{ or} \\ i+1 & \text{otherwise.} \end{cases}$$

Therefore

$$v_i(G)-v_{-i}(G)+i=\begin{cases} i & i=1, h-1, h+1, 2h-1, \text{ or} \\ -i & \text{otherwise.} \end{cases}$$

which implies $\{v_i(G)-v_{-i}(G)+i : i\in\mathbb{Z}_n\}=\mathbb{Z}_n$. By Theorem 2.3, $v(G)$ is a symmetric starter of an ODC of $K_{n,n}$ by $G=C_4\cup S_{n-4}\cup(n-1)K_1$.

Note that the i-th graph isomorphic to $G=C_4\cup S_{n-4}\cup(n-1)K_1$ has the following edges: $\{i_0,(i+h-1)_1\}, \{(i+h-1)_1, (i+h)_0\}, \{(i+h)_0, (i+h+1)_1\}, \{(i+h+1)_1, i_0\}$ and $\{(i+j)_0, (i+1)_1\}$ where $j=1, 3, 4, 5,\ldots, m-2, m-1, m+1, m+2,\ldots, n-1$. Figure 11 shows a symmetric starter of an ODC of $K_{10,10}$ by $G=C_4\cup S_6\cup 9K_1$.

Lemma 4.2. *For each integer $n\geq 9$, there exists a symmetric starter of an ODC of $K_{n,n}$ by $G=C_8\cup S_{n-8}\cup(n-1)K_1$.*

Proof. Define the vector $v(G)$ as follows:

$$v_i(G)=\begin{cases} 0 & i=2,\ 4, \text{ or} \\ 2 & i=1,\ 3, \text{ or} \\ 8 & i=n-3,\ n-4, \text{ or}, \\ 4 & i=n-1,\ n-2, \text{ or} \\ 1 & \text{otherwise.} \end{cases}$$

Hence

$$v_{-i}(G)=\begin{cases} 0 & i=n-2,\ n-4, \text{ or} \\ 2 & i=n-1,\ n-3, \text{ or} \\ 8 & i=3,\ 4, \text{ or} \\ 4 & i=1,\ 2, \text{ or} \\ 1 & \text{otherwise.} \end{cases}$$

Therefore

$$v_i(G)-v_{-i}(G)+i=\begin{cases} -i & i=1,\ 2,\ 3,\ 4,\ n-1,\ n-2,\ n-3,\ n-4, \text{ or} \\ i & \text{otherwise.} \end{cases}$$

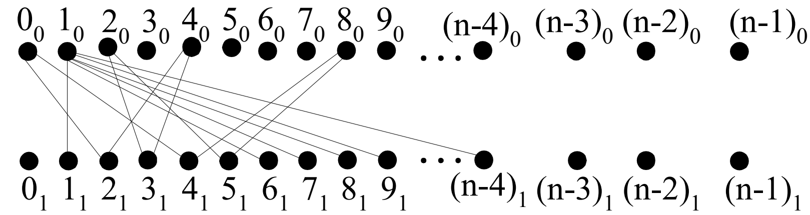

Figure 12: A symmetric starter of an ODC of $K_{n,n}$ by $G = C_8 \cup S_{n-8} \cup (n-1)K_1$.

which implies $\{v_i(G) - v_{-i}(G) + i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$. By Theorem 2.3 $v(G)$ is a symmetric starter of an ODC of $K_{n,n}$ by $G = C_8 \cup S_{n-8} \cup (n-1)K_1$.

As shown in Figure 12, note that the i-th graph isomorphic to $G = C_8 \cup S_{n-8} \cup (n-1)K_1$ has the following edges:

$$\{\{(i+j)_0, (i+2)_1\} : j = 0, 4\} \cup \{\{(i+j)_0, (i+3)_1\} : j = 2, 4\} \cup$$
$$\{\{(i+j)_0, (i+5)_1\} : j = 2, 8\} \cup \{\{(i+j)_0, (i+4)_1\} : j = 0, 8\} \cup$$
$$\{\{(i+1)_0, (i+j)_1\} : 1, 6, 7, 8, \ldots, n-4\}.$$

Proof of Theorem 1.2. For $m = 4, 8$ and $m < n$, Lemmas 4.1 to 4.2 provide symmetric starters of an ODC of $K_{n,n}$ by $G = C_m \cup S_{n-m} \cup (n-1)K_1$ with respect to $\mathbb{Z}_n$.

In addition to the above results, the following result can be deduced.

Proposition 4.3. *For each integer $n \geq 7$, there exists a symmetric starter of an ODC of $K_{n,n}$ by $G = C_6 \cup S_1 \cup S_{n-7} \cup (n-2)K_1$.*

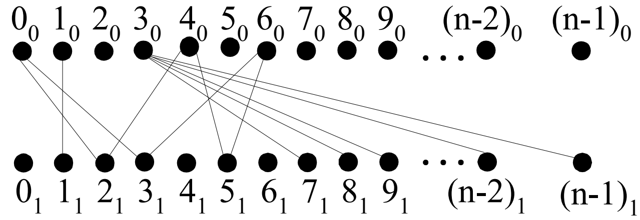

Figure 13: A symmetric starter of an ODC of $K_{n,n}$ by $G = C_6 \cup S_1 \cup S_{n-7} \cup (n-2)K_1$.

Proof. For each integer $n \geq 7$, the vector $v(G)$ defined below is a symmetric starter of an ODC of $K_{n,n}$ by $G = C_6 \cup S_1 \cup S_{n-7} \cup (n-2)K_1$.

$$v_i(G) = \begin{cases} 1 & i = 0, \text{ or} \\ 4 & i = 1, n-2, \text{ or} \\ 0 & i = 2, 3, \text{ or} \\ 6 & i = n-3, n-1, \text{ or} \\ 3 & \text{otherwise.} \end{cases}$$

Hence

$$v_{-i}(G)=\begin{cases}1 & i=0,\text{ or}\\ 4 & i=n-1,\ 2,\text{ or}\\ 0 & i=n-3,\ n-2,\text{ or}\\ 6 & i=1,\ 3,\text{ or}\\ 3 & \text{otherwise.}\end{cases}$$

Therefore

$$v_i(G)-v_{-i}(G)+i=\begin{cases}-i & i=1,\ 2,\ 3,\ n-1,\ n-2,\ n-3,\text{ or}\\ i & \text{otherwise,}\end{cases}$$

which implies $\{v_i(G)-v_{-i}(G)+i : i\in\mathbb{Z}_n\}=\mathbb{Z}_n$. By Theorem 2.3 the vector $v(G)$ is a symmetric starter of an ODC of $K_{n,n}$ by $G=C_6\cup S_1\cup S_{n-7}\cup(n-2)K_1$. As shown in Figure 13, note that the *i*-th graph isomorphic to $G=C_6\cup S_1\cup S_{n-7}\cup(n-2)K_1$ has the following edges:

$$\left\{\{(i+j)_0,\ (i+2)_1\} : j=0,\ 4\right\}\cup\left\{\{(i+j)_0,\ (i+5)_1\} : j=4,\ 6\right\}\cup$$
$$\left\{\{(i+j)_0,\ (i+3)_1\} : j=0,\ 6\right\}\cup\left\{(i+1)_0,\ (i+1)_1\right\}\cup$$
$$\left\{\{(i+3)_0,\ (i+j)_1\} : j=7,\ 8,\ 9,\ \ldots,\ n-1\right\}.$$

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