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Slice Hyperholomorphic Schur Analysis

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Prologue

Functions analytic and contractive in the open unit disk (also known as Schur functions) have applications to, and connections with, a host of domains, such as classical analysis (via for instance the theory of orthogonal polynomials), linear system theory, inverse scattering, signal processing and operator models, to name a few. Schur analysis can be defined as a collection of problems related to Schur functions (and to related classes of functions, such as functions analytic in the open unit disk and with a real positive part there) and their applications to these various fields. A key role in Schur analysis is played by reproducing kernel Hilbert spaces of functions, of the kind introduced by de Branges and Rovnyak (see e.g. [105, 106, 156]), associated to these functions.

It is of interest to consider Schur analysis in various other settings. Extensions have been made for instance to the case of several complex variables (the Schur-Agler classes; see [6, 7, 82, 83]), the case of upper triangular operators, [42, 159, 161], the case of compact Riemann surfaces [74, 75, 90, 91, 267, 268], and function theory on trees, see [49, 92], to name a few.

The purpose of this book is to define and study the counterpart of Schur functions and Schur analysis in the slice hyperholomorphic setting. There are at least two motivations for such a study, both having in the background the desire to replace the complex numbers by the quaternions. One motivation comes from the theory of linear systems and signal processing, see e.g. [209], [243]. Another motivation is to define new tools and problems in hypercomplex analysis inspired from the complex setting (for instance Nevanlinna-Pick interpolation [20] and the characteristic operator functions [32] to name two of them).

To set the work in perspective, it is well to mention a few words on classical Schur analysis. Given a Schur function, say s, the kernel

$$K_s(z,w) = \frac{1-s(z)\overline{s(w)}}{1-z\overline{w}} \tag{1}$$

is positive definite in $\mathbb{D}$. Equivalently, the multiplication by s is a contraction from the Hardy space of the unit disk $\mathrm{H}^2(\mathbb{D})$ into itself (such functions s are called Schur functions or Schur multipliers). The reproducing kernel Hilbert spaces associated to the kernels (1) were studied in details, also in the case of operator-valued functions, by de Branges and Rovnyak; see [105] and [106, Appendix]. These spaces allow, among other things, to develop one of the methods to solve interpolation problems in the class of Schur functions; See [25, 169] and see [26, 86, 81, 176, 250] for a sample of other methods. In each extension of Schur analysis it is in particular important to identify the correct notion of Schur function, and define the associated Hardy space (if possible) and the de Branges Rovnyak spaces.

In the setting of hypercomplex analysis, Schur functions have been considered in at least three different directions, namely:

1. In the setting of hypercomplex functions, Fueter series play then a key role. See for instance the papers [67, 69, 70, 71].

2. In the setting of bicomplex numbers. See [56].
3. In the setting of slice hyperholomorphic functions. See [20, 32, 33, 34, 35, 39] and [2] in the case of several quaternionic variables.

The present book is an introduction to Schur analysis in this latter setting. Such a study was recently initiated in a number of papers, of which we mention [20, 32, 33, 34, 35, 39]. The purpose of this work is to present in a systematic way the results presented in these papers, together with some necessary preliminaries, as well as a number of new results. See the paragraph entitled *note* later in the section.

The book is divided into three parts, namely *Classical Schur analysis*, *Quaternionic analysis*, and *Quaternionic Schur analysis*, and we now briefly outline their contents:

Classical Schur analysis: This part is for the convenience of readers from the quaternionic analysis community. There are a number of works on Schur analysis and its applications, of which we mention (in a non exhaustive way) in particular the books of Constantinescu [153], of Bakonyi and Constantinescu [79], and of Dym [169]. We also mention [11, 10], and, in the indefinite case, the survey [44]. We focus in particular on the notion of matrix-valued rational functions and their realizations, and on reproducing kernel Pontryagin spaces. We review the Schur algorithm and some of its applications. A Schur function can be seen as the reflection coefficient function of a so-called discrete first order system. We also briefly discuss the theory of these systems, and define in particular the scattering function and the asymptotic equivalence matrix function. Matrix-valued rational functions which take unitary values (with respect to a possibly indefinite metric; these functions are also called J-unitary rational functions) on the imaginary axis or the circle play an important role, and we survey their main properties.

Quaternionic analysis: We begin this part by providing some background material on quaternions, quaternionic polynomials and matrices with quaternionic entries. This material can be considered as classical and can be found for example in [231, 248, 276]. Because of the noncommutativity new features appear with respect to the complex case. In some cases, it is useful to translate the quaternionic formalism into the complex one by using a map which transform a quaternion into a 2×2 complex matrix or, analogously, a quaternionic matrix into a complex matrix of double size. Then we consider quaternionic functional analysis, with emphasis on Krein spaces. Several classical results in functional analysis extended to the quaternionic setting have appeared just in recent times, see e.g. [32, 37]. Although most of the classical proofs can be repeated or easily adapted to the quaternionic case, it is however useful to have the results collected here. Then we introduce the class of slice hyperholomorpic functions, both in the scalar (see [144, 188]) and operator-valued cases and we discuss the Hardy space of the unit ball and of the half-space, and the corresponding Blaschke products, see [32, 34, 35] Most of the material in this part is new, including the study of the Wiener algebra [29].
We also discuss the basic facts on the quaternionic functional calculus based on the S-spectrum, see [130, 132, 144], which is the basis to introduce the notion of realization

in this framework. We then discuss slice hyperholomorphic kernels and we extend the notion of Hardy space to the operator-valued case.

Quaternionic Schur analysis: We discuss some highlights of quaternionic Schur analysis, both for operator-valued and quaternionic-valued functions. We first define operator-valued generalized Schur functions and operator-valued generalized Herglotz functions, and characterize these functions in terms of realizations. The Hilbert space case is of special importance, and we also discuss the counterpart of Beurling's theorem in the present setting. The above classes consist of functions slice hyperholomorphic in the open unit ball of the quaternions. The analogs of these classes for the right half-space are also introduced and characterized in terms of realizations. We then turn to the case of matrix-valued functions and study rational functions, and their minimal realizations. Special emphasis is given on the counterpart of J-unitary rational functions. The theory of first order discrete systems provides examples of such functions. We also consider the analogs of some of the classical interpolation problems in the present setting, both in the scalar and operator-valued cases.

Note: We began our study of Schur analysis in the slice hyperholomorphic setting, in part with various coauthors some four years ago, and the material we presented in this book is largely new. In particular, the material in Chapter 9 and 11 appears for the first time, and one can find also new results in Chapters 7, 8 and 10. More precisely, in Chapter 9 we develop the theory of slice hyperholomophic rational functions and in Chapter 11 we consider a general one-sided interpolation problem in the operator-valued setting. In Chapter 7, we develop the operator-valued version of the Hardy space of the unit ball. In Chapter 8 we present in particular a Beurling-Lax theorem, and study Bohr's theorem in the present setting. Chapter 10 contains some new material on interpolation (the scalar Carathéodory-Fejér problem) and new results on first order discrete systems. However, the theory is still under development and as we solve problems, new challenges in quaternionic Schur analysis and its various applications arise. It provides the ground to develop new directions of research. We hope that the reader will take some of these challenges.

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Contents

Part I

Classical Schur analysis

The first part of this book is essentially intended to the readers from hypercomplex analysis, and in general, for people not necessarily familiar with the main aspects of Schur analysis. It provides motivation for the third part of the book, where counterparts of the notions and results of Part I are considered in the slice hyperholomorphic setting.

In the first chapter we discuss some classical interpolation (or extension) problems (namely the Nehari and the Carathéodory-Toeplitz problems) which play an important role in modern operator theory, see for instance [204, 205]. It should be noted that these problems and their solutions had a large influence in modern signal processing and optimal control theory (see for instance [155, 177, 218]).

To that purpose we also need to recall some aspects of the theory of indefinite inner product spaces, and in particular Krein and Pontryagin spaces. Moreover we review the main properties of the Wiener algebra and the theory of realization of matrix-valued rational functions.

We also introduce the various classes of (possibly operator-valued) meromorphic functions which appear in Schur analysis. We remark that these definitions (for instance J-contractive functions, see [245]) originated from operator theory and Schur analysis, and lead to new problems in classical function theory. As the reader will see in the sequel, such interactions and links occur also in the slice hyperholomorphic setting.

Finally we review the Schur algorithm and in particular its connections to the theory of first order discrete systems. These connections allow to make links with important notions such as the scattering matrix and the theory of layered medium (see for instance [153, 154]).

Chapter 1

Preliminaries

In this chapter we present some definitions and results which play an important role in Schur analysis. In particular we discuss indefinite inner product spaces, reproducing kernel spaces and two extension problems.

1.1 Some history

Albeit its title, this section has no historical pretense but we only wish to mention some key steps in the development of Schur analysis. In 1917-1918, and motivated by the trigonometric moment problem, Schur gave a new characterization of functions analytic and contractive in the open unit disk $\mathbb{D}$, see [257, 258]. In the sequel, we will denote the family of these functions by $\mathcal{S}$ and call them Schur functions. Rather than defining a function $s \in \mathcal{S}$ by its Taylor coefficients at the origin, Schur introduced a (possibly finite) family $\rho_0, \rho_1, \ldots$ of numbers in the open unit disk $\mathbb{D}$ and a family $s_0, s_1, \ldots$ of elements in $\mathcal{S}$ by the recursions $s_0(z) = s(z)$ and

$$\begin{aligned} \rho_n &= s_n(0), \\ s_{n+1}(z) &= \begin{cases} \dfrac{s_n(z) - s_n(0)}{z(1 - s_n(z)\overline{s_n(0)})}, & \text{for, } z \neq 0 \\ \dfrac{s_n'(0)}{1 - |s_n(0)|^2}, & \text{for } z = 0. \end{cases} \end{aligned} \tag{1.1}$$

The function $z \mapsto \frac{s_n(z)-s_n(0)}{1-s_n(z)\overline{s_n(0)}}$ belongs to $\mathcal{S}$ and vanishes at the origin. Hence, Schwarz' lemma insures that $s_{n+1} \in \mathcal{S}$ when $\rho_n \in \mathbb{D}$, while the maximum modulus principle forces s_{n+1} to be a unitary constant when $|s_{n+1}(0)| = 1$. The recursion (1.1), called the Schur algorithm, then stops at the index n. As proved by Schur, the recursion ends after a finite number of steps if and only if the function s is a finite Blaschke product, that is, if s is of the form

$$s(z) = c \prod_{t=0}^{n} \frac{z - a_t}{1 - z\overline{a_t}}, \tag{1.2}$$

where c belongs to the unit circle $\mathbb{T}$ and $a_1,\ldots,a_n \in \mathbb{D}$, while the sequence does not end if $|s_n(0)| < 1$, for any $n \in \mathbb{N}$.

The function s is uniquely obtained from the sequence $\rho_0, \rho_1, \ldots$ where, in case of a finite sequence, the last number is on $\mathbb{T}$. More precisely (and when $s_1 \not\equiv 0$), rewriting for $n = 0$ equation (1.1) as

$$s(z) = \frac{\rho_0 + zs_1(z)}{1 + z\overline{\rho_0}s_1(z)} = \rho_0 + \cfrac{(1-|\rho_0|^2)z}{\overline{\rho_0}z + \cfrac{1}{s_1(z)}} \tag{1.3}$$

one obtains the partial fraction expansion (see Wall's book [271, Theorem 77.1, p. 285])

$$s(z) = \rho_0 + \cfrac{(1-|\rho_0|^2)z}{\overline{\rho_0}z + \cfrac{1}{\rho_1 + \cfrac{(1-|\rho_1|^2)z}{\overline{\rho_1}z + \cfrac{1}{\ddots}}}} \tag{1.4}$$

The numbers $\rho_0, \rho_1, \ldots$ are called the Schur parameters of s and they can be expressed in terms of the Taylor coefficients. When they are computed from the Taylor coefficients of the associated function $\varphi(z) = \frac{1-s(z)}{1+s(z)}$ they are called Verblunsky parameters. They are then connected with the orthogonal polynomials associated with the positive measure μ appearing in the Herglotz integral representation of φ:

$$\varphi(z) = ia + \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t), \quad \text{(where } a \in \mathbb{R}\text{)}, \tag{1.5}$$

and to signal processing. See Kailath's survey paper [221].

More generally, the main characters of the topic are (possibly operator-valued) functions meromorphic in the disk or in a half-plane, and taking contractive values or having a positive real part there (possibly with respect to an indefinite metric), or with associated kernels having a finite number of negative squares (see Definition 1.2.9 for the latter).

The study of such functions occurred along the years in a number of instances, of which we mention:

1. The trigonometric moment problem.
2. Classical function theory.
3. The characteristic operator function of a close to self-adjoint operator.
4. The theory of linear systems and digital signal processing.
5. The theory of direct and inverse problems associated to first order discrete systems.

Schur functions have been extended to various settings, from several complex variables to Riemann surfaces, upper triangular operators and more and, as already mentioned, the purpose of this book is to define and study the counterpart of Schur functions in the slice hyperholomorphic setting.

Remark 1.1.1. Although an important part of the book deals with scalar or matrix-valued functions, some of the definitions and results are given in the setting of operator-valued functions.

1.2 Krein spaces, Pontryagin spaces and negative squares

Spaces endowed with an indefinite metric play an important role in the sequel, and we here review some definitions and results. When considering Schur analysis in the quaternionic setting these results need to be appropriately extended, see Chapter 5.

Consider a vector space $\mathscr{V}$ over the complex numbers, endowed with an Hermitian form $[\cdot,\cdot]$. Given two linear subspaces $\mathscr{V}_1$ and $\mathscr{V}_2$ such that $\mathscr{V}_1 \cap \mathscr{V}_2 = \{0\}$, we denote their direct sum by$\mathscr{V}_1 \oplus \mathscr{V}_2$. Two elements $v,w \in \mathscr{V}$ are called orthogonal with respect to this form if $[v,w] = 0$. Two linear subspaces $\mathscr{V}_1$ and $\mathscr{V}_2$ are orthogonal if every element of the first is orthogonal to every element of the second. We use the notation $\mathscr{V}_1[+]\mathscr{V}_2$ to denote the orthogonal sum. If moreover $\mathscr{V}_1 \cap \mathscr{V}_2 = \{0\}$, the sum (which is then also direct) is denoted by

$$\mathscr{V}_1[\oplus]\mathscr{V}_2.$$

Definition 1.2.1. The space $\mathscr{V}$ endowed with the Hermitian form $[\cdot,\cdot]$ is called a Krein space if there exist two subspaces $\mathscr{V}_+$, $\mathscr{V}_-$ such that $\mathscr{V}$ can be written as an orthogonal and direct sum

$$\mathscr{V} = \mathscr{V}_+[\oplus]\mathscr{V}_-, \tag{1.6}$$

where $(\mathscr{V}_+,[\cdot,\cdot])$ and $(\mathscr{V}_-,-[\cdot,\cdot])$ are both Hilbert spaces. The space $\mathscr{V}$ is called a Pontryagin space if $\mathscr{V}_-$ is finite dimensional. The dimension of the vector space $\mathscr{V}_-$ is called the index of the Pontryagin space.

The decomposition (1.6), called fundamental decomposition, will not be unique, unless one of its component equals the subspace $\{0\}$. The so-called signature operator or fundamental symmetry is the operator $J_{\mathscr{V}} : \ \mathscr{V} \to \mathscr{V}$ such that

$$J_{\mathscr{V}}(v) = J_{\mathscr{V}}(v_+ + v_-) = v_+ - v_-, \qquad v_\pm \in \mathscr{V}_\pm.$$

The Krein space $\mathscr{V}$ becomes a Hilbert space when endowed with the inner product:

$$\langle v,w\rangle = [v_+,w_+] - [v_-,w_-], \tag{1.7}$$

where $v = v_+ + v_-$ and $w = w_+ + w_-$ belong to $\mathscr{V}$ and $v_\pm, w_\pm \in \mathscr{V}_\pm$. The inner product (1.7) depends on the given decomposition, but all the resulting norms are equivalent, and this defines the topology of the Krein space (see [98, p. 102]; the proof of this fact is given in the quaternionic setting in Chapter 5; see Theorem 5.8.5). The notion of continuity and of convergence are with respect to this topology. Given two Krein spaces $\mathscr{V}$, $\mathscr{W}$ we denote by $\mathbf{B}(\mathscr{V},\mathscr{W})$ the set of continuous linear operators from $\mathscr{V}$ to $\mathscr{W}$ and when $\mathscr{W} = \mathscr{V}$ we will use the symbol $\mathbf{B}(\mathscr{V})$. By $I_{\mathscr{V}}$ we denote the identity operator on $\mathscr{V}$.

Remark 1.2.2. In the finite dimensional case, let $\mathscr{V} = \mathbb{C}^m$ and let $J \in \mathbb{C}^{m\times m}$ be a signature matrix. Then $\mathscr{V}$ endowed with the Hermitian form

$$[u,v] = v^*Ju, \quad u,v \in \mathbb{C}^m,$$

is a Pontryagin space and a fundamental symmetry is given by the map $u \mapsto Ju$.

The notion of adjoint of an operator can be done with respect to the Hilbert space inner product or with respect to the Krein space inner product. More precisely, let $(\mathscr{V}, \langle\cdot,\cdot\rangle_{\mathscr{V}})$, $(\mathscr{W}, \langle\cdot,\cdot\rangle_{\mathscr{W}})$ be Hilbert spaces. Given $A \in \mathbf{B}(\mathscr{V},\mathscr{W})$ its adjoint is the unique operator $A^* \in \mathbf{B}(\mathscr{W},\mathscr{V})$ such that

$$\langle Af,g\rangle_{\mathscr{W}} = \langle f,A^*g\rangle_{\mathscr{V}}, \qquad f \in \mathscr{V}, g \in \mathscr{W}.$$

Definition 1.2.3. Let $(\mathscr{V}, [\cdot,\cdot]_{\mathscr{V}})$ and $(\mathscr{W}, [\cdot,\cdot]_{\mathscr{W}})$ be two Krein spaces. Given $A \in \mathbf{B}(\mathscr{V},\mathscr{W})$ its adjoint is the unique operator $A^{[*]} \in \mathbf{B}(\mathscr{W},\mathscr{V})$ such that

$$[Af,g]_{\mathscr{W}} = [f,A^{[*]}g]_{\mathscr{V}}, \qquad f \in \mathscr{V}, g \in \mathscr{W}.$$

The Krein spaces $\mathscr{V}$ and $\mathscr{W}$ are Hilbert spaces when endowed with the Hermitian forms

$$\langle f_1,f_2\rangle_{\mathscr{V}} = [f_1,J_{\mathscr{V}}f_2]_{\mathscr{V}} \quad \text{and} \quad \langle g_1,g_2\rangle_{\mathscr{W}} = [g_1,J_{\mathscr{W}}g_2]_{\mathscr{W}} \quad f_1,f_2 \in \mathscr{V}, \quad g_1,g_2 \in \mathscr{W}. \tag{1.8}$$

The Hilbert space adjoint A^* (with respect to the inner products (1.8)) and the Krein space adjoint $A^{[*]}$ are related by the formula

$$A^* = J_{\mathscr{V}}A^{[*]}J_{\mathscr{W}} \tag{1.9}$$

where $J_{\mathscr{V}}$ and $J_{\mathscr{W}}$ are associated to some fundamendal decompositions of $\mathscr{V}$ and $\mathscr{W}$. To prove (1.9) note that

$$\begin{aligned}[Af,g]_{\mathscr{W}} &= \langle J_{\mathscr{W}}Af,g\rangle \\ &= \langle f,A^*J_{\mathscr{W}}g\rangle \\ &= [f\,,J_{\mathscr{V}}A^*J_{\mathscr{W}}g]_{\mathscr{V}}.\end{aligned} \tag{1.10}$$

Definition 1.2.4. An operator $A \in \mathbf{B}(\mathscr{V},\mathscr{W})$ is said to be:
(1) isometric if $A^{[*]}A = I_{\mathscr{V}}$;
(2) coisometric if $AA^{[*]} = I_{\mathscr{W}}$;
(3) unitary if it is isometric and coisometric;
(4) a contraction if $[Af,Af]_{\mathscr{W}}] \le [f,f]_{\mathscr{V}}$, $f \in \mathscr{V}$.

Definition 1.2.5. A subspace $\mathscr{L}$ of a Krein space $\mathscr{V}$, $[\cdot,\cdot]$ is said to be nonpositive if $[f,f] \le 0$ for all $f \in \mathscr{L}$. It is said uniformly negative if there exists $\delta > 0$ such that $[f,f] \le -\delta\|f\|^2$, for all $f \in \mathscr{L}$, where $\|f\|$ denotes the norm associated to one of the fundamental decompositions. It is said maximal nonpositive (resp. maximal uniformly negative) if it is nonpositive (resp. uniformly negative) and not properly contained in a subspace of $\mathscr{V}$ having the same property.

Analogous definitions, with reversed inequalities can be given in the case of subspaces (maximal) nonnegative, (maximal) uniformly positive.

We will mention in Chapter 5 the main results from the theory of indefinite inner product spaces needed, in the quaternionic setting, in Schur analysis. Here we content ourselves to mention five important results which are used in Schur analysis. We refer to the book [216] by Iohvidov, Krein and Langer for proofs of the first three ones, to [259], [262] for the fourth one, and to [72] for the fifth.

Theorem 1.2.6. *The adjoint of a contraction between Pontryagin spaces of same index is a contraction.*

Theorem 1.2.7. *A contraction between Pontryagin spaces of same index has a maximal strictly negative invariant subspace.*

Theorem 1.2.8. *A densely defined contractive relation between Pontryagin spaces of same index extends to the graph of an everywhere defined contraction.*

Before stating the last two results alluded to above, we need two definitions. The first definition introduces the notions of negative squares and kernels. It is given in the general case in which the coefficient space is a Krein space. Note that, in the sequel, we will often use the symbol $\mathscr{K}$ to denote a Krein space (which will often play the role of a coefficient space) and $\mathscr{P}$ to denote a Pontryagin space:

Definition 1.2.9. Let Ω be some set and let $\mathscr{K}$ be a Krein space. The $\mathbf{B}(\mathscr{K})$-valued function $K(z,w)$ defined on $\Omega\times\Omega$ is said to have κ negative squares if it is Hermitian

$$K(z,w) = K(w,z)^{[*]}, \quad \forall z,w\in\Omega$$

and if for every choice of $N\in\mathbb{N}$, $c_1,\ldots,c_N\in\mathscr{K}$ and $w_1,\ldots,w_N\in\Omega$ the $N\times N$ Hermitian matrix with (u,v)-entry equal to

$$[K(w_u,w_v)c_v,c_u]_{\mathscr{K}}$$

has at most κ strictly negative eigenvalues, and exactly κ such eigenvalues for some choice of $N,c_1,\ldots,c_N,w_1,\ldots,w_N$.
The function is called positive definite if $\kappa=0$, that is if all the above Hermitian matrices are nonnegative (remark that this standard terminology is a bit unfortunate. Note also that one uses also the term *kernel* rather that function).

We will refer to the function K as kernel. An important related notion is the one of reproducing kernel Pontryagin space.

Definition 1.2.10. Let Ω be some set and let $\mathscr{K}$ be a Krein space, and let $\mathscr{P}$ be a Pontryagin space of $\mathscr{K}$-valued functions defined on Ω. Then $\mathscr{P}$ is called a reproducing kernel Pontryagin space if there exists a $\mathbf{B}(\mathscr{K})$-valued function $K(z,w)$ with the following two properties: For every $c\in\mathscr{K}$, $w\in\Omega$ and $F\in\mathscr{P}$,
(1) The function $z\mapsto K(z,w)c$ belongs to $\mathscr{P}$.
(2) It holds that

$$[F,K(\cdot,w)c]_{\mathscr{P}} = [F(w),c]_{\mathscr{K}}. \tag{1.11}$$

The function $K(z,w)$ is called the reproducing kernel of the space. It is Hermitian. We say that K has finite rank if the associated reproducing kernel Pontryagin space is finite dimensional. By Riesz' representation theorem it is uniquely defined. The following theorem relates the two above definitions, and originates with the work of Aronszajn [76, 77] in the case of positive definite kernels and Schwartz [259] and Sorjonen [262] in the case of negative squares. See also [47, Theorem 1.1.3, p. 7] for a proof.

Theorem 1.2.11. *Let $\Omega \subseteq \mathbb{C}$ be some set and let $\mathscr{K}$ be a Krein space. There is a one-to-one correspondence between reproducing kernel Pontryagin spaces of $\mathscr{K}$-valued functions defined on Ω and $\mathbf{B}(\mathscr{K})$-valued functions having a finite number of negative squares in Ω.*

Remark 1.2.12. The function $K(z,w)$ has κ negative squares in Ω if and only if it can be written as a difference $K(z,w) = K_+(z,w) - K_-(z,w)$, where both $K_+(z,w)$ and $K_-(z,w)$ are positive definite in Ω and moreover K_- has rank κ.
The above theorem fails for reproducing kernel Krein spaces. Schwartz proved that there is an onto (but not one-to-one) correspondence between reproducing kernel Krein spaces and Hermitian functions which are differences of positive definite functions on Ω. See [259] and [9] for counterexamples.

Definition 1.2.13. Let A be a self-adjoint operator in a Pontryagin space $\mathscr{P}$. We say that A has κ negative squares, and write $\kappa = \nu_-(A)$ if the function $K(f,g) = [Af,g]_{\mathscr{P}}$ has κ negative squares.

Theorem 1.2.14. *Let A be a bounded, self-adjoint operator from the Pontryagin space $\mathscr{P}$ into itself, which has a finite number of negative squares. Then, there exists a Pontryagin space $\mathscr{P}_1$ with $\mathrm{ind}_{\mathscr{P}_1} = \nu_-(A)$, and a bounded right linear operator T from $\mathscr{P}$ into $\mathscr{P}_1$ such that*

$$A = T^{[*]}T.$$

1.3 The Wiener algebra

The Wiener algebra of the unit circle was introduced in the thirties of the previous century by Wiener in [275] and plays an important role in harmonic analysis. It lies between the algebra of rational functions analytic on the unit circle $\mathbb{T}$ and $L^\infty(\mathbb{T})$. Later on, it has been realized that the fact that its elements are continuous on the unit circle and the algebra structure makes a number of problems such as the Nehari extension problem, see Section 1.4, or the Carathéodory-Fejér interpolation problem, see for Theorem 1.5.2) best understood in its setting.

Definition 1.3.1. The Wiener algebra of the unit circle $\mathscr{W}^{r\times r}$ (we will write $\mathscr{W}$ when $r = 1$) consists of the functions of the form

$$f(e^{it}) = \sum_{u\in\mathbb{Z}} f_u e^{iut}$$

where $f_u \in \mathbb{C}^{r\times r}$ and

$$\sum_{u\in\mathbb{Z}} \|f_u\| < \infty, \tag{1.12}$$

where $\|\cdot\|$ denotes the operator norm.

There is also a version of the Wiener algebra for the real line. Since we will not consider it here, we will just write *Wiener algebra* rather than the more precise *Wiener algebra of the circle*.

Remark 1.3.2. It is useful to note that the product is jointly continuous in the two variables in the Wiener algebras. It is also useful to note that the function $t \mapsto (f(e^{it}))^*$ belongs to $\mathscr{W}^{r\times r}$ when f belongs to $\mathscr{W}^{r\times r}$.

The space $\mathscr{W}^{r\times r}$ with pointwise multiplication and norm (1.12) is a Banach algebra of functions continuous on the unit circle. Rational functions without poles on the unit circle belong to $\mathscr{W}$. This can be seen using the partial fraction expansion of the given function. We note that any rational function without poles on $\mathbb{T}$ is in the space $L^\infty(\mathbb{T})$.

We denote by $\mathscr{W}_+^{r\times r}$ (resp. $\mathscr{W}_-^{r\times r}$) the subalgebras of functions f for which $f_u = 0$ for $u < 0$ (resp. $f_u = 0$ for $u > 0$). Elements of $\mathscr{W}_+^{r\times r}$ are analytic in the open unit disk, and continuous in the closed unit disk, while elements of $\mathscr{W}_-^{r\times r}$ are analytic in the exterior of the closed unit disk, and continuous in the complement of the open unit disk.

The celebrated Wiener-Lévy theorem, see [275], [235, Théorème V, p. 10], characterizes invertible elements of $\mathscr{W}$. In the case of matrix-valued functions it takes the form:

Theorem 1.3.3. *A function $f \in \mathscr{W}^{r\times r}$ is invertible in this algebra if and only if it is pointwise invertible:*

$$\det f(e^{it}) \neq 0, \quad \forall t \in \mathbb{R}.$$

Similarly, $f \in \mathscr{W}_+^{r\times r}$ (resp. in $\mathscr{W}_-^{r\times r}$) is invertible in $\mathscr{W}_+^{r\times r}$ (resp. in $\mathscr{W}_-^{r\times r}$) if and only if $\det f(z) \neq 0$ for all z in the closed unit disk (resp. in the complement of the open unit disk).

An important notion is that of Wiener-Hopf factorization:

Definition 1.3.4. The function $f \in \mathscr{W}^{r\times r}$ admits a left (resp. right) Wiener-Hopf factorization if it can be written as

$$f = f_+ f_- \quad (\text{resp. } f = f_- f_+)$$

where f_+ and its inverse belong to $\mathscr{W}_+^{r\times r}$ and f_- and its inverse belong to $\mathscr{W}_-^{r\times r}$.

This notion plays an important role in a number of topics, of which we mention (in the setting of the continuous Wiener algebra) singular integral equations and convolution integral equations. See for instance [207]. In the present work we will see an example of such factorization for the scattering matrix function; see Theorem 3.3.7.

1.4 The Nehari extension problem

The Nehari extension problem originates with Nehari's paper [239] and is as follows:

Problem 1.4.1. *Given complex numbers $\dots, f_{-2}, f_{-1}, f_0$, find a necessary and sufficient condition for numbers $f_1, f_2, \dots$ to exist such that the a priori formal series $f(e^{it}) = \sum_{n\in\mathbb{Z}} f_n e^{int}$ is such that*

$$\sup_{t\in[0,2\pi]} |\sum_{n\in\mathbb{Z}} f_n e^{int}| < \infty.$$

Of particular interest is the case where the above supremum is strictly less than 1; another case of interest is when the function f takes almost everywhere unitary values on the unit circle. Nehari proved that the problem is solvable if and only if the infinite Hankel matrix

$$\Gamma = \begin{pmatrix} f_0 & f_{-1} & \cdots \\ f_{-1} & f_{-2} & \cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{pmatrix}$$

defines a bounded operator from ℓ_2 into itself. Nehari did not describe the set of all solutions of these various versions of the problem. This was addressed later, for example in the works of Adamyan, Arov and Krein (see [3, 4, 5]), Dym and Gohberg [170, 171], Arov and Dym [78] and others. A description of the set of all strictly contractive solutions in the setting of the Wiener algebra, using the band method, can be found in [205, Chapter XXXV.4, p. 956].

Nehari's problem has applications in H^∞-control theory; see for instance [162], [176, p. 247], [178].

The problem is better understood if one considers a family of Nehari extension problems, rather than an isolated one. More precisely, and in the setting of the Wiener algebra, we set:

Problem 1.4.2. *Given $n \in \mathbb{N}_0$, and given $r \times r$ matrices $f_u, u = -n, -n-1, \dots$ such that*

$$\sum_{u=-n}^{-\infty} \|f_u\| < \infty$$

find a necessary and sufficient condition for $r \times r$ matrices $f_u, u = -n+1, -n+2, \dots$ to exist, such that

$$\sum_{u=-n+1}^{\infty} \|f_u\| < \infty$$

and

$$W(e^{it})(W(e^{it}))^* < I_r, \quad t \in [0, 2\pi]$$

where $W(e^{it}) = \sum_{u\in\mathbb{Z}} f_n e^{iut}$.

To solve this problem one needs first to introduce the block Hankel operator

$$\Gamma_n = \begin{pmatrix} f_{-n} & f_{-n-1} & \cdots \\ f_{-n-1} & f_{-n-2} & \cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{pmatrix}, \quad n = 0, 1, \ldots \tag{1.13}$$

To present the solution of Problem 1.4.2 we first need some preliminary definitions (see [205]): Let

$$e = \begin{pmatrix} I_r \\ 0 \\ 0 \\ \vdots \end{pmatrix} \in \ell_2^{r\times r},$$

consider the solutions $a_n, b_n, c_n, d_n \in \ell_2^{r\times r}$ of the equations

$$\begin{pmatrix} I_{\ell_2^{r\times r}} & -\Gamma_n \\ -\Gamma_n^* & I_{\ell_2^{r\times r}} \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} e \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_{\ell_2^{r\times r}} & -\Gamma_n \\ -\Gamma_n^* & I_{\ell_2^{r\times r}} \end{pmatrix} \begin{pmatrix} c_n \\ d_n \end{pmatrix} = \begin{pmatrix} 0 \\ e \end{pmatrix}, \tag{1.14}$$

and set

$$\begin{aligned} \alpha_n(z) &= a_{n0} + a_{n1}z^{-1} + \cdots \\ \beta_n(z) &= b_{n0} + b_{n1}z^{-1} + \cdots \\ \gamma_n(z) &= c_{n0} + c_{n1}z + \cdots \\ \delta_n(z) &= d_{n0} + d_{n1}z + \cdots \end{aligned} \tag{1.15}$$

It is known that $\alpha_n^{\pm} \in \mathscr{W}_-$, $\beta_n \in \mathscr{W}_-$ and $\delta_n^{\pm} \in \mathscr{W}_+$ and $\gamma_n \in \mathscr{W}_+$.

Theorem 1.4.3. *A necessary and sufficient condition for the Nehari extension problem to be solvable is that $\|\Gamma_n\| < 1$. When this condition holds, the set of all solutions is described as follows. A function f is a solution of the Nehari extension problem if and only if it can be written as*

$$f(z) = (\alpha_n(z)a_{n0}^{-1/2}e(z) + \beta_n(z)d_{n0}^{-1/2})(\gamma_n(z)a_{n0}^{-1/2}e(z) + \delta_n(z)d_{n0}^{-1/2})^{-1},$$

where e varies in the functions of $\mathscr{W}^{r\times r}$ taking strictly contractive values on the unit circle.

The functions

$$H_n(z)\begin{pmatrix} a_{n0}^{-1/2} & 0 \\ 0 & d_{n0}^{-1/2} \end{pmatrix}$$

with

$$H_n(z) = \begin{pmatrix} \alpha_n(z) & \beta_n(z) \\ \gamma_n(z) & \delta_n(z) \end{pmatrix} \tag{1.16}$$

have a specific boundary property, called J_0-unitarity (with J_0 as in (1.22)). Since we consider these facts only in the rational case we postpone the discussion to Chapter 2. For more information we refer to [204, 205]. Furthermore, the functions $H_n(z)$ form a discrete linear system. See Section 3.3 for the definition and discussion of this aspect.

We conclude this section with connecting a special case of the Nehari extension problem with the Carathéodory-Fejér problem interpolation for Schur functions (see Problem 3.2.1). More precisely, assume that in Problem 1.4.1 we have $f_{-N-1} = f_{-N-2} = \cdots = 0$. Then $f(z) = \sum_{n=-N}^{\infty} f_n z^n$ satisfies $\sup_{z\in\mathbb{T}} |f(z)| \le 1$ if and only if the function $z^N f(z)$ is a Schur function, and thus the problem is then equivalent to finding all Schur functions whose Taylor series at the origin begins with $\sum_{n=0}^{N} f_{n-N} z^n$.

1.5 The Carathéodory-Toeplitz extension problem

The Carathéodory-Toeplitz extension problem can be set as follows:

Problem 1.5.1. *Given $r \times r$ matrices $t_{-n}, t_{-(n-1)}, \ldots, t_0, t_1, \ldots, t_n$ such that $t_j = t_{-j}^*$, $j = 0, 1, \ldots, n$:*
(1) *Find a necessary and sufficient condition for matrices $t_{n+1}, \ldots,$ to exist such that all Toeplitz matrices $T_m = (t_{i-j})_{i,j=0,\ldots,m}$ satisfy*

$$T_m > 0, \quad m = n+1, n+2, \ldots.$$

(2) *Describe the set of all solutions when this condition holds.*

In the scalar case, one way to solve this problem is by one-step extensions. Given $T_m > 0$ find all $t_{m+1} \in \mathbb{C}$ such that

$$T_{m+1} = \begin{pmatrix} & & & t_{m+1} \\ & & & t_m \\ & T_m & & \vdots \\ & & & t_1 \\ \overline{t_{m+1}} & \overline{t_m} & \cdots \quad \overline{t_1} & t_0 \end{pmatrix} > 0.$$

Using Schur complements it is easy to check that the t_{m+1} varies in the open disk with center $a_m T_{m-1}^{-1} b_m$ and radius $\sqrt{(t_0 - b_m^* T_{m-1}^{-1} b_m)(t_0 - a_m T_{m-1}^{-1} a_m^*)}$, where

$$a_m = \begin{pmatrix} t_1 & \cdots & t_m \end{pmatrix} \quad \text{and} \quad b_m = \begin{pmatrix} t_m \\ \vdots \\ t_1 \end{pmatrix}.$$

It follows from this analysis that, in the scalar case, a necessary and sufficient condition for Problem 1.5.1 to have a solution is that $T_n > 0$. This condition is still necessary and sufficient in the matrix-valued case. Assuming now this condition, we present a solution to Problem 1.5.1 in terms of a linear fractional transformation. We set

$$T_n^{-1} = (\gamma_{ij}^{(n)})_{i,j=1,n}$$

where the blocks $\gamma_{ij}^{(n)} \in \mathbb{C}^{r\times r}$. We set (see [168, p. 80])

$$\begin{aligned}
A_n(z) &= \sum_{\ell=0}^{n} z^\ell \gamma_{\ell 0}^{(n)},\\
C_n(z) &= \sum_{\ell=0}^{n} z^\ell \gamma_{\ell n}^{(n)},\\
A_n^\circ(z) &= 2I_r - \sum_{\ell=0}^{n} p_\ell(z)\gamma_{\ell 0}^{(n)},\\
C_n^\circ(z) &= \sum_{\ell=0}^{n} p_\ell(z)\gamma_{\ell n}^{(n)},
\end{aligned}$$

where $p_\ell(z) = z^\ell t_0 + 2\sum_{s=1}^{\ell} z^{\ell-s} t_s^*$.

In order to describe the solutions of the problem it is necessary to first associate to a (potential solution) the function

$$\Phi(z) = t_0 + 2\sum_{u=1}^{\infty} t_{-u} z^n.$$

The conditions $T_m > 0$ for $m = n, n+1, \dots$ force the matrices t_m to be uniformly bounded in norm, and thus $\Phi(z)$ converges in $\mathbb{D}$. Furthermore, we have the formula

$$\frac{\Phi(z)+\Phi(w)^*}{2(1-z\overline{w})} = \sum_{n,m=0}^{\infty} z^n \overline{w}^m t_{n-m}, \quad z, w \in \mathbb{D}. \tag{1.17}$$

The function Φ will be analytic and with a positive real part in the open unit disk if and only if the sequence $t_0, \dots, t_n, t_{n+1}, \dots$ is a solution to the given extension problem. Hence the given sequence is a solution to the Carathéodory-Toeplitz extension problem if and only if the corresponding function Φ is such that

$$\Phi(z) = \underbrace{t_0 + 2t_{-1}z + \cdots + 2t_{-n}z^n}_{\text{fixed}} + 2t_{-(n+1)}z^{n+1} + \cdots$$

This is the Carathéodory-Fejér interpolation problem for matrix-valued Carathéodory functions (see also Problem 3.2.1). The next theorem describes the set of all solutions in the Wiener algebra, and is taken from Dym's paper [168].

Theorem 1.5.2. *The linear fractional transformation*

$$\Phi(z) = (A_n^\circ(z) - zC_n^\circ(z)G(z))(A_n(z) + zC_n(z)G(z))^{-1}$$

describes the set of all solutions of the Carathéodory-Fejér problem which belong to the Wiener class $\mathscr{W}^{r\times r}$ when G varies in the class of elements of $\mathscr{W}_+^{r\times r}$ which are moreover strictly contractive on the unit circle.

1.6 Various classes of functions and realization theorems

In this section we introduce various families of meromorphic functions and associated reproducing kernel Pontryagin spaces which play an important role in Schur analysis and in the present book. The parallel section in the slice hyperholomorphic setting is Section 8.1. The reader should be aware that some other important families are not or barely considered here, both in the classical and in the quaternionic setting. For instance we will not study here Hilbert spaces of entire functions of the type introduced by de Branges, partially in collaboration with Rovnyak (see [156, 172, 173]).

Definition 1.6.1. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two Pontryagin spaces of same index. The $\mathbf{B}(\mathscr{P}_1,\mathscr{P}_2)$-valued function S meromorphic in an open subset Ω of the unit disk is called a generalized Schur function if the kernel

$$\frac{I_{\mathscr{P}_2}-S(z)S(w)^{[*]}}{1-z\overline{w}} \tag{1.18}$$

has a finite number κ of negative squares in Ω.
The class containing such functions is denoted by $\mathcal{S}_\kappa(\mathscr{P}_1,\mathscr{P}_2)$ or simply $\mathcal{S}_\kappa(\mathscr{P})$ when $\mathscr{P}_1=\mathscr{P}_2=\mathscr{P}$.

One can use the Hilbert space structures of the coefficient spaces associated to some preassigned fundamental decomposition, with associated signature operators J_1 and J_2. Then, (1.18) takes the form

$$\frac{J_2-S(z)J_1S(w)^*}{1-z\overline{w}}. \tag{1.19}$$

Such functions S appear as characteristic operator functions of operators, and also as coefficient matrices allowing to describe the solutions of some underlying problem in terms of a linear fractional transformation.

Definition 1.6.2. Let $\mathscr{P}$ be a Pontryagin space. The $\mathbf{B}(\mathscr{P})$-valued function Φ meromorphic in an open subset Ω of the open unit disk is called a generalized Carathéodory function if the kernel

$$\frac{\Phi(z)+\Phi(w)^{[*]}}{1-z\overline{w}} \tag{1.20}$$

has a finite number, say κ, of negative squares in Ω.
The class containing such functions is denoted by $\mathcal{C}_\kappa(\mathscr{P}_1,\mathscr{P}_2)$ or simply $\mathcal{C}_\kappa(\mathscr{P})$ when $\mathscr{P}_1=\mathscr{P}_2=\mathscr{P}$.

Such functions Φ appear in particular in operator models for pairs of unitary operators.
In Definitions 1.6.1 and 1.6.2 one could consider Krein spaces as coefficient spaces, but the main realization theorems hold only in the case where the coefficient spaces are Pontryagin spaces of same index.

Definition 1.6.3. Let $\mathscr{P}$ be a Pontryagin space. The pair of $\mathbf{B}(\mathscr{P})$-valued functions (E_+, E_-) analytic in some open subset of the extended complex plane symmetric with respect to the unit circle is called a de Branges pair if the kernel

$$\frac{E_+(z)E_+(w)^{[*]} - E_-(z)E_-(w)^{[*]}}{1 - z\overline{w}}$$

has a finite number of negative squares in Ω.

An important tool in the arguments in Schur analysis in the operator-valued case is a factorization result for positive kernels of the form

$$\frac{A(z)A(w)^{[*]} - B(z)B(w)^{[*]}}{1 - z\overline{w}}$$

where A and B are analytic and operator-valued. This factorization is originally due to Leech, see [232]. For the case of bounded operator-valued analytic functions we refer to [249, Theorem 2, p. 134] and [250, p. 107] (these last works are based on the commutant lifting theorem).

These various kernels, and the associated reproducing kernel Pontryagin spaces, can also be considered in the open right half-plane $\mathbb{C}_r$, when the denominator $1 - z\overline{w}$ is replaced by $2\pi(z + \overline{w})$ (the 2π factor is to make easier the use of Cauchy's formula in $\mathbb{C}_r$). The kernel corresponding to (1.20) plays then an important role in models for pairs of self-adjoint operators.

In every case the associated reproducing kernel Pontryagin space is a state space for a realization of the given function. We give two examples in Theorems 1.6.4 and 1.6.6.

These various kernels, and others, are part of a general family of kernels. To describe this family, we first consider a pair (a, b) of functions analytic in some connected open set Ω and such that the sets

$$\Omega_+ = \{z \in \Omega\,;\, |b(z)| < |a(z)|\} \quad \text{and} \quad \Omega_- = \{z \in \Omega\,;\, |b(z)| > |a(z)|\}$$

are both non-empty. Then

$$\Omega_0 = \{z \in \Omega\,;\, |b(z)| = |a(z)|\}$$

is also non-empty (this is a simple, but nice, exercise in complex variable, see for instance [12, Exercise 4.1.12, p. 148]). Let

$$\rho_w(z) = a(z)\overline{a(w)} - b(z)\overline{b(w)}.$$

The kernel $\dfrac{1}{\rho_w(z)}$ is positive definite in Ω_+. For instance, the case where

$$a(z) = \frac{1+z}{\sqrt{2}} \quad \text{and} \quad b(z) = \frac{1-z}{\sqrt{2}}$$

corresponds to the case of the open right half-plane, $\Omega_+ = \mathbb{C}_r$.

Let now $J \in \mathbf{B}(\mathscr{P}_1)$ be a signature operator, and X be a $\mathbf{B}(\mathscr{P}_1, \mathscr{P}_2)$-valued function analytic in an open subset U of Ω_+. The kernels described above are all of the form

$$K(z,w) = \frac{X(z)JX(w)^*}{\rho_w(z)}. \tag{1.21}$$

A general Schur algorithm for such kernels (in the matrix-valued case) has been developed in a series of papers which includes [54, 55]. The related one point interpolation problem was studied in [45] when K is complex-valued (as opposed to matrix-valued).

The functions defined above and the associated reproducing kernel Pontryagin spaces play an important role in operator theory and related topics. These applications originate with the works of de Branges and Rovnyak, see [105, 106]. We mention in particular the following applications:

1. Operator models
2. Prediction theory of Gaussian stochastic processes.
3. Inverse scattering problem.
4. Interpolation problems for Schur functions.

In the present work we consider the counterparts, in the setting of slice hyperholomorphic functions, of some of these kernels individually. The general theory of reproducing kernel Pontryagin spaces with reproducing kernel (1.21) uses interpolation of operator-valued Schur functions and a factorization theorem for analytic functions due to Leech (see [233, 220] for the latter).

The case of finite dimensional spaces is of particular importance. Early (and sometimes implicit) instances of the corresponding functions (1.19) with J_1 and J_2 equal to

$$J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.22}$$

appear in Schur's papers [257, 258] and in the paper [116].

We now conclude with the realization theorems mentioned above. Let F be an operator-valued function analytic in a neighborhood of the origin. A realization of F centered at the origin is an expression of the form

$$F(z) = D + zC(I - zA)^{-1}B,$$

where $D = F(0)$ and where A, B and C are operators between appropriate spaces. The space where the operator A acts is called the state space of the realization. Functions associated to kernels defined in Section 1.6 admit realizations in terms of the associated reproducing kernel spaces.

Theorem 1.6.4. *Let $S \in \mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2)$, where $\mathscr{P}_1$ and $\mathscr{P}_2$ are Pontryagin spaces with same index κ, and let $\mathscr{P}(S)$ be the associated reproducing kernel Pontryagin space with reproducing kernel* (1.18)*. Then*

$$S(z) = D + zC(I_{\mathscr{P}(S)} - zA)^{-1}B,$$

where the operator matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \; : \; \mathscr{P}(S) \oplus \mathscr{P}_1 \longrightarrow \mathscr{P}(S) \oplus \mathscr{P}_2 \tag{1.23}$$

is defined by

$$\begin{aligned} Af(z) &= \begin{cases} \dfrac{f(z) - f(0)}{z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \\ Bc_1(z) &= \frac{S(z) - S(0)}{z} c_1, \\ Cf &= f(0), \\ Dc_1 &= S(0)c_1, \end{aligned} \tag{1.24}$$

where $c_1 \in \mathscr{P}_1$. Furthermore, the realization is coisometric and observable, meaning that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{[*]} = \begin{pmatrix} I_{\mathscr{P}(S)} & 0 \\ 0 & I_{\mathscr{P}_2} \end{pmatrix} \quad \text{and} \quad \bigcap_{n=0}^{\infty} \ker CA^n = \{0\}. \tag{1.25}$$

A proof of Theorem 1.6.4 can be found in [47]. We do not repeat it here since we give a proof of the quaternionic counterpart of it in the sequel. It is useful to recall the formula

$$K_S(z,w) = C(I - zA)^{-1}(I - \overline{w}A^{[*]})^{-1}C^{[*]}$$

for the reproducing kernel of $\mathscr{P}(S)$.

Definition 1.6.5. The operator A in (1.24) is called the backward-shift operator, and is usually denoted by R_0:

$$R_0 f(z) = \begin{cases} \dfrac{f(z) - f(0)}{z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{1.26}$$

for an operator-valued function analytic in a neighborhood of the origin.

Often, as in the next theorem, we do not write out the value at the origin, and simply set $R_0 f(z) = \dfrac{f(z) - f(0)}{z}$. We now recall the realization result for generalized Carathéodory functions.

Theorem 1.6.6. *Let $\Phi \in \mathcal{C}_\kappa(\mathscr{P})$, where $\mathscr{P}$ is a Pontryagin space, and let $\mathscr{P}(\Phi)$ be the associated reproducing kernel Pontryagin space with reproducing kernel* (1.18)*. Then*

$$\Phi(z) = D + zC(I_{\mathscr{P}(\Phi)} - zA)^{-1}B,$$

where

$$\begin{aligned} Af(z) &= \frac{f(z)-f(0)}{z}, \\ Cf &= f(0), \\ Bc(z) &= \frac{\Phi(z)-\Phi(0)}{z}c, \\ Dc &= \frac{1}{2}(\Phi(0)-\Phi(0)^*)c. \end{aligned}$$

Furthermore A is coisometric and the pair (C,A) is observable,

$$AA^{[*]} = I \quad \textit{and} \quad \bigcap_{n=0}^{\infty} \ker CA^n = \{0\}. \tag{1.27}$$

Remark 1.6.7. In the above theorems the realization is the celebrated backward-shift realization. This realization appears also in the next section in Theorem 2.1.1 and in other places in the book. Realizations for the analogous functions defined in the right half-plane are more involved. We will not recall them for the complex-valued case, but present them in later sections in the setting of slice hyperholomorphic functions.

Chapter 2

Rational functions

Since this book is intended to (at least) two different audiences we recall in the present chapter the main features of realization theory for matrix-valued rational functions (that is, of matrices whose entries are quotient of polynomials). Note that realization of elements in certain classes of operator-valued analytic functions have been considered in Section 1.6.

2.1 Rational functions and minimal realizations

The theory of realization of rational, and more generally analytic and possibly operator-valued functions, plays an important role in classical operator theory and in related fields. The starting point is the following result, a proof of which is outlined after Definition 2.1.2.

Theorem 2.1.1. *Let r be a $\mathbb{C}^{n\times m}$-valued rational function analytic at the origin. Then r can be written in the form*

$$r(z) = D + zC(I_N - zA)^{-1}B \tag{2.1}$$

where $D = r(0)$ and $(A,B,C) \in \mathbb{C}^{N\times N} \times \mathbb{C}^{N\times m} \times \mathbb{C}^{n\times N}$ for some $N \in \mathbb{N}$.

Definition 2.1.2. Expression (2.1) is called a realization of r centered at the origin. The realization is called minimal if N is minimal.

We have:

Proposition 2.1.3. *A realization is minimal if and only if the following two conditions hold:*

1. *The pair (C,A) is observable, meaning that*

$$\bigcap_{u=0}^{N-1} \ker CA^u = \{0\} \tag{2.2}$$

and

2. *The pair (A,B) is controllable, meaning that*

$$\bigcup_{u=0}^{N-1} \operatorname{ran} A^u B = \mathbb{C}^N. \tag{2.3}$$

Note that condition (2.2) is a special case of the second condition in (1.25) when A is a $N\times N$ matrix, as is seen by using Cayley-Hamilton theorem.

A minimal realization is unique up to a similarity matrix, that is up to a transformation of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} S & 0 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} S^{-1} & 0 \\ 0 & I_m \end{pmatrix}$$

where $S \in \mathbb{C}^{N\times N}$ is invertible. We will not give a proof of these last facts, but will focus on Theorem 2.1.1 instead.

There are various ways to prove Theorem 2.1.1. One is given in [12, Exercise 7.5.3, p. 329] (see in particular the hints given after the exercise). We here outline a proof which exhibits an important realization, called the backward-shift realization. This realization has also the advantage (in the rational case) to be minimal.

Outline of the proof of Theorem 2.1.1: The backward-shift realization. We note that the linear span $\mathscr{M}(r)$ of the functions

$$z \mapsto (R_0^u rc)(z), \quad c \in \mathbb{C}^m \quad \text{and} \quad u = 1,2,\ldots,$$

where R_0 is defined by (1.26), is R_0-invariant by construction, and is finite dimensional since r is rational. To verify this last point, assume without loss of generality that r is scalar-valued, and consider the partial fraction expansion of r. It is a sum of a polynomial and of functions of the form $\frac{1}{(z+a)^t}$ where $a \in \mathbb{C}$ and $t \in \mathbb{N}$. The finite dimensionality claim follows from the formulas

$$R_0\left(\frac{1}{(z+a)^t}\right) = \sum_{u=1}^{t-1} \frac{1}{(z+a)^{t-u}a^u} \quad \text{and} \quad R_0 z^t = z^{t-1}.$$

In particular the function $R_0 rc$ belongs to $\mathscr{M}(r)$ for $c \in \mathbb{C}^m$. It is then easy to check that (2.1) is in force with A,B,C and D as in Theorem 1.6.4. □

As a consequence we have:

Theorem 2.1.4. *Let r be a $\mathbb{C}^{n\times m}$-valued rational function analytic at infinity. Then r can be written in the form*

$$r(z) = D + C(zI_N - A)^{-1}B \tag{2.4}$$

where $D = r(\infty)$ and $(A,B,C) \in \mathbb{C}^{N\times N} \times \mathbb{C}^{N\times m} \times \mathbb{C}^{n\times N}$ for some $N \in \mathbb{N}$.

It is of interest to pass from a realization (2.1) to a realization (2.4). Starting from (2.1) and assuming A invertible this is done as follows:

$$\begin{aligned} r(z) &= D+zC(I_N-zA)^{-1}B \\ &= D+zCA^{-1}(A^{-1}-zI_N)^{-1}B \\ &= D+CA^{-1}(A^{-1}-zI_N)^{-1}(zI_N-A^{-1}+A^{-1})B \\ &= D-CA^{-1}B-CA^{-1}(zI_N-A^{-1})^{-1}A^{-1}B. \end{aligned} \tag{2.5}$$

By renaming the various matrices we have (2.4).

We now recall two important formulas related to inverse and product of realizations, see for instance [94]. We provide the proof since the argument is the same when the matrices have quaternionic entries and z is restricted to be real. See Theorems 9.1.4 and 9.1.6. We will use these formulas in such a context in various places in the book, see for instance the proof of Theorem 9.1.8.

Proposition 2.1.5. *Let*

$$r(z)=D+zC(I-zA)^{-1}B,$$

where A,B,C and D are matrices with entries in $\mathbb{C}$ and of appropriate sizes, be a realization of the rational function r, and assume that D is invertible. Then,

$$r(z)^{-1}=D^{-1}-zD^{-1}C(I-zA^{\times})^{-1}BD^{-1}, \tag{2.6}$$

with

$$A^{\times}=A-BD^{-1}C. \tag{2.7}$$

Proof. Write

$$r(z)=D(I+zD^{-1}C(I-zA)^{-1}B).$$

The formula is then a consequence of the well known formula

$$(I-ab)^{-1}=I+a(I-ba)^{-1}b \tag{2.8}$$

(where a,b are matrices of appropriate sizes) with

$$a=-zD^{-1}C \quad \text{and} \quad b=(I-zA)^{-1}B.$$

Then

$$\begin{aligned} &(I+zD^{-1}C(I-zA)^{-1}B)^{-1}= \\ &\qquad =I-zD^{-1}C(I+(I-zA)^{-1}BzD^{-1}C)^{-1}(I-zA)^{-1}B \\ &\qquad =I-zD^{-1}C(I-zA^{\times})^{-1}B. \end{aligned}$$

□

Remark 2.1.6. Formulas (2.6) and (2.5) can be written as involution maps on matrices as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{pmatrix},$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A^{-1} & A^{-1}B \\ -CA^{-1} & D - CA^{-1}B \end{pmatrix}.$$

Remark 2.1.7. The transformations $r \mapsto r^{-1}$ and $A \mapsto A^{\times}$ have a flavor of perturbation theory. This is indeed the case when r has a positive real part in a half-plane or in the disk, see [105, 107] and [59, 60, 214].

Proposition 2.1.8. *Let*

$$r_j(z) = D_j + zC_j(I_{N_j} - zA_j)^{-1}B_j, \quad j = 1,2,$$

be two functions admitting realizations of the form (2.1). *Let r_1, r_2 be $\mathbb{C}^{m\times n}$ and $\mathbb{C}^{n\times u}$-valued, respectively. Then the $\mathbb{C}^{m\times u}$-valued function $r_1 r_2$ can be written in the form* (2.1), *with $D = D_1 D_2$ and*

$$A = \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1D_2 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & D_1C_2 \end{pmatrix}. \tag{2.9}$$

When considering matrix-valued functions, addition is a particular case of multiplication: given r_1 and r_2 two $\mathbb{C}^{n\times m}$-valued functions we have

$$r_1(z) + r_2(z) = \begin{pmatrix} r_1(z) & I_n \end{pmatrix} \begin{pmatrix} I_m \\ r_2(z) \end{pmatrix}.$$

Thus, as a corollary of Proposition 2.1.8 we obtain the realization formula for the sum $r_1 + r_2$ (see for instance [12, (7.5.5) p. 330]):

$$\begin{aligned} &r_1(z) + r_2(z) = \\ &= (D_1 + D_2) + z\begin{pmatrix} C_1 & C_2 \end{pmatrix} \left(I_{N_1+N_2} - z\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \end{aligned} \tag{2.10}$$

2.2 Minimal factorization

Let r_1 and r_2 be two $\mathbb{C}^{n\times n}$-valued rational functions analytic at the origin. The factorization $r = r_1 r_2$ of the r into a product of two other $\mathbb{C}^{n\times n}$-valued rational functions r_1 and r_2 if called minimal if

$$\deg r = \deg r_1 + \deg r_2.$$

Minimal factorizations were characterized in [93, 94]. To present next Theorem 2.2.2, we first recall the notion of supporting projection.

Definition 2.2.1. Let r be a $\mathbb{C}^{n\times n}$-valued rational function analytic at the origin and assume $r(0)$ invertible. Let $r(z) = D + zC(I_N - zA)^{-1}B$ be a minimal realization of r. Let $\mathscr{M}$ and $\mathscr{N}$ be a pair of subspaces of $\mathbb{C}^N$ such that

$$A\mathscr{M} \subset \mathscr{M} \quad \text{and} \quad A^{\times}\mathscr{N} \subset \mathscr{N},$$

and assume

$$\mathbb{C}^N = \mathscr{M} + \mathscr{N}, \quad \mathscr{M} \cap \mathscr{N} = \{0\}.$$

The projection π from $\mathbb{C}^N$ on $\mathscr{M}$ parallel to $\mathscr{N}$ is called a supporting projection.

Theorem 2.2.2. *Let r be a $\mathbb{C}^{n\times n}$-valued rational function analytic at the origin and assume $r(0)$ invertible. Let $r(z) = D + zC(I_N - zA)^{-1}B$ be a minimal realization of r. Then, $r = r_1 r_2$ is a minimal factorization if and only if there exists a supporting projection π and invertible matrices D_1 and D_2 such that*

$$r_1(z) = D_1 + Cz(I_N - zA)^{-1}(I_N - \pi)BD_2^{-1}, \tag{2.11}$$

$$r_2(z) = D_2 + zD_1^{-1}C\pi(I_N - zA)^{-1}B. \tag{2.12}$$

We note that a rational matrix function may lack non-trivial minimal (square) factorizations, as the classical example

$$r(z) = \begin{pmatrix} 1 & z^2 \\ 0 & 1 \end{pmatrix}. \tag{2.13}$$

Indeed a minimal realization of the function r is given by

$$r(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left(I - z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and so

$$A^{\times} = A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The matrix A has only one non-trivial invariant subspace, and so there exists no non trivial supporting projection.

As we will explain in the sequel, a large part of the theory of realization of rational functions extend to the setting of slice hyperholomorphic functions. In this short section we point out in particular a point, the notion of degree, which do not, seemingly, extend to the quaternionic setting. Not surprisingly this difference pertains to the notion of singularities. In the complex case, given a rational function with pole at the point $z_0 \in \mathbb{C}$ one considers the Laurent expansion at z_0

$$r(z) = \sum_{u=1}^{q} \frac{r_{-u}}{(z - z_0)^u} + s(z),$$

where s is analytic at z_0.

Definition 2.2.3. The local degree at z_0 is the rank of the upper triangular block Toeplitz matrix

$$H=\begin{pmatrix} r_{-q} & r_{-q+1} & \cdots & \cdots & r_{-1} \\ 0 & r_{-q} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & & r_{-q} \end{pmatrix}$$

and similarly for the point ∞, see [94, p. 77].
The McMillan degree of r is then equal, by definition, to the sum of the local degrees over the Riemann sphere.

It is worthwhile to note that the McMillan degree is the dimension of any minimal realization.

In a minimal realization centered at infinity, that is of the form (2.4), the spectrum of the main operator A coincides with the poles of the given rational function.

We also note, and we elaborate on this point later, that the notion of J-unitarity cannot be extended in a straightforward way, since point-evaluation is not multiplicative in the slice hyperholomorphic setting.

2.3 Rational functions J-unitary on the imaginary line

Let $J\in\mathbb{C}^{n\times n}$ be a signature matrix

$$J=J^*=J^{-1},$$

and set

$$K_\Theta(z,w)=\frac{J-\Theta(z)J\Theta(w)^*}{z+\overline{w}},$$

where Θ is analytic in some subset $\Omega(\Theta)$ of the plane. We let $\mathscr{P}(\Theta)$ denote the linear span of the functions

$$z\mapsto K_\Theta(z,w)c$$

when w runs through $\Omega(\Theta)$ and c runs through $\mathbb{C}^n$.

The following result is taken from [57, Theorem 2.1, p. 179]. The case of the real line is considered in [43, §5]. We give a proof of this result in the quaternionic setting in Section 9.

Theorem 2.3.1. *Let Θ be a $\mathbb{C}^{n\times n}$-valued function analytic at infinity, with minimal realization*

$$\Theta(z)=D+C(zI-A)^{-1}B.$$

Then the following are equivalent:
(1) *The function Θ is J-unitary on the imaginary line, meaning that*

$$\Theta(z)J\Theta(z)^* = J, \quad \forall z \in \Omega(\Theta) \cap i\mathbb{R}. \tag{2.14}$$

(2) *The space $\mathscr{P}(\Theta)$ is finite dimensional.*
(3) *D is J-unitary (that is $D^*JD = J$) and there exists a (uniquely defined invertible) Hermitian matrix H such that*

$$\begin{aligned} &A^*H + HA = -C^*JC, \\ &B = -H^{-1}C^*JD. \end{aligned} \tag{2.15}$$

This theorem can be proved in two different ways. One can use the finite dimensional reproducing kernel Pontryagin space with reproducing kernel $K_\Theta(z,w)$ (see [52]) as state space for the backward shift realization. One can also rewrite (2.14) as

$$\Theta(z)^{-1} = J\Theta(-\overline{z})^*J$$

and use the fact that the corresponding two minimal realizations are similar. See [57]. Furthermore, one can prove the formulas

$$\Theta(z) = (I_n - C(zI-A)^{-1}H^{-1}C^*J)D, \tag{2.16}$$

$$\frac{J - \Theta(z)J\Theta(w)^*}{z+\overline{w}} = C(zI-A)^{-1}H^{-1}(wI-A)^{-*}C^*. \tag{2.17}$$

Definition 2.3.2. The rational function Θ will be called J-inner when $H > 0$.

Remark 2.3.3. An important problem for J-unitary rational functions is the characterization of minimal factorizations, where both factors are themselves J-unitary. In the positive case (that is, when the function is J-inner), elementary factors have been characterized by Potapov. See [245]. In the general case, minimal factorizations have been characterized in [57].

When $J = I_m$, a special factorization exists and Θ turns out to be a finite Blaschke product. The following is a very particular case of a result of Krein and Langer, see [227].

Theorem 2.3.4. *A rational function Θ is unitary on the real line if and only if it can be written as $\Theta = B_1B_2^{-1}$ where B_1 and B_2 are (finite) Blaschke products.*

Remark 2.3.5. We note that rational J-inner functions play a key role in interpolation theory for functions analytic and contractive in the open right half-plane (Schur functions of the right half-plane). More generally, rational J-unitary functions play a role in interpolation theory for generalized Schur functions (of the right half-plane). They also appear in the theory of canonical differential systems with rational spectral data, see for instance [64, 206].

In the previous analysis the special case

$$J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \quad \text{and} \quad \Theta(z) = \begin{pmatrix} I_n & \Phi(z) \\ 0 & I_n \end{pmatrix}$$

where $\Phi(z)$ is a $\mathbb{C}^{n\times n}$-valued rational function leads to the following results (see [57, Theorem 4.1, p. 210]).

Theorem 2.3.6. *Let Φ be a $\mathbb{C}^{n\times n}$-rational function analytic at infinity, and let $\Phi(z) = D + C(zI_N - A)^{-1}B$ be a minimal realization of Φ. Then the following are equivalent:*
(1) *Φ takes anti self-adjoint values on the imaginary line.*
(2) *The matrix D is anti self-adjoint and there exists an invertible Hermitian matrix H such that*

$$\begin{aligned} A^*H + HA &= 0, \\ C &= B^*H. \end{aligned}$$

We note the formula

$$\frac{\Phi(z) + \Phi(w)^*}{z + \overline{w}} = C(zI_N - A)^{-1}H^{-1}(wI_N - A)^{-*}C^*, \quad z, w \in \rho(A), \tag{2.18}$$

where $\rho(A)$ is the resolvent set of A. The matrix H in the theorem is uniquely defined from the given realization, and is called the associated Hermitian matrix to the given realization.

2.4 Rational functions J-unitary on the circle

Let $J \in \mathbb{C}^{n\times n}$ be a signature matrix and let Θ be a rational $\mathbb{C}^{n\times n}$-valued function, with domain of definition $\Omega(\Theta)$. Let

$$K_\Theta(z, w) = \frac{J - \Theta(z)J\Theta(w)^*}{1 - z\overline{w}},$$

and let $\mathscr{P}(\Theta)$ denote the linear span of the functions

$$z \mapsto K_\Theta(z, w)c$$

when w runs through $\Omega(\Theta)$ and c runs through $\mathbb{C}^n$. The space $\mathscr{P}(\Theta)$ can of course be infinite dimensional, as the example $n = 1$, $J = 1$ and $\Theta = 0$ illustrates trivially. The next theorem characterizes the case where $\mathscr{P}(\Theta)$ is finite dimensional (see [52, 57]). Note that we consider the case of functions analytic at the origin, while Theorem 2.3.1 considered, for the imaginary line case, functions analytic at infinity.

Theorem 2.4.1. *Let Θ be a $\mathbb{C}^{n\times n}$-valued function analytic at the origin and at infinity, with minimal realization*

$$\Theta(z) = D + zC(I - zA)^{-1}B.$$

Then the following are equivalent:
(1) *The function Θ is J-unitary on the unit circle, meaning that*

$$\Theta(z)J\Theta(z)^* = J, \quad \forall z \in \Omega(\Theta) \cap \mathbb{T}. \tag{2.19}$$

(2) *The space $\mathscr{P}(\Theta)$ is finite dimensional.*
(3) *There exists a (uniquely defined invertible) Hermitian matrix H such that*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix}.$$

This theorem can be proved as follows. One can rewrite (2.19) as

$$\Theta(z)^{-1} = J\Theta\left(\frac{1}{\overline{z}}\right)^* J$$

and compare minimal realizations. Note that in this approach one needs to assume Θ analytic at infinity as well, or equivalently, Θ invertible at the origin. One can also use the finite dimensional reproducing kernel Pontryagin space with reproducing kernel K_Θ as a state space for the backward shift realization. Finally, one can specialize the results of [47] for Θ for which the associated space $\mathscr{P}(\Theta)$ is finite dimensional, of dimension m. A formula for Θ is given by

$$\Theta(z) = I_n - (1 - z\overline{w_0})C(I_m - zA)^{-1}H^{-1}(I_m - w_0A)^{-*}C^*J. \tag{2.20}$$

where $w_0 \in \mathbb{T}$ is such that $(I_m - w_0A)$ is invertible. We also note the formula

$$C(I_m - zA)^{-1}H^{-1}(I_m - wA)^{-*}C^* = \frac{J - \Theta(z)J\Theta(w)^*}{1 - z\overline{w}}, \tag{2.21}$$

where z and w are such that the matrices $(I_m - zA)$ and $(I_m - wA)$ are invertible,
Blaschke products and rational J-inner functions are defined in a way similar to Definition 2.3.2. We note that Theorem 2.3.4 was proved first in the setting of the disk.

The counterpart of Theorem 2.3.6 is:

Theorem 2.4.2. *Let Φ be a $\mathbb{C}^{n\times n}$-rational function analytic at the origin, and let $\Phi(z) = D + zC(I_N - zA)^{-1}B$ be a minimal realization of Φ. Then the following are equivalent:*
(1) *Φ takes anti self-adjoint values on the unit circle.*
(2) *The matrix D is anti self-adjoint and there exists an invertible Hermitian matrix H such that*

$$A^*HA = H,$$
$$C = B^*H.$$

As in Theorem 2.3.6 the matrix H in the theorem is uniquely defined from the given realization, and is called the associated Hermitian matrix to the given realization. We have the formulas

$$\begin{aligned}\Phi(z) &= D+\frac{1}{2}C(I_N-zA)^{-1}(I_N+zA)H^{-1}C^* \\ &= D+\frac{1}{2}CH^{-1}C^*+zC(I_N-zA)^{-1}H^{-1}C,\end{aligned} \tag{2.22}$$

and

$$\frac{\Phi(z)+\Phi(w)^*}{1-z\overline{w}} = C(I_N-zA)^{-1}H^{-1}(I_N-wA)^{-*}C^*, \tag{2.23}$$

for z and w in the domain of analyticity of Φ.

Remarks 2.3.3 and 2.3.5 hold also for the circle case. With J_0 as in (1.22), rational J_0-inner function play a key role in interpolation theory for functions analytic and contractive in the open unit disk (Schur functions). More generally, rational J_0-unitary function play a role in interpolation theory for generalized Schur functions; see for instance [116, 158, 14]. They also appear in the theory of first order discrete systems with rational spectral data. This corresponds to the case where the f_n in Problem 1.4.2 are of the form

$$f_n = ca^n b,$$

where a, b and c are matrices of suitable sizes. The asymptotic equivalence function is J_0-unitary on the unit circle, with $J_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_p\end{pmatrix}$ (for $p=q=1$ this is (1.22)), while the scattering function takes unitary values on the unit circle. See Section 3.3 for the definition of these functions. At this stage we contend ourselves by extracting an explicit example of rational J-unitary function from this theory. To that purpose, let H_n be defined by (1.16). Then the entries of H_n are given by the formulas

$$\alpha_n(z) = I_p + ca^n z(zI-a)^{-1}(I-\Delta\Omega_n)^{-1}\Delta a^{*n}c^*, \tag{2.24}$$

$$\beta_n(z) = ca^n z(zI-a)^{-1}(I-\Delta\Omega_n)^{-1}b, \tag{2.25}$$

$$\gamma_n(z) = b^*(I-za^*)^{-1}(I-\Omega_n\Delta)^{-1}a^{*n}c^*, \tag{2.26}$$

$$\delta_n(z) = I_p + b^*(I-za^*)^{-1}(I-\Omega_n\Delta)^{-1}\Omega_n b, \tag{2.27}$$

where Ω_n and Δ are the solutions of the Stein equations

$$\begin{aligned}\Delta - a\Delta a^* &= bb^*,\\ \Omega_n - a^*\Omega_n a &= a^{*n}c^*ca^n.\end{aligned}$$

Let

$$\begin{aligned}t_n &= I_p + ca^n(I-\Delta\Omega_n)^{-1}\Delta a^{*n}c^*,\\ u_n &= I_p + b^*(I-\Omega_n\Delta)^{-1}\Omega_n b.\end{aligned}$$

Then, the matrices t_n and u_n are strictly positive.

Theorem 2.4.3. *The function*

$$H_n(z)\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix}$$

is J_0–unitary on the unit circle, with minimal realization

$$H_n(z)\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix} =$$
$$= D_n \cdot \begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix} + C_n(zI-A)^{-1}B_n \cdot \begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix}.$$

where

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix},$$
$$B_n = \begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix} \cdot \begin{pmatrix} (I-\Delta\Omega_n)^{-1}\Delta & (I-\Delta\Omega_n)^{-1} \\ -(I-\Omega_n\Delta)^{-1} & -(I-\Omega_n\Delta)^{-1}\Omega_n \end{pmatrix} \cdot \begin{pmatrix} a^{*n}c^* & 0 \\ 0 & b \end{pmatrix},$$
$$C_n = \begin{pmatrix} ca^n & 0 \\ 0 & b^* \end{pmatrix},$$
$$D_n = \begin{pmatrix} I_p + ca^n(I-\Delta\Omega_n)^{-1}\Delta a^{*n}c^* & ca^n(I-\Delta\Omega_n)^{-1}b \\ 0 & I_p \end{pmatrix}.$$

The Hermitian matrix associated to this realization is given by

$$X_n = \begin{pmatrix} -\Omega_n & -I \\ -I & -a\Delta a^* \end{pmatrix}.$$

A proof in the rational case was given in [58] (this last paper is devoted to the scalar case, but the proof given there is still valid in the matrix case). We repeat this proof in Section 10.6 for the case of matrices with quaternionic entries. See Theorem 10.7.5.

Chapter 3

Schur analysis

As discussed in Chapter 1, functions analytic and contractive in the open unit disk, the Schur functions, are part of classical mathematics, as is illustrated by the works of Schur [257, 258], Takagi [265, 266] and Bloch [97], to name a few. They play an important role in numerous areas of mathematics, and can be characterized in a number of ways. By the name *Schur analysis* one means a collection of problems pertaining to Schur functions in function theory, operator theory and related fields. In this chapter we gather the main aspects in this setting useful to the reader in preparation for the quaternionic generalization presented in this book.

3.1 The Schur algorithm

Via an iterative procedure now called the Schur algorithm, Schur associated in 1917 to a function $s \in \mathcal{S}$ a (possibly finite) sequence of numbers in the open unit disk (with additionally a number of modulus one if the sequence is finite), which uniquely characterizes s. An important, but not so well known, consequence of the Schur algorithm is a proof of the power expansion for an analytic function without using integration. See [152, 273].

Recall (see for instance [210, pp. 67-68]) that a function f analytic and bounded in the open unit disk admits a multiplicative representation $f(z) = i(z)o(z)$ into an inner function $i(z)$ and an outer function $o(z)$, the inner function being itself a product of a constant c of modulus 1, a Blaschke product $b(z)$ and of a singular inner function $j(z)$: thus,

$$f(z) = cb(z)j(z)o(z), \qquad z \in \mathbb{D}$$

with

$$\begin{aligned} b(z) &= z^p \prod_{n\in\mathbb{J}} \frac{\overline{z_n}}{z_n}\frac{z_n - z}{1-\overline{z_n}z}, \\ j(z) &= \exp\left(-\frac{1}{2\pi}\int_{[0,2\pi]}\frac{e^{it}+z}{e^{it}-z}d\mu(t)\right), \\ o(z) &= \exp\left(\frac{1}{2\pi}\int_{[0,2\pi]}\frac{e^{it}+z}{e^{it}-z}\ln|f(e^{it})|dt\right). \end{aligned}$$

In these expressions, $p \in \mathbb{N}_0$, $\mathbb{J} \subset \mathbb{N}$ and the points $z_n \in \mathbb{D} \setminus \{0\}$. Furthermore, $d\mu$ is a finite singular positive measure. When the function at hand is a Schur function, the outer part is also a Schur function.

The above representation is fundamental in function theory and in operator theory, and admits generalizations to the matrix-valued and operator-valued cases. See [245, 202, 203]. On the other hand, it does not seem to be the best tool to solve classical interpolation problems such as the Carathéodory-Fejér interpolation problem. See Definition 3.2.1 for the latter.

The recursion

$$\begin{aligned} \rho_n &= s_n(0), \\ s_{n+1}(z) &= \begin{cases} \dfrac{s_n(z)-s_n(0)}{z(1-s_n(z)\overline{s_n(0)})}, & \text{for, } z \neq 0 \\ \dfrac{s_n'(0)}{1-|s_n(0)|^2}, & \text{for } z = 0, \end{cases} \end{aligned} \tag{3.1}$$

is called Schur algorithm. It can be rewritten in a projective way as

$$zk_n(z)\begin{pmatrix}1 & -s_{n+1}(z)\end{pmatrix} = \begin{pmatrix}1 & -s_n(z)\end{pmatrix}\begin{pmatrix}1 & \rho_n \\ \overline{\rho_n} & 1\end{pmatrix}\begin{pmatrix}z & 0 \\ 0 & 1\end{pmatrix}, \quad n = 0,1,\ldots, \tag{3.2}$$

where $k_n(z)$ is analytic and invertible in the open unit disk.

Let now

$$\theta_n(z) = \frac{1}{\sqrt{1-|\rho_n|^2}}\begin{pmatrix}1 & \rho_n \\ \overline{\rho_n} & 1\end{pmatrix}\begin{pmatrix}z & 0 \\ 0 & 1\end{pmatrix}, \quad n = 0,1,\ldots,$$

and let

$$\Theta_n(z) = \theta_0(z)\cdots\theta_n(z). \tag{3.3}$$

With $K_n(z) = k_0(z)\cdots k_n(z)$, the recursion (3.1) can be rewritten as

$$z^{n+1}K_{n+1}(z)\begin{pmatrix}1 & -s_{n+1}(z)\end{pmatrix} = \begin{pmatrix}1 & -s(z)\end{pmatrix}\Theta_n(z), \tag{3.4}$$

for $n = 0, 1, \ldots$. This last form is conducive to important generalizations, in particular to a projective form of the Schur algorithm. See [159, 163, 234].

Relating properties of $s \in \mathcal{S}$ and of its sequence of coefficients $\rho_0, \rho_1, \ldots$ leads to deep and interesting problems. Let

$$\Theta_n(z) = \begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{pmatrix}.$$

We note that

$$s(z) = \frac{A_n(z)s_{n+1}(z) + B_n(z)}{C_n(z)s_{n+1}(z) + D_n(z)}. \tag{3.5}$$

Furthermore, when one replaces in the linear fractional transformation (3.5) the function s_{n+1} by an arbitrary Schur function, one obtains a description of *all* Schur functions whose first $n+1$ Taylor coefficients coincide with the first $n+1$ Taylor coefficients of s, that is the description of all the solutions to a corresponding Carathéodory-Fejér interpolation problem.

Recall (see (1.22)) that $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The matrix-valued polynomial function (3.3) is J_0-inner, meaning that

$$\Theta_n(z) J_0 \Theta(z)^* \begin{cases} \leq J_0, & z \in \mathbb{D}, \\ = J_0, & z \in \mathbb{T}. \end{cases}$$

More generally, the solutions of most of the classical interpolation problems in the Schur class can be described in terms of a linear fractional transformation associated to a J-inner function or to a J-unitary function.

We note that (3.3) is an example for $J = J_0$ of a minimal factorization of a J-unitary rational functions into elementary J-unitary factors.

3.2 Interpolation problems

In this section we briefly define three interpolation problems, in the scalar case. Their quaternionic counterparts are considered in Chapter 10. They are special instances of much more general problems, an example of which is given, in the quaternionic setting, in Chapter 11. We begin with the Carathéodory-Fejér interpolation problem.

Problem 3.2.1. *The Carathéodory-Fejér interpolation problem: Given $a_0, \ldots, a_N \in \mathbb{C}$, find a necessary and sufficient condition for a Schur function s to exist such that*

$$s(z) = \underbrace{a_0 + a_1 z + \cdots + a_N z^N}_{\text{fixed}} + a_{N+1} z^{N+1} + \cdots,$$

and describe the set of all solutions when this condition is in force.

Next we have:

Problem 3.2.2. *The Nevanlinna-Pick interpolation problem: Given N pairs*

$$(z_1,w_1),(z_2,w_2),\dots,(z_N,w_N)$$

in $\mathbb{D}\times\overline{\mathbb{D}}$, find a necessary and sufficient condition for a Schur function s to exist such that

$$s(z_i)=w_i,\quad i=1,\dots,N$$

and describe the set of all solutions when this condition is in force.

If one of the interpolation values, say w_i, lies on the unit circle, the maximum modulus principle implies that there is at most one solution, which is $s(z)\equiv w_i$.

In the previous two problems the interpolation nodes are inside $\mathbb{D}$. The case where they are chosen on the unit circle is much more complicated. The main reason is that Schur functions have nontangential boundary values almost everywhere. But even if one restricts to rational solutions, difficulties remain. This is best explained by mentioning Carathéodory's theorem (see for instance [113, pp. 203-205]), [255, p. 48]). We write the result for a radial limit, but the result holds in fact for a non tangential limit.

Theorem 3.2.3. *Let s be a Schur function and let e^{it_0} be a point on the unit circle such that*

$$\liminf_{\substack{r\to 1\\ r\in(0,1)}}\frac{1-|s(re^{it_0})|}{1-r}<\infty. \tag{3.6}$$

Then, the limits

$$c=\lim_{\substack{r\to 1\\ r\in(0,1)}} s(re^{it_0})\quad \text{and}\quad \lim_{\substack{r\to 1\\ r\in(0,1)}}\frac{1-s(re^{it_0})\overline{c}}{1-r} \tag{3.7}$$

exist, and the second one is positive.

This result plays an important role in the classical boundary interpolation problem for Schur functions. See for instance [46, 88, 99, 256]. We note also that, conversely, conditions (3.7) imply (3.6), as follows from the identity

$$\frac{1-|s(re^{it_0})|^2}{1-r^2}=\frac{1-s(re^{it_0})\overline{c}}{(1-r)(1+r)}+(s(re^{it_0})\overline{c})\frac{1-c\overline{s(re^{it_0})}}{(1-r)(1+r)}. \tag{3.8}$$

Problem 3.2.4. *The boundary Nevanlinna-Pick interpolation problem: Given N pairs*

$$(z_1,w_1),(z_2,w_2),\dots,(z_N,w_N)$$

in $\mathbb{T}^2$, and given N positive numbers $\kappa_1,\dots,\kappa_N$, find necessary and sufficient conditions for a Schur function s to exist such that

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} s(rz_i)=w_i,\quad i=1,\dots,N,$$

$$\lim_{\substack{r\to 1\\ r\in(0,1)}}\frac{1-|s(rz_i)|^2}{1-r^2}\le\kappa_i,\quad i=1,\dots,N,$$

and describe the set of all solutions when these conditions are in force.

We refer to [225]for a solution of this problem, and to [99] for a solution in the setting of generalized Schur functions.

3.3 First order discrete systems

Given a sequence $\{\rho_n\}$ of points in the open unit disk, one associates to it expressions of the form

$$X_{n+1}(z) = \begin{pmatrix} 1 & -\rho_n \\ -\overline{\rho_n} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} X_n(z), \quad n = 0, 1, \ldots \tag{3.9}$$

or of the form

$$Z_{n+1}(z) = Z_n(z) \begin{pmatrix} 1 & \rho_n \\ \overline{\rho_n} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \quad n = 0, 1, \ldots \tag{3.10}$$

where the ρ_n are in the open unit disk. Note that

$$\begin{aligned} Z_{n+1}(z) \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} X_{n+1}(1/z) &= (1 - |\rho_n|^2) Z_n(z) \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} X_n(1/z) \\ &= (\prod_{u=0}^{n+1} (1 - |\rho_u|^2)) Z_0(z) \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} X_0(1/z). \end{aligned}$$

These expressions are called *first order discrete systems*. They are motivated by the projective form (3.2) of the Schur algorithm. They appear also in the theory of layered medium. See [112, 111]. The first recursion plays an important role in the solution of the Nehari extension problem, while the second one appears in the Carathéodory-Toeplitz extension problem.

Associated to first order discrete systems are a number of functions meromorphic in the open unit disk, called the *characteristic spectral functions* of the system. Associated inverse and direct problems consist in finding the coefficients ρ_n from these functions and conversely. Direct and inverse problems have a long history, and they were studied in the rational case in a series of papers which includes [58, 61, 62, 63]. Not surprisingly the Schur function $R(z)$ with Schur coefficients $\rho_1, \rho_2, \ldots$ is one of these functions. It is then called the reflection coefficient function. The inverse problem associated to R is then solved via the Schur algorithm.

Besides this function, there are also four other functions, namely:

1. The asymptotic equivalence matrix function $V(z)$.
2. The scattering function $S(z)$.

3. The spectral function $W(z)$.
4. The Weyl function $N(z)$.

Discrete first order systems have also been considered in the matrix-valued case, and this is the setting we will need in the quaternionic setting. In the matrix-valued case, one considers expressions of the form

$$X_{n+1}(z) = \begin{pmatrix} I_m & \alpha_n \\ \beta_n & I_m \end{pmatrix}^* \begin{pmatrix} zI_m & 0 \\ 0 & I_m \end{pmatrix} X_n(z), \quad n = 0, 1, 2, \dots \tag{3.11}$$

or of the related form

$$Z_{n+1}(z) = Z_n(z) \begin{pmatrix} I_m & -\alpha_n \\ -\beta_n & I_m \end{pmatrix}^* \begin{pmatrix} zI_m & 0 \\ 0 & I_m \end{pmatrix}, \quad n = 0, 1, 2, \dots \tag{3.12}$$

where the α_n and β_n, (with $n = 0, 1, \dots$) are strict contractions in $\mathbb{C}^{m\times m}$, subject to the following condition: there exists a sequence of block diagonal matrices $\Delta_n \in \mathbb{C}^{2m\times 2m}$, $n = 0, 1, 2, \dots$, such that

$$\begin{pmatrix} I_m & \alpha_n \\ \beta_n & I_m \end{pmatrix} J\Delta_n \begin{pmatrix} I_m & \alpha_n \\ \beta_n & I_m \end{pmatrix}^* = J\Delta_{n-1}, \quad n = 1, 2, \dots \tag{3.13}$$

where $J = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}$.

Definition 3.3.1. The sequence (α_n, β_n) is called Δ–admissible for the given sequence of block diagonal matrices.

To define the characteristic spectral functions we first need:

Theorem 3.3.2. *Let (α_n, β_n) be a Δ–admissible sequence for some sequence of block diagonal matrices $\Delta = (\Delta_n)$ and assume that $\lim_{n\to\infty} \Delta_n$ exists and is equal to I_{2m}, and that, moreover:*

$$\sum_{n=1}^{\infty} (\|\alpha_n\| + \|\beta_n\|) < \infty. \tag{3.14}$$

Then the first order discrete system (3.9) *has a unique solution $X_n(z)$ such that*

$$\lim_{n\to\infty} \begin{pmatrix} z^{-n} I_m & 0 \\ o & I_m \end{pmatrix} X_n(z) = \begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix}, \quad |z| = 1.$$

The proof of this theorem and of the following results in the section can be found in [63] to which we send the reader for proofs. In the quaternionic setting, we give a proof of Theorem 3.3.2; see Theorem 10.6.2 there. The other results are considered in Section 10.7 in the rational case.

Definition 3.3.3. The function $Y(z) = X_0(z)^{-1}$ is called the asymptotic equivalence matrix function associated to the discrete system.

Theorem 3.3.4. *The asymptotic equivalence matrix function belongs to* $\mathscr{W}^{2m\times 2m}$ *and has the following properties:*
(a) Y_{11} *and* Y_{12} *belong to* $\mathscr{W}_-^{m\times m}$*, and* Y_{11} *is invertible in* $\mathscr{W}_-^{m\times m}$*. Furthermore,*

$$Y_{11}(\infty) = I_m \quad \textit{and} \quad Y_{12}(\infty) = 0.$$

(b) Y_{21} *and* Y_{22} *belong to* $\mathscr{W}_+^{m\times m}$*, and* Y_{22} *is invertible in* $\mathscr{W}_+^{m\times m}$*. Furthermore,*

$$Y_{21}(0) = 0 \quad \textit{and} \quad Y_{22}(0) = I_m.$$

We define now the scattering function. An important role is played by the $\mathbb{C}^{2m\times m}$–valued solution $A_n(z)$ to (3.9) such that

$$\begin{pmatrix} I_m & -I_m \end{pmatrix} A_0(z) = 0 \quad \text{and} \quad \lim_{n\to\infty} \begin{pmatrix} 0 & I_m \end{pmatrix} A_n(z) = I_m, \quad |z| = 1.$$

Theorem 3.3.5. *The system* (3.9) *has a unique* $\mathbb{C}^{2m\times m}$*–valued solution* $A_n(z)$ *with the following properties:*
(a) $\begin{pmatrix} I_m & -I_m \end{pmatrix} A_0(z) = 0$*, and*
(b) $\begin{pmatrix} 0 & I_m \end{pmatrix} A_n(z) = I_m + o(n), \quad |z| = 1.$
It then holds that

$$\begin{pmatrix} I_m & 0 \end{pmatrix} A_n(z) = z^n S(z) + o(n)$$

where $S(z) = (Y_{11}(z) + Y_{12}(z))(Y_{21}(z) + Y_{22}(z))^{-1}$.

Definition 3.3.6. The function

$$S(z) = (Y_{11}(z) + Y_{12}(z))(Y_{21}(z) + Y_{22}(z))^{-1}.$$

is called the scattering matrix function associated to the given first order discrete system.

Theorem 3.3.7. *The scattering matrix function has the following properties: it is in the Wiener algebra, takes unitary values on the unit circle, and admits a Wiener–Hopf factorization:*

$$S(z) = S_-(z)S_+(z),$$

where $S_-(z) = (Y_{11}(z) + Y_{12}(z))$ *and its inverse are in* $\mathscr{W}_-^{m\times m}$ *and* $S_+(z) = (Y_{21}(z) + Y_{22}(z))^{-1}$ *and its inverse are in* $\mathscr{W}_+^{m\times m}$.

To introduce the reflection coefficient function we first define for

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathbb{C}^{2m\times 2m} \quad \text{and} \quad X \in \mathbb{C}^{m\times m}$$

the linear fractional transformation $T_M(X)$:

$$T_M(X) = (M_{11}X + M_{12})(M_{21}X + M_{22})^{-1}. \tag{3.15}$$

Recall that $T_{M_1M_2(X)} = T_{M_1}(T_{M_2}(X))$ when all expressions make sense. We also recall that $T_M(X)$ is contractive when M is J_0–contractive and X is contractive.

Consider now the system

$$\Psi_n(z)\Delta_n^{-1/2} = \Psi_{n-1}(z)\Delta_{n-1}^{-1/2}\begin{pmatrix} I_m & \alpha_n \\ \beta_n & I_m \end{pmatrix}\begin{pmatrix} zI_m & 0 \\ 0 & I_m \end{pmatrix}, \quad n = 1,2,\dots \tag{3.16}$$

and $\Psi_0(z) = I_m$.

Definition 3.3.8. Let (α_n, β_n) be a Δ–admissible sequence for some associated sequence of diagonal matrices $\Delta = (\Delta_n)$. Let Ψ_n be defined by (3.16). Then the reflection coefficient matrix function associated to the first order discrete system (3.9) is:

$$R(z) = \lim_{n\to\infty} T_{\Psi_n(z)}(0).$$

Theorem 3.3.9. *Let $Y(z) = (Y_{\ell j}(z))_{\ell,j=1,2}$ be the asymptotic equivalence matrix function. Then,*

$$R(z) = \frac{1}{z}Y_{21}(\overline{z})^*(Y_{22}(\overline{z}))^{-*} = \frac{1}{z}(Y_{11}(1/z))^{-1}Y_{12}(1/z), \quad |z| = 1. \tag{3.17}$$

Furthermore, the reflection coefficient function is analytic and contractive in the open unit disk and takes strictly contractive values on the unit circle.

We now define the Weyl function under the hypothesis that the series

$$\sum_{\ell=0}^{\infty} \frac{\|\beta_\ell\|}{|z^\ell|} \tag{3.18}$$

converges for $1-\varepsilon < |z| \le 1$ (for some $\varepsilon > 0$). In the following theorem, M_n denotes the solution of the discrete system (3.9) subject to the initial condition $M_0(z) = I_2$.

Theorem 3.3.10. *Under hypothesis* (3.18) *the Weyl function is the unique function $N(z)$ defined in $1-\varepsilon < |z| < 1+\varepsilon$ and such that the sequence*

$$M_n(z)\begin{pmatrix} I_m & I_m \\ I_m & -I_m \end{pmatrix}\begin{pmatrix} iN(\overline{z})^* \\ I_m \end{pmatrix} \tag{3.19}$$

has its entries in ℓ_2 for at least one z in the open unit disk. Furthermore, $N(z)$ is given by the formula

$$N(z) = i(I_m - zR(z))(I_m + zR(z))^{-1}. \tag{3.20}$$

We conclude this section by mentioning:

Theorem 3.3.11. *The following formula holds for the asymptotic equivalence matrix function:*

$$Y(z) = \frac{1}{2}\begin{pmatrix} S_-(z)(I_m - iN(1/z)) & S_-(z)(I_m + iN(1/z)) \\ S_+(z)^{-1}(I_m - iN(\overline{z})^*) & S_+(z)^{-1}(I_m + iN(\overline{z})^*) \end{pmatrix}.$$

In Sections 10.6 and 10.7 we consider the counterpart of these systems in the quaternionic setting, and in particular in the rational case.

3.4 The Schur algorithm and reproducing kernel spaces

We first give some background to provide motivation for the results presented in this section. Recall that we denote by R_0 the backward-shift operator:

$$R_0 f(z) = \frac{f(z) - f(0)}{z},$$

where f is analytic in a neighborhood of the origin. Beurling's theorem gives a characterization of closed subspaces of $\mathrm{H}^2(\mathbb{D})$ invariant under the operator M_z of multiplication by z. These are exactly spaces of the form $j\mathrm{H}^2(\mathbb{D})$, where j is an inner function. Since $M_z^* = R_0$ in $\mathrm{H}^2(\mathbb{D})$ Beurling's theorem can be seen as the characterization of R_0-invariant subspaces of the Hardy space $\mathrm{H}^2(\mathbb{D})$ as being the spaces $\mathrm{H}^2(\mathbb{D}) \ominus j\mathrm{H}^2(\mathbb{D})$. Equivalently, these are the reproducing kernel Hilbert spaces with reproducing kernel

$$K_j(z,w) = \frac{1 - j(z)\overline{j(w)}}{1 - z\overline{w}}.$$

When replacing j inner by s analytic and contractive in the open unit disk, the kernel

$$K_s(z,w) = \frac{1 - s(z)\overline{s(w)}}{1 - z\overline{w}} \tag{3.21}$$

is still positive definite in the open unit disk, but it is more difficult to characterize the reproducing kernel Hilbert spaces $\mathscr{H}(s)$ with reproducing kernel $K_s(z,w)$. Allowing for s not necessarily scalar valued, de Branges gave a characterization of $\mathscr{H}(s)$ spaces in [104, Theorem 11, p. 171]. This result was extended in [47, Theorem 3.1.2, p. 85] to the case of Pontryagin spaces.

The Schur algorithm leads to a representation of a Schur function s in the form (3.15)

$$s = \frac{ae+b}{ce+d} \overset{\text{def.}}{=} T_\Theta(e),$$

where $\Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a $\mathbb{C}^{2\times 2}$-valued polynomial which is J_0-inner and e is another Schur function. In fact, and as proved in [14] in the wider context of generalized Schur functions, the Schur algorithm provides all representations of s of the form (3.15) which are J_0-inner polynomials. The problem of finding all representations of s of the previous form is called the inverse spectral problem.

More generally, one can ask for all these representations of s, where Θ is J_0-inner but not necessarily of polynomial form. This is the inverse scattering problem, and can be solved using the theory of de Branges-Rovnyak spaces. Before mentioning the key result toward this approach we recall that, given a J_0-inner function Θ analytic in $\Omega(\Theta) \subset \mathbb{D}$, the kernel

$$K_\Theta(z,w) = \frac{J_0 - \Theta(z) J_0 \Theta(w)^*}{1 - z\overline{w}}$$

is positive definite in $\Omega(\Theta)$. We denote by $\mathscr{H}(\Theta)$ the associated reproducing kernel Hilbert space.

The key in the approach to inverse scattering using de Branges-Rovnyak spaces is the following result, which appears (in the setting of operator-valued functions) in [105].

Theorem 3.4.1. *Let s be a Schur function. Then the J_0-inner function Θ is such that $s = T_\Theta(e)$ for some Schur function e if and only if the map*

$$\tau(f) = \begin{pmatrix} 1 & -s \end{pmatrix} f \tag{3.22}$$

is a contraction from $\mathscr{H}(\Theta)$ into $\mathscr{H}(s)$.

The idea behind the proof of this result (and of its counterpart in the quaternionic setting; see for instance (11.19)) is the decomposition

$$K_s(z,w) = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \frac{J_0 - \Theta(z)J_0\Theta(w)^*}{1 - z\overline{w}} \begin{pmatrix} 1 \\ -\overline{s(w)} \end{pmatrix} + $$

$$+ \begin{pmatrix} 1 & -s(z) \end{pmatrix} \frac{\Theta(z)J_0\Theta(w)^*}{1 - z\overline{w}} \begin{pmatrix} 1 \\ -\overline{s(w)} \end{pmatrix}$$

of the positive definite kernel K_s into a sum of two kernels.

In [50, 51] the inverse scattering problem was solved by going from s to the function

$$\varphi = \frac{1-s}{1+s}.$$

In the case of functions analytic in the open unit disk, the idea is to use the integral representation (1.5) of φ. The solutions of the inverse scattering problem are then expressed in terms of the backward shift invariant subspaces of analytic functions inside $\mathbf{L}_2(d\mu)$.

A similar analysis holds for functions analytic and contractive in a half-plane, say the upper half-plane $\mathbb{C}_+$. Then the inverse spectral problem consists in finding entire functions Θ, J-inner in $\mathbb{C}_+$. See the works of de Branges [156], Krein [226], and Dym and McKean [173].

Part II

Quaternionic analysis

In this part of the book we provide the necessary preliminaries on quaternions, and we review some basic facts on matrices and polynomials in this framework. This material is largely classical and we refer the reader to [231, 248, 276] for more information. We also present a chapter on quaternionic functional analysis. The majority of the results in the chapter is new and it is largely taken from [37], while Hilbert and Pontryagin spaces have also been treated in [68, 193]. We then introduce the notion of slice hyperholomorphic function and in Chapter 6 we provide some results for this function theory which useful in this work. In particular, we study the Hardy spaces on the unit ball and half space as well as the Blaschke products, see also the original sources [20, 32, 34].
We note that some other important function spaces of slice hyperholomorphic functions have been studied in the recent years namely the Bloch, Besov, Dirichlet space, see [115], the Fock space, see [40], and the Bergman spaces which are treated in [115, 123, 124, 125, 126, 127]. Since they are not considered in this work, we refer the reader to the original sources.
The notion of operator-valued slice hyperholomorphic function studied in Chapter 7 is more recent and it is also related to the so-called S-functional calculus. This calculus is based on the Cauchy formula for slice hyperholomorphic functions and it is the natural generalization of the Riesz-Dunford functional calculus for quaternionic operators, see [133, 135, 136, 137, 138]. We note that the continuous version of this functional calculus is studied in [193].
A suitable modification of the S-functional calculus also applies to n-tuples of linear operators, see [28, 118, 121, 122, 130, 132, 134, 142].
We remark for the interested reader that the theory of slice hyperholomorphic functions is nowadays quite well developed, see the papers [95, 96, 119, 120, 157, 185, 186, 187, 188, 189, 192, 190, 191, 263, 264], while some approximation results are proved in [181, 182, 183]. There is also a Clifford algebra valued analogue of these functions, called slice monogenic, and we refer the reader to [120, 128, 129, 131, 134, 141, 149, 143, 150, 145, 146, 147, 148, 151]. Finally, the generalization to functions with values in an alternative real algebra is treated in [195, 196, 197, 198, 199, 200].

Chapter 4

Finite dimensional preliminaries

In this chapter we discuss the finite dimensional aspects of quaternionic analysis which are needed in the sequel. In the first section we survey the main properties of quaternions. Then we consider quaternionic polynomials and we discuss their zeros. In the third section we discuss quaternionic matrices and basic definitions such as adjoint, transpose and inverse. We introduce the map χ which to any quaternionic square matrix associates a complex matrix of double size. In particular this map is also defined for quaternions. We discuss the eigenvalue problem and show that it is associated with the notion of left, right spectrum and with the so-called S-spectrum. Finally, we present the Jordan decomposition of a matrix. In the fourth and last section we consider some matrix equations which appear in the sequel.

4.1 Some preliminaries on quaternions

The set of quaternions, denoted by $\mathbb{H}$ in honor of Hamilton who introduced this set of numbers, contains elements of the form

$$p = x_0 + x_1 i + x_2 j + x_3 k,$$

where the three imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $ki = -ik = j$, $jk = -kj = i$. The sum and the product of two quaternions $p = x_0 + x_1 i + x_2 j + x_3 k$, $q = y_0 + y_1 i + y_2 j + y_3 k$ are defined by

$$\begin{aligned} p+q &= (x_0+y_0) + (x_1+y_1)i + (x_2+y_2)j + (x_3+y_3)k \\ pq &= (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) + (x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)i + \\ &\quad + (x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1)j + (x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)k. \end{aligned}$$

With respect to these operations $\mathbb{H}$ turns out to be a skew field. The element

$$\bar{p} = x_0 - x_1 i - x_2 j - x_3 k$$

is called the conjugate of p and the expression

$$\sqrt{p\bar{p}} = \sqrt{\bar{p}p} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

is called the modulus of the quaternion p and is denoted by $|p|$. Given $p = x_0 + x_1 i + x_2 j + x_3 k$, its real (or scalar) part x_0 will be denoted also by $\mathrm{Re}(p)$ while $x_1 i + x_2 j + x_3 k$ is the imaginary part of p, denoted also by $\mathrm{Im}(p)$.

Proposition 4.1.1. *Let $p, q \in \mathbb{H}$. The following properties are immediate:*

(1) $\overline{pq} = \bar{q}\bar{p}$;

(2) $|pq| = |p||q|$;

(3) $|p+q| \leq |p| + |q|$ *and* $|p-q| \geq ||p| - |q||$;

(4) if $p \neq 0$, $p^{-1} = \dfrac{\bar{p}}{|p|^2}$;

(5) if $pq \neq 0$, $(pq)^{-1} = q^{-1}p^{-1}$.

Let

$$\mathbb{S} = \{p = x_1 i + x_2 j + x_3 k \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\};$$

then $\mathbb{S}$ is a 2-dimensional sphere in $\mathbb{H}$ identified with $\mathbb{R}^4$. Any element $\mathrm{I} \in \mathbb{S}$ satisfies $\mathrm{I}^2 = -1$ and thus will be called *imaginary unit*.

Remark 4.1.2. To $p = x_0 + x_1 i + x_2 j + x_3 k$ with $\mathrm{Im}(p) \neq 0$ one associates the imaginary unit I_p defined by $\mathrm{I}_p = \frac{\mathrm{Im}(p)}{|\mathrm{Im}(p)|}$. Moreover, $p = |p|(\cos\varphi + \mathrm{I}_p \sin\varphi)$ where

$$\cos\varphi = \frac{\mathrm{Re}(p)}{|p|}, \qquad \sin\varphi = \frac{|\mathrm{Im}(p)|}{|p|}.$$

In the sequel, we will make use of the following definition.

Definition 4.1.3. Let $p \in \mathbb{H}$. The set of elements

$$[p] = \left\{ qpq^{-1} \text{ when } q \text{ runs through } \mathbb{H} \setminus \{0\} \right\}$$

is called the sphere associated to p.

Observe that $[p]$ contains just p if and only if $p \in \mathbb{R}$.
We have:

Lemma 4.1.4. *Two points belong to the same sphere if and only if they have same real part and same absolute value.*

Proof. If $p' \in [p]$ then $p' = qpq^{-1}$. By taking the absolute value of both sides and using point (2) in Proposition 4.1.1 we immediately have $|p'| = |p|$. Moreover

$$\mathrm{Re}(p') = \frac{1}{2}(p' + \bar{p}') = \frac{1}{2|q|^2}(qp\bar{q} + q\bar{p}\bar{q}) = \frac{1}{2}q(p+\bar{p})q^{-1} = \mathrm{Re}(p).$$

Conversely, if p' has same real part and modulus as p then

$$p' = \mathrm{Re}(p') + I_{p'}|\mathrm{Im}(p')| = \mathrm{Re}(p) + I_{p'}|\mathrm{Im}(p)|.$$

Then the proof is completed by direct computations to show that the equation $p'q = qp$ has always a nonzero solution q. □

Let us write

$$p = x_0 + x_1 i + x_2 j + x_3 k = z_1 + z_2 j \in \mathbb{H},$$

with

$$z_1 = x_0 + ix_1 \text{ and } z_2 = x_2 + ix_3 \in \mathbb{C},$$

where we identify $\mathbb{C}$ with the subset of $\mathbb{H}$ given by the elements of the form $x+iy$, $x, y \in \mathbb{R}$. Let $\chi : \mathbb{H} \to \mathbb{C}^{2\times 2}$ be the map, see [219],

$$\chi(p) = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}. \tag{4.1}$$

The map χ allows to translate problems from the quaternionic to the complex matricial setting. Then $\mathbb{H}$ can be identified with a subset of the ring $\mathbb{C}^{2\times 2}$ which is in fact a skew field.

Proposition 4.1.5. *The map $\chi : \mathbb{H} \to \mathbb{C}^{2\times 2}$ is an injective homomorphism of rings, i.e.*

$$\chi(p+q) = \chi(p) + \chi(q), \qquad \chi(pq) = \chi(p)\chi(q).$$

Lemma 4.1.6. *Let $p \in \mathbb{H} \setminus \mathbb{R}$. If x is a solution of*

$$xp = \overline{p}x, \tag{4.2}$$

then it is purely imaginary.

Proof. The conjugate of (4.2) is

$$\overline{x}p = \overline{p}\,\overline{x}. \tag{4.3}$$

Adding (4.2) and (4.3) we obtain

$$\mathrm{Re}(x)p = \overline{p}\mathrm{Re}(x).$$

Since p is not real we get that $\mathrm{Re}(x) = 0$. □

Lemma 4.1.7. *Let p and q be quaternions of modulus* 1. *Then, the equation*

$$ph - hq = 0, \tag{4.4}$$

where $h \in \mathbb{H}$, has the only solution $h = 0$ if and only if $\mathrm{Re}(p) \neq \mathrm{Re}(q)$, that is, if and only if $[p] \cap [q] = \emptyset$.

Proof. If (4.4) has a solution $h \neq 0$, then $p = hqh^{-1}$ and so p and q are in the same sphere. So a necessary condition for (4.4) to have only $h = 0$ as solution is that $[p] \cap [q] = \emptyset$. We now show that this condition is also sufficient. Let $p = z_1 + z_2 j$ and $q = w_1 + w_2 j$, where $z_1, z_2, w_1, w_2 \in \mathbb{C}$. Since $\mathrm{Re}(p) \neq \mathrm{Re}(q)$ we have

$$\mathrm{Re}(z_1) \pm i\sqrt{1-(\mathrm{Re}(z_1))^2} \neq \mathrm{Re}(w_1) \pm i\sqrt{1-(\mathrm{Re}(w_1))^2}. \tag{4.5}$$

Using the map χ and the fact that it is a ring homomorphism, equation (4.4) becomes

$$\chi(p)\chi(h) - \chi(h)\chi(q) = 0. \tag{4.6}$$

The eigenvalues of $\chi(p)$ are the solutions of

$$\lambda^2 - 2(\mathrm{Re}(z_1))\lambda + 1 = 0,$$

that is, $\lambda = \mathrm{Re}(z_1) \pm i\sqrt{1-(\mathrm{Re}(z_1))^2}$, and similarly the eigenvalues of $\chi(q)$ are of the form $\mathrm{Re}(w_1) \pm i\sqrt{1-(\mathrm{Re}(w_1))^2}$. By a well known result on matrix equations (see e.g., Corollary 4.4.7 in [211]), equation (4.6) has only the solution $\chi(h) = 0$ if and only if $\lambda - \mu \neq 0$ for all possible choices of eigenvalues of $\chi(p)$ and $\chi(q)$, and this condition holds in view of (4.5). So the only solution of (4.6) is $h = 0$. □

Remark 4.1.8. Lemma 4.1.7 still holds when p and q have the same modulus.

4.2 Polynomials with quaternionic coefficients

In this section we study polynomials with quaternionic coefficients. Because of the non-commutativity of quaternions, one can consider polynomials with coefficients on one side (either left or right) or on both sides or even polynomials which are sum of monomials of the form $a_0 p a_1 p \cdots p a_n$ where a_ℓ are the coefficients and p is the indeterminate. To our purposes, it will be of interest to consider one sided polynomials. These are very well known in the literature, see for example [231], and are examples of slice hyperholomorphic functions. Thus we will consider polynomials with coefficients on the right and the set of such polynomials in the variable p shall be denoted by $\mathbb{H}[p]$. One peculiarity with quaternions, is that second degree polynomials may have an infinite number of roots. This is readily seen with p^2+1 whose roots are all the elements in the sphere $\mathbb{S}$. But this a quite general situation as we shall see below.
Let us recall that two polynomials $f(p) = p^n a_n + \ldots + p a_1 + a_0$, $g(p) = p^n b_n + \ldots + p b_1 + b_0$ with quaternionic coefficients (more in general with coefficients in a division ring, see [231]) can be added in the standard way and multiplied using a suitable product, denoted by $\star$, by taking the convolutions of the coefficients:

$$(f \star g)(p) = \sum_{s=0}^{m+n} p^s c_s, \qquad c_s = \sum_{\ell+r=s} a_\ell b_r.$$

This product is associative, distributive with respect to the sum and noncommutative. Note that the $\star$-product can also be written as:

$$(f\star g)(p)=\sum_{r=0}^{m} p^r f(p) b_r$$

The evaluation $f(p_0)$ of a polynomial $f(p)=\sum_{\ell=0}^{n} p^\ell a_\ell$ at a point p_0 is defined to be the quaternion $f(p_0)=\sum_{\ell=0}^{n} p_0^\ell a_\ell$. The evaluation $e_{p_0}:\mathbb{H}[p]\to\mathbb{H}$ is not a ring homomorphism, in fact $(f\star g)(p_0)\neq f(p_0)g(p_0)$. Instead, we have the following result:

Proposition 4.2.1. *Let $f(p)=\sum_{\ell=0}^{n} p^\ell a_\ell$, $g(p)=\sum_{r=0}^{m} p^m b_m\in\mathbb{H}[p]$ and let $f(p_0)\neq 0$. Then*

$$(f\star g)(p_0)=f(p_0)g(f(p_0)^{-1}p_0 f(p_0)).$$

Thus if $(f\star g)(p_0)=0$ and $f(p_0)\neq 0$ then $g(f(p_0)^{-1}p_0 f(p_0))=0$.

Proof. From the above formula, if $f(p_0)\neq 0$ we have

$$\begin{aligned}(f\star g)(p_0)&=\sum_{r=0}^{m} p_0^r f(p_0) b_r=\sum_{r=0}^{m} f(p_0)(f(p_0)^{-1}p_0^r f(p_0))b_r\\&=f(p_0)\sum_{r=0}^{m}(f(p_0)^{-1}p_0 f(p_0))^r b_r=f(p_0)g(f(p_0)^{-1}p_0 f(p_0)).\end{aligned}$$

□

In particular, the above result applies in the case of quadratic polynomials. Consider

$$p^2-p(\alpha+\beta)+\alpha\beta=(p-\alpha)\star(p-\beta),$$

and assume that $\beta\neq\overline{\alpha}$. Then the left factor $p-\alpha$ gives the root α while the second root is not β but instead $(\beta-\overline{\alpha})^{-1}\beta(\beta-\overline{\alpha})$. If $\beta=\overline{\alpha}$ the situation is quite different and it is illustrated below (with s instead of α).

Definition 4.2.2. The polynomial $Q_s(p)=p^2-2\mathrm{Re}(s)p+|s|^2$ is the so-called minimal (or companion) polynomial associated with the sphere $[s]$.

Lemma 4.2.3. *The polynomial $Q_s(p)$ vanishes exactly at the points on the sphere $[s]$.*

Proof. Let $s=a+\mathrm{I}b$ so that $\mathrm{Re}(s)=a$ and $|s|^2=a^2+b^2$. An easy calculation shows that $a+Jb$ for any $J\in\mathbb{S}$ is a zero of $Q_s(p)=0$. The fact that there are no zeros of $Q_s(p)=0$ outside the sphere $[s]$ can be shown using Lemma 4.1.7. Indeed, assume there is a zero $p\notin[s]$, then

$$|s|^2-2\mathrm{Re}(s)p+p^2=s(\overline{s}-p)-(\overline{s}-p)p=0, \tag{4.7}$$

from which we deduce that $|s|=|p|$. Using Remark 4.1.8 we have that (4.7) has only the solution $p=\overline{s}$ since p and s are assumed on different spheres. But this contradicts our assumption. □

The polynomial ring $\mathbb{H}[p]$ is Euclidean, both on the left and on the right, i.e. it allows right and left division (in general with remainder). In fact, for every two polynomials $f(p)$ and $d(p)$, with $d(p)$ nonzero, there exist $q(p)$, $r(p) \in \mathbb{H}[p]$ such that

$$f(p) = d(p) \star q(p) + r(p), \quad \text{with } \deg r(p) < \deg d(p) \text{ or } r(p) = 0$$

and similarly for the right division.
Moreover, we have:

Proposition 4.2.4. *We have $Q_s(p) = Q_{s'}(p)$ if and only if $[s] = [s']$. If Q_s divides a polynomial $f(p)$ then $f(p) = 0$ for every $p \in [s]$. Otherwise, at most one element in $[s]$ is a zero of f.*

Theorem 4.2.5. *(1) A quaternion α is a zero of a (nonzero) polynomial $f \in \mathbb{H}[p]$ if and only if the polynomial $p - \alpha$ is a left divisor of $f(p)$, i.e. $f(p) = (p-\alpha) \star g(p)$.*

(2) If $f(p) = (p-\alpha_1) * \ldots * (p-\alpha_n) \in \mathbb{H}[p]$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{H}$, $\alpha_{j+1} \neq \overline{\alpha_j}$ then α_1 is a zero of f and every other zero of f is in the equivalence class of α_i, $i = 2, \ldots, n$.

(3) If f has two distinct zeros in an equivalence class $[\alpha]$, then all the elements in $[\alpha]$ are zeros of f.

Remark 4.2.6. Assume that $f(p) \in \mathbb{H}[p]$ factors as

$$f(p) = (p-\alpha_1) \star \cdots \star (p-\alpha_n), \quad \alpha_{j+1} \neq \bar{\alpha}_j, \ \ j = 1, \ldots, n-1,$$

and assume that $\alpha_j \in [\alpha_1]$ for all $j = 2, \ldots, n$. Then the only root of $f(p)$ is $p = \alpha_1$, see [242, Lemma 2.2.11], [244, p. 519] the decomposition in linear factors is unique, and α_1 is the only root of f.
Assume that $[\alpha_j]$ is a spherical zero. Then, for any $a_j \in [\alpha_j]$ we have

$$p^2 + 2\mathrm{Re}(\alpha_j)p + |\alpha_j|^2 = (p-a_j) \star (p-\bar{a}_j) = (p-\bar{a}_j) \star (p-a_j)$$

thus showing that both a_j and $\bar{a}_j$ are zeroes of multiplicity 1. So we can say that the (points of the) sphere $[\alpha_j]$ have multiplicity 1. Thus the multiplicity of a spherical zero $[\alpha_j]$ equals the exponent of $p^2 + 2\mathrm{Re}(\alpha_j)p + |\alpha_j|^2$ in a factorization of $f(p)$.

The discussion in the previous remark justifies the following:

Definition 4.2.7. Let

$$f(p) = (p-\alpha_1) \star \cdots \star (p-\alpha_n) \star g(p), \quad \alpha_{j+1} \neq \bar{\alpha}_j, \ \ j = 1, \ldots, n-1, \ g(p) \neq 0 \text{ for } p \in [\alpha_1].$$

We say that $\alpha_1 \in \mathbb{H} \setminus \mathbb{R}$ is a zero of f of *multiplicity* 1 if $\alpha_j \notin [\alpha_1]$ for $j = 2, \ldots, n$.
We say that $\alpha_1 \in \mathbb{H} \setminus \mathbb{R}$ is a zero of f of *multiplicity* $n \geq 2$ if $\alpha_j \in [\alpha_1]$ for all $j = 2, \ldots, n$.
We say that $\alpha_1 \in \mathbb{R}$ is a zero of f of *multiplicity* $n \geq 1$ of f if $f(p) = (p-\alpha_1)^n g(p)$ with $g(\alpha_1) \neq 0$. Assume now that $f(p)$ contains the factor $(p^2 + 2\mathrm{Re}(\alpha_j)p + |\alpha_j|^2)$ so that $[\alpha_j]$ is a zero of $f(p)$. We say that the *multiplicity of the spherical zero* $[\alpha_j]$ is m_j if m_j is the maximum of the integers m such that $(p^2 + 2\mathrm{Re}(\alpha_j)p + |\alpha_j|^2)^m$ divides $f(p)$.

4.3 Matrices with quaternionic entries

Matrices with quaternionic entries have been the subject of numerous studies; see for instance [276]. They arise in slice hyperholomorphic Schur analysis in at least three key places.

(a) Hermitian matrices occur in the definition of positive definite kernels and of kernels having a finite number of negative squares (see Definition 5.10.1).

(b) General matrices appear in realization theory of slice hyperholomorphic rational functions.

(c) Matrix equations appear in the theory of structured rational matrices. See equations (4.16) and (4.17) for instance.

The present section is built having in view these cases. We begin with some definitions on elements in $\mathbb{H}^{m\times n}$. Let $A=(a_{ij})\in\mathbb{H}^{m\times n}$ and $q\in\mathbb{H}$. The addition of matrices is defined componentwise and the product is the standard product of matrices. Then $\mathbb{H}^{m\times n}$ becomes a right (or left) linear space over $\mathbb{H}$ by defining

$$Aq=(a_{ij}q) \quad \text{or} \quad qA=(qa_{ij}).$$

The following properties are immediate:

(1) $A(Bq)=(AB)q$, for every $A,B\in\mathbb{H}^{n\times n}$ and $q\in\mathbb{H}$;

(2) $A(pq)=(Ap)q$, for every $A\in\mathbb{H}^{n\times n}$ and $p,q\in\mathbb{H}$;

(3) $(Aq)B=A(qB)$, for every $A,B\in\mathbb{H}^{n\times n}$ and $q\in\mathbb{H}$.

Definition 4.3.1. Let $A=(a_{ij})\in\mathbb{H}^{m\times n}$. The conjugate of A is the matrix $\overline{A}=(\overline{a_{ij}})\in\mathbb{H}^{m\times n}$. The transpose of A is the matrix $A^T=(a_{ji})\in\mathbb{H}^{n\times m}$ and the adjoint is $A^*=(\overline{A})^T=(\overline{a_{ji}})\in\mathbb{H}^{n\times m}$.
A matrix $A=(a_{ij})\in\mathbb{H}^{n\times n}$ is invertible if there is a matrix in $\mathbb{H}^{n\times n}$, denoted by A^{-1}, such that $AA^{-1}=A^{-1}A=I$, where I denotes the identity matrix (that will also be denoted by I_n when it is important to show the size of the matrix); it is said normal if $AA^*=A^*A$, Hermitian if $A^*=A$ and unitary if $AA^*=I$.

The proof of the following theorem is immediate:

Theorem 4.3.2. *Let $A\in\mathbb{H}^{m\times n}$, $B,C\in\mathbb{H}^{n\times n}$ and $D\in\mathbb{H}^{n\times t}$. Then*

(1) $(AD)^*=D^*A^*$,

(2) if B,C are both invertible then $(BC)^{-1}=C^{-1}B^{-1}$;

(3) if B is invertible then $(B^*)^{-1}=(B^{-1})^*$.

However some classical properties possessed by complex matrices do not hold in the quaternionic setting. In fact we have:

Remark 4.3.3. Let $A\in\mathbb{H}^{m\times n}$, $D\in\mathbb{H}^{n\times t}$. Then, in general (and assuming A square and invertible for the third and fourth claims)

(1) $\overline{(AD)} \neq \bar{A}\bar{D}$;

(2) $(AD)^T \neq D^T A^T$;

(3) $(\bar{A})^{-1} \neq \overline{(A^{-1})}$;

(4) $(A^T)^{-1} \neq (A^{-1})^T$.

In fact consider

$$A = \begin{pmatrix} i & j \\ 0 & k \end{pmatrix} \qquad D = \begin{pmatrix} i & 1 \\ 0 & k \end{pmatrix}.$$

Then

$$\overline{(AD)} = \begin{pmatrix} -1 & -2i \\ 0 & -1 \end{pmatrix}$$

while

$$\bar{A}\bar{D} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover

$$(AD)^T = \begin{pmatrix} -1 & 0 \\ 2i & -1 \end{pmatrix}, \qquad D^T A^T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally

$$(\bar{A})^{-1} = \begin{pmatrix} i & -1 \\ 0 & k \end{pmatrix}, \qquad \overline{(A^{-1})} = \begin{pmatrix} i & 1 \\ 0 & k \end{pmatrix}$$

$$(A^T)^{-1} = \begin{pmatrix} -i & 0 \\ -1 & -k \end{pmatrix}, \qquad (A^{-1})^T = \begin{pmatrix} -i & 0 \\ 1 & -k \end{pmatrix}$$

which show (3) and (4).

We can extend the map χ in (4.1) to matrices and define another map, still denoted by χ, such that $\chi: \mathbb{H}^{n\times n} \to \mathbb{C}^{2n\times 2n}$. Given a matrix $A \in \mathbb{H}^{n\times n}$, we can write it in the form $A = A_1 + A_2 j$ where $A_1, A_2 \in \mathbb{C}^{2n\times 2n}$. Then we set

$$\chi(A) = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}. \tag{4.8}$$

The matrix $\chi(A)$ is called *complex adjoint matrix*. The properties of the map χ are illustrated below. Their proofs follow from direct computations.

Proposition 4.3.4. *Let $A, B \in \mathbb{H}^{n\times n}$, then*

(1) $\chi(A+B) = \chi(A) + \chi(B)$*;*

(2) $\chi(AB) = \chi(A)\chi(B)$*;*

(3) $\chi(I_n) = I_{2n}$*;*

(4) $\chi(A^*) = \chi(A)^*$*;*

(5) if A is invertible, $\chi(A^{-1}) = \chi(A)^{-1}$.

A matrix A can be considered a linear operator acting on the right linear space $\mathbb{H}^{n\times 1}$ of columns with n quaternionic components. We will denote this space by $\mathbb{H}^n$. Given $A \in \mathbb{H}^{n\times n}$ there are two possibilities to define eigenvalues:

Definition 4.3.5. Let $A \in \mathbb{H}^{n\times n}$ and let $\lambda \in \mathbb{H}$. The quaternion λ is a left eigenvalue of A if $Av = \lambda v$ for some $v \neq 0$, while it is a right eigenvalue of A if $Av = v\lambda$ for some $v \neq 0$.

The set of right (resp. left) eigenvalues of A is called right (resp. left) spectrum and is denoted by $\sigma_r(A)$ (resp. $\sigma_l(A)$).
Right eigenvalues are the most used in the literature. If λ is a right eigenvalue, all the elements in the sphere $[\lambda]$ are right eigenvalues, in fact if $Av = v\lambda$ then for any $q \neq 0$

$$A(vq) = (Av)q = v\lambda q = vq(q^{-1}\lambda q)$$

so if λ is a right eigenvalue, all the elements in the sphere $[\lambda]$ are right eigenvalues.
The right eigenvalues are then either real or spheres. Real points can be considered as spheres reduced to one point. Thus we have the following result:

Theorem 4.3.6. *Any matrix $A \in \mathbb{H}^{n\times n}$ has n spheres of right eigenvalues, if each of them is counted with its multiplicity.*

Remark 4.3.7. The left and right spectrum are not related. To give an example, let us consider the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}.$$

Then $\sigma_l(A) = \{1, k\}$ while $\sigma_r(A) = \{1\} \cup \mathbb{S}$.
When A has real entries, the left and the right spectrum coincide.

Theorem 4.3.8. *Let $A \in \mathbb{H}^{n\times n}$. The following are equivalent:*

(1) A is invertible;

(2) $Ax = 0$ has a unique solution $x = 0$;

(3) $\chi(A)$ is invertible;

(4) 0 is neither a right nor a left eigenvalue of A.

Proof. It is immediate that condition (1) implies (2). To show that (2) and (3) are equivalent, let us write $A = A_1 + A_2 j$. Let $x = x_1 + x_2 j$ be such that $Ax = 0$ then

$$Ax = (A_1 + A_2 j)(x_1 + x_2 j) = (A_1x_1 - A_2\bar{x}_2) + (A_1x_2 + A_2\bar{x}_1)j = 0$$

that is

$$A_1x_1 - A_2\bar{x}_2 = 0 \qquad A_1x_2 + A_2\bar{x}_1 = 0.$$

These two conditions are equivalent to

$$\begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix}\begin{pmatrix} x_1 \\ -\bar{x}_2 \end{pmatrix} = \chi(A)\begin{pmatrix} x_1 \\ -\bar{x}_2 \end{pmatrix} = 0.$$

Thus $Ax = 0$ has only the trivial solution if and only if $\chi(A)y = 0$, where $y = \begin{pmatrix} x_1 \\ -\bar{x}_2 \end{pmatrix}$, has only the trivial solution if and only if $\chi(A)$ is invertible.
Assume that (3) holds and let

$$\chi(A)^{-1} = \begin{pmatrix} B_1 & B_2 \\ -\bar{B}_2 & \bar{B}_1 \end{pmatrix}.$$

By setting $B = B_1 + B_2 j$ it is easy to check that $BA = I$ and so (1) holds.
To show that condition (3) implies (4), assume that $\lambda = 0$ is a left or a right eigenvalue. Then $Ax = 0$ for a nonzero vector x and so by the equivalence between (2) and (3), $\chi(A)$ would not be invertible. Conversely, if (4) holds then $Ax = 0$ has only trivial solutions and thus (3) is in force. □

Brenner has shown in his paper [108] that for any matrix $A \in \mathbb{H}^{n\times n}$ there exists a unitary matrix U such that U^*AU is upper triangular. Moreover if T is a triangular matrix, its diagonal elements are right eigenvalues of A. Moreover, every quaternion similar to a diagonal element of T is a right eigenvalue of T. In each sphere of eigenvalues $[\lambda]$, where $\lambda = \lambda_0 + I\lambda_1$ we can chose a so-called *standard eigenvalue* which is the element in $[\lambda]$ of the form $\lambda = \lambda_0 + i\lambda_1$ with $\lambda_1 \geq 0$ (this is always possible by changing i with $-i$ if necessary). The above discussion can be made more precise in the following result:

Theorem 4.3.9. *Let $A \in \mathbb{H}^{n\times n}$ and let $\lambda_{10} + i\lambda_{11}, \dots, \lambda_{n0} + i\lambda_{n1}$ be its n standard eigenvalues. Then there exists a unitary matrix U such that U^*AU is upper triangular and its diagonal entries are the standard eigenvalues of A.*

The next result corresponds to the spectral theorem for Hermitian matrices. It plays an important role and it allows, in particular, to define functions and kernels with a finite number of negative squares (see Definition 5.10.1).

Theorem 4.3.10. *The matrix $A \in \mathbb{H}^{n\times n}$ with standard eigenvalues $\lambda_{10} + i\lambda_{11}, \dots, \lambda_{n0} + i\lambda_{n1}$ is normal if and only if there exists a unitary matrix such that*

$$U^*AU = \mathrm{diag}(\lambda_{10} + i\lambda_{11}, \dots, \lambda_{n0} + i\lambda_{n1}).$$

The matrix A is Hermitian if and only if all the eigenvalues are real, namely $\lambda_{i1} = 0$ for all $i = 1, \dots, n$.

The following is an application of the definition of the map χ and of the previous result.

Proposition 4.3.11. *Let $A \in \mathbb{H}^{n\times n}$. Then:*

(1) A is Hermitian if and only if $\chi(A)$ is Hermitian;

(2) the Hermitian matrix A has signature (ν_+, ν_-, ν_0) if and only if $\chi(A)$ has signature $(2\nu_+, 2\nu_-, 2\nu_0)$;

(3) the Hermitian matrix A is positive if and only if $\chi(A)$ is positive.

Proof. To prove (1), assume that A is Hermitian. By Proposition 4.3.4 we deduce $\chi(A) = \chi(A^*) = \chi(A)^*$ and so $\chi(A)$ is Hermitian. Conversely, let $\chi(A)$ be Hermitian. Proposition 4.3.4 yields $\chi(A)^* = \chi(A^*)$ and so $A = A^*$. To show (2) and (3) we use Theorem 4.3.10 and the properties of the map χ. We then have

$$\chi(A) = \chi(U)^*\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_n)\chi(U)$$

from which the statements follow. □

The following result is used in the proof of Lemma 8.2.1 pertaining to the Potapov-Ginzburg transform. It related to the Hadamard (that is, entrywise) product of positive functions. In the case of two matrices with complex entries it appears in [48, Lemma 2.1, p. 20]. Note that in the statement Q can be singular. The result in [48, Lemma 2.1, p. 20] is proved for the case of complex numbers, but extends to the quaternionic case, as is seen by using the map χ defined in (4.1) and Lemma 4.3.11. For completeness we present the proof of the proposition.

Proposition 4.3.12. *. Let $x_1, \ldots, x_N$ be N different positive strictly numbers, and let $Q \in \mathbb{H}^{N\times N}$ be a positive matrix such that $Q_{ii} > 0$ for $i = 1, \ldots, N$. Then the matrix P with (j,k) entry equal to $\frac{Q_{jk}}{x_j+x_k}$ is strictly positive.*

Proof. We follow the argument in the proof of [48, Lemma 2.1, p. 20]. By a Cayley transform we replace the denominators $x_j + x_k$ by $1 - y_j y_k$, where $y_1, \ldots, y_N \in (-1, 1)$. We can then write

$$P = \sum_{u=0}^{\infty} D^u Q D^u,$$

where $D \in \mathbb{R}^{N\times N}$ is the diagonal matrice with entries $x_1, \ldots, x_N$. Thus P is positive since each of the matrices $D^u Q D^u$ is positive. Let $\xi \in \mathbb{H}^N$ be such that $P\xi = 0$. The positivity of the various matrices implies that

$$QD^u\xi = 0, \quad u = 0, 1, \ldots$$

and hence $Qp(D)\xi = 0$ for any polynomial with real entries. The choice

$$p(x) = \frac{\prod_{\substack{a=1\\ a\neq j}}^{N}(x - x_a)}{\prod_{\substack{a=1\\ a\neq j}}^{N}(x_j - x_a)}$$

leads to $p(D) = \mathrm{diag}\,(0, 0, \ldots, 1, 0, \ldots 0)$, where the 1 is at the j-th place. The condition $Qp(D)\xi = 0$ implies that $Q_{jj}\xi_j = 0$ and so $\xi_j = 0$,and so $\xi = 0$. □

Proposition 4.3.13. *Let*

$$M = \begin{pmatrix} m_{11} & b \\ b^* & D \end{pmatrix} \in \mathbb{H}^{n\times n}$$

be an Hermitian positive matrix. Then, M is invertible if and only if

$$m_{11} > 0 \quad \textit{and} \quad D - \frac{b^*b}{m_{11}} > 0.$$

In the above statement, recall that m_{11} is real since M is Hermitian. The matrix $D-\dfrac{b^*b}{m_{11}}$ is called the *Schur complement* of m_{11} in M.

Proof. Assume first M invertible. Since $M \geq 0$ we have

$$\begin{pmatrix} m_{11} & m_{1k} \\ m_{k1} & m_{kk} \end{pmatrix} \geq 0$$

for $k = 2, \dots, n$. Therefore for every $p \in \mathbb{H}$

$$m_{11} + m_{1k}p + \overline{p}m_{k1} + |p|^2 m_{kk} \geq 0, \quad \forall p \in \mathbb{H}.$$

Assume now $m_{11} = 0$. Then the above inequality forces $m_{1k} = 0$ and in particular M will have its first line (and column) equal to 0 and therefore M will not be invertible. We thus assume $m_{11} > 0$. The formula (recall that m_{11} is a positive number)

$$\begin{pmatrix} m_{11} & b \\ b^* & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{b^*}{m_{11}} & I_{n-1} \end{pmatrix} \begin{pmatrix} m_{11} & 0 \\ 0 & D-\dfrac{b^*b}{m_{11}} \end{pmatrix} \begin{pmatrix} 1 & \dfrac{b}{m_{11}} \\ 0 & I_{n-1} \end{pmatrix}$$

allows to conclude. That same formula allows to go backwards and prove the converse direction. □

Besides the left and right spectrum it is useful to introduce another notion of spectrum:

Definition 4.3.14. Given a matrix $A \in \mathbb{H}^{n\times n}$ the S-spectrum of A, is defined as

$$\sigma_S(A) = \{p \in \mathbb{H} \mid A^2 - 2(\operatorname{Re} p)A + |p|^2 I_n \text{ is not invertible}\}. \tag{4.9}$$

By its definition it is clear that if $\lambda \in \sigma_S(A)$ then all the elements in $[\lambda]$ belong to $\sigma_S(A)$. In fact, a stronger result holds:

Proposition 4.3.15. *Let $A \in \mathbb{H}^{n\times n}$ then $\sigma_r(A) = \sigma_S(A)$.*

Proof. If $p \in \mathbb{R}$, $A^2 - 2(\operatorname{Re} p)A + |p|^2 I_n = (A - pI_n)^2$ and $\sigma_r(A) \cap \mathbb{R} = \sigma_S(A) \cap \mathbb{R}$. If $p \notin \mathbb{R}$ and $p \in \sigma_S(A)$ then $p = a + \mathrm{J}b$, for $\mathrm{J} \in \mathbb{S}$. So $A^2 - 2aA + (a^2+b^2)I_n$ is not invertible and there exists $v \neq 0$ such that $(A^2 - 2aA + (a^2+b^2)I)v = 0$. If $Av = v(a + \mathrm{J}b)$ then $a + \mathrm{J}b \in \sigma_r(A)$ and we have the statement; otherwise $Av - v(a + \mathrm{J}b)$ is nonzero. Rewriting $(A^2 - 2aA + (a^2+b^2)I_n)v = 0$ as

$$A(Av - v(a + \mathrm{J}b)) = (Av - v(a + \mathrm{J}b))(a - \mathrm{J}b)$$

we deduce that $a - \mathrm{J}b$, and so the whole sphere $[a + \mathrm{J}b]$, belongs to $\sigma_r(A)$. Conversely, assume that $a + \mathrm{J}b \in \sigma_r(A)$, and so $Av = v(a + \mathrm{J}b)$, for some $v \neq 0$. Then $a + \mathrm{J}b$ and the whole 2-sphere $[a + \mathrm{J}b]$ belong to $\sigma_S(A)$. Indeed

$$A^2v - 2aAv + (a^2+b^2)v = v(a + \mathrm{J}b)^2 - 2av(a + \mathrm{J}b) + (a^2+b^2)v = 0$$

and so $(A^2 - 2aA + (a^2+b^2)I)$ is not invertible. Thus $a + \mathrm{J}b \in \sigma_S(A)$. □

Remark 4.3.16. Note however that the eigenvalue equation $Av = v\lambda$ is not associated to a right linear operator, in fact the multiplication on the right by a quaternion is obviously not linear. The operator $A^2 - 2(\operatorname{Re} p)A + |p|^2 I_n$ is right linear on $\mathbb{H}^n$, thus it is the linear operator associated with $\sigma_r(A)$.

Given a polynomial $f(p) = \sum_{n=0}^{r} p^n a_n$ (or $f(p) = \sum_{n=0}^{r} a_n p^n$) then we can define $f(A) = \sum_{n=0}^{r} A^n a_n$ (or $f(A) = \sum_{n=0}^{r} a_n A^n$).
If the coefficients a_n are real then $f(p) = \sum_{n=0}^{r} p^n a_n = \sum_{n=0}^{r} a_n p^n$. From now on, we will assume $a_n \in \mathbb{R}$. Moreover, for any $A, P \in \mathbb{H}^{m\times m}$, P invertible, we have

$$f(P^{-1}AP) = P^{-1} f(A)P.$$

Since the real vector space $\mathbb{H}^{m\times m}$ is finite dimensional the powers A^n of A cannot be linearly independent and so there exists a polynomial f with real coefficients such that $f(A) = 0$. We denote by $m_A(p)$ the monic polynomial with real coefficients such that

$$m_A(A) = 0, \tag{4.10}$$

and m_A has minimal degree. It can be shown, using the same arguments as in the classical complex case, that m_A is unique. The polynomial m_A will be called minimal polynomial of A.
We can then factorize the minimal polynomial $m_A(p)$ as

$$m_A(p) = \prod_{j=1}^{\nu}(p - a_j)^{r_j} \prod_{j=1}^{\mu}(p^2 + (\operatorname{Re} b_j)p + |b_j|^2)^{s_j} \tag{4.11}$$

where $a_j, b_j, c_j \in \mathbb{R}$, $r_j s_j \in \mathbb{N}$ and the polynomials $p^2 + b_j p + c_j$ do not have real roots. Thus the roots of $m_A(p)$ are the real numbers a_j with multiplicity r_j and the spheres $[b_j]$ with multiplicity s_j. Let us denote by $f_j(p)$ any of the polynomials $(p - a_j)$ or $(p^2 + (\operatorname{Re} b_j)p + |b_j|^2)$, $j = 1, \dots, \nu + \mu$ and let $n_j = r_j$ if $f_j(p) = p - a_j$ or $n_j = s_j$ if $f_j(p) = p^2 + (\operatorname{Re} b_j)p + |b_j|^2$.
The root subspaces are defined as:

$$\mathcal{R}_j(A) = \left\{ v \in \mathbb{H}^{n\times 1} \,:\, f_j(p)^{n_j} v = 0 \right\}, \qquad j = 1, \dots, \nu + \mu.$$

Remark 4.3.17. It is immediate to verify that $A\mathcal{R}_j(A) \subseteq \mathcal{R}_j(A)$, i.e. the root subspaces $\mathcal{R}_j(A)$ are A-invariant.

We now mention the following result whose proof can be found in [248]:

Proposition 4.3.18. *Let $A \in \mathbb{H}^{n\times n}$. Then the sets $\mathcal{R}_j(A)$ are right subspaces of $\mathbb{H}^n$ and $\mathbb{H}^n$ is the direct sum of $\mathcal{R}_j(A)$, $j = 1, \dots, \nu + \mu$.*
If $\mathcal{V}$ is an A-invariant subspace of $\mathbb{H}^n$ then

$$\mathcal{V} = \oplus_{j=1}^{\nu+\mu} (\mathcal{R}_j(A) \cap \mathcal{V}).$$

In order to state the Jordan decomposition of a matrix we need some more definitions.

Definition 4.3.19. Let $A, B \in \mathbb{H}^{n\times n}$. We say that A is similar to B if there exists an invertible matrix $P \in \mathbb{H}^{n\times n}$ such that $B = P^{-1}AP$.

Since the relation of similarity is symmetric we will simply say that A and B are similar.

Definition 4.3.20. A Jordan block is a $s\times s$ matrix of the form

$$J_s(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

where $\lambda \in \mathbb{H}$.

The following result is stated without proof. We refer the reader to [274] and [248].

Theorem 4.3.21. *Any $A \in \mathbb{H}^{n\times n}$ is similar to a Jordan matrix $\oplus_{m=1}^{r} J_{s_m}(\lambda_m)$ where $J_{s_m}(\lambda_m)$ denotes the upper triangular $s_m \times s_m$ Jordan block and λ_m is a standard eigenvalue of A. A Jordan matrix is unique up to the permutation of the Jordan blocks and up to replacing λ_m by any element in $[\lambda_m]$ and it is called Jordan form of A.*

4.4 Matrix equations

The following lemma will be used in the study of first order discrete linear systems with quaternionic entries. See Section 10.6.

Lemma 4.4.1. *Let $p, q \in \mathbb{H}$ and assume that*

$$\begin{pmatrix} 1 & p \\ q & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & -d_2 \end{pmatrix} \begin{pmatrix} 1 & p \\ q & 1 \end{pmatrix}^* = \begin{pmatrix} d_3 & 0 \\ 0 & -d_4 \end{pmatrix} \tag{4.12}$$

where d_1, d_2, d_3 and d_4 are strictly positive. Then qp and pq are real and belong to $[0,1)$.

Proof. Equation (4.12) is equivalent to

$$d_1 - |p|^2 d_2 = d_3, \tag{4.13}$$

$$qd_1 = d_2\overline{p}, \tag{4.14}$$

$$|q|^2 d_1 - d_2 = -d_4. \tag{4.15}$$

If $p = 0$ (resp. $q = 0$) then (4.14) leads to $q = 0$ (resp. $p = 0$). We now assume $p \neq 0$. Then,

$$d_2 = qd_1\overline{p}^{-1},$$

and plugging this equality in (4.13) gives

$$d_1 - p\overline{p}qd_1\overline{p}^{-1} = d_3.$$

Thus $d_1(1 - qp) = d_3$ and the conclusion for qp follows since $d_1 > 0$ and $d_3 > 0$. The conclusion for pq is obtained by interchanging the roles of p and q. □

We will encounter equations of the form

$$P - A^*PA = C^*JC \tag{4.16}$$

(called Stein equations) and

$$A^*P + PA = C^*JC, \tag{4.17}$$

(called Lyapunov equation), where the various symbols denote quaternionic matrices. Before proving results on matrix equations we first present a lemma in the special case of Jordan blocks. For more information on matrix equations we refer the reader to [248].

Lemma 4.4.2. *Let λ, ρ be such that $[\lambda] \neq [\rho]$. Then the equation*

$$J_r(\lambda)X = XJ_s(\rho), \qquad X \in \mathbb{H}^{r\times s}$$

has only the solution $X = 0$.

Proof. Let $X = [x_{ij}]$, $x_{ij} \in \mathbb{H}$. By writing explicitly the scalar equations corresponding to the matrix equation $J_r(\lambda)X = XJ_s(\rho)$ we obtain

$$\begin{aligned}
\lambda x_{r1} &= x_{r1}\rho \\
\lambda x_{r2} &= x_{r1} + x_{r2}\rho \\
&\dots \\
\lambda x_{rs} &= x_{r,s-1} + x_{rs}\rho.
\end{aligned}$$

We know that $\lambda x_{r1} = x_{r1}\rho$ admits just the solution $x_{r1} = 0$ since λ and ρ belong to different spheres. By substituting in the second equation and by iterating the procedure, we obtain the statement. □

Remark 4.4.3. It is immediate that if $[\lambda] \in \sigma_r(A)$ then $[\lambda] \in \sigma_r(A^*)$. In fact by taking the inner product in $\mathbb{H}^{m\times 1}$ given by $\langle x, y\rangle := y^*x$, and assuming that $Ax = x\lambda$, $A^*x = x\rho$ we have

$$\langle A^*x, x\rangle = \langle x, Ax\rangle = \langle x, x\lambda\rangle = \overline{\lambda}\|x\|^2,$$

and

$$\langle A^*x, x\rangle = \langle x\rho, x\rangle = \rho\|x\|^2,$$

from which we deduce $\overline{\lambda} = \rho$. If $\lambda \in \mathbb{R}$ this equality translates into $\lambda = \rho$. If λ is not real then the whole sphere $[\lambda]$ consists of eigenvalues and so is $[\rho]$. In all the cases, $[\lambda] = [\rho]$.

Theorem 4.4.4. *Let $A, B \in \mathbb{H}^{n\times n}$. Then the equation*

$$AX - XB = C$$

has a unique solution for every $C \in \mathbb{H}^{n\times n}$ if and only if

$$\sigma_r(A) \cap \sigma_r(B) = \emptyset.$$

Proof. The given equation can be translated into a linear system in $4n^2$ real unknowns which admits a unique solution if and only if the equation $AX - XB = 0$ has $X = 0$ as its unique solution. We can always assume that A, B are in Jordan form, if not it is sufficient the consider the maps $A \mapsto P^{-1}AP$, $B \mapsto Q^{-1}BQ$, $X \mapsto P^{-1}XQ$. So we assume $A = \oplus_{j=1}^{k} J_{r_j}(\lambda_j)$, $B = \oplus_{j=1}^{h} J_{s_j}(\rho_j)$. Write $X = [X_{i,j}]$ where $X_{i,j} \in \mathbb{H}^{r_j \times s_\ell}$. Then we obtain the equations

$$J_{r_i}(\lambda_i)X_{i,j} + X_{i,j}J_{s_j}(\rho_j) = 0$$

which, since for all j, ℓ $[\lambda_j] \neq [\rho_\ell]$, has only the trivial solution by Lemma 4.4.2. To show the converse, let us assume that there is $\lambda \in \sigma_S(A) \cap \sigma_S(B)$. Then there exist suitable $x, y \in \mathbb{H}^{m\times 1}$ such that $Ax = x\lambda$, $B^*y = y\overline{\lambda}$ from which we deduce $y^*B = \lambda y^*$. By setting $X = xy^*$ we have

$$AX - XB = Axy^* - xy^*B = x\lambda y^* - x\lambda y^* = 0$$

so $X = xy^*$ is a nontrivial solution of the given equation, which is a contradiction. □

The following corollary is a special case of the previous result:

Corollary 4.4.5. *Let $A \in \mathbb{H}^{n\times n}$. Then the equation*

$$A^*X + XA = C$$

has a unique solution for every $C \in \mathbb{H}^{n\times n}$ if and only if

$$\sigma_r(A^*) \cap \sigma_r(-A) = \emptyset.$$

Remark 4.4.6. If λ is a real eigenvalue of A^* then $-\lambda$ is eigenvalue of $-A$ and if λ is a non real eigenvalue of A^* then $-\overline{\lambda}$ is an eigenvalue of $-A$. It is then clear that $A^*, -A$ have disjoint spectrum if their real eigenvalues are different and their nonreal eigenvalues are not purely imaginary.

Theorem 4.4.7. *Let $A \in \mathbb{H}^{n\times n}$. Then the equation*

$$X - A^*XA = C$$

has a solution for every $C \in \mathbb{H}^{n\times n}$ if and only if

$$\lambda\rho \neq 1 \tag{4.18}$$

for all $\lambda \in \sigma_r(A)$, $\rho \in \sigma_r(A^*)$.

Proof. The statement holds true if it is valid in the case $C = 0$, so we consider $A^*XA - X = 0$ and we show that it has only the trivial solution if and only if (4.18) holds.
Suppose that (4.18) is in force. As in the proof of Theorem 4.4.4, we can assume that $A = J_r(\lambda)$, $A^* = J_s(\rho)$ and so we consider $J_s(\rho)XJ_r(\lambda) = X$. If $\lambda \neq 0$ then $J_r(\lambda)$ is invertible, the equation becomes $J_s(\rho)X = X(J_r(\lambda))^{-1}$. Since the eigenvalues of $(J_r(\lambda))^{-1}$ correspond to $[\lambda^{-1}]$, by Theorem 4.4.4 this equation has only the trivial solution. We conclude similarly if $\rho \neq 0$. So let us suppose that $\rho = \lambda = 0$. We have, by using iteratively the equation $J_s(\rho)XJ_r(\lambda) = X$:

$$X = J_s(\rho)XJ_r(\lambda) = (J_s(\rho))^2X(J_r(\lambda))^2 = \cdots = (J_s(\rho))^kX(J_r(\lambda))^k = 0$$

for $k \geq \min\{s, r\}$.
We now show the converse and so we assume that the equation $A^*XA - X = 0$ has only the solution $X = 0$. Suppose, by absurd that there exist $\lambda \in \sigma_r(A)$, $\rho \in \sigma_r(A^*)$ such that $\lambda\rho = 1$. We assume that A and A^* are in Jordan form $A = \oplus_{j=1}^{k} J_{r_j}(\lambda_j)$, $A^* = \oplus_{j=1}^{h} J_{s_j}(\rho_j)$ where it is not reductive to take $\lambda_1 = \lambda$, $\rho_1 = \rho$. We write $X = [X_{i,j}]$ where $X_{i,j} \in \mathbb{H}^{r_j \times s_\ell}$. Then we obtain the equation

$$J_{r_1}(\lambda_1)X_{1,1}J_{s_1}(\rho_1) - X_{1,1} = 0$$

which can be written as

$$J_{r_1}(\lambda_1)X_{1,1} = X_{1,1}(J_{s_1}(\rho_1))^{-1}$$

and this last equation has nontrivial solutions, which is a contradiction. □

In particular, denoting by $\mathbb{B}$ the unit ball of $\mathbb{H}$ and by $\partial\mathbb{B}$ its boundary, we have:

Proposition 4.4.8. *Let $p, q \in \partial\mathbb{B}$ be such that $[p] \cap [q] = \emptyset$, and let $h \in \mathbb{H}$. Then the unique solution of the equation $x - px\overline{q} = h$ is given by*

$$x = (h - \overline{p}h\overline{q})\,(1 - 2\mathrm{Re}(p)\overline{q} + \overline{q}^2)^{-1}. \tag{4.19}$$

Proof. The solution is unique in view of Theorem 4.4.4. It remains then to check directly that (4.19) answers the question. More precisely, and since $\overline{q}$ and $(1 - 2\mathrm{Re}(p)\overline{q} + |p|^2\overline{q}^2)$ commute and $|p| = 1$, we can write

$$\begin{aligned} x - px\overline{q} &= (h - \overline{p}h\overline{q})\,(1 - 2\mathrm{Re}(p)\overline{q} + |p|^2\overline{q}^2)^{-1} - \\ &\quad - \left(ph\overline{q} - |p|^2h\overline{q}^2\right)(1 - 2\mathrm{Re}(p)\overline{q} + \overline{q}^2)^{-1} \\ &= \left(h - 2\mathrm{Re}(p)h\overline{q} + |p|^2h\overline{q}^2\right)(1 - 2\mathrm{Re}(p)\overline{q} + \overline{q}^2)^{-1} \\ &= h. \end{aligned}$$

□

The notion of observability and controllability still make sense in the quaternionic setting.

Definition 4.4.9. The pair $(A, B) \in \mathbb{H}^{n\times n} \times \mathbb{H}^{n\times m}$ is called controllable if

$$\cup_{u=0}^{\infty} \mathrm{ran}\, A^u B = \mathbb{H}^m.$$

Definition 4.4.10. The pair $(C, A) \in \mathbb{H}^{n\times m} \times \mathbb{H}^{m\times m}$ is called observable if

$$\cap_{u=0}^{\infty} \ker CA^u = \{0\}.$$

Note that in the above union and intersection one can replace ∞ by the degree of the minimal polynomial m_A (see (4.10)). Another way to see that is to remark that the functions

$$n \mapsto \dim(\cup_{u=0}^{n} \mathrm{ran}\, A^u B) \quad \text{and} \quad n \mapsto \dim(\cap_{u=0}^{n} \ker CA^u)$$

have values in $\{0, \dots, m\}$ and are respectively increasing and decreasing.
It is interesting to note that the quaternionic setting enlights the fact that the use of the Cayley-Hamilton theorem (which does not hold in this framework) is not needed.

Proposition 4.4.11. *Let* $(C,A) \in \mathbb{H}^{n\times m} \times \mathbb{H}^{m\times m}$ *be an observable pair of matrices such that* $\sigma_S(A) \subset \mathbb{B}$. *Then,*

$$P = \sum_{u=0}^{\infty} A^{*u} C^* J C A^u \tag{4.20}$$

is the unique solution of (4.16).

Proof. Assume that X is a solution of (4.16). Then for every N,

$$X = \sum_{u=0}^{N} A^{*u} C^* J C A^u + A^{*(N+1)} X A^{N+1}.$$

The condition $\sigma_S(A) \subset \mathbb{B}$ implies that the series (4.20) converges absolutely, and it follows that $X = P$. □

Chapter 5

Quaternionic functional analysis

To develop Schur analysis in the slice hyperholomophic setting, a number of facts and results from quaternionic functional analysis for which no references were available are needed. These are developed in the present chapter, largely taken from [37]. Most of the results can be proved with the same arguments as in the classical proofs since they do not rely on specific properties of complex numbers that do not hold for quaternions. In some cases we repeat here the arguments to show that indeed they carry out.

5.1 Quaternionic locally convex vector spaces

We will work in quaternionic right linear spaces $\mathscr{V}$ on $\mathbb{H}$ in which are defined the operations of sum and scalar multiplication on the right. This is a particular case of linear space over a (skew) field which is well known in the literature. The following results are for example in [100, Théorème 1 and Proposition 4, Ch. 2, §7]:

Theorem 5.1.1. (1) *Every right quaternionic vector space has a basis.*
(2) *Every (right) linear subspace of a quaternionic vector space has a direct complement.*

From now on, if not otherwise stated, when we will write quaternionic linear spaces we will mean quaternionic right linear spaces (i.e. we will omit to write ”right”).

Definition 5.1.2. Let $\mathscr{V}$, $\mathscr{W}$ be quaternionic linear spaces and let $T : \mathscr{V} \to \mathscr{W}$ be such that

$$T(u\alpha + v\beta) = T(u)\alpha + T(v)\beta, \qquad \forall u, v \in \mathscr{V},\ \forall \alpha, \beta \in \mathbb{H}.$$

Then T is called a (right) linear map. The set of linear maps from $\mathscr{V}$ to $\mathscr{W}$ is denoted by $\mathbf{L}(\mathscr{V}, \mathscr{W})$ or by $\mathbf{L}(\mathscr{V})$ when $\mathscr{W} = \mathscr{V}$. Note that $\mathbf{L}(\mathscr{V}, \mathscr{W})$ has no linear structure, unless $\mathscr{W}$ is two sided.
When $\mathscr{W} = \mathbb{H}$, T is called functional.

Definition 5.1.3. Let $\mathscr{V}$ be a quaternionic linear space. A semi-norm is defined as a map $p : \mathscr{V} \to \mathbb{R}$ such that

$$p(v_1 + v_2) \leq p(v_1) + p(v_2), \quad \forall v_1, v_2 \in \mathscr{V}, \tag{5.1}$$

and

$$p(vc) = |c| p(v), \quad \forall v \in \mathscr{V} \text{ and } c \in \mathbb{H}. \tag{5.2}$$

It is immediate that (5.2) implies that $p(0) = 0$ and (5.1) implies

$$0 = p(v - v) \leq 2p(v),$$

so that a semi-norm has values in $\mathbb{R}^+$.

In the sequel, we will make use of the quaternionic version of the Hahn-Banach theorem and of Corollary 5.1.7 below:

Theorem 5.1.4 (Hahn-Banach). *Let $\mathscr{V}_0$ be a subspace of a right quaternionic linear space $\mathscr{V}$. Suppose that p is a seminorm on $\mathscr{V}$ and let ϕ be a linear functional on $\mathscr{V}_0$ such that*

$$|\langle \phi, v \rangle| \leq p(v), \qquad \forall v \in \mathscr{V}_0. \tag{5.3}$$

Then ϕ extends to a linear functional Φ on $\mathscr{V}$ satisfying the estimate (5.3) for all $v \in \mathscr{V}$.

Let p be a semi-norm and set

$$U_{v_0}(p, \alpha) = \{v \in \mathscr{V} \mid p(v - v_0) < \alpha\}.$$

A family $\{p_\gamma\}_{\gamma \in \Gamma}$ of semi-norms on $\mathscr{V}$ indexed by some set Γ defines a topology on $\mathscr{V}$, in which a subset $U \subseteq \mathscr{V}$ is said to be open if and only if for every $v_0 \in U$ there are $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $\varepsilon > 0$ such that $v \in U_{v_0}(p_{\gamma_j}, \varepsilon)$, $j = 1, \ldots, n$, implies $v \in U$.

Following standard arguments, one can easily use (5.1) and (5.2) to verify that when $\mathscr{V}$ is endowed with the topology induced by a family of semi-norms, it is a locally convex space. Also the converse is true in fact we have, see [37]:

Proposition 5.1.5. *A topological quaternionic vector space is locally convex if and only if the topology is defined by a family of semi-norms.*

Definition 5.1.6. A locally convex quaternionic linear space $\mathscr{V}$ is a Fréchet space if it is complete with respect to a (translation invariant) metric. If the metric is induced by a norm then we say that $\mathscr{V}$ is a Banach space.

Corollary 5.1.7. *Let $\mathscr{V}$ be a quaternionic Banach space and let $v \in \mathscr{V}$. If $\langle \phi, v \rangle = 0$ for every linear continuous functional ϕ in $\mathscr{V}'$, then $v = 0$.*

We now state the quaternionic counterpart of some classical results. We begin with a result which implies the principle of uniform boundedness.

Theorem 5.1.8. *For each $a \in A$, where A is a set, let S_a be a continuous map of a quaternionic Fréchet space $\mathscr{V}$ into a quaternionic Fréchet space $\mathscr{W}$, which satisfies the following properties*

(1) $|S_a(u+w)| \leq |S_a(u)| + |S_a(w)|$, $\forall u, w \in \mathscr{V}$,

(2) $|S_a(w\alpha)| = |S_a(w)\alpha|$, $\forall w \in \mathscr{V}$, $\forall \alpha \geq 0$.

If, for each $u \in \mathscr{V}$*, the set* $\{S_a v\}_{a\in A}$ *is bounded, then* $\lim_{v\to 0} S_a v = 0$ *uniformly in* $a \in A$.

Definition 5.1.9. Let $\mathscr{V}$, $\mathscr{W}$ be normed spaces. A linear operator T is said to be bounded (or continuous) if

$$\|T\| := \sup_{\|v\|=1} \|Tv\| < \infty.$$

The set of linear, bounded operators from $\mathscr{V}$ to $\mathscr{W}$ is denoted by $\mathbf{B}(\mathscr{V},\mathscr{W})$ or by $\mathbf{B}(\mathscr{V})$ when $\mathscr{W} = \mathscr{V}$.

In the special case of linear maps, Theorem 5.1.8 becomes the following result:

Theorem 5.1.10 (Principle of uniform boundedness)**.** *For each* $a \in A$*, where* A *is a set, let* T_a *be continuous linear map of a quaternionic Fréchet space* $\mathscr{V}$ *into a quaternionic Fréchet space* $\mathscr{W}$*. If, for each* $v \in \mathscr{V}$*, the set* $\{T_a v\}_{a\in A}$ *is bounded, then* $\lim_{v\to 0} T_a v = 0$ *uniformly in* $a \in A$.

The same result can be also formulated in the setting of quaternionic Banach spaces.

Theorem 5.1.11. *Let* $\mathscr{V}$ *and* $\mathscr{W}$ *be two quaternionic Banach spaces and let* $\{T_a\}_{a\in A}$ *be bounded linear maps from* $\mathscr{V}$ *to* $\mathscr{W}$*. Suppose that* $\sup_{a\in A} \|T_a v\| < \infty$ *for any* $v \in \mathscr{V}$*. Then*

$$\sup_{a\in A} \|T_a\| < \infty.$$

Another classical result which generalizes to the quaternionic setting is the open mapping theorem:

Theorem 5.1.12 (Open mapping theorem)**.** *Let* $\mathscr{V}$ *and* $\mathscr{W}$ *be two quaternionic Fréchet spaces, and let* T *be a linear continuous quaternionic map from* $\mathscr{V}$ *onto* $\mathscr{W}$*. Then the image of every open set is open.*

Theorem 5.1.13 (Banach continuous inverse theorem)**.** *Let* $\mathscr{V}$ *and* $\mathscr{W}$ *be two quaternionic Fréchet spaces and let* $T : \mathscr{V} \to \mathscr{W}$ *be a linear continuous quaternionic map which is one-to-one and onto. Then* T *has a linear continuous inverse.*

Definition 5.1.14. Let $\mathscr{V}$ and $\mathscr{W}$ be two quaternionic Fréchet spaces. Suppose that T is a quaternionic operator whose domain $\mathscr{D}(T)$ is a linear manifold contained in $\mathscr{V}$ and whose range belongs to $\mathscr{W}$. The graph of T consists of all point (v, Tv), with $v \in \mathscr{D}(T)$, in the product space $\mathscr{V} \times \mathscr{W}$.

Definition 5.1.15. We say that T is a closed operator if its graph is closed in $\mathscr{V} \times \mathscr{W}$.

In an equivalent way, we say that T is closed if $v_n \in \mathscr{D}(T)$, $v_n \to v$, $Tv_n \to y$ imply that $v \in \mathscr{D}(T)$ and $Tv = y$.
The proof of the following theorem can be found in [101, Corollaire 5, p. I.19] and also in [37, Theorem 3.9].

Theorem 5.1.16 (Closed graph theorem)**.** *Let $\mathscr{V}$ and $\mathscr{W}$ be two quaternionic Fréchet spaces. Let $T:\mathscr{V}\to\mathscr{W}$ be a linear closed quaternionic operator. Then T is continuous.*

In the sequel we will use a consequence of the Ascoli-Arzelà theorem that we state in this lemma, which can be proved as in Corollary 9 p. 267 in [166].

Lemma 5.1.17 (Corollary of Ascoli-Arzelá theorem)**.** *Let $\mathscr{G}_1$ be a compact subset of a topological group $\mathscr{G}$ and let $\mathscr{K}$ be a bounded subset of the space of continuous functions $C(\mathscr{G}_1)$. Then $\mathscr{K}$ is conditionally compact if and only if for every $\varepsilon>0$ there is a neighborhood $\mathscr{U}$ of the identity in $\mathscr{G}$ such that $|f(t)-f(s)|<\varepsilon$ for every $f\in\mathscr{K}$ and every pair $s,t\in\mathscr{G}_1$ with $t\in\mathscr{U}s$.*

Definition 5.1.18. We say that a quaternionic topological space $\mathscr{T}$ has the fixed point property if for every continuous mapping $T:\mathscr{T}\to\mathscr{T}$ there exists $u\in\mathscr{T}$ such that $u=T(u)$.

To show our result we need the following Lemmas:

Lemma 5.1.19. *Let $\mathfrak{C}$ be the subset of $\ell^2(\mathbb{H})$ defined by*

$$\mathfrak{C}=\{\{\xi_n\}\in\ell^2(\mathbb{H})\ :\ |\xi_n|\leq 1/n,\quad \forall n\in\mathbb{N}\}.$$

Then $\mathfrak{C}$ has the fixed point property.

Lemma 5.1.20. *Let $\mathscr{K}$ be a compact convex subset of a locally convex linear quaternionic space $\mathscr{V}$ and let $T:\mathscr{K}\to\mathscr{K}$ be continuous. If $\mathscr{K}$ contains at least two points, then there exists a proper closed convex subset $\mathscr{K}_1\subset\mathscr{K}$ such that $T(\mathscr{K}_1)\subseteq\mathscr{K}_1$.*

Theorem 5.1.21 (Schauder-Tychonoff)**.** *A compact convex subset of a locally convex quaternionic linear space has the fixed point property.*

5.2 Quaternionic inner product spaces

In this section we consider quaternionic inner product spaces, their decomposition and ortho-complemented subspaces. The main source for this part is our paper [37].

Definition 5.2.1. Let $\mathscr{V}$ be a quaternionic vector space. The map

$$[\cdot,\cdot]\ :\ \mathscr{V}\times\mathscr{V}\quad\longrightarrow\quad\mathbb{H}$$

is called an inner product if it is a (right) sesquilinear form:

$$[v_1c_1,v_2c_2]=\overline{c_2}[v_1,v_2]c_1,\quad\forall v_1,v_2\in\mathscr{V},\text{ and }c_1,c_2\in\mathbb{H},$$

and Hermitian:

$$[v,w]=\overline{[w,v]},\quad\forall v,w\in\mathscr{V}.$$

When the space $\mathscr{V}$ is two-sided, we require moreover that

$$[f, cg] = [\overline{c}f, g], \quad c \in \mathbb{H}, \quad f, g \in \mathscr{V}. \tag{5.4}$$

We will call the pair $(\mathscr{V}, [\cdot,\cdot])$ (or the space $\mathscr{V}$ for short when the form is understood from the context) a (right) quaternionic indefinite inner product space. The form is called positive (or non-negative) if $[v,v] \geq 0$ for all $v \in \mathscr{V}$.

Definition 5.2.2. A linear subspace $\mathscr{M} \subset \mathscr{V}$ is called positive if $[m,m] \geq 0$ for all $m \in \mathscr{M}$. It is called strictly positive if the inequality is strict for all $m \neq 0$. Similarly, $\mathscr{M}$ is called negative if $[m,m] \leq 0$ for all $m \in \mathscr{M}$ and strictly negative if the inequality is strict for all $m \neq 0$.

Two vectors $v, w \in \mathscr{V}$ are orthogonal if $[v,w] = 0$. An element $v \in \mathscr{V}$ such that $[v,v] = 0$ is said to be neutral and the set of neutral elements of $\mathscr{V}$ forms the so-called neutral part of $\mathscr{V}$. For $\mathscr{L} \subset \mathscr{V}$ we set

$$\mathscr{L}^{[\perp]} = \{v \in \mathscr{V} \ : \ [v,w] = 0, \ \forall w \in \mathscr{L}\}.$$

The definition of $\mathscr{L}^{[\perp]}$ makes sense even when $\mathscr{L}$ is simply a subset of $\mathscr{V}$, not necessarily a subspace, and the set $\mathscr{L}^{[\perp]}$, called the *orthogonal companion* of $\mathscr{L}$, is always a subspace of $\mathscr{V}$.

Definition 5.2.3. Let $\mathscr{L} \subseteq \mathscr{V}$ be a linear subspace of $\mathscr{V}$. The subspace $\mathscr{L}^0 = \mathscr{L} \cap \mathscr{L}^{[\perp]}$ is called isotropic part of $\mathscr{L}$.

We have:

$$\mathscr{L} \subset \left(\mathscr{L}^{[\perp]}\right)^{[\perp]} \overset{\text{def.}}{=} \mathscr{L}^{[\perp\perp]}. \tag{5.5}$$

When $\mathscr{V} \cap \mathscr{V}^{[\perp]} \neq \{0\}$ we say that $\mathscr{V}$ is degenerate.
The quaternionic inner product space $\mathscr{V}$ is decomposable if it can be written as a direct and orthogonal sum

$$\mathscr{V} = \mathscr{V}_+ [\oplus] \mathscr{V}_- [\oplus] \mathscr{N} \tag{5.6}$$

where $\mathscr{V}_+$ is a strictly positive subspace, $\mathscr{V}_-$ is a strictly negative subspace, and $\mathscr{N}$ is a neutral subspace.

Proposition 5.2.4. *Assume that* (5.6) *holds. Then $\mathscr{N} = \mathscr{V} \cap \mathscr{V}^{[\perp]}$ (namely $\mathscr{N}$ is the isotropic part of $\mathscr{V}$).*

The representation (5.6) is called a fundamental decomposition. A quaternionic inner product space need not be decomposable (see for instance [98, Example 11.3, p. 23] for an example in the complex setting, which still holds in the quaternionic case), and the decomposition will not be unique, unless one of the spaces $\mathscr{V}_\pm$ is trivial. A precise characterization of the decompositions is given in the following results.

Lemma 5.2.5. *Let $\mathscr{V}$ denote a quaternionic inner product space, and let $\mathscr{V}^0$ be its isotropic part. Let $\mathscr{V}_1$ be a direct complement of $\mathscr{V}^0$. Then $\mathscr{V}_1$ is nondegenerate and the direct sum decomposition*

$$\mathscr{V} = \mathscr{V}^0[\oplus]\mathscr{V}_1 \tag{5.7}$$

holds.

Proof. Let $v \in \mathscr{V}^0 \cap \mathscr{V}_1$ be such that

$$[v, v_1] = 0, \quad \forall v_1 \in \mathscr{V}_1.$$

By definition of the isotropic part, we have

$$[v, v_0] = 0, \quad \forall v_0 \in \mathscr{V}^0.$$

Since $\mathscr{V}_1$ is a direct complement of $\mathscr{V}^0$ in $\mathscr{V}$, we have $v \in \mathscr{V}^0$, and so $v = 0$ since $\mathscr{V}^0 \cap \mathscr{V}_1 = \{0\}$. Then equality (5.7) follows. □

Proposition 5.2.6.
(1) *Let $\mathscr{V} = \mathscr{V}_1[\oplus]\mathscr{V}_2$ denote an orthogonal direct decomposition of the indefinite inner product quaternionic vector space $\mathscr{V}$, where $\mathscr{V}_1$ is positive and $\mathscr{V}_2$ is maximal strictly negative. Then, $\mathscr{V}_1$ is maximal positive.*
(2) *The space orthogonal to a maximal positive subspace is negative.*
(3) *The space orthogonal to a maximal strictly positive subspace is negative.*

Proof.
(1) Let $\mathscr{W}_1 \supset \mathscr{V}_1$ be a positive subspace of $\mathscr{V}$ containing $\mathscr{V}_1$ and let $v \in \mathscr{W}_1 \setminus \mathscr{V}_1$. Since $\mathscr{V} = \mathscr{V}_1[\oplus]\mathscr{V}_2$, we can write $v = v_1 + v_2$, where $v_1 \in \mathscr{V}_1$ and $v_2 \in \mathscr{V}_2$. Then, we have that $v_2 = v - v_1 \in \mathscr{W}_1$ since $\mathscr{W}_1$ is a subspace. On the other hand, $v_2 \neq 0$ (otherwise $v \in \mathscr{V}_1$) and so $[v_2, v_2] < 0$. This contradicts the assumption that $\mathscr{W}_1$ is positive.

(2) Let $\mathscr{L}$ be a maximal positive subspace of $\mathscr{V}$, and let $v \in \mathscr{L}^{[\perp]}$. We have three cases:

(a) If $v \notin \mathscr{L}$ and $[v, v] = 0$, there is nothing to prove.

(b) If $v \notin \mathscr{L}$ and $[v, v] > 0$, then the space spanned by v and $\mathscr{L}$ is positive, contradicting the maximality of $\mathscr{L}$. So $[v, v] \leq 0$.

(c) If $v \in \mathscr{L}$. Then, $v \in \mathscr{L} \cap \mathscr{L}^{[\perp]}$, and so $[v, v] = 0$, which is what we wanted to prove.

(3) Let now $\mathscr{L}$ be a maximal positive definite subspace of $\mathscr{V}$, and let $v \in \mathscr{L}^{[\perp]}$ a nonzero element. If $[v, v] \leq 0$ there is nothing to prove. If $[v, v] > 0$, the space spanned by v and $\mathscr{L}$ is strictly positive, contradicting the maximality of $\mathscr{L}$. □

To state and prove next result, we recall that if $\mathscr{V}$ is a right quaternionic vector space and $\mathscr{V}_1 \subset \mathscr{V}$ is a (right) linear subspace of $\mathscr{V}$, the quotient space $\mathscr{V}/\mathscr{V}_1$ endowed with

$$(v + \mathscr{V}_1)q = vq + \mathscr{V}_1$$

is also a right quaternionic vector space. The symbol $v + \mathscr{V}_1$ denotes the equivalence class of $v \in \mathscr{V}_1$ in the quotient space $\mathscr{V}/\mathscr{V}_1$.

Definition 5.2.7. The linear subspace $\mathscr{L}$ of $\mathscr{V}$ is ortho-complemented if $\mathscr{V}$ is spanned by the sum of $\mathscr{L}$ and $\mathscr{L}^{[\perp]}$.

Theorem 5.2.8. *Let $\mathscr{V}$ denote a quaternionic inner product space. Then the subspace $\mathscr{L}$ is ortho-complemented if and only if the following two conditions hold:*
(1) *The isotropic part of $\mathscr{L}$ is included in the isotropic part of $\mathscr{V}$.*
(2) *The image under the inclusion map*

$$\begin{aligned}\iota:\ \mathscr{L}/\mathscr{L}^0 &\to \mathscr{V}/\mathscr{V}^0\\ \iota(\ell+\mathscr{L}^0) &= \ell+\mathscr{V}_0\end{aligned}$$

of the quotient space $\mathscr{L}/\mathscr{L}^0$ is ortho-complemented in $\mathscr{V}/\mathscr{V}^0$.

Proof. Assume that $\mathscr{L}$ is ortho-complemented: by definition $\mathscr{V}=\mathscr{L}[+]\mathscr{L}^{[\perp]}$. The formula

$$[v+\mathscr{V}^0, w+\mathscr{V}^0]_q \overset{\text{def.}}{=} [v,w] \tag{5.8}$$

defines a nondegenerate indefinite inner product on $\mathscr{V}/\mathscr{V}^0$. It is immediate to verify that the inner product (5.8) preserves orthogonality, and thus

$$\mathscr{V}/\mathscr{V}^0 = (\mathscr{L}/\mathscr{V}^0)[+](\mathscr{L}^{[\perp]}/\mathscr{V}^0).$$

Since $\mathscr{L}$, $\mathscr{V}^0$ are subspaces of the quaternionic vector space $\mathscr{V}$, and $\mathscr{V}^0\subset\mathscr{V}$, the map ι from $\mathscr{L}/\mathscr{L}^0$ into $\mathscr{V}/\mathscr{V}_0$ is well defined. Let us show that ι is one-to-one, so that

$$(\mathscr{L}/\mathscr{V}^0) = \iota(\mathscr{L}/\mathscr{L}^0),$$

and this will conclude the proof of the direct assertion. To this end, observe that every v in $\mathscr{V}$ can be written as

$$v=\ell+m,\quad \ell\in\mathscr{L},\quad m\in\mathscr{L}^{[\perp]}.$$

If $\ell_0\in\mathscr{L}^0$ then

$$[\ell_0,v]=[\ell_0,\ell]+[\ell_0,m]=0$$

and thus $\mathscr{L}^0\subset\mathscr{V}^0$. Since

$$\mathscr{V}^0\cap\mathscr{L}\subset\mathscr{L}^0, \tag{5.9}$$

one deduces that the map ι is well defined and one-to-one and so (2) holds.

Conversely we assume now that (1) and (2) hold. We prove that $\mathscr{L}$ is ortho-complemented. The condition (1) insures that the map ι is well defined and the inclusion (5.9) holds by definition of $\mathscr{V}_0$. Thus ι is one-to-one. The property (2) shows that for every $v\in\mathscr{V}$ there exist $\ell\in\mathscr{L}$ and $m\in\mathscr{L}^{[\perp]}$ such that

$$v+\mathscr{V}^0=\ell+\mathscr{V}^0+m+\mathscr{V}^0.$$

Thus we have $v=\ell+m+v_0$ and this concludes the proof since $\mathscr{V}^0\cap\mathscr{L}\subset\mathscr{L}^0\subset\mathscr{L}^{[\perp]}$. □

Next result holds for a nondegenerate inner product space.

Proposition 5.2.9. *Let $\mathscr{V}$ be a quaternionic nondegenerate inner product space. Then:*
(1) *Every ortho-complemented subspace is nondegenerate.*
(2) *Let $\mathscr{L} \subset \mathscr{V}$ be ortho-complemented. Then $\mathscr{L} = \mathscr{L}^{[\perp\perp]}$.*

Proof. Property (1) follows directly from Theorem 5.2.8 (1) since

$$\mathscr{L} \cap \mathscr{L}^{[\perp]} \subset \mathscr{V}^0 = \{0\}.$$

To show (2), we observe that

$$\mathscr{L} \subset \mathscr{L}^{[\perp\perp]}. \tag{5.10}$$

We assume that $\mathscr{L}$ is ortho-complemented. Let $v \in \mathscr{L}^{[\perp\perp]}$, and let

$$v = v_1 + v_2, \quad v_1 \in \mathscr{L}, \quad \text{and} \quad v_2 \in \mathscr{L}^{[\perp]}.$$

Then, in view of (5.10), $v_2 = v - v_1 \in \mathscr{L}^{[\perp\perp]}$, and so $v_2 \in \mathscr{L}^{[\perp]} \cap \mathscr{L}^{[\perp\perp]}$. We have

$$\mathscr{L}^{[\perp]}[+]\mathscr{L}^{[\perp\perp]} = \mathscr{V}.$$

In fact if $\mathscr{L}$ is ortho-complemented, also $\mathscr{L}^{[\perp]}$ is also ortho-complemented by formula (5.5). This implies that $v_2 = 0$ since $\mathscr{V}$ is nondegenerate. We conclude that equality holds in (5.10). $\square$

Proposition 5.2.10. *Let $\mathscr{V}$ be a quaternionic inner product space, and let $\mathscr{L}$ be a positive definite subspace of $\mathscr{V}$. There exists a fundamental decomposition of $\mathscr{V}$ with $\mathscr{V}_+ = \mathscr{L}$ if and only if $\mathscr{L}$ is maximal positive definite and ortho-complemented.*

Proof. Let us assume that there exists a fundamental decomposition of $\mathscr{V}$ with $\mathscr{V}_+ = \mathscr{L}$, namely $\mathscr{V} = \mathscr{L}[\oplus]\mathscr{V}_-[\oplus]\mathscr{V}^0$, where $\mathscr{V}_-$ is negative definite and $\mathscr{V}^0$ is the isotropic part of $\mathscr{V}$. Then $\mathscr{L}$ is ortho-complemented. To show that $\mathscr{L}$ is maximal, assume that $\mathscr{M} \supset \mathscr{L}$ be a positive definite subspace containing $\mathscr{L}$ and let $v \in \mathscr{M}$, with decomposition

$$v = v_+ + v_- + n, \quad v_+ \in \mathscr{L}, \quad v_- \in \mathscr{L}_-, \quad n \in \mathscr{V}^0.$$

By linearity, $v - v_+ = v_- + n \in \mathscr{M}$. But

$$[v - v_+, v - v_+] = [v_-, v_-] + [n, n] < 0,$$

unless $v_- = 0$. But then $[v - v_+, v - v_+] = 0$ implies $v = v_+$ (and so $n = 0$) since $\mathscr{M}$ is positive definite. Thus $v = v_+$ and $\mathscr{L} = \mathscr{M}$. Thus, $\mathscr{L}$ is maximal positive definite.

Conversely, if $\mathscr{L}$ is ortho-complemented, then $\mathscr{V} = \mathscr{L}[+]\mathscr{L}^{[\perp]}$ and, since $\mathscr{L}$ is positive definite, the latter sum is direct, that is, $\mathscr{V} = \mathscr{L}[\oplus]\mathscr{L}^{[\perp]}$.

Since $\mathscr{L}$ is maximal positive definite, we have that $\mathscr{L}^{[\perp]}$ is negative. Indeed, neither $\mathscr{L}^{[\perp]} \setminus \mathscr{L}$ nor $\mathscr{L}^{[\perp]} \cap \mathscr{L}$ contain positive vectors v since in the first case the space spanned by v and $\mathscr{L}$ would be positive, contradicting the maximality of $\mathscr{L}$ and in the second case we would have $[v, v] = 0$ contradicting the positivity of v. Using Lemma 5.2.5 we can write $\mathscr{L}^{[\perp]}$ as a direct orthogonal sum of a negative definite space and of an isotropic space $\mathscr{N}$. Finally, the isotropic part $\mathscr{N}$ of $\mathscr{L}^{[\perp]}$ is the isotropic part of $\mathscr{V}$. $\square$

5.3 Quaternionic Hilbert spaces. Main properties

Quaternionic Hilbert spaces have been treated in the literature in several papers, see e.g. [68, 193, 213, 246, 270]. Our study here starts from the definition of right quaternionic pre-Hilbert space, then moves to the notion of Hilbert space and we show how it is possible to make a right Hilbert space a two-sided Hilbert space, once that a Hilbert basis has been selected.

Definition 5.3.1. A quaternionic linear space $\mathscr{H}$ is said to be a quaternionic pre-Hilbert space if it is a quaternionic linear space endowed with an $\mathbb{H}$–valued form $[\cdot,\cdot]$ which is sesquilinear, Hermitian and positive.

Remark 5.3.2. In this book we will denote usually inner product in Hilbert spaces by the symbol $\langle\cdot,\cdot\rangle$ rather than $[\cdot,\cdot]$. Thus two elements belonging to a quaternionic pre-Hilbert space $\mathscr{H}$ will be called orthogonal if $\langle f,g\rangle = 0$. Given a subset $\mathscr{M}$ of $\mathscr{H}$ we define $\mathscr{M}^\perp$ as

$$\mathscr{M}^\perp = \{v \in \mathscr{H}\ :\ \langle v,m\rangle = 0,\ \forall m \in \mathscr{M}\}.$$

Let us define

$$\|v\| = \sqrt{\langle v,v\rangle}.$$

Observe that $\langle vq,vq\rangle = |q|^2\|v\|^2$. Then $\|\cdot\|$ is a norm for which the Cauchy-Schwarz inequality holds:

$$|\langle u,v\rangle|^2 \le \|u\|^2\,\|v\|^2. \tag{5.11}$$

Definition 5.3.3. A quaternionic pre-Hilbert space is said to be a quaternionic Hilbert space if

$$\|v\| = \sqrt{\langle v,v\rangle}$$

defines a norm for which $\mathscr{H}$ is complete.

We note that every quaternionic pre-Hilbert space has a completion, as follows from [101, 10, p. I.6].
If $\mathscr{H}$ is a quaternionic Hilbert space and $\mathscr{M} \subseteq \mathscr{H}$ is a closed subspace then

$$\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp.$$

The Riesz representation theorem for continuous functionals holds, see [102, p.24]:

Theorem 5.3.4. *Let $\mathscr{H}$ be a quaternionic Hilbert space with quaternionic inner product $\langle\cdot,\cdot\rangle$. Let φ be a continuous right linear functional. Then there is a uniquely defined element $u_\varphi \in \mathscr{H}$ such that*

$$\varphi(v) = \langle v,u_\varphi\rangle, \quad \forall v \in \mathscr{H}.$$

Using the Riesz representation theorem one can introduce the notion of adjoint of a linear operator.

Definition 5.3.5. Let $(\mathscr{H}_1, \langle\cdot,\cdot\rangle_1)$, $(\mathscr{H}_2, \langle\cdot,\cdot\rangle_2)$ be two quaternionic Hilbert spaces and $A:\ \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded (right) linear operator. Then there exists a uniquely defined bounded (right) linear operator

$$A^*:\ \mathscr{H}_2\ \to\ \mathscr{H}_1$$

such that for any $u \in \mathscr{H}_1$ and $v \in \mathscr{H}_2$

$$\langle u, A^*v\rangle_1 = \langle Au, v\rangle_2$$

The operator A^* is called the adjoint of A.

We define the norm of a linear bounded operator as

$$\|A\| = \sup_{u\in\mathscr{H}_1, u\neq 0} \frac{\|Au\|_2}{\|u\|_1}.$$

The following result is proved as in case of complex Hilbert spaces, and is used in particular in proving the quaternionic version of Bohr's inequality. See Theorem 8.7.4 for the latter.

Proposition 5.3.6. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two right quaternionic Hilbert spaces and let $A \in \mathbf{B}(\mathscr{H}_1, \mathscr{H}_2)$. Then,*

$$\|A\| = \|A^*\| \quad \textit{and} \quad \|AA^*\| = \|A\|^2. \tag{5.12}$$

We now discuss an example. Given a quaternionic Hilbert space $\mathscr{H}$, we associate with $u \in \mathscr{H}$ the operator M_u of multiplication by u on the left defined by

$$M_u q = uq, \quad q \in \mathbb{H}. \tag{5.13}$$

u We have

$$M_u^*(v) = \langle v, u\rangle, \quad v \in \mathscr{H}.$$

Indeed, for $q \in \mathbb{H}$,

$$\langle M_u^*(v), q\rangle = \langle v, uq\rangle = \overline{q}\langle v, u\rangle.$$

In particular we have

$$M_u^* M_v = \langle v\,,\,u\rangle. \tag{5.14}$$

We now introduce the notion of Hilbert basis in the quaternionic setting, see [193, 269, 270]. We first state the following result whose proof follows as in the complex case, see [252]:

Proposition 5.3.7. *Let $\mathscr{H}$ be a quaternionic Hilbert space and let N be a subset of $\mathscr{H}$ such that for all $u, u' \in N$, $\langle u, u'\rangle = 0$ if $u \neq u'$ and $\langle u, u\rangle = 1$. The following conditions are equivalent:*

a) For every $w, v \in \mathscr{H}$, the series $\sum_{u\in N}\langle w,u\rangle\langle u,v\rangle$ converges absolutely and:

$$\langle w,v\rangle = \sum_{u\in N}\langle w,u\rangle\langle u,v\rangle.$$

b) For every $v \in \mathscr{H}$, it holds:

$$\|v\|^2 = \sum_{u\in N}|\langle u,v\rangle|^2.$$

c) $N^\perp = \{0\}$.

d) The linear subspace of $\mathscr{H}$ consisting of all finite right linear combinations of elements of N with quaternionic coefficients is dense in $\mathscr{H}$.

Proposition 5.3.8. *Every quaternionic Hilbert space $\mathscr{H}$ admits a subset N, called Hilbert basis, such that for $u,u' \in N$, $\langle u,u'\rangle = 0$ if $u \neq u'$ and $\langle u,u\rangle = 1$, and N satisfies one of the equivalent conditions in Proposition 5.3.7. Two such sets have the same cardinality.*
If N is a Hilbert basis of $\mathscr{H}$, then every $v \in \mathscr{H}$ can be uniquely decomposed as:

$$v = \sum_{u\in N} u\langle u,v\rangle,$$

where the series $\sum_{u\in N} u\langle u,v\rangle$ converges absolutely in $\mathscr{H}$.

The theory we have developed for quaternionic linear operators works in a two-sided quaternionic linear space, namely, in a linear space which is endowed not only with a notion of multiplication by a scalar on the right but also on the left. If we are assigned a right linear Hilbert space $\mathscr{H}$ it is possible to endow it with a notion of scalar multiplication on the left, though in a non-canonical way which depends on the choice of a Hilbert basis. Let us fix a Hilbert basis N of $\mathscr{H}$. We define the left scalar multiplication of $\mathscr{H}$ induced by N as the map $\mathbb{H}\times\mathscr{H}\to\mathscr{H}$ given by $(q,v)\mapsto qv$ where

$$qv := \sum_{u\in N} uq\langle u,v\rangle \quad q\in\mathbb{H},\ v\in\mathscr{H}. \tag{5.15}$$

The following result, see [193], follows with easy computations.

Proposition 5.3.9. *The left product defined in (5.15) satisfies the following properties.*

(1) $q(w+v) = qw+qv$ and $q(vp) = (qv)p$, for every $v,w\in\mathscr{H}$ and $q,p\in\mathbb{H}$.

(2) $\|qv\| = |q|\,\|v\|$, for every $v\in\mathscr{H}$ and $q\in\mathbb{H}$.

(3) $q(q'v) = (qq')v$, for every $v\in\mathscr{H}$ and $q,q'\in\mathbb{H}$.

(4) $\langle\bar{q}w,v\rangle = \langle w,qv\rangle$, for every $w,v\in\mathscr{H}$ and $q\in\mathbb{H}$.

(5) $rv = vr$, for every $v\in\mathscr{H}$ and $r\in\mathbb{R}$.

(6) $qu = uq$, for every $u\in N$ and $q\in\mathbb{H}$.

As a consequence, for every $q \in \mathbb{H}$, the map $L_q : \mathscr{H} \to \mathbb{H}$, sending v into qv, belongs to $\mathbf{B}(\mathscr{H})$. The map $\mathscr{L}_N : \mathbb{H} \to \mathbf{B}(\mathscr{H})$, defined by setting

$$\mathscr{L}_N(q) := L_q,$$

is a norm–preserving real algebra homomorphism, with the additional properties:

$$L_r v = vr, \quad r \in \mathbb{R},\ v \in \mathscr{H}, \tag{5.16}$$

and

$$(L_q)^* = L_{\overline{q}}, \quad q \in \mathbb{H}. \tag{5.17}$$

Let $\mathscr{H}$ denote a right quaternionic Hilbert space. The bounded operator A from $\mathscr{H}$ into itself is called positive (we will also say positive semi-definite) if

$$\langle Ah, h\rangle_{\mathscr{H}} \geq 0, \quad \forall h \in \mathscr{H}.$$

The following two theorems deal with squareroots of positive operators. The first one can be proved via the quaternionic spectral theorem; see [37], or using power series expansions of the function $\sqrt{1-z}$ for $|z| < 1$; see [65, Lemma 2.2, p. 670]. We give only the proof of the second theorem, and refer to these papers for more information on the first theorem.

Theorem 5.3.10. *A bounded positive operator in a right quaternionic Hilbert space has a positive squareroot.*

Theorem 5.3.11. *Let A be a positive operator on a a right quaternionic Hilbert space. Then, there exists a right quaternionic Hilbert space $\mathscr{H}_A$ and a bounded linear T operator from $\mathscr{H}$ into $\mathscr{H}_A$ such that* $\ker T^* = \{0\}$ *and*

$$A = T^*T. \tag{5.18}$$

Proof. The proof follows the case of complex Hilbert and Banach spaces; see [73, Theorem 2.2 p. 703] for the latter. For $u, v \in \mathscr{H}$ the inner product

$$\langle Au, v\rangle_{\mathscr{H}} = \langle u, Av\rangle_{\mathscr{H}}$$

depends only on Au and Av. We endow the range of A endowed with the inner product $\langle \cdot, \cdot \rangle_A$:

$$\langle Au, Av\rangle_A = \langle Au, v\rangle_{\mathscr{H}}. \tag{5.19}$$

This makes the operator range $\operatorname{ran} A$ a right quaternionic pre-Hilbert space. Let $\mathscr{H}_A$ denote its completion to a right quaternionic Hilbert space. The map T from $\mathscr{H}$ into $\mathscr{H}_A$ defined by $Tu = Au$ is bounded since

$$\|Tu\|_A^2 = \langle Au, u\rangle_{\mathscr{H}} \leq \|A\| \cdot \|u\|_{\mathscr{H}}^2.$$

We compute its adjoint, first on elements of the form Au. Let $h \in \mathscr{H}$. We have

$$\begin{aligned}\langle T^*(Au), h\rangle_{\mathscr{H}} &= \langle Au, Th\rangle_A \\ &= \langle Au, Ah\rangle_A \\ &= \langle Au, h\rangle_{\mathscr{H}}.\end{aligned}$$

Thus $T^*(Au) = Au$, and by continuity we have $T^*g = g$ for all $g \in \mathscr{H}_A$. Finally we note that

$$T^*Th = T^*(Ah) = Ah, \quad h \in \mathscr{H}$$

and so (5.18) is in force. To conclude we note that T^* is injective by construction. □

We conclude with a theorem on tensor products of quaternionic Hilbert spaces. To deepen the topic, see for instance [212, 213, 246]. Our discussion is based on [65, §3]. Let $\mathscr{G}$ and $\mathscr{H}$ be quaternionic Hilbert spaces on the right and on the left, respectively. Their algebraic tensor product has only a group structure in general. However, under suitable assumptions, it is possible to define an inner product in order to obtain a Hilbert space, see [65, Theorem 3.1].

Theorem 5.3.12. *Let $\mathscr{H}$ be a separable two sided quaternionic Hilbert space whose inner product satisfies* (5.4)*, and let $\mathscr{G}$ be a separable right quaternionic Hilbert space. Then, the tensor product $\mathscr{G} \otimes_{\mathbb{H}} \mathscr{H}$ endowed with the inner product*

$$\langle g_1 \otimes h_1, g_2 \otimes h_2\rangle_{\mathscr{G}\otimes_{\mathbb{H}}\mathscr{H}} = \langle \langle g_1, g_2\rangle_{\mathscr{G}} h_1, h_2\rangle_{\mathscr{H}}$$

is a right quaternionic Hilbert space.

5.4 Partial majorants

A standard reference for special topologies, called partial majorants, studied in this section is [98, Chapter III] in the complex case. For the quaternionic case we refer the reader to [37] which is the main source for this section. We begin by proving a simple fact (which, in general, is not guaranteed in a vector space over any field):

Lemma 5.4.1. *Let $\mathscr{V}$ be a quaternionic inner product space and let $w \in \mathscr{V}$. The maps*

$$v \mapsto \ p_w(v) = |[v, w]|, \quad v \in \mathscr{V} \tag{5.20}$$

are semi-norms.

Proof. Property (5.1) is evident. Property (5.2) comes from the fact that the absolute value is multiplicative in $\mathbb{H}$:

$$p_w(vc) = |[vc, w]| = |[v, w]c| = |[v, w]| \cdot |c| = |c| p_w(v).$$

□

Next definition is classical:

Definition 5.4.2. The weak topology on $\mathscr{V}$ is the smallest topology such that all the semi-norms (5.20) are continuous.

Definition 5.4.3. (1) A topology on the quaternionic indefinite inner product space $\mathscr{V}$ is called a partial majorant if it is locally convex and if all the maps

$$v \quad \mapsto \quad [v,w], \quad w \in \mathscr{V}, \tag{5.21}$$

are continuous.
(2) A partial majorant is called admissible if every continuous linear functional from $\mathscr{V}$ to $\mathbb{H}$ is of the form $v \mapsto [v,w_0]$ for some $w_0 \in \mathscr{V}$.

Next result relates the weak topology with partial majorants.

Theorem 5.4.4. *The weak topology of an inner product space is a partial majorant. A locally convex topology is a partial majorant if and only if it is stronger than the weak topology.*

Proof. To prove that the weak topology of an inner product space is a partial majorant, we have to show that in the weak topology the maps (5.21) are continuous. For any $\varepsilon > 0$, and for any $v_0, w \in \mathscr{V}$ the inequality $|[v,w]-[v_0,w]| < \varepsilon$ is equivalent to $p_w(v-v_0) < \varepsilon$ and the set $\{v \in \mathscr{V} \,:\, p_w(v-v_0) < \varepsilon\}$ is a neighborhood $U_{v_0}(p_w,\varepsilon)$ of v_0. Thus the weak topology is a partial majorant.
Let us now consider another locally convex topology stronger than the weak topology. As we already know, the inequality $|[v,w]-[v_0,w]| < \varepsilon$ holds for $v \in U_{v_0}(p_w,\varepsilon)$ which is also an open set in the stronger topology. So any locally convex topology stronger than the weak topology is a partial majorant. Finally, we consider a partial majorant. Let $v_0, w_1, \ldots, w_n \in \mathscr{V}$, let $\varepsilon > 0$. Then, by definition, there are neighborhoods U_ℓ of w_ℓ, $\ell = 1,\ldots,n$ such that for any $v \in U_\ell$ the inequality $|[v,w_\ell]-[v_0,w_\ell]| < \varepsilon$, i.e. $p_{w_\ell}(v-v_0) < \varepsilon$ holds. Thus any w which belongs to the neighborhood of v_0 defined by $\cap_{\ell=1}^n U_\ell$ belongs to $U_{v_0}(p_{w_\ell},\varepsilon)$ and the statement follows. □

From this result we obtain:

Corollary 5.4.5. *Every partial majorant of a nondegenerate inner product space $\mathscr{V}$ is Hausdorff.*

Proof. Any open set in the weak topology is also open in the partial majorant topology. The weak topology is Hausdorff if it separates points, i.e. if and only if for every $w \in \mathscr{V}$ the condition $p_w(v) = |[v,w]| = 0$ implies $v = 0$. But this is the case since $\mathscr{V}$ is nondegenerate. □

Proposition 5.4.6. *If a topology is a partial majorant of the quaternionic inner product space $\mathscr{V}$ then the orthogonal companion of every subspace is closed.*

Proof. Let $\mathscr{L}$ be a subspace of $\mathscr{V}$ and let $\mathscr{L}^{[\perp]}$ be its orthogonal companion. We show that the complement $(\mathscr{L}^{[\perp]})^c$ of $\mathscr{L}^{[\perp]}$ is an open set. Let v_0 be in $(\mathscr{L}^{[\perp]})^c$; then there is $w \in \mathscr{L}$ such that $[v_0,w] \neq 0$. By continuity, there exists a neighborhood U of v_0 such that $[v,v_0] \neq 0$ for all $v \in U$, thus $(\mathscr{L}^{[\perp]})^c$ is open. □

Corollary 5.4.7. *If a topology is a partial majorant of a nondegenerate inner product space $\mathscr{V}$ then every ortho-complemented subspace of $\mathscr{V}$ is closed.*

Proof. Let $\mathscr{L}$ be a subspace of $\mathscr{V}$ and let $\mathscr{L}^{[\perp]}$ be its orthogonal companion. Then $\mathscr{L}^{[\perp\perp]}$ is closed by Proposition 5.4.6 and since $\mathscr{L}^{[\perp\perp]} = \mathscr{L}$ by Proposition 5.2.9, the statement follows. □

Corollary 5.4.8. *Let τ be a partial majorant of the quaternionic inner product $\mathscr{V}$ and assume that $\mathscr{V}$ is nondegenerate. Then the components of any fundamental decomposition are closed with respect to τ.*

Proof. This is a consequence of the previous corollary, since the two components are ortho-complemented. □

Theorem 5.4.9. *Let $\mathscr{V}$ be a nondegenerate quaternionic inner product space and let τ_1 and τ_2 be two Fréchet partial majorants of $\mathscr{V}$. Then, $\tau_1 = \tau_2$.*

Proof. Let τ be the topology $\tau_1 \cup \tau_2$. Then, following the proof of Theorem 3.3. p. 63 in [98], we show that τ is a Fréchet topology stronger than τ_1 and τ_2. Let us consider the two topological vector spaces $\mathscr{V}$ endowed with τ and $\mathscr{V}$ endowed with τ_1 and the identity map acting between them. By the closed graph theorem, see Theorem 5.1.16, we have that the identity map takes closed sets to closed sets and so τ_1 is stronger than τ. A similar argument holds by considering τ_2 and thus $\tau = \tau_1 = \tau_2$. □

We now consider the case in which a partial majorant τ is defined by a norm $\|\cdot\|$ on a nondegenerate inner product space $\mathscr{V}$, and we define

$$\|v\|' \overset{\text{def.}}{=} \sup_{\|w\|\le 1} |[v,w]|, \qquad v \in \mathscr{V}. \tag{5.22}$$

Then it can be verified that $\|\cdot\|'$ is a norm called *polar* of the norm $\|\cdot\|$. As in the proof of Lemma 5.4.1 the crucial fact is that the modulus is multiplicative in $\mathbb{H}$. The topology τ' induced by $\|\cdot\|'$ is called the polar of the topology τ.
The definition (5.22) implies

$$|[v, \frac{w}{\|w\|}]| \le \sup_{w\in\mathscr{V}} |[v, \frac{w}{\|w\|}]| \le \sup_{\|w\|\le 1} |[v,w]| = \|v\|', \tag{5.23}$$

from which we deduce the inequality $|[v,w]| \le \|v\|'\|w\|$. Thus the polar of a partial majorant is a partial majorant since (5.21) holds and thus one can define $\tau'' \overset{\text{def.}}{=} (\tau')'$ and so on, iteratively.

Proposition 5.4.10. *Let $\mathscr{V}$ be a nondegenerate inner product space.*

(1) If τ_1 and τ_2 are normed partial majorants of $\mathscr{V}$ and τ_1 is weaker than τ_2, then τ_2' is weaker than τ_1'.

(2) If τ is a normed partial majorant of $\mathscr{V}$, then its polar τ' is a normed partial majorant on $\mathscr{V}$. Furthermore, $\tau'' \le \tau$, and $\tau''' = \tau'$.

Proof. Let τ_1, τ_2 be induced by the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively and let us assume that $\tau_1 \leq \tau_2$. Then there exists $\lambda > 0$ such that $\lambda \|w\|_2 \leq \|w\|_1$ for all $w \in \mathscr{V}$ and so, if we take $\|w\|_1 \leq 1$ we have

$$\sup_{\|w\|_1 \leq 1} |[v,w]| \leq \sup_{\|w\|_2 \leq 1} |[v, \lambda w]| = \lambda \sup_{\|w\|_2 \leq 1} |[v,w]|,$$

so that $\tau_2' \leq \tau_1'$.
Moreover we have $\sup_{\|y\|' \leq 1} |[x,y]| \leq \|x\|$ and so $\tau'' \leq \tau$. Let us now use this inequality by replacing τ by τ' and we get $\tau''' \leq \tau'$. By using point (1) applied to $\tau_1 = \tau''$ and $\tau_2 = \tau$ we obtain the reverse inequality and so $\tau''' = \tau'$. □

Among the partial majorants there are the admissible topologies (see Definition 5.4.3). The next result shows that an admissible topology which is also metrizable is uniquely defined. In order to prove this fact, we recall that given a quaternionic vector space $\mathscr{V}$, its so-called conjugate $\mathscr{V}^*$ is defined to be the quaternionic vector space in which the additive group coincides with $\mathscr{V}$ and whose multiplication by a scalar is given by $(c,v) \mapsto v\bar{c}$. An inner product $(\cdot,\cdot)$ in $\mathscr{V}^*$ can be assigned by $(v,w) \stackrel{\text{def.}}{=} [w,v] = \overline{[v,w]}$.

Theorem 5.4.11. *Let τ_1, τ be admissible topologies on a quaternionic inner product space $\mathscr{V}$. If τ_1 is given by a countable family of semi-norms, then τ_1 is stronger than τ. Moreover, no more than one admissible topology of $\mathscr{V}$ is metrizable.*

Proof. Assume that τ_1 and τ are given by the families of semi-norms $\{p_i\}$, $i \in \mathbb{N}$, and $\{q_\gamma\}$, $\gamma \in \Gamma$, respectively. Suppose that τ_1 is not stronger than τ. Then there exists an open set in τ that does not contain any open set in τ_1. In particular, it does not contain

$$\{v \in \mathscr{V} \mid p_i(v) < \frac{1}{n},\ i = 1,\dots,n\} \quad \text{for some } n \in \mathbb{N}.$$

Thus, there exists a sequence $\{v_n\} \subset \mathscr{V}$ such that $p_i(v_n) < \dfrac{1}{n}$ but $\max_{k=1,\dots,m} q_{\gamma_k}(v_n) = q_{\gamma_j}(v_n) \geq \varepsilon$ for some $\varepsilon > 0$. By choosing $w_n = nv_n$ we have

$$\max_{i=1,\dots,n} p_i(w_n) < 1, \qquad q_{\gamma_j}(w_n) \geq n\varepsilon, \quad n \in \mathbb{N}. \tag{5.24}$$

Let us consider the subspace of $\mathscr{V}$ given by $\mathscr{L} = \{v \in \mathscr{V} \mid q_{\gamma_j}(v) = 0\}$ and the quotient $\hat{\mathscr{L}} \stackrel{\text{def.}}{=} \mathscr{V}/\mathscr{L}$. We can endow $\hat{\mathscr{L}}$ with the norm $\|\hat{v}\| \stackrel{\text{def.}}{=} q_{\gamma_j}(v)$, for $\hat{v} = v + \mathscr{L} \in \hat{\mathscr{L}}$. Let $\hat{\varphi}:\ \hat{\mathscr{L}} \to \mathbb{H}$ be a linear function which is also continuous:

$$|\hat{\varphi}(\hat{v})| \leq \|\hat{\varphi}\|\,\|\hat{v}\|, \quad \hat{v} \in \hat{\mathscr{L}}.$$

Then the formula $\varphi(v) \stackrel{\text{def.}}{=} \hat{\varphi}(\hat{v})$, $v \in \mathscr{V}$, $v \in \hat{v}$, defines a linear and continuous function on $\mathscr{V}$ since

$$|\varphi(v)| \leq \|\hat{\varphi}\|\,\|\hat{v}\| = \|\hat{\varphi}\| q_{\gamma_j}(v).$$

Thus φ is continuous in the topology τ and since τ is admissible, $\varphi(v) = [v, w_0]$ for some suitable $w_0 \in \mathscr{V}$. We conclude that φ is also continuous in the topology τ_1. So for some $r \in \mathbb{N}$ and $\delta > 0$ we have

$$|\varphi(v)| \leq \frac{1}{\delta} \max_{i=1,\ldots,r} p_i(v), \quad v \in \mathscr{V}.$$

This last inequality together with (5.24) give $|\varphi(w_n)| < 1/\delta$ for $n > r$. So the sequence $\{\hat{\varphi}(\hat{w}_n)\}$ is bounded for any $\hat{\varphi}$ fixed in the conjugate space $\hat{\mathscr{L}}^*$ of the normed space $\hat{\mathscr{L}}$. However, we can look at $\hat{\varphi}(\hat{w}_n)$ as the value of the functional $\hat{w}_n$ acting on the elements of the Banach space $\hat{\mathscr{L}}^*$. Since we required that $|\hat{\varphi}(\hat{v})| \leq \|\hat{\varphi}\| \, \|\hat{v}\|$, for $\hat{v} \in \hat{\mathscr{L}}$ the functional $\hat{w}_n$ is continuous. By the quaternionic version of the Hahn-Banach theorem, we deduce that $\|\hat{w}_n\| = q_{\gamma_j}(\hat{w}_n)$. From (5.24), more precisely from $q_{\gamma_j}(w_n) \geq n\varepsilon$, we obtain a contradiction with the principle of uniform boundedness, see Theorem 5.1.10. □

5.5 Majorant topologies and inner product spaces

The material in this section can be found, in the complex case, in [98, Chapter IV]. The source for this section is our paper [37].

Definition 5.5.1. A locally convex topology on $(\mathscr{V}, [\cdot,\cdot])$ is called a majorant if the inner product is jointly continuous in this topology. It is called a complete majorant if it is metrizable and complete. It is called a normed majorant if it is defined by a single semi-norm or norm, and a Banach majorant if it is moreover complete with respect to this norm. It is called a Hilbert majorant if it is a complete normed majorant, and the underlying norm is defined by an inner product.

Remark 5.5.2. The norm defining a Banach majorant (and hence the inner product defining a Hilbert majorant) is not unique. But Theorem 5.1.12 implies that any two such norms are equivalent.

Proposition 5.5.3.
(1) *Given a majorant, there exists a weaker majorant defined by a single semi-norm.*
(2) *A normed partial majorant τ on the nondegenerate inner product space $\mathscr{V}$ is a majorant if and only if it is stronger than its polar: $\tau' \leq \tau$.*

Proof. (1) From the definition of a majorant, there exist semi-norms $p_1, \ldots, p_N$ and $\varepsilon > 0$ such that

$$|[u,v]| \leq 1, \quad \forall u, v \in U,$$

where

$$U = \left\{ v \in \mathscr{V} \,;\, p_j(v) \leq \varepsilon, \ j = 1, \ldots N \right\}.$$

It follows that the inner product is jointly continuous with respect to the semi-norm $\max_{j=1,\ldots N} p_j$.

(2) Bearing in mind thr definition of polar τ', see (5.22), we have that $\tau' \leq \tau$ if and only if the identity map from $(\mathscr{V}, \tau)$ into $(\mathscr{V}, \tau')$ is continuous. This happens if and only if there exists $k > 0$ such that

$$\|v\|' \leq k\|v\|, \quad \forall v \in \mathscr{V}. \tag{5.25}$$

This in turn holds if and only if

$$|[v,u]| \leq k\|v\|, \quad \forall v,u \in \mathscr{V} \quad \text{with} \quad \|u\| \leq 1. \tag{5.26}$$

The result follows since any such $u \neq 0$ is such that $\|u\| \leq 1$ if and only if it can be written as $\frac{w}{\|w\|}$, for some $w \neq 0 \in \mathscr{V}$. □

Proposition 5.5.4. *Let $\mathscr{V}$ be a nondegenerate inner product space, admitting a normed majorant. Then there exists a weaker normed majorant which is self-polar.*

Proof. We closely follow and sketch the proof of [98, p. 85]. The key is that the polar norm (defined in (5.22)) is still a norm in the quaternionic case. By possibly renormalizing, we can assume that

$$|[u,v]| \leq \|u\|\|v\|, \quad u,v \in \mathscr{V}, \tag{5.27}$$

where $\|\cdot\|$ denotes a norm defining the majorant. Define a sequence of norms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ by $\|\cdot\|_1 = \|\cdot\|$ and

$$\|u\|_{n+1} = \left(\frac{1}{2}(\|u\|_n^2 + (\|u\|_n')^2)\right)^{\frac{1}{2}}, \quad n = 1,2,\ldots, \tag{5.28}$$

where we recall that $\|\cdot\|'$ denotes the polar norm of $\|\cdot\|$; see (5.22). By induction, one shows that each $\|\cdot\|_n$ satisfies (5.27) and that the sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is decreasing, and thus defining a semi-norm $\|\cdot\|_\infty = \lim_{n\to\infty}\|\cdot\|_n$. One readily shows that $\|\cdot\|_\infty \geq \frac{1}{\sqrt{2}}\|\cdot\|_1'$, and hence $\|\cdot\|_\infty$ is a norm, and a majorant since it also satisfies (5.27) by passing to the limit the corresponding inequality for $\|\cdot\|_n$.

We now show that the topology defined by $\|\cdot\|_\infty$ is self-polar. We first note that the sequence of polars $(\|\cdot\|_n')_{n\in\mathbb{N}}$ is increasing, and bounded by the polar $\|\cdot\|_\infty'$. Set $\|\cdot\|_e = \lim_{n\to\infty}\|\cdot\|_n'$. Applying inequality (5.23) to $\|\cdot\|_n$ and taking limits leads to

$$|[u,v]| \leq \|u\|_e\|v\|_\infty, \quad u,v \in \mathscr{V}.$$

Thus $\|\cdot\|_\infty' \leq \|\cdot\|_e$, and we get that $\|\cdot\|_\infty' = \|\cdot\|_e$. Letting $n \to \infty$ in (5.28), we get $\|\cdot\|_\infty = \|\cdot\|_\infty'$. □

Proposition 5.5.5. *Let $(\mathscr{V}, [\cdot,\cdot])$ be a quaternionic nondegenerate inner product space. Then a partial majorant is a minimal majorant if and only if it is normed and self-polar.*

Proof. Assume first that the given partial majorant τ is a minimal majorant. By item (1) of Proposition 5.5.3 there is a weaker majorant τ_a defined by a single semi-norm. Moreover by Corollary 5.4.5 any partial majorant (and in particular any majorant) is Hausdorff, and so the τ_a is Hausdorff and the above semi-norm is in fact a norm. By Proposition 5.5.4

there exists a self-polar majorant τ_∞ which is weaker that τ_1. The minimality of τ implies that $\tau_\infty = \tau$.

Conversely, assume that the given partial majorant τ is normed and self-polar. Then τ is a majorant in view of item (2) of Proposition 5.5.3. Assume that $\tau_a \leq \tau$ is another majorant. Then, by part (2) in Lemma 5.5.3, $\tau_a \geq \tau'_a$, and by item (1) of Proposition 5.4.10 we have $\tau'_a \geq \tau'$. This ends the proof since τ is self-polar. □

Theorem 5.5.6. *Let $\mathscr{V}$ be a quaternionic nondegenerate inner product space, and let τ be an admissible topology which is moreover a majorant. Then τ is minimal, it defines a Banach topology and is the unique admissible majorant on $\mathscr{V}$. Finally, τ is stronger than any other admissible topology on $\mathscr{V}$.*

In next proposition we introduce an operator, called Gram operator, which will play an important role in the sequel. Recall that Hilbert majorants have been defined in Definition 5.5.1.

Proposition 5.5.7. *Let $(\mathscr{V}, [\cdot,\cdot])$ be a quaternionic inner product space, admitting a Hilbert majorant, with associated inner product $\langle\cdot,\cdot\rangle$, and corresponding norm $\|\cdot\|$. There exists a linear continuous operator G, self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle$, and such that*

$$[v,w] = \langle v, Gw\rangle, \quad v,w \in \mathscr{V}.$$

Proof. The existence of G follows from Riesz' representation theorem for continuous functionals, which still holds in quaternionic Hilbert spaces (see [102, p. 36], [212, Theorem II.1, p. 440]); the fact that G is Hermitian follows from the fact that the form $[\cdot,\cdot]$ is Hermitian. In the complex case, an everywhere defined Hermitian operator in a Hilbert space is automatically bounded; rather than proving the counterpart of this fact in the quaternionic setting we note, as in [98, p. 88], that there exists a constant k such that

$$|[u,w]| \leq k\|u\|\cdot\|v\|, \quad \forall u,v \in \mathscr{V}. \tag{5.29}$$

The boundedness of G follows from (5.29) and $[v,Gv] = \|Gv\|^2$. □

The operator G in the preceding result is called *Gram operator*. The semi-norm

$$v \mapsto \|Gv\| \tag{5.30}$$

defines a topology called the Mackey topology. As we remarked after Definition 5.5.1 the inner product defining a given Hilbert majorant is not unique, and so to every inner product will correspond a different Gram operator.

Proposition 5.5.8. *The Mackey topology is admissible and is independent of the choice of the inner product defining the Hilbert majorant.*

Proof. The uniqueness will follow from Theorem 5.4.11 once we know that the topology, say τ_G, associated to the semi-norm (5.30) is admissible. From the inequality

$$|[u,v]| = \langle Gu, v\rangle \leq \|Gu\|\cdot\|v\|$$

we see that τ_G is a partial majorant. To show that it is admissible, consider a linear functional f continuous with respect to τ_G. There exists $k > 0$ such that

$$|f(u)| \le k\|Gu\|, \quad \forall u \in \mathscr{V}.$$

Thus the linear relation

$$(kGu, f(u)), \quad u \in \mathscr{V}$$

is the graph of a contraction, say T,

$$T(Gu) = \frac{1}{k} f(u), \quad \forall u \in \mathscr{V},$$

in the pre-Hilbert space (ran G) $\times \mathbb{H}$, the latter being endowed with the inner product

$$\langle (Gu, p), (Gv, q) \rangle_{\mathscr{V} \times \mathbb{H}} = \langle Gu, Gv \rangle + \overline{q}p = [Gu, v] + \overline{q}p.$$

The operator T admits a contractive extension to all of $\mathscr{V} \times \mathbb{H}$, and by Riesz representation theorem, there exists $f_0 \in \mathscr{V}$ such that

$$T(u) = \langle u, f_0 \rangle, \quad \forall u \in \mathscr{V}.$$

Thus

$$f(u) = kT(Gu) = k\langle Gu, f_0 \rangle = [u, kf_0],$$

which ends the proof. □

Let $\mathscr{L}$ be a subspace of a quaternionic inner product space $(\mathscr{V}, [\cdot,\cdot])$. Assume that $\mathscr{V}$ admits an Hilbert majorant with associated inner product $\langle \cdot,\cdot \rangle$ and associated norm $\|\cdot\|$. We denote by $P_{\mathscr{L}}$ the orthogonal projection onto $\mathscr{L}$ in the Hilbert space $(\mathscr{V}, \langle \cdot,\cdot \rangle)$, and we set

$$G_{\mathscr{L}} = P_{\mathscr{L}} G\big|_{\mathscr{L}}. \tag{5.31}$$

Proposition 5.5.9. *Let $\mathscr{V}$ be a quaternionic inner product space, admitting an Hilbert majorant, let $\mathscr{L}$ be a closed subspace of $\mathscr{V}$ and let $G_{\mathscr{L}}$ be defined by* (5.31). *Then:*
(1) *An element $v \in \mathscr{V}$ admits a projection onto $\mathscr{L}$ if and only if*

$$P_{\mathscr{L}} v \in \text{ran } G_{\mathscr{L}}. \tag{5.32}$$

(2) *$\mathscr{L}$ is ortho-complemented in $(\mathscr{V}, [\cdot,\cdot])$ if and only if*

$$\text{ran } P_{\mathscr{L}} G = \text{ran } G_{\mathscr{L}}.$$

Proof. (1) The vector $v \in V$ has a (not necessarily unique) projection, say w on $\mathscr{L}$ if and only if

$$[v - w, u] = 0, \quad \forall u \in \mathscr{L},$$

that is, if and only if

$$\langle G(v - w), u \rangle = 0, \quad \forall u \in \mathscr{L}.$$

This last condition is equivalent to $P_{\mathscr{L}} Gv = G_{\mathscr{L}} w$, which is equivalent to (5.32).

(2) The second claim is equivalent to the fact that every element admits a projection on $\mathscr{L}$, and therefore follows from (1). □

The following result on the decomposability of a inner product space is based on the spectral theorem for Hermitian operators.

Theorem 5.5.10. *Let* $(\mathscr{V},[\cdot,\cdot])$ *be a quaternionic inner product space, admitting a Hilbert majorant. Then* $\mathscr{V}$ *is decomposable, and there exists a fundamental decomposition such that all three components and any sum of two of them are complete with respect to the Hilbert majorant.*

Proof. As in the proof of the corresponding result in the complex case (see [98, p. 89] we apply the spectral theorem (see [37, Theorem 8.1]) to the Gram operator G associated to the form $[\cdot,\cdot]$, and write G as:

$$G=\int_{-\infty}^{+\infty}\lambda dE(\lambda),$$

where the spectral measure is continuous and its support is finite since G is bounded. We then set

$$\mathscr{V}_-=E(0^-)\mathscr{V},\quad \mathscr{V}_0=(E(0)-E(0^-))\mathscr{V},\quad \text{and}\quad \mathscr{V}_+=(I-E(0))\mathscr{V}.$$

We have

$$\mathscr{V}=\mathscr{V}_-[\oplus]\mathscr{V}_0[\oplus]\mathscr{V}_+.$$

Each of the components and each sum of pairs of components of this decomposition is an orthogonal companion, and therefore closed for the Hilbert majorant in view of Proposition 5.4.6. □

Let $\mathscr{V}$ be a quaternionic inner product space which is decomposable and nondegenerate, and let

$$\mathscr{V}=\mathscr{V}_+[\oplus]\mathscr{V}_-, \tag{5.33}$$

where $\mathscr{V}_+$ is a strictly positive subspace and $\mathscr{V}_-$ is a strictly negative subspace. The map

$$J(v)=v_+-v_-$$

is called the associated fundamental symmetry. Note that $J(Jv)=v$, thus J is invertible and $J=J^{-1}$. It is readily seen that

$$[v,w]=[Jv,Jw],\quad v,w\in\mathscr{V}. \tag{5.34}$$

If, as in the proof of the previous result, we set

$$P_\pm(v)=v_\pm, \tag{5.35}$$

we have $J=P_+-P_-$.

Theorem 5.5.11. *Let* $\mathscr{V}$ *be a decomposable and nondegenerate quaternionic inner product space, and let* (5.6) *be a fundamental decomposition of* $\mathscr{V}$*, and let*

$$\langle v,w\rangle_J\overset{\text{def.}}{=}[Jv,w],\quad v,w\in\mathscr{V}.$$

Then,

$$\langle v,w\rangle_J = [v,Jw] = [v_+,w_+] - [v_-,w_-], \tag{5.36}$$

$$[v,w] = \langle v,Jw\rangle_J = \langle Jv,w\rangle_J, \tag{5.37}$$

and $(\mathscr{V},\langle\cdot,\cdot\rangle_J)$ *is a pre-Hilbert space. Furthermore, with* $\|v\|_J^2 = [v,Jv]$*, it holds*

$$|[v,w]|^2 \le \|v\|_J^2\|w\|_J^2, \quad v,w\in\mathscr{V}. \tag{5.38}$$

Proof. The first claim follows from the fact that both $\mathscr{V}_+$ and $\mathscr{V}_-$ are positive definite. In a quaternionic pre-Hilbert space, the Cauchy-Schwarz inequality holds and this implies (5.38) since

$$|[v,w]|^2 = |\langle v,Jw\rangle_J|^2 \le \|v\|_J^2\|Jw\|_J^2.$$

Equations (5.36) and (5.34) imply that $\|w\|_J = \|Jw\|_J$, and this ends the proof. □

Remark 5.5.12. Let $\mathscr{V}$ be a quaternionic, nondegenerate, inner product vector space admitting a fundamental decomposition of the form $\mathscr{V} = \mathscr{V}_+[\oplus]\mathscr{V}_-$ and let J be the associated fundamental symmetry. Then $\mathscr{V}_+$ is J-orthogonal to $\mathscr{V}_-$, i.e. $\langle v_+,w_-\rangle_J = 0$ for every $v_+\in\mathscr{V}_+$ and $w_-\in\mathscr{V}_-$, as one can see from formula (5.36).

The topology defined by the norm $\|\cdot\|_J$ is called the decomposition majorant belonging to the given fundamental decompostion.

In the next result the majorant is a Banach majorant rather than a Hilbert majorant and the space in nondegenerate.

Proposition 5.5.13. *Let* $(\mathscr{V},[\cdot,\cdot])$ *be a quaternionic nondegenerate inner product space, admitting a Banach majorant* τ *and a decomposition majorant* τ_1*. Then,* $\tau_1\le\tau$*.*

Proof. Let $\mathscr{V} = \mathscr{V}_+[\oplus]\mathscr{V}_-$ be a fundamental decomposition of $\mathscr{V}$. By Corollary 5.4.8 the space $\mathscr{V}_+$ is closed in the topology τ. Let P_+ denote the map

$$P_+v = v_+ \tag{5.39}$$

where $v = v_+ + v_-$ is the decomposition of $v\in\mathscr{V}$ along the given fundamental decomposition of $\mathscr{V}$. We claim that the graph of P_+ is closed, when $\mathscr{V}$ is endowed with the topology τ. Indeed, if $(v_n)_{n\in\mathbb{N}}$ is a sequence converging (in the topology τ) to $v\in\mathscr{V}$ and such that the sequence $((v_n)_+)_{n\in\mathbb{N}}$ converges to $z\in\mathscr{V}_+$ also in the topology τ. Since the inner product is continuous with respect to τ we have for $w\in\mathscr{V}_+$

$$\begin{aligned}[z-v_+,w] &= \lim_{n\to\infty}[(v_n)_+,w] - [v_+,w]\\ &= \lim_{n\to\infty}[v_n,w] - [v_+,w] = [v,w]-[v_+,w] = [v-v_+,w] = 0\end{aligned}$$

and so $z = v_+$. By the closed graph theorem (see Theorem 5.1.16) P_+ is continuous. The same holds for the operator $P_-v = v_-$ and so the operator

$$Jv = v_+ - v_- \tag{5.40}$$

is continuous from $(\mathscr{V},\tau)$ onto $(\mathscr{V},\tau)$. Recall now that $[Jv,v]$ is the square of the J-norm defining τ_1. We have

$$[Jv,v] \le k\|Jv\|\cdot\|v\|,$$

where $\|\cdot\|$ denotes a norm defining τ. The consinutity of J implies

$$[Jv,v] \le k\|Jv\|\cdot\|v\| \le k_1\|v\|^2.$$

It follows that the inclusion map is continuous from $(\mathscr{V},\tau)$ into $(\mathscr{V},\tau_1)$, and so $\tau_1 \le \tau$. □

Proposition 5.5.14. *Every decomposition majorant is a minimal majorant.*

Proof. A decomposition majorant is also a partial majorant and is normed, with associated J-norm $\|u\|_J = [Ju,u]$, where J is associated to the decomposition $J(v) = v_+ - v_-$. Thus, using Proposition 5.5.5, to prove the minimality it is enough to show that $\|u\|_J$ is self-polar. This fact follows from

$$\|u\|_J' = \sup_{\|v\|_J \le 1} |[Ju,v]| = \sup_{\|v\|_J \le 1} |[u,Jv]| = \|u\|_J.$$

□

We can now address the problem of the uniqueness of a minimal majorant.

Proposition 5.5.15. *Let $(\mathscr{V},[\cdot,\cdot])$ be a quaternionic inner product space, admitting a decomposition*

$$\mathscr{V} = \mathscr{V}_+[\oplus]\mathscr{V}_-, \tag{5.41}$$

where $\mathscr{V}_+$ is positive definite and $\mathscr{V}_-$ is negative definite. Assume that $(\mathscr{V}_+,[\cdot,\cdot])$ (resp. $(\mathscr{V}_-,-[\cdot,\cdot])$) is complete. Then, so is $(\mathscr{V}_-,-[\cdot,\cdot])$ (resp. $(\mathscr{V}_+,[\cdot,\cdot])$), and $(\mathscr{V},[\cdot,\cdot])$ has a unique minimal majorant.

Proof. The topology τ defines a fundamental decomposition, and an associated minimal majorant $\|\cdot\|_J$. See Proposition 5.5.14. Let τ be another minimal majorant. By Proposition 5.5.5 it is normed and self-polar and so there is a norm $\|\cdot\|$ and $k_1 > 0$ such that

$$\|v_+\| \le k_1 \sup_{\substack{y\in\mathscr{V}_+ \\ \|y\|\le 1}} |[v_+,y]|.$$

Using the uniform boundedness we find $k_2 > 0$ such that

$$|[v_+,y]| \le k_2[v_+,v_+], \quad \forall y \text{ such that } \|y\| \le 1.$$

Hence, with $C = k_1k_2$,

$$\|v_+\| \le C[v_+,v_+], \quad \forall v_+ \in \mathscr{V}_+. \tag{5.42}$$

Let now $v \in \mathscr{V}$ with decomposition $v = v_+ + v_-$, where $v_\pm \in \mathscr{V}_\pm$. Since τ is a normed majorant, there exists C_1 such that

$$\|v_+\|^2 \le C[v_+,v_+] = C[v_+,v] \le CC_1\|v_+\|\cdot\|v\|.$$

Hence

$$\|v\|_J^2 = [Jv, v] \leq C_1\|Jv\| \cdot \|v\| = C_1\|2v_+ - v\| \cdot K\|v\|^2$$

for an appropriate $K > 0$. The identity map is therefore continuous from $(\mathscr{V}, \tau)$ onto $(\mathscr{V}, \|\cdot\|_J)$. Since τ is defined by a single norm, it follows that the identity map is also continuous from $(\mathscr{V}, \|\cdot\|_J)$ onto $(\mathscr{V}, \tau)$ and this ends the proof. □

Definition 5.5.16. The space $\mathscr{V}_+$ (resp. $\mathscr{V}_-$) is called intrinsically complete when $(\mathscr{V}_+, [\cdot,\cdot])$ (resp. $(\mathscr{V}_-, -[\cdot,\cdot])$) is complete.

Proposition 5.5.17. *Let* $(\mathscr{V}, [\cdot,\cdot])$ *be a quaternionic inner product space, admitting a decomposition of the form* (5.33)*, and with associated fundamental symmetry* J*. Then:*

(1) Let $\mathscr{L}$ *denote a positive subspace of* $\mathscr{V}$*. Then, the operator* $P_+\big|_{\mathscr{L}}$ *and its inverse are* τ_J *continuous.*

(2) Given another decomposition of the form (5.33)*, the positive (resp. negative) components are simultaneously intrinsically complete.*

Proof. To prove the result we follow [98, pp. 93-94]. Let $\mathscr{L}$ be a positive subspace of $\mathscr{V}$ and let $v \in \mathscr{L}$. By recalling (5.36), (5.39), where $v = v_+ + v_-$ is the decomposition of v with respect to the fundamental decomposition $\mathscr{V} = \mathscr{V}_+[\oplus]\mathscr{V}_-$, we have:

$$\|v\|_J^2 = \|P_+v\|_J^2 + \|P_-v\|_J^2.$$

Since $\mathscr{V}_+$ and $\mathscr{V}_-$ are J-orthogonal, see Remark 5.5.12, we then have

$$[v, v] = \|P_+v\|_J^2 - \|P_-v\|_J^2$$

and so, since $\mathscr{L}$ is positive,

$$\|v\|_J^2 = 2\|P_+v\|_J^2 - [v, v] \leq 2\|P_+v\|_J^2.$$

It is immediate that $\|P_+v\|_J^2 \leq \|v\|_J^2$ and so we conclude that both P_+ and its inverse are τ_J continuous as stated in point (1).

To show point (2), we assume that there is another fundamental decomposition $\mathscr{V} = \mathscr{V}'_+[\oplus]\mathscr{V}'_-$. If we suppose that $\mathscr{V}'_+$ is intrinsically complete, then Proposition 5.5.15 implies that $\mathscr{V}'_+$ is complete with respect to the decomposition majorant corresponding to the decomposition $\mathscr{V} = \mathscr{V}_+[\oplus]\mathscr{V}_-$. Part (1) of the statement implies that also $P^+\mathscr{V}'_+$ is complete in this topology and so it is intrinsically complete. If $P^+\mathscr{V}'_+ = \mathscr{V}_+$ there is nothing to prove. Otherwise there exists a non-zero $\tilde{v} \in \mathscr{V}_+$ orthogonal to $P^+\mathscr{V}'_+$ so $\tilde{v}$ is orthogonal to $\mathscr{V}'_+$. Then the subspace $\mathscr{U}$ spanned by $\tilde{v}$ and $\mathscr{V}'_+$ is positive. Indeed, for a generic nonzero element $u = \tilde{v} + \tilde{v}'$ ($\tilde{v}' \in \mathscr{V}'_+$) we have

$$[u, u] = [\tilde{v} + \tilde{v}', \tilde{v} + \tilde{v}'] = [\tilde{v}, \tilde{v}] + [\tilde{v}', \tilde{v}'] > 0.$$

This implies that $\mathscr{U}$ is a proper extension of $\mathscr{V}'_+$ which is absurd by Proposition 5.2.10. This completes the proof. □

5.6 Quaternionic Hilbert spaces. Weak topology

As it is well known, the unit ball in a normed space which is infinite dimensional cannot be compact in the norm topology. However, in a Hilbert space (and more in general in a Banach space) it is possible to consider a topology weaker than the norm topology such that the unit ball becomes compact. This result is classically known as Banach-Alaoglu theorem (or Banach-Alaoglu-Bourbaki theorem), see [166]. This theorem holds also in the quaternionic case. Indeed the classical proof, as we will see below, does not make use of any specific property of the complex numbers that is not possessed by the quaternions. We begin by recalling some definitions. Let $\mathscr{H}$ be a right quaternionic Hilbert space. We endow $\mathscr{H}$ with the so-called weak topology, in which a fundamental system of neighborhood of an element $u_0 \in \mathscr{H}$ is given by the sets

$$U_{\varepsilon,v_1,\dots,v_k}(u_0) = \{u \in \mathscr{H} \ : \ |\langle u - u_0, v_i\rangle| < \varepsilon, \ i = 1,\dots,k\},$$

where $\varepsilon > 0$, $v_1,\dots,v_k \in \mathscr{H}$.
We have the following result:

Theorem 5.6.1 (Banach-Alaoglu)**.** *Let $\mathscr{H}$ be a quaternionic Hilbert space. The closed unit ball of $\mathscr{H}$ is weakly compact.*

Proof. Let $B = B_{\mathscr{H}}$ denote the closed unit ball centered at 0, i.e. the set of $u \in \mathscr{H}$ such that $\|u\| \le 1$. For any $u \in \mathscr{H}$ let $D_u = \{q \in \mathbb{H} \ : \ |q| \le \|u\|\}$ and $D = \prod_{u\in\mathscr{H}} D_u$. Consider the map $\eta : \ B \to D$ such that to each $v \in B$ it associates the element $\eta(v) = \langle u, v\rangle \in D$, when u varies in B. The map τ is a homeomorphism of B endowed with the weak topology into D with the product topology. In fact, $\tau(v_1) = \tau(v_2)$ implies $\langle u, v_1\rangle = \langle u, v_2\rangle$ for all $u \in \mathscr{H}$ i.e. $v_1 = v_2$. The continuity follows from the fact that $v_k \to v$ in the weak topology if and only if $\langle u, v_k\rangle \to \langle u, v\rangle$ for all $u \in \mathscr{H}$ and so $\eta(v_k) \to \eta(v)$. Since D is a product of compact sets, by Tychonoff's theorem it is compact in the product topology and so to prove our result, it will be enough to show that $\eta(B)$ is closed, and therefore compact, in D. To this end, we note that by the Riesz theorem and by the Schwarz inequality, the range of η consists of the elements in D that are linear functionals ξ on $\mathscr{H}$ of norm less than or equal 1. Consider now the sets

$$E(q_1,q_2,u_1,u_2) =$$
$$= \{\xi \in D \ : \ \xi(u_1q_1 + u_2q_2) = \xi(u_1)q_1 + \xi(u_2)q_2, \ u_1,u_2 \in \mathscr{H}, q_1,q_2 \in \mathbb{H}\}.$$

The range of η in D is the intersection of all the sets of the form $E(q_1,q_2,u_1,u_2)$. Since the functions $\xi \mapsto \xi(u_1)q_1 + \xi(u_2)q_2, \ u_1,u_2 \in \mathscr{H}$ are continuous on D with the product topology, the sets $E(q_1,q_2,u_1,u_2)$ are closed and so is their intersection. The statement follows. □

In the sequel we will also need to show that the unit ball of the quaternionic Hilbert space $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$ (and in particular the space of of right linear operators from $\mathscr{H}$ to itself) is compact with respect to the weak topology. To this purpose we will now assume that $\mathscr{H}_1$ is a right linear space over $\mathbb{H}$ while $\mathscr{H}_2$ is a two sided vector space, so that $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$ is

a left linear space over $\mathbb{H}$.
Let us recall, see [166], that the weak topology on $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$ is defined by the family of seminorms $\{p_{u,v}\}$ given by
$$p_{u,v}(T)=|\langle Tu,v\rangle_{\mathscr{H}_2}|$$
and a fundamental system of neighborhood of 0 is of the form
$$U_{\varepsilon,u_1,\dots,u_k,v_1,\dots,v_k}(0)=\{T\ :\ |\langle Tu_1,v_1\rangle|<\varepsilon,\dots,|\langle Tu_k,v_k\rangle|<\varepsilon\},$$
where $u_i\in\mathscr{H}_1, v_i\in\mathscr{H}_2$, $i=1,\dots,k$ and $\varepsilon>0$.
It is immediate that a net $\{T_\alpha\}$ converges weakly to T if and only if $\langle T_\alpha u,v\rangle_{\mathscr{H}_2}\to\langle Tu,v\rangle_{\mathscr{H}_2}$ for all $u\in\mathscr{H}_1$, $v\in\mathscr{H}_2$.

Theorem 5.6.2. *The closed unit ball of* $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$ *is compact in the weak topology.*

Proof. From Theorem 5.6.1 we know that the closed unit ball $B=B_{\mathscr{H}_2}$ of $\mathscr{H}_2$ is weakly compact. By Tychonoff's theorem also the product $\prod_{u\in B}B$ is compact with respect to the product topology. Let T be an element in the unit ball of $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$ with $\|T\|=1$. Let us define the map
$$\Lambda:\ \mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)\to\prod_{u\in B}B$$
by setting $\Lambda(T)=\{Tu\}_{u\in B}$. This map is injective from the closed unit ball $\tilde{B}=B_{\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)}$ of $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$ and it is an homeomorphism onto its image in $\prod_{u\in B}B$ with respect to the weak topology. We now show that the image of Λ is closed and therefore compact.
Let $\{T_\alpha\}$ be a net in $\tilde{B}$ such that $\Lambda(T_\alpha)\to\Psi\in\prod_{u\in B}B$. It is immediate to verify that Ψ is right linear. The element Ψ is of the form $\{\Psi_u\}_{u\in B}$; for the sake of clarity, we will write $\Psi(u)$ instead of Ψ_u. Let us define a right linear operator T by setting $T(0)=0$ and $T(u)=\Psi(u\|u\|^{-1})\|u\|$ for $u\neq 0$. Then $\Lambda(T)=\Psi$ and so Ψ belongs to the image of Λ which is therefore closed. □

Theorem 5.6.3. *In a separable quaternionic Hilbert space, the weak topology of the closed unit ball is metrizable.*

Proof. We follow the proof of [208, p. 14]. Since the closed unit ball B of $\mathscr{H}$ is weakly compact by Theorem 5.6.1 it is enough to prove that there exists a countable basis for the weak topology of B. Let us consider a subset $\{u_n\}_{n\in\mathbb{N}}$ dense in B and the set of basic neighborhoods in B given by
$$U_{1/q,u_1,\dots,u_k}(u_\ell)=\{u\in B\ :\ |\langle u-u_\ell,u_j\rangle|<\frac{1}{q},\ j=1,\dots,k\},$$
where $q,\ell,k\in\mathbb{N}$. We now prove that if $\tilde{u}$ is any element in B and if $U_{\varepsilon,v_1,\dots,v_k}(\tilde{u})$ is any of the fundamental neighborhoods of $\tilde{u}$ which are in B, then there exist $q,\ell,k\in\mathbb{N}$ such that
$$\tilde{u}\in U_{1/q,u_1,\dots,u_k}(u_\ell)\subset U_{\varepsilon,v_1,\dots,v_h}(\tilde{u}).$$
The proof of this fact follows using standard arguments. In fact we have a chain of inequalities that leads to
$$|\langle u-\tilde{u},v_i\rangle|\leq|\langle u-u_\ell,u_j\rangle|+\|u_\ell-\tilde{u}\|\|u_j\|+\|u-\tilde{u}\|\|v_i-u_j\|.$$

By choosing q such that $1/q < \varepsilon/3$, choose an index $j = J_i$ such that $\|v_i - u_{J_i}\| < 1/2q$ and k such that $J_i \leq k$ for $i = 1, \ldots, r$ for some $r \in \mathbb{N}$, and p such that $\|u_\ell - \tilde{u}\| < 1/(qm)$ where m denotes the maximum of $\|u_i\|$ when $i = 1, \ldots k$. Then we have:

$$|\langle u - u_\ell, u_j \rangle| < 1/q < \varepsilon/3,$$

$$\|u_\ell - \tilde{u}\| \|u_j\| < \frac{1}{qm} m < \varepsilon/3$$

and

$$\|u - \tilde{u}\| \|v_i - h_j\| < 2\frac{1}{2q} < \varepsilon/3$$

it follows that any $u \in U_{1/q, u_1, \ldots, u_k}(u_\ell)$ belongs to $U_{\varepsilon, v_1, \ldots, v_h}(\tilde{u})$. □

We now turn to a convergence theorem.

Proposition 5.6.4. *Let $(A_n)_{n\in\mathbb{N}}$ be an increasing family of positive bounded operators in the quaternionic Hilbert space $\mathscr{H}$, and assume that*

$$\lim_{n\to\infty} \langle A_n f, f \rangle < \infty, \quad \forall f \in \mathscr{H}.$$

Then $(A_n)_{n\in\mathbb{N}}$ converges strongly to a bounded self-adjoint operator.

Proof. We follow [8, p. 98-99], and first we remark that the quaternionic polarization identity

$$\begin{aligned}4\langle A_n f, g \rangle = {} & \langle A_n(f+g), f+g \rangle - \langle A_n(f-g), f-g \rangle + i\langle A_n(f+gi), f+gi \rangle \\ & - i\langle A_n(f-gi), f-gi \rangle + i\langle A_n(f-gj), f-gj \rangle k - i\langle A_n(f+jg), f+gj \rangle k \\ & + \langle A_n(f+gk), f+gk \rangle k - \langle A_n(f-gk), f-gk \rangle k\end{aligned}$$

implies that

$$\lim_{n\to\infty} \langle A_n f, g \rangle$$

exists for all $f, g \in \mathscr{H}$. Theorem 5.1.10 applied to the maps

$$g \mapsto \langle g, A_n f \rangle$$

gives that $\sup_{n\in\mathbb{N}} \|A_n f\| < \infty$. That same theorem now applied to the maps

$$f \mapsto A_n f$$

implies that $M = \sup_{n\in\mathbb{N}} \|A_n\| < \infty$.

Furthermore, by Theorem 5.3.10 for $\|f\| = 1$, we have

$$\|\sqrt{B} f\|^2 = \langle Bf, f \rangle \leq \|B\|$$

so that $\|\sqrt{B}\|^2 \leq \|B\|$. Thus for a positive operator B we can write

$$\|Bf\|^2 \leq \|\sqrt{B}\|^2 \cdot \|\sqrt{B}f\|^2 \leq \|B\| \cdot \langle Bf, f\rangle.$$

Applying this inequality to $B = A_m - A_n$ with $m \geq n$, we have

$$\begin{aligned}\|A_m f - A_n f\| &\leq \|\sqrt{A_m - A_n}\|^2 \cdot \langle A_m f - A_n f, f\rangle \\ &\leq \|\sqrt{A_m}\|^2 \cdot \langle A_m f - A_n f, f\rangle \\ &\leq M\langle A_m f - A_n f, f\rangle\end{aligned}$$

which allows to conclude the proof. □

5.7 Quaternionic Pontryagin spaces

A first study of quaternionic Pontryagin spaces appears in [68], but a general theory of operators in Pontryagin spaces (especially the structure of contractions) remains to be done. We need the quaternionic versions of a number of results in the setting of Pontryagin spaces. We mention in particular the fact that the adjoint of a contraction is a contraction (see Theorem 5.7.8), an invariant subspace theorem for contractions in Pontryagin spaces (see Theorem 5.7.9) and a theorem on contractive relations in Pontryagin spaces. See Theorem 5.7.10.

We begin with the following definition:

Definition 5.7.1. The indefinite inner product quaternionic space $(\mathscr{P}, [\cdot,\cdot])$ is called a Pontryagin space if it can be written as

$$\mathscr{P} = \mathscr{P}_+ \oplus \mathscr{P}_-, \tag{5.43}$$

where:
(1) Both $(\mathscr{P}_+, [\cdot,\cdot])$ and $(\mathscr{P}_-, -[\cdot,\cdot])$ are quaternionic Hilbert spaces.
(2) The sum (5.43) is direct and orthogonal, meaning that $\mathscr{P}_+ \cap \mathscr{P}_- = \{0\}$ and

$$[f_+, f_-] = 0, \quad \forall f_+ \in \mathscr{P}_+ \text{ and } f_- \in \mathscr{P}_-.$$

(3) $\mathscr{P}_-$ is finite dimensional.

The decomposition (5.43) of a Pontryagin space $\mathscr{P}$ is obviously not unique when one of the components is not trivial. Both the spaces $\mathscr{P}_+$ and $\mathscr{P}_-$ are Hilbert spaces and they can both be, in particular, of finite dimension.
Each element in $\mathscr{P}$ can be decomposed, in a non unique way, as $f = f_+ + f_-$ where $(f_+, f_-) \in \mathscr{P}_+ \times \mathscr{P}_-$. This decomposition is called a *fundamental decomposition*.
We note that $\mathscr{P}$ endowed with the inner product

$$\langle f, g\rangle = [f_+, g_+] - [f_-, g_-] \tag{5.44}$$

is a quaternionic Hilbert space and we set $\|f\| = \sqrt{\langle f, f\rangle}$.

Remark 5.7.2. When the third condition is removed in Definition 5.7.1 the space is called a Krein space.

Remark 5.7.3. Remark 1.2.2 still holds here for a J with real entries. This latter requirement allows condition (5.4) to hold in spite of the noncommutatvity of the quaternions.

Proposition 5.7.4. *The form $\langle\cdot,\cdot\rangle$ is continuous with respect to the topology defined by (5.44). More precisely:*

$$|\langle f,g\rangle|^2 \le \|f\|^2\|g\|^2.$$

Proof. The Cauchy-Schwarz inequality (5.11) implies

$$\|[f_\pm,g_\pm]\|^2 \le [f_\pm,f_\pm][g_\pm,g_\pm],$$

and the triangle inequality gives

$$\begin{aligned}|\langle f,g\rangle|^2 &\le (|[f_+,g_+]|+|[f_-,g_-]|)^2\\ &\le \left(\sqrt{[f_+,f_+]}\sqrt{[g_+,g_+]}+\sqrt{-[f_-,f_-]}\sqrt{-[g_-,g_-]}\right)^2.\end{aligned}$$

Using again the Cauchy-Schwarz inequality one obtains

$$\begin{aligned}|\langle f,g\rangle|^2 &\le ([f_+,f_+]-[f_-,f_-])([g_+,g_+]-[g_-,g_-])\\ &= \|f\|^2\|g\|^2\end{aligned}$$

and the result follows. □

Proposition 5.7.5. *Let $\mathscr{P}=\mathscr{P}_+\oplus\mathscr{P}_-$ be a fundamental decomposition of $\mathscr{P}$. Then*

$$\begin{aligned}\mathscr{P}_+ &= \mathscr{P}_-^{[\perp]},\\ \mathscr{P}_- &= \mathscr{P}_+^{[\perp]}.\end{aligned}$$

Proof. The inclusion $\mathscr{P}_-\subseteq\mathscr{P}_+^{[\perp]}$ is obvious. Assume that there exists $h\in\mathscr{P}_+^{[\perp]}\setminus\mathscr{P}_-$ and let $h=h_++h_-$ with $h_\pm\in\mathscr{P}_\pm$. Then, since $h\in\mathscr{P}_+^{[\perp]}$ we have $[h,f_+]=0$ for all $f_+\in\mathscr{P}_+$, moreover $[h_-,f_+]=0$ so we deduce $[h_+,f_+]=0$ and thus $h_+=0$. We conclude that $h\in\mathscr{P}_+\cap\mathscr{P}_-$ and so $h=0$ and $\mathscr{P}_-\subseteq\mathscr{P}_+^{[\perp]}$. The other equality can be proved similarly. □

Proposition 5.7.6. *Let $\mathscr{P}$ be a quaternionic Pontryagin space and let $\mathscr{P}=\mathscr{P}_+\oplus\mathscr{P}_-$ be a fundamental decomposition. Then $\mathscr{P}_+$ (resp. $\mathscr{P}_-$) is a maximal strictly positive subspace (resp. maximal strictly negative).*

Proof. Let $\mathscr{L}$ be a strictly positive subspace of $\mathscr{P}$ such that $\mathscr{P}_+\subset\mathscr{L}$. Let $h\in\mathscr{L}\setminus\mathscr{P}_+$ and $h=h_++h_-$ be the decomposition of h where $h_\pm\in\mathscr{P}_\pm$. Then, reasoning as in the proof of Proposition 5.7.5, $h_-=0$ thus $h=h_+\in\mathscr{P}_+$. □

Corollary 5.7.7. *The dimension of $\mathscr{P}_-$ is the same for all the decompositions.*

Proof. This follows from the fact that all the subspaces $\mathscr{P}_-$ are maximal strictly negative and since they have finite dimension, they all have the same dimension. □

Theorem 5.7.8. *Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two quaternionic Pontryagin spaces of the same index, and let T be a contraction from $\mathscr{P}_1$ to $\mathscr{P}_2$. Then $T^{[*]}$ is a contraction from $\mathscr{P}_2$ to $\mathscr{P}_1$.*

Proof. Write (see [47, (1.3.14), p. 26])

$$\begin{pmatrix} I_{\mathscr{P}_1} & 0 \\ T & I_{\mathscr{P}_2} \end{pmatrix} \begin{pmatrix} I_{\mathscr{P}_1} & 0 \\ 0 & I_{\mathscr{P}_2} - TT^{[*]} \end{pmatrix} \begin{pmatrix} I_{\mathscr{P}_1} & T^{[*]} \\ 0 & I_{\mathscr{P}_2} \end{pmatrix} = \\ = \begin{pmatrix} I_{\mathscr{P}_1} & T^{[*]} \\ 0 & I_{\mathscr{P}_2} \end{pmatrix} \begin{pmatrix} I_{\mathscr{P}_1} - T^{[*]}T & 0 \\ 0 & I_{\mathscr{P}_2} \end{pmatrix} \begin{pmatrix} I_{\mathscr{P}_1} & 0 \\ T & I_{\mathscr{P}_2} \end{pmatrix}. \tag{5.45}$$

Thus (and with ν_- defined as in Definition 1.2.13), we have:

$$\nu_-(I_{\mathscr{P}_2} - TT^{[*]}) + \nu_-(\mathscr{P}_1) = \nu_-(I_{\mathscr{P}_1} - T^{[*]}T) + \nu_-(\mathscr{P}_2). \tag{5.46}$$

We have $I_{\mathscr{P}_1} - T^{[*]}T \geq 0$, and so $\nu_-(I_{\mathscr{P}_1} - T^{[*]}T) = 0$. □

Theorem 5.7.9. *A contraction in a quaternionic Pontryagin space has a unique maximal invariant negative subspace, and it is one-to-one on it.*

Proof. The proof will follow the lines of the analogous proof given in [165]. We recall the main lines for the sake of completeness. Let A be a contraction in the Pontryagin space $\mathscr{P}$. Let us recall a well known fact in the theory of linear fractional transformations (see for instance [169] for more details). Let $\mathscr{P} = \mathscr{P}_+ \oplus \mathscr{P}_-$ be a fundamental decomposition of $\mathscr{P}$ and let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be the block decomposition of A along $\mathscr{P}_+ \oplus \mathscr{P}_-$. Since A is a contraction we have

$$A_{21}A_{21}^* - A_{22}A_{22}^* \leq -I,$$

and it follows that A_{22}^{-1} and $A_{22}^{-1}A_{21}$ are strict contractions. Thus the map

$$L(X) = (A_{11}X + A_{12})(A_{21}X + A_{22})^{-1}$$

is well defined, and sends the closed unit ball B_1 of $\mathbf{B}(\mathscr{P}_, \mathscr{P}_+)$ into itself. To show that A has a maximal negative invariant subspace we have to show that the map L is continuous in the weak operator topology from B_1 into itself. Since B_1 is compact in this topology (and of course convex) the Schauder-Tychonoff theorem (see Theorem 5.1.21) implies that L has a unique fixed point, say X. To conclude one notes (see Theorem [165, 1.3.10]) that the space spanned by the elements

$$f + Xf, \quad f \in \mathscr{P}_- \tag{5.47}$$

is then negative. It is maximal negative because X cannot have a kernel (any f such that $Xf = 0$ will lead to a strictly positive element of (5.47)). □

The following result is the quaternionic version of a theorem of Shmulyan, which is the key to the approach to the study of generalized Schur functions in [47].

Theorem 5.7.10. *Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two quaternionic Pontryagin spaces of the same index, and let $R \subset \mathscr{P}_1 \times \mathscr{P}_2$ be a densely defined contractive relation. Then, R extends to the graph of a contraction from $\mathscr{P}_1$ into $\mathscr{P}_2$.*

Proof. We follow the lines of the proof of [47, p. 29-30] and we divide the proof into steps.
STEP 1: *The domain of the relation contains a maximum negative subspace.*

Indeed, every dense linear subspace of a right quaternionic Pontryagin space of index $\kappa > 0$ contains a κ dimensional strictly negative subspace. See [68, Theorem 12.8 p. 470]. Let $\mathscr{V}_-$ be such a subspace of the domain of R.

STEP 2: *The relation R restricted to $\mathscr{V}_-$ has a zero kernel, moreover the image of $\mathscr{V}_-$ is a strictly negative subspace of $\mathscr{P}_2$ of dimension κ.*

Let $(v_1, v_2) \in R$ with $v_1 \in \mathscr{V}_-$. The fact that R is a contraction gives

$$[v_2, v_2]_2 \leq [v_1, v_1]_1 \leq 0.$$

The second inequality is strict when $v_1 \neq 0$, thus the image of $\mathscr{V}_-$ is a strictly negative subspace of $\mathscr{P}_2$. Next, let (v, w) and $(\widetilde{v}, w)$ be in R, with $v, \widetilde{v} \in \mathscr{V}_-$ and $w \in \mathscr{P}_2$. Then, we have $(v - \widetilde{v}, 0) \in R$. Since R is contractive we have

$$[0,0]_2 \leq [v - \widetilde{v}, v - \widetilde{v}]_1.$$

This forces $v = \widetilde{v}$ since $\mathscr{V}_-$ is strictly negative, and proves the second step.

STEP 3: *The relation R is the graph of a densely defined contraction.*

Let $\mathscr{V}_-$ be as in the first two steps, and take $v_1, \ldots, v_\kappa$ a basis of $\mathscr{V}_-$. Then, there are uniquely defined vectors $w_1, \ldots, w_\kappa \in \mathscr{P}_2$ such that $(v_i, w_i) \in R$ for $i = 1, \ldots, \kappa$. Set $\mathscr{W}_-$ to be the linear span of $w_1, \ldots, w_\kappa$. By Step 2 and since the spaces $\mathscr{P}_1$ and $\mathscr{P}_2$ have the same negative index we deduce

$$\dim \mathscr{V}_- = \dim \mathscr{W}_- = \mathrm{ind}_- \mathscr{P}_1 = \mathrm{ind}_- \mathscr{P}_2,$$

so there exist fundamental decompositions

$$\mathscr{P}_1 = \mathscr{V}_- + \mathscr{V}_+ \quad \text{and} \quad \mathscr{P}_2 = \mathscr{W}_- + \mathscr{W}_+,$$

where $(\mathscr{V}_+, [\cdot,\cdot]_1)$ and $(\mathscr{W}_+, [\cdot,\cdot]_2)$ are right quaternionic Hilbert spaces. Now we show that if $(0, w) \in R$ then $w = 0$. Let us write $w = w_- + w_+$ where $w_- \in \mathscr{W}_-$ and $w_+ \in \mathscr{W}_+$. Let $w_- = \sum_{j=1}^{\kappa} w_j q_j$, $q_j \in \mathbb{H}$, and set $v_- = \sum_{j=1}^{\kappa} v_j q_j$. Then, $(v_-, w_-) \in R$ and

$$(0, w) = (v_-, w_-) + (-v_-, w_+).$$

It follows that $(-v_-, w_+) \in R$. Since R is contractive, we have

$$[w_+, w_+]_2 \leq [v_-, v_-]_1,$$

and so $[w_+, w_+]_2 \leq 0$. Thus $w_+ = 0$. It follows that $(0, w_-) \in R$ and so $w_- = 0$ because, by Step 2, R is one-to-one on $\mathscr{V}_-$.

STEP 4: *The relation R extends to the graph of an everywhere defined contraction.*

This fact is well known in the complex case, see [47, Theorem 1.4.1 p. 27]. We follow the arguments there. We consider the orthogonal projection from $\mathscr{P}_2$ onto $\mathscr{W}_-$. Let T be the densely defined contraction having the relation R as a graph. There exist $\mathbb{H}$-valued right linear functionals $c_1, \ldots, c_\kappa$, defined on the domain of R, and such that

$$Tv = \sum_{n=1}^{\kappa} w_n c_n(v) + w_+,$$

where $w_+ \in \mathscr{W}_+$ satisfies $[f_n, w_+]_2 = 0$ for $n = 1, 2, \ldots, \kappa$. Assume that c_1 is not bounded on its domain, let v_+ be such that $c_1(v_+) = 1$, and let v_n be vectors in $\mathscr{V}_+$ such that $c_1(v_n) = 1$ for $n \geq 1$, and $\lim_{n\to\infty}[v_+ - v_n, v_+ - v_n]_1 = 0$. Then v_+ belongs to the closure of $\ker c_1$ and so, we have that the closure of $\ker c_1 = \mathscr{V}_+$. Thus $\ker c_1$ contains a strictly negative subspace of dimension κ, which we denote by $\mathscr{K}_-$. For $v \in \mathscr{K}_-$, we have

$$Tv = \sum_{n=2}^{\kappa} w_n c_n(v),$$

since $v \in \ker c_1$. This contradicts STEP 2 and the proof of the theorem is complete. □

The following result is very useful to study convergence of sequences in Pontryagin spaces. We state it without proof and we refer the reader to [68, Proposition 12.9, p. 471]. This result implies, in particular, that in a reproducing kernel Pontryagin space convergence is equivalent to convergence of the self-inner product together with pointwise convergence.

Proposition 5.7.11. *Let $(\mathscr{P}, [\cdot,\cdot])$ denote a quaternionic right Pontryagin space. The sequence f_n of elements in $\mathscr{P}$ tends to $f \in \mathscr{P}$ if and only if the following two conditions hold:*

$$\lim_{n\to\infty}[f_n, f_n] = [f, f],$$

and

$$\lim_{n\to\infty}[f_n, g] = [f, g] \quad \text{for } g \text{ in a dense subspace of } \mathscr{P}.$$

5.8 Quaternionic Krein spaces

In this section we will study quaternionic Krein spaces following our paper [37] which, in turns, follows Bognar's book [98, Chapter V]. As in the classical case, they are characterized by the fact that they are inner product spaces nondegenerate, decomposable and complete. We will show that the scalar product associated to the decomposition gives a norm, and so a topology, which does not depend on the chosen decomposition. We will also study ortho-complemented subspaces of a Krein space and we will prove that they are closed subspaces which are Krein spaces themselves.

Definition 5.8.1. The indefinite inner product quaternionic space $(\mathscr{K}, [\cdot,\cdot])$ is called a Krein space if it can be written as

$$\mathscr{K} = \mathscr{K}_+ \oplus \mathscr{K}_-, \tag{5.48}$$

where:
(1) Both $(\mathscr{K}_+, [\cdot,\cdot])$ and $(\mathscr{K}_-, -[\cdot,\cdot])$ are quaternionic Hilbert spaces.
(2) The sum (5.48) is direct and orthogonal, meaning that $\mathscr{K}_+ \cap \mathscr{K}_- = \{0\}$ and

$$[f_+, f_-] = 0, \quad \forall f_+ \in \mathscr{K}_+ \text{ and } f_- \in \mathscr{K}_-.$$

The decomposition of a Krein space is obviously not unique when one of the components is not trivial. Both the spaces $\mathscr{K}_+$ and $\mathscr{K}_-$ are Hilbert spaces and they can be, in particular, of finite dimension.

Krein spaces are characterized in the next result:

Proposition 5.8.2. *A Krein space is nondegenerate and decomposable, and $\mathscr{K}_\pm$ are intrinsically complete. Every other decomposition of $\mathscr{K}$ is of the form (5.48).*

Proof. A Krein space is obviously decomposable by its definition and nondegenerate by Proposition 5.2.4.
By Theorem 5.5.17 (2), given (5.48) and any other fundamental decomposition $\mathscr{K} = \mathscr{K}'_+[\oplus]\mathscr{K}'_-$ if $\mathscr{K}_+$ is intrinsically complete so is $\mathscr{K}'_+$ (and similarly for $\mathscr{K}'_-$). □

Proposition 5.8.3. *A nondegenerate, decomposable, quaternionic inner product space $\mathscr{K}$ is a Krein space if and only if for every associated fundamental symmetry J, $\mathscr{K}$ endowed with the inner product $\langle v, w\rangle_J \overset{\text{def.}}{=} [v, Jw]$ is a Hilbert space.*

Proof. Let $\mathscr{K}$ be a nondegenerate, decomposable, quaternionic inner product space, i.e. $\mathscr{K} = \mathscr{K}'_+[\oplus]\mathscr{K}'_-$. If $\mathscr{K}$ is a Krein space then the associated fundamental symmetry $J = P^+ - P^-$ makes it into a pre-Hilbert space, see Theorem 5.5.11. The fact that $\mathscr{K}$ is complete follows from the fact that both $\mathscr{K}_\pm$ are complete. To prove the converse, let us assume that given a fundamental symmetry J the inner product $\langle v, w\rangle_J \overset{\text{def.}}{=} [v, Jw]$ makes $\mathscr{K}$ a Hilbert space. The intrinsic norm in $\mathscr{K}_+$ is obtained by restricting the J-inner product to $\mathscr{K}_+$. Any Cauchy sequence in $\mathscr{K}_+$ converges to an element in $\mathscr{K}$ and it is immediate that this element belongs to $\mathscr{K}_+$. □

Next result gives a necessary a sufficient condition in order that a quaternionic inner product space is a Krein space.

Theorem 5.8.4. *Let $\mathscr{K}$ be a quaternionic vector space with inner product $[\cdot,\cdot]$. Then $\mathscr{K}$ is a Krein space if and only if :*

(1) $[\cdot,\cdot]$ has a Hilbert majorant τ with associated inner product $\langle\cdot,\cdot\rangle$ and norm $\|v\| = \sqrt{\langle v,v\rangle}$;

(2) the Gram operator such that $[v,w] = \langle v, Gw\rangle$ is completely invertible.

Proof. We follow repeat the main arguments in the proof of Theorem V, 1.3 in [98]. Assume that $\mathscr{K}$ is a Krein space and denote by J the fundamental symmetry associated to the chosen decomposition (5.48). Define a norm using the J-inner product $\langle\cdot,\cdot\rangle_J$ and let τ_J be the corresponding topology which is a decomposition majorant by Proposition 5.5.15 and a Hilbert majorant. Since

$$[v,w] = [v,J^2w] = \langle v,Jw\rangle_J,$$

the Gram operator of $[\cdot,\cdot]$ with respect to $\langle\cdot,\cdot\rangle_J$ is equal to J, and and is J is completely invertible. We now prove part (2) of the statement. By Theorem 5.4.9 there is only one Hilbert majorant, thus if there are two positive inner products $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ whose associated norms define the Hilbert majorant, then the two norms must be equivalent. Reasoning as in [98], the two Gram operators G_j, $j = 1,2$ of $[\cdot,\cdot]$ with respect to $\langle\cdot,\cdot\rangle_j$, $j = 1,2$ are both completely invertible if and only if one of them is so. Since we have previously shown that (2) holds for $G_1 = J$ then (2) holds for any other Gram operator.
Let us show the converse and assume that (1) and (2) hold. Then by Theorem 5.5.10 $\mathscr{K}$ is decomposable and nondegenerate thus, by Proposition 5.2.4, it admits a decomposition of the form (5.48). By Proposition 5.8.3, $\mathscr{K}$ is a Krein space if for every chosen decomposition the J-inner product makes $\mathscr{K}$ a Hilbert space or, equivalently, if τ_J coincides with J. First of all we observe that since G is completely invertible, by the closed graph theorem we have that the Mackey topology coincides with τ. By Theorem 5.5.8 we deduce that τ is an admissible majorant and by Theorem 5.5.6 τ is also a minimal majorant and so $\tau \leq \tau_J$. However we know from Proposition 5.5.13 that $\tau_J \leq \tau$ and the conclusion follows. □

Theorem 5.8.5. *In a Krein space all the J-norms are equivalent.*

Proof. From Proposition 5.5.15 it follows that all the decomposition majorants are equivalent, or in other words, that all the J-norms are equivalent. □

The J-norms are called natural norms on $\mathscr{K}$, and define a Hilbert majorant called the strong topology of $\mathscr{K}$.

As an immediate consequence of Theorem 5.8.4 we have:

Corollary 5.8.6. *The strong topology of $\mathscr{K}$ equals the Mackey topology.*

In the sequel we will always consider a Krein space $\mathscr{K}$ endowed with the strong topology $\tau_M(\mathscr{K})$.

Proposition 5.8.7. *The strong topology $\tau_M(\mathscr{K})$ of the Krein space $\mathscr{K}$ is an admissible majorant.*

Proof. We know from Proposition 5.5.8 that the Mackey topology is admissible and the fact that it is an admissible majorant is ensured by (5.38). □

Assume now that $\mathscr{L}$ is a subpace of a Krein space $\mathscr{K}$. In general, it is not true that $\mathscr{L}$ is decomposable. However, if $\mathscr{L}$ is closed, then Proposition 5.8.3 together with Theorem 5.5.10 show that $\mathscr{L}$ is decomposable and its components and the sum of any two of them are closed.
Next result describes under which conditions a closed subspace of a Krein space is a Krein space.

Theorem 5.8.8. *Let $\mathscr{K}$ be a quaternionic Krein space. A subspace $\mathscr{L}$ of $\mathscr{K}$ is ortho-complemented if and only if it is closed and it is a Krein space itself.*

Proof. We assume that $\mathscr{L}$ is ortho-complemented. Then Corollary 5.4.7 shows that $\mathscr{L}$ is closed. By Theorem 5.8.4 $\mathscr{K}$ has a Hilbert majorant and thus we can use the condition given in Proposition 5.5.9 (2). Let us denote by $G_{\mathscr{L}}$ the Gram operator defined by $[v,w] = \langle v, G_{\mathscr{L}} w\rangle_J$, for $v,w \in \mathscr{L}$, where J denotes the fundamental symmetry of $\mathscr{K}$ associated with the chosen decomposition. By Theorem 5.8.4, the Gram operator G is completely invertible and thus, by Proposition 5.5.9 (2) $\mathscr{L}$ is ortho-complemented if and only if $\operatorname{ran}(G_{\mathscr{L}}) = \mathscr{L}$ but, since $G_{\mathscr{L}}$ is J-symmetric, this is equivalent to $G_{\mathscr{L}}$ completely invertible and so, again by Theorem 5.8.4 to the fact that $\mathscr{L}$ is a Krein space.
The converse directly follows from the previous argument using Proposition 5.5.9 (2) and Theorem 5.8.4. □

Given a definite subspace $\mathscr{L}$ of a Krein space $\mathscr{K}$, it is clear that the intrinsic topology $\tau_{int}(\mathscr{L})$ is weaker than the topology induced by the strong topology $\tau_M(\mathscr{K})$ induces on $\mathscr{L}$. Thus we give the following definition:

Definition 5.8.9. A subspace $\mathscr{L}$ of a Krein space $\mathscr{K}$ is said to be uniformly positive (resp. negative) if $\mathscr{L}$ is positive definite (resp. negative definite) and $\tau_{int}(\mathscr{L}) = \tau_M(\mathscr{K})|\mathscr{L}$.

Note that the second condition amounts to require that $\mathscr{L}$ is uniformly positive if $[v,v] \geq c\|v\|_J^2$ for $v \in \mathscr{L}$ (resp. $\mathscr{L}$ is uniformly negative if $[v,v] \leq -c\|v\|_J^2$ for $v \in \mathscr{L}$) where c is a positive constant.
We have the following result:

Theorem 5.8.10. *Let $\mathscr{K}$ be a Krein space.*

(1) A closed definite subspace $\mathscr{L}$ of $\mathscr{K}$ is intrinsically complete if and only if it is uniformly definite.

(2) A semi-definite subspace $\mathscr{L}$ of $\mathscr{K}$ is ortho-complemented if and only if it is closed and uniformly definite (either positive or negative).

Proof. Point (1) follows from the fact that Proposition 5.8.3 and the closed graph theorem imply that a closed and definite subspace $\mathscr{L}$ is intrinsically complete if and only if $\tau_{int}(\mathscr{L}) = \tau_M(\mathscr{K})|\mathscr{L}$ i.e. if and only if $\mathscr{L}$ is uniformly definite.
By Proposition 5.8.2 and Theorem 5.8.8, a subspace $\mathscr{L}$ is ortho-complemented if and only if it is closed, definite and intrinsically complete, i.e. if and only if $\mathscr{L}$ is uniformly definite (either positive or negative). Point (2) follows. □

As a consequence of the previous two theorems we have the following result:

Theorem 5.8.11. *Let $\mathscr{K}$ denote a quaternionic Krein space, and let $\mathscr{M}$ be a closed uniformly positive subspace of $\mathscr{K}$. Then, $\mathscr{M}$ is a Hilbert space and is ortho-complemented in $\mathscr{K}$: one can write*

$$\mathscr{K} = \mathscr{M} \oplus \mathscr{M}^{[\perp]},$$

and $\mathscr{M}^{[\perp]}$ is a Krein subspace of $\mathscr{K}$.

Proof. The space is a Hilbert space by (1) of the previous theorem. That it is ortho-complemented follows then from Theorem 5.8.8. □

Remark 5.8.12. We note that formula (1.9)

$$A^* = J_{\mathscr{V}} A^{[*]} J_{\mathscr{W}}$$

relating the Krein space adjoint and the Hilbert space adjoint of a linear bounded operators between two Krein spaces $\mathscr{V}$ and $\mathscr{W}$ still holds in the quaternionic setting. The proof is the same.

5.9 Positive definite functions and reproducing kernel quaternionic Hilbert spaces

In preparation to Section 5.10 where quaternionic reproducing kernel Pontryagin spaces are considered, we here present the main aspects of quaternionic reproducing kernel Hilbert spaces. First a definition:

Definition 5.9.1. Let Ω be some set and let $\mathscr{K}$ be a two-sided quaternionic Krein space. The $\mathbf{B}(\mathscr{K})$-valued function $K(p,q)$ defined on $\Omega \times \Omega$ is called positive definite if is Hermitian

$$K(p,q) = K(q,p)^{[*]}, \quad \forall p,q \in \Omega$$

and if for every choice of $N \in \mathbb{N}$, $c_1, \ldots, c_N \in \mathscr{K}$ and $w_1, \ldots, w_N \in \Omega$ the $N \times N$ Hermitian matrix with (u,v)-entry equal to

$$[K(w_u, w_v)c_v, c_u]_{\mathscr{K}}$$

is positive (as in Definition 1.2.9, note this standard terminology is a bit unfortunate. Note also that one uses also the term *kernel* rather that function).

Definition 5.9.2. Let $\mathscr{K}$ be a right-sided quaternionic Krein space. A kernel $K(p,q)$ defined on $\Omega\times\Omega$ with values in $\mathbf{B}(\mathscr{K})$ is said to be a reproducing kernel for a Hilbert space $\mathscr{H}$ if for any $p\in\Omega$ and any $f\in\mathscr{K}$ the following properties hold:

(1) The function $p\to K(p,q)f$ belongs to $\mathscr{H}$;

(2) $\langle g(\cdot),K(\cdot,q)f\rangle_{\mathscr{H}}=[g(q),f]_{\mathscr{K}}$ for every $q\in\Omega$ and every $g\in\mathscr{H}$.

If such a function $K(\cdot,\cdot)$ exists, $\mathscr{H}$ is called a reproducing kernel Hilbert space.

As in the classical case, there is a one-to-one correspondence between $\mathbf{B}(\mathscr{K})$-valued function $K(p.q)$ positive definite on Ω and reproducing kernel Hilbert spaces of $\mathscr{K}$-valued functions defined on Ω. The finite dimensional case is of special interest, and is considered in the following theorem:

Theorem 5.9.3. *Let Ω be some set and let $\mathscr{K}$ be a two-sided quaternionic Krein space. Let $K(p,q)$ be a $\mathbf{B}(\mathscr{K})$-valued function positive definite on Ω, and let $\mathscr{H}(K)$ be the associated reproducing kernel quaternionic Hilbert space. Then $\mathscr{H}(K)$ is of finite dimension if and only if there exist a finite dimensional right-sided quaternionic Hilbert space $\mathscr{G}$ and a $\mathbf{B}(\mathscr{K},\mathscr{G})$-valued function $G(p)$ such that*

$$K(p,q)=G(q)^*G(p). \tag{5.49}$$

Proof. One can take $\mathscr{G}=\mathscr{H}(K)$. Then the equality

$$[K(p,q)c,d]_{\mathscr{K}}=\langle K(\cdot,q)c,K(\cdot,p)d\rangle_{\mathscr{H}(K)}$$

shows that the application $G(p)\in\mathbf{B}(\mathscr{K},\mathscr{H}(K))$ defined by

$$G(p)c=K(\cdot,p)c,\quad c\in\mathscr{K},$$

satisfies (5.49). □

When $\mathscr{K}$ is finite dimensional of dimension N, then (5.49) can be rewritten as

$$F(p)P^{-1}F(q)^* \tag{5.50}$$

where now F is a matrix-valued function with columns $f_1,\ldots,f_N$ being a basis of $\mathscr{H}(K)$ and P is the Gram matrix, that is the $N\times N$ Hermitian matrix with (j,k) entry

$$P_{jk}=\langle f_k,f_j\rangle_{\mathscr{H}(K)}.$$

When the basis is orthonormal (5.50) becomes:

$$K(p,q)=\sum_{j=1}^{N}f_j(p)f_j(q)^*. \tag{5.51}$$

We will also need the following result, well known in the complex case. We refer to [77, 259] for more information and to [174] for connections with operator ranges.

Theorem 5.9.4. *Let Ω be some set and let $\mathscr{K}$ be a two-sided quaternionic Krein space. Let $K_1(p,q)$ and $K_2(p,q)$ be two $\mathbf{B}(\mathscr{K})$-valued functions positive definite in a set Ω and assume that the corresponding reproducing kernel Hilbert spaces have a zero intersection. Then the sum*

$$\mathscr{H}(K_1+K_2)=\mathscr{H}(K_1)+\mathscr{H}(K_2)$$

is orthogonal.

Proof. Let $K=K_1+K_2$. The linear relation in $\mathscr{H}(K)\times(\mathscr{H}(K_1)\times\mathscr{H}(K_2))$ spanned by the pairs

$$(K(\cdot,q)c,(K_1(\cdot,q)c,K_2(\cdot,q)c)),\quad c\in\mathscr{K}\text{ and }q\in\Omega,$$

is densely defined and isometric. It therefore extends to the graph of an everywhere defined isometry. From

$$\begin{aligned}\langle T^*(f_1,f_2),K(\cdot,q)c\rangle_{\mathscr{H}(K)}&=\langle(f_1,f_2),T(K(\cdot,q)c)\rangle_{\mathscr{H}(K_1)\times\mathscr{H}(K_2)}\\&=\langle f_1,K_1(\cdot,q)c\rangle_{\mathscr{H}(K_1)}+\langle f_2,K_2(\cdot,q)c\rangle_{\mathscr{H}(K_2)}\\&=[f_1(q)+f_2(q),c]_{\mathscr{K}},\quad c\in\mathscr{K}\text{ and }q\in\Omega,\end{aligned}$$

we see that $\ker T^*=\{0\}$ since $\mathscr{H}(K_1)\cap\mathscr{H}(K_2)=\{0\}$. Thus T is unitary and the result follows easily. ☐

Proposition 5.9.5. *With $\mathscr{K}$ as above, let $K(p,q)$ be a $\mathbf{B}(\mathscr{K})$-valued function positive in a set Ω. Then, the linear span of the functions*

$$p\mapsto K(p,q)h,\quad q\in\Omega\text{ and }h\in\mathscr{K}$$

is dense in $\mathscr{H}(K)$.

Proof. As in the classical complex case, this is a direct consequence of the reproducing kernel property. ☐

5.10 Negative squares and reproducing kernel quaternionic Pontryagin spaces

A right linear bounded operator A from the right Pontryagin space $\mathscr{P}$ into itself is called self-adjoint if

$$[Af,g]_{\mathscr{P}}=[f,Ag]_{\mathscr{P}},\quad\forall f,g\in\mathscr{P}.$$

The structure of Hermitian quaternionic matrices (see Theorem 4.3.10) allows to extend Definition 1.2.9 (that is the number $\nu_-(A)$) to the case of quaternionic spaces. We denote by $\nu_-(A)$ the (possibly infinite) number of negative squares of the function $K(f,g)=[Af,g]_{\mathscr{P}}$. A version of the following theorem was proved in the complex case in [72, Theorem 3.4, p. 456]. In that definition, the coefficient space $\mathscr{K}$ is a Krein space. Note that in this book, $\mathscr{K}$ will mainly be a Hilbert space or a Pontryagin space. In the statement, $^{[*]}$ denotes the Krein space adjoint. The definition makes sense in view of the spectral theorem for Hermitian quaternionic matrices (see Theorem 4.3.10 for the latter).

Definition 5.10.1. Let Ω be some set and let $\mathcal{K}$ be a right-sided quaternionic Krein space. The $\mathbf{B}(\mathcal{K})$-valued function $K(p,q)$ defined on $\Omega\times\Omega$ is said to be a kernel. We say that $K(p,q)$ has κ-negative squares if it is Hermitian

$$K(p,q)=K(q,p)^{[*]},\quad \forall p,q\in\Omega$$

and if for every choice of $N\in\mathbb{N}$, $c_1,\ldots,c_N\in\mathcal{K}$ and $q_1,\ldots,q_N\in\Omega$ the $N\times N$ Hermitian matrix with (u,v)-entry equal to

$$[K(q_u,q_v)c_v,c_u]_{\mathcal{K}}$$

has at most κ strictly negative eigenvalues, and exactly κ such eigenvalues for some choice of $N,c_1,\ldots,c_N$ and $q_1,\ldots,q_N$. Eigenvalues are counted with their multiplicities.

Definition 5.10.2. Let $\mathcal{K}$ be a right-sided quaternionic Krein space. A kernel $K(p,q)$ defined on $\Omega\times\Omega$ with values in $\mathbf{B}(\mathcal{K})$ is said to be a reproducing kernel for a Pontryagin space $\mathcal{P}$ of $\mathcal{K}$-valued functions defined on Ω if for any $q\in\Omega$ and any $c\in\mathcal{K}$ the following properties hold:

(1) The function $p\to K(p,q)c$ belongs to $\mathcal{P}$;

(2) $[g(\cdot),K(\cdot,q)c]_{\mathcal{P}}=[g(q),c]_{\mathcal{K}}$ for every $q\in\Omega$, every $c\in\mathcal{K}$ and every $g\in\mathcal{P}$.

If such a function $K(\cdot,\cdot)$ exists, $\mathcal{P}$ is called a reproducing kernel Pontryagin space.

As a consequence of point (2) in the above definition we have:

Lemma 5.10.3. *Let $\mathcal{K}$ be a two sided quaternionic Krein space, and let $\mathcal{P}(K)$ be a reproducing kernel Pontryagin space of $\mathcal{K}$-valued functions defined on the set Ω, and with reproducing kernel $K(p,q)$. For $p_0\in\Omega$, let $G_{p_0}g=g(p_0)$. Then*

$$\left(G_{p_0}^{[*]}c\right)(p)=K(p,p_0)c,\quad p\in\Omega \text{ and } c\in\mathcal{K}. \tag{5.52}$$

Theorem 1.2.11 still holds in the quaternionic setting, namely:

Theorem 5.10.4. *Let Ω be some set and let $\mathcal{K}$ be a two-sided quaternionic Krein space. There is a one-to-one correspondence between quaternionic reproducing kernel Pontryagin spaces of $\mathcal{K}$-valued functions on Ω and $\mathbf{B}(\mathcal{K})$-valued functions which have a finite number of negative squares on Ω.*

Proof. Let $K:\Omega\times\Omega\to\mathbf{B}(\mathcal{K})$ be a function with κ negative squares. Let us denote by $\overset{\circ}{\mathcal{P}}(K)$ the linear span of the functions of the form $p\mapsto K(p,q)a$ where $q\in\Omega$ and $a\in\mathcal{K}$. The inner product

$$[K(\cdot,q)a,K(\cdot,p)b]_{\overset{\circ}{\mathcal{P}}(K)}=[K(p,q)a,b]_{\mathcal{K}},\quad a,b\in\mathcal{K},$$

is well defined and for any $f \in \overset{\circ}{\mathscr{P}}(K)$ the following reproducing property holds

$$[f(p),b]_{\mathscr{K}} = [f(\cdot),K(\cdot,p)b]_{\overset{\circ}{\mathscr{P}}(K)}$$

for all $f \in \overset{\circ}{\mathscr{P}}(K)$. By Corollary 5.7.7, any maximal strictly negative subspace of $\overset{\circ}{\mathscr{P}}(K)$ has dimension κ. Let $\mathscr{N}_-$ be such a subspace. Since it is finite dimensional it is a reproducing kernel space. By Theorem 5.9.3 there is a finite dimensional right Hilbert space $\mathscr{G}$ and a function F from Ω into $\mathbf{B}(\mathscr{K},\mathscr{G})$ such that:

$$K_{\mathscr{N}_-}(p,q) = F(q)^*F(p).$$

Let us write

$$\overset{\circ}{\mathscr{P}}(K) = \mathscr{N}_- + \mathscr{N}_-^{[\perp]},$$

where $\mathscr{N}_-^{[\perp]}$ is a quaternionic pre-Hilbert space. Then $\mathscr{N}_-^{[\perp]}$ has reproducing kernel

$$K_{\mathscr{N}_-^{[\perp]}}(p,q) = K(p,q) - F(q)^*F(p). \tag{5.53}$$

Since $K_{\mathscr{N}_-^{[\perp]}}(p,q)$ is a positive definite kernel, the space $\mathscr{N}_-^{[\perp]}$ has a unique completion as a reproducing kernel Hilbert space with kernel (5.53). Let us denote by $\mathscr{N}_+$ this completion. Let $\mathscr{P}(K) = \mathscr{N}_+ + \mathscr{N}_-$ with the inner product

$$[f,f] = [f_+,f_+]_{\mathscr{N}_+} + [f_-,f_-]_{\mathscr{N}_-}, \quad f = f_+ + f_-.$$

It is not difficult to verify that $\mathscr{P}(K)$ is a quaternionic reproducing kernel Pontryagin space with kernel $K(p,q)$. We have to prove its uniqueness. If there exists another quaternionic reproducing kernel Pontryagin space with kernel $K(p,q)$, say $\mathscr{P}'$, then $\overset{\circ}{\mathscr{P}}(K)$, $\mathscr{N}_-$, $\mathscr{N}_-^{[\perp]}$are isometrically included in $\mathscr{P}'$. Thus $\mathscr{N}_-^{[\perp]}$ is dense in $\mathscr{P}' \ominus \mathscr{N}_-$ and its closure is isometrically included in $\mathscr{P}'$. So $\mathscr{P}(K)$ is isometrically included in $\mathscr{P}'$ and equality follows with standard arguments, see [68, 77]. □

Theorem 5.10.5. *With $\mathscr{K}$ and Ω as above, a $\mathbf{B}(\mathscr{K})$-valued function $K(p,q)$ defined on Ω has at most κ negative squares if and only if it can be written as $K(p,q) = K_+(p,q) - K_-(p,q)$ where both K_+ and K_- are positive definite, and where moreover K_- is of finite rank κ. It has exactly κ negative squares if moreover $\mathscr{P}(K_+) \cap \mathscr{P}(K_-) = \{0\}$.*

Proof. Let $K(p,q)$ be the reproducing kernel of the reproducing kernel Pontryagin space $\mathscr{P}(K)$, where $\mathscr{P}(K) = \mathscr{P}_+ \oplus \mathscr{P}_-$ is a fundamental decomposition. Let $K_\pm(p,q)$ be such that for every $q \in \Omega$, and every $a \in \mathscr{K}$ the function $p \mapsto K(p,q)a$ decomposes as

$$K(p,q)a = K_+(p,q)a - K_-(p,q)a.$$

The functions K_+, K_- are positive in Ω and they are the reproducing kernels of $\mathscr{P}_+$ and $\mathscr{P}_-$ respectively and since $\dim \mathscr{P}_- = \kappa$, the function K_- has finite rank κ by Theorem

5.9.3. To show the converse, let us assume that $K(p,q) = K_+(p,q) - K_-(p,q)$ where K_- has finite rank. Then K_- has a finite number of negative squares. Thus there exists a Pontryagin space $\mathscr{P}(K)$ and the rest follows from considering the above decomposition of K and any fundamental decomposition of $\mathscr{P}(K)$. $\square$

We conclude with a factorization theorem which uses the notion of finite number of negative squares.

Theorem 5.10.6. *Let A be a bounded right linear self-adjoint operator from the quaternionic Pontryagin space $\mathscr{P}$ into itself, which has a finite number of negative squares. Then, there exists a quaternionic Pontryagin space $\mathscr{P}_1$ with* $\operatorname{ind}\mathscr{P}_1 = \nu_-(A)$*, and a bounded right linear operator T from $\mathscr{P}$ into $\mathscr{P}_1$ such that* $\ker(T^{[*]}) = \{0\}$ *and*

$$A = T^{[*]}T.$$

Proof. The proof follows that of [72, Theorem 3.4, p. 456], slightly adapted to the present setting. Since A is Hermitian, the formula

$$[Af, Ag]_A = [Af, g]_{\mathscr{P}}$$

defines a Hermitian form on the range of A. Since $\nu_-(A) = \kappa$, there exists $N \in \mathbb{N}$ and $f_1, \ldots, f_N \in \mathscr{P}$ such that the Hermitian matrix M with (ℓ, j) entry $[Af_j, f_\ell]_{\mathscr{P}}$ has exactly κ strictly negative eigenvalues. Let $v_1, \ldots, v_\kappa$ be the corresponding eigenvectors, with strictly negative eigenvalues $\lambda_1, \ldots, \lambda_\kappa$. Theorem 4.3.10 implies, in particular, that v_j and v_k are orthogonal when $\lambda_j \neq \lambda_k$. Moreover, we can assume that vectors corresponding to a given eigenvalue are orthogonal. Then,

$$v_s^* M v_t = \lambda_t \delta_{ts}, \quad t, s = 1, \ldots, N. \tag{5.54}$$

In view of the linearity property $[fa, gb]_A = \bar{b}[f,g]_A a$ and setting

$$v_t = \begin{pmatrix} v_{t1} \\ v_{t2} \\ \vdots \\ v_{tN} \end{pmatrix}, \quad t = 1, \ldots, N,$$

we see that (5.54) can be rewritten as

$$[F_s, F_t]_A = \lambda_t \delta_{ts}, \quad \text{with} \quad F_s = \sum_{k=1}^{N} Af_k v_{sk}, \quad t, s = 1, \ldots, N.$$

The space $\mathscr{M}$ spanned by $F_1, \ldots, F_N$ is strictly negative, and it has an ortho-complement in $(\operatorname{ran} A, [\cdot,\cdot]_A)$, say $\mathscr{M}^{[\perp]}$, which is a right quaternionic pre-Hilbert space. The space $\operatorname{ran} A$ endowed with the quadratic form

$$\langle m+h, m+h \rangle_A = -[m,m]_A + [h,h]_A, \quad m \in \mathscr{M},\ h \in \mathscr{M}^{[\perp]},$$

is a pre-Hilbert space, and we denote by $\mathscr{P}_1$ its completion. We note that $\mathscr{P}_1$ is defined only up to an isomorphism of Hilbert spaces. We denote by ι the injection from ran A into $\mathscr{P}_1$ such that

$$\langle f,f\rangle_A = \langle \iota(f),\iota(f)\rangle_{\mathscr{P}_1}.$$

We consider the decomposition $\mathscr{P}_1 = \iota(\mathscr{M})\oplus\iota(\mathscr{M})^{\perp}$, and endow $\mathscr{P}_1$ with the indefinite inner product

$$[\iota(m)+h,\iota(m)+h]_{\mathscr{P}_1} = [m,m]_A + \langle h,h\rangle_{\mathscr{P}_1}.$$

See [216, Theorem 2.5, p. 20] for the similar argument in the complex case. Let us define

$$Tf = \iota(Af), \quad f\in\mathscr{P}.$$

We now prove that T is a bounded right linear operator from $\mathscr{P}$ into $\iota(\text{ran } A)\subset\mathscr{P}_1$. Indeed, let $(f_n)_{n\in\mathbb{N}}$ denote a sequence of elements in $\mathscr{P}$ converging (in the topology of $\mathscr{P}$) to $f\in\mathscr{P}$. Since ran$A$ is dense in $\mathscr{P}_1$, using Proposition 5.7.11 it is therefore enough to prove that:

$$\lim_{n\to\infty}[Tf_n,Tf_n]_{\mathscr{P}_1} = [Tf,Tf]_{\mathscr{P}_1},$$

and

$$\lim_{n\to\infty}[Tf_n,Tg]_{\mathscr{P}_1} = [Tf,Tg]_{\mathscr{P}_1}, \quad \forall g\in\mathscr{P}.$$

By definition of the inner product, the first equality amounts to

$$\lim_{n\to\infty}[Af_n,f_n]_{\mathscr{P}} = [Af,f]_{\mathscr{P}},$$

which is true since A is continuous, and similarly for the second claim. Therefore T has an adjoint operator, which is also continuous. The equalities (with $f,g\in\mathscr{P}$)

$$\begin{aligned}
[f,T^{[*]}Tg]_{\mathscr{P}} &= [Tf,Tg]_{\mathscr{P}_1}\\
&= [Tf,\iota(Ag)]_{\mathscr{P}_1}\\
&= [\iota(Af),\iota(Ag)]_{\mathscr{P}_1}\\
&= [Af,Ag]_A\\
&= [f,Ag]_{\mathscr{P}}
\end{aligned}$$

show that $T^{[*]}T = A$. $\square$

Chapter 6

Slice hyperholomorphic functions

In the first section of this chapter we give a brief survey of the theory of slice hyperholomorphic functions, and we provide just the basic results needed in the sequel. The theory is much more developed and we refer the reader to the Introduction to Part II of the present work for the references. Then, the Sections from 6.2 to 6.5 focus on the functions slice hyperholomorphic in the unit ball. We define in particular the Hardy space of the ball, Schur functions and Blaschke factors. We also study linear fractional transformations and introduce the Wiener algebra of the ball. The last two sections, namely 6.6 and 6.7 are devoted to the case of functions slice hyperholomorphic in the half-space of quaternions with real positive part. We discuss in particular the Hardy space and Blaschke factors.

6.1 The scalar case

The class of slice hyperholomorphic functions with quaternionic values was originally introduced in [189]. It is one of the possible sets of functions generalizing to the quaternionic setting the class of holomorphic functions in the complex plane. The main property possessed by the set of slice hyperholomorphic functions is that it contains polynomials and converging power series of the quaternionic variable, provided that the coefficients are written on the same side (either left or right). We present in this section an overview of these functions and we refer the reader to [144] or [188] for the missing proofs and more details.

Definition 6.1.1. Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f:\ \Omega \to \mathbb{H}$ be a real differentiable function. Let $\mathrm{I} \in \mathbb{S}$ and let f_{I} be the restriction of f to the complex plane $\mathbb{C}_{\mathrm{I}} := \mathbb{R} + \mathrm{I}\mathbb{R}$ passing through 1 and I; denote by $x + \mathrm{I}y$ an element in $\mathbb{C}_{\mathrm{I}}$.

1. We say that f is a (left) slice hyperholomorphic function (or slice regular) if, for

every $\mathrm{I}\in\mathbb{S}$, we have:

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{I}\frac{\partial}{\partial y}\right)f_{\mathrm{I}}(x+\mathrm{I}y)=0.$$

2. We say that f is right slice regular function (or right slice hyperholomorphic) if, for every $\mathrm{I}\in\mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x}f_{\mathrm{I}}(x+\mathrm{I}y)+\frac{\partial}{\partial y}f_{\mathrm{I}}(x+\mathrm{I}y)\mathrm{I}\right)=0.$$

The class of left slice hyperholomorphic functions on Ω is denoted by $\mathscr{R}(\Omega)$ or by $\mathscr{R}(\Omega,\Omega')$, if it is necessary to specify that the range is $\Omega'\subseteq\mathbb{H}$. Analogously, the class of right slice hyperholomorphic functions on Ω is denoted by $\mathscr{R}^R(\Omega)$ or by $\mathscr{R}^R(\Omega,\Omega')$.
It is immediate that $\mathscr{R}(\Omega)$ (resp. $\mathscr{R}^R(\Omega)$) is a right (resp. left) linear space on $\mathbb{H}$.
If we fix $\mathrm{I},\mathrm{J}\in\mathbb{S}$, $\mathrm{I}\perp\mathrm{J}$ then we can write the restriction f_{I} of a function f to the complex plane $\mathbb{C}_{\mathrm{I}}\ni z=x+\mathrm{I}y$ in terms of its real or complex components

$$f_{\mathrm{I}}(z)=f_0(z)+f_1(z)\mathrm{I}+f_2(z)\mathrm{J}+f_3(z)\mathrm{IJ}=F(z)+G(z)\mathrm{J}. \tag{6.1}$$

If f is slice hyperholomorphic in Ω then it is readily seen that the two functions $F,G:\Omega\cap\mathbb{C}_{\mathrm{I}}\to\mathbb{C}_{\mathrm{I}}$ are holomorphic. This property is known as splitting lemma. This splitting is highly non canonical, as it depends on the choices of I and J.

Definition 6.1.2. Let $f:\Omega\subseteq\mathbb{H}\to\mathbb{H}$ and let $p_0\in U$ be a nonreal point, $p_0=u_0+\mathrm{I}v_0$. Let f_{I} be the restriction of f to the plane $\mathbb{C}_{\mathrm{I}}$. Assume that

$$\lim_{p\to p_0,\,p\in\mathbb{C}_{\mathrm{I}}}(p-p_0)^{-1}(f_{\mathrm{I}}(p)-f_{\mathrm{I}}(p_0)) \tag{6.2}$$

exists. Then we say that f admits left slice derivative in p_0. If p_0 is real, assume that

$$\lim_{p\to p_0,\,p\in\mathbb{C}_{\mathrm{I}}}(p-p_0)^{-1}(f_{\mathrm{I}}(p)-f_{\mathrm{I}}(p_0)) \tag{6.3}$$

exists, equal to the same value, for all $\mathrm{I}\in\mathbb{S}$. Then we say that f admits a left slice derivative in p_0. If f admits a left slice derivative for every $p_0\in\Omega$, then we say that f admits a left slice derivative in Ω or, for short, that f is *left slice differentiable* in Ω.

It is possible to give an analogous definition for right slice differentiable functions: it is sufficient to multiply $(p-p_0)^{-1}$ on the right. In this case we will speak of right slice hyperhomolomorphic functions. In this paper, we will speak of slice differentiable functions or slice hyperholomorphic functions when we are considering them on the left, while we will specify if we consider the analogous notions on the right.

We have the following result:

Proposition 6.1.3. *Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f : \Omega \subseteq \mathbb{H} \to \mathbb{H}$ be a real differentiable function. Then f is slice hyperholomorphic on Ω if and only if it admits a slice derivative on Ω.*

Proof. Let $f \in \mathscr{R}(\Omega)$. Then we write its restriction to the complex plane $\mathbb{C}_{\mathrm{I}}$ as $f_{\mathrm{I}}(p) = F(p) + G(p)\mathrm{J}$ where $\mathrm{J} \in \mathbb{S}$ is orthogonal to I, p belongs to $\mathbb{C}_{\mathrm{I}}$ and $F, G : \Omega \cap \mathbb{C}_{\mathrm{I}} \to \mathbb{C}_{\mathrm{I}}$ are holomorphic functions. Let p_0 be a nonreal quaternion and let $p_0 \in \Omega \cap \mathbb{C}_{\mathrm{I}}$. Then we have

$$\begin{aligned}\lim_{p \to p_0,\, p \in \mathbb{C}_{\mathrm{I}}} &(p - p_0)^{-1}(f_{\mathrm{I}}(p) - f_{\mathrm{I}}(p_0)) = \\ &= \lim_{p \to p_0,\, p \in \mathbb{C}_{\mathrm{I}}} (p - p_0)^{-1}(F(p) + G(p)\mathrm{J} - F(p_0) - G(p_0)\mathrm{J}) \\ &= F'(p_0) + G'(p_0)\mathrm{J}\end{aligned} \tag{6.4}$$

so the limit exists and f admits slice derivative at every nonreal point $p_0 \in \Omega$. If p_0 is real then the same reasoning shows that the limit in (6.4) exists on each complex plane $\mathbb{C}_{\mathrm{I}}$. Since f is slice hyperholomorphic at p_0 we have

$$F'(p_0) + G'(p_0)\mathrm{J} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{I}\frac{\partial}{\partial y}\right)(F + G\mathrm{J})(p_0) = \frac{\partial}{\partial x} f(p_0)$$

and so the limit exists on $\mathbb{C}_{\mathrm{I}}$ for all $\mathrm{I} \in \mathbb{S}$ equal to $\frac{\partial}{\partial x} f(p_0)$.
Conversely, assume that f admits a slice derivative in Ω. By (6.2) and (6.3), the function f_{I} admits derivative on $\Omega \cap \mathbb{C}_{\mathrm{I}}$ for all $\mathrm{I} \in \mathbb{S}$. Let us write $f_{\mathrm{I}}(p) = F(p) + G(p)\mathrm{J}$, where $F, G : \Omega \cap \mathbb{C}_{\mathrm{I}} \to \mathbb{C}_{\mathrm{I}}$, $p = x + \mathrm{I}y$ and J is orthogonal to I. We deduce that both F and G admit complex derivative and thus they are in the kernel of the Cauchy Riemann operator $\partial_x + \mathrm{I}\partial_y$ for all $\mathrm{I} \in \mathbb{S}$ as well as f_{I}. Thus f is slice hyperholomorphic. □

Remark 6.1.4. The terminology of Definition 6.1.2 is consistent with the notion of slice derivative $\partial_s f$ of f, see [144], which is defined by:

$$\partial_s(f)(p) = \begin{cases} \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} f_{\mathrm{I}}(x + \mathrm{I}y) - \mathrm{I}\dfrac{\partial}{\partial y} f_{\mathrm{I}}(x + \mathrm{I}y)\right) & \text{if } p = x + \mathrm{I}y,\ y \neq 0, \\ \\ \dfrac{\partial f}{\partial x}(p) & \text{if } p = x \in \mathbb{R}. \end{cases}$$

Analogously to what happens in the complex case, for any slice hyperholomorphic function we have

$$\partial_s(f)(x + \mathrm{I}y) = \partial_x(f)(x + \mathrm{I}y).$$

It is immediate that when $f \in \mathscr{R}(\Omega)$, also $\partial_s(f) \in \mathscr{R}(\Omega)$.
Using the splitting lemma and the corresponding result in the complex case, one can also prove the following theorem (see [192]):

Theorem 6.1.5. *Let $B(0, r)$ be the ball with center at the origin and radius $r > 0$. A function $f : B(0, r) \to \mathbb{H}$ is slice hyperholomorphic if and only if it has a series expansion* *of the form*

$$f(q) = \sum_{n\geq 0} q^n \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(0)$$

converging on $B(0,r)$.

More in general, one can prove the following (see [144]):

Theorem 6.1.6. *Let* f *be a function slice hyperholomorphic in an annular domain of the form* $A = \{p \in \mathbb{H} \mid R_1 < |p| < R_2\}$, $0 < R_1 < R_2$. *Then* f *admits the following unique Laurent expansion*

$$f(p) = \sum_{m=-\infty}^{+\infty} p^m a_m \tag{6.5}$$

where $a_m = \frac{1}{m!}\frac{\partial^m}{\partial x^m} f(0)$ *if* $m \geq 0$ *and* $a_m = \frac{1}{2\pi}\int_{\partial B(0,R_1')\cap\mathbb{C}_{\mathrm{I}_p}} q^{m-1} dq_{\mathrm{I}_p} f(q)$ *if* $m < 0$.

These results can be generalized without efforts to the case of functions slice hyperholomorphic on balls with center at a real point.
The previous discussion justifies the following claim that we state without proof, see [144], [196]:

Proposition 6.1.7. *Slice hyperholomorphic functions in an open set* $\Omega \subseteq \mathbb{H}$ *are infinitely differentiable, moreover real analytic in* Ω.

An important feature of slice hyperholomorphic functions is that, on a suitable class of open sets which are described below, they can be reconstructed by knowing their values on a complex plane $\mathbb{C}_{\mathrm{I}}$ by the so-called representation formula.

Definition 6.1.8. Let Ω be a domain in $\mathbb{H}$. We say that Ω is a slice domain (s-domain for short) if $\Omega \cap \mathbb{R}$ is non empty and if $\Omega \cap \mathbb{C}_{\mathrm{I}}$ is a domain in $\mathbb{C}_{\mathrm{I}}$ for all $\mathrm{I} \in \mathbb{S}$. We say that Ω is axially symmetric if, for all $p \in \Omega$, the 2-sphere $[p]$ is contained in Ω. See Definition 4.1.3.

Theorem 6.1.9 (Identity principle)**.** *Let* $f : \Omega \to \mathbb{H}$ *be a slice hyperholomorphic function on an s-domain* Ω. *Denote by* $Z_f = \{p \in \Omega : f(p) = 0\}$ *the zero set of* f. *If there exists* $\mathrm{I} \in \mathbb{S}$ *such that* $\mathbb{C}_{\mathrm{I}} \cap Z_f$ *has an accumulation point, then* $f \equiv 0$ *on* Ω.

Proof. We have, see (6.1), $f_{\mathrm{I}}(z) = F(z) + G(z)\mathrm{J}$ and since $\mathbb{C}_{\mathrm{I}} \cap Z_f$ has an accumulation point we deduce that both F, and G are identically zero on $\Omega \cap \mathbb{C}_{\mathrm{I}}$ and, in particular f_{I} and so f vanishes on the intersection of Ω with the real axis. We now show that f vanishes at any other point $p \in \Omega$. In fact, $p \in \mathbb{C}_{\mathrm{I}_p}$ and f_{I_p} vanishes on $\Omega \cap \mathbb{C}_{\mathrm{I}_p}$ at the points on the real axis, namely f_{I_p} vanishes on a set which has an accumulation point. So f_{I_p} vanishes on $\Omega \cap \mathbb{C}_{\mathrm{I}_p}$ and thus it vanishes at p. □

Next two results were originally proved in [192]:

Theorem 6.1.10. *[Maximum modulus principle] Let* $f : \Omega \to \mathbb{H}$ *be a slice hyperholomorphic function where* Ω *is a slice domain. If* $|f|$ *has a relative maximum at a point* $p_0 \in \Omega$, *then* f *is constant in* Ω.

Proof. If f vanishes at p_0 then $|f(p)|$ has maximum value 0 and so it vanishes everywhere. So let us assume $f(p_0) \neq 0$. By possibly changing the basis of $\mathbb{H}$ we can assume that $f(p_0)$ is real. Assume that $p_0 \in \mathbb{C}_{\mathrm{I}}$ and use the splitting lemma to write $f_{\mathrm{I}}(p) = F(p) + G(p)\mathrm{J}$. Then for all $p \in \Omega \cap \mathbb{C}_{\mathrm{I}}$ we have

$$|F(p)|^2 = |f_{\mathrm{I}}(p)|^2 \geq |f_{\mathrm{I}}(z)|^2 = |F(z)|^2 + |G(z)|^2 \geq |F(z)|^2.$$

So $|F|$ has a relative maximum at p_0 and so one can apply the maximum modulus principle for holomorphic functions and thus F is constant, so $F \equiv f(p_0)$ from which one deduces $G(z) = 0$ for $z \in \Omega \cap \mathbb{C}_{\mathrm{I}}$. By the identity principle $f \equiv f(p_0)$ in Ω. □

The analog of the classical Schwarz lemma holds in this framework:

Lemma 6.1.11. *(Schwarz) Let $f : \mathbb{B} \to \mathbb{B}$, be a slice hyperholomorphic function such that $f(0) = 0$. Then, for every $p \in \mathbb{B}$,*

$$|f(p)| \leq |p| \tag{6.6}$$

and

$$|\partial_s f(0)| \leq 1. \tag{6.7}$$

Moreover, for $p \neq 0$, equality holds in (6.6) *and* (6.7) *if and only if $f(p) = pa$ for some $a \in \partial\mathbb{B}$.*

Proof. By hypothesis f admits power series expansion of the form $f(p) = \sum_{n=1}^{\infty} p^n a_n$ since $f(0) = 0$. The function

$$g(p) = p^{-1} f(p) = \sum_{n \geq 0} p^n a_{n+1}$$

is slice hyperholomorphic on $\mathbb{B}$ since its radius of convergence is the same as the radius of convergence of f. Let $p \in \mathbb{B}$, be such that $|p| < r < 1$. The maximum principle implies that

$$|g(p)| \leq \sup_{|w|=r} |g(w)| = \sup_{|w|=r} \frac{|f(w)|}{|w|} \leq \frac{1}{r}.$$

Letting $r \to 1$ one obtains that $|g(p)| \leq 1$ on $\mathbb{B}$ and so $|p^{-1} f(p)| \leq 1$ from which the first assertion follows and since $\partial_s f(0) = g(0)$, we immediately have that $|\partial_s f(0)| \leq 1$.
We now assume that equality holds in (6.6) for some $p \in \mathbb{B}$. Then for such p, we have

$$\frac{|f(p)|}{|p|} = |g(p)| = 1$$

and by the maximum principle we obtain that $g(p) = a$ for all $p \in \mathbb{B}$, for a suitable $a \in \partial\mathbb{B}$. Therefore we conclude that $p^{-1} f(p) = a$, and thus $f(p) = pa$. Similarly, if $|\partial_s f(0)| = 1$, we obtain that $|g(0)| = 1$ and the thesis follows. □

The following formula is a crucial tool when dealing with slice hyperholomorphic functions (see [120, 131]):

Theorem 6.1.12 (Representation formula)**.** *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain. Let f be a left slice hyperholomorphic function on $\Omega \subseteq \mathbb{H}$. Then the following equality holds for all $p = x + \mathrm{J}y \in \Omega$:*

$$f(p) = f(x+\mathrm{J}y) = \frac{1}{2}\Big[f(z) + f(\bar{z})\Big] + \frac{1}{2}\mathrm{J}\,\mathrm{I}\Big[f(\bar{z}) - f(z)\Big], \tag{6.8}$$

where $z := x + \mathrm{I}y$, $\bar{z} := x - \mathrm{I}y \in \Omega \cap \mathbb{C}_\mathrm{I}$. Let f be a right slice hyperholomorphic function on $\Omega \subseteq \mathbb{H}$. Then the following equality holds for all $p = x + \mathrm{J}y \in \Omega$:

$$f(x+\mathrm{J}y) = \frac{1}{2}\Big[f(z) + f(\bar{z})\Big] + \frac{1}{2}\Big[f(\bar{z}) - f(z)\Big]\mathrm{I}\,\mathrm{J}. \tag{6.9}$$

Proof. We prove the result in the case of left slice hyperholomorphic functions as the other case will follow similarly. If $y = 0$ is real, the formula trivially holds, so let us assume $y \neq 0$. Let us define the following function $\psi : \Omega \to \mathbb{H}$

$$\psi(x+\mathrm{J}y) = \frac{1}{2}\Big[f(x+\mathrm{I}y) + f(x-\mathrm{I}y) + \mathrm{J}\,\mathrm{I}[f(x-\mathrm{I}y) - f(x+\mathrm{I}y)]\Big].$$

When $\mathrm{I} = \mathrm{J}$ we have

$$\psi_\mathrm{I}(x+\mathrm{I}y) = \psi(x+\mathrm{I}y) = f(x+\mathrm{I}y) = f_\mathrm{I}(x+\mathrm{I}y).$$

We now prove that ψ is slice hyperholomorphic on Ω, so that the first part of the assertion will follow from the identity principle. Let us compute $\frac{\partial}{\partial x}\psi(x+\mathrm{I}y)$ and $\frac{\partial}{\partial y}\psi(x+\mathrm{I}y)$ where we will use the fact that f is slice hyperholomorphic on Ω:

$$\begin{aligned}
2\frac{\partial}{\partial x}\psi(x+\mathrm{J}y) &= \frac{\partial}{\partial x}\Big[f(x+\mathrm{I}y) + f(x-\mathrm{I}y) + \mathrm{J}\,\mathrm{I}[f(x-\mathrm{I}y) - f(x+\mathrm{I}y)]\Big] \\
&= \frac{\partial}{\partial x}f(x+\mathrm{I}y) + \frac{\partial}{\partial x}f(x-\mathrm{I}y) + \mathrm{J}\,\mathrm{I}[\frac{\partial}{\partial x}f(x-\mathrm{I}y) - \frac{\partial}{\partial x}f(x+\mathrm{I}y)] \\
&= -\mathrm{I}\frac{\partial}{\partial y}f(x+\mathrm{I}y) + \mathrm{I}\frac{\partial}{\partial y}f(x-\mathrm{I}y) + \mathrm{J}\,\mathrm{I}[\mathrm{I}\frac{\partial}{\partial y}f(x-\mathrm{I}y) + \mathrm{I}\frac{\partial}{\partial y}f(x+\mathrm{I}y)] \\
&= -\mathrm{I}\frac{\partial}{\partial y}f(x+\mathrm{I}y) + \mathrm{I}\frac{\partial}{\partial y}f(x-\mathrm{I}y) - \mathrm{J}[\frac{\partial}{\partial y}f(x-\mathrm{I}y) + \frac{\partial}{\partial y}f(x+\mathrm{I}y)] \\
&= -\mathrm{J}\frac{\partial}{\partial y}\Big[f(x+\mathrm{I}y) + f(x-\mathrm{I}y) + \mathrm{J}\,\mathrm{I}[f(x-yJ) - f(x+yJ)]\Big] = -2\mathrm{J}\frac{\partial}{\partial y}\psi(x+\mathrm{J}y)
\end{aligned}$$

i.e.

$$\frac{1}{2}(\frac{\partial}{\partial x} + \mathrm{J}\frac{\partial}{\partial y})\psi(x+\mathrm{J}y) = 0. \tag{6.10}$$

□

Corollary 6.1.13. *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain. Let f be a left slice hyperholomorphic function on $\Omega \subseteq \mathbb{H}$. Then $f(x+\mathrm{J}y) = \alpha(x,y) + \mathrm{J}\beta(x,y)$ where $\alpha(x,-y) = \alpha(x,y)$, $\beta(x,-y) = -\beta(x,y)$ and the pair α, β satisfies the Cauchy-Riemann system.*

Proof. It is sufficient to set

$$\alpha(x,y) = \frac{1}{2}\Big[f(x+\mathrm{I}y) + f(x-\mathrm{I}y)\Big]$$

and

$$\beta(x,y) = \frac{1}{2}\mathrm{I}\Big[f(x-\mathrm{I}y) - f(x+\mathrm{I}y)\Big].$$

Using the Representation Formula, one shows that $f(x+\mathrm{I}y)+f(x-\mathrm{I}y) = f(x+\mathrm{K}y)+f(x-\mathrm{K}y)$ and similarly that $f(x-\mathrm{I}y)-f(x+\mathrm{I}y) = f(x-\mathrm{K}y)-f(x+\mathrm{K}y)$ where K is any other imaginary unit in $\mathbb{S}$. The other properties follow. □

Remark 6.1.14. The class of functions defined on axially symmetric open set $\Omega \subseteq \mathbb{H}$ which are of the form $f(x+\mathrm{J}y) = \alpha(x,y)+\mathrm{J}\beta(x,y)$ where $\alpha(x,-y) = \alpha(x,y)$, $\beta(x,-y) = -\beta(x,y)$ and the pair α, β satisfies the Cauchy-Riemann system, is the class of the so-called slice regular functions introduced and studied by Ghiloni and Perotti in [196]. On axially symmetric s-domain they coincide with the slice hyperholomorphic functions introduced in this section. Note that in [196] the authors treat a more general case, namely the case of functions with values in a real alternative algebra.

The Representation Formula allows to extend any function $f:\ \widetilde{\Omega} \subseteq \mathbb{C}_\mathrm{I} \to \mathbb{H}$, defined on an s-domain $\widetilde{\Omega}$ symmetric with respect to the real axis and in the kernel of the corresponding Cauchy-Riemann operator, to a function $f:\ \Omega \subseteq \mathbb{H} \to \mathbb{H}$ slice hyperholomorphic where Ω is the smallest axially symmetric open set in $\mathbb{H}$ containing $\widetilde{\Omega}$. Using the above notations, the extension is obtained by means of the *extension operator*

$$\mathrm{ext}(f)(p) := \frac{1}{2}\Big[f(z)+f(\bar{z})\Big] + \frac{1}{2}\mathrm{J}\,\mathrm{I}\Big[f(\bar{z})-f(z)\Big], \quad z,\bar{z} \in \widetilde{\Omega} \cap \mathbb{C}_\mathrm{I},\ p \in \Omega. \tag{6.11}$$

When a function f satisfies

$$\frac{\partial}{\partial x} f_\mathrm{I} + \frac{\partial}{\partial y} f_\mathrm{I}\mathrm{I} = 0$$

it is possible to extend it to a right slice hyperholomorphic function using the formula

$$\mathrm{ext}(f)(p) := \frac{1}{2}\Big[f(z)+f(\bar{z})\Big] + \frac{1}{2}\Big[f(\bar{z})-f(z)\Big]\mathrm{I}\,\mathrm{J}, \quad z,\bar{z} \in \widetilde{\Omega} \cap \mathbb{C}_\mathrm{I},\ p \in \Omega. \tag{6.12}$$

Now we briefly discuss the composition of two slice hyperholomorphic functions which, in general, does not give a slice hyperholomorphic function. Consider, for example, the functions $f(p) = p^2$ and $g(p) = p - p_0$. Then $(f \circ g)(p) = (p-p_0)^2$ is not slice hyperholomorphic if $p_0 \in \mathbb{H} \setminus \mathbb{R}$. However, we can guarantee that the composition $f \circ g$ is slice hyperholomorphic when g belongs to a suitable subclass of functions that we define below.

Definition 6.1.15. Let Ω be an axially symmetric open set in $\mathbb{H}$. We say that a slice hyperholomorphic function $f : \Omega \to \mathbb{H}$ is quaternionic intrinsic if $f : \Omega \cap \mathbb{C}_\mathrm{I} \to \mathbb{C}_\mathrm{I}$ for all $\mathrm{I} \in \mathbb{S}$. We denote this class of function by $\mathscr{N}(\Omega)$.

Remark 6.1.16. Another characterization of quaternionic intrinsic functions is that, for every $p \in \Omega$, where Ω is an axially symmetric open set in $\mathbb{H}$, they satisfy $f(\bar{p}) = \overline{f(p)}$. Moreover, all quaternionic power series with real coefficients are examples of quaternionic intrinsic functions. In particular, all transcendental functions like exponential, logarithm, sine, cosine are of this type.

Moreover we have:

Proposition 6.1.17. *Let Ω, Ω' be open sets in $\mathbb{H}$ and let Ω be axially symmetric. Let $g \in \mathcal{N}(\Omega)$, $f \in \mathcal{R}(\Omega')$ and $g(\Omega) \subseteq \Omega'$. Then $f \circ g \in \mathcal{R}(\Omega)$.*

Proof. Since $g \in \mathcal{N}(\Omega)$, by Corollary 6.1.13, we have $g(x+\mathrm{I}y) = \alpha(x,y) + \mathrm{I}\beta(x,y)$ where α, β are real valued functions. Then the statement follows by direct computation. □

Given two slice hyperholomorphic functions f, g, they can be multiplied using a binary operation called the $\star$-product, such that $f \star g$ is a slice hyperholomorphic function. Similarly, given two right slice hyperholomorphic functions, we can define their $\star$-product. When it is necessary to distinguish between them we will write $\star_l$ or $\star_r$ according to the fact that we are using the left or the right slice hyperholomorphic product. When there is no subscript, we will mean that we are considering the left $\star$-product.

Definition 6.1.18. Let $f, g \in \mathcal{R}(\Omega)$ and let $f_{\mathrm{I}}(z) = F(z) + G(z)\mathrm{J}$, $g_{\mathrm{I}}(z) = H(z) + K(z)\mathrm{J}$ be their restrictions to the complex plane $\mathbb{C}_{\mathrm{I}}$. Assume that Ω is an axially symmetric s-domain. We define the function $f_{\mathrm{I}} \star g_{\mathrm{I}} : \Omega \cap \mathbb{C}_{\mathrm{I}} \to \mathbb{H}$ as

$$(f_{\mathrm{I}} \star g_{\mathrm{I}})(z) = [F(z)H(z) - G(z)\overline{K(\bar{z})}] + [F(z)K(z) + G(z)\overline{H(\bar{z})}]\mathrm{J}, \tag{6.13}$$

and

$$(f \star g)(q) = \mathrm{ext}(f_{\mathrm{I}} \star g_{\mathrm{I}})(q).$$

If f, g are right slice hyperholomorphic, then with the above notations we have $f_{\mathrm{I}}(z) = F(z) + \mathrm{J}G(z)$, $g_{\mathrm{I}}(z) = H(z) + \mathrm{J}L(z)$ and

$$\begin{aligned}(f_{\mathrm{I}} \star_r g_{\mathrm{I}})(z) &:= (F(z) + \mathrm{J}G(z)) \star_r (H(z) + \mathrm{J}L(z)) \\ &= (F(z)H(z) - \overline{G(\bar{z})}L(z)) + \mathrm{J}(G(z)H(z) + \overline{F(\bar{z})}L(z)),\end{aligned} \tag{6.14}$$

and $f \star_r g = \mathrm{ext}(f_{\mathrm{I}} \star_r g_{\mathrm{I}})$.

We note that $(f_{\mathrm{I}} \star g_{\mathrm{I}})(z)$ (resp. $(f_{\mathrm{I}} \star_r g_{\mathrm{I}})(z)$) is obviously a holomorphic map and hence we can consider its unique slice hyperholomorphic extension (resp. right slice hyperholomorphic extension) to Ω.

Remark 6.1.19. Let $f(p) = \sum_{k=0}^{\infty} p^k f_k$, $g(p) = \sum_{k=0}^{\infty} p^k g_k$ be two sice hyperholomorphic functions in $B(0,r)$. Their $\star$-product coincides with the classical convolution multiplication

$$(f \star g)(p) = \sum_{k=0}^{\infty} p^k \cdot \left(\sum_{r=0}^{k} f_r g_{k-r} \right) \tag{6.15}$$

used e.g. in [175].

Remark 6.1.20. Let $f(p)=\sum_{k=0}^{\infty}p^k a_k$ and $g(p)=\sum_{k=0}^{\infty}p^k b_k$. If f has real coefficients then $f\star g=g\star f=f\cdot g$.

Pointwise multiplication and $\star$-multiplication are different, but they can be related as in the following result, originally proved in [120]:

Proposition 6.1.21. *Let $\Omega\subseteq\mathbb{H}$ be an axially symmetric s-domain, $f,g:\Omega\to\mathbb{H}$ be slice hyperholomorphic functions. If $f(p)\neq 0$ then*

$$(f\star g)(p)=f(p)g(f(p)^{-1}pf(p)), \tag{6.16}$$

while if $f(p)=0$ then $(f\star g)(p)=0$.

Proof. Let $\mathrm{I}\in\mathbb{S}$ be any element in $\mathbb{S}$, $q=x+\mathrm{I}y$. If $f(x+\mathrm{I}y)=0$ the conclusion follows so let us assume $f(x+\mathrm{I}y)\neq 0$. Then

$$f(x+\mathrm{I}y)^{-1}(x+\mathrm{I}y)f(x+\mathrm{I}y)=x+yf(x+\mathrm{I}y)^{-1}\mathrm{I}f(x+\mathrm{I}y)$$

and $f(x+\mathrm{I}y)^{-1}\mathrm{I}f(x+\mathrm{I}y)\in\mathbb{S}$. By applying the representation formula (6.1.12) to the function g, we have (with obvious notations for the derivatives)

$$\begin{aligned}g(f(q)^{-1}qf(q))&=g(x+yf(x+\mathrm{I}y)^{-1}\mathrm{I}f(x+\mathrm{I}y))\\&=\frac{1}{2}\{g(x+\mathrm{I}y)+g(x-\mathrm{I}y)-f(x+\mathrm{I}y)^{-1}\mathrm{I}f(x+\mathrm{I}y)[\mathrm{I}g(x+\mathrm{I}y)-\mathrm{I}g(x-\mathrm{I}y).]\}\end{aligned}$$

Let us set

$$\begin{aligned}\psi(q):&=f(q)g(f(q)^{-1}qf(q))\\&=\frac{1}{2}\{f(x+\mathrm{I}y)[g(x+\mathrm{I}y)+g(x-\mathrm{I}y)]-\mathrm{I}f(x+\mathrm{I}y)[\mathrm{I}g(x+\mathrm{I}y)-\mathrm{I}g(x-\mathrm{I}y)]\}.\end{aligned}$$

If we prove that the function $\psi(q)$ is regular, then our assertion will follow by the Identity principle. In fact formula (6.16) holds on a small open ball of Ω centered at a real point where the functions admit a power series expansion, see the proof of Proposition 4.2.1. We have:

$$\begin{aligned}\frac{\partial}{\partial x}\psi(x+\mathrm{I}y)&=\frac{1}{2}\{f_x(x+\mathrm{I}y)[g(x+\mathrm{I}y)+g(x-\mathrm{I}y)]-\mathrm{I}f_x(x+\mathrm{I}y)[\mathrm{I}g(x+\mathrm{I}y)-\mathrm{I}g(x-\mathrm{I}y)]\}\\&\quad+\frac{1}{2}\{f(x+\mathrm{I}y)[g_x(x+\mathrm{I}y)+g_x(x-\mathrm{I}y)]-\mathrm{I}f(x+\mathrm{I}y)[\mathrm{I}g_x(x+\mathrm{I}y)-\mathrm{I}g_x(x-\mathrm{I}y)]\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{I}\frac{\partial}{\partial y}\psi(x+\mathrm{I}y)&=\frac{1}{2}\{\mathrm{I}f_y(x+\mathrm{I}y)[g(x+\mathrm{I}y)+g(x-\mathrm{I}y)]+f_y(x+\mathrm{I}y)[\mathrm{I}g(x+\mathrm{I}y)-\mathrm{I}g(x-\mathrm{I}y)]\}\\&\quad+\frac{1}{2}\{\mathrm{I}f(x+\mathrm{I}y)[g_y(x+\mathrm{I}y)+g_y(x-\mathrm{I}y)]+f(x+\mathrm{I}y)[\mathrm{I}g_y(x+yI)-\mathrm{I}g_y(x-\mathrm{I}y)]\}.\end{aligned}$$

Using the fact that

$$f_x(x+\mathrm{I}y)+\mathrm{I}f_y(x+\mathrm{I}y)=g_x(x+\mathrm{I}y)+\mathrm{I}g_y(x+\mathrm{I}y)=g_x(x-\mathrm{I}y)-\mathrm{I}g_y(x-\mathrm{I}y)=0,$$

we obtain that $(\frac{\partial}{\partial x}+\mathrm{I}\frac{\partial}{\partial y})\psi(x+\mathrm{I}y)=0$ and the statement now follows from the arbitrarity of I. □

Remark 6.1.22. In the sequel, we will consider functions $k(p,q)$ left slice hyperholomorphic in p and right slice hyperholomorphic in $\bar{q}$. When taking the $\star$-product of a function $f(p)$ slice hyperholomorphic in the variable p with such a function $k(p,q)$, we will write $f(p)\star k(p,q)$ meaning that the $\star$-product is taken with respect to the variable p; similarly, the $\star_r$-product of $k(p,q)$ with functions right slice hyperholomorphic in the variable $\bar{q}$ is always taken with respect to $\bar{q}$.

Let Ω be an axially symmetric open set. If f is left slice hyperholomorphic in $q\in\Omega$ then $\overline{f(q)}$ is right slice hyperholomorphic in $\bar{q}$. This fact follows immediately from $(\partial_x+\mathrm{I}\partial_y)f_{\mathrm{I}}(x+\mathrm{I}y)=0$, since by conjugation we get $\overline{f_{\mathrm{I}}(x+\mathrm{I}y)}(\partial_x-\mathrm{I}\partial_y)=0$ for all $\mathrm{I}\in\mathbb{S}$.

Lemma 6.1.23. *Let Ω be an axially symmetric s-domain and let $f,g:\Omega\to\mathbb{H}$ be two left slice hyperholomorphic functions. Then*

$$\overline{f\star_l g}=\overline{g}\star_r\overline{f},$$

where $\star_l$, $\star_r$ are the left and right $\star$-products with respect to q and $\bar{q}$, respectively.

Proof. Let $f_{\mathrm{I}}(z)=F(z)+G(z)\mathrm{J}$, $g_{\mathrm{I}}(z)=H(z)+L(z)\mathrm{J}$ be the restrictions of f and g to the complex plane $\mathbb{C}_{\mathrm{I}}$, respectively. The functions F, G, H, L are holomorphic functions of the variable $z\in\Omega\cap\mathbb{C}_{\mathrm{I}}$ which exist by the splitting lemma and J is an element in the sphere $\mathbb{S}$ orthogonal to I. The $\star_r$-product of the two right slice hyperholomorphic functions $\overline{g}$ and $\overline{f}$ in the variable $\overline{q}$ is defined as the unique right slice hyperholomorphic function whose restriction to complex plane $\mathbb{C}_{\mathrm{I}}$ is given by

$$(\overline{H(z)}-\mathrm{J}\,\overline{L(z)})\star_r(\overline{F(z)}-\mathrm{J}\,\overline{G(z)}):=(\overline{H(z)}\,\overline{F(z)}-L(\bar{z})\overline{G(z)})-\mathrm{J}(\overline{L(z)}\,\overline{F(z)}+H(\bar{z})\overline{G(z)}).$$

Thus, comparing with (6.13), it is clear that

$$\overline{f_{\mathrm{I}}\star_l g_{\mathrm{I}}}=\overline{g_{\mathrm{I}}}\star_r\overline{f_{\mathrm{I}}},$$

and the statement follows by taking the unique right slice hyperholomorphic extension. □

Definition 6.1.24. Let $f\in\mathscr{R}(\Omega)$ and let $f_{\mathrm{I}}(z)=F(z)+G(z)\mathrm{J}$. We define

$$f^c_{\mathrm{I}}(z)=\overline{F(\bar{z})}-G(z)\mathrm{J}$$

and we set $f^c(p)=\mathrm{ext}(f^c)(p)$. The function f^c is called the slice hyperholomorphic conjugate of f. We then define

$$f^s_{\mathrm{I}}(z)=(f_{\mathrm{I}}\star f^c_{\mathrm{I}})(z)=(f^c_{\mathrm{I}}\star f_{\mathrm{I}})(z)=F(z)\overline{F(\bar{z})}+G(z)\overline{G(\bar{z})} \tag{6.17}$$

and the function $f^s(p)=\mathrm{ext}(f^s)(p)$ is called the symmetrization (or normal form) of f.

It is not difficult to show that if $f\in\mathscr{R}(\Omega)$ then $f^s\in\mathscr{N}(\Omega)$.
It is also useful to note that when $f(p)=\sum_{k=0}^{\infty}p^k f_k$ then $f^c(p)=\sum_{k=0}^{\infty}p^k\overline{f_k}$.
In this case

$$(f\star f^c)(p)=\sum_{n=0}^{\infty}p^n c_n,\qquad c_n=\sum_{r=0}^{n}f_r\overline{f_{n-r}} \tag{6.18}$$

where the coefficients c_n are real numbers. In fact

$$\overline{c_n} = \overline{\sum_{r=0}^{n} f_r \overline{f_{n-r}}} = \sum_{r=0}^{n} f_{n-r} \overline{f_r} = c_n.$$

Proposition 6.1.25. *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f, g \in \mathscr{R}(\Omega)$. Then:*

(1) $(f^c)^c = f$;

(2) $(f \star g)^c = g^c \star f^c$.

Proof. Equality (1) follows from trivial computations, so we prove (2). We show that the two functions $(f \star g)^c$ and $g^c \star f^c$ coincide on a complex plane (so the needed equality follows from the identity principle). Using the notation introduced above, let us write $f_{\mathrm{I}}(z) = F(z) + G(z)\mathrm{J}$ and $g_{\mathrm{I}}(z) = H(z) + L(z)\mathrm{J}$. We have

$$(f \star g)_{\mathrm{I}}(z) = f_{\mathrm{I}}(z) \star g_{\mathrm{I}}(z) = (F(z)H(z) - G(z)\overline{L(\bar{z})}) + (F(z)L(z) + G(z)\overline{H(\bar{z})})\mathrm{J}$$

so, by definition of $(f \star g)^c$, we have

$$(f \star g)^c_{\mathrm{I}}(z) = (\overline{H(\bar{z})}\,\overline{F(\bar{z})} - L(z)\overline{G(\bar{z})}) - (F(z)L(z) + G(z)\overline{H(\bar{z})})\mathrm{J}$$

and

$$\begin{aligned}(g^c \star f^c)_{\mathrm{I}}(z) &= (\overline{H(\bar{z})} - L(z)\mathrm{J}) \star (\overline{F(\bar{z})} - G(z)\mathrm{J}) \\ &= (\overline{H(\bar{z})}\,\overline{F(\bar{z})} - L(z)\overline{G(\bar{z})}) - (\overline{H(\bar{z})}G(z) + L(z)F(z))\mathrm{J}\end{aligned}$$

the two expressions coincide since the functions F, G, H, L are $\mathbb{C}_{\mathrm{I}}$-valued and thus they commute. □

The inverse of a function $f \in \mathscr{R}(\Omega)$ with respect to the $\star$-product can be computed as follows:

Definition 6.1.26. Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f : \Omega \to \mathbb{H}$ be a slice hyperholomorphic function. We define the function $f^{-\star}$ as

$$f^{-\star}(p) := (f^s(p))^{-1} f^c(p).$$

If $f : \Omega \to \mathbb{H}$ is a right slice hyperholomorphic function, we define the function $f^{-\star_r}$ as

$$f^{-\star}(q) := f^c(q)(f^s(q))^{-1}.$$

We have the following properties: (analogous properties hold for the $\star_r$-product):

(1) $(f^{-\star})^{-\star} = f$;

(2) $(f \star g)^{-\star} = g^{-\star} \star f^{-\star}$.

The function $1-pq$, where $p,q\in\mathbb{H}$ can be considered as a left slice hyperholomorphic function in the variable p and right slice hyperholomorphic function in the variable q. Observe that its $\star$-inverse with respect to p is:

$$(1-pq)^{-\star}=(1-2\mathrm{Re}(q)p+p^2|q|^2)^{-1}(1-p\bar{q}). \tag{6.19}$$

Proposition 6.1.27. *The function defined in* (6.19) *is slice hyperholomorphic in p and right slice hyperholomorphic in q.*

Proof. The function $(1-pq)^{-\star}$ is slice hyperholomorphic in p by construction. To show the second assertion, let us set $q=x+\mathrm{I}y$. We have:

$$\begin{aligned}
&\frac{\partial}{\partial x}(1-2px+p^2(x^2+y^2))^{-1}(1-p(x-\mathrm{I}y))\\
&+\frac{\partial}{\partial y}(1-2px+p^2(x^2+y^2))^{-1}(1-p(x-\mathrm{I}y))\mathrm{I}=\\
&=(1-2px+p^2(x^2+y^2))^{-2}(2p-2p^2x)(1-p(x-\mathrm{I}y))-(1-2px+p^2(x^2+y^2))^{-1}p\\
&-(1-2px+p^2(x^2+y^2))^{-2}(2yp^2)(1-p(x-\mathrm{I}y))\mathrm{I}-(1-2px+p^2(x^2+y^2))^{-1}p=\\
&=(1-2px+p^2(x^2+y^2))^{-2}[p(2-2xp)(1-p(x-\mathrm{I}y))-(1-2px+p^2(x^2+y^2))p\\
&-2yp^2(1-p(x-\mathrm{I}y))\mathrm{I}-(1-2px+p^2(x^2+y^2))p]=\\
&=(1-2px+p^2(x^2+y^2))^{-2}[(2p-2xp^2)(1-p(x-\mathrm{I}y))\\
&-2yp^2(1-p(x-\mathrm{I}y))\mathrm{I}-2(1-2px+p^2(x^2+y^2))p]=0.
\end{aligned} \tag{6.20}$$

□

The following result will be useful to justify a notation we will use in the sequel:

Proposition 6.1.28. *For any $p,q\in\mathbb{H}$ such that p,q^{-1} (and similarly, p^{-1},q) do not belong to the same sphere the following equality holds*

$$(1-\bar{p}q)(1-2\mathrm{Re}(p)q+|p|^2q^2)^{-1}=(1-2\mathrm{Re}(q)p+|q|^2p^2)^{-1}(1-p\bar{q}),$$

in other words

$$(1-pq)^{-\star_r}=(1-pq)^{-\star} \tag{6.21}$$

where the $\star_r$-inverse is computed in the variable q and the $\star$-inverse is computed in the variable p.

Proof. The proof follows by proving, with direct computations, that

$$(1-2\mathrm{Re}(q)p+|q|^2p^2)(1-\bar{p}q)=(1-p\bar{q})(1-2\mathrm{Re}(p)q+|p|^2q^2).$$

□

Remark 6.1.29. Another way to prove Proposition 6.1.28 is to observe that the function

$$(1-pq)^{-\star_r} = (1-\bar{p}q)(1-2\mathrm{Re}(p)q+|p|^2q^2)^{-1}$$

is right slice hyperholomorphic in q by construction and computations similar to those in the proof of Proposition 6.1.27 show that it is left slice hyperholomorphic in p. For this reason, by the identity principle, as a slice hyperholomorphic function in p, it coincides with the function in (6.19), i.e.

$$(1-pq)^{-\star} = (1-pq)^{-\star_r}, \tag{6.22}$$

(where the $\star$-inverse is computed with respect to p and the $\star_r$-inverse is computed with respect to q).

Remark 6.1.30. Throughout the book, when dealing with functions in two variables p, q they will be (left) slice hyperholomorphic in p and right slice hyperholomorphic (or anti slice hyperholomorphic) in q.

Below we will use the notation $\overline{\partial}_{\mathrm{I}}$ for the Cauchy-Riemann operator on the plane $\mathbb{C}_{\mathrm{I}}$.

Lemma 6.1.31. *Let f, g be quaternionic valued, continuously (real) differentiable functions on an open set $\Omega\cap\mathbb{C}_{\mathrm{I}}$ of the complex plane $\mathbb{C}_{\mathrm{I}}$. Then, for every open $W\subset\Omega\cap\mathbb{C}_{\mathrm{I}}$ whose boundary consists of a finite number of piecewise smooth, closed curves, we have*

$$\int_{\partial W} g\,ds_{\mathrm{I}}f = 2\int_W ((g\overline{\partial}_{\mathrm{I}})f+g(\overline{\partial}_{\mathrm{I}}f))d\sigma,$$

where $s=x+\mathrm{I}y\in\mathbb{C}_{\mathrm{I}}$, $ds_{\mathrm{I}}=-\mathrm{I}ds$ and $d\sigma=dx\wedge dy$.

Proof. Let $\mathrm{J}\in\mathbb{S}$ be orthogonal to I and let us consider $\mathrm{I},\mathrm{J},\mathrm{IJ}$ as a basis of $\mathbb{H}$. Then we write $f(s)=f_0(s)+f_1(s)\mathrm{J}$, $g(s)=g_0(s)+\mathrm{J}g_1(s)$ where $f_i(s),g_i(s)$, $i=0,1$ are suitable $\mathbb{C}_{\mathrm{I}}$-valued functions. Stokes' theorem applied to these complex functions gives

$$\begin{aligned}
&\int_{\partial W} g\,ds_{\mathrm{I}}f = \int_{\partial W}(g_0(s)+\mathrm{J}g_1(s))ds_{\mathrm{I}}(f_0(s)+f_1(s)\mathrm{J})\\
&=\int_{\partial W} g_0f_0ds_{\mathrm{I}}+g_0f_1ds_{\mathrm{I}}\mathrm{J}+\mathrm{J}g_1f_0ds_{\mathrm{I}}+\mathrm{J}g_1f_1ds_{\mathrm{I}}\mathrm{J}\\
&=\int_W \partial_x(g_0f_0)d\sigma+\partial_y(g_0f_0)\mathrm{I}d\sigma+\partial_x(g_0f_1)d\sigma\mathrm{J}+\partial_y(g_0f_1)\mathrm{I}d\sigma\mathrm{J}+\\
&+\mathrm{J}\partial_x(g_1f_0)d\sigma+\mathrm{J}\partial_y(g_1f_0)\mathrm{I}d\sigma+\mathrm{J}\partial_x(g_1f_1)d\sigma\mathrm{J}+\mathrm{J}\partial_y(g_1f_1)\mathrm{I}d\sigma\mathrm{J}.
\end{aligned}$$

Moreover, with direct computations we have

$$\begin{aligned}
&\partial_x(g_0f_0)+\partial_y(g_0f_0)\mathrm{I}+\mathrm{J}\partial_x(g_1f_0)+\mathrm{J}\partial_y(g_1f_0)\mathrm{I}\\
&=(\partial_x(g_0)+\partial_y(g_0)\mathrm{I})f_0+\mathrm{J}(\partial_x(g_1)+\partial_y(g_1)\mathrm{I})f_0\\
&+g_0(\partial_x(f_0)+\partial_y(f_0)\mathrm{I})+\mathrm{J}g_1(\partial_x(f_0)+\partial_y(f_0)\mathrm{I})\\
&=(g_0\overline{\partial}_{\mathrm{I}}+\mathrm{J}g_1\overline{\partial}_{\mathrm{I}})f_0\\
&+(g_0+\mathrm{J}g_1)(\overline{\partial}_{\mathrm{I}}f_0)=2(g\overline{\partial}_{\mathrm{I}})f_0+2g(\overline{\partial}_{\mathrm{I}}f_0)
\end{aligned}$$

and similarly

$$\partial_x(g_0f_1)\mathrm{J}+\partial_y(g_0f_1)\mathrm{IJ}+\mathrm{J}\partial_x(g_1f_1)\mathrm{J}+\mathrm{J}\partial_y(g_1f_1)\mathrm{IJ}=2(g\overline{\partial}_{\mathrm{I}})f_1\mathrm{J}+2g(\overline{\partial}_{\mathrm{I}}f_1)\mathrm{J}.$$

Therefore we have

$$\int_{\partial W} g ds_{\mathrm{I}} f = 2\int_W (g\overline{\partial}_{\mathrm{I}})f_0 d\sigma + g(\overline{\partial}_{\mathrm{I}}f_0)d\sigma + (g\overline{\partial}_{\mathrm{I}})f_1\mathrm{J}d\sigma + g(\overline{\partial}_{\mathrm{I}}f_1)\mathrm{J}d\sigma$$
$$= 2\int_W ((g\overline{\partial}_{\mathrm{I}})f + g(\overline{\partial}_{\mathrm{I}}f))d\sigma.$$

□

A consequence of the previous Lemma is the following:

Corollary 6.1.32. *Let f and g be a left slice hyperholomorphic and a right slice hyperholomorphic function, respectively, on an open set $\Omega\subset\mathbb{H}$. For any $\mathrm{I}\in\mathbb{S}$ and every open $W\subset\Omega\cap\mathbb{C}_{\mathrm{I}}$ whose boundary consists of a finite number of piecewise smooth, closed curves, we have*

$$\int_{\partial W} g(s)ds_{\mathrm{I}}f(s)=0.$$

Theorem 6.1.33 (Cauchy integral formula)**.** *Let $\Omega\subseteq\mathbb{H}$ be an axially symmetric s-domain whose boundary $\partial(\Omega\cap\mathbb{C}_{\mathrm{I}})$ is a union of a finite number of rectifiable Jordan arcs. Let $f\in\mathscr{R}(\Omega)$ and, for any $\mathrm{I}\in\mathbb{S}$, set $ds_{\mathrm{I}}=-\mathrm{I}ds$. Then for every $p\in\Omega$ we have:*

$$f(p)=\frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_{\mathrm{I}})} S_L^{-1}(s,p)ds_{\mathrm{I}}f(s), \tag{6.23}$$

where

$$S_L^{-1}(s,p)=-(p^2-2\mathrm{Re}(s)p+|s|^2)^{-1}(p-\overline{s}).$$

Moreover, the value of the integral depends neither on Ω nor on the imaginary unit $\mathrm{I}\in\mathbb{S}$.

Proof. The integral does not depend on the open set Ω, by Corollary 6.1.32 since the function

$$S_L^{-1}(s,p)=-(p^2-2\mathrm{Re}(s)p+|s|^2)^{-1}(p-\overline{s})=(s-p)^{-\star}$$

(the $\star$-inverse is computed with respect to p) is right slice hyperholomorphic with respect to the variable s.
Let us show that the integral does not depend on the choice of the imaginary unit $\mathrm{I}\in\mathbb{S}$. Let $p=x+\mathrm{I}_p y\in\Omega$, then the set of the zeroes of the function $p^2-2\mathrm{Re}(s)p+|s|^2=0$ consists of a real point (of multiplicity two) or a 2-sphere. If the zeroes are not real, on any complex plane $\mathbb{C}_{\mathrm{I}}$ we find the two zeroes $s_{1,2}=x\pm\mathrm{I}y$. When the singularity is a real number, the integral reduces to the classical Cauchy integral formula for holomorphic maps. Thus we consider the case of nonreal zeroes and we calculate the residues about the points s_1 e s_2. Let us start with $s_1=x+\mathrm{I}y$. We set $s=x+y\mathrm{I}+\varepsilon e^{\mathrm{I}\theta}$, $\mathrm{Re}(s)=x+\varepsilon\cos\theta$, so that $ds_{\mathrm{I}}=-\mathrm{I}[\varepsilon\mathrm{I}e^{\mathrm{I}\theta}]d\theta=\varepsilon e^{\mathrm{I}\theta}d\theta$, and

$$|s|^2=x^2+2x\varepsilon\cos\theta+\varepsilon^2+y^2+2y\varepsilon\sin\theta.$$

We now compute the integral which appears at the right hand side of (6.23) along the circle with center at s_1 and radius $\varepsilon > 0$ on the plane $\mathbb{C}_\mathrm{I}$:

$$2\pi\mathscr{I}_1^\varepsilon = \int_0^{2\pi} -(-2p\varepsilon\cos\theta + 2x\varepsilon\cos\theta + \varepsilon^2 + 2y\varepsilon\sin\theta)^{-1}(p-(x-y\mathrm{I}+\varepsilon e^{-\mathrm{I}\theta}))\varepsilon e^{\mathrm{I}\theta}$$
$$\cdot\, d\theta f(x+y\mathrm{I}+\varepsilon e^{\mathrm{I}\theta})$$
$$= \int_0^{2\pi} -(-2p\cos\theta + 2x\cos\theta + \varepsilon + 2y\sin\theta)^{-1}(p-(x-y\mathrm{I}+\varepsilon e^{-\mathrm{I}\theta}))e^{\mathrm{I}\theta} d\theta f(x+y\mathrm{I}+\varepsilon e^{\mathrm{I}\theta}).$$

For $\varepsilon \to 0$ we obtain $\mathscr{I}_1^0$ namely the residue at s_1

$$2\pi\mathscr{I}_1^0 = \int_0^{2\pi}(2p\cos\theta - 2x\cos\theta - 2y\sin\theta)^{-1}(y\mathrm{I}_p + y\mathrm{I})e^{\mathrm{I}\theta} d\theta f(x+y\mathrm{I})$$
$$= \frac{1}{2}\int_0^{2\pi}(y\cos\theta\mathrm{I}_p - y\sin\theta)^{-1}(y\mathrm{I}_p + y\mathrm{I})e^{\mathrm{I}\theta} d\theta f(x+y\mathrm{I})$$
$$= -\frac{1}{2y^2}\int_0^{2\pi}(y\cos\theta\mathrm{I}_p + y\sin\theta)(y\mathrm{I}_p + y\mathrm{I})[\cos\theta + \mathrm{I}\sin\theta] d\theta f(x+y\mathrm{I})$$
$$= -\frac{1}{2y^2}\int_0^{2\pi}[(y\mathrm{I}_p)^2\cos\theta + y^2\sin\theta\mathrm{I}_p + y^2\cos\theta\mathrm{I}_p\mathrm{I} + y^2\sin\theta\mathrm{I}][\cos\theta + \mathrm{I}\sin\theta] d\theta f(x+y\mathrm{I})$$
$$= -\frac{1}{2}\int_0^{2\pi}[-\cos\theta + \sin\theta\mathrm{I}_p + \cos\theta\mathrm{I}_p\mathrm{I} + \sin\theta\mathrm{I}][\cos\theta + \mathrm{I}\sin\theta] d\theta f(x+y\mathrm{I})$$
$$= -\frac{1}{2}\int_0^{2\pi}[-\cos^2\theta - \cos\theta\sin\theta\mathrm{I} + \cos\theta\sin\theta\mathrm{I}_p + \sin^2\theta\mathrm{I}_p\mathrm{I} + \cos^2\theta\mathrm{I}_p\mathrm{I}$$
$$-\cos\theta\sin\theta\mathrm{I}_p + \cos\theta\sin\theta\mathrm{I} - \sin^2\theta] d\theta f(x+y\mathrm{I})$$
$$= -\frac{1}{2}\int_0^{2\pi}\Big(-1 + \mathrm{I}_p\mathrm{I}\Big) d\theta f(x+y\mathrm{I}) = \pi\Big(1 - \mathrm{I}_p\mathrm{I}\Big) f(x+y\mathrm{I}).$$

So we obtain

$$\mathscr{I}_1^0 = \frac{1}{2}\Big[1 - \mathrm{I}_p\mathrm{I}\Big] f(x+y\mathrm{I}),$$

and with analogous calculation we have that the residue about s_2 is

$$\mathscr{I}_2^0 = \frac{1}{2}\Big(1 + \mathrm{I}_p\mathrm{I}\Big) f(x-y\mathrm{I}).$$

By the classical residues theorem in the complex plane $\mathbb{C}_\mathrm{I}$

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_\mathrm{I})} S^{-1}(s,p)\, ds_\mathrm{I} f(s) = \mathscr{I}_1^0 + \mathscr{I}_2^0,$$

and the statement now follows from the Representation Formula (see Theorem 6.1.12).
□

Remark 6.1.34. There is an analogous formula for right slice hyperholomorphic functions $f \in \mathscr{R}^R(\Omega)$, in fact using the notations in Theorem 6.1.33 we have

$$f(p) = \frac{1}{2\pi} \int_{\partial(\Omega \cap \mathbb{C}_\mathrm{I})} f(s) ds_\mathrm{I} \, S_R^{-1}(s,p), \tag{6.24}$$

where

$$S_R^{-1}(s,p) = -(p-\bar{s})(p^2 - 2\mathrm{Re}(s)p + |s|^2)^{-1}(p-\bar{s}).$$

We note that the function $s-p$ is both left and right slice hyperholomorphic in p and $S_R^{-1}(s,p) = (s-p)^{-\star_r}$ where the $\star_r$-inverse is again computed with respect to p.
Direct computations show that:

Proposition 6.1.35. *For any s,p such that $p^2 - 2\mathrm{Re}(s)p + |s|^2 \neq 0$ the following equality holds*

$$S_L^{-1}(s,p) = -S_R^{-1}(p,s).$$

We conclude this section by recalling some basic facts about the zeros of a slice hyperholomorphic function.

Proposition 6.1.36. *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f:\ \Omega \to \mathbb{H}$ be a slice hyperholomorphic function, not identically zero. Then:*

(1) Every zero of f is a a zero of f^s.

(2) The zeros of f^s are isolated real points and/or isolated spheres.

Proof. Point (1) follows from the fact that $f^s = f \star f^c$ and Theorem 6.1.21.
Let us prove (2). By Remark 6.1.16, f^s is quaternionic intrinsic and not identically zero. So if $f^s(x_0 + \mathrm{I}y_0) = 0$ then also $f^s(x_0 - \mathrm{I}y_0) = 0$ and by the Representation Formula $f^s(x_0 + \mathrm{J}y_0) = 0$ for any other $\mathrm{J} \in \mathbb{S}$. The real zeros of f^s and the spheres of zeros must be isolated, otherwise f^s would have non isolated zeros on the intersection $\Omega \cap \mathbb{C}_\mathrm{I}$ for any $\mathrm{I} \in \mathbb{S}$ and, by the identity principle, f^s would be identically zero, which is a contradiction. □

Theorem 6.1.37 (Structure of the Zero Set)**.** *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f:\ \Omega \to \mathbb{H}$ be a slice hyperholomorphic function. Suppose that f does not vanish identically. Then, if the zero set of f is nonempty, it consists of the union of isolated 2-spheres and/or isolated points.*

Proof. Let $p_0 = x_0 + \mathrm{J}y_0$ be a zero of f. By Corollary 6.1.13, we know that $f(x+\mathrm{I}y) = \alpha(x,y) + \mathrm{I}\beta(x,y)$, thus

$$f(p_0) = f(x_0 + \mathrm{J}y_0) = \alpha(x_0,y_0) + \mathrm{J}\beta(x_0,y_0) = 0.$$

If $\beta(x_0,y_0) = 0$ then also $\alpha(x_0,y_0) = 0$ so $f(x_0 + \mathrm{I}y_0) = 0$ for every choice of an imaginary unit $I \in \mathbb{S}$ thus the whole sphere defined by p_0 is solution of the equation $f(p) = 0$. If $\beta(x_0,y_0) \neq 0$ then it is an invertible element in $\mathbb{H}$. In this case, $\alpha(x_0,y_0) \neq 0$ otherwise we would get $\mathrm{J}\beta(x_0,y_0) = 0$ which is absurd. Since the inverse of $\beta(x_0,y_0)$ is unique, it is also unique the element $\mathrm{J} = -\alpha(x_0,y_0)\beta(x_0,y_0)^{-1}$. Then p_0 is the only solution of $f(p) = 0$ on the sphere defined by p_0. The isolated zeros of f and the spherical zeros are isolated, otherwise f^s would have non isolated zeros. □

The following result guarantees that the zeros can be factored out.

Proposition 6.1.38. *Let Ω be an axially symmetric s-domain and let $f:\Omega\to\mathbb{H}$ be a slice hyperholomorphic function. An element $p_0\in\Omega$ is a zero of f if and only if there exists a slice hyperholomorphic function $g:\Omega\to\mathbb{H}$ such that*

$$f(p)=(p-p_0)\star g(p).$$

Moroever, f vanishes on $[p_0]$ if and only if there exists a slice hyperholomorphic function $h:\Omega\to\mathbb{H}$ such that

$$f(p)=(p^2-2\mathrm{Re}(p_0)p+|p_0|^2)h(p).$$

Proof. Let p_0 be an isolated zero. Then it belongs to a complex plane $\mathbb{C}_{\mathrm{I}}$ (unique if p_0 is not real). Use the splitting lemma to write $f_{\mathrm{I}}(z)=F(z)+G(z)\mathrm{J}$. Since $f_{\mathrm{I}}(p_0)=0$ also $F(p_0)=0$ and $G(p_0)=0$ and since F,G are holomorphic functions we can factor $(z-p_0)$ out and write $F(z)=(z-p_0)F_1(z)$, $G(z)=(z-p_0)G_1(z)$. Then consider $(z-p_0)(F_1(z)+G_1(z)\mathrm{J})$ for $z\in\Omega\cap\mathbb{C}_{\mathrm{I}}$. The result follows by extending this function to the whole Ω as $(p-p_0)\star g(p)$. The extended function coincides with f by the identity principle. The second part of the statement follows by applying the preceding discussion first to the factors $p-p_0$ and then $p-\bar{p}_0$. □

The notion of multiplicity of an isolated zero p_0 or of a spherical zero can be given as we did in the case of polynomials. So we refer the reader to Definition 4.2.7 where f and g are, in this case, two slice hyperholomorphic functions.
Let us now introduce the notion of singularity of a function, following [264].
Let Ω be an axially symmetric s-domain and let $p_0\in\Omega$. Let f be a function which, in a subset of Ω containing p_0, can be written in the form $f(p)=\sum_{n=-\infty}^{+\infty}(p-p_0)^{\star n}a_n$ where $a_n\in\mathbb{H}$.
We have:

Definition 6.1.39. A function f has a pole at the point p_0 if there exists $m\geq 0$ such that $a_{-k}=0$ for $k>m$. The minimum of such m is called the order of the pole;
If p is not a pole then we call it an essential singularity for f;
f has a removable singularity at p_0 if it can be extended in a neighborhood of p_0 as a slice hyperholomorphic function.

Note the following important fact: a function f has a pole at p_0 if and only if its restriction to a complex plane has a pole. Note that there can be poles of order 0: let us consider for example the function $(p+I)^{-\star}=(p^2+1)^{-1}(p-I)$. It has a pole of order 0 at the point $-I$ which, however, is not a removable singularity.

Definition 6.1.40. Let Ω be an axially symmetric s-domain in $\mathbb{H}$. We say that a function $f:\Omega\to\mathbb{H}$ is slice hypermeromorphic in Ω if f is slice hyperholomorphic in $\Omega'\subset\Omega$ such that $\Omega\setminus\Omega'$ has no point limit in Ω and every point in $\Omega\setminus\Omega'$ is a pole.

The functions which are slice hypermeromorphic are called semi-regular in [264] and for these functions we have the following result, proved in [264, Proposition 7.1, Theorem 7.3]:

Proposition 6.1.41. *Let Ω be an axially symmetric s-domain in $\mathbb{H}$ and let $f,g:\Omega\to\mathbb{H}$ be slice hyperholomorphic. Then the function $f^{-\star}\star g$ is slice hypermeromorphic in Ω. Conversely, any slice hypermeromorphic function on Ω can be locally expressed as $f^{-\star}\star g$ for suitable f and g.*

Since $f^{-\star}=(f\star f^c)^{-1}f^c$ it is then clear that the poles of a slice hypermeromorphic function occur in correspondence to the zeros of the function $f\star f^c$ and so they are isolated spheres, possibly reduced to real points.
We end this section with a consequence of Runge theorem proved in [146]. In the statement $\overline{\mathbb{C}}_I$ denotes the extended complex plane $\mathbb{C}_I$. We note that Runge theorem has been proved by taking the class of functions $f(x+\mathrm{J}y)=\alpha(x,y)+\mathrm{J}\beta(x,y)$ where $\alpha(x,-y)=\alpha(x,y)$, $\beta(x,-y)=-\beta(x,y)$ and the pair α,β satisfies the Cauchy-Riemann system on axially symmetric open sets (not necessarily s-domains). See Remark 6.1.14.

Theorem 6.1.42. *Let K be an axially symmetric compact set such that $\overline{\mathbb{C}}_I\setminus(K\cap\mathbb{C}_I)$ is connected for all $I\in\mathbb{S}$. Let f be slice regular in the open set Ω with $\Omega\supset K$. Then there exists a sequence $\{P_n\}$ of polynomials such that $P_n(q)\to f(q)$ uniformly on K.*

It is immediate that if K is an axially symmetric compact set, then $\overline{\mathbb{H}}\setminus K$ is connected if and only if $\overline{\mathbb{C}}_I\setminus(K\cap\mathbb{C}_I)$ is connected for all $I\in\mathbb{S}$.

6.2 The Hardy space of the unit ball

The quaternionic Hardy space $\mathrm{H}^2(\mathbb{B})$ of the unit ball $\mathbb{B}$ space (or simply H^2 when it appears as a subscript) is defined by mimicking the analogous definition in the complex case: it contains square summable (left) slice hyperholomorphic power series, in other words:

$$\mathrm{H}^2(\mathbb{B})=\left\{f(p)=\sum_{k=0}^{\infty}p^kf_k:\ \|f\|^2_{\mathrm{H}^2}:=\sum_{k=0}^{\infty}|f_k|^2<\infty\right\}.$$

The space $\mathrm{H}^2(\mathbb{B})$ can be endowed with the inner product

$$\langle f,\,g\rangle=\sum_{k=0}^{\infty}\bar{g}_kf_k\quad\text{if}\quad f(p)=\sum_{k=0}^{\infty}p^kf_k,\quad g(p)=\sum_{k=0}^{\infty}p^kg_k,\tag{6.25}$$

from which it follows that

$$\|f\|_{\mathrm{H}^2}=(\sum_{k=0}^{\infty}|f_k|^2)^{\frac{1}{2}}.$$

Proposition 6.2.1. *The norm of $f\in\mathrm{H}^2(\mathbb{B})$ can also be computed as*

$$\|f\|^2_{\mathrm{H}^2}=\sup_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{\mathrm{I}\theta})|^2d\theta\tag{6.26}$$

where the value of the integral does not depend on the choice of $\mathrm{I}\in\mathbb{S}$.

Proof. By integrating the (uniformly converging on compact sets) power series expressing f, see (6.25), for a fixed $\mathrm{I} \in \mathbb{S}$, we obtain

$$\int_0^{2\pi} |f(re^{\mathrm{I}\theta})|^2 d\theta = \int_0^{2\pi} \left(\sum_{j,k=0}^{\infty} r^{k+j} \overline{f}_k e^{\mathrm{I}(j-k)\theta} f_j \right) d\theta$$
$$= \sum_{j,k=0}^{\infty} r^{k+j} \overline{f}_k \left(\int_0^{2\pi} e^{\mathrm{I}(j-k)\theta} d\theta \right) f_j = 2\pi \cdot \sum_{n=0}^{\infty} r^{2n} |f_n|^2.$$

The latter formula implies the statement. We observe that the supremum in formula (6.26) can be replaced by the limit as $r \to 1$. □

It is immediate to verify the following:

Proposition 6.2.2. $\mathrm{H}^2(\mathbb{B})$ *is a right quaternionic Hilbert space on* $\mathbb{H}$.

The space $\mathrm{H}^2(\mathbb{B})$ can also be characterized as the reproducing kernel Hilbert space with reproducing kernel

$$k_{\mathrm{H}^2}(p,q) = \sum_{n=0}^{\infty} p^n \bar{q}^n. \tag{6.27}$$

Thus the function $k_{\mathrm{H}^2}(\cdot,q)$ belongs to $\mathrm{H}^2(\mathbb{B})$ for every $q \in \mathbb{B}$ and for any function $f \in \mathrm{H}^2(\mathbb{B})$ as in (6.25),

$$\langle f, k_{\mathrm{H}^2}(\cdot,q) \rangle_{\mathrm{H}^2} = \sum_{k=0}^{\infty} q^k f_k = f(q). \tag{6.28}$$

The kernel $k_{\mathrm{H}^2}(p,q)$ can be computed in closed form:

Proposition 6.2.3. *The sum of the series* $\sum_{n=0}^{+\infty} p^n \bar{q}^n$ *is the function* $k(p,q)$ *given by*

$$k_{\mathrm{H}^2}(p,q) = (1 - 2\mathrm{Re}(q)p + |q|^2 p^2)^{-1}(1 - pq) = (1 - \bar{p}\bar{q})(1 - 2\mathrm{Re}(p)\bar{q} + |p|^2 \bar{q}^2)^{-1}. \tag{6.29}$$

The kernel $k_{\mathrm{H}^2}(p,q)$ *is defined for* $p \notin [q^{-1}]$ *for* $q \neq 0$ *or, equivalently, for* $q \notin [p^{-1}]$ *for* $p \neq 0$. *Moreover:*

(1) $k_{\mathrm{H}^2}(p,q)$ *is left slice hyperholomorphic in p and right slice hyperholomorphic in q;*

(2) $\overline{k_{\mathrm{H}^2}(p,q)} = k_{\mathrm{H}^2}(q,p)$.

Proof. The proof of the first equality is an application of the extension operator (6.11) applied to the function $(1 - z\bar{q})^{-1}$ of the complex variable z which gives

$$k_{\mathrm{H}^2}(p,q) = (1 - 2\mathrm{Re}(q)p + |q|^2 p^2)^{-1}(1 - pq). \tag{6.30}$$

If we apply to the function of the complex variable w: $(1 - p\bar{w})^{-1}$ the right extension operator (6.12), we obtain

$$(1 - \bar{p}\bar{q})(1 - 2\mathrm{Re}(p)\bar{q} + |p|^2 \bar{q}^2)^{-1}. \tag{6.31}$$

The function (6.31) is right slice hyperholomorphic in the variable $\bar{q}$ by construction and it is slice hyperholomorphic in the variable p in its domain of definition, since it is the product of a slice hyperholomorphic function and a polynomial with real coefficients. By the identity principle it coincides with the function in (6.30) which is slice hyperholomorphic in p by construction. Thus point (1) follows. Point (2) follows from the equalities

$$\begin{aligned}\overline{k(q,p)} &= \overline{[(1-\bar{p}\bar{q})(1-2\mathrm{Re}(p)\bar{q}+|p|^2\bar{q}^2)^{-1}]}\\ &= (1-2\mathrm{Re}(p)q+|p|^2\bar{q}^2)^{-1}(1-qp) = k(q,p).\end{aligned}$$

□

The next result shows that the limit of a function $f \in \mathrm{H}^2(\mathbb{B})$ on the boundary of $\mathbb{B}$ exists almost everywhere.

Proposition 6.2.4. *If* $\lim_{r\to 1}|f(re^{\mathrm{I}\theta})| = 1$, *for all* I *fixed in* $\mathbb{S}$, *then for all* $g \in \mathscr{R}(\mathbb{B})$ *continuous in* $\overline{\mathbb{B}}$ *we have*

$$\lim_{r\to 1}|f\star g(re^{\mathrm{I}\theta})| = |g(e^{\mathrm{I}'\theta})|, \qquad a.e.$$

where $\theta \in [0,2\pi)$, *and* $\mathrm{I}' \in \mathbb{S}$ *depends on* θ *and* f.

Proof. Let $b = f(re^{\mathrm{I}\theta})$ and write $b = Re^{\mathrm{J}\alpha}$ for suitable R, J, α. By hypothesis, we can assume that $b \neq 0$ when $r \to 1$, thus b^{-1} exists. We have

$$\begin{aligned} b^{-1}re^{\mathrm{I}\theta}b &= e^{-\mathrm{J}\alpha}(re^{\mathrm{I}\theta})e^{\mathrm{J}\alpha} = r(\cos\alpha - \mathrm{J}\sin\alpha)(\cos\theta+\mathrm{I}\sin\theta)(\cos\alpha+\mathrm{J}\sin\alpha)\\ &= r(\cos\theta + \mathrm{I}\cos^2\alpha\sin\theta - \mathrm{JI}\cos\alpha\sin\alpha\sin\theta + \mathrm{IJ}\cos\alpha\sin\alpha\sin\theta - \mathrm{JIJ}\sin^2\alpha\sin\theta)\\ &= r(\cos\theta + \cos\alpha e^{-\mathrm{J}\alpha}\mathrm{I}\sin\theta + e^{-\mathrm{J}\alpha}\mathrm{IJ}\sin\alpha\sin\theta)\\ &= r(\cos\theta + e^{-\mathrm{J}\alpha}\mathrm{I}e^{\mathrm{J}\alpha}\sin\theta) = r(\cos\theta + \mathrm{I}'\sin\theta),\end{aligned}$$

where $\mathrm{I}' = e^{-\mathrm{J}\alpha}\mathrm{I}e^{\mathrm{J}\alpha}$. Finally, we have

$$\lim_{r\to 1}|f\star g(re^{\mathrm{I}\theta})| = \lim_{r\to 1}|f(re^{\mathrm{I}\theta})g(b^{-1}re^{\mathrm{I}\theta}b)| = \lim_{r\to 1}|g(re^{\mathrm{I}'\theta})| = |g(e^{\mathrm{I}'\theta})|.$$

□

In the following theorem, the function S has *a priori* no properties besides being defined on $\mathbb{B}$. The positivity of the kernel $K_S(p,q)$ implies that S is slice hyperholomorphic (see also Theorems 8.4.2 and 8.4.4).

Theorem 6.2.5. *Let* $S : \mathbb{B} \to \mathbb{H}$. *The following are equivalent:*

(1) S *is slice hyperholomorphic on* $\mathbb{B}$ *and* $|S(p)| \leq 1$ *for all* $p \in \mathbb{B}$.

(2) The operator M_S *of left* $\star$*–multiplication by* S

$$M_S : f \mapsto S\star f \tag{6.32}$$

is a contraction on $\mathrm{H}^2(\mathbb{B})$, *that is,* $\|S\star f\|_{\mathrm{H}^2} \leq \|f\|_{\mathrm{H}^2}$ *for all* $f \in \mathrm{H}^2(\mathbb{B})$.

(3) The kernel

$$K_S(p,q) = \sum_{k=0}^{\infty} p^k (1 - S(p)\overline{S(q)})\bar{q}^k \tag{6.33}$$

is positive on $\mathbb{B}\times\mathbb{B}$.

(4) $S \in \mathscr{R}(\mathbb{B})$ and $I_{n+1} - L_n L_n^* \geq 0$ for all $n \geq 0$ where L_n is the lower triangular Toeplitz matrix given by

$$L_n = \begin{bmatrix} S_0 & 0 & \dots & 0 \\ S_1 & S_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ S_n & \dots & S_1 & S_0 \end{bmatrix}, \quad \textit{where} \quad S(p) = \sum_{k=0}^{\infty} p^k S_k. \tag{6.34}$$

Proof. First of all, we define the operator M_S of multiplication by S by setting, for any $f(p) = \sum_{k=0}^{\infty} p^k f_k$:

$$(S \star f)(p) = \sum_{k=0}^{\infty} p^k S(p) f_k. \tag{6.35}$$

(Note that M_S can be defined without assuming assume any hypothesis on S). If M_S maps $\mathrm{H}^2(\mathbb{B})$ into itself, then the function $S = M_S 1$ belongs to $\mathrm{H}^2(\mathbb{B})$ and thus it is slice hyperholomorphic.

Let us show that $(2) \Longrightarrow (3)$. To this end, assume that $M_S : \mathrm{H}^2(\mathbb{B}) \to \mathrm{H}^2(\mathbb{B})$ is a contraction. From (6.35) and (6.27) we obtain

$$M_S k_{\mathrm{H}^2}(\cdot, q) = \sum_{j=0}^{\infty} p^k S(p) \bar{q}^k$$

which, together with reproducing kernel property (6.28), yields

$$\begin{aligned} (M_S^* k_{\mathrm{H}^2}(\cdot,q))(p) &= \langle M_S^* k_{\mathrm{H}^2}(\cdot,q), k_{\mathrm{H}^2}(\cdot,p)\rangle_{\mathrm{H}^2} \\ &= \langle k_{\mathrm{H}^2}(\cdot,q), S \star k_{\mathrm{H}^2}(\cdot,p)\rangle_{\mathrm{H}^2} = \sum_{k=0}^{\infty} p^k \overline{S(q)} \bar{q}^k. \end{aligned} \tag{6.36}$$

Consequently, we have

$$\langle (I - M_S M_S^*) k_{\mathrm{H}^2}(\cdot,q), k_{\mathrm{H}^2}(\cdot,p)\rangle_{\mathrm{H}^2} = \sum_{k=0}^{\infty} p^k (1 - S(p)\overline{S(q)})\bar{q}^k.$$

We deduce that for any function $f \in \mathrm{H}^2(\mathbb{B})$ of the form

$$f = \sum_{i=1}^{r} k_{\mathrm{H}^2}(\cdot, p_i)\alpha_i, \qquad r \in \mathbb{N},\ p_i \in \mathbb{B},\ \alpha_i \in \mathbb{H}, \tag{6.37}$$

one has

$$\begin{aligned}\langle (I-M_SM_S^*)f,f\rangle_{\mathrm{H}^2} &= \langle f,f\rangle_{\mathrm{H}^2}-\langle M_S^*f,M_S^*f\rangle_{\mathrm{H}^2}\\ &=\sum_{i,j=1}^{r}\overline{\alpha}_i k_{\mathrm{H}^2}(p_i,p_j)\alpha_j-\sum_{i,j=1}^{r}\sum_{k=0}^{\infty}\overline{\alpha}_i p_i^k S(p_i)\overline{S(p_j)}\overline{p}_i^k\alpha_j\\ &=\sum_{i,j=1}^{r}\overline{\alpha}_i K_S(p_i,p_j)\alpha_j. \end{aligned}\tag{6.38}$$

As M_S is a contraction, the inner product on the left hand side of (6.38) is nonnegative, so the quadratic form on the right hand side of (6.38) is nonnegative and thus K_S is a positive kernel.

To show that $(3)\Longrightarrow(2)$ we assume that the kernel (6.33) is positive on $\mathbb{B}\times\mathbb{B}$. Then we note that the function on the right side of (6.36) belongs to $\mathrm{H}^2(\mathbb{B})$ for each fixed $q\in\mathbb{B}$ and that its norm equals

$$\frac{|S(q)|^2}{1-|q|^2}.$$

Thus we can define the operator $T:\mathrm{H}^2(\mathbb{B})\to\mathrm{H}^2(\mathbb{B})$ by setting

$$T:\,k_{\mathrm{H}^2}(\cdot,q)\mapsto\sum_{k=0}^{\infty}p^k\overline{S(q)}\overline{q}^k$$

with subsequent extension by linearity to functions f of the form (6.37). Since such functions are dense in $\mathrm{H}^2(\mathbb{B})$, they extend by continuity to all of $\mathrm{H}^2(\mathbb{B})$. Using this density and (6.38) with T instead of M_S^*, one obtains that T is a contraction on $\mathrm{H}^2(\mathbb{B})$. We then compute its adjoint obtaining $T^*f=S\star f=M_Sf$. Since T is a contraction on $\mathrm{H}^2(\mathbb{B})$, its adjoint M_S is a contraction as well.

Let us now prove that $(3)\Longrightarrow(1)$. If the kernel K_S is positive on $\mathbb{B}\times\mathbb{B}$, then

$$0\le K_S(q,q)=\sum_{k=0}^{\infty}q^k(1-|S(q)|^2)\overline{q}^k=\frac{1-|S(q)|^2}{1-|q|^2}$$

and therefore, $|S(q)|\le 1$ for every $q\in\mathbb{B}$. On the other hand, by $(3)\Longrightarrow(2)$, we know that the operator M_S maps $\mathrm{H}^2(\mathbb{B})$ into itself and thus $S=M_S1\in\mathrm{H}^2(\mathbb{B})\subset\mathscr{R}(\mathbb{B})$.

To show that $(1)\Longrightarrow(2)$ let us assume that $S\in\mathscr{R}(\mathbb{B},\overline{\mathbb{B}})$, i.e., S is slice hyperholomorphic and such that $|S(p)|\le 1$ for all $p\in\mathbb{B}$. By (6.16), we have for every $f\in\mathrm{H}^2(\mathbb{B})$ and every $\mathrm{I}\in\mathbb{S}$,

$$\begin{aligned}\|f\star S\|_{\mathrm{H}^2}^2 &= \sup_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi}|f\star S(re^{\mathrm{I}\theta})|^2d\theta\\ &=\sup_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{\mathrm{I}\theta})S(f(re^{\mathrm{I}\theta})^{-1}re^{\mathrm{I}\theta}f(re^{\mathrm{I}\theta})|^2d\theta\\ &\le\sup_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{\mathrm{I}\theta})|^2d\theta=\|f\|_{\mathrm{H}^2}^2. \end{aligned}\tag{6.39}$$

Let S^c and f^c be the slice hyperholomorphic conjugates of S and f, respectively. By the properties of the conjugate of a function and since $\|f\|_{\mathrm{H}^2} = \|f^c\|_{\mathrm{H}^2}$, then (6.39) gives

$$\|S^c \star f^c\|_{\mathrm{H}^2} = \|(f\star S)^c\|_{\mathrm{H}^2} = \|f\star S\|_{\mathrm{H}^2} \leq \|f\|_{\mathrm{H}^2} = \|f^c\|_{\mathrm{H}^2}. \tag{6.40}$$

Thus the operator $M_{S^c} : f \mapsto S^c \star f$ is a contraction on $\mathrm{H}^2(\mathbb{B})$. Using (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1), we deduce that $S^c \in \mathscr{R}(\mathbb{B},\overline{\mathbb{B}})$. By applying (6.40) to S^c we conclude that the operator $M_{(S^c)^c} = M_S$ is a contraction on $\mathrm{H}^2(\mathbb{B})$.

The proof of (2) $\Longleftrightarrow$ (4) mimics that of (2) $\Longleftrightarrow$ (3), but instead of functions of the form (6.37) we consider polynomials with coefficients written on the right. Let us assume (2). Calculations analogous to those in (6.36) show that for S with the Taylor series as in (6.34),

$$M_S^* : p^k \mapsto \sum_{j=0}^{k} p^j \overline{S}_{k-j} \quad \text{for all} \quad k\geq 0$$

which extends by linearity to

$$M_S^* : f(p) = \sum_{k=0}^{n} p^k f_k \mapsto \sum_{k=0}^{n} p^k \left(\sum_{j=k}^{n} \overline{S}_{j-k} f_j\right).$$

By setting $\mathbf{f} := \begin{bmatrix} f_0 & f_1 & \dots & f_n\end{bmatrix}^T$ we obtain an analog of (6.38) in terms of the matrix L_n from (6.34):

$$\|f\|^2_{\mathrm{H}^2} - \|M_S^* f\|^2_{\mathrm{H}^2} = \sum_{k=0}^{n} |f_k|^2 - \sum_{k=0}^{n}\left|\sum_{j=k}^{n} \overline{S}_{j-k} f_j\right|^2 = \mathbf{f}^* \left(I_{n+1} - L_n L_n^*\right)\mathbf{f}. \tag{6.41}$$

If M_S is a contraction on $\mathrm{H}^2(\mathbb{B})$, the last expression is nonnegative for every $\mathbf{f}\in\mathbb{H}^{n+1}$ and therefore the matrix $I_{n+1} - L_n L_n^*$ is positive semidefinite. Conversely, if this matrix is positive semidefinite for each $n\geq 1$, then (6.41) shows that M_S^* acts contractively (in $\mathrm{H}^2(\mathbb{B})$-metric) on any polynomial. But since the polynomials are dense in $\mathrm{H}^2(\mathbb{B})$, the operators M_S^* and M_S are contractions on the whole $\mathrm{H}^2(\mathbb{B})$. $\square$

Definition 6.2.6. A function $S : \mathbb{B}\to\mathbb{H}$ satisfying any of the equivalent conditions in the preceding theorem is called a Schur function or a Schur multiplier.

We now collect some properties of Schur multipliers. Note that property (3) will be proved in a more general form in Theorem 7.5.1.

Theorem 6.2.7. *Let S_1, S_2 and S be Schur multipliers depending on the quaternionic variable p. Then:*

(1) $M_{S_1}M_{S_2} = M_{S_1\star S_2}$;

(2) $M_S M_p = M_p M_S = M_{pS}$;

(3) $\left(M_S^* k_{\mathrm{H}^2}(\cdot,q)\right)(p) = \sum_{k=0}^{\infty} p^k \overline{S(q)}\bar{q}^k$.

Proof. Equality (1) follows from the associativity of the $\star$-product. The second property is a consequence of Remark 6.1.20 and of equality (1). Property (3) follows from the computations (6.36) in the proof of the previous theorem. □

Corollary 6.2.8. *Let S be slice hyperholomorphic in $\mathbb{B}$ and bounded there in modulus. Then M_S is a bounded operator from $\mathrm{H}^2(\mathbb{B})$ into itself.*

Proof. It suffices to apply the previous theorem to $\dfrac{S}{\sup_{p\in\mathbb{B}}|S(p)|}$. □

6.3 Blaschke products (unit ball case)

In classical analysis Blaschke factors and Blaschke products play an important role in the study of invariant subspaces and interpolation; see for instance [167, 252]. In this section we study Blaschke products in the present setting.

Definition 6.3.1. Let $a\in\mathbb{H}$, $|a|<1$. The function

$$B_a(p)=(1-p\bar{a})^{-\star}\star(a-p)\frac{\bar{a}}{|a|} \tag{6.42}$$

is called a Blaschke factor at a.

Remark 6.3.2. Using Proposition 6.1.21, $B_a(p)$ can be rewritten in terms of the pointwise multiplication as

$$B_a(p)=(1-\tilde{p}\bar{a})^{-1}(a-\tilde{p})\frac{\bar{a}}{|a|}$$

where $\tilde{p}=(1-pa)^{-1}p(1-pa)$.

The following result is immediate, since it follows from the definition:

Proposition 6.3.3. *Let $a\in\mathbb{H}$, $|a|<1$. The Blaschke factor B_a is a slice hyperholomorphic function in $\mathbb{B}$.*

As one expects, $B_a(p)$ has only one zero at $p=a$ and analogously to what happens in the case of the zeros of a function, the product of two Blaschke factors of the form $B_a(p)\star B_{\bar{a}}(p)$ gives the Blaschke factor with zeros at the sphere $[a]$. Thus we give the following definition:

Definition 6.3.4. Let $a\in\mathbb{H}$, $|a|<1$. The function

$$B_{[a]}(p)=(1-2\mathrm{Re}(a)p+p^2|a|^2)^{-1}(|a|^2-2\mathrm{Re}(a)p+p^2) \tag{6.43}$$

is called Blaschke factor at the sphere $[a]$.

Theorem 6.3.5. *Let $a\in\mathbb{H}$, $|a|<1$. The Blaschke factor B_a has the following properties:*

(1) it takes the unit ball $\mathbb{B}$ to itself;

(2) it takes the boundary of the unit ball to itself;

(3) it has a unique zero for $p = a$.

Proof. First observe that by setting $\lambda(p) = 1 - p\bar{a}$ we can write

$$(1 - p\bar{a})^{-\star} = (\lambda^c(p) \star \lambda(p))^{-1} \lambda^c(p).$$

Applying formula (6.16) to the products $\lambda^c(p) \star \lambda(p)$ and $\lambda^c(p) \star (a - p)$, the Blaschke factor (6.42) may be written as

$$\begin{aligned} B_a(p) &= (\lambda^c(p) \star \lambda(p))^{-1} \lambda^c(p) \star (a - p) \frac{\bar{a}}{|a|} = (\lambda^c(p)\lambda(\tilde{p}))^{-1} \lambda^c(p)(a - \tilde{p}) \frac{\bar{a}}{|a|} \\ &= \lambda(\tilde{p})^{-1}(a - \tilde{p}) \frac{\bar{a}}{|a|} = (1 - \tilde{p}\bar{a})^{-1}(a - \tilde{p}) \frac{\bar{a}}{|a|}, \end{aligned} \tag{6.44}$$

where $\tilde{p} = \lambda^c(p)^{-1} p \lambda^c(p)$. Thus $B_a(p) = (1 - \tilde{p}\bar{a})^{-1}(a - \tilde{p}) \dfrac{\bar{a}}{|a|}$. Let us show that $|p| = |\tilde{p}| < 1$ implies $|B_a(p)|^2 < 1$. The latter inequality is equivalent to

$$|a - \tilde{p}|^2 < |1 - \tilde{p}\bar{a}|^2$$

which is also equivalent to

$$|a|^2 + |p|^2 < 1 + |a|^2 |p|^2. \tag{6.45}$$

Then (6.45) is equivalent to $(|p|^2 - 1)(1 - |a|^2) < 0$ and it holds when $|p| < 1$. When $|p| = 1$ we set $p = e^{\mathrm{I}\theta}$, so that $\tilde{p} = e^{\mathrm{I}'\theta}$ by the proof of Corollary 6.2.4; we have

$$|B_a(e^{\mathrm{I}\theta})| = |1 - e^{\mathrm{I}'\theta}\bar{a}|^{-1} |a - e^{\mathrm{I}'\theta}| \frac{|\bar{a}|}{|a|} = |e^{-\mathrm{I}'\theta} - \bar{a}|^{-1} |a - e^{\mathrm{I}'\theta}| = 1.$$

Finally, from (6.44) it follows that $B_a(p)$ has only one zero that comes from the factor $a - \tilde{p}$. Moreover $B_a(a) = (1 - \tilde{a}\bar{a})^{-1}(a - \tilde{a}) \frac{\bar{a}}{|a|}$ where $\tilde{a} = (1 - a^2)^{-1} a (1 - a^2) = a$ and thus $B_a(a) = 0$. □

The next lemma contains a useful calculation:

Lemma 6.3.6. *Let $a \in \mathbb{B}$. Then, it holds that*

$$B_a(\overline{a})\overline{a} = \overline{a} B_a(\overline{a}). \tag{6.46}$$

Proof. We have

$$\begin{aligned} B_a(p) &= \Big(\sum_{n=0}^{\infty} p^n \overline{a}^n\Big) \star (a - p) \frac{\overline{a}}{|a|} \\ &= \sum_{n=0}^{\infty} (p^n \overline{a}^n a - p^{n+1} \overline{a}^n) \frac{\overline{a}}{|a|} \\ &= |a| + \sum_{n=0}^{\infty} p^{n+1} \overline{a}^{n+1} \Big(|a| - \frac{1}{|a|}\Big). \end{aligned} \tag{6.47}$$

Finally, (6.46) is a direct consequence of the last equality. □

Theorem 6.3.7. *Let B_a be a Blaschke factor. The operator*

$$M_a\,:\, f\mapsto B_a\star f$$

is an isometry from $\mathrm{H}^2(\mathbb{B})$ into itself. An element of $\mathrm{H}^2(\mathbb{B})$ belongs to the range of M_a if and only if it vanishes at a, the range of M_a has codimension one and the space $\mathrm{H}^2(\mathbb{B})\ominus M_a\mathrm{H}^2(\mathbb{B})$ is spanned by the function $(1-p\overline{a})^{-\star}$.

Proof. We first consider $f(p)=p^uh$ and $g(p)=p^vk$ where $u,v\in\mathbb{N}_0$ and $h,k\in\mathbb{H}$, and show that

$$\langle B_a\star f,B_a\star g\rangle=\delta_{uv}\overline{k}h. \tag{6.48}$$

Using calculation (6.47), and with f and g as above, we have

$$(B_a\star f)(p)=p^uh|a|+\sum_{n=0}^{\infty}p^{n+1+u}\overline{a}^{n+1}(|a|-\frac{1}{|a|})h$$

and

$$(B_a\star g)(p)=p^vk|a|+\sum_{n=0}^{\infty}p^{n+1+v}\overline{a}^{n+1}(|a|-\frac{1}{|a|})k. \tag{6.49}$$

If $u=v$ we have

$$\langle B_a\star f,B_a\star g\rangle=\overline{k}h\left(|a|^2+\sum_{n=0}^{\infty}|a|^{2n+2}(|a|-\frac{1}{|a|})^2\right)=\overline{k}h=\langle f,g\rangle.$$

To compute $\langle f,g\rangle$ we assume that $u<v$. Then, in view of (6.49) we have

$$\langle p^uh|a|,B_a\star g\rangle=0.$$

So

$$\begin{aligned}
\langle B_a\star f,B_a\star g\rangle&=\langle\sum_{n=0}^{\infty}p^{n+1+u}\overline{a}^{n+1}\left(|a|-\frac{1}{|a|}\right)h,p^v|a|k\rangle+\\
&\quad+\langle\sum_{n=0}^{\infty}p^{n+1+u}\overline{a}^{n+1}\left(|a|-\frac{1}{|a|}\right)h,\sum_{m=0}^{\infty}p^{m+1+v}\overline{a}^{m+1}\left(|a|-\frac{1}{|a|}\right)k\rangle\\
&=|a|\overline{k}\overline{a}^{v-u}\left(|a|-\frac{1}{|a|}\right)h+\\
&\quad+\langle\sum_{m=0}^{\infty}p^{m+1+v}\overline{a}^{m+1+v-u}\left(|a|-\frac{1}{|a|}\right)h,\sum_{m=0}^{\infty}p^{m+1+v}\overline{a}^{m+1}\left(|a|-\frac{1}{|a|}\right)k\rangle\\
&=|a|\overline{k}\overline{a}^{v-u}\left(|a|-\frac{1}{|a|}\right)h+\overline{k}\left(|a|-\frac{1}{|a|}\right)^2\overline{a}^{v-u}\frac{|a|^2}{1-|a|^2}h\\
&=0\\
&=\langle f,g\rangle.
\end{aligned}$$

The case $v < u$ is considered by symmetry of the inner product. Hence, (6.48) holds for polynomials. By continuity, and a corollary of the Runge theorem, see Theorem 6.1.42, it holds for all $f \in \mathrm{H}^2(\mathbb{B})$.

We now characterize the range of M_a. It is clear that any element in the range vanishes at a. Conversely, if $f \in \mathrm{H}^2(\mathbb{B})$ is such that $f(a) = 0$ then $f(p) = (p-a) \star g(p)$, where g is slice hyperholomorphic in $\mathbb{B}$ (see Proposition 6.1.38). The result follows since

$$f(p) = (p-a) \star g(p) = B_a(p) \star (1 - p\bar{a}) \star g(p).$$

The last claim follows from the decomposition

$$f(p) = \left(f(p) - (1-p\bar{a})^{-\star} \frac{f(a)}{1-|a|^2}\right) + \left((1-p\bar{a})^{-\star} \frac{f(a)}{1-|a|^2}\right)$$

□

Theorem 6.3.8. *Let $\{a_j\} \subset \mathbb{B}$, $j = 1, 2, \dots$ be a sequence of nonzero quaternions and assume that $\sum_{j \geq 1}(1 - |a_j|) < \infty$. Then the function*

$$B(p) := \Pi^{\star}_{j \geq 1} (1 - p\bar{a}_j)^{-\star} \star (a_j - p) \frac{\bar{a}_j}{|a_j|}, \tag{6.50}$$

where $\Pi^\star$ denotes the $\star$-product, converges uniformly on the compact subsets of $\mathbb{B}$ and defines a slice hyperholomorphic function.

Proof. Let $\alpha_j(p) := B_{a_j}(p) - 1$. Using Remark 6.3.2 we have the chain of equalities:

$$\begin{aligned} \alpha_j(p) =& B_{a_j}(p) - 1 = (1 - \tilde{p}\bar{a}_j)^{-1}(a_j - \tilde{p}) \frac{\bar{a}_j}{|a_j|} - 1 \\ =& (1 - \tilde{p}\bar{a}_j)^{-1} \left[(a_j - \tilde{p}) \frac{\bar{a}_j}{|a_j|} - (1 - \tilde{p}\bar{a}_j) \right] \\ =& (1 - \tilde{p}\bar{a}_j)^{-1} \left[(|a_j| - 1) \left(1 + \tilde{p} \frac{\bar{a}_j}{|a_j|} \right) \right]. \end{aligned}$$

Thus, if $|p| < 1$ and recalling that $|\tilde{p}| = |p|$, we have

$$|\alpha_j(p)| \leq 2(1 - |p|)^{-1}(1 - |a_j|)$$

and since $\sum_{j=1}^{\infty}(1 - |a_j|) < \infty$ then $\overset{\star}{\Pi}_{j \geq 1} \alpha_j(p) = \alpha_1(p) \star \alpha_2(p) \cdots$ converges in $\mathbb{B}$ and the statement follows. □

Theorem 5.16 in [34] assigns a Blaschke product having zeroes at a given set of points a_j with multiplicities n_j, $j \geq 1$ and at spheres $[c_i]$ with multiplicities m_i, $i \geq 1$, where the multiplicities are meant as exponents of the factors $(p - a_j)$ and $(p^2 - \mathrm{Re}(a_j)p + |a_j|^2)$, respectively. In view of Definition 4.2.7, the polynomial $(p - a_j)^{\star n_j}$ is not the unique polynomial having a zero at a_j with the given multiplicity n_j, thus the Blaschke product $\prod_{j=1}^{\star n_j} B_{a_j}$ is not the unique Blaschke product having zero at a_j with multiplicity n_j.

We give below a form of Theorem 5.16 in [34] in which we use the notion of multiplicity in Definition 4.2.7:

Theorem 6.3.9. *A Blaschke product having zeroes at the set*

$$Z = \{(a_1, n_1), \dots, ([c_1], m_1), \dots\}$$

where $a_j \in \mathbb{B}$, a_j have respective multiplicities $n_j \geq 1$, $a_j \neq 0$ for $j = 1, 2, \dots$, $[a_i] \neq [a_j]$ if $i \neq j$, $c_i \in \mathbb{B}$, the spheres $[c_j]$ have respective multiplicities $m_j \geq 1$, $j = 1, 2, \dots$, $[c_i] \neq [c_j]$ if $i \neq j$ and

$$\sum_{i,j \geq 1} \Big(n_i(1 - |a_i|) + 2m_j(1 - |c_j|) \Big) < \infty \tag{6.51}$$

is of the form

$$\prod_{i \geq 1} (B_{[c_i]}(p))^{m_i} \prod_{i \geq 1}^{\star} \prod_{j=1}^{\star n_i} (B_{\alpha_{ij}}(p)), \tag{6.52}$$

where $n_j \geq 1$, $\alpha_{11} = a_1$ and α_{ij} are suitable elements in $[a_i]$ for $j = 2, 3, \dots$.

Proof. The fact that (6.51) ensure the convergence of the product follows from Theorem 6.3.8. The zeroes of the pointwise product $\prod_{i \geq 1} (B_{[c_i]}(p))^{m_i}$ correspond to the given spheres with their multiplicities. Let us consider the product:

$$\prod_{i=1}^{\star n_1} (B_{\alpha_{i1}}(p)) = B_{\alpha_{11}}(p) \star B_{\alpha_{12}}(p) \star \cdots \star B_{\alpha_{1n_1}}(p).$$

From the definition of multiplicity, see Definition 4.2.7,this product admits a zero at the point $\alpha_{11} = a_1$ and it is a zero of multiplicity 1 if $n_1 = 1$; if $n_1 \geq 2$, the other zeroes are $\tilde{\alpha}_{12}, \dots, \tilde{\alpha}_{1n_1}$ where $\tilde{\alpha}_{1j}$ belong to the sphere $[\alpha_{1j}] = [a_1]$. This fact can be seen directly using formula (6.16). Thus a_1 is a zero of multiplicity n_1. Let us now consider $r \geq 2$ and

$$\prod_{j=1}^{\star n_r} (B_{\alpha_{rj}}(p)) = B_{\alpha_{r1}}(p) \star \cdots \star B_{\alpha_{rn_r}}(p), \tag{6.53}$$

and set

$$B_{r-1}(p) := \prod_{i \geq 1}^{\star (r-1)} \prod_{j=1}^{\star n_i} (B_{\alpha_{ij}}(p)).$$

Then

$$B_{r-1}(p) \star B_{\alpha_{r1}}(p) = B_{r-1}(p) B_{\alpha_{r1}}(B_{r-1}(p)^{-1} p B_{r-1}(p))$$

has a zero at a_r if and only if $B_{\alpha_{r1}}(B_{r-1}(a_r)^{-1} a_r B_{r-1}(a_r)) = 0$, i.e. if and only if $\alpha_{r1} = B_{r-1}(a_r)^{-1} a_r B_{r-1}(a_r)$. If $n_r = 1$ then a_r is a zero of multiplicity 1 while if $n_r \geq 2$, all the other zeroes of the product (6.53) belongs to the sphere $[a_r]$ thus the zero a_r has multiplicity n_r. This completes the proof. □

To prove the following important result which will be used in Chapter 10, we need to define the notion of degree of a finite Blaschke product. Recalling that the Blaschke factor $B_{[a]}(p)$ at the sphere $[a]$ can be seen as $B_{[a]}(p) = B_a(p) \star B_{\bar{a}}(p)$, the degree of a Blaschke product is defined to be the number of factors of the form B_q, $q \in \mathbb{H}$ appearing in it.

Proposition 6.3.10. *Let $B(p)$ be a Blaschke product as in* (6.52)*. Then* $\dim(\mathscr{H}(B)) = \deg B$.

Proof. Let us rewrite $B(p)$ as

$$B(p) = \prod_{i=1}^{r}(B_{c_i}(p) \star B_{\bar{c}_i}(p))^{m_i} \prod_{i=1}^{\star s} \prod_{j=1}^{\star n_i} (B_{\alpha_{ij}}(p)) = \prod_{j=1}^{\star d} B_{\beta_j}(p),$$

where $d = \deg B$. Let us first observe that in the case in which the factors B_{β_j} are such that no three of the quaternions β_j belong to the same sphere, then the statement follows from the fact that $\mathscr{H}(B_{\beta_j})$ is the span of $(1-p\bar{\beta}_j)^{-\star}$. Moreover $(1-p\bar{\beta}_1)^{-\star}, \dots, (1-p\bar{\beta}_d)^{-\star}$ are linearly independent in the Hardy space $\mathrm{H}^2(\mathbb{B})$, see [19, Remark 3.1]. So we now assume that $d \geq 3$ and at least three among the β_j's belong the same sphere. We proceed by induction. Assume that $d = 3$ and $\beta_1, \beta_2, \beta_3$ belong to the same sphere. Since

$$\begin{aligned} K_B(p,q) = \sum_{n=0}^{\infty} p^n(1-B(p)B(q)^*)\bar{q}^n &= \sum_{n=0}^{\infty} p^n(1-B_{\beta_1}(p)B_{\beta_1}(q)^*)\bar{q}^n \\ &+ B_{\beta_1}(p) \star \sum_{n=0}^{\infty} p^n(1-B_{\beta_2}(p)B_{\beta_2}(q)^*)\bar{q}^n \star_r B_{\beta_1}(q)^* \\ &+ B_{\beta_1}(p) \star B_{\beta_2}(p) \star \sum_{n=0}^{\infty} p^n(1-B_{\beta_3}(p)B_{\beta_3}(q)^*)\bar{q}^n \star_r B_{\beta_1}(q)^* \star_r B_{\beta_1}(q)^* \end{aligned}$$

we have

$$\mathscr{H}(B_{\beta_1}) + B_{\beta_1} \star \mathscr{H}(B_{\beta_2}) + B_{\beta_1} \star B_{\beta_2} \star \mathscr{H}(B_{\beta_3}). \tag{6.54}$$

Now note that $\mathscr{H}(B_{\beta_1})$ is spanned by $f_1(p) = (1-p\bar{\beta}_1)^{-\star}$, while $B_{\beta_1} \star \mathscr{H}(B_{\beta_2})$ is spanned by $f_2(p) = B_{\beta_1}(p) \star (1-p\bar{\beta}_2)^{-\star}$ and, finally, $B_{\beta_1} \star B_{\beta_2} \star \mathscr{H}(B_{\beta_3})$ is spanned by $f_3(p) = B_{\beta_1}(p) \star B_{\beta_2}(p) \star (1-p\bar{\beta}_3)^{-\star}$. By using the reproducing property of f_1 we have $\langle f_1, f_2\rangle = 0$ and $\langle f_1, f_3\rangle = 0$. Observe that

$$\langle f_2, f_3\rangle = \langle (1-p\bar{\beta}_2)^{-\star}, B_{\beta_2}(p) \star (1-p\bar{\beta}_3)^{-\star}\rangle = 0$$

since the left multiplication by $B_{\beta_1}(p)$ is an isometry in $\mathrm{H}^2(\mathbb{B})$ and by the reproducing property of $(1-p\bar{\beta}_2)^{-\star}$. So f_1, f_2, f_3 are orthogonal in $\mathrm{H}^2(\mathbb{B})$ and so they are linearly independent. We conclude that the sum (6.54) is direct and has dimension 3. Now assume that the assertion hold when $d = n$ and there in $B(p)$ are at least three Blaschke factors at points on the same sphere. We show that the assertion holds for $d = n+1$. We generalize the above discussion by considering

$$\begin{aligned} &(\mathscr{H}(B_{\beta_1}) + B_{\beta_1} \star \mathscr{H}(B_{\beta_2}) + \cdots + B_{\beta_1} \star \cdots \star B_{\beta_{n-1}} \star \mathscr{H}(B_{\beta_n}) + \cdots + \\ &+ B_{\beta_1} \star \cdots \star B_{\beta_n} \star \mathscr{H}(B_{\beta_{n+1}}). \end{aligned} \tag{6.55}$$

Let us denote, as before, by $f_1(p) = (1-p\bar{\beta}_1)^{-\star}$ a generator of $\mathscr{H}(B_{\beta_1})$ and by $f_j(p) = B_{\beta_1} \star \cdots \star B_{\beta_{j-1}} \star (1-p\bar{\beta}_j)^{-\star}$ a generator of $B_{\beta_1} \star \cdots \star B_{\beta_{j-1}} \star \mathscr{H}(B_{\beta_j})$, $j = 1, \dots, n+1$.

By the induction hypothesis, the sum of the first n terms is direct so we show that $[f_j, f_{n+1}] = 0$ for $j = 1, \ldots, n$. This follows, as before, from the fact that the multiplication by a Blaschke factor is an isometry and by the reproducing property. The statement follows. □

6.4 The Wiener algebra

An important topic both in harmonic and complex analysis is the Wiener algebra that we have discussed in the classical case in Chapter 1. Here we extend this notion to the quaternionic case. As we shall see, the methods to prove the results are different from those used in the complex case, due to the noncommutativity of the quaternions. Here we will treat the case of the discrete Wiener algebra, while the continuous case has been considered in the joint paper with David Kimsey [29] (which is also the source of this section).

The set which will correspond to the Wiener algebra is introduced in the following definition.

Definition 6.4.1. Let $\mathscr{W}_{\mathbb{H}}$ be the set of functions of the form

$$f(p) = \sum_{u \in \mathbb{Z}} p^u f_u, \tag{6.56}$$

where the f_u, $u \in \mathbb{Z}$, are quaternions such that

$$\sum_{u \in \mathbb{Z}} |f_u| < \infty.$$

It is important to note that the elements of $\mathscr{W}_{\mathbb{H}}$ are continuous functions on the unit sphere $\partial\mathbb{B}$, and an element $f \in \mathscr{W}_{\mathbb{H}}$ is the sum of two slice hyperholomorphic functions: the function $\sum_{u \geq 0} p^u f_u$ which is slice hyperholomorphic in $|p| < 1$ and continuous in $|p| \leq 1$, and $\sum_{u<0} p^u f_u$ which is slice hyperholomorphic in $|p| > 1$ and continuous in $|p| \geq 1$.
We endow the set $\mathscr{W}_{\mathbb{H}}$ with the multiplication given by the convolution of the coefficients (compare with 6.15 and Section 4.2)

$$(f \star g)(p) = \sum_{u \in \mathbb{Z}} p^u \Big(\sum_{k \in \mathbb{Z}} f_{u-k} g_k\Big). \tag{6.57}$$

Also in this setting we have the formula (6.16)

$$(f \star g)(p) = f(p) g(f(p)^{-1} p f(p))$$

which holds for $f(p) \neq 0$ while for $f(p) = 0$ we have $(f \star g)(p) = 0$. The proof of this formula follows the lines of the proof of Proposition 4.2.1.

Proposition 6.4.2. *The set $\mathscr{W}_{\mathbb{H}}$ endowed with the $\star$-multiplication is a real Banach algebra. The $\star$-product is in particular jointly continuous in the two variables.*

Proof. The assertion follows from the chain of inequalities:

$$\sum_{u\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} f_{u-k}g_k| \le \sum_{u\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |f_{u-k}|\cdot|g_k| = (\sum_{u\in\mathbb{Z}} |f_u|)(\sum_{k\in\mathbb{Z}} |g_k|).$$

□

Definition 6.4.3. We will call $\mathscr{W}_{\mathbb{H}}$ the quaternionic Wiener algebra.

In contrast to the complex case (see Remark 1.3.2) the quaternionic Wiener algebra is not closed under conjugation. To palliate this we define for $f(p) = \sum_{u\in\mathbb{Z}} p^u f_u \in \mathscr{W}_{\mathbb{H}}^{r\times r}$:

$$f^c(p) = \sum_{u\in\mathbb{Z}} p^u f_u^*. \tag{6.58}$$

Proposition 6.4.4. *Let $f \in \mathscr{W}_{\mathbb{H}}^{r\times r}$. Then $f \star f^c$ has self-adjoint coefficients. When $r = 1$ we have $f^c \star f = f \star f^c$.*

Proof. Let $(f\star f^c)(p) = \sum_{u\in\mathbb{Z}} p^u g_u$. The claims follow from the formula

$$c_u = \sum_{k\in\mathbb{Z}} f_{u-k} f_u^*,$$

(see also formula (6.18)) for the case $n = 1$. □

Let $\mathrm{I},\mathrm{J} \in \mathbb{S}$ be any two orthogonal elements. Any quaternion p can be written as $p = z + w\mathrm{J}$, where z, w complex numbers belonging to the complex plane $\mathbb{C}_{\mathrm{I}} = \{x + \mathrm{I}y : x, y \in \mathbb{R}\}$. In this section, we will sometimes write $\mathbb{C}$ instead of $\mathbb{C}_{\mathrm{I}}$, for short. Let $f \in \mathscr{W}_{\mathbb{H}}$ and consider the restriction of f to the unit circle, by writing $f(e^{\mathrm{I}t}) = a(e^{\mathrm{I}t}) + b(e^{\mathrm{I}t})\mathrm{J}$. Applying the map χ defined in (4.1) to $f(e^{\mathrm{I}t})$ we have

$$\chi(f(e^{\mathrm{I}t})) = \begin{pmatrix} a(e^{\mathrm{I}t}) & b(e^{\mathrm{I}t}) \\ -\overline{b(e^{\mathrm{I}t})} & \overline{a(e^{\mathrm{I}t})} \end{pmatrix}. \tag{6.59}$$

The functions

$$a(e^{\mathrm{I}t}) = \sum_{u\in\mathbb{Z}} e^{\mathrm{I}ut} a_u \quad \text{and} \quad b(e^{\mathrm{I}t}) = \sum_{u\in\mathbb{Z}} e^{\mathrm{I}ut} b_u \tag{6.60}$$

belong to the classical Wiener algebra $\mathscr{W} = \mathscr{W}^{1\times 1}$ and so

$$\chi(f(e^{\mathrm{I}t})) \in \mathscr{W}^{2\times 2}. \tag{6.61}$$

The map χ is not multiplicative with respect to the $\star$-product, in fact in general

$$\chi((f\star g)(e^{\mathrm{I}t})) \ne \chi((f)(e^{It}))\chi((g)(e^{\mathrm{I}t})).$$

For this reason, we now introduce another map depending, as the map χ, on the choice of the two orthogonal imaginary units $\mathrm{I},\mathrm{J} \in \mathbb{S}$. This map is denoted by $\omega_{\mathrm{I},\mathrm{J}}$ or ω, for simplicity, and acts from the values of functions (belonging to a given set of functions) to the set of 2×2 matrices whose elements are complex-valued functions when restricted to the complex plane $\mathbb{C}$. The map ω is defined by

$$(\omega(f))(z) = \begin{pmatrix} a(z) & b(z) \\ -\overline{b(\bar{z})} & \overline{a(\bar{z})} \end{pmatrix}, \quad z \in \overline{\mathbb{B}\cap\mathbb{C}}. \tag{6.62}$$

The map ω is multiplicative with respect to the $\star$-product, as we prove in the following result.

Lemma 6.4.5. *Let $f,g \in \mathscr{W}_{\mathbb{H}}$. Then it is immediate that*

$$(\omega(f\star g))(z) = (\omega(f))(z)(\omega(g))(z), \quad z \in \overline{\mathbb{B}\cap\mathbb{C}}. \tag{6.63}$$

Proof. Let $f(p) = p^n a$ and $g(p) = p^m b$ for $n,m \in \mathbb{Z}$ and $a,b \in \mathbb{H}$. Then,

$$(f\star g)(p) = p^{n+m}ab,$$

and we have

$$(\omega(f))(z) = z^n\chi(a), \qquad (\omega(g))(z) = z^m\chi(b)$$

and

$$(\omega(f\star g))(z) = z^{n+m}\chi(ab).$$

Thus we have

$$(\omega(f\star g))(z) = z^{n+m}\chi(ab) = z^{n+m}\chi(a)\chi(b) = (\omega(f))(z)(\omega(g))(z).$$

In general, if $f(p) = \sum_{n\in\mathbb{Z}} p^n a_n \in \mathscr{W}_{\mathbb{H}}$, then

$$(\omega(f))(z) = \sum_{n\in\mathbb{Z}} z^n\chi(a_n) \in \mathscr{W}^{2\times 2},$$

and if $g(p) = \sum_{n\in\mathbb{Z}} p^n b_n \in \mathscr{W}_{\mathbb{H}}$, then

$$\begin{aligned} ((\omega(f))(z))((\omega(g))(z)) &= \left(\sum_{n\in\mathbb{Z}} z^n\chi(a_n)\right)\left(\sum_{n\in\mathbb{Z}} z^n\chi(b_n)\right) \\ &= \sum_{u\in\mathbb{Z}} z^u \left(\sum_{n+m=u} \chi(a_n)\chi(b_m)\right) \\ &= \sum_{u\in\mathbb{Z}} z^u \chi\left(\sum_{n+m=u} a_n b_m\right) \\ &= (\omega(f\star g))(z). \end{aligned}$$

This concludes the proof. □

Remark 6.4.6. As we already observed in [29] the functions $a(e^{\mathrm{I}t})$, $b(e^{\mathrm{I}t})$ belong to $\mathscr{W}$ and so $\omega(f)(z)$ belongs to $\mathscr{W}^{2\times 2}$.

Theorem 6.4.7. *Let $f \in \mathscr{W}_{\mathbb{H}}$. The following are equivalent:*

(1) f is invertible in $\mathscr{W}_{\mathbb{H}}$;

(2) Let I *be any fixed element in* $\mathbb{S}$*, then* $(\det \omega(f))(z) \neq 0$ *for all* $z \in \partial\mathbb{B} \cap \mathbb{C}_{\mathrm{I}}$;

(3) The function f does not vanish on $\partial\mathbb{B}$.

Proof. The fact that $(1) \Longrightarrow (2)$ follows from (6.63): in fact, if there exists $g \in \mathscr{W}_{\mathbb{H}}$ such that $f \star g = 1$ then

$$I_2 = (\omega(f)(e^{\mathrm{I}t}))(\omega(g)(e^{\mathrm{I}t}))$$

and so $\omega(f)(e^{\mathrm{I}t})$ is invertible for every $t \in [0, 2\pi)$.

Conversely, to show $(2) \Longrightarrow (1)$, we know from Remark 6.4.6 that $\omega(f)(z) \in \mathscr{W}^{2\times 2}$. By the classical matricial Wiener-Lévy theorem the condition $\det \omega(f)(e^{\mathrm{I}t}) \neq 0$ for all $t \in [0, 2\pi)$ implies that $\omega(f)$ is invertible in $\mathscr{W}^{2\times 2}$. Let $G \in \mathscr{W}^{2\times 2}$ be such that $\omega(f)G = I_2$. The matrix G can be computed using the formula for the inverse of a 2×2 and so G is of the form

$$G(e^{\mathrm{I}t}) = \begin{pmatrix} \dfrac{c(e^{\mathrm{I}t})}{-d(e^{-\mathrm{I}t})} & \dfrac{d(e^{\mathrm{I}t})}{c(e^{-\mathrm{I}t})} \end{pmatrix},$$

where $c(e^{-\mathrm{I}t}) = \overline{a(e^{-\mathrm{I}t})}/(\det \omega(f))(e^{\mathrm{I}t})$ and $d(e^{-\mathrm{I}t}) = -b(e^{\mathrm{I}t})/(\det \omega(f))(e^{\mathrm{I}t})$ so that they belong to $\mathscr{W}$ and we can write

$$c(e^{\mathrm{I}t}) = \sum_{n\in\mathbb{Z}} e^{\mathrm{I}nt} c_n \quad \text{and} \quad d(e^{\mathrm{I}t}) = \sum_{n\in\mathbb{Z}} e^{\mathrm{I}nt} d_n.$$

The function defined by

$$g(e^{\mathrm{I}t}) = \sum_{n\in\mathbb{Z}} e^{\mathrm{I}nt}(c_n + d_n \mathrm{J}),$$

belongs to $\mathscr{W}_{\mathbb{H}}$. Since $\omega(f)\omega(g) = 1$ on the unit circle, if $f(p) = \sum_{n\in\mathbb{Z}} p^n f_n$ and $g(p) = \sum_{n\in\mathbb{Z}} p^n g_n$, we have

$$\chi\left(\sum_{n+m=u} f_n g_m\right) = \begin{cases} 0 \text{ if } u \neq 0, \\ 1 \text{ if } u = 0. \end{cases}$$

Hence $(f \star g)(e^{\mathrm{I}t}) = 1$ and so $f \star g = 1$ everywhere, since the latter is uniquely determined by its values on the unit circle.

We now show the equivalence between (2) and (3). With the notation in (6.62) and some computations, we have that for some fixed $i \in \mathbb{S}$

$$(\det \omega(f))(z) = a(z)\overline{a(\bar{z})} + b(z)\overline{b(\bar{z})} = \sum_{u\in\mathbb{Z}} z^u (\sum_{k\in\mathbb{Z}} a_{u-k}\overline{a_k} + b_{u-k}\overline{b_k}),$$

where the functions a and b are defined by (6.60).

Given $f(p)=\sum_{u\in\mathbb{Z}} p^u f_u \in \mathscr{W}_{\mathbb{H}}$ we define the function $f^c(p)=\sum_{u\in\mathbb{Z}} p^u \overline{f_u}$ by mimicking the definition in the slice hyperholomorphic case (compare with Definition 6.1.24 and the note below it). Obviously, the function $f^c(p)$ belongs to $\mathscr{W}_{\mathbb{H}}$. Consider now the function

$$(f\star f^c)(p)=\sum_{u\in\mathbb{Z}} p^u(\sum_{k\in\mathbb{Z}} f_{u-k}\overline{f_k}).$$

An immediate computation shows that $f\star f^c$ has real coefficients, thus the zeros of its restriction $(f\star f^c)|_{\mathbb{C}}(z)$ to $\mathbb{C}$ has zeros which are real points and/or complex conjugate points. Since the representation formula, see Theorem 6.1.12, is valid for power series, it follows that $(f\star f^c)(p)$ has zero set consisting of real points and/or spheres. By formula (6.16), we have that $(f\star f^c)(p_0)=0$ implies $f(p_0)=0$ or $f(p_0)\neq 0$ and $f^c(f(p_0)^{-1}p_0 f(p_0))=0$ and the element $f(p_0)^{-1}p_0 f(p_0)$ belongs to $[p_0]$. We now claim that the zeros of f^c on $[p_0]$ are in one-to-one correspondence with the zeros of f on $[p_0]$. To see this fact, let us write

$$f(x+\mathrm{I}y)=\sum_{u\in\mathbb{Z}}(x+\mathrm{I}y)^u f_u=\sum_{u\in\mathbb{Z}}(s_u(x,y)+\mathrm{I}t_u(x,y))f_u=s(x,y)+\mathrm{I}t(x,y)$$

where the functions s_u, t_u are real valued, and

$$f^c(x+\mathrm{I}y)=\sum_{u\in\mathbb{Z}}(x+\mathrm{I}y)^u \bar{f}_u=\sum_{u\in\mathbb{Z}}(s_u(x,y)+\mathrm{I}t_u(x,y))\bar{f}_u=\overline{s(x,y)}+\mathrm{I}\overline{t(x,y)}.$$

If f vanishes at all points of $[p_0]$ then $s(x_0,y_0)=t(x_0,y_0)=0$ and so also f^c vanishes at all points of $[p_0]$. If p_0 is the only zero of f belonging to $[p_0]$ then $t(x_0,y_0)\neq 0$. An immediate computation shows that $f^c(x_0+\tilde{\mathrm{I}}y_0)=0$ where $\tilde{\mathrm{I}}=-\overline{t(x_0,y_0)}\mathrm{I}\,\overline{t(x_0,y_0)}^{-1}$. The converse follows in an analogous way, since $(f^c)^c=f$. We conclude that if f^c has a zero belonging to the sphere $[p_0]$ also f must have a zero belonging to the same sphere. Now we observe that since $f_u=a_u+b_u\mathrm{J}$ and $\overline{f_u}=\overline{a_u}-b_u\mathrm{J}$ we have

$$\begin{aligned}(f\star f^c)|_{\mathbb{C}}(z)&=\sum_{u\in\mathbb{Z}} p^u(\sum_{k\in\mathbb{Z}} f_{u-k}\overline{f_k})=\sum_{u\in\mathbb{Z}} p^u(\sum_{k\in\mathbb{Z}}(a_{u-k}+b_{u-k}\mathrm{J})(\bar{a}_k-b_k\mathrm{J}))\\&=\sum_{u\in\mathbb{Z}} p^u\{\sum_{k\in\mathbb{Z}}(a_{u-k}\bar{a}_k+b_{u-k}\bar{b}_k+(b_{u-k}a_k-a_{u-k}b_k)J)\}\end{aligned}$$

and so $(\det\omega(f))(z)=(f\star f^c)|_{\mathbb{C}}(z)$.
If (2) holds then f cannot have zeros on $\partial\mathbb{B}$ otherwise, if $f(p_0)=0$ we have that $f\star f^c$ vanishes on $[p_0]$ and in particular on $[p_0]\cap\mathbb{C}$ and so $\det(\omega(f))$ vanishes on $\partial\mathbb{B}\cap\mathbb{C}$. Conversely, if f does not vanish on $\partial\mathbb{B}$ neither f^c vanishes there and so $f\star f^c$ does not have zeros on $\partial\mathbb{B}$ and thus $\det(\omega(f))$ does not vanish on $\partial\mathbb{B}\cap\mathbb{C}$. ☐

Remark 6.4.8. It is important to point out that condition (2) holds on a fixed plane while condition (3) refers to the whole boundary of $\mathbb{B}$.
Consider, for example, $f(p)=p-j$. Then f does not have any zero on the complex plane $\mathbb{C}_i$. However, $\det(\omega(f))(e^{\pm i\pi})=0$.

Definition 6.4.9. We denote by $\mathscr{W}_{\mathbb{H},+}$ (resp. $\mathscr{W}_{\mathbb{H},-}$) the set of elements $f(p)=\sum_{n\in\mathbb{Z}} p^n f_n\in\mathscr{W}_{\mathbb{H},}$ for which $f_n=0$ for $n<0$ (resp. for $n>0$).

Remark 6.4.10. It is immediate that $\mathscr{W}_{\mathbb{H},+}$ and $\mathscr{W}_{\mathbb{H},-}$ are subalgebras of $\mathscr{W}_{\mathbb{H}}$.

We now address the question of invertibility of an element in $\mathscr{W}_{\mathbb{H},+}$. Note that there are functions invertible in $\mathscr{W}_{\mathbb{H}}$, but not in $\mathscr{W}_{\mathbb{H},+}$. As an example, consider $f(p) = p$: it is invertible in $\mathscr{W}_{\mathbb{H}}$, but not in $\mathscr{W}_{\mathbb{H},+}$.
The characterization of functions invertible in $\mathscr{W}_{\mathbb{H},+}$ is the following.

Theorem 6.4.11. *Let $f \in \mathscr{W}_{\mathbb{H},+}$. The following are equivalent*:

(1) The function f is invertible in $\mathscr{W}_{\mathbb{H},+}$;

(2) Let I *be any fixed element in* $\mathbb{S}$, *then* $(\det \omega(f))(z) \neq 0$ *for all* $z \in \overline{\mathbb{B} \cap \mathbb{C}_{\mathrm{I}}}$;

(3) The function f does not vanish on $\overline{\mathbb{B}}$.

Proof. The proof follows the one of Theorem 6.4.7 so we only provide details for the implication $(2) \Longrightarrow (1)$.
First of all, we note that $\omega(f) \in \mathscr{W}_{+}^{2\times 2}$. By the result for invertiblity of matrix-valued functions in the Wiener algebra $\mathscr{W}_{+}^{2\times 2}$ (see Chapter 1), the condition $\det(\omega(f))(z) \neq 0$ for $z \in \overline{\mathbb{B} \cap \mathbb{C}}$ implies that $\omega(f)$ is invertible in $\mathscr{W}_{+}^{2\times 2}$. Let $G \in \mathscr{W}_{+}^{2\times 2}$ be such that $\omega(f)G = I_2$ on $\overline{\mathbb{B} \cap \mathbb{C}}$. As in the proof of Theorem 6.4.7 we get that

$$G(z) = \begin{pmatrix} \frac{c(z)}{-d(\bar{z})} & \frac{d(z)}{c(z)} \end{pmatrix},$$

where $c(z) = \overline{a(\bar{z})}\det(\omega(f))(z)$ and $d(z) = -b(z)\det(\omega(f))(z)$. Thus, we may write $c(z) = \sum_{n=0}^{\infty} z^n c_n$ and $d(z) = \sum_{n=0}^{\infty} z^n d_n$ and if we set $g(z) = \sum_{n=0}^{\infty} z^n (c_n + d_n \mathrm{J})$, then $g \in \mathscr{W}_{\mathbb{H},+}$. The rest of the argument follows as in the proof of Theorem 6.4.7. □

Definition 6.4.12. We say that $f \in \mathscr{W}_{\mathbb{H}}$ is strictly positive if

$$\omega(f)(e^{\mathrm{I}t}) > 0, \quad t \in [0, 2\pi).$$

The previous definition depends on $\mathrm{I} \in \mathbb{S}$. We now give an intrinsic characterization of strictly positive elements of $\mathscr{W}_{\mathscr{H}}$.

Theorem 6.4.13. *$f \in \mathscr{W}_{\mathbb{H}}$ is strictly positive if and only if it can be written as $f = f_+ \star f_+^c$, where f_+ is an invertible element of $\mathscr{W}_{\mathbb{H},+}$.*

Proof. By the classical Wiener-Hopf theory, there is an element $A \in \mathscr{W}_{+}^{2\times 2}$, unique up to a right multiplicative unitary constant, such that $A^{-1} \in \mathscr{W}_{+}^{2\times 2}$ and

$$\omega(f)(e^{\mathrm{I}t}) = A(e^{\mathrm{I}t})A(e^{\mathrm{I}t})^*.$$

Let us set

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.64}$$

We have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{a(e^{-\mathrm{I}t})} & \overline{b(e^{-\mathrm{I}t})} \\ -b(e^{\mathrm{I}t}) & a(e^{\mathrm{I}t}) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{a(e^{\mathrm{I}t})}{-b(e^{-\mathrm{I}t})} & \frac{b(e^{\mathrm{I}t})}{a(e^{-\mathrm{I}t})} \end{pmatrix}$$

from which it follows

$$J_1\overline{\omega(f)(e^{-\mathrm{I}t})}J_1^* = \omega(f)(e^{\mathrm{I}t}).$$

The function

$$B(e^{\mathrm{I}t}) = J_1\overline{A(e^{-\mathrm{I}t})}J_1^*$$

is an invertible element of $\mathscr{W}_+^{2\times 2}$. Thus, for a unitary constant U we can write

$$A(e^{\mathrm{I}t}) = J_1\overline{A(e^{-\mathrm{I}t})}J_1^*U. \tag{6.65}$$

Moreover, $A(1)$ is invertible since $A(e^{\mathrm{I}t})$ is invertible in $\mathscr{W}_+^{2\times 2}$. By replacing $A(z)$ by $A(z)A(1)^*(A(1)A(1)^*)^{-1/2}$ we can always choose $A(1)>0$, in other words, we can assume that $A(1)$ is a positive definite matrix. This forces $U = I_2$. In fact, let $A(1) = \begin{pmatrix} a & \overline{c} \\ c & d \end{pmatrix}$. Then

$$A(1) = J_1\overline{A(1)}J_1^*$$

and (6.65) leads to $U = I_2$. By setting $A(z) = \sum_{n=0}^{\infty} z^n A_n$ we have

$$A_n = J_1\overline{A_n}J_1^*,$$

and so

$$A(e^{\mathrm{I}t}) = \omega(a_+)(e^{\mathrm{I}t})$$

where $a_+(p) = \sum_{n=0}^{\infty} p^n a_n$ and $\chi(a_n) = A_n$. □

6.5 The Hardy space of the open half-space

Let $\mathbb{C}_r$ be the complex open right half plane. The function $\dfrac{1}{2\pi(z+\overline{w})}$ is positive in $\mathbb{C}_r$, and its associated reproducing kernel Hilbert space is the Hardy space $\mathrm{H}^2(\mathbb{C}_r)$. The counterpart of $\mathbb{C}_r$ is the open right half-space of the quaternions with real positive part which will be denoted by $\mathbb{H}_+$. In order to define the Hardy space in this framework, we set $\Pi_{+,\mathrm{I}} = \mathbb{H}_+\cap\mathbb{C}_\mathrm{I}$ and define (see the paper [32] with Izchak Lewkowicz)

$$\mathrm{H}^2(\Pi_{+,\mathrm{I}}) = \{f\in\mathscr{R}(\mathbb{H}_+)\ :\ \int_{-\infty}^{+\infty}|f_\mathrm{I}(\mathrm{I}y)|^2dy<\infty\},$$

where $f(\mathrm{I}y)$ denotes the nontangential value of f at $\mathrm{I}y$. These values exist almost everywhere, in fact by the splitting lemma, any $f\in\mathrm{H}^2(\Pi_{+,\mathrm{I}})$ restricted to a complex plane $\mathbb{C}_\mathrm{I}$ can be written as $f_\mathrm{I}(x+\mathrm{I}y) = F(x+\mathrm{I}y)+G(x+\mathrm{I}y)\mathrm{J}$ where J is any element in $\mathbb{S}$ orthogonal to I, and F,G are $\mathbb{C}_\mathrm{I}$-valued holomorphic functions. Since the nontangential values of F and G exist almost everywhere at $\mathrm{I}y$, also the nontangential value of f exists at $\mathrm{I}y$ a. e. on $\Pi_{+,\mathrm{I}}$ and $f_\mathrm{I}(\mathrm{I}y) = F(\mathrm{I}y)+G(\mathrm{I}y)\mathrm{J}$ almost everywhere.

It is also possible to set

$$\mathrm{H}^2(\Pi_{+,\mathrm{I}}) = \{f\in\mathscr{R}(\mathbb{H}_+)\ :\ \sup_{x>0}\int_{-\infty}^{+\infty}|f_\mathrm{I}(x+\mathrm{I}y)|^2dy<\infty\}.$$

The formula

$$f_{\mathrm{I}}(x+\mathrm{I}y)=F(x+\mathrm{I}y)+G(x+\mathrm{I}y)\mathrm{J}$$

gives

$$|f_{\mathrm{I}}(x+\mathrm{I}y)|^2=|F(x+\mathrm{I}y)|^2+|G(x+\mathrm{I}y)|^2.$$

Thus we have

$$\begin{aligned}\sup_{x>0}\int_{-\infty}^{+\infty}|f_{\mathrm{I}}(x+\mathrm{I}y)|^2dy&=\sup_{x>0}\int_{-\infty}^{+\infty}|F(x+\mathrm{I}y)|^2dy+\sup_{x>0}\int_{-\infty}^{+\infty}|G(x+\mathrm{I}y)|^2dy\\&=\int_{-\infty}^{+\infty}|F(\mathrm{I}y)|^2dy+\int_{-\infty}^{+\infty}|G(\mathrm{I}y)|^2dy\\&=\int_{-\infty}^{+\infty}|f_{\mathrm{I}}(\mathrm{I}y)|^2dy.\end{aligned}\tag{6.66}$$

We then equip the quaternionic right linear space $\mathrm{H}^2(\Pi_{+,\mathrm{I}})$ with the scalar product

$$\langle f,g\rangle_{\mathrm{H}^2(\Pi_{+,\mathrm{I}})}=\int_{-\infty}^{+\infty}\overline{g_{\mathrm{I}}(\mathrm{I}y)}f_{\mathrm{I}}(\mathrm{I}y)dy,$$

where $f_{\mathrm{I}}(\mathrm{I}y)$, $g_{\mathrm{I}}(\mathrm{I}y)$ denote the nontangential values of f,g at $\mathrm{I}y$ on $\Pi_{+,\mathrm{I}}$. This scalar product gives the norm

$$\|f\|_{\mathrm{H}^2(\Pi_{+,\mathrm{I}})}=\left(\int_{-\infty}^{+\infty}|f_{\mathrm{I}}(\mathrm{I}y)|^2dy\right)^{\frac{1}{2}},$$

which is finite, by our assumptions.

Proposition 6.5.1. *Let f be slice hyperholomorphic in $\mathbb{H}_+$ and let $f\in\mathrm{H}^2(\Pi_{+,\mathrm{I}})$ for some $\mathrm{I}\in\mathbb{S}$. Then for all $\mathrm{J}\in\mathbb{S}$ the following inequalities hold*

$$\frac{1}{2}\|f\|_{\mathrm{H}^2(\Pi_{+,\mathrm{I}})}\leq\|f\|_{\mathrm{H}^2(\Pi_{+,\mathrm{J}})}\leq 2\|f\|_{\mathrm{H}^2(\Pi_{+,\mathrm{I}})}.$$

Proof. The Representation formula implies the inequality

$$|f(x+\mathrm{J}y)|\leq|f(x+\mathrm{I}y)|+|f(x-\mathrm{I}y)|,$$

which yields

$$|f(x+\mathrm{J}y)|^2\leq 2(|f(x+\mathrm{I}y)|^2+|f(x-\mathrm{I}y)|^2).\tag{6.67}$$

Using also (6.66) and (6.67) we deduce

$$\begin{aligned}\|f\|^2_{\mathrm{H}^2(\Pi_{+,\mathrm{J}})}=\int_{-\infty}^{+\infty}|f_{\mathrm{J}}(\mathrm{J}y)|^2dy&=\sup_{x>0}\int_{-\infty}^{+\infty}|f_{\mathrm{J}}(x+\mathrm{J}y)|^2dy\\&\leq\sup_{x>0}\int_{-\infty}^{+\infty}2(|f_{\mathrm{I}}(x+\mathrm{I}y)|^2+f_{\mathrm{I}}(x-\mathrm{I}y)|^2)dy\\&=4\int_{-\infty}^{+\infty}|f_{\mathrm{I}}(\mathrm{I}y)|^2dy\end{aligned}$$

and so $\|f\|^2_{\mathrm{H}^2(\Pi_{+,\mathrm{J}})} \leq 4\|f\|^2_{\mathrm{H}^2(\Pi_{+,\mathrm{I}})}$. By changing the role of J and I we obtain the reverse inequality and the statement follows. □

An immediate consequence is:

Corollary 6.5.2. *A function $f \in \mathrm{H}^2(\Pi_{+,\mathrm{I}})$ for some $\mathrm{I} \in \mathbb{S}$ if and only if $f \in \mathrm{H}^2(\Pi_{+,\mathrm{J}})$ for all $\mathrm{J} \in \mathbb{S}$.*

We now introduce the Hardy space of the half space $\mathbb{H}_+$:

Definition 6.5.3. We define $\mathrm{H}^2(\mathbb{H}_+)$ as the space of functions $f \in \mathscr{R}(\mathbb{H}_+)$ such that

$$\sup_{\mathrm{I}\in\mathbb{S}} \int_{-\infty}^{+\infty} |f(\mathrm{I}y)|^2 dy < \infty. \tag{6.68}$$

We have:

Proposition 6.5.4. *The function*

$$k(p,q) = (\bar{p}+\bar{q})(|p|^2 + 2\mathrm{Re}(p)\bar{q} + \bar{q}^2)^{-1} \tag{6.69}$$

is slice hyperholomorphic in p and $\bar{q}$ on the left and on the right, respectively in its domain of definition, i.e. for $p \notin [-q]$. The restriction of $\frac{1}{2\pi}k(p,q)$ to $\mathbb{C}_\mathrm{I}\times\mathbb{C}_\mathrm{I}$ coincides with $k_{\Pi_+}(z,w)$. Moreover we have the equality:

$$k(p,q) = (|q|^2 + 2\mathrm{Re}(q)p + p^2)^{-1}(p+q). \tag{6.70}$$

Proof. By taking z on the same complex plane as q, we can obtain $k(p,q)$ as the left slice hyperholomorphic extension in z of $k_q(z) = k(z,q)$. The function we obtain turns out to be also right slice hyperholomorphic in $\bar{q}$. The second equality follows by taking the right slice hyperholomorphic extension in $\bar{q}$ and observing that it is left slice hyperholomorphic in p. □

Proposition 6.5.5. *The kernel $\frac{1}{2\pi}k(p,q)$ is reproducing, i.e. for any $f \in \mathrm{H}^2(\mathbb{H}_+)$ and $\mathrm{I} \in \mathbb{S}$,*

$$f(p) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(p,\mathrm{I}y) f(\mathrm{I}y) dy.$$

Proof. Let $q = u + \mathrm{I}_q v$ and let $p = u + \mathrm{I}v \in [q] \cap \mathbb{C}_\mathrm{I}$. Then

$$f(p) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(p,\mathrm{I}y) f(\mathrm{I}y) dy, \qquad f(\bar{p}) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(\bar{p},\mathrm{I}y) f(\mathrm{I}y) dy.$$

The extension formula applied to $k_{\mathrm{I}y}(p) = k(p,\mathrm{I}y)$ proves the statement. □

The following result is a property which will be used in the sequel:

Proposition 6.5.6. *The kernel $k(p,q)$ satisfies*

$$pk(p,q) + k(p,q)\overline{q} = 1.$$

Proof. From the expression (6.69), and since q commutes with $(|p|^2+2\mathrm{Re}(p)\bar{q}+\bar{q}^2)^{-1}$, we have

$$p(\bar{p}+\bar{q})(|p|^2+2\mathrm{Re}(p)\bar{q}+\bar{q}^2)^{-1}+(\bar{p}+\bar{q})(|p|^2+2\mathrm{Re}(p)\bar{q}+\bar{q}^2)^{-1}\bar{q}$$
$$=(|p|^2+p\bar{q}+\bar{p}\bar{q}+\bar{q}^2)(|p|^2+2\mathrm{Re}(p)\bar{q}+\bar{q}^2)^{-1}=1.$$

□

We conclude with another representation of the reproducing kernel of the Hardy space $\mathrm{H}^2(\Pi_{+,\mathrm{I}})$, and take the opportunity to define the quaternionic fractional Hardy space. For $\nu>-1$ we have for $x,y\in(0,\infty)$

$$\int_0^\infty e^{-(x+y)t}t^\nu dt=\frac{\Gamma(\nu+1)}{(x+y)^{\nu+1}}. \tag{6.71}$$

The function

$$K_\nu(p,q)=\int_0^\infty e^{-pt}t^\nu e^{-\bar{q}t}dt \tag{6.72}$$

is the slice hyperholomorphic extension of (6.71) in p and $\bar{q}$ to $\mathrm{H}^2(\Pi_{+,\mathrm{I}})\times\mathrm{H}^2(\Pi_{+,\mathrm{I}})$. We note that

$$\chi(e^{-pt})=e^{-\chi(p)t}$$

and so the integral (6.72) does converge for $p,q\in\mathbb{H}_+$. By uniqueness of the slice hyperholomorphic extension, and setting $\nu=0$ we have

$$\int_0^\infty e^{-pt}e^{-\bar{q}t}dt=(|q|^2+2\mathrm{Re}(q)p+p^2)^{-1}(p+q).$$

When $\nu\neq 0$, and following the complex case, the reproducing kernel Hilbert space associated to the function $K_\nu(p,q)$ will be called the fractional Hardy space. This space plays a key role in harmonic analysis, see [164], and in the theory of self-similar systems, see [237].

6.6 Blaschke products (half-space case)

In this section, which is based on the paper [32] with Izchak Lewkowicz, we study the Blaschke factors in the half space $\mathbb{H}_+$. We begin by giving the definition:

Definition 6.6.1. For $a\in\mathbb{H}_+$ set

$$b_a(p)=(p+\bar{a})^{-\star}\star(p-a).$$

The function $b_a(p)$, which is defined outside the sphere $[-a]$, is called Blaschke factor at a in the half space $\mathbb{H}_+$.

Remark 6.6.2. By definition, a Blaschke factor b_a is obviously slice hyperholomorphic. It has a zero at $p=a\in\mathbb{R}_+$ and a sphere of poles at $[-a]$. The sphere reduces to the point $p=-a$ when $a\in\mathbb{R}$.

The following result characterizes the convergence of a Blaschke product.

Theorem 6.6.3. *Let* $\{a_j\}\subset\mathbb{H}_+$, $j=1,2,\dots$ *be a sequence of quaternions such that* $\sum_{j\geq 1}\mathrm{Re}(a_j)<\infty$. *Then the function*

$$B(p):=\Pi^{\star}_{j\geq 1}(p+\bar{a}_j)^{-\star}\star(p-a_j), \tag{6.73}$$

converges uniformly on the compact subsets of $\mathbb{H}_+$ *and it is slice hyperholomorphic.*

Proof. We follow the proof of the corresponding result in the complex case, in fact the result follows from inequalities involving moduli. Reasoning as in Remark 6.3.2, we rewrite $b_a(p)$ in terms of the pointwise multiplication as

$$(p+\bar{a}_j)^{-\star}\star(p-a_j)=(\tilde{p}+\bar{a}_j)^{-1}(\tilde{p}-a_j) \tag{6.74}$$

where $\tilde{p}=\lambda^c(p)^{-1}p\lambda^c(p)$ and $\lambda^c(p)=p+a_j$ (note that $\lambda^c(p)\neq 0$ for $p\notin[-a_j]$) and so

$$(p+\bar{a}_j)^{-\star}\star(p-a_j)=(\tilde{p}+\bar{a}_j)^{-1}(\tilde{p}-a_j)=1-2\mathrm{Re}(a_j)(\tilde{p}+\bar{a}_j)^{-1}. \tag{6.75}$$

By taking the modulus of the right hand side of (6.73), using (6.75), and reasoning as in the complex case, we conclude that the Blaschke product converges if and only if $\sum_{j=1}^{\infty}\mathrm{Re}(a_j)<\infty$. The function defined by (6.73) is slice hyperholomorphic since it is the uniform limit of the sequence of slice hyperholomorphic functions

$$p\ \mapsto\ \Pi^{\star N}_{j=1}(p+\bar{a}_j)^{-\star}\star(p-a_j).$$

□

Recalling that the zeros of a slice hyperholomorphic function are either isolated points or spheres, as in the case of the unit ball, we have we have also the Blaschke factors associated to spheres. A product of the form

$$b_a(p)\star b_{\bar{a}}(p)=((p+\bar{a})^{-\star}\star(p-a))\star((p+a)^{-\star}\star(p-\bar{a}))$$

can be rewritten as

$$b_a(p)\star b_{\bar{a}}(p)=(p^2+2\mathrm{Re}(a)p+|a|^2)^{-1}(p^2-2\mathrm{Re}(a)p+|a|^2),$$

and it has the sphere $[a]$ as set of zeros. Thus, it is convenient to introduce the following:

Definition 6.6.4. For $a\in\mathbb{H}_+$ set

$$b_{[a]}(p)=(p^2+2\mathrm{Re}(a)p+|a|^2)^{-1}(p^2-2\mathrm{Re}(a)p+|a|^2).$$

The function $b_a(p)$, which is defined for $p\notin[-a]$ is called Blaschke factor at the sphere $[a]$ in the half space $\mathbb{H}_+$.

Note that the definition is well posed since it does not depend on the choice of the point a inside the sphere of zeros. Theorem 6.6.3 yields:

Corollary 6.6.5. *Let* $\{c_j\} \subset \mathbb{H}_+$, $j = 1,2,\dots$ *be a sequence of quaternions such that* $\sum_{j\geq 1} \mathrm{Re}(c_j) < \infty$. *Then the function*

$$B(p) := \Pi_{j\geq 1}(p^2 + 2\mathrm{Re}(c_j)p + |c_j|^2)^{-1}(p^2 - 2\mathrm{Re}(c_j)p + |c_j|^2), \qquad (6.76)$$

converges uniformly on the compact subsets of $\mathbb{H}_+$.

Proof. It is sufficient to write $B(p) = \prod_{j\geq 1} b_{[c_j]}(p) = \prod_{j\geq 1} b_{c_j}(p) \star b_{\bar{c}_j}(p)$ and to observe that $2\sum_{j\geq 1} \mathrm{Re}(c_j) < \infty$ by hypothesis. □

We have:

Theorem 6.6.6. *A Blaschke product having zeros at the set*

$$Z = \{(a_1,\mu_1),(a_2,\mu_2),\dots,([c_1],\nu_1),([c_2],\nu_2),\dots\}$$

where $a_j \in \mathbb{H}_+$, a_j *have respective multiplicities* $\mu_j \geq 1$, $[a_i] \neq [a_j]$ *if* $i \neq j$, $c_i \in \mathbb{H}_+$, *the spheres* $[c_j]$ *have respective multiplicities* $\nu_j \geq 1$, $j = 1,2,\dots$, $[c_i] \neq [c_j]$ *if* $i \neq j$ *and*

$$\sum_{i,j\geq 1} \Big(\mu_j(1-|a_j|) + 2\nu_i(1-|c_i|)\Big) < \infty$$

is given by

$$\prod_{i\geq 1}(b_{[c_i]}(p))^{\nu_i} \prod_{j\geq 1}^{\star} \prod_{k=1}^{\star\mu_j}(b_{a_{jk}}(p))^{\star\mu_j}, \qquad (6.77)$$

where $a_{11} = a_1$ *and* $a_{jk} \in [a_j]$ *are suitably chosen elements,* $k = 1,2,3,\dots,\mu_j$.

Proof. The fact that the Blaschke product (6.77) converges and defines a slice hyperholomorphic function is guaranteed by Theorem 6.6.3 and its Corollary 6.6.5. Then observe that the product

$$\prod_{i=1}^{\star\mu_1}(B_{a_{i1}}(p)) = B_{a_{11}}(p) \star B_{a_{12}}(p) \star \dots \star B_{a_{1\mu_1}}(p) \qquad (6.78)$$

admits a zero at the point $a_{11} = a_1$ and it is a zero of multiplicity 1 if $n_1 = 1$; if $n_1 \geq 2$, the other zeros are $\tilde{a}_{12},\dots,\tilde{a}_{1n_1}$ where $\tilde{a}_{1j}$ belongs to the sphere $[a_{1j}] = [a_1]$. Thus $\tilde{a}_{12},\dots,\tilde{a}_{1n_1}$ all coincide with a_1 (otherwise if there was another zero in $[a_1]$ different from a_1, the wholw sphere $[a_1]$ consists of zeros) which is the only zero of the product (6.78) and it has multiplicity μ_1. Let now $r \geq 2$ and let us consider

$$\prod_{j=1}^{\star\mu_r}(B_{a_{rj}}(p)) = B_{a_{r1}}(p) \star \dots \star B_{a_{rn_r}}(p), \qquad (6.79)$$

and set

$$B_{r-1}(p) := \prod_{i\geq 1}^{\star(r-1)} \prod_{k=1}^{\star\mu_j}(B_{a_{jk}}(p)).$$

Then we have:

$$B_{r-1}(p)\star B_{a_{r1}}(p)=B_{r-1}(p)B_{a_{r1}}(B_{r-1}(p)^{-1}pB_{r-1}(p))$$

has a zero at a_r if and only if $B_{a_{r1}}(B_{r-1}(a_r)^{-1}a_rB_{r-1}(a_r))=0$, i.e. if and only if $a_{r1}=B_{r-1}(a_r)^{-1}a_rB_{r-1}(a_r)$. If $n_r=1$ then a_r is a zero of multiplicity 1 while if $\mu_r\geq 2$, all the other zeros of the product (6.79) belong to the sphere $[a_r]$ thus the zero a_r has multiplicity μ_r. □

We now prove that the operator of multiplication by a Blaschke factor is an isometry. To this end we need the following preliminary result:

Lemma 6.6.7. *Let $f\in \mathrm{H}^2(\mathbb{H}_+)$. Then $\|f\|_{\mathrm{H}^2(\mathbb{H}_+)}=\|f^c\|_{\mathrm{H}^2(\mathbb{H}_+)}$.*

Proof. By definition we have

$$\|f\|^2_{\mathrm{H}^2(\Pi_{+,\mathrm{I}})}=\int_{-\infty}^{+\infty}|f_{\mathrm{I}}(\mathrm{I}y)|^2dy=\int_{-\infty}^{+\infty}(|F(\mathrm{I}y)|^2+|G(\mathrm{I}y)|^2)dy$$

and

$$\begin{aligned}\|f^c\|^2_{\mathrm{H}^2(\Pi_{+,\mathrm{I}})}&=\int_{-\infty}^{+\infty}|f^c_{\mathrm{I}}(\mathrm{I}y)|^2dy=\int_{-\infty}^{+\infty}(|\overline{F(-\mathrm{I}y)}|^2+|G(\mathrm{I}y)|^2)dy\\&=\int_{-\infty}^{+\infty}(|F(-\mathrm{I}y)|^2+|G(\mathrm{I}y)|^2)dy.\end{aligned}$$

So we deduce that $\|f\|^2_{\mathrm{H}^2(\Pi_{+,\mathrm{I}})}=\|f^c\|^2_{\mathrm{H}^2(\Pi_{+,\mathrm{I}})}$ and the statement follows by taking the supremum for $\mathrm{I}\in\mathbb{S}$. □

Theorem 6.6.8. *Let b_a be a Blaschke factor. The operator*

$$M_{b_a}\,:\,f\mapsto b_a\star f$$

is an isometry from $\mathrm{H}^2(\mathbb{H}_+)$ into itself.

Proof. Recall that, by (6.74), we can write $b_a(p)=(\tilde{p}+\bar{a})^{-\star}(\tilde{p}-a)$ for $\tilde{p}=\lambda^c(p)^{-1}p\lambda(p)$. Let us set $\tilde{p}=\mathrm{I}y$ where $\mathrm{I}\in\mathbb{S}$. We have

$$|b_a(\mathrm{I}y)|=|(\mathrm{I}y+\bar{a})^{-1}(\mathrm{I}y-a)|=|-(\mathrm{I}y+\bar{a})^{-1}\overline{(\mathrm{I}y+\bar{a})}|=1$$

and, with similar computations, $|b^c_a(\mathrm{I}y)|=1$. By the property (2) of the conjugate of a function (see Definition 6.1.24). we have $(f\star g)^c=g^c\star f^c$. So, in order to compute $\|b_a\star f\|_{\mathrm{H}^2(\mathbb{H}_+)}$, where $f\in\mathrm{H}^2(\mathbb{H}_+)$, we compute instead $\|(b_a\star f)^c\|^2_{\mathrm{H}^2(\mathbb{H}_+)}$. Note that $(f^c\star b^c_a)(x+\mathrm{I}y)=0$ where $f^c(x+\mathrm{I}y)=0$, i.e. on a set of isolated points on $\Pi_{+,\mathrm{I}}$ while, if $q=f^c(x+\mathrm{I}y)\neq 0$, $(f^c\star b^c_a)(x+\mathrm{I}y)=f^c(x+\mathrm{I}y)b^c_a(q^{-1}(x+\mathrm{I}y)q)$, see [144, Proposition

4.3.22], where $q^{-1}(x+\mathrm{I}y)q = x+\mathrm{I}'y$. Thus we have $(f^c \star b_a^c)(\mathrm{I}y) = f^c(\mathrm{I}y)b_a^c(\mathrm{I}'y)$ almost everywhere and

$$\begin{aligned}
\|b_a \star f\|^2_{\mathrm{H}^2(\mathbb{H}_+)} &= \|(b_a \star f)^c\|^2_{\mathrm{H}^2(\mathbb{H}_+)} \\
&= \sup_{\mathrm{I}\in\mathbb{S}} \int_{-\infty}^{+\infty} |(f^c \star b_a^c)(\mathrm{I}y)|^2 dy \\
&= \sup_{\mathrm{I}\in\mathbb{S}} \int_{-\infty}^{+\infty} |f^c(\mathrm{I}y)b_a^c(\mathrm{I}'y)|^2 dy \\
&= \sup_{\mathrm{I}\in\mathbb{S}} \int_{-\infty}^{+\infty} |f^c(\mathrm{I}y)|^2 |b_a^c(\mathrm{I}'y)|^2 dy \\
&= \sup_{\mathrm{I}\in\mathbb{S}} \int_{-\infty}^{+\infty} |f^c(\mathrm{I}y)|^2 dy \\
&= \|f^c\|^2_{\mathrm{H}^2(\mathbb{H}_+)}.
\end{aligned}$$

By the previous lemma, we have $\|f^c\|^2_{\mathrm{H}^2(\mathbb{H}_+)} = \|f\|^2_{\mathrm{H}^2(\mathbb{H}_+)}$ and this concludes the proof. □

Chapter 7

Operator-valued slice hyperholomorphic functions

In this chapter we introduce slice hyperholomorphic functions with values in a quaternionic Banach space. As in the complex case, there are two equivalent notions, namely weak and strong slice hyperholomorphicity. In order to properly define a multiplication between slice hyperholomorphic functions, we give a third characterization in terms of the Cauchy-Riemann system. Functions with values in a quaternionic Banach space can also be obtained by using the so-called S-functional calculus. This calculus is associated with the notions of S-spectrum and S-resolved which are introduced and studied. We also present some hyperholomorphic extension results and, finally, we study the Hilbert space valued quaternionic Hardy space of the ball and backward-shift invariant subspaces.

7.1 Definition and main properties

In the sequel, we denote by $\mathscr{X}$ a left quaternionic Banach space and by $\mathscr{X}^*$ its dual, i.e. the set of bounded, left linear maps from $\mathscr{X}$ to $\mathbb{H}$. We introduce and study the class $\mathscr{R}(\Omega,\mathscr{X})$ of functions defined on an open set $\Omega\subseteq\mathbb{H}$ with values in $\mathscr{X}$ which are slice hyperholomorphic. In order to get a linear structure also on $\mathscr{R}(\Omega,\mathscr{X})$ we also assume that $\mathscr{X}$ is two sided quaternionic vector space, so that the function space $\mathscr{R}(\Omega,\mathscr{X})$ turns out to be a right vector space over $\mathbb{H}$.
The definition below is based on the notion of slice derivative, see Definition 6.1.2. It appeared originally in [32].

Definition 7.1.1. Let $\mathscr{X}$ be a two sided quaternionic Banach space and let $\mathscr{X}^*$ be its dual. Let Ω be an open set in $\mathbb{H}$.
A function $f:\Omega\to\mathscr{X}$ is said to be *weakly slice hyperholomorphic* in Ω if Λf admits slice derivative in Ω for every $\Lambda\in\mathscr{X}^*$.

A function $f:\Omega\to\mathscr{X}$ is said to be *strongly slice hyperholomorphic* in Ω if

$$\lim_{p\to p_0,p\in\mathbb{C}_{\rm I}}(p-p_0)^{-1}(f_{\rm I}(p)-f_{\rm I}(p_0)) \tag{7.1}$$

exists in the topology of $\mathscr{X}$ in case $p_0\in\Omega$ is nonreal and $p_0\in\mathbb{C}_{\rm I}$ and if

$$\lim_{p\to p_0,p\in\mathbb{C}_{\rm I}}(p-p_0)^{-1}(f_{\rm I}(p)-f_{\rm I}(p_0)) \tag{7.2}$$

exists in the topology of $\mathscr{X}$ for every $\rm I\in\mathbb{S}$, equal to the same value, in case $p_0\in\Omega$ is real.

The following lemma can be proved with the same arguments used for its analog in the complex case (see e.g. [247], p. 189).

Lemma 7.1.2. *Let $\mathscr{X}$ be a two sided quaternionic Banach space. Then a sequence $\{v_n\}$ is Cauchy if and only if $\{\Lambda v_n\}$ is Cauchy uniformly for $\Lambda\in\mathscr{X}^*$, $\|\Lambda\|\leq 1$.*

Theorem 7.1.3. *A function on $\Omega\subseteq\mathbb{H}$ is weakly slice hyperholomorphic if and only if it is strongly slice hyperholomorphic.*

Proof. Since any $\Lambda\in\mathscr{X}^*$ is continuous, every strongly slice hyperholomorphic function is weakly slice hyperholomorphic.
To show the converse, let f be a weakly slice hyperholomorphic function on $p_0\in\Omega$. Assume that $p_0\in\mathbb{C}_{\rm I}$. Then, for any $\Lambda\in\mathscr{X}^*$ and any $\rm I\in\mathbb{S}$, we can choose $\rm J\in\mathbb{S}$ such that J is orthogonal to I, and write

$$(\Lambda f)_{\rm I}(p)=(\Lambda f)_{\rm I}(x+{\rm I}y)=F_\Lambda(x+{\rm I}y)+G_\Lambda(x+{\rm I}y){\rm J}$$

where $F_\Lambda,G_\Lambda:\mathbb{C}_{\rm I}\to\mathbb{C}_{\rm I}$. The limit $\lim_{p\to p_0,\,p\in\mathbb{C}_{\rm I}}(p-p_0)^{-1}((\Lambda f)_{\rm I}(p)-(\Lambda f)_{\rm I}(p_0))$ exists, and so the limits

$$\lim_{p\to p_0,\,p\in\mathbb{C}_{\rm I}}(p-p_0)^{-1}(F_\Lambda(p)-F_\Lambda(p_0)) \qquad \lim_{p\to p_0,\,p\in\mathbb{C}_{\rm I}}(p-p_0)^{-1}(G_\Lambda(p)-G_\Lambda(p_0))$$

exist. We deduce that the functions F_Λ and G_Λ are holomorphic on $\Omega\cap\mathbb{C}_{\rm I}$ and so they admit a Cauchy formula on the plane $\mathbb{C}_{\rm I}$, computed on a circle γ, contained in $\mathbb{C}_{\rm I}$, whose interior contains p_0 and is contained in Ω. Note that if p_0 is real we can pick any complex plane $\mathbb{C}_{\rm I}$ while if $p_0\in\mathbb{H}\setminus\mathbb{R}$ then $\mathbb{C}_{\rm I}$ is uniquely determined. For any increment h in $\mathbb{C}_{\rm I}$ we compute

$$\begin{aligned}\Lambda(h^{-1}&(f_{\rm I}(p_0+h)-f_{\rm I}(p_0))-\partial_s\Lambda(f_{\rm I}x(p_0))=\\ &=\frac{1}{2\pi}\int_\gamma\left[h^{-1}\left(\frac{1}{p-(p_0+h)}-\frac{1}{p-p_0}\right)-\frac{1}{(p-p_0)^2}\right]dp_{\rm I}\Lambda(f_{\rm I}(p)),\end{aligned}$$

where $dp_{\rm I}=(dx+{\rm I}dy)/{\rm I}$. Since $\Lambda(f_{\rm I}(p))$ is continuous on γ which is compact, we have that $|\Lambda(f_{\rm I}(p))|\leq C_\Lambda$ for all $p\in\gamma$. The maps $f(p):\mathscr{X}^*\to\mathbb{H}$ are pointwise bounded at each Λ, thus $\sup_{p\in\gamma}\|f_{\rm I}(p)\|\leq C$ by the uniform boundedness theorem. Thus

$$\begin{aligned}\big|\Lambda(h^{-1}&(f_{\rm I}(p_0+h)-f_{\rm I}(p_0))-\partial_s\Lambda(f_{\rm I}(p_0))\big|\leq\\ &\leq\frac{C}{2\pi}\|\Lambda\|\int_\gamma\left|\left(\frac{1}{p-(p_0+h)}-\frac{1}{p-p_0}\right)-\frac{1}{(p-p_0)^2}\right|dp_{\rm I},\end{aligned}$$

so $h^{-1}(f_I(p_0+h)-f_I(p_0))$ is uniformly Cauchy for $\|\Lambda\|\leq 1$ and by Lemma 7.1.2 it converges in $\mathscr{X}$. Thus f admits slice derivative at every $p_0\in\Omega$ and so it is strongly slice hyperholomorphic in Ω. □

We can also extend the notion of slice hypermeromorphic function to the case of $\mathscr{X}$-valued functions:

Definition 7.1.4. Let $\mathscr{X}$ be a two-sided quaternionic Banach space.. We say that a function $f:\Omega\to\mathscr{X}$ is (weakly) slice hypermeromorphic if for any $\Lambda\in\mathscr{X}^*$ the function $\Lambda f:\Omega\to\mathbb{H}$ is slice hypermeromorphic in Ω.

Note that the previous definition means, in particular, that $f:\Omega'\to\mathscr{X}$ is slice hyperholomorphic, and the points belonging to $\Omega\setminus\Omega'$ are the poles of f and $\Omega\setminus\Omega'$ has no point limit in Ω.
We now show that weakly slice hyperholomorphic (and so slice hyperholomorphic functions) functions are those functions whose restrictions to any complex plane $\mathbb{C}_I$ are in the kernel of the Cauchy-Riemann operator $\partial_x+I\partial_y$.

Proposition 7.1.5. *Let $\mathscr{X}$ be a two sided quaternionic Banach space.*
A real differentiable function $f:\Omega\subseteq\mathbb{H}\to\mathscr{X}$ is weakly slice hyperholomorphic if and only if $(\partial_x+I\partial_y)f_I(x+Iy)=0$ for all $I\in\mathbb{S}$.

Proof. Assume that f is weakly slice hyperholomorphic. Then, for every nonreal $p_0\in\Omega$, $p_0\in\mathbb{C}_I$, where $I=\mathrm{Im}p_0/|\mathrm{Im}p_0|$, we can compute the limit (7.1) for the function Λf_I by taking $p=p_0+h$ with $h\in\mathbb{R}$ and for $p=p_0+Ih$ with $h\in\mathbb{R}$. We obtain, respectively, $\partial_x f_I\Lambda(p_0)$ and $-I\partial_y\Lambda f_I(p_0)$ which coincide. Consequently, $(\partial_x+I\partial_y)\Lambda f_I=\Lambda(\partial_x+I\partial_y)f_I=0$ for any $\Lambda\in\mathscr{X}^*$ and the statement follows by the Hahn-Banach theorem. If p_0 is real, then the statement follows by an analogous argument where now I varies in $\mathbb{S}$. Conversely, if f_I satisfies the Cauchy-Riemann on $\Omega\cap\mathbb{C}_I$ then $\Lambda((\partial_x+I\partial_y)f_I(x+Iy))=0$ for all $\Lambda\in\mathscr{X}^*$ and all $I\in\mathbb{S}$. Since Λ is linear and continuous we can write $(\partial_x+I\partial_y)\Lambda f_I(x+Iy)=0$ and thus the function $\Lambda f_I(x+Iy)$ is in the kernel of $\partial_x+I\partial_y$ for all $\Lambda\in\mathscr{X}^*$ or, equivalently by Proposition 6.1.3, it admits slice derivative. Thus at every $p_0\in\Omega\cap C_I$ we have

$$\lim_{p\to p_0, p\in\mathbb{C}_I}(p-p_0)^{-1}(\Lambda f_I(p)-\Lambda f_I(p_0))=\lim_{p\to p_0, p\in\mathbb{C}_I}\Lambda((p-p_0)^{-1}(f_I(p)-f_I(p_0))),$$

for all $\Lambda\in\mathscr{X}^*$. So f is weakly slice hyperholomorphic. □

Since the class of weakly and strongly slice hyperholomorphic functions coincide, from now on we will refer to them simply as slice hyperholomorphic functions.
The following result follows with trivial computations:

Proposition 7.1.6. *Let $\mathscr{X}$ be a two sided quaternionic Banach space. The set of slice hyperholomorphic functions defined on $\Omega\subseteq\mathbb{H}$ and with values in $\mathscr{X}$ is a right quaternionic vector space denoted by $\mathscr{R}(\Omega,\mathscr{X})$.*

Proposition 7.1.7 (Identity principle)**.** *Let $\mathscr{X}$ be a two sided quaternionic Banach space, Ω be an s-domain and let $f,g:\Omega\subseteq\mathbb{H}\to\mathscr{X}$ be two slice hyperholomorphic functions. If $f=g$ on a set $Z\subseteq\Omega\cap\mathbb{C}_{\mathrm{I}}$ having an accumulation point, for some $\mathrm{I}\in\mathbb{S}$, then $f=g$ on Ω.*

Proof. The assumption $f=g$ on Z implies $\Lambda f=\Lambda g$ on Z for every $\Lambda\in\mathscr{X}^*$ so the slice hyperholomorphic function $\Lambda(f-g)$ is identically zero not only on Z but also on Ω, by the identity principle for quaternionic valued slice hyperholomorphic functions. The Hahn-Banach theorem yields $f-g=0$ on Ω. □

The Cauchy formula is valid for slice hyperholomorphic functions with values in a quaternionic Banach space:

Theorem 7.1.8 (Cauchy formulas)**.** *Let $\mathscr{X}$ be a two sided quaternionic Banach space and let W be an open set in $\mathbb{H}$. Let $\overline{\Omega}\subset W$ be an axially symmetric s-domain, and let $\partial(\Omega\cap\mathbb{C}_{\mathrm{I}})$ be the union of a finite number of rectifiable Jordan curves for every $\mathrm{I}\in\mathbb{S}$. Set $ds_{\mathrm{I}}=ds/\mathrm{I}$. If $f:W\to\mathscr{X}$ is a left slice hyperholomorphic function, then, for $p\in\Omega$, we have*

$$f(p)=\frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_{\mathrm{I}})}S_L^{-1}(s,p)\,ds_{\mathrm{I}}\,f(s), \tag{7.3}$$

if $f:W\to\mathscr{X}$ is a right slice hyperholomorphic, then, for $p\in\Omega$, we have

$$f(p)=\frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_{\mathrm{I}})}f(s)\,ds_{\mathrm{I}}\,S_R^{-1}(s,p), \tag{7.4}$$

and the integrals (7.3), (7.4) do not depend on the choice of the imaginary unit $\mathrm{I}\in\mathbb{S}$ nor on $\Omega\subset W$.

Proof. Since weakly slice hyperholomorphic functions are strongly slice hyperholomorphic functions, they are also continuous functions, so the validity of the formulas (7.3), (7.4) follows as in point (b) p. 80 [253]. □

Slice hyperholomorphic functions on Ω with values in $\mathscr{X}$ can be defined in another way. Consider the set of functions of the form $f(p)=f(x+\mathrm{I}y)=\alpha(x,y)+\mathrm{I}\beta(x,y)$ where $\alpha,\beta:\Omega\to\mathscr{X}$ depend only on x,y, are real differentiable, satisfy the Cauchy-Riemann equations $\partial_x\alpha-\partial_y\beta=0$, $\partial_y\alpha+\partial_x\beta=0$ and, in order to have well posedness of the function f, we assume $\alpha(x,-y)=\alpha(x,y)$, $\beta(x,-y)=-\beta(x,y)$. Observe that if $p=x$ is a real quaternion, then I is not uniquely defined but the hypothesis that β is odd in the variable y implies $\beta(x,0)=0$. We will denote the class of function of this form by $\mathcal{O}(\Omega,\mathscr{X})$.

Theorem 7.1.9. *Let Ω be an axially symmetric s-domain in $\mathbb{H}$, and let $\mathscr{X}$ be a two sided quaternionic Banach space. Then $\mathscr{R}(\Omega,\mathscr{X})=\mathcal{O}(\Omega,\mathscr{X})$.*

Proof. The inclusion $\mathcal{O}(\Omega,\mathscr{X})\subseteq\mathscr{R}(\Omega,\mathscr{X})$ is clear: any function $f\in\mathcal{O}(\Omega,\mathscr{X})$ is real differentiable and such that f_{I} satisfies $(\partial_x+\mathrm{I}\partial_y)f_{\mathrm{I}}=0$ (note that this implication does not need any hypothesis on the open set Ω). Conversely, assume that $f \in \mathscr{R}(\Omega, \mathscr{X})$. Let us show that

$$f(x+\mathrm{I}y) = \frac{1}{2}(1-\mathrm{IJ})f(x+\mathrm{J}y) + \frac{1}{2}(1+\mathrm{IJ})f(x-\mathrm{J}y).$$

If we consider real quaternions, i.e. $y=0$ the formula holds. For nonreal quaternions, set

$$\phi(x+\mathrm{I}y) = \frac{1}{2}(1-\mathrm{IJ})f(x+\mathrm{J}y) + \frac{1}{2}(1+\mathrm{IJ})f(x-\mathrm{J}y).$$

Then, using the fact that f is slice hyperholomorphic, it is immediate that $(\partial_x + \mathrm{I}\partial y)\phi(x+\mathrm{I}y) = 0$ and so ϕ is slice hyperholomorphic. Since $\phi = f$ on $\Omega \cap \mathbb{C}_\mathrm{I}$ then it coincides with f on Ω by the Identity principle. By writing

$$f(x+\mathrm{I}y) = \frac{1}{2}\left[(f(x+\mathrm{J}y) + f(x-\mathrm{J}y) + \mathrm{IJ}(f(x-\mathrm{J}y) - f(x+\mathrm{J}y))\right]$$

and setting $\alpha(x,y) = \frac{1}{2}(f(x+\mathrm{J}y) + f(x-\mathrm{J}y))$, $\beta(x,y) = \frac{1}{2}\mathrm{J}(f(x-\mathrm{J}y) - f(x+\mathrm{J}y))$ we have that $f(x+\mathrm{I}y) = \alpha(x,y) + \mathrm{I}\beta(x,y)$. Reasoning as in Corollary 6.1.13 we can prove that α, β do not depend on I. It is then an easy computation to verify that α, β satisfy the above assumptions. □

Using this other description of $\mathscr{X}$-valued slice hyperholomorphic functions, we can now define a notion of product which is inner in the set of slice hyperholomorphic functions on Ω. To this purpose, we need an additional structure on the two sided quaternionic Banach space $\mathscr{X}$. Suppose that in $\mathscr{X}$ is defined a multiplication which is associative, distributive with respect to the sum in $\mathscr{X}$. Moreover, suppose that $q(x_1x_2) = (qx_1)x_2$ and $(x_1x_2)q = x_1(x_2q)$ for all $q \in \mathbb{H}$ and for all $x_1, x_2 \in \mathscr{X}$. Then we say that $\mathscr{X}$ is a two sided quaternionic Banach algebra. We say that the algebra $\mathscr{X}$ is with unity if $\mathscr{X}$ has a unity with respect to the product.

Definition 7.1.10. Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain and let $f, g : \Omega \to \mathscr{X}$ be slice hyperholomorphic functions with values in a two sided quaternionic Banach algebra $\mathscr{X}$. Let $f(x+\mathrm{I}y) = \alpha(x,y) + \mathrm{I}\beta(x,y)$, $g(x+\mathrm{I}y) = \gamma(x,y) + \mathrm{I}\delta(x,y)$. Then we define

$$(f \star g)(x+\mathrm{I}y) := (\alpha\gamma - \beta\delta)(x,y) + \mathrm{I}(\alpha\delta + \beta\gamma)(x,y). \tag{7.5}$$

It can be easily verified that, by its construction, the function $f \star g$ is slice hyperholomorphic.

Remark 7.1.11. Let us consider the case in which Ω is a ball with center at a real point (let us assume at the origin for simplicity). Then it is immediate to verify, using standard techniques, that $f \in \mathscr{R}(\Omega, \mathscr{X})$ if and only if it admits power series expansion $f(p) = \sum_{n=0}^\infty p^n f_n$, $f_n \in \mathscr{X}$ converging in Ω.

Remark 7.1.12. If Ω is a ball with center at the origin and if f, g admit power series expansion of the form $f(p) = \sum_{n=0}^\infty p^n f_n$, $g(p) = \sum_{n=0}^\infty p^n f_n$, $f_n, g_n \in \mathscr{X}$ for all n, then

$$(f \star g)(p) := \sum_{n=0}^{\infty} p^n (\sum_{r=0}^{n} f_r g_{n-r}).$$

Remark 7.1.13. Note that for $Y=(y_{u,v})_{u,v=1}^N$ and $Z=(z_{u,v})_{u,v=1}^N$ we define $Y\star Z$ to be the $N\times N$ matrix whose (u,v) entry is given by $\sum_{t=1}^N y_{u,t}\star z_{t,v}$.
In case we consider right slice hyperholomorphic functions, the class $\mathcal{O}^R(\Omega,\mathscr{X})$ consists of functions of the form $f(x+\mathrm{I}y)=\alpha(x,y)+\beta(x,y)\mathrm{I}$ where α,β satisfy the assumptions discussed above. The right slice product, denoted by $\star_r$, is defined below.

Definition 7.1.14. Let $\Omega\subseteq\mathbb{H}$ be an axially symmetric s-domain and let $f,g:\Omega\to\mathscr{X}$ be right slice hyperholomorphic functions with values in a two sided quaternionic Banach algebra $\mathscr{X}$. Let $f(x+\mathrm{I}y)=\alpha(x,y)+\beta(x,y)\mathrm{I}$, $g(x+\mathrm{I}y)=\gamma(x,y)+\delta(x,y)\mathrm{I}$. We define

$$(f\star_r g)(x+\mathrm{I}y):=(\alpha\gamma-\beta\delta)(x,y)+(\alpha\delta+\beta\gamma)(x,y)\mathrm{I}. \tag{7.6}$$

Lemma 6.1.23 can be generalized to this setting and in fact, using (7.5) and (7.6), it is immediate to verify with direct computations the validity of the following formula:

$$(f\star g)^*=g^*\star_r f^*. \tag{7.7}$$

7.2 *S*-spectrum and *S*-resolvent operator

In Chapter 4 we discussed the left and the right eigenvalue problem for matrices and we showed that the former is associated to a right linear quaternionic operator, while the latter is not, since the right multiplication with a quaternion is clearly not a right linear operator. The right linear operator associated with a right eigenvalue problem for a matrix A is then $A^2-2\mathrm{Re}(p)A+|p|^2I_n$ and the quaternions p for which this matrix is not invertible is the so-called S-spectrum. This notion can be generalized to any quaternionic, bounded linear operator as follows.
In the sequel, unless otherwise specified, $\mathscr{V}$ is either a right quaternionic Banach space or a right quaternionic Hilbert space. The symbol $\mathbf{B}(\mathscr{V})$ denotes the set of bounded, quaternionic, right linear maps $T:\ \mathscr{V}\to\mathscr{V}$. When we will need a linear structure on $\mathbf{B}(\mathscr{V})$, we will assume that $\mathscr{V}$ is a two-sided linear space.

Definition 7.2.1. Let $T\in\mathbf{B}(\mathscr{V})$. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T)=\{s\in\mathbb{H}\ :\ T^2-2\,\mathrm{Re}\,(s)T+|s|^2I\ \text{ is not invertible in } \mathbf{B}(\mathscr{V})\}$$

where I denotes the identity operator. Its complement

$$\rho_S(T)=\mathbb{H}\setminus\sigma_S(T)$$

is called the S-resolvent set.

More in general, let T be a linear operator from its domain $\mathscr{D}(T)\subseteq\mathscr{V}$ to $\mathscr{V}$ and let us set

$$Q_s(T):=T^2-2\,\mathrm{Re}\,(s)T+|s|^2I.$$

Then $Q_s(T):\mathscr{D}(T^2)\to\mathscr{V}$.
The definition of S-spectrum can be made more precise by dividing it into three subsets, as described below. This definition appeared originally in [193].

Definition 7.2.2. Let $\mathscr{V}$ be as above and let $T : \mathscr{D}(T) \longrightarrow \mathscr{V}$ be a linear operator. The S-resolvent set of T is the set $\rho_S(T) \subset \mathbb{H}$ of the quaternions s such that the three following conditions hold:

(1) $\operatorname{Ker}(Q_s(T)) = \{0\}$;

(2) $Q_s(T)^{-1} : \operatorname{ran}(Q_s(T)) \longrightarrow \mathscr{D}(T^2)$ is bounded;

(3) $\operatorname{ran}(Q_s(T))$ is dense in $\mathscr{V}$.

The S-spectrum $\sigma_S(T)$ of T is defined by setting $\sigma_S(T) := \mathbb{H} \setminus \rho_S(T)$. It decomposes into three disjoint subsets as follows:

(1) the point S-spectrum of T:

$$\sigma_{pS}(T) := \{s \in \mathbb{H} \ : \ \operatorname{Ker}(Q_s(T)) \neq \{0\}\};$$

(2) the continuous S-spectrum of T:

$$\sigma_{cS}(T) := \left\{s \in \mathbb{H} \ : \ \operatorname{Ker}(Q_s(T)) = \{0\}, \overline{\operatorname{ran}(Q_s(T))} = \mathscr{V}, Q_s(T)^{-1} \notin \mathbf{B}(\mathscr{V})\right\};$$

(3) the residual S-spectrum of T:

$$\sigma_{rS}(T) := \left\{s \in \mathbb{H} \ : \ \operatorname{Ker}(Q_s(T)) = \{0\}, \overline{\operatorname{ran}(Q_s(T))} \neq \mathscr{V}\right\}.$$

As in the classical case, we have the following result:

Theorem 7.2.3 (Compactness of S-spectrum)**.** *Let $T \in \mathbf{B}(\mathscr{V})$. Then the S-spectrum $\sigma_S(T)$ is a compact nonempty set.*

Definition 7.2.4. The *S-spectral radius of T* $r_S(T) \in \mathbb{R}^+ \cup \{+\infty\}$ is defined as:

$$r_S(T) := \sup\left\{|s| \in \mathbb{R}^+ \,\middle|\, s \in \sigma_S(T)\right\}.$$

To compute the spectral radius we first need the following preliminary results.

Lemma 7.2.5. *Let $n \in \mathbb{N}$ and q, $s \in \mathbb{H}$. Let*

$$P_{2n}(q) := q^{2n} - 2\operatorname{Re}(s^n)\, q^n + |s^n|^2.$$

Then

$$\begin{aligned} P_{2n}(q) &= Q_{2n-2}(q)(q^2 - 2\operatorname{Re}(s)\, q + |s|^2) \\ &= (q^2 - 2\operatorname{Re}(s)\, q + |s|^2) Q_{2n-2}(q), \end{aligned} \tag{7.8}$$

where $Q_{2n-2}(q)$ is a polynomial of degree $2n-2$ in q.

Proof. First of all we observe that

$$P_{2n}(s) = s^{2n} - 2\mathrm{Re}(s^n)s^n + |s^n|^2 = s^{2n} - (s^n + \overline{s^n})s^n + s^n\overline{s^n} = 0.$$

If, in the coefficients of the polynomial $P_{2n}(p)$, we substitute s by any other element s' on the same 2-sphere, we observe that the polynomial $P_{2n}(q)$ does not change, and so $P_{2n}(s') = 0$. We conclude that the whole 2-sphere defined by s is solution to the equation $P_{2n}(q) = 0$. The assertion follows from the fact that $(q^2 - 2\,\mathrm{Re}(s)\,q + |s|^2)$ is a real coefficient factor of P_{2n}. □

Lemma 7.2.6. *Let $n \in \mathbb{N}$ and q, $s \in \mathbb{H}$. Let λ_j, $j = 0, 1, \dots, n-1$ be the solutions of $\lambda^n = s$ in the complex plane $\mathbb{C}_{\mathrm{I}_s}$. Then*

$$q^{2n} - 2\,\mathrm{Re}(s)\,q^n + |s|^2 = \prod_{j=0}^{n-1}(q^2 - 2\,\mathrm{Re}(\lambda_j)\,q + |\lambda_j|^2). \tag{7.9}$$

Proof. We solve $\lambda^n = s$ in the complex plane $x + \mathrm{I}_s y$ containing $s = s_0 + \mathrm{I}_s s_1$. This equation admits exactly n solutions $\lambda_j = \lambda_{j0} + \mathrm{I}_s\lambda_{j1}$, $j = 0, 1, \dots, n-1$ in $\mathbb{C}_{\mathrm{I}_s}$. Let $s' = s_0 + \mathrm{I}s_1$, $\mathrm{I} \in \mathbb{S}$ be any element in the 2-sphere $[s]$. Then the solutions to the equation $\lambda^n = s'$ are $\lambda'_j = \lambda_{j0} + \mathrm{I}\lambda_{j1}$, $j = 0, 1, \dots, n-1$, $\mathrm{I} \in \mathbb{S}$. Let us consider

$$P_{2n}(q) = q^{2n} - 2\,\mathrm{Re}(s)\,q^n + |s|^2.$$

Then $q = \lambda_j$ is a root of $P_{2n}(q) = 0$, in fact

$$P_{2n}(\lambda_j) = \lambda_j^{2n} - 2\,\mathrm{Re}(s)\,\lambda_j^n + |s|^2 = s^2 - 2\,\mathrm{Re}(s)\,s + |s|^2 = 0.$$

If we substitute s by $s' \in [s]$, P_{2n} is unchanged and it is immediate that $P_{2n}(\lambda'_j) = 0$ when I varies in $\mathbb{S}$. This proves that the roots of $P_{2n}(q) = 0$ are the 2-spheres $[\lambda_j]$, $j = 0, \dots, n-1$. The statement follows from Proposition 6.1.38. □

Theorem 7.2.7. *Let $\mathcal{V}$ be a two sided quaternionic Banach space and let $T \in \mathbf{B}(\mathcal{V})$. Then*

$$\sigma_S(T^n) = (\sigma_S(T))^n = \{s^n \in \mathbb{H} : s \in \sigma_S(T)\}.$$

Proof. From the definition of S-spectrum we have

$$\sigma_S(T^n) = \{s \in \mathbb{H} \ : \ T^{2n} - 2\ \mathrm{Re}(s)T^n + |s|^2 I \quad \text{is not invertible in}\, \mathbf{B}(\mathcal{V})\}.$$

From Lemma 7.2.5 and Theorem 7.3.7, it follows that $T^{2n} - 2\mathrm{Re}[s^n]T^n + |s^n|^2 I$ can be factorized as

$$T^{2n} - 2\mathrm{Re}(s^n)T^n + |s^n|^2 I = Q_{2n-2}(T)(T^2 - 2\mathrm{Re}(s)T + |s|^2 I).$$

Consequently, if $T^2 - 2\mathrm{Re}(s)T + |s|^2 I$ is not injective also $T^{2n} - 2\mathrm{Re}(s^n)T^n + |s^n|^2 I$ is not injective. This proves that $(\sigma_S(T))^n \subseteq \sigma_S(T^n)$. To show the converse, we consider $p \in \sigma_S(T^n)$. Lemma 7.2.6 and Theorem 7.3.7 give

$$T^{2n} - 2\mathrm{Re}(p)T^n + |p|^2 I = \prod_{j=0}^{n-1}(T^2 - 2\mathrm{Re}(\lambda_j)T + |\lambda_j|^2 I).$$

If $T^{2n}-2\mathrm{Re}(p)T^n+|p|^2I$ is not invertible then at least one of the operators $T^2-2\mathrm{Re}(\lambda_j)T+|\lambda_j|^2I$ for some j is not invertible. This proves that $\sigma_S(T^n)\subseteq(\sigma_S(T))^n$. □

Theorem 7.2.8. *Let $\mathscr{V}$ be a two sided quaternionic Banach space, let $T\in\mathbf{B}(\mathscr{V})$, and let $r_S(T)$ be its S-spectral radius. Then*

$$r_S(T)=\lim_{n\to\infty}\|T^n\|^{1/n}.$$

Proof. For every $s\in\mathbb{H}$ such that $|s|>r_S(T)$ the series $\sum_{n\geq 0}T^ns^{-1-n}$ converges in $\mathbf{B}(\mathscr{V})$ to the S-resolvent operator $S_L^{-1}(s,T)$ (we reason analogously for $\sum_{n\geq 0}s^{-1-n}T^n$). So the sequence T^ns^{-1-n} is bounded in the norm of $\mathbf{B}(\mathscr{V})$ and

$$\limsup_{n\to\infty}\|T^n\|^{1/n}\leq r_S(T). \tag{7.10}$$

Theorem 7.2.7 implies $\sigma_S(T^n)=(\sigma_S(T))^n$, so we have

$$(r_S(T))^n=r_S(T^n)\leq\|T^n\|,$$

from which we get

$$r_S(T)\leq\liminf\|T^n\|^{1/n}. \tag{7.11}$$

From (7.10), (7.11) we obtain

$$r_S(T)\leq\liminf_{n\to\infty}\|T^n\|^{1/n}\leq\limsup_{n\to\infty}\|T^n\|^{1/n}\leq r_S(T). \tag{7.12}$$

The chain of inequalities (7.12) also proves the existence of the limit. □

According to the terminology already introduced in the case of matrices, see Definition 4.3.5, if $Tu=us$ for some $s\in\mathbb{H}$ and $u\in\mathscr{V}$, $u\neq 0$ then u is called right eigenvector of T with right eigenvalue s. The following proposition has been proved in [144].

Proposition 7.2.9. *Let T be a bounded quaternionic linear operator acting on a quaternionic, two sided, Banach space $\mathscr{V}$. Then, for $\|T\|<|s|$*

$$\sum_{n=0}^{\infty}T^ns^{-1-n}=-(T^2-2\mathrm{Re}(s)T+|s|^2I)^{-1}(T-\bar{s}I) \tag{7.13}$$

and

$$\sum_{n=0}^{\infty}s^{-1-n}T^n=-(T-\bar{s}I)(T^2-2\mathrm{Re}(s)T+|s|^2I)^{-1}, \tag{7.14}$$

where I denotes the identity operator on $\mathscr{V}$.

Definition 7.2.10. Let $\mathscr{V}$ be a two sided quaternionic Banach space, $T\in\mathbf{B}(\mathscr{V})$ and $s\in\rho_S(T)$. We define the left S-resolvent operator as

$$S_L^{-1}(s,T):=-(T^2-2\mathrm{Re}\,(s)T+|s|^2I)^{-1}(T-\bar{s}I), \tag{7.15}$$

and the right S-resolvent operator as

$$S_R^{-1}(s,T):=-(T-\bar{s}I)(T^2-2\mathrm{Re}\,(s)T+|s|^2I)^{-1}. \tag{7.16}$$

The operators $S_L^{-1}(s,T)$ and $S_R^{-1}(s,T)$ satisfy the following relations

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = I$$

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = I$$

called left and right S-resolvent equation, respectively, which can be verified by direct computations. In the paper [28] the S-resolvent equation has been proved:

Theorem 7.2.11. *Let $\mathscr{V}$ be a two sided quaternionic Banach space, $T \in \mathbf{B}(\mathscr{V})$ and s and $p \in \rho_S(T)$. Then we have*

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = ((S_R^{-1}(s,T) - S_L^{-1}(p,T))p - \bar{s}(S_R^{-1}(s,T) - S_L^{-1}(p,T)))(p^2 - 2s_0 p + |s|^2)^{-1}. \tag{7.17}$$

Moreover, the resolvent equation can also be written as

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = (s^2 - 2p_0 s + |p|^2)^{-1}(s(S_R^{-1}(s,T) - S_L^{-1}(p,T)) - (S_R^{-1}(s,T) - S_L^{-1}(p,T))\overline{p}). \tag{7.18}$$

Theorem 7.2.12. *(Structure of the S-spectrum) Let $T \in \mathbf{B}(\mathscr{V})$ and let $p \in \sigma_S(T)$. Then all the elements of the sphere $[p]$ belong to $\sigma_S(T)$.*

Proof. The fact that the operator $(T^2 - 2\mathrm{Re}\,(p)T + |p|^2 I)$ is not invertible depends only on the real numbers $\mathrm{Re}(p), |p|$. Therefore all the elements in the sphere $[p]$ belong to the S-spectrum of T. □

Let us now assume that $\mathscr{V}$ is a quaternionic Hilbert space, that we will denote by $\mathscr{H}$. The following properties can be proved exactly as in the complex case.

Proposition 7.2.13. *Let $T : \mathscr{D}(T) \to \mathscr{H}$, $S : \mathscr{D}(S) \to \mathscr{H}$ be linear quaternionic operators with domain dense in $\mathscr{H}$. Then:*

(1) If $T \subset S$ then $S^* \subset T^*$;

(2) $T \subset (T^*)^*$ and $T = (T^*)^*$ if $T \in \mathbf{B}(\mathscr{H})$;

(3) Assume that $T \in \mathbf{B}(\mathscr{H})$. If T is bijective and $T^{-1} \in \mathbf{B}(\mathscr{H})$ then

$$T^*(T^{-1})^* = (T^{-1})^* T^* = I.$$

(4) Assume that $T \in \mathbf{B}(\mathscr{H})$. Then T is bijective and $T^{-1} \in \mathbf{B}(\mathscr{H})$ if and only if T^* is bijective and $(T^*)^{-1} \in \mathbf{B}(\mathscr{H})$. Moreover, $(T^*)^{-1} = (T^{-1})^*$.

We now prove some results on the S-spectrum.

Proposition 7.2.14. *Let $\mathscr{H}$ be a quaternionic Hilbert space and let $T \in \mathbf{B}(\mathscr{H})$. Then $\sigma_S(T) = \sigma_S(T^*)$.*

Proof. First note that $Q_s(T)^* = Q_s(T^*)$. By point (4) in Proposition 7.2.13, we get $\rho_S(T) = \rho_S(T^*)$ from which we deduce $\sigma_S(T) = \sigma_S(T^*)$. □

Theorem 7.2.15. *Let $\mathscr{H}$ be a quaternionic Hilbert space. Let $T : \mathscr{D}(T) \to \mathscr{H}$ be a linear operator such that $\mathscr{D}(T)$ is dense in $\mathscr{H}$. Then*

(1) Let T be a normal, bounded operator. Then $\sigma_{pS}(T) = \sigma_{pS}(T^*)$, $\sigma_{rS}(T) = \sigma_{rS}(T^*) = \emptyset$, $\sigma_{cS}(T) = \sigma_{cS}(T^*)$.

(2) Let T be a self adjoint operator. Then $\sigma_S(T) \subset \mathbb{R}$ and $\sigma_{rS}(T) = \emptyset$.

(3) Let T be a anti-self adjoint operator. Then $\sigma_S(T) \subset \mathrm{Im}(\mathbb{H})$ and $\sigma_{rS}(T) = \emptyset$.

(4) Let T be a bounded and unitary operator. Then $\sigma_S(T) \subset \{p \in \mathbb{H} \; : \; |p| = 1\}$.

(5) Let T be a bounded, anti-self adjoint operator and unitary. Then $\sigma_S(T) = \sigma_{pS}(T) = \mathbb{S}$.

7.3 The functional calculus

In this section we introduce the quaternionic functional calculus based on the slice hyperholomorphic functions. This calculus is the natural generalization of the Riesz-Dunford functional calculus for quaternionic operators but it also applies to n-tuples of linear operators, see [28, 118, 121, 122, 130, 132, 134, 142]. The quaternionic version of the calculus, which we treat in the Chapter, was originally developed in the papers [133, 135, 136, 137, 138]. There is also a continuous version of the quaternionic functional calculus, based on the S-spectrum, which has been studied in [193].
We begin by introducing the class of open sets for which we can define the calculus.

Definition 7.3.1. Let $\mathscr{V}$ be a two sided quaternionic Banach space, $T \in \mathbf{B}(\mathscr{V})$ and let $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain that contains the S-spectrum $\sigma_S(T)$ and such that $\partial(\Omega \cap \mathbb{C}_{\mathrm{I}})$ is union of a finite number of continuously differentiable Jordan curves for every $\mathrm{I} \in \mathbb{S}$. We say that Ω is a T-admissible open set.

Definition 7.3.2. Let $\mathscr{V}$ be a two sided quaternionic Banach space, $T \in \mathbf{B}(\mathscr{V})$ and let W be an open set in $\mathbb{H}$.

(i) A function $f \in \mathscr{R}^L(W)$ is said to be locally left hyperholomorphic on $\sigma_S(T)$ if there exists a T-admissible domain $\Omega \subset \mathbb{H}$. We will denote by $\mathscr{R}^L_{\sigma_S(T)}$ the set of locally left hyperholomorphic functions on $\sigma_S(T)$.

(ii) A function $f \in \mathscr{R}^R(W)$ is said to be locally right hyperholomorphic on $\sigma_S(T)$ if there exists a T-admissible domain $\Omega \subset \mathbb{H}$ such that $\overline{\Omega} \subset W$. We will denote by $\mathscr{R}^R_{\sigma_S(T)}$ the set of locally right hyperholomorphic functions on $\sigma_S(T)$.

Theorem 7.3.3. *Let $\mathscr{V}$ be a two sided quaternionic Banach space and $T \in \mathbf{B}(\mathscr{V})$. Let $\Omega \subset \mathbb{H}$ be a T-admissible domain and set $ds_{\mathrm{I}} = -ds\mathrm{I}$. Then the integrals*

$$\frac{1}{2\pi} \int_{\partial(\Omega \cap \mathbb{C}_{\mathrm{I}})} S_L^{-1}(s,T) \, ds_{\mathrm{I}} \, f(s), \quad f \in \mathscr{R}^L_{\sigma_S(T)} \tag{7.19}$$

and

$$\frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s,T), \quad f\in\mathscr{R}^R_{\sigma_S(T)} \tag{7.20}$$

do not depend on the choice of the imaginary unit $I\in\mathbb{S}$ *nor on* Ω*.*

Proof. We will prove that (7.19) does not depend on the choice of the imaginary unit $I\in\mathbb{S}$ nor on Ω. The proof for (7.20) works similarly. We begin by observing that the function $S_L^{-1}(s,q)=(s-q)^{-\star}$ (where left $\star$-inverse is computed with respect to q) is right slice hyperholomorphic in the variable s so we can replace q with an operator $T\in\mathbf{B}(\mathscr{V})$ in the Cauchy formula (7.3). Let us define

$$\mathscr{V}_{\mathbb{R}}=\{v\in\mathscr{V} \mid qv=vq, \ \ \forall q\in\mathbb{H}\}.$$

Two right linear operators T, T' coincide if and only if they coincide on $\mathscr{V}_{\mathbb{R}}$. In fact, consider $v\in\mathscr{V}$ then $v=v_0+v_1i+v_2j+v_3k$ with $v_\ell\in\mathscr{V}_{\mathbb{R}}$. If T, T' coincide on $\mathscr{V}_{\mathbb{R}}$ then

$$\begin{aligned}T(v)&=T(v_0+v_1i+v_2j+v_3k)=T(v_0)+T(v_1)i+T(v_2)j+T(v_3)k\\&=T'(v_0)+T'(v_1)i+T'(v_2)j+T'(v_3)k=T'(v),\end{aligned}$$

and if T and T' coincide, they coincide in particular on $\mathscr{V}_{\mathbb{R}}$. Then, for any linear and continuous functional $\phi\in\mathscr{V}'_{\mathbb{R}}$, consider the duality $\langle\phi,S_L^{-1}(s,T)v\rangle$, for $v\in\mathscr{V}_{\mathbb{R}}$ and define the function

$$g_{\phi,v}(s):=\langle\phi,S_L^{-1}(s,T)v\rangle, \quad \text{for} \quad v\in\mathscr{V}, \quad \phi\in\mathscr{V}'.$$

It can be verified by direct computations that the function $g_{\phi,v}$ is right slice hyperholomorphic on $\rho_S(T)$ in the variable s. Moreover, $g_{\phi,v}(s)\to 0$ as $s\to\infty$ we have that $g_{\phi,v}$ is slice hyperholomorphic also at infinity.
We also have that for any $v\in\mathscr{V}_{\mathbb{R}}$ and any $\phi\in\mathscr{V}'_{\mathbb{R}}$

$$\begin{aligned}\langle\phi,\frac{1}{2\pi}\left[\int_{\partial(\Omega\cap\mathbb{C}_I)} S_L^{-1}(s,T)ds_I f(s)\right]v\rangle &=\\ &=\frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_I)}\langle\phi,S_L^{-1}(s,T)v\rangle ds_I f(s) \quad (7.21)\\ &=\frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_I)} g_{\phi,v}(s)ds_I f(s).\end{aligned}$$

Suppose that Ω is a T-admissible open set such that $\partial(\Omega\cap\mathbb{C}_I)$ does not cross the S-spectrum of T for every $I\in\mathbb{S}$. The fact that, for fixed $I\in\mathbb{S}$, the integral

$$\frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_I)} g_{\phi,v}(s)ds_I\, f(s) \tag{7.22}$$

does not depend on Ω follows from the Cauchy theorem 6.1.33. By the Hahn-Banach theorem also the integral (7.19) does not depend on Ω. We now prove that the integral (7.22) does not depend on $I\in\mathbb{S}$. Since $g_{\phi,v}$ is a right slice hyperholomorphic function on $\rho_S(T)$, we can consider an open set Ω' such that $\overline{\Omega}'\subset\rho_S(T)$, $\Omega'\cap\mathbb{R}\neq\emptyset$ and $[q]\subset\Omega'$

whenever $q \in \Omega'$. We assume that $\partial(\Omega' \cap \mathbb{C}_{\mathrm{I}})$ consists of a finite number of continuously differentiable Jordan curves $\forall \mathrm{I} \in \mathbb{S}$ and that $\partial\Omega \subset \Omega'$ where Ω is an open set as above so, in particular, Ω contains $\mathrm{Re}\,(s)$ whenever $s \in \Omega$. Choose $\mathrm{J} \in \mathbb{S}$, $\mathrm{J} \neq \mathrm{I}$ and represent $g_{\phi,v}(s)$ by the Cauchy integral formula (7.4) as

$$g_{\phi,v}(s) = -\frac{1}{2\pi}\int_{\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})^-} g_{\phi,v}(t)\, dt_{\mathrm{J}}\, S_R^{-1}(s,t) \tag{7.23}$$

where the boundary $\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})^-$ is oriented clockwise to include the points $\mathrm{Re}\,(s) \in \partial(\Omega \cap \mathbb{C}_{\mathrm{J}})$ (recalling that the singularities of $S_L^{-1}(s,t)$ correspond to the 2-sphere $\mathrm{Re}\,(s)$) and to exclude the points belonging to the S-spectrum of T.
We now substitute the expression of $g_{\phi,v}(s)$ in (7.23) into (7.22) and taking into account the orientation of $\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})^-$ we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(\Omega \cap \mathbb{C}_{\mathrm{I}})} g_{\phi,v}(s)\, ds_{\mathrm{I}}\, f(s)\\
&= \frac{1}{2\pi}\int_{\partial(\Omega \cap \mathbb{C}_{\mathrm{I}})} \Big[\frac{1}{2\pi}\int_{\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})} g_{\phi,v}(t)\, dt_{\mathrm{J}}\, S_L^{-1}(s,t)\Big]\, ds_{\mathrm{I}}\, f(s)\\
&= \frac{1}{2\pi}\int_{\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})} g_{\phi,v}(t)\, dt_{\mathrm{J}}\Big[\frac{1}{2\pi}\int_{\partial(\Omega \cap \mathbb{C}_{\mathrm{I}})} S_L^{-1}(s,t)\, ds_{\mathrm{I}}\, f(s)\Big]
\end{aligned} \tag{7.24}$$

where we have used the Fubini theorem. Now observe that $\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})$ consists of a finite number of Jordan curves inside and outside $\Omega \cap \mathbb{C}_{\mathrm{J}}$, but the integral

$$\frac{1}{2\pi}\int_{\partial(\Omega \cap \mathbb{C}_{\mathrm{I}})} S_L^{-1}(s,t)\, ds_{\mathrm{I}}\, f(s)$$

equals $f(t)$ for those $t \in \partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})$ belonging to $\Omega \cap \mathbb{C}_{\mathrm{J}}$. Thus we obtain:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})} g_{\phi,v}(t)\, dt_{\mathrm{J}}\Big[\frac{1}{2\pi}\int_{\partial(\Omega \cap \mathbb{C}_{\mathrm{I}})} S_L^{-1}(s,t)\, ds_{\mathrm{I}}\, f(s)\Big]\\
&= \frac{1}{2\pi}\int_{\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})} g_{\phi,v}(t)\, dt_{\mathrm{J}} f(t).
\end{aligned} \tag{7.25}$$

So from (7.24) and (7.25) we have

$$\frac{1}{2\pi}\int_{\partial(\Omega \cap \mathbb{C}_{\mathrm{I}})} g_{\phi,v}(s)\, ds_{\mathrm{I}}\, f(s) = \frac{1}{2\pi}\int_{\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})} g_{\phi,v}(t)\, dt_{\mathrm{J}} f(t). \tag{7.26}$$

Now observe that $\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})$ is positively oriented and surrounds the S-spectrum of T. By the independence of the integral on the open set, we can substitute $\partial(\Omega' \cap \mathbb{C}_{\mathrm{J}})$ by $\partial(\Omega \cap \mathbb{C}_{\mathrm{J}})$ in (7.26) and we obtain

$$\frac{1}{2\pi}\int_{\partial(\Omega \cap \mathbb{C}_{\mathrm{I}})} g_{\phi,v}(s)\, ds_{\mathrm{I}}\, f(s) = \frac{1}{2\pi}\int_{\partial(\Omega \cap \mathbb{C}_{\mathrm{J}})} g_{\phi,v}(t)\, dt_{\mathrm{J}}\, f(t),$$

that is

$$\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_{\mathrm{I}})} S_L^{-1}(s,T)ds_{\mathrm{I}}\, f(s)\right] v\rangle$$
$$= \langle \phi, \left[\frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_{\mathrm{J}})} \langle \phi, S_L^{-1}(t,T)dt_{\mathrm{J}}\, f(t)\right] v\rangle, \quad \text{for all } v\in\mathcal{V}_{\mathbb{R}},\ \phi\in\mathcal{V}_{\mathbb{R}}',\ \mathrm{I},\mathrm{J}\in\mathbb{S}.$$

Again by the Hahn-Banach theorem, the integral (7.19) does not depend on $\mathrm{I}\in\mathbb{S}$. □

Definition 7.3.4 (Quaternionic functional calculus)**.** Let $\mathcal{V}$ be a two sided quaternionic Banach space and $T\in\mathbf{B}(\mathcal{V})$. Let $\Omega\subset\mathbb{H}$ be a T-admissible domain and set $ds_{\mathrm{I}} = -ds\mathrm{I}$. We define

$$f(T) = \frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_{\mathrm{I}})} S_L^{-1}(s,T)\, ds_{\mathrm{I}}\, f(s), \ \text{ for } f\in\mathcal{R}^L_{\sigma_S(T)}, \tag{7.27}$$

and

$$f(T) = \frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_{\mathrm{I}})} f(s)\, ds_{\mathrm{I}}\, S_R^{-1}(s,T), \ \text{ for } f\in\mathcal{R}^R_{\sigma_S(T)}. \tag{7.28}$$

Remark 7.3.5. Thanks to the functional calculus we can define functions of an operator T. If we consider the function $(1-pq)^{-\star_r}$ and we use the functional calculus, we can define $(1-pT)^{-\star_r}$. Note that for $p\neq 0$

$$(1-pT)^{-\star_r} = p^{-1}S_R(p,T),$$

moreover

$$(1-pT)^{-\star_r} = \sum_{n\geq 0} p^n T^n \qquad \text{for } |p|\|T\|<1.$$

For the sake of simplicity, and in view of (6.22), in the sequel we will simply write $(1-sT)^{-\star}$.

The following results are used to study some of the properties of the quaternionic functional calculus for bounded linear operators.

Proposition 7.3.6. *Let $\Omega\subset\mathbb{H}$ be an open set.*

(1) Let $f\in\mathcal{N}(\Omega)$, $g\in\mathcal{R}^L(\Omega)$, then $fg\in\mathcal{R}^L(\Omega)$.

(2) Let $f\in\mathcal{N}(\Omega)$, $g\in\mathcal{R}^R(\Omega)$, then $gf\in\mathcal{R}^R(\Omega)$.

(3) Let $f,g\in\mathcal{N}(\Omega)$, then $fg=gf$ and $fg\in\mathcal{N}(\Omega)$.

Theorem 7.3.7. *Let $\mathcal{V}$ be a two sided quaternionic Banach space and $T\in\mathbf{B}(\mathcal{V})$.*

(1) If $f\in\mathcal{N}_{\sigma_S(T)}$ and $g\in\mathcal{R}^L_{\sigma_S(T)}$, then $(fg)(T)=f(T)g(T)$.

(2) If $f\in\mathcal{N}_{\sigma_S(T)}$ and $g\in\mathcal{R}^R_{\sigma_S(T)}$, then $(gf)(T)=g(T)f(T)$.

We conclude this section by pointing out that the quaternionic functional calculus allows to extend to the quaternionic setting the theory of groups and semigroups of linear operators, see [27, 38, 135, 201]. The S-spectrum turned out to be the correct object also for the quaternionic version of the spectral theorem, see the recent papers [30, 31, 194]. We also note that the Fueter-Sce mapping theorem written in integral form gives rise to an integral transform that maps slice hyperholomorphic functions into monogenic functions of axial type. Using this integral transform it is possible to define a monogenic functional calculus, see [36, 117, 139, 140].

7.4 Two results on slice hyperholomorphic extension

In this section we prove two extension results which will be crucial in the next Chapters. In particular, Proposition 7.4.2 applies whenever we deal with operators from a quaternionic Hilbert space to itself, and the space is right-sided but not necessarily two-sided.

Proposition 7.4.1. *Let $\mathscr{H}$ be a right quaternionic Hilbert space and let F be a $\mathbf{B}(\mathscr{H})$-valued slice hyperholomorphic function in some open set Ω which intersects the real line. Assume that $F(x)$ is boundedly invertible for $x \in (a,b) \subset \Omega \cap \mathbb{R}$. Then there is a slice hyperholomorphic inverse to F in an open subset of Ω.*

Proof. Without loss of generality we assume that $0 \in (a,b)$. Viewing F as a power series in the real variable x with operator coefficients and since $F(0)$ is boundedly invertible, $(F(x))^{-1}$ can be expressed as an absolutely convergent power series in x near the origin, say in $|x| < r$ for some $r > 0$. For the scalar case, see for instance [114, p. 22-23]. The proof is the same in the operator quaternionic case when replacing the absolute values by operator norms. Replacing x by a quaternionic variable p (and putting the powers of p on the left) we obtain a slice hyperholomorphic inverse of F in $|p| < r$. □

We now give some applications of this proposition. A first application of the above proposition is related to linear fractional trasformations. Recall first that linear fractional transformations play an important role in Schur analysis, starting by the Schur algorithm itself. See (1.1) and (1.4). In the setting of slice hyperholomorphic functions, let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two-sided quaternionic Hilbert spaces and let Ω be a s-domain. Furthermore let

$$M(p) = \begin{pmatrix} A(p) & B(p) \\ C(p) & D(p) \end{pmatrix} : \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{pmatrix}$$

be slice hyperholomorphic. We define the associated linear frational transformation as

$$T_M(e) = (A \star e + B) \star (C \star e + D)^{-\star}, \tag{7.29}$$

where e is an $\mathbf{B}(\mathscr{H}_1, \mathscr{H}_2)$-valued slice hyperholomorphic function such that $C \star e + D$ is invertible in a real neighborhood of a point of the real line. Using slice hyperholomorphic extension we obtain the semi-group property

$$T_{M_1}(T_{M_2}(e)) = T_{M_1 \star M_2}(e) \tag{7.30}$$

wherever both sides are defined, first in a real neighborhood of a point of the real line.

The second application is as follows: Let $\mathscr{H}$ be a *two-sided* quaternionic Hilbert space, and let A be a right linear bounded operator from $\mathscr{H}$ into itself. Then the expression

$$\sum_{n=0}^{\infty} p^n A^n$$

makes sense for p in a neighborhood of the origin. When $\mathscr{H}$ is one-sided, we have the following useful result:

Proposition 7.4.2. *Let A be a bounded linear operator from a right-sided quaternionic Hilbert $\mathscr{H}$ space into itself, and let G be a bounded linear operator from $\mathscr{H}$ into $\mathscr{Q}$, where $\mathscr{Q}$ is a two sided quaternionic Hilbert space. The slice hyperholomorphic extension of $G(I-xA)^{-1}$, $1/x \in \rho_S(A) \cap \mathbb{R}$, is*

$$(G-\overline{p}GA)(I-2\mathrm{Re}(p)A+|p|^2A^2)^{-1},$$

and it is defined for $1/p \in \rho_S(A)$.

Proof. First we observe that for $|x|\|A\| < 1$ we have $G(I-xA)^{-1} = \sum_{n=0}^{\infty} x^n GA^n$. Let us now take $p \in \mathbb{H}$ such that$|p|\|A\| < 1$. Then, the slice hyperholomorphic extension of the series $\sum_{n=0}^{\infty} x^n GA^n$ is $\sum_{n=0}^{\infty} p^n GA^n$ this is immediate since it is a converging power series in p with coefficients on the right. To show that

$$\sum_{n=0}^{\infty} p^n GA^n = (G-\overline{p}GA)(I-2\mathrm{Re}(p)A+|p|^2A^2)^{-1} \tag{7.31}$$

we prove instead the equality

$$(\sum_{n=0}^{\infty} p^n GA^n)(I-2\mathrm{Re}(p)A+|p|^2A^2) = (G-\overline{p}GA).$$

Computing the left hand side, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} p^n GA^n - 2\sum_{n=0}^{\infty} \mathrm{Re}(p)p^n GA^{n+1} + \sum_{n=0}^{\infty} |p|^2 p^n GA^{n+2} \\
&= G + (p-2\mathrm{Re}(p))GA + (p^2 - 2p\mathrm{Re}(p) + |p|^2)\sum_{n=0}^{\infty} p^n GA^{n+2} \\
&= G - \bar{p}GA,
\end{aligned}$$

where we have used the identity $p^2 - 2p\mathrm{Re}(p) + |p|^2 = 0$ and this shows the assertion using the identity principle. □

Remark 7.4.3. In analogy with the matrix case we will write, with an abuse of notation in this case, $G \star (I-pA)^{-\star}$ instead of the expression $(G-\overline{p}GA)(I-2\mathrm{Re}(p)A+|p|^2A^2)^{-1}$.

For an illustration of formula (7.31) see for instance Remark 8.8.2.

Proposition 7.4.4. *With the notation in Remark 7.4.3 we have the following equalities*

$$(D+pC\star(I-pA)^{-\star}B)^{-\star}=D^{-1}-pD^{-1}C\star(I-p(A-BD^{-1}C))^{-\star}BD^{-1}, \qquad (7.32)$$

and

$$\begin{aligned}(D_1+pC_1\star(I-pA_1)^{-\star}B_1)\star(D_2+pC_2\star(I-pA_2)^{-\star}B_2)=\\ =D_1D_2+p\begin{pmatrix}C_1 & D_1C_2\end{pmatrix}\star\left(I-p\begin{pmatrix}A_1 & B_1C_2\\ 0 & A_2\end{pmatrix}\right)^{-\star}\begin{pmatrix}B_1D_2\\ B_2\end{pmatrix}.\end{aligned} \qquad (7.33)$$

Proof. When p is real, the $\star$-product is replaced by the operator product (or matrix product in the finite dimensional case) and formulas (7.32) and (7.33) are then well known as special instances of the complex setting (and after identifying $\mathbb{H}$ with a space of 2×2 complex matrices); see Propositions 2.1.5 and 2.1.8. See also e.g. [87] for more information. By taking the slice-hyperholomorphic extension we obtain the required result. □

7.5 Slice hyperholomorphic kernels

In the following results, $\mathscr{K}_1$ and $\mathscr{K}_2$ are two-sided quaternionic Krein spaces and $K_1(p,q)$ and $K_2(p,q)$ are two kernels $\mathbf{B}(\mathscr{K}_1)$-valued and and $\mathbf{B}(\mathscr{K}_2)$-valued, respectively. The two kernels are left slice hyperholomorphic in p and right slice hyperholomophic in $\overline{q}$, for p,q in some axially symmetric s-domain Ω and both have the same finite number of negative squares in Ω. We denote by $\mathscr{P}(K_1)$ and $\mathscr{P}(K_2)$ the associated reproducing kernel Pontryagin spaces.

Theorem 7.5.1. *Let S be a slice hyperholomorphic $\mathbf{B}(\mathscr{K}_1,\mathscr{K}_2)$-valued function and assume that the operator*

$$M_S\ :\ f\ \mapsto S\star f \qquad (7.34)$$

is bounded from $\mathscr{P}(K_1)$ into $\mathscr{P}(K_2)$. Then,

$$M_S^*(K_2(\cdot,q_2)c_2))(q_1)=(K_1(q_1,\cdot)\star_r S(\cdot)^*c_2)(q_2).$$

Proof. To prove the result, it suffices to observe that:

$$\langle (M_S^*(K_2(\cdot,q_2)c_2))(q_1),c_1\rangle_{\mathscr{K}_1}=\overline{\langle (S\star K_1(\cdot,q_1)c_1)(q_2),c_2\rangle_{\mathscr{K}_2}}.$$

□

As a consequence we have, in case of positive definite kernels, the following result.

Proposition 7.5.2. *Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two-sided quaternionic Krein spaces and let S be a $\mathbf{B}(\mathscr{K}_1,\mathscr{K}_2)$-valued slice hyperholomorphic function defined on an axially symmetric s-domain Ω. Let $K_1(p,q)$ and $K_2(p,q)$ be positive definite kernels in Ω which are $\mathbf{B}(\mathscr{K}_1)$-*

and $\mathbf{B}(\mathscr{K}_2)$*-valued, respectively, and slice hyperholomorphic in the variable* p *in* Ω*. Then, the multiplication operator* M_S *is bounded and with norm less or equal to* k *if and only if the function*

$$K_2(p,q) - \frac{1}{k^2} S(p) \star K_1(p,q) \star_r S(q)^{[*]} \tag{7.35}$$

is positive definite on Ω*.*

Proof. We observe that by the operator-valued version of Lemma 6.1.23 (and replacing conjugation by operator adjoint) we have

$$(S(q) \star K_1(q,p))^{[*]} = K_1(p,q) \star_r S^{[*]}(q),$$

and so

$$M_S^*(K_2(\cdot,q)d) = K_1(\cdot,q) \star_r S^{[*]}(q)d.$$

The positivity of (7.35) follows from the positivity of the operator $k^2 I - M_S M_S^*$. Conversely, if (7.35) is positive, the standard argument shows that $\|M_S\| \leq k$. □

When $\mathscr{K}_1 = \mathscr{K}_2$, the case $S = I_N$ leads to:

Corollary 7.5.3. *In the notation of the following theorem, the space* $\mathscr{H}(K_1)$ *is contractively included in* $\mathscr{H}(K_2)$ *if and only if* $K_2 - K_1$ *is positive definite in* Ω*.*

We note that the result itself holds for general kernels, not necessarily slice hyperholomorphic.

Example. Let us consider the case in which $K_1 = K_2$ and equal to the kernel K of the form

$$K(p,q) = \sum_{n=0}^{\infty} p^n \overline{q}^n \alpha_n, \quad \alpha_n \in (0,\infty), \ \forall n \in \mathbb{N}.$$

Then

$$S(p) \star K(p,q) = \sum_{n=0}^{\infty} p^n S(p) \overline{q}^n \alpha_n$$

and

$$(S(p) \star K(p,q))^* = \sum_{n=0}^{\infty} q^n S(p)^* \overline{p}^n \alpha_n,$$

from which one obtains

$$S(q) \star (S(p) \star K(p,q))^* = S(q) \star \sum_{n=0}^{\infty} q^n S(p)^* \overline{p}^n \alpha_n = S(q) \star K(q,p) \star_r S(p)^*.$$

We now show that if a kernel $K(p,q)$ is positive and slice hyperholomorphic in p, then the reproducing kernel Hilbert space associated to it consists of slice hyperholomorphic functions.

Theorem 7.5.4. *Let $\mathscr{K}$ be a two-sided quaternionic Krein space and let $K(p,q)$ be a $\mathbf{B}(\mathscr{K})$-valued function on an open set $\Omega \subset \mathbb{H}$. Let $\mathscr{H}(K)$ be the associated reproducing kernel quaternionic Hilbert space. Assume that for all $q \in \Omega$ the function $p \mapsto K(p,q)$ is slice hyperholomorphic. Then the entries of the elements of $\mathscr{H}(K)$ are also slice hyperholomorphic.*

Proof. We consider the case of $\mathbb{H}$-valued function since the general case works in a similar way. Let $f \in \mathscr{H}(K)$, $p,q \in \Omega$ and $\varepsilon \in \mathbb{R} \setminus \{0\}$ sufficiently small. We have

$$\frac{1}{\varepsilon}(K(p,q+\varepsilon)-K(p,q)) = \frac{1}{\varepsilon}\overline{(K(q+\varepsilon,p)-K(q,p))}.$$

Consider $(u+\mathrm{I}v, x+\mathrm{I}y) \in \mathbb{C}_{\mathrm{I}} \times \mathbb{C}_{\mathrm{I}}$. We have that

$$\frac{\partial K(p,q)}{\partial x} = \frac{\partial \overline{K(q,p)}}{\partial u}.$$

In a similar way we have:

$$\frac{1}{\varepsilon}(K(p,q+\mathrm{I}\varepsilon)-K(p,q)) = \frac{1}{\varepsilon}\overline{(K(q+\mathrm{I}\varepsilon,p)-K(q,p))},$$

from which we deduce

$$\frac{\partial K(p,q)}{\partial y} = \frac{\partial \overline{K(q,p)}}{\partial v}.$$

The two families

$$\left\{\frac{1}{\varepsilon}(K(p,q+\varepsilon)-K(p,q))\right\}_{\varepsilon\in\mathbb{R}\setminus\{0\}}, \qquad \left\{\frac{1}{\varepsilon}(K(p,q+\mathrm{I}\varepsilon)-K(p,q))\right\}_{\varepsilon\in\mathbb{R}\setminus\{0\}},$$

are uniformly bounded in norm and so they have weakly convergent subsequences which converge to $\dfrac{\partial K(p,q)}{\partial x}$ and $\dfrac{\partial K(p,q)}{\partial y}$, respectively. Moreover we have

$$\frac{1}{\varepsilon}(f(p+\varepsilon)-f(p)) = \langle f(\cdot), \frac{1}{\varepsilon}(K(\cdot,p+\varepsilon)-K(\cdot,p))\rangle_{\mathscr{H}(K)}$$

and

$$\frac{1}{\varepsilon}(f(p+\mathrm{I}\varepsilon)-f(p)) = \langle f(\cdot), \frac{1}{\varepsilon}(K(\cdot,p+\mathrm{I}\varepsilon)-K(\cdot,p))\rangle_{\mathscr{H}(K)}.$$

Thus we can write

$$\frac{\partial f}{\partial u}(p) = \langle f(\cdot), \frac{\partial K(\cdot,p)}{\partial x}\rangle_{\mathscr{H}(K)},$$

and

$$\frac{\partial f}{\partial v}(p) = \langle f(\cdot), \frac{\partial K(\cdot,p)}{\partial y}\rangle_{\mathscr{H}(K)}.$$

To show that the function f is slice hyperholomorphic, we take its restriction to any complex plane $\mathbb{C}_{\mathrm{I}}$ and we show that it is in the kernel of the Cauchy-Riemann operator:

$$\begin{aligned}\frac{\partial f}{\partial u}+\mathrm{I}\frac{\partial f}{\partial v} &= \langle f, \frac{\partial K(\cdot,q)}{\partial x}\rangle_{\mathscr{H}(K)} + \mathrm{I}\langle f(\cdot), \frac{\partial K(\cdot,q)}{\partial y}\rangle_{\mathscr{H}(K)} \\ &= \langle f, \frac{\partial K(\cdot,q)}{\partial x} - \frac{\partial K(\cdot,q)}{\partial y}\mathrm{I}\rangle_{\mathscr{H}(K)} \\ &= \langle f, \overline{\frac{\partial K(q,\cdot)}{\partial u}+\mathrm{I}\frac{\partial K(q,\cdot)}{\partial v}}\rangle_{\mathscr{H}(K)} = 0\end{aligned}$$

since the kernel $K(q,p)$ is slice hyperholomorphic in the first variable q. □

As a consequence of this result, we have the following theorem which is the operator-valued version of Theorem 5.10.4. Although the coefficient space is taken to be a Krein space, the proof goes in the same way and is omitted.

Theorem 7.5.5. *Let $\mathscr{K}$ be a quaternionic two-sided Krein space, and let $K(p,q)$ be a $\mathbf{B}(\mathscr{K})$-valued kernel, left slice hyperholomorphic in p and right slice hyperholomophic in $\overline{q}$, for p,q in some axially symmetric s-domain Ω and having a finite number of negative squares in Ω. Then there exists a unique reproducing kernel Pontryagin space of $\mathscr{K}$-valued left slice hyperholomorphic functions, with reproducing kernel $K(p,q)$.*

We conclude this section with some propositions pertaining to kernels of the form $K(p,q)=\sum_{n,m=0}^{\infty}p^n a_{n,m}\overline{q}^m$, where $a_{n,m}=a_{m,n}^*\in\mathbb{H}^{N\times N}$. It is immediate that $K(p,q)$ is a function slice hyperholomorphic in p and right slice hyperholomorphic in $\bar{q}$; moreover the assumption on the coefficients $a_{n,m}$ implies that $K(p,q)$ is Hermitian.

Proposition 7.5.6. *Let $(a_{n,m})_{n,m\in\mathbb{N}_0}$ denote a sequence of $N\times N$ quaternionic matrices such that $a_{n,m}=a_{m,n}^*$, and assume that the power series*

$$K(p,q)=\sum_{n,m=0}^{\infty}p^n a_{n,m}\overline{q}^m$$

converges in a neighborhood V of the origin. Then the following are equivalent:
(1) *The function $K(p,q)$ has κ negative squares in V.*
(2) *All the finite matrices $A_\mu \stackrel{\text{def.}}{=} (a_{n,m})_{n,m=0,\dots\mu}$ have at most κ strictly negative eigenvalues, and exactly κ strictly negative eigenvalues for at least one $\mu\in\mathbb{N}_0$.*

Proof. Let $r>0$ be such that the ball $B(0,r)$ is contained in V. Let I, J be two units in the unit sphere of purely imaginary quaternions $\mathbb{S}$. Then we have:

$$a_{n,m}=\frac{1}{4r^{n+m}\pi^2}\iint_{[0,2\pi]^2}e^{-\mathrm{I}nt}K(re^{\mathrm{I}t},re^{\mathrm{J}s})e^{\mathrm{J}ms}dtds.$$

This expression does not depend on the choice of I and J. Furthermore, we can take $\mathrm{I}=\mathrm{J}$

and so:

$$A_\mu = \frac{1}{4r^{n+m}\pi^2}\iint_{[0,2\pi]^2}\begin{pmatrix} I_N \\ e^{-\mathrm{J}t}I_N \\ \vdots \\ e^{-\mathrm{J}\mu t}I_N\end{pmatrix} K(re^{\mathrm{J}t}, re^{\mathrm{J}s})\begin{pmatrix} I_N & e^{\mathrm{J}s}I_N & \cdots & e^{\mathrm{J}\mu s}I_N\end{pmatrix} dtds.$$

Write

$$K(p,q) = K_+(p,q) - F(p)F(q)^*,$$

where F is $\mathbb{H}^{N\times\kappa}$-valued. The function F is built from functions of the form $p \mapsto K(p,q)$ for a finite number of q's, and so is a continuous function of p, and so is $K_+(p,q)$. See [47, pp. 8-9] for the argument in the complex setting, which is valid also in the present case. Thus

$$A_\mu = A_{\mu,+} - A_{\mu,-}$$

where

$$A_{\mu,+} = \frac{1}{4r^{n+m}\pi^2}\iint_{[0,2\pi]^2}\begin{pmatrix} I_N \\ e^{-\mathrm{J}t}I_N \\ \vdots \\ e^{-\mathrm{J}\mu t}I_N\end{pmatrix} K_+(re^{\mathrm{J}t}, re^{\mathrm{J}s})\begin{pmatrix} I_N & e^{\mathrm{J}s}I_N & \cdots & e^{\mathrm{J}\mu s}I_N\end{pmatrix} dtds,$$

$$A_{\mu,-} = \frac{1}{4r^{n+m}\pi^2}\iint_{[0,2\pi]^2}\begin{pmatrix} I_N \\ e^{-\mathrm{J}t}I_N \\ \vdots \\ e^{-\mathrm{J}\mu t}I_N\end{pmatrix} F(re^{\mathrm{J}t})F(re^{\mathrm{J}s})^*\begin{pmatrix} I_N & e^{\mathrm{J}s}I_N & \cdots & e^{\mathrm{J}\mu s}I_N\end{pmatrix} dtds.$$

These two expressions show that A_μ has at most κ strictly negative eigenvalues.

To prove the converse, assume that all the matrices A_μ have at most κ strictly negative eigenvalues. Let us now define

$$K_\mu(p,q) = \sum_{n,m=0}^{\mu} p^m a_{n,m}\overline{q}^m.$$

Then, K_μ has at most κ negative squares, as is seen by writing A_μ as a difference of two positive matrices, one of rank κ. Since, pointwise, we have

$$K(p,q) = \lim_{\mu\to\infty} K_\mu(p,q),$$

we deduce that the function $K(p,q)$ has at most κ negative squares.

To conclude, it remains to be proved that the number of negative squares of $K(p,q)$ and A_μ is the same. Assume that $K(p,q)$ has κ negative squares, but that the A_μ have at most $\kappa' < \kappa$ strictly negative eigenvalues. Then, the argument above shows that $K(p,q)$ would have at most κ' negative squares, which contradicts the hypothesis. The other direction is proved in a similar way. □

The following results are consequences of the previous proposition:

Proposition 7.5.7. *In the notation of the preceding proposition, the number of negative squares is independent of the neighborhood* V.

Proof. This follows from the fact that the coefficients $a_{n,m}$ do not depend on the given neighborhood. □

Proposition 7.5.8. *Assume that* $K(p,q)$ *is* $\mathbb{H}^{N\times N}$*-valued and has* κ *negative squares in* V *and let* $\alpha(p)$ *be a* $\mathbb{H}^{N\times N}$*-valued slice hyperholomorphic function and such that* $\alpha(0)$ *is invertible. Then the kernel*

$$B(p,q) = \alpha(p) \star K(p,q) \star_r \alpha(q)^* \tag{7.36}$$

has κ *negative squares in* V.

Proof. Let $K(p,q) = \sum_{n,m=0}^{\infty} p^n a_{n,m} \overline{q}^m$ and $\alpha(p) = \alpha_0 + p\alpha_1 + \cdots$. The $\mu \times \mu$ main block matrix B_μ corresponding to the power series (7.36) equals

$$B_\mu = L A_\mu L^*,$$

where

$$L = \begin{pmatrix} \alpha_0 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & 0 & \cdots & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & \cdots \\ \vdots & \vdots & & & \\ \alpha_\mu & \alpha_{\mu-1} & \cdots & \alpha_1 & \alpha_0 \end{pmatrix}.$$

Since we assumed the invertibility of $\alpha_0 = \alpha(0)$, the signatures of A_μ and B_μ are the same for every $\mu \in \mathbb{N}_0$. By Proposition 7.5.6 it follows that the kernels K and B have the same number of negative squares. □

Remark 7.5.9. We remark that the above results still hold in the setting of operator-valued functions.

7.6 The space $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ and slice backward-shift invariant subspaces

Theorem 5.3.12 allows us to introduce the vector version of the space $\mathrm{H}^2(\mathbb{B})$. Given a separable two sided quaternionic Hilbert space $\mathscr{H}$ we define the space $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B}) :=$

$\mathrm{H}^2(\mathbb{B})\otimes\mathscr{H}$ and identify it with the space of $\mathscr{H}$-valued power series f with finite $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$-norm:

$$\mathrm{H}^2_{\mathscr{H}}(\mathbb{B}) = \left\{ f(p) = \sum_{n=0}^{\infty} p^n f_n : \ \|f\|^2_{\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathscr{H}} < \infty \right\}. \tag{7.37}$$

Definition 7.6.1. The operator $M_p : \mathrm{H}^2_{\mathscr{H}}(\mathbb{B}) \to \mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ of slice hyperholomorphic multiplication by p is defined by:

$$M_p : \sum_{n=0}^{\infty} p^n f_n \mapsto \sum_{n=0}^{\infty} p^{n+1} f_n. \tag{7.38}$$

Remark 7.6.2. The definition of the $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$-norm yields that M_p is an isometry and an inner-product calculation shows that its adjoint is given by

$$M_p^* : \sum_{n=0}^{\infty} p^n f_n \mapsto \sum_{n=0}^{\infty} p^n f_{n+1}. \tag{7.39}$$

Furthermore,

$$\|M_p^{*n} f\|^2_{\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})} = \sum_{j=0}^{\infty} \|f_{n+j}\|^2_{\mathscr{H}} < \infty$$

tends to zero as $n \to \infty$ whenever $\sum_{n=0}^{\infty} \|f_n\|^2_{\mathscr{H}} < \infty$, that is, whenever f belongs to $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$. Thus, the powers of M_p^* tend to zero strongly which means by definition that M_p^* is a *strongly stable operator*.

We note that the right-side of (7.39) still makes sense when we do not necessarily work in the metric of $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$, provided the function f is slice hyperholomorphic in a neighborhood of the origin. We set then

$$R_0 f(p) = \sum_{n=1}^{\infty} p^{n-1} f_n. \tag{7.40}$$

More in general, we give the following definition:

Definition 7.6.3. Let $\mathscr{X}$ be a right quaternionic Hilbert space and let $x_0 \in \mathbb{R}$. Let $f \in \mathscr{R}(\Omega, \mathscr{X})$ and let

$$f(p) = \sum_{n=0}^{\infty} (p - x_0)^n f_n, \qquad f_n \in \mathscr{X}.$$

We define the operator R_{x_0} by

$$(R_{x_0} f)(p) = (p - x_0)^{-1}(f(p) - f(x_0)) \stackrel{\text{def.}}{=} \begin{cases} \sum_{n=1}^{\infty} (p - x_0)^{n-1} f_n, & p \neq x_0, \\ f_1, & p = x_0. \end{cases} \tag{7.41}$$

When $x_0 = 0$, R_0 is called backward-shift operator.

Subspaces of $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ which are invariant under M_p or under M_p^* admit representation similar to those in the classical setting. To present them we need some preliminaries.

Definition 7.6.4. Let $\mathscr{X}$ and $\mathscr{H}$ be two right quaternionic Hilbert spaces, and let $A \in \mathbf{B}(\mathscr{X})$ and $C \in \mathbf{B}(\mathscr{X}, \mathscr{H})$. The pair (C,A) is called *contractive* if

$$A^*A + C^*C \leq I_{\mathscr{X}} \tag{7.42}$$

and it is called *isometric* if

$$A^*A + C^*C = I_{\mathscr{X}}. \tag{7.43}$$

Furthermore, the pair (C,A) is called *output-stable* if the observability operator $\mathscr{O}_{C,A} : \mathscr{X} \to \mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ defined by

$$\mathscr{O}_{C,A} :\ x \mapsto C \star (I_{\mathscr{X}} - pA)^{-\star} x = \sum_{n=0}^{\infty} p^n C A^n x \tag{7.44}$$

is bounded from $\mathscr{X}$ into $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$. Finally, the pair (C,A) is called *observable* if the operator $\mathscr{O}_{C,A}$ is injective.

Definition 7.6.5. If the pair (C,A) is output-stable, one can define the *observability Gramian*

$$\mathscr{G}_{C,A} := \mathscr{O}_{C,A}^* \mathscr{O}_{C,A} = \sum_{n=0}^{\infty} A^{*n} C^* C A^n. \tag{7.45}$$

The representation (7.45) follows from the definition of the $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$-inner product. Convergence of this series (in the weak and, therefore, in the strong operator topology; see Proposition 5.6.4) is equivalent to the output stability of the pair (C,A). It follows directly from the series representation (7.45) that $\mathscr{G}_{C,A}$ satisfies the Stein identity

$$\mathscr{G}_{C,A} - A^* \mathscr{G}_{C,A} A = C^* C. \tag{7.46}$$

Proposition 7.6.6. *Let the pair (C,A) be output-stable and let $\mathscr{O}_{C,A} : \mathscr{X} \to \mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ be as in* (7.44)*. Then $\mathscr{O}_{C,A}^* :\ \mathbf{H}^2_{\mathscr{H}}(\mathbb{B}) \to \mathscr{X}$ is defined by*

$$\mathscr{O}_{C,A}^* f = \sum_{k=0}^{\infty} A^{*k} C^* f_k \quad \textit{if} \quad f(p) = \sum_{k=0}^{\infty} p^k f_k. \tag{7.47}$$

Proof. Making use of the power series representation (7.44) for $\mathscr{O}_{C,A}$ and the definition of inner product in $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ we get for every $x \in \mathscr{X}$

$$\langle \mathscr{O}_{C,A}^* f, x \rangle_{\mathscr{X}} = \langle f,\ \mathscr{O}_{C,A} x \rangle_{\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})} = \sum_{k=0}^{\infty} \left\langle f_k,\ C A^k x \right\rangle_{\mathscr{H}} = \left\langle \sum_{k=0}^{\infty} A^{*k} C^* f_k,\ x \right\rangle_{\mathscr{X}}$$

which proves formula (7.47). It follows from the same computation that the series in the formula (7.47) for $\mathscr{O}_{C,A}^* f$ converges in the weak operator topology, and hence in view of Proposition 5.6.4 also in the the strong topology. ☐

Proposition 7.6.7. *If the pair (C,A) is contractive, then it is output stable and $\mathscr{G}_{C,A} \le I_{\mathscr{X}}$. If (C,A) is isometric and A is strongly stable, then $\mathscr{G}_{C,A} = I_{\mathscr{X}}$.*

Proof. We have from (7.42)

$$A^{*k}C^*CA^k \le A^{*k}A^k - A^{*k+1}A^{k+1}.$$

Summing up the latter equalities for $k = 0,\dots,n$ gives

$$\sum_{k=0}^{n} A^{*k}C^*CA^k \le I_{\mathscr{X}} - A^{*n+1}A^{n+1} \le I_{\mathscr{X}}.$$

Passing to the limit as $n \to \infty$ in the latter inequality and making use of (7.45) we conclude that $\mathscr{G}_{C,A} \le I_{\mathscr{X}}$ and in particular, (C,A) is output stable. If we start with (7.43) rather than with (7.42) we get

$$\sum_{k=0}^{n} A^{*k}C^*CA^k = I_{\mathscr{X}} - A^{*n+1}A^{n+1}.$$

Passing to the limit as $n \to \infty$ in the latter equality and taking into account that A is strongly stable (so that the second term on the right tends to zero strongly) we conclude that $\mathscr{G}_{C,A} = I_{\mathscr{X}}$. □

Proposition 7.6.8. *Let (C,A) be an output-stable pair. Then the intertwining relation*

$$M_p^* \mathscr{O}_{C,A} = \mathscr{O}_{C,A} A \tag{7.48}$$

holds and therefore, the linear manifold $\operatorname{ran}\mathscr{O}_{C,A}$ *is M_p^*-invariant.*

Proof. For every $x \in \mathscr{X}$, we have from (7.39) and (7.44)

$$M_p^* \mathscr{O}_{C,A} x = \sum_{n=0}^{\infty} p^n CA^{n+1} x = \left(\sum_{n=0}^{\infty} p^n CA^n\right) Ax = \mathscr{O}_{C,A} Ax$$

which proves (7.48). □

The manifold $\mathscr{N} = \operatorname{ran}\mathscr{O}_{C,A}$ need not be closed in the metric of $\mathrm{H}^2_{\mathcal{H}}(\mathbb{B})$. However, it becomes a Hilbert space with respect to the lifted norm $\|\mathscr{O}_{C,A}x\|_{\mathscr{N}} = \|Qx\|_{\mathscr{X}}$ where Q is the orthogonal projection of $\mathscr{X}$ onto the observability subspace $\mathscr{X} \ominus \operatorname{Ker}\mathscr{O}_{C,A}$. Since $\operatorname{Ker}\mathscr{O}_{C,A}$ is A-invariant we may let C' and A' to be restrictions of C and A to the observability subspace and then conclude that $\operatorname{ran}\mathscr{O}_{C,A} = \operatorname{ran}\mathscr{O}_{C',A'}$. Since the pair (C',A') is observable, we may assume from the very beginning that the given output-stable pair (C,A) is observable.

Proposition 7.6.9. *Let (C,A) be an observable output-stable pair. Then the manifold $\mathscr{N} = \operatorname{ran}\mathscr{O}_{C,A}$ with the lifted norm*

$$\|\mathscr{O}_{C,A}x\|_{\mathscr{N}} = \|x\|_{\mathscr{X}} \tag{7.49}$$

is the reproducing kernel Hilbert space with reproducing kernel

$$K_{C,A}(p,q) = C \star (I - pA)^{-\star} \left(C \star (I - qA)^{-\star}\right)^*. \tag{7.50}$$

Proof. For $x \in \mathscr{X}$ and $y \in \mathscr{H}$, we have

$$\begin{aligned}
\langle (\mathscr{O}_{C,A}x)(q),\, y\rangle_{\mathscr{H}} &= \left\langle C\star(I-qA)^{-\star}x,\, y\right\rangle_{\mathscr{H}} \\
&= \left\langle x,\, \left(C\star(I-qA)^{-\star}\right)^* y\right\rangle_{\mathscr{X}} \\
&= \left\langle \mathscr{O}_{C,A}x,\, \mathscr{O}_{C,A}\left(C\star(I-qA)^{-\star}\right)^* y\right\rangle_{\mathscr{N}} \\
&= \langle \mathscr{O}_{C,A}x,\, K_{C,A}(\cdot,q)y\rangle_{\mathscr{N}}
\end{aligned}$$

which means that $K_{C,A}(p,q)$ of the form (7.50) is indeed the reproducing kernel for $\mathscr{N}$. □

Theorem 7.6.10. *Let (C,A) be a contractive observable pair and let $\mathscr{N} = \operatorname{ran}\mathscr{O}_{C,A}$ be given the lifted norm* (7.49) *(equivalently, let $\mathscr{N} = \mathscr{H}(K_{C,A})$ be the reproducing kernel Hilbert space with reproducing kernel $K_{C,A}$ given in* (7.50)*). Then*

(1) $\mathscr{N}$ is R_0-invariant.

(2) $\mathscr{N}$ is contractively included in $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$.

(3) The following difference-quotient inequality holds

$$\|R_0 f\|^2_{\mathscr{N}} \le \|f\|^2_{\mathscr{N}} - \|f(0)\|^2_{\mathscr{H}} \quad \text{for all } f \in \mathscr{N}. \tag{7.51}$$

Conversely, if $\mathscr{M}$ is a quaternionic Hilbert space contractively included in $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ which is R_0-invariant and for which the difference-quotient inequality (7.51) *holds, then there is a contractive observable pair (C,A) such that $\mathscr{N} = \operatorname{ran}\mathscr{O}_{C,A} = \mathscr{H}(K_{C,A})$. In particular, $\mathscr{M}$ is contractively included in $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$.*

Proof. Let us assume that the pair (C,A) is contractive and observable. By Proposition 7.6.7, (C,A) is output stable and $\mathscr{G}_{C,A} \le I_{\mathscr{X}}$. Therefore for a generic element $f = \mathscr{O}_{C,A}x$ in $\mathscr{N}$ we have

$$\|f\|^2_{\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})} = \|\mathscr{O}_{C,A}x\|^2_{\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})} = \langle \mathscr{G}_{C,A}x,\, x\rangle_{\mathscr{X}} \le \|x\|^2_{\mathscr{X}} = \|\mathscr{O}_{C,A}x\|^2_{\mathscr{N}} = \|f\|^2_{\mathscr{N}}. \tag{7.52}$$

Thus, $\|f\|_{\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})} \le \|f\|_{\mathscr{N}}$ for every $f \in \mathscr{N}$ which means that $\mathscr{N}$ is contractively included in $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$. The M_p^* invariance follows from Proposition 7.6.8. Finally, due to the contractivity (7.42) and the intertwining relation (7.48), and since for $f = \mathscr{O}_{C,A}x$, we have $f(0) = Cx$,

$$\begin{aligned}
\|M_p^* f\|^2_{\mathscr{N}} + \|f(0)\|^2_{\mathscr{H}} &= \|M_p^*\mathscr{O}_{C,A}x\|^2_{\mathscr{N}} + \|Cx\|^2_{\mathscr{H}} \\
&= \|\mathscr{O}_{C,A}Ax\|^2_{\mathscr{N}} + \|Cx\|^2_{\mathscr{H}} \\
&= \|Ax\|^2_{\mathscr{X}} + \|Cx\|^2_{\mathscr{H}} \le \|x\|^2_{\mathscr{X}} = \|\mathscr{O}_{C,A}x\|^2_{\mathscr{N}} = \|f\|^2_{\mathscr{N}}
\end{aligned}$$

which proves (7.51).

Conversely, let us assume that $\mathscr{N}$ is a subspace of $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ enjoying properties (1) and (3) from the first part of the theorem. Then the operator $A = R_0|_{\mathscr{N}}$ maps $\mathscr{N}$ into itself. We thus define $C : \mathscr{N} \to \mathscr{H}$ and $A : \mathscr{N} \to \mathscr{N}$ by

$$A = R_0|_{\mathscr{N}} \quad \text{and} \quad C : f \to f(0). \tag{7.53}$$

Then the pair (C,A) is contractive by (7.51). By (7.39), we have

$$CA^n f = (R_0^n f)(0) = f_n \quad \text{for every} \quad f(p) = \sum_{j=0}^{\infty} p^n f_n \in \mathscr{N}$$

and therefore

$$f(p) = \sum_{j=0}^{\infty} p^n CA^n f = \mathscr{O}_{C,A} f.$$

Thus, the observability operator $\mathscr{O}_{C,A}$ with (C,A) defined as above equals the identity map on $\mathscr{N}$. Therefore, $\ker \mathscr{O}_{C,A}$ is trivial so that the pair (C,A) is observable. The space $\mathscr{N} = \operatorname{ran} \mathscr{O}_{C,A} = \mathscr{H}(K_{C,A})$ is contractively included into $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ by the first part of the proof (since the pair (C,A) is contractive and observable). □

Remark 7.6.11. We remark that the theorem in this section are written more precisely using Schur functions in the next chapter; see in particular Section 8.4. We also note that condition (3) in the theorem implies (2). In the sequel we prove a more general result in the setting of Pontryagin spaces, where only (1) and (3) are in force. See Theorem 8.6.1. The case of isometrically included R_0-invariant subspaces of $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ is of special interest (and then in particular, $R_0 = M_p^*$).

Theorem 7.6.12. *A subspace $\mathscr{N} \subseteq \mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ is M_p^*-invariant and isometrically included in $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ if and only if there exists a Hilbert space $\mathscr{X}$ and an isometric pair $(C,A) \in \mathbf{B}(\mathscr{X},\mathscr{H}) \times \mathbf{B}(\mathscr{X})$ with A strongly stable such that $\mathscr{N} = \operatorname{ran} \mathscr{O}_{C,A} = \mathscr{H}(K_{C,A})$.*

Proof. If the pair (C,A) is isometric and A is strongly stable, then $\mathscr{G}_{C,A} = I_{\mathscr{X}}$, by Remark 7.38. Then it follows from calculation (7.52) that $\|f\|_{\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})} = \|f\|_{\mathscr{N}}$ for evey $f \in \mathscr{N} = \mathscr{O}_{C,A}$. The M_p^*-invariance of $\mathscr{N}$ follows by Proposition 7.6.8 and completes the proof of the "if" part.

Conversely, for an M_p^*-invariant closed subspace $\mathscr{N}$ of $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ we define the operators A and C as in (7.53). As in the proof of Theorem 7.6.10, we show that the observability operator $\mathscr{O}_{C,A}$ equals the identity map on $\mathscr{N}$. Since the metric of $\mathscr{N}$ coincides with that of $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ and since $M_p^* : \mathrm{H}^2_{\mathscr{H}}(\mathbb{B}) \to \mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ is strongly stable, its restriction A (to the invariant subspace $\mathscr{N}$) is also strongly stable. It remains to demonstrate that the pair (C,A) is isometric. To this end, observe that for every $f \in \mathscr{N}$,

$$\begin{aligned}
\|f\|^2_{\mathscr{N}} = \|f\|^2_{\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})} &= \|M_p^* f\|^2_{\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})} + \|f(0)\|^2_{\mathscr{H}} \\
&= \|M_p^* f\|^2_{\mathscr{N}} + \|f(0)\|^2_{\mathscr{H}} \\
&= \|Af\|^2_{\mathscr{N}} + \|Cf\|^2_{\mathscr{H}},
\end{aligned}$$

which is equivalent to (7.43). □

Part III

Quaternionic Schur analysis

In this third and last part of the book, which consists of four chapters, we discuss various aspects of Schur analysis in the slice hyperholomorphic setting. In Chapter 8, we discuss realization of the counterpart of Schur functions and related classes, in the operator-valued case. We also consider the Beurling-Lax theorem in the present setting, as well as a number of function theoretic questions (such as slice hyperholomorphic extension). An important role in Schur analysis is played rational functions, especially with symmetry properties. These are studied in Chapter 9. We define and study in particular the counterpart of matrix-valued rational functions taking unitary values on the imaginary line or the unit circle. Here the imaginary line and the unit circle are replaced by the space of purely imaginary quaternions and the unit ball of the quaternions, respectively. In Chapter 10 we focus on two topics. First, we consider some interpolation problems for scalar Schur (slice hyperholomorphic) functions. Next, we outline the theory of first order discrete systems in the present setting. In the last chapter we study a general one-sided interpolation problem for vector-valued functions (resp. operator-valued functions) in the setting of the Hardy space (resp. for Schur multipliers).

A large part of the material here is based on the unpublished manuscript [19] and on the papers [1], [32].

Chapter 8

Reproducing kernel spaces and realizations

The tools developed in the previous chapters allow us to define and study in the operator-valued case the various families of functions appearing in classical Schur analysis. In this section we obtain realization formulas for these functions. These formulas in turn have important consequences, such as existence of slice hyperholomorphic extensions and results in function theory such as an extension of Bohr's inequality. Recall that all two-sided quaternionic vector spaces are assumed to satisfy condition (5.4). An important tool in this chapter is Shmulyan' theorem on densely defined contractive relations between Pontryagin spaces with the same index, see Theorem 5.7.10, and this forces us to take for coefficients spaces two-sided quaternionic Pontryagin spaces with the same index, and not Krein spaces. The rational case, studied in the following chapter, corresponds to the settnig where both the coefficient spaces and the reproducing kernel Pontryagin spaces associated to the various functions are finite dimensional.

8.1 The various classes of functions

We now describe the counterparts of the classes mentioned in Section 1.6. These functions, and the associated reproducing kernel Pontryagin spaces, form the building blocks of Schur analysis. In the sequel we consider some of their applications, but a lot of aspects remain to be developed.

The quaternionic Pontryagin spaces $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}$ appearing in the definitions are coefficient spaces, and are assumed to be *two-sided* vector spaces. On the other hand, the associated reproducing kernel Pontryagin spaces appearing in the various realizations will be *right-sided.*

Let us begin by recalling that $\sum_{t=0}^{\infty} p^t \bar{q}^t$ can be written in closed form as

$$\sum_{t=0}^{\infty} p^t \bar{q}^t = (1-p\bar{q})^{-\star},$$

where we are taking the (left) $\star$-inverse of the function $1-p\bar{q}$ with respect to the variable p. Recall that

$$(1-p\bar{q})^{-\star} = (1-2\mathrm{Re}(q)p+|q|^2p^2)^{-1}(1-pq),$$

and note that, for $q \neq 0$, the right hand side is defined for all $p \notin [q^{-1}]$.

Since the functions we consider are, respectively, left slice hyperholomorphic in p and right slice hyperholomorphic in $\bar{q}$, when considering the $\star$-multiplication we will always assume that it is computed with respect to p while the $\star_r$-multiplication is computed with respect to $\bar{q}$.

We have:

Definition 8.1.1. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two quaternionic two-sided Pontryagin spaces of same index. The $\mathbf{B}(\mathscr{P}_1,\mathscr{P}_2)$-valued function S is called a generalized Schur function of the unit ball if it is slice hyperholomorphic in some axially symmetric open subset Ω of the unit ball and if the kernel

$$K_S(p,q) = \left(I_{\mathscr{P}_2} - S(p)S(q)^{[*]}\right) \star (1-p\bar{q})^{-\star}$$

has a finite number, say κ, of negative squares in Ω.

Remark 8.1.2. Note that

$$K_S(p,q) = \left(I_{\mathscr{P}_2} - S(p)S(q)^{[*]}\right) \star (1-p\bar{q})^{-\star} = \sum_{t=0}^{\infty} p^t \left(I_{\mathscr{C}_2} - S(p)S(q)^{[*]}\right)\bar{q}^t,$$

and that, in particular,

$$K_S(p,q) - pK_S(p,q)\bar{q} = I_{\mathscr{P}_2} - S(p)S(q)^{[*]}. \tag{8.1}$$

The same kernel can be written also noting that the series $\sum_{t=0}^{\infty} p^t \bar{q}^t$ defines a function right slice hyperholomorphic in $\bar{q}$ and so one could write the sum of that series also as

$$(1-p\bar{q})^{-\star_r} = (1-\bar{p}\bar{q})(1-2\mathrm{Re}(p)\bar{q}+|p|^2\bar{q}^2)^{-1}$$

where the $\star_r$-inverse is computed with respect to the variable $\bar{q}$. We then have

$$K_S(p,q) = (1-p\bar{q})^{-\star_r} \star_r \left(I_{\mathscr{P}_2} - S(p)S(q)^{[*]}\right).$$

We will denote the class of such functions by the symbol $\mathcal{S}_\kappa(\mathscr{P}_1,\mathscr{P}_2,\mathbb{B})$. The dependence on Ω is not stressed out in the notation since, as we will see in Section 8.3 (see Theorem 8.3.6), any such function S has a unique slice hypermeromorphic extension to the open unit ball $\mathbb{B}$.

Definition 8.1.3. Let $\mathscr{P}$ be a quaternionic two-sided Pontryagin space. The $\mathbf{B}(\mathscr{P})$-valued function Φ slice hyperholomorphic in some axially symmetric open subset Ω of the unit ball is called a generalized Carathéodory function of the ball if the

$$K_\Phi(p,q) = \left(\Phi(p) + \Phi(q)^{[*]}\right) \star (1 - p\bar{q})^{-\star} \tag{8.2}$$

has a finite number, say κ, of negative squares in Ω.

We note that

$$K_\Phi(p,q) = \sum_{t=0}^{\infty} p^t \left(\Phi(p) + \Phi(q)^{[*]}\right) \bar{q}^t. \tag{8.3}$$

We will denote the class of such functions by the symbol $\mathcal{C}_\kappa(\mathscr{P},\mathbb{B})$. As for Schur functions, the dependence on Ω is not stressed out in the notation because of the existence of a unique slice hypermeromorphic extension to the ball; see Section 8.7.

The two next families of kernels pertain to the half-space. The above remark on slice hypermeromorphic extensions also hold in these two cases (see Sections 8.8 and 8.9 respectively). First recall the definition of the kernel (6.29):

$$k(p,q) = (p+\bar{q})^{-\star} = (p^2 + 2\mathrm{Re}(q)p + |q|^2)^{-1}(p+q).$$

Definition 8.1.4. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two quaternionic two-sided Pontryagin spaces of same index. The $\mathbf{B}(\mathscr{P}_1,\mathscr{P}_2)$-valued function S is called a generalized Schur function of the half-space if it is slice-hyperholomorphic in some axially symmetric open subset Ω of the open right half-space $\mathbb{H}_+$ and if the kernel

$$K_S(p,q) = I_{\mathscr{P}_2} k(p,q) - S(p) \star k(p,q) \star_r S(q)^{[*]}$$

has a finite number, say κ, of negative squares in Ω. We will denote the class of such functions by the symbol $\mathcal{S}_\kappa(\mathscr{P}_1,\mathscr{P}_2,\mathbb{H}_+)$.

Definition 8.1.5. Let $\mathscr{P}$ be a quaternionic two-sided Pontryagin space. The $\mathbf{B}(\mathscr{P})$-valued function Φ slice hyper-holomorphic in some axially symmetric open subset Ω of the open right half-space $\mathbb{H}_+$ is called a generalized Herglotz function if

$$K_\Phi(p,q) = \Phi(p) \star k(p,q) + k(p,q) \star_r \Phi(q)^{[*]}$$

has a finite number, say κ, of negative squares in Ω.

We will denote the class of such functions by the symbol $\mathcal{H}_\kappa(\mathscr{P},\mathbb{H}_+)$. It is useful to note the equation

$$pK_\Phi(p,q) + K_\Phi(p,q)\overline{q} = \Phi(p) + \Phi(q)^{[*]} \tag{8.4}$$

satisfied by the kernel $K_\Phi(p,q)$.

These form the four main families of functions which we consider here. Two important class of functions are defined in terms of pairs. We mention them for completeness, but will not treat them in this book.

Definition 8.1.6. Let $\mathscr{P}$ be a quaternionic two-sided Pontryagin space. The pair (E_+,E_-) of $\mathbf{B}(\mathscr{P})$-valued functions slice hyperholomorphic in some axially symmetric open subset Ω of $\mathbb{H}_+$ is called a generalized de Branges pair of the half-space if the kernel

$$K_{E_+,E_-}(p,q) = E_+(p)\star k(p,q)I_{\mathscr{P}}\star_r E_+(q)^* - E_-(p)\star k(p,q)I_{\mathscr{P}}\star_r E_-(q)^*, \qquad (8.5)$$

has a finite number, say κ, of negative squares in Ω.

Definition 8.1.7. Let $\mathscr{P}$ be a quaternionic two-sided Pontryagin space. The pair (E_+,E_-) of $\mathbf{B}(\mathscr{P})$-valued functions slice hyperholomorphic in some axially symmetric open subset Ω of the open ball $\mathbb{B}$ is called a generalized de Branges pair if the kernel

$$K_{E_+,E_-}(p,q) = \sum_{t=0}^{\infty} p^t(E_+(p)E_+(q)^* - E_-(p)E_-(q)^*)\overline{q}^t \qquad (8.6)$$

has a finite number, say κ, of negative squares in Ω.

We denote by $\mathcal{B}_\kappa(\mathscr{P},\mathbb{H}_+)$ and $\mathcal{B}_\kappa(\mathscr{P},\mathbb{B})$ the corresponding families of pairs. We note that the problem of hypermeromorphic extension is more involved for pairs.

8.2 The Potapov-Ginzburg transform

The Potapov-Ginzburg transform allows to reduce the case where coefficient spaces $\mathscr{P}_1$ and $\mathscr{P}_2$ are Pontryagin spaces (of the same index) to the case of Hilbert spaces. We begin with a lemma. A proof in the classical case can be found in [47, Lemma 4.4.3, p. 164] but we provide an argument for completeness. First a remark: a matrix $A \in \mathbb{H}^{m\times m}$ is not invertible if and only if there exists $c \neq 0 \in \mathbb{H}^m$ such that $c^*A = 0$. This fact can be seen for instance from [230, Theorem 7, p. 202], where it is shown that a matrix over a division ring has row rank equal to the column rank, or [272, Corollary 1.1.8].

Lemma 8.2.1. *Let $\varphi(p,q)$ denote either of the kernels $(1-p\overline{q})^{-\star}$ or $k_{\mathrm{H}^2}(p,q)$, see (6.29). Let T be a $\mathbb{H}^{m\times m}$-valued function slice hyperholomorphic in an axially symmetric s-domain Ω which intersect $(-1,1)$ in the first case, and the positive real line in the second case, and such that the kernel*

$$T(p)\star\varphi(p,q)\star_r T(q)^* - \varphi(p,q)I_m$$

has a finite number of negative squares, say κ, in Ω. Then T is invertible in Ω, with the possible exception of a countable number of spheres.

Proof. We consider the first case. The second case is treated in the same way. We first show that T is invertible on $\Omega\cap\mathbb{R}_+$ with the possible exception of a countable number of points. Let $x_1,\ldots,x_M$ be zeros of $T(p)$. Then, there exist vectors $c_1,\ldots,c_M \in \mathbb{H}^m$ such that

$$c_j^*T(x_j) = 0, \quad j = 1,\ldots,M.$$

Thus

$$m_{jk} = c_j^* K(x_j, x_k) c_k = -\frac{c_j^* c_k}{x_j + x_k}.$$

To conclude, we use to Proposition 4.3.12 to see that the $M \times M$ matrix with (j,k) entry m_{jk} is strictly negative, and so $M \le k$. $\square$

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two-sided quaternionic Pontryagin spaces with the same index, with associated fundamental symmetries J_1 and J_2. Given $S \in \mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2)$ we denote by

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \tag{8.7}$$

its block decomposition according to the fundamental decompositions defined by J_1 and J_2. In the statement of the following theorem, we denote by I_{2+} the identity of the positive space in the fundamental decomposition of $\mathscr{P}_2$.

Theorem 8.2.2. *Let $S \in \mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2)$, defined in an axially symmetric s-domain Ω as in Lemma 8.2.1, and with decomposition* (8.7)*. Then the function S_{22} is $\star$-invertible in Ω, with the possible exception of a countable number of spheres. Let*

$$A(p) = \begin{pmatrix} I_{2+} & S_{12}(p) \\ 0 & S_{22}(p) \end{pmatrix} \quad \text{and} \quad \Sigma(p) = \begin{pmatrix} S_{11} - S_{12} \star S_{22}^{-\star} \star S_{21} & S_{12} \star S_{22}^{-\star} \\ S_{22}^{-\star} \star S_{21} & S_{22}^{-\star} \end{pmatrix}(p). \tag{8.8}$$

Then,

$$\begin{aligned} &J_2 \star \varphi(p,q) - S(p) \star \varphi(p,q) \star_r J_1 S(q)^* = \\ &\quad = A(p) \star (\varphi(p,q) - \Sigma(p) \star \varphi(p,q) \star_r \Sigma(q)^*) \star_r A(q)^*, \end{aligned} \tag{8.9}$$

and the kernel

$$\varphi(p,q) - \Sigma(p) \star \varphi(p,q) \star_r \Sigma(q)^* \tag{8.10}$$

has a finite number of negative squares on the domain of definition of Σ in Ω and hence has a slice hyperholomorphic extension to the whole of the right half-space, with the possible exception of a finite number of spheres.

The function Σ is called the Potapov-Ginzburg transform of S; see e.g. [48, (i), p. 25].

Proof of Theorem 8.2.2. To show that S_{22} is $\star$-invertible, we note that

$$\begin{pmatrix} 0 & I \end{pmatrix} (J_2 \star \varphi(p,q) - S(p) \star \varphi(p,q) \star_r J_1 S(q)^*) \begin{pmatrix} 0 \\ I \end{pmatrix} = S_{22}(p) \star \varphi(p,q) \star_r S_{22}(q)^* - \varphi(p,q) I_m.$$

This last kernel has therefore a finite number of negative squares, and Lemma 8.2.1 allows to conclude that S_{22} is $\star$-invertible, and the definition of the Potapov-Ginzburg transform makes sense.

When p and q are real, the $\star$-product is replaced by the pointwise product and the (8.9) then follow from [47, p. 156]. The case of $p \in \Omega$ follows by slice hyperholomorphic extension. The claim on the number of negative squares of (8.10) follows

$$\begin{aligned}\varphi(p,q) - \Sigma(p) \star \varphi(p,q) \star_r \Sigma(q)^* = \\ = A(p)^{-\star} \star (J_2 \star \varphi(p,q) - S(p) \star \varphi(p,q) \star_r J_1 \star S(q)^*) \star_r (A(q)^*)^{-\star_r},\end{aligned} \tag{8.11}$$

and from an application of [35, Proposition 5.3]. □

8.3 Schur and generalized Schur functions of the ball

In this section we study generalized Schur functions of the ball, and in particular characterize them in terms of realization (see Theorem 8.3.6). We first show that there exist indeed generalized Schur functions (see Theorem 8.3.1). In the section, as in the whole book, expressions such as $C \star (I - pA)^{-\star}$ (where A acts on a *right-sided* quaternionic vector space) are understood as in Proposition 7.4.2.

Theorem 8.3.1. *Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two-sided quaternionic Pontryagin spaces of the same index, and let $\mathscr{P}$ be a right-sided quaternionic Pontryagin space. Assume that the operator matrix*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathscr{P} \\ \mathscr{P}_1 \end{pmatrix} \to \begin{pmatrix} \mathscr{P} \\ \mathscr{P}_2 \end{pmatrix} \tag{8.12}$$

is a Pontryagin-space contraction. Then the function

$$S(p) = D + pC \star (I_{\mathscr{P}} - pA)^{-\star} B \tag{8.13}$$

is slice hypermeromorphic in $\mathbb{B}$ and belongs to $\mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$, where $\kappa \leq \operatorname{ind}(\mathscr{P})$.

Definition 8.3.2. The representation (8.13) is called a *realization* of the function S. A realization is called contractive, isometric, coisometric or unitary if the corresponding colligation operator U is contractive, isometric, coisometric or unitary.

Theorem 8.3.1 thus asserts that there exist generalized Schur functions with contractive realizations. The fact that every generalized Schur function has a coisometric realization will be proved in Theorem 8.3.6. In the Hilbert space setting, unitary realizations are considered in Theorem 8.4.10.

Proof of Theorem 8.3.1: We divide the proof in a number of steps. We first prove that S is a generalized Schur function defined in a neighborhood of the origin. We then show that it is in fact slice hypermeromorphic.

STEP 1: *The function S given by* (8.13) *belongs to some class* $\mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$.

The operator A is bounded and thus the function $C\star(I_{\mathscr{P}}-pA)^{-\star}$ is an operator-valued function slice hyperholomorphic in some axially symmetric neighborhood Ω of the origin. We now show that for S of the form (8.13), the associated kernel equals

$$K_S(p,q)=\underbrace{C\star(I_{\mathscr{P}}-pA)^{-\star}(C\star(I_{\mathscr{P}}-qA)^{-\star})^{[*]}}_{\text{has at most (ind }\mathscr{P}\text{) negative squares}}+\underbrace{\sum_{n=0}^{\infty}\Lambda_n(p)(I-UU^{[*]})\Lambda_n(q)^{[*]}}_{\text{positive kernel since }U\text{ is contractive}},\quad (8.14)$$

where $p,q\in\Omega$, and

$$\Lambda_n(p)=p^n\begin{pmatrix}pC\star(I_{\mathscr{P}}-pA)^{-\star} & I_{\mathscr{P}_2}\end{pmatrix}.$$

Let $X(p,q)$ denote the left side of (8.14). To show that $X(p,q)=K_S(p,q)$ we show that

$$X(p,q)-pX(p,q)\overline{q}=I_{\mathscr{P}_2}-S(p)S(q)^{[*]}. \quad (8.15)$$

The result will follow since this equation has a unique solution for any given $p,q\in\Omega$, as is seen by iterating it. We can define:

$$\Gamma(p)=C\star(I_{\mathscr{P}}-pA)^{-\star}. \quad (8.16)$$

Since $p\Lambda_n(p)=\Lambda_{n+1}(p)$ and

$$p\Gamma(p)A+C=\Gamma(p)\star p+C=C\star(I_{\mathscr{P}}-pA)^{-\star}\star(pA+I_{\mathscr{P}}-pA)=\Gamma(p),$$

we have:

$$\begin{aligned}X(p,q)-pX(p,q)\overline{q}&=\Gamma(p)\Gamma(q)^{[*]}-p\Gamma(p)\Gamma(q)^{[*]}\overline{q}+\Lambda_0(p)\Lambda_0(q)^{[*]}\\&=\Gamma(p)\Gamma(q)^{[*]}-p\Gamma(p)\Gamma(q)^{[*]}\overline{q}+p\Gamma(p)\Gamma(q)^{[*]}\overline{q}+I_{\mathscr{P}_2}-\\&\quad-\begin{pmatrix}p\Gamma(p)A+C & p\Gamma(p)B+D\end{pmatrix}\begin{pmatrix}q\Gamma(q)A+C & q\Gamma(q)B+D\end{pmatrix}^{[*]}\\&=\Gamma(p)\Gamma(q)^{[*]}+I_{\mathscr{P}_2}-\Gamma(p)\Gamma(q)^{[*]}-(p\Gamma(p)B+D)(q\Gamma(q)B+D)^{[*]}\\&=I_{\mathscr{P}_2}-S(p)S(q)^{[*]}\end{aligned}$$

since $S(p)=p\Gamma(p)B+D$.

STEP 2: *The function S defined by* (8.13) *in an axially symmetric domain containing the origin admits a uniquely defined slice hypermeromorphic extension to $\mathbb{B}$.*

We first suppose that $\mathscr{P}_1$ and $\mathscr{P}_2$ are Hilbert spaces. Then A is a contraction in the Pontryagin space $\mathscr{P}(S)$. Thus, it admits a maximal strictly negative invariant subspace, say $\mathscr{M}$ (see [165, Theorem 1.3.11] for the complex case and Theorem 5.7.9 for the quaternionic case). Writing

$$\mathscr{P}(S)=\mathscr{M}[+]\mathscr{M}^{[\perp]},$$

the operator matrix representation of A is upper triangular with respect to this decomposition where

$$B=\begin{pmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{pmatrix}.$$

The operator A_{22} is a contraction from the Hilbert space $\mathscr{M}^{[\perp]}$ into itself, and so the operator $(I_{\mathscr{M}^{[\perp]}} - pA)$ is invertible for every $p \in \mathbb{B}$. The operator A_{11} is a contraction from the finite dimensional anti-Hilbert space $\mathscr{M}$ onto itself, and so has right eigenvalues *outside the open unit ball*. So the operator $I - pA_{11}$, is invertible in $\mathbb{B}$, at the possible exception of a finite number of sphere since, see [276, Corollary 5.2, p. 39], a $n \times n$ quaternionic matrix has exactly n right eigenvalues (counting multiplicity) up to equivalence (in other words, it has exactly n spheres of eigenvalues). Thus S has only a finite number of sphere of poles in $\mathbb{B}$.

The case where $\mathscr{P}_1$ and $\mathscr{P}_2$ are Pontryagin spaces (of the same index) follows from the definition of the Potapov-Ginzburg transform.

□

As a direct corollary of (8.14) we have:

Corollary 8.3.3. *In the notation of the previous theorem:*
(1) *It holds that*

$$K_S(p,q) = \sum_{n=0}^{\infty} p^n \left(I_{\mathscr{P}_2} - S(p)S(q)^{[*]}\right) \overline{q}^n, \tag{8.17}$$

where p, q run through the points in $\mathbb{B}$ at which S is hyperholomorphic.
(2) *If U is moreover coisometric we have*

$$C \star (I_{\mathscr{P}} - pA)^{-\star} (C \star (I_{\mathscr{P}} - qA)^{-\star})^{[*]} = \sum_{n=0}^{\infty} p^n \left(I_{\mathscr{P}_2} - S(p)S(q)^{[*]}\right) \overline{q}^n. \tag{8.18}$$

Proof. To prove the first claim it suffices to iterate (8.14). The sum converge since $I_{\mathscr{P}_2} - S(p)S(q)^{[*]}$ is a bounded operator and $p, q \in \mathbb{B}$. The second formula is then clear. □

We note that (8.15) suggests an equivalent definition of the class $\mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$. The function S slice hyperholomorphic in some axially symmetric open subset Ω of $\mathbb{B}$ containing the origin is a generalized Schur function if there is a $\mathbf{B}(\mathscr{P}_2)$-valued function $K(p,q)$ with a finite number of negative squares in Ω and such that

$$I_{\mathscr{P}_2} - S(p)S(q)^{[*]} = K(p,q) - pK(p,q)\overline{q}, \quad p, q \in \Omega. \tag{8.19}$$

This equation can be rewritten as an equality

$$I_{\mathscr{P}_2} + pK(p,q)\overline{q} = K(p,q) + S(p)S(q)^{[*]}, \quad p, q \in \Omega,$$

which induce an isometry between Pontryagin spaces of same index. This idea is called the *lurking isometry method*; see [89].

In Theorem 8.3.6 we associate to a generalized Schur function a coisometric realization. We adapt the arguments of [47] (see in particular p. 50 there) to the present setting and follow the paper [32]. The strategy of the proof is as follows. Let S be a Schur multiplier, defined on the set Ω, and let $\mathscr{P}(S)$ be the associated reproducing kernel quaternionic

Pontryagin space of $\mathscr{P}_2$-valued functions and with reproducing kernel $K_S(p,q)$. As in the classical case, we want to show that $\mathscr{P}(S)$ is the state space, in the present setting, of a coisometric realization of S. We define a densely defined linear relation R in $(\mathscr{P}(S)\oplus\mathscr{P}_2)\times(\mathscr{P}(S)\oplus\mathscr{P}_1)$. We show that this relation is isometric. Using the quaternionic version of a theorem of Shmulyan (see Theorem 5.7.10), we see that R extends to the graph of a contraction between Pontryagin spaces of same index. The adjoint of this contraction gives the realization of S.

Before stating the theorem we give two definitions.

Definition 8.3.4. Let $S\in\mathcal{S}_\kappa(\mathscr{P}_1,\mathscr{P}_2,\mathbb{B})$. The reproducing kernel Pontryagin space $\mathscr{P}(S)$ of $\mathscr{P}_2$-valued functions with reproducing kernel K_S is called the *de Branges-Rovnyak space* associated with S.

Definition 8.3.5. The realization is called observable (or, closely outer-connected) if the pair (C,A) is observable.

The de Branges-Rovnyak space serves as state space in a closely outer-connected coisometric realization, as is explained in the following theorem. There exist also isometric and unitary realizations in terms of de Branges-Rovnyak spaces, but we will not consider them here.

Theorem 8.3.6. *Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two-sided quaternionic Pontryagin space of the same index, say k, and let Ω be an axially symmetric domain containing the origin. The $\mathbf{B}(\mathscr{P}_1,\mathscr{P}_2)$-valued function S slice hyperholomorphic in Ω is the restriction to Ω of a uniquely defined generalized Schur function if and only if it there is a right Pontryagin space $\mathscr{P}$ and a coisometric operator matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathscr{P}\oplus\mathscr{P}_1 \longrightarrow \mathscr{P}\oplus\mathscr{P}_2 \tag{8.20}$$

such that

$$S(p) = D + pC\star(I_{\mathscr{P}} - pA)^{-\star}B. \tag{8.21}$$

The realization is unique up to a unitary similarity operator when the pair (C,A) is observable.

Conversely, any function of the form (8.21) *belongs to a class $\mathcal{S}_{\kappa'}(\mathscr{P}_1,\mathscr{P}_2,\mathbb{B})$ for some $\kappa'\le\kappa$, and $\kappa=\kappa'$ when the realization is closely outer-connected, that is when the pair (C,A) is observable.*

Outline of the proof of Theorem 8.3.6: Consider the linear relation R defined by the right linear span in $(\mathscr{P}(S)\oplus\mathscr{P}_2)\times(\mathscr{P}(S)\oplus\mathscr{P}_1)$ of the elements of the form

$$\left\{\begin{pmatrix} K_S(p,q)\overline{q}u \\ \overline{q}v \end{pmatrix}, \begin{pmatrix} (K_S(p,q)-K_S(p,0))u + K_S(p,0)\overline{q}v \\ (S(q)^{[*]}-S(0)^{[*]})u + S(0)^{[*]}\overline{q}v \end{pmatrix}\right\}, \tag{8.22}$$

with $p\in\Omega$ and $u,v\in\mathscr{P}_2$. Note that the definition of (8.22) takes into account that $\mathscr{P}_2$ is in particular left-sided.

We claim that the relation R is densely defined and isometric. It will follow that R can be extended in a unique way to the graph of an isometric operator from $(\mathscr{P}(S) \oplus \mathscr{P}_2)$ into $(\mathscr{P}(S) \oplus \mathscr{P}_1)$. This operator (or more precisely its adjoint) will give the realization. We first prove some preliminary lemmas and then give the proof of the theorem along the above lines.

Lemma 8.3.7. *The relation R is isometric and densely defined.*

Proof. We first check that the linear relation is isometric. We want to prove that

$$\begin{aligned}
&[K_S(p,q_1)\overline{q_1}u_1, K_S(p,q_2)\overline{q_2}u_2]_{\mathscr{P}(S)} + [\overline{q_1}v_1, \overline{q_2}v_2]_{\mathscr{P}_2} = \\
&\qquad = [(K_S(p,q_1)-K_S(p,0))u_1, (K_S(p,q_2)-K_S(p,0))u_2]_{\mathscr{P}(S)} + \\
&\qquad\quad + [K_S(p,0)\overline{q_1}v_1, (K_S(p,q_2)-K_S(p,0))u_2]_{\mathscr{P}(S)} + \\
&\qquad\quad + [(K_S(p,q_1)-K_S(p,0))u_1, K_S(p,0)\overline{q_2}v_2]_{\mathscr{P}(S)} + \\
&\qquad\quad + [K_S(p,0)\overline{q_1}v_1, K_S(p,0)\overline{q_2}v_2]_{\mathscr{P}(S)} \\
&\qquad\quad + [(S(q_1)^{[*]}-S(0)^{[*]})u_1, (S(q_2)^{[*]}-S(0)^{[*]})u_2]_{\mathscr{P}_1} + \\
&\qquad\quad + [(S(q_1)^{[*]}-S(0)^{[*]})u_1, S(0)^{[*]}\overline{q_2}v_2]_{\mathscr{P}_1} + \\
&\qquad\quad + [S(0)^{[*]}\overline{q_1}v_1, (S(q_2)^{[*]}-S(0)^{[*]})u_2]_{\mathscr{P}_1} + \\
&\qquad\quad + [S(0)^{[*]}\overline{q_1}v_1, S(0)^{[*]}\overline{q_2}v_2]_{\mathscr{P}_1}
\end{aligned} \tag{8.23}$$

for all choices of $u_1, u_2, v_1, v_2 \in \mathscr{P}_2$ and q_1, q_2 in Ω. Note that the above expressions make sense since $\mathscr{P}_2$ is assumed two-sided. This equality is equivalent to check four equalities (the last two equalities are really equivalent, and so it is only necessary to check one of them), namely

$$\begin{aligned}
&[K_S(p,q_1)\overline{q_1}u_1, K_S(p,q_2)\overline{q_2}u_2]_{\mathscr{P}(S)} \\
&\quad = [(K_S(p,q_1)-K_S(p,0))u_1, (K_S(p,q_2)-K_S(p,0))u_2]_{\mathscr{P}(S)} + \\
&\qquad + [(S(q_1)^{[*]}-S(0)^{[*]})u_1, (S(q_2)^{[*]}-S(0)^{[*]})u_2]_{\mathscr{P}_1} \\
[\overline{q_1}v_1, \overline{q_2}v_2]_{\mathscr{P}_2} &= [K_S(p,0)\overline{q_1}v_1, K_S(p,0)\overline{q_2}v_2]_{\mathscr{P}(S)} \\
0 &= [K_S(p,0)\overline{q_1}v_1, (K_S(p,q_2)-K_S(p,0))u_2]_{\mathscr{P}(S)} + \\
&\quad + [S(0)^{[*]}\overline{q_1}v_1, (S(q_2)^{[*]}-S(0)^{[*]})u_2]_{\mathscr{P}_1} \\
0 &= [(K_S(p,q_1)-K_S(p,0))u_1, K_S(p,0)\overline{q_2}v_2]_{\mathscr{P}(S)} + \\
&\quad [(S(q_1)^{[*]}-S(0)^{[*]})u_1, S(0)^{[*]}\overline{q_2}v_2]_{\mathscr{P}_1}.
\end{aligned} \tag{8.24}$$

These equalities in turn are readily verified using the reproducing kernel property. We check the first one and leave the others to the reader. We thus want to check that

$$\begin{aligned}
[K_S(q_2,q_1)\overline{q_1}u_1, \overline{q_2}u_2]_{\mathscr{P}_2} &= [K_S(q_2,q_1)u_1, u_2]_{\mathscr{P}_2} - [K_S(q_2,0)u_1, u_2]_{\mathscr{P}_2} - \\
&\quad - [K_S(0,q_1)u_1, u_2]_{\mathscr{P}_2} + [K_S(0,0)u_1, u_2]_{\mathscr{P}_2} + \\
&\quad + [(S(q_1)^{[*]}-S(0)^{[*]})u_1, (S(q_2)^{[*]}-S(0)^{[*]})u_2]_{\mathscr{P}_1}.
\end{aligned}$$

Using property (5.4) of the inner product in $\mathscr{P}_2$, this is equivalent to prove that

$$q_2K_S(q_2,q_1)\overline{q_1} = K_S(q_2,q_1) - K_S(q_2,0) - K_S(0,q_1) + \\ + K_S(0,0) + (S(q_2)-S(0))(S(q_1)^{[*]} - S(0)^{[*]}).$$

This is a direct consequence of (8.1).

To check that R has dense domain, let $(f,w) \in \mathscr{P}(S)\times\mathscr{P}_2$ be orthogonal to the domain of R. Then, using the reproducing kernel property and property (5.4) in $\mathscr{P}_2$ we have

$$[qf(q),u]_{\mathscr{P}_2} + [w,v]_{\mathscr{P}_2} = 0, \quad \forall q\in\Omega, \text{ and } u,v\in\mathscr{P}_2.$$

This forces $w=0$ and $f(p)=0$ for $p\neq 0$. Since the kernel is slice hyperholomorphic in p and $\overline{q}$ we get that $f(0)=0$ too. ☐

By Shmulyan's theorem (see Theorem 5.7.10) R extends to the graph of an everywhere defined isometry. Let us denote by

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{[*]} : \begin{pmatrix}\mathscr{P}(S)\\ \mathscr{P}_2\end{pmatrix} \longrightarrow \begin{pmatrix}\mathscr{P}(S)\\ \mathscr{P}_1\end{pmatrix} \tag{8.25}$$

its extension to all of $\mathscr{P}(S)\oplus\mathscr{P}_2$.

Lemma 8.3.8. *The following formulas hold:*

$$\begin{aligned}
(Af)(p) &= \begin{cases} p^{-1}(f(p)-f(0)), & p\neq 0\\ f_1, & p=0,\end{cases}\\
(Bv)(p) &= \begin{cases} p^{-1}(S(p)-S(0))v, & p\neq 0\\ s_1v, & p=0,\end{cases}\\
Cf &= f(0),\\
Dv &= S(0)v.
\end{aligned} \tag{8.26}$$

Proof. We first compute the operator A. Let $q\in\Omega$ and $u\in\mathscr{P}_2$. We have

$$A^{[*]}(K_S(\cdot,q)\overline{q})u = (K_S(\cdot,q)-K_S(\cdot,0))\,u.$$

Hence, for $f\in\mathscr{P}(S)$ it holds that:

$$\begin{aligned}
[f,A^{[*]}(K_S(\cdot,q)\overline{q})u]_{\mathscr{P}(S)} &= [f,(K_S(\cdot,q)-K_S(\cdot,0))u]_{\mathscr{P}(S)}\\
&= [u,(f(q)-f(0))]_{\mathscr{P}_2},
\end{aligned}$$

on the one hand, and

$$\begin{aligned}
[f,A^{[*]}(K_S(\cdot,q)\overline{q})u]_{\mathscr{P}(S)} &= [Af,K_S(\cdot,q)\overline{q}u]_{\mathscr{P}(S)}\\
&= [qu,(Af)(q)]_{\mathscr{P}_2},
\end{aligned}$$

on the other hand. Hence

$$q(Af)(q) = f(q) - f(0).$$

Similarly we have

$$B^{[*]}(K_S(\cdot,q)\overline{q}u) = (S(q)^{[*]} - S(0)^{[*]})u,$$

so that we can write for $v \in \mathscr{P}_1$ on the one hand

$$[v, B^{[*]}(K_S(\cdot,q)\overline{q}u)]_{\mathscr{P}_1} = [(S(q)-S(0))v, u]_{\mathscr{P}_2},$$

and on the other hand

$$[v, B^{[*]}(K_S(\cdot,q)\overline{q}u)]_{\mathscr{P}_1} = [Bv, K_S(\cdot,q)\overline{q}u]_{\mathscr{P}(S)} = [q(Bv)(q), u]_{\mathscr{P}_2},$$

and hence the formula for B. To compute C we note that

$$C^{[*]}(\overline{q}u) = K_S(\cdot,0)\overline{q}u$$

for every q and $u \in \mathscr{P}_2$. So, for $f \in \mathscr{P}(S)$ we have:

$$\begin{aligned}[pCf, u]_{\mathscr{P}_2} &= [Cf, \overline{p}u]_{\mathscr{P}_2}\\ &= [f, K_S(\cdot,0)\overline{q}u]_{\mathscr{P}(S)}\\ &= [qf(0), u]_{\mathscr{P}_2}\end{aligned}$$

and hence $Cf = f(0)$. Finally, it is clear that $D = S(0)$. □

With these results at hand we turn to the proof of the realization theorem.

Proof of Theorem 8.3.6:

STEP 1: *A function $S \in \mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$ admits a realization of the required form.*

Starting from a function $S \in \mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$ we build the operator matrix (8.26) (note the pair (C,A) in is closely outer connected). Since $0 \in \Omega$ the elements of $\mathscr{P}(S)$ are slice hyperholomorphic at the origin, a function $f \in \mathscr{P}(S)$ admits a power series expansion

$$f(p) = \sum_{n=0}^{\infty} p^n f_n.$$

We have the formulas

$$f_n = CA^n f, \quad n = 0,1,2,\ldots$$

so that

$$f(p) = C \star (I - pA)^{-\star} f, \quad f \in \mathscr{P}(S).$$

Applying these formulas to the function Bu, where $u \in \mathscr{P}_1$, we obtain

$$(S(p) - S(0))u = pC \star (I - pA)^{-\star} Bu,$$

which is the required realization.

The converse statement follows from Theorem 8.3.1 since S is given by a realization of the form (8.21), the associated operator matrix being coisometric. For the same reason the following step also follows from that theorem.

STEP 2: *A function* $S \in \mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$ *admits a slice hypermeromorphic extension to* $\mathbb{B}$.

STEP 3: *The realization* (8.21) *is unique up to a unitary similarity operator when the pair* (C, A) *is assumed observable.*

Let

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \, : \, \mathscr{P} \oplus \mathscr{P}_1 \longrightarrow \mathscr{P} \oplus \mathscr{P}_2$$

and

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \, : \, \mathscr{P}' \oplus \mathscr{P}_1 \longrightarrow \mathscr{P}' \oplus \mathscr{P}_2$$

be two closely outer-connected coisometric realizations of S, with state spaces right quaternionic Pontryagin spaces $\mathscr{P}$ and $\mathscr{P}'$ respectively. From (8.18) we have

$$U_1(p)(U_1(q))^{[*]} = U_2(p)(U_2(p))^{[*]},$$

where U_1 and U_2 are built as in (8.12) from the present realizations. It follows that

$$C_1 A_1^n (A_1^m)^{[*]} C_1^{[*]} = C_2 A_2^n (A_2^m)^{[*]} C_2^{[*]}, \quad \forall n, m \in \mathbb{N}_0.$$

In view of the presumed outer-connectedness, the relation in $\mathscr{P} \times \mathscr{P}'$ defined by

$$((A_1^{[*]})^m C_1^{[*]} u, (A_2^{[*]})^m C_2^{[*]} u), \quad u \in \mathbb{H}, \quad m \in \mathbb{N}_0,$$

is a densely defined isometric relation with dense range. It is thus the graph of a unitary map U such that:

$$U \left((A_1^{[*]})^m C_1^{[*]} u \right) = (A_2^{[*]})^m C_2^{[*]} u, \quad m \in \mathbb{N}_0, \quad \text{and} \quad u \in \mathbb{H}.$$

Setting $m = 0$ leads to $UC_1^{[*]} = C_2^{[*]}$, that is

$$C_1 = C_2 U. \tag{8.27}$$

With this equality, writing

$$(UA_1^{[*]})((A_1^{[*]})^m C_1) = A_2^{[*]}(A_2^{[*]})^m C_2^{[*]} = A_2^{[*]} U U^{[*]} (A_2^{[*]})^m C_2^{[*]} = (A_2^{[*]} U)((A_1^{[*]})^m C_1),$$

and taking into account that both pairs (C_1, A_1) and (C_2, A_2) are closely outer-connected, we obtain $A_1 U^* = U^* A_2$, that is

$$UA_1 = A_2 U. \tag{8.28}$$

Since clearly $D_1 = D_2 = S(0)$, it remains only to prove that $UB_1 = B_2$. This follows from the equalities (where we use (8.27) and (8.28))

$$S_n = C_1A_1^{n-1}B_1 = C_2A_2^{n-1}B_2 = C_1A_1^{n-1}U^*B_2, \quad n = 1,2,\ldots$$

and from the fact that (C_1, A_1) is closely outer connected. $\square$

We note that we have followed the arguments in [47] suitably adapted to the present case. In particular the proof of the uniqueness is adapted from that of [47, Theorem 2.1.3, p. 46].

As a corollary of Theorem 8.3.6 we have:

Corollary 8.3.9. *In the notation and with the hypothesis of Theorem 8.3.6, let S be in* $\mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$. *Then,*

$$[R_0f, R_0f]_{\mathscr{P}(S)} \le [f, f]_{\mathscr{P}(S)} - [f(0), f(0)]_{\mathscr{P}_2}, \quad f \in \mathscr{P}(S). \tag{8.29}$$

Proof. Consider the backward-shift realization (8.26). Since the corresponding matrix U is a coisometry between Pontryagin spaces of the same index, the adjoint $U^{[*]}$ is also a contraction (see Theorem 5.7.8) and we have

$$A^*A + C^*C \le I_{\mathscr{P}_2}$$

which is (8.29) since $A = R_0$ and C is the point evaluation at the origin. $\square$

The converse statement holds. Inequality (8.29) does characterizes the $\mathscr{P}(S)$ spaces. See Section 8.6.

Theorem 8.3.6 gives a characterization of *all* Schur multipliers. A simple example is given by the choice

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \overline{a} & \sqrt{1-|a|^2} \\ \sqrt{1-|a|^2} & -a \end{pmatrix},$$

where $a \in \mathbb{B}$. The corresponding Schur multiplier $s_a(p)$ is

$$s_a(p) = -a + (1-|a|^2)p(1-p\overline{a})^{-\star} = (p-a) \star (1-p\overline{a})^{-\star},$$

and is the counterpart, up to right unitary multiplicative factor, of the elementary Blaschke factor (6.42) introduced in Section 6.3. The corresponding space $\mathscr{P}(s_a)$ is one dimensional and spanned by the function $(1-p\overline{a})^{-\star}$.

In the next section we focus on the Hilbert space case, and, as a transition, we conclude this section with a result from the Hilbert space setting. In the statement and proofs we stick to the Pontryagin space notations.

Proposition 8.3.10. *Assume that $\kappa = 0$ and that both $\mathscr{P}_1$ and $\mathscr{P}_2$ are Hilbert spaces. The reproducing kernel K_S can be written as*

$$\begin{aligned}[K_S(p,q)u,v]_{\mathscr{P}_2} &= [(C\star(I-pA)^{-\star})^{[*]}u\,,\,(C\star(I-qA)^{-\star})^{[*]}v]_{\mathscr{P}(S)}\\ &= [\left(\sum_{n=0}^{\infty} p^n CA^n A^{[*]n} C^* \overline{q}^n\right) u, v]_{\mathscr{P}_2}.\end{aligned} \tag{8.30}$$

Proof. Indeed, A is now a contraction in a Hilbert space, and both $K_S(p,q)$ and

$$U(p)(U(q))^{[*]} = \left(\sum_{n=0}^{\infty} p^n CA^n A^{[*]n} C^{[*]} \overline{q}^n\right)$$

satisfy equation (8.19):

$$X - pX\overline{q} = I_{\mathscr{P}_2} - S(p)S(q)^{[*]}.$$

□

Finally we conclude this section with an easy corollary of equation (8.19). The proof is a direct consequence of the definition, or using the previous proposition (although here we are in the Pontryagin space setting) since the spectral radius of A is strictly less than 1.

Corollary 8.3.11. *Let $S \in \mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$, and let $p_1, \ldots, p_N \in \mathbb{B}$ be points in the neighborhood of which S is slice hyperholomorphic. Let*

$$A = \mathrm{diag}\,(p_1 I_{\mathscr{P}_2}, \ldots, p_N I_{\mathscr{P}_2}), \quad C = \begin{pmatrix} I_{\mathscr{P}_2} & I_{\mathscr{P}_2} & \cdots & I_{\mathscr{P}_2} \\ S(p_1)^{[*]} & S(p_2)^{[*]} & \cdots & S(p_N)^{[*]} \end{pmatrix},$$

and

$$J_0 = \begin{pmatrix} I_{\mathscr{P}_2} & 0 \\ 0 & -I_{\mathscr{P}_1} \end{pmatrix}.$$

Then the block operator matrix with (u,v) entry equal to $K_S(p_u, p_v)$, $u, v = 1, \ldots, N$ is the unique solution of the equation

$$X - A^{[*]}XA = C^{[*]}J_0C$$

8.4 Contractive multipliers, inner multipliers and Beurling-Lax theorem

In this section we specialize the results of Section 8.3 in the case where both $\mathscr{P}_1$ and $\mathscr{P}_2$ are two-sided quaternionic Hilbert spaces (we will denote these spaces now by $\mathscr{H}_1$ and $\mathscr{H}_2$) and $\kappa = 0$, that is the space $\mathscr{P}(S)$ is a Hilbert space, which we now denote by $\mathscr{H}(S)$. Furthermore we use the notation A^* rather than $A^{[*]}$ for the adjoint of an operator between quaternionic Hilbert spaces.

There are three main differences with the non positive case.

(1) First, there is the characterization of Schur functions as contractive multipliers (with respect to the $\star$-product) betwen Hardy spaces.

(2) Next, and in a way analogous to the classical case, positivity implies slice hyperholomorphicity. In the case of negative squares, one has to assume the function hyperholomorphic in some domain to begin with, in order to insure a slice hypermeromorphic extension (see Theorem 8.3.6).

(3) Finally we can give a Beurling-Lax theorem. See Theorem 8.4.12.

Definition 8.4.1. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two-sided quaternionic Hilbert spaces. The $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$-valued function S slice-hyperholomorphic in $\mathbb{B}$ is called a Schur multiplier if the multiplication operator

$$M_S:\ f\mapsto S\star f \tag{8.31}$$

is a contraction from $\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$ into $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$.

We denote by $\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ the set of $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$-valued Schur multipliers. The following result has been proved in [33] in the case $\mathscr{H}_1=\mathscr{H}_2=\mathbb{H}$. It allows to connect the classes defined in the previous section to $\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$, and show that

$$\mathcal{S}_0(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})=\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B}). \tag{8.32}$$

The proof of the general case goes in the same way, and we only recall the main ideas.

Theorem 8.4.2. *The $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$-valued function S defined in $\mathbb{B}$ is a Schur multiplier if and only if the $\mathbf{B}(\mathscr{H}_2,\mathscr{H}_2)$-valued kernel*

$$K_S(p,q)=\sum_{n=0}^{\infty}p^n(I_{\mathscr{H}_2}-S(p)S(q)^*)\overline{q}^n \tag{8.33}$$

is positive definite on $\mathbb{B}$.

Proof. In one direction one uses the formula

$$(M_S^*(K(\cdot,q)u))(p)=\sum_{n=0}^{\infty}p^nS(q)^*\overline{q}^nu.$$

In the other direction, using the positivity of the kernel (8.33) one builds the relation R_S in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})\times\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$ spanned by the pairs

$$\left(\sum_{n=0}^{\infty}p^n\overline{q}^nu\,,\,\sum_{n=0}^{\infty}p^n(S(q))^*\overline{q}^nu\right),\quad q\in\mathbb{B},\text{ and }u\in\mathscr{H}_2. \tag{8.34}$$

The domain of R_S is dense, and the positivity of the kernel implies that R_S is a contraction. Thus R_S extends to the graph of an everywhere defined contraction, whose adjoint is M_S. In particular $Sv\in\mathrm{H}^2_{\mathscr{H}_2}$ for every $v\in\mathscr{H}_1$, and so S is slice hyperholomorphic. $\square$

In general a Schur multiplier will not take contractive values, as the example (see also (9.30)), where more details are given)

$$U(p)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\star\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}p & i\\ pi & 1\end{pmatrix}$$

shows. The operator of slice multiplication by U is an isometry from $\mathrm{H}^2(\mathbb{B})\times\mathrm{H}^2(\mathbb{B})$ into itself, but U does not take contractive, let alone unitary, values on the unit sphere. Indeed for p of modulus 1 one has:

$$U(p)U(p)^*=\begin{pmatrix}1 & \frac{i-pi\overline{p}}{2}\\ \frac{pi\overline{p}-i}{2} & 1\end{pmatrix}.$$

For instance the choice $p=j$ leads to

$$U(j)U(j)^*=\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix},$$

which is not unitary. More generally

$$U(p)U(p)^*-I_2=\begin{pmatrix}0 & 1 & \frac{i-pi\overline{p}}{2}\\ \frac{pi\overline{p}-i}{2} & 0\end{pmatrix},$$

which is a signed matrix.

On the other hand one has:

Corollary 8.4.3. *Let S be a Schur multiplier. Then $\|S(x)\|\le 1$ for $x\in(-1,1)$.*

Proof. It suffices to set $p=q=x\in(-1,1)$ in (8.33). □

The following theorem strengthens Theorem 8.4.2, and implies in particular that a function *defined* in $\mathbb{B}$ but for which the kernel K_S is positive, is in fact slice hyperholomorphic. Recall that such a result does not hold in the case of negative squares, as the example

$$s(z)=\begin{cases}0, & z\neq 0\\ 1, & z=0\end{cases}$$

already shows in the case of complex numbers; see for instance [47, p. 82].

Theorem 8.4.4. *Let S be a $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$-valued function defined in an open subset $\Omega\subset\mathbb{B}$. Assume that the kernel K_S is positive in Ω. Then, S extends to a Schur multiplier.*

Proof. The relation (8.34), with $q\in\Omega$ and $u\in\mathscr{H}_2$, is densely defined and contractive in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})\times\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$. It extends to the graph of a contraction, say X, whose adjoint is given by the formula

$$X^*\left(\sum_{n=0}^{\infty}p^n\overline{q}^n d\right)=p^nS(p)\overline{q}^n d,\quad q\in\Omega,\quad d\in\mathscr{H}_1.$$

The choice $q=0$ gives

$$X^*(d)=S(p)d,$$

and so $S(p)d$ is the restriction to Ω of a function slice hyperholomorphic in $\mathbb{B}$. Denoting still by S this extension, the fact that X is a contraction implies that S is a Schur multiplier. □

Remark 8.4.5. We note that Theorem 8.4.4 gives in fact a necessary and sufficient condition for a Schur multiplier to exist with preassigned values in a given set Ω.

Using the Potapov-Ginzburg transform we have:

Theorem 8.4.6. *Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two quaternionic Pontryagin spaces of the same index, and let S be a $\mathbf{B}(\mathscr{P}_1,\mathscr{P}_2)$-valued function defined in an open subset $\Omega\subset\mathbb{B}$. Assume that the kernel K_S is positive in Ω. Then, S has a uniquely defined slice hypermeromorphic extension to $\mathbb{B}$.*

Remark 8.4.7. The set Ω in the previous theorems need not be open, or even need not have an accumulation point. Then, the asserted extension is unique.

As we have already seen, for every contractive multiplier $S\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$, the associated de Branges-Rovnyak space $\mathscr{H}(S)$ is contractively included in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$. It is natural to ask for what contractive multipliers S the space $\mathscr{H}(S)$ is *isometrically* included in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$. The answer is given in Theorem 8.4.9 below.

Definition 8.4.8. The Schur multiplier $S\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ is called *inner* if the multiplication operator $M_S:\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})\to\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ defined in (8.31) is a partial isometry and it is called *strongly inner* if M_S is an isometry.

Theorem 8.4.9. *Let S be in $\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$. The following are equivalent:*

(1) S is inner.

(2) The de Branges-Rovnyak space $\mathscr{H}(S)$ is isometrically included in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$.

(3) S admits a coisometric realization

$$S(p)=D+pC\star(I_{\mathscr{H}(S)}-pA)^{-\star}B=D+\sum_{k=0}^{\infty}p^{k+1}CA^kB \tag{8.35}$$

with isometric pair (A,C) and strongly stable state space operator A.

Proof. Let S be a contractive multiplier and let $\mathscr{H}(S)$ be the associated de Branges-Rovnyak space. For any $h\in\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$, we have

$$\|(I-M_SM_S^*)h\|^2_{\mathscr{H}(S)} = \langle(I-M_SM_S^*)h,\,h\rangle_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})}, \tag{8.36}$$

$$\|(I-M_SM_S^*)h\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} = \langle(I-M_SM_S^*)h,\,(I-M_SM_S^*)h\rangle_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})}, \tag{8.37}$$

where (8.36) follows from (5.19).

If S is inner, then the multiplication operator $M_S : \mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B}) \to \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ is a partial isometry. Then the operators $M_S M_S^*$ and $I_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} - M_S M_S^*$ are orthogonal projections. Then the inner products in the right side in (8.37) are equal and we conclude that the norms of $\mathscr{H}(S)$ and $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ coincide on all elements of the form $(I - M_S M_S^*)h$. Since these elements are dense in $\mathscr{H}(S)$ (see Proposition 5.9.5), the latter space is isometrically included in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$.

Conversely, if $\mathscr{H}(S)$ is isometrically included in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$, then the left hand side norms in (8.37) are equal, so that the right side inner products are also equal. Since they are equal for every $h \in \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$, we conclude that $I - M_S M_S^*$ is an orthogonal projection so that M_S is a partial isometry, so that S is inner. We thus showed the equivalence $(1) \Leftrightarrow (2)$.

To prove the implication $(2) \Rightarrow (3)$, let us assume again that $\mathscr{H}(S)$ is isometrically included in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ and let us consider the backward shift (coisometric) realization of S with operators A, B, C, D defined as in (8.26). In the current situation, A is strongly stable and the pair (C,A) is isometric (see the proof of Theorem 7.6.12).

Finally, let S admit a realization as in part (3). Since the realization is coisometric, $K_S(p,q) = K_{C,A}(p,q)$ by formula (8.14). Therefore $\mathscr{H}(S) = \mathscr{H}(K_{C,A}) = \operatorname{ran} \mathscr{O}_{C,A}$ and the latter space is isometrically included in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ (by Theorem 7.6.12) since A is strongly stable and the pair (C,A) is isometric. This proves the implication $(3) \Rightarrow (2)$ and completes the proof of the theorem. □

Theorem 8.4.10. *A Schur multiplier $S \in \mathcal{S}(\mathscr{H}_1, \mathscr{H}_2, \mathbb{B})$ is strongly inner if and only if it admits a unitary realization* (8.35) *with a strongly stable state space operator A.*

Proof. Since S is strongly inner, it is also inner and then by Theorem 8.4.9, it admits a coisometric realization (8.35) with state space $\mathscr{X}$ and with isometric pair (A,C) and strongly stable state space operator A. We thus have that the colligation operator (8.12) is coisometric. Let us show that it is also isometric. The latter is equivalent to the following three equalities:

$$A^*A + C^*C = I_{\mathscr{X}}, \quad A^*B + C^*D = 0, \quad B^*B + D^*D = I_{\mathscr{H}_2}. \tag{8.38}$$

The first equality holds since the pair (C,A) is isometric. Since the operator

$$I - U^*U = \begin{pmatrix} I - A^*A - C^*C & -A^*B - C^*D \\ -B^*A - D^*C & I - B^*B - D^*D \end{pmatrix}$$

is positive semidefinite and the $(1,1)$-block entry equals zero, the off-diagonal block also equals zero which gives the second equality in (8.38). Recall that $\mathscr{G}_{C,A} = I_{\mathscr{X}}$ by Remark 7.6.7 (since the pair (C,A) is isometric and A is strongly stable). Since the operator M_S is an isometry from $\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$ to $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$, we have in particular,

$$\|Su\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} = \|u\|^2_{\mathscr{H}_1} \quad \text{for every} \quad u \in \mathscr{H}_1. \tag{8.39}$$

Making use of (8.35) and the definition of the inner product in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$, and taking into account that $\mathscr{G}_{C,A} = I_{\mathscr{X}}$, we get

$$\begin{aligned}
\|Su\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} &= \|Du\|^2_{\mathscr{H}_1} + \sum_{k=0}^{\infty} \|CA^k Bu\|^2_{\mathscr{H}_2} \\
&= \langle D^*Du, u\rangle_{\mathscr{H}_1} + \sum_{k=0}^{\infty} \langle B^*A^{*k}C^*CA^k Bu, u\rangle_{\mathscr{H}_1} \\
&= \langle D^*Du, u\rangle_{\mathscr{H}_1} + \left\langle B^*\left(\sum_{k=0}^{\infty} A^{*k}C^*CA^k\right) Bu, u\right\rangle_{\mathscr{H}_1} \\
&= \langle (D^*D + B^*B)u, u\rangle_{\mathscr{H}_1}.
\end{aligned}$$

Combining the latter equality (holding for every $u \in \mathscr{H}_1$) with (8.39) gives the third equality in (8.38). Thus, the colligation operator U is isometric and therefore (since it is coisometric from the very beginning) it is unitary. This completes the proof of the "only if" part.

To prove the "if" part, let us assume that S admits a unitary realization (8.35) with a strongly stable state space operator A. Then $\mathscr{G}_{C,A} = I_{\mathscr{X}}$ (by Remark 7.6.7) and equality (8.39) holds due to the third relation in (8.38). Moreover, since M_p acts as an isometry on $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ we actually have

$$\|M_p^k Su\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} = \|M_p^k u\|^2_{\mathscr{H}_1} \quad \text{for all} \quad u \in \mathscr{H}_1,\ k \geq 0. \tag{8.40}$$

Let us show that $M_p^n Su$ is orthogonal (in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$) to $M_p^m Sv$ for every $u, v \in \mathscr{H}_1$ and any nonnegative integers $m \neq n$. Assuming without loss of generality that $n > m$ we get, again making use of (8.35),

$$\begin{aligned}
\langle M_p^n Su, M_p^m Sv\rangle_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} &= \langle CA^{n-m-1}Bu, Dv\rangle_{\mathscr{H}_2} + \sum_{j=0}^{\infty} \langle CA^{n-m+j}Bu, CA^j Bv\rangle_{\mathscr{H}_2} \\
&= \langle D^*CA^{n-m-1}Bu, v\rangle_{\mathscr{H}_1} + \left\langle B^*\left(\sum_{j=0}^{\infty} A^{*j}C^*CA^j\right) A^{n-m}Bu, v\right\rangle_{\mathscr{H}_1} \\
&= \big\langle (D^*C + B^*\mathscr{G}_{C,A}A)A^{n-m-1}Bu, v\big\rangle_{\mathscr{H}_1} = 0
\end{aligned} \tag{8.41}$$

where we used the second relation in (8.38) and equality $\mathscr{G}_{C,A} = I_{\mathscr{X}}$ for the last step.

Let us now take a quaternionic polynomial $f(p) = \sum_{j=0}^{N} p^j u_j$ with the coefficients $u_j \in \mathscr{H}_1$.

Then we have by (8.40) and (8.41),

$$\begin{aligned}\|S\star f\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} &= \left\|\sum_{j=0}^{\infty} M_p^j S u_j\right\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} \\ &= \sum_{j=0}^{\infty}\left\|M_p^j S u_j\right\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} = \sum_{j=0}^{\infty}\|u_j\|^2_{\mathscr{H}_1} = \|f\|^2_{\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})}.\end{aligned}$$

Since polynomials are dense in $\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$, the equality $\|S\star f\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} = \|f\|^2_{\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})}$ holds for every $f\in\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$. Therefore, the operator $M_S : \mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})\to\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ is isometric so that the Schur multiplier S is inner. □

Corollary 8.4.11. *Let $\mathscr{N}$ be a right quaternionic Hilbert space which is isometrically included in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ and which is M_p^*-invariant. Then there exists a two sided quaternionic Hilbert space $\mathscr{H}_1$ and a strongly inner multiplier $S\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ such that $\mathscr{N}=\mathscr{H}(S)$.*

Proof. Define the operators $A:\mathscr{N}\to\mathscr{N}$ and $C:\mathscr{N}\to\mathscr{H}_2$ by formulas (7.53). Then A is strongly stable and the pair (C,A) is isometric (see the proof of Theorem 8.4.9). For this pair we have $\mathscr{N}=\operatorname{Ran}\mathscr{O}_{C,A}=\mathscr{H}(K_{C,A})$, by Theorem 7.6.12. Extend the isometric operator $\begin{pmatrix}A\\C\end{pmatrix}:\mathscr{N}\to\mathscr{N}\oplus\mathscr{H}_2$ to a unitary operator U of the form (8.12). By Theorem 5.3.11, there exists an injective operator $\begin{pmatrix}B\\D\end{pmatrix}:\mathscr{N}\to\mathscr{N}\oplus\mathscr{H}_2$ solving the factorization problem

$$\begin{pmatrix}B\\D\end{pmatrix}\begin{pmatrix}B^* & D^*\end{pmatrix}=\begin{pmatrix}I_{\mathscr{N}} & 0\\ 0 & I_{\mathscr{H}_2}\end{pmatrix}-\begin{pmatrix}A\\C\end{pmatrix}\begin{pmatrix}A^* & C^*\end{pmatrix}.$$

We then define S by formula (8.35). Then S is a strongly inner multiplier by Theorem 8.4.9 and equality $K_S(p,q)=K_{C,A}(p,q)$ holds since U is unitary. Thus $\mathscr{N}=\mathscr{H}(K_{C,A})=\mathscr{H}(S)$ as we wanted. □

We now turn to the Beurling-Lax theorem.

Theorem 8.4.12. *Let $\mathscr{M}$ be a closed M_p-invariant subspace of $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$. Then there exists a right quaternionic Hilbert space $\mathscr{H}_1$ and a strongly inner multiplier $S\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ such that $\mathscr{M}=S\star\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$.*

Proof. The orthogonal complement $\mathscr{M}^\perp$ is M_p^*-invariant. By Corollary 8.4.11, there is a right quaternionic Hilbert space $\mathscr{H}_1$ and a strongly inner multiplier $S\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,)$ such that $\mathscr{M}^\perp=\mathscr{H}(S)$. Since $M_SM_S^*$ and $I-M_SM_S^*$ are both orthogonal projections in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$, it follows that the orthogonal complement of $\mathscr{M}^\perp=\mathscr{H}(S)$ is $\mathscr{M}=S\star\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$. □

8.5 A theorem on convergence of Schur multipliers

Taking a converging subsequence of bounded analytic functions from a given sequence of such functions is an important tool in Schur analysis. Unfortunately, Montel's theorem does not hold in the case of vector-valued function. Still we can obtain such a subsequence in our setting, using the Banach-Alaoglu theorem and the metrizability of the closed unit ball of $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$. See Chapter 5 for the latter results. This is the topic of the present section. The main result, Theorem 8.5.1, plays an important role in the sequel in the interpolation problem for operator-valued Schur multipliers.

Theorem 8.5.1. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two-sided quaternionic Hilbert spaces, and let $(S_n)_{n\in\mathbb{N}}$ be a sequence of Schur multipliers in $\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$. Then there is $S\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $M_{S_{n_k}}$ tends to M_S in the weak operator topology, that is:*

$$\lim_{k\to\infty}\langle M_{S_{n_k}}f,g\rangle_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})}=\langle M_Sf,g\rangle_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})},$$

for every $f\in\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$ and $g\in\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$. In particular, for every $p\in\mathbb{B}$ and $h_1\in\mathscr{H}_1$, $h_2\in\mathscr{H}_2$,

$$\lim_{k\to\infty}\langle S_{n_k}(p)h_1,h_2\rangle_{\mathscr{H}_2}=\langle S(p)h_1,h_2\rangle_{\mathscr{H}_2}.$$

Proof. The closed unit ball of $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$ is weakly compact (see Theorem 5.6.1) and metrizable (see Theorem 5.6.3) and therefore there exists an operator $T\in\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$ of norm less or equal to 1 and such that, via a subsequence $(n_k)_{k\in\mathbb{N}}$

$$\lim_{k\to\infty}M_{S_{n_k}}=T$$

in the weak operator topology. In particular, for $q\in\mathbb{B}$ and $f\in\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$ and $h_2\in\mathscr{H}_2$ we have:

$$\begin{aligned}\langle (Tf)(q),h_2\rangle_{\mathscr{H}_2}&=\langle Tf,I_{\mathscr{H}_2}(1-p\overline{q})^{-\star}h_2\rangle_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})}\\&=\lim_{k\to\infty}\langle S_{n_k}\star f,(I_{\mathscr{H}_2}-p\overline{q})^{-\star}h_2\rangle_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})}\\&=\lim_{k\to\infty}\langle (S_{n_k}\star f)(q),h_2\rangle_{\mathscr{H}_2}.\end{aligned}$$

Setting $f(p)\equiv h_1$ we have

$$\langle (Th_1)(q),h_2\rangle_{\mathscr{H}_2}=\lim_{k\to\infty}\langle S_{n_k}(q)h_1,h_2\rangle_{\mathscr{H}_2}. \tag{8.42}$$

The $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$-valued function S defined by $S(q)h_1=(Th_1)(q)$ is slice hyperholomorphic. We claim that $T=M_S$. To check this, we first take f to be a polynomial:

$$f(p)=\sum_{n=0}^{N}p^nf_n.$$

Then, and using the property (5.4) to go from the second to the third line, we have

$$\begin{aligned}\langle (Tf)(q),h_2\rangle &= \lim_{k\to\infty}\langle (S_{n_k}\star f)(q),h_2\rangle_{\mathscr{H}_2}\\ &= \sum_{n=0}^{N}\lim_{k\to\infty}\langle q^nS_{n_k}(q)f_n,h_2\rangle_{\mathscr{H}_2}\\ &= \sum_{n=0}^{N}\lim_{k\to\infty}\langle S_{n_k}(q)f_n,\overline{q}^nh_2\rangle_{\mathscr{H}_2}\\ &= \sum_{n=0}^{N}\langle Tf_n(q),\overline{q}^nh_2\rangle_{\mathscr{H}_2},\end{aligned}$$

where we have used (8.42), and so, using once more (5.4) we can write

$$\begin{aligned}\langle (Tf)(q),h_2\rangle &= \sum_{n=1}^{N}\langle S(q)f_n,\overline{q}^nh_2\rangle_{\mathscr{H}_2}\\ &= \sum_{n=1}^{N}\langle q^nS(q)f_n,h_2\rangle_{\mathscr{H}_2}\\ &= \langle (S\star f)(q),h_2\rangle_{\mathscr{H}_2}.\end{aligned}$$

The case of general f is done by approximation. More precisely, let $f\in \mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$, with power series expansion

$$f(p)=\sum_{n=0}^{\infty}p^nf_n,$$

and let $f_N(p)=\sum_{n=0}^{N}p^nf_n$. Since T is continuous, Tf_N tends to Tf in the norm of $\mathbf{H}^2_{\mathscr{H}_1}(\mathbb{B})$, and in particular it tends weakly to Tf, and (by taking inner product with $g(p)=I_{\mathscr{H}_2}(1-p\overline{q})^{-\star}h_2)$, we have

$$\begin{aligned}\langle Tf(q),h_2\rangle_{\mathscr{H}_2} &= \lim_{N\to\infty}\langle Tf_N(q),h_2\rangle_{\mathscr{H}_2}\\ &= \lim_{N\to\infty}\sum_{n=0}^{N}\langle q^nS(q)f_n,h_2\rangle_{\mathscr{H}_2}\\ &= \sum_{n=0}^{\infty}\langle q^nS(q)f_n,h_2\rangle_{\mathscr{H}_2}\quad \text{(since the limit exists).}\end{aligned}$$

So the weak limit of Tf_N is

$$\sum_{n=0}^{\infty}q^nS(q)f_n. \tag{8.43}$$

Since the sequence $(Tf_N)_{N\in\mathbb{N}}$ has a strong limit, (8.43) is the strong limit of Tf_N and so is equal to Tf, i.e. $Tf=S\star f$. □

8.6 The structure theorem

In Section 8.4 we gave a characterization of closed R_0-invariant subspaces of $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$, see Corollary 8.4.11. We also gave a version of the Beurling-Lax theorem; see Theorem 8.4.12. Furthermore, the $\mathscr{P}(S)$ spaces were shown in Corollary 8.3.9 to satisfy the inequality

$$[R_0 f, R_0 f]_{\mathscr{P}(S)} \leq [f,f]_{\mathscr{P}(S)} - [f(0), f(0)]_{\mathscr{P}_2}, \quad f \in \mathscr{P}(S).$$

The theorem below is a converse of this corollary, and is the analog of de Branges' result in the slice hyperholomorphic setting, in which the backward-shift operator R_0 is now defined as

$$R_0 f(p) = p^{-1}(f(p) - f(0)) = (f(p) - f(0)) \star_\ell p^{-1}.$$

In order to prove the result, we will be in need of a fact which is direct consequence of Lemma 6.1.23: if f, g are two left slice hyperholomorphic functions then (see formula (7.7))

$$(f \star_l g)^* = g^* \star_r f^*.$$

Theorem 8.6.1. *Let $\mathscr{P}_2$ be a two-sided Pontryagin space, and let $\mathscr{M}$ be a Pontryagin space of index κ, whose elements are $\mathscr{P}_2$-valued functions slice hyperholomorphic in a spherical neighborhood Ω of the origin, and invariant under the operator R_0. Assume moreover that*

$$[R_0 f, R_0 f]_{\mathscr{M}} \leq [f,f]_{\mathscr{M}} - [f(0), f(0)]_{\mathscr{P}_2}. \tag{8.44}$$

Then, there exist a Pontryagin space $\mathscr{P}_1$ of the same index as $\mathscr{P}_2$ and a function $S \in \mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$ such that the elements of $\mathscr{M}$ are the restrictions to Ω of the elements of $\mathscr{P}(S)$.

Proof. We follow the proof in [47, Theorem 3.1.2, p. 85]. Let $\mathscr{P}_1 = \mathscr{M} \oplus \mathscr{P}_2$, and denote by C the point evaluation at the origin. We divide the proof into a number of steps.

STEP 1: *Let $p \in \Omega$ and $f \in \mathscr{M}$. Then,*

$$f(p) = C \star (I - pR_0)^{-\star} f. \tag{8.45}$$

STEP 2: *The reproducing kernel of $\mathscr{M}$ is given by*

$$K(p,q) = C \star (I - pR_0)^{-\star} \left(C \star (I - qR_0)^{-\star} \right)^{[*]}.$$

STEP 3: *Let E denote the operator*

$$E = \begin{pmatrix} R_0 \\ C \end{pmatrix} : \quad \mathscr{M} \longrightarrow \mathscr{P}_2.$$

There exist a quaternionic Pontryagin space $\mathscr{P}_1$ with $\mathrm{ind}_{\mathscr{P}_1}=\mathrm{ind}_{\mathscr{P}_2}$, and a bounded right linear operator T from $\mathscr{M}$ into $\mathscr{P}_1$ such that

$$I_{\mathscr{M}}-EE^{[*]}=T^{[*]}T. \tag{8.46}$$

Applying (5.45) (see the proof of Theorem 5.7.8) with $\mathscr{M}=\mathscr{P}_1$ we obtain

$$\nu_-(I_{\mathscr{P}_2}-EE^{[*]})+\nu_-(\mathscr{M})=\nu_-(I_{\mathscr{M}}-E^{[*]}E)+\nu_-(\mathscr{P}_2).$$

Equation (8.44) can be rewritten as $I-E^{[*]}E\geq 0$, and in particular $\nu_-(I-E^{[*]}E)=0$. Thus

$$\nu_-(I_{\mathscr{P}_2}-EE^{[*]})=\nu_-(\mathscr{P}_2),$$

and we obtain the factorization (8.46) by using Theorem 5.10.6.

We set

$$T^{[*]}=\begin{pmatrix}B\\D\end{pmatrix}:\ \mathscr{P}_1\longrightarrow\mathscr{M}\oplus\mathscr{P}_2,$$

and

$$V=\begin{pmatrix}R_0 & B\\ C & D\end{pmatrix}.$$

Let us define the function

$$S(p)=D+pC\star(I_{\mathscr{M}}-pR_0)^{-\star}B.$$

STEP 4: *We have that*

$$I_{\mathscr{P}_2}-S(p)S(q)^{[*]}=C\star(I-pR_0)^{-\star}\star(I-p\overline{q})((I-qR_0)^{-\star})^{[*]}\star_r C^{[*]}.$$

The computation is as the ones giving formula (8.14). □

8.7 Carathéodory and generalized Carathéodory functions

Formula (1.5) has extension to the case of generalized Carathéodory functions, see the works [217, 215] of Iohvidov and Krein. The formula takes into account the spectral structure of an isometry acting in a Pontryagin space, and is quite involved. On the other hand, a nicer formula can be given in form of a realization, as is stated in the following theorem, originally proved in [34].

We use the formula to prove a version of Bohr's inequality for operator-valued Schur multipliers.

Theorem 8.7.1. *Let $\mathscr{P}$ denote a two-sided Pontryagin space, and let Ω be an axially symmetric s-domain. The $\mathbf{B}(\mathscr{P})$-valued function Φ slice hyperholomorphic in Ω is the restriction to Ω of a generalized Carathéodory function if and only if there is a right Pontryagin space $\mathscr{P}_\kappa$, a unitary operator $U \in \mathbf{B}(\mathscr{P}_k)$ and a map $C \in \mathbf{B}(\mathscr{P}_k, \mathscr{P})$ such that*

$$\Phi(p) = \frac{1}{2} C \star (I - pU)^{-\star} \star (I + pU) \star C^{[*]} + \frac{\Phi(0) - \Phi(0)^{[*]}}{2}. \tag{8.47}$$

Under the condition $\cap_{n=0}^{\infty} \ker CU^n = \{0\}$ the pair (C, U) is unique up to a unitary map.

Remark 8.7.2. As before, $C \star (I - pU)^{-\star} \star (I + pU)$ is understood using Proposition 7.4.2.

Proof. As in previous realization results we build a linear relation and use Shmulyan's theorem to get the realization. We denote by $\mathscr{L}(\Phi)$ the reproducing kernel right quaternionic Pontryagin space of functions slice hyperholomorphic in Ω, with reproducing kernel $K_\Phi(p,q)$, and proceed in a number of steps. As usual, we denote the identity operator by I without specifying the space on which it acts. We also note that the relation R below appears in the setting of complex numbers in [73, Proof of Theorem 5.2, p. 708].

STEP 1: *The linear relation consisting of the pairs $(F, G) \in \mathscr{L}(\Phi) \times \mathscr{L}(\Phi)$ with*

$$F(p) = \sum_{j=1}^{n} K_\Phi(p, p_j) \overline{p_j} b_j, \quad \text{and} \quad G(p) = \sum_{j=1}^{n} K_\Phi(p, p_j) b_j - K_\Phi(p, 0) \left(\sum_{\ell=1}^{n} b_\ell \right),$$

where n varies in $\mathbb{N}$, $p_1, \ldots, p_n \in \Omega \subset \mathbb{H}$ and $b_1, \ldots, b_n \in \mathscr{P}$ is densely defined and isometric.

The reproducing kernel property implies that the domain of R is dense. Furthermore we note that $\overline{p_j} b_j$ is well defined since $\mathscr{P}$ is a two sided quaternionic vector space. To check that R is isometric we need to verify that:

$$[F, F]_{\mathscr{L}(\Phi)} = [G, G]_{\mathscr{L}(\Phi)}. \tag{8.48}$$

We have (and here we follow in particular the computations done in [34])

$$\begin{aligned} [F, F]_{\mathscr{L}(\Phi)} &= [\sum_{j=1}^{n} K_\Phi(p, p_j) \overline{p_j} b_j, \sum_{k=1}^{n} K_\Phi(p, p_k) \overline{p_k} b_k]_{\mathscr{L}(\Phi)} \\ &= \sum_{j,k=1}^{n} [K_\Phi(p_k, p_j) \overline{p_j} b_j, \overline{p_k} b_k]_{\mathscr{P}} \\ &= \sum_{\ell=1}^{\infty} \sum_{j,k=1}^{n} b_k^* p_k^{\ell+1} [(\Phi(p_k) + \Phi(p_j)^{[*]}) \overline{p_j}^{\ell+1} b_j, \overline{p_k}^{\ell+1} b_k]_{\mathscr{P}}. \end{aligned}$$

In a similar way, the inner product $[G,G]_{\mathscr{L}(\Phi)}$ is computed as follows, with $b=\sum_{\ell=1}^n b_\ell$.

$$\begin{aligned}
[G,G]_{\mathscr{L}(\Phi)} = &\sum_{j,k=1}^n [K_\Phi(p_k,p_j)b_j,b_k]_{\mathscr{P}}-\\
&-2\mathrm{Re}\left(\sum_{k=1}^n [K_\Phi(p_k,0)b,b_k]_{\mathscr{P}}\right)+[K_\Phi(0,0)b,b]_{\mathscr{P}}\\
=&\sum_{\ell=1}^\infty\sum_{j,k=1}^n [(\Phi(p_k)+\Phi(p_j)^{[*]})\overline{p_j}^\ell b_j,\overline{p_k}^\ell b_k]_{\mathscr{P}}-\\
&-\sum_{k=1}^n [(\Phi(p_k)+\Phi(0)^{[*]})b,b_k]_{\mathscr{P}}+[(\Phi(0)+\Phi(0)^{[*]})b,b]_{\mathscr{P}}.
\end{aligned}$$

Equation (8.48) follows readily from these equalities. The domain of R is dense. Thus by Shmulyan's theorem (Theorem 5.7.10 above), R is the graph of a densely defined isometry, which extends to an isometry to all of $\mathscr{L}(\Phi)$. We denote by T this extension.

STEP 2: *We have* $T^{[*]}=R_0$.

Indeed, let $f\in\mathscr{L}(\Phi)$, $h\in\mathscr{P}$ and $p\in\Omega$. We have:

$$\begin{aligned}
[(T^{[*]}f)(p),h]_{\mathscr{P}} &= [T^{[*]}f,K_\Phi(\cdot,p)\overline{p}h]_{\mathscr{L}(\Phi)}\\
&=[f\,,\,T(k_\Phi(\cdot,p)h)]_{\mathscr{L}(\Phi)}\\
&=[f\,,\,K_\Phi(\cdot,p)h-K_\Phi(\cdot,0)h]_{\mathscr{L}(\Phi)}\\
&=[f(p)-f(0),h]_{\mathscr{P}}.
\end{aligned}$$

STEP 3: *The realization formula* (8.47) *holds.*

An easy induction shows that $f_\ell = CR_0^\ell f$. Hence,

$$f(p)=\sum_{\ell=0}^\infty p^\ell CR_0^\ell f = C\star(I-pR_0)^{-\star}f.$$

Applying this formula to the function $C^*h=K_\Phi(\cdot,0)h$ where $h\in\mathscr{P}$ we obtain:

$$(\Phi(p)+\Phi(0)^{[*]})h = C\star(I-pR_0)^{-\star}C^*h \quad\text{and}\quad \Phi(0)+\Phi(0)^{[*]}=CC^*h,$$

from which (8.47) follows.

STEP 4: *Conversely, every function of the form* (8.47) *is in a class* $\mathscr{P}_{\kappa'}(\mathscr{P},\mathbb{B})$ *for some* $\kappa'\le\kappa$. *If the pair* (C,U) *is observable, then* $\kappa=\kappa'$.

From (8.47) we obtain

$$\Phi(p)+\Phi(q)^{[*]} = C\star(I-pU)^{-\star}\star(1-p\overline{q})\star_r((I-qU)^{-\star})^{[*]}\star_r C^{[*]}, \tag{8.49}$$

and the reproducing kernel of $\mathscr{L}(\Phi)$ is equal to

$$K_\Phi(p,q) = C\star(I-pU)^{-\star}((I-qU)^{-\star})^{[*]}\star_r C^{[*]},$$

since, in view of (8.49), the right side of the above equation satisfies

$$K_\Phi(p,q) - pK_\Phi(p,q)\overline{q} = \Phi(p) + \Phi(q)^{[*]}.$$

When the pair (C,U) is observable, $\mathscr{L}(\Phi)$ consists of the functions of the form

$$f(p) = C\star(I-pU)^{-\star}\xi, \quad \xi\in\mathscr{P},$$

with the inner product

$$[f,g]_{\mathscr{L}(\Phi)} = [\xi,\eta]_{\mathscr{P}} \quad (\text{with} \quad g(p) = C\star(I-pU)^{-\star}\eta),$$

and so the kernel K_Φ has exactly κ negative squares.

The last steps (namely uniqueness up to similarity of the realization when the pair (C,U) is observable and hypermeromorphic extension to) are proved in a way similar as in the proofs given in Section 8.3 for the corresponding facts. □

Corollary 8.7.3. *When $\kappa = 0$ and the coefficient space $\mathscr{P}$ is a Hilbert space, the function Φ has a slice hyperholomorphic extension to all of $\mathbb{B}$.*

Proof. This follows from (8.47) since the operator U is then a contraction in Hilbert space, and thus has its S-spectrum inside the closed unit ball of $\mathbb{H}$. Thus $(I-pU)^{-\star}$ exists for all $p\in\mathbb{B}$. □

We now turn to an operator-valued quaternionic version of Bohr's inequality. In the statement and in the proof, we have set

$$\operatorname{Re} S_0 = \frac{S_0+S_0^*}{2},$$

and similarly for other operators.

Theorem 8.7.4. *Let $\mathscr{H}$ be a two-sided quaternionic Hilbert space, and let S be a $\mathbf{B}(\mathscr{H})$-valued Schur multiplier, with expansion*

$$S(p) = \sum_{j=0}^\infty p^j S_j, \quad S_j\in\mathbf{B}(\mathscr{H}),\ j=0,1,\ldots$$

Then

$$\sum_{j=1}^\infty \|p^jS_j\| \le \|I_n - \operatorname{Re} S_0\|, \quad |p| < \frac{1}{3}, \tag{8.50}$$

and in particular

$$\sum_{j=0}^\infty \|p^jS_j\| \le \|S_0\| + \|I_n - \operatorname{Re} S_0\|, \quad |p| < \frac{1}{3}. \tag{8.51}$$

Proof. We proceed in a number of steps.

STEP 1: *Assume the function S to be a $\mathbf{B}(\mathscr{H})$-valued Schur multiplier. Then $\Phi = I - S$ is a Herglotz multiplier, meaning that*

$$\operatorname{Re} M_\Phi \geq 0.$$

Indeed, we have $\|M_S\| \leq 1$ and so, using (5.3.6), $\|\frac{M_S+M_S^*}{2}\| \leq 1$, and so the self-adjoint operator $\operatorname{Re} M_\Phi$ satisfies

$$\operatorname{Re} M_\Phi = I - \frac{M_S + M_S^*}{2} \geq 0.$$

Now a direct application of Theorem 8.7.1 to $I - S$ leads to:

STEP 2: *There exists a right quaternionic Hilbert space $\mathscr{X}$, an operator $C \in \mathbf{B}(\mathscr{X}, \mathscr{H})$ and a unitary operator $U \in \mathbf{B}(\mathscr{X})$ such that*

$$I - S_0 = \frac{1}{2}CC^* + \frac{S_0^* - S_0}{2}, \tag{8.52}$$

$$S_{j+1} = -CU^{j+1}C^*, \quad j \geq 0. \tag{8.53}$$

STEP 3: *It holds that*

$$\|C\|^2 = 2\|I_N - \operatorname{Re} S_0\|.$$

Indeed, the first equation in (8.52) gives $CC^* = 2(I - \operatorname{Re} S_0)$. Using Proposition 5.3.6 we get $\|C\|^2 = 2\|I - \operatorname{Re} S_0\|$.

STEP 4: *It holds that*

$$\|S_{j+1}\| \leq 2\|I_N - \operatorname{Re} S_0\|, \quad j = 0, 1, \ldots$$

This follows from the previous step and the second equation in (8.52) since $\|C\| = \|C^*\|$.

STEP 5: (8.50) *holds.*

Indeed, for $|p| \leq \frac{1}{3}$

$$\begin{aligned}\sum_{j=0}^{\infty} \|p^{j+1} S_{j+1}\| &\leq 2\|I_N - \operatorname{Re} S_0\| \left(\sum_{j=0}^{\infty} \frac{1}{3^{j+1}} \right) \\ &\leq 2\|I_N - \operatorname{Re} S_0\| \frac{1}{3} \frac{1}{1 - \frac{1}{3}} \\ &= \|I_N - \operatorname{Re} S_0\|.\end{aligned}$$

□

Remark 8.7.5. In the scalar case, we can always suppose that $S_0 \geq 0$, and then the norm becomes absolute value, and we have

$$|S_0| + |1 - S_0| = S_0 + 1 - \operatorname{Re} S_0 = 1,$$

and we get back to Bohr's result.

In the matrix-valued case, even when $S_0 \geq 0$ we may have $\|S_0\| + \|I_N - \operatorname{Re} S_0\| > 1$ as is illustrated by the example

$$S_0 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

Then $\|S_0\| = 0.9$ and $\|I_2 - S_0\| = 0.9$, and $\|S_0\| + \|I_N - \operatorname{Re} S_0\| = 1.8$.

Remark 8.7.6. The same proof works in the complex case. See the work [241] of Paulsen and Singh. Then, Schur multipliers and contractive operator-valued functions coincide, in opposition to the quaternionic case.

8.8 Schur and generalized functions of the half-space

We now characterize elements in the classes $\mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{H}_+)$ (see Definition 8.1.4) in terms of realization. Here too, we build a densely linear isometric relation (inspired from the work [21]) and use Shmulyan's theorem. We note that we could also obtain a realization by making the change of variable $q = (p - x_0)(p + x_0)^{-1}$ (which keeps slice hyperholomorphicity since x_0 is real; see Proposition 6.1.17). The function $\widetilde{S}(q) = S(p)$ belongs to $\mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{B})$, and one can apply the results of Section 8.3 to it to get a realization for $\widetilde{S}$ and then for S. But the present approach is more characteristic of the half-space case and in particular the realization has state space the de Branges-Rovnyak space.

The change of variable $q = (p - x_0)(p + x_0)^{-1}$ will be of key importance in the next section, to "guess" the form of the realizations of Herglotz functions.

Theorem 8.8.1. *Let x_0 be a strictly positive real number. A function S slice hyperholomorphic in an axially symmetric s-domain Ω containing x_0 is the restriction to Ω of an element of $\mathcal{S}_\kappa(\mathscr{P}_1, \mathscr{P}_2, \mathbb{H}_+)$ if and only if it can be written as*

$$S(p) = H - (p - x_0)G \star ((p + x_0)I - (p - x_0)A)^{-\star} F, \tag{8.54}$$

where A is a linear bounded operator in a right-sided quaternionic Pontryagin space $\mathscr{P}_\kappa$ of index κ, and, with $B = -(I + x_0 A)$, the operator matrix

$$\begin{pmatrix} B & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathscr{P}_\kappa \\ \mathscr{P}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathscr{P}_\kappa \\ \mathscr{P}_2 \end{pmatrix}$$

is coisometric. In particular S has a unique slice hypermeromorphic extension to $\mathbb{H}_+$. Furthermore, when the pair (G, A) is observable, the realization is unique up to a unitary isomorphism of Pontryagin right quaternionic spaces.

Remark 8.8.2. As in the previous realization results, by (8.54) is meant

$$S(p) = H - (p-x_0)\left(G - (\overline{p}-x_0)(\overline{p}+x_0)^{-1}GA\right)\times \\ \times\left(\frac{|p-x_0|^2}{|p+x_0|^2}A^2 - 2\mathrm{Re}\left(\frac{p-x_0}{p+x_0}\right)A + I\right)^{-1}F. \tag{8.55}$$

See Proposition 7.4.2.

Proof of Theorem 8.8.1. We proceed in a number of steps, and follow [32]. We first prove in Steps 1-8 that a realization of the asserted type exists with $\mathscr{P}_\kappa = \mathscr{P}(S)$. In fact we obtain the backward shift realization with main operator R_{x_0} (see Definition 7.6.3 for the latter).

Following the analysis in [21, pp. 51-52], but taking into account the lack of commutativity, we define a relation R in $(\mathscr{P}(S)\oplus\mathscr{P}_2)\times(\mathscr{P}(S)\oplus\mathscr{P}_1)$ by the linear span of the vectors

$$\left(\begin{pmatrix} K_S(\cdot,q)(x_0-\overline{q})u \\ (x_0-\overline{q})v\end{pmatrix}, \begin{pmatrix} K_S(\cdot,q)(x_0+\overline{q})u - 2x_0K_S(\cdot,x_0)u + \sqrt{2x_0}K_S(\cdot,x_0)(x_0-\overline{q})v \\ \sqrt{2x_0}(S(q)^{[*]} - S(x_0)^{[*]})u + S(x_0)^{[*]}(x_0-\overline{q})v\end{pmatrix}\right),$$

where q runs through Ω and u,v through $\mathscr{P}_2$ and show that the relation is isometric. In the computation the equalities

$$k(x_0,x_0) = \frac{1}{2x_0} \quad \text{and} \quad K_S(x_0,x_0) = \frac{1}{2x_0}\left(I_{\mathscr{P}_2} - S(x_0)S(x_0)^{[*]}\right), \tag{8.56}$$

and

$$K_S(x_0,q_2)(x_0+\overline{q_2}) = I_{\mathscr{P}_2} - S(x_0)S(q_2)^{[*]}, \tag{8.57}$$

are important, as well as the equality (see Proposition 6.5.6)

$$q_1k(q_1,q_2) + k(q_1,q_2)\overline{q_2} = 1. \tag{8.58}$$

STEP 1: *The relation R extends to the graph of an isometry.*

We first check that R has a dense domain. Let $\begin{pmatrix} f \\ w\end{pmatrix} \in (\mathscr{P}(S)\oplus\mathscr{P}_2)$ be orthogonal to $\mathrm{Dom}\,R$. Then, for all $q\in\Omega$ and $u,v\in\mathscr{P}_2$, and using property (5.4) for the inner product in $\mathscr{P}_2$, we have

$$[(x_0-q)f(q),u]_{\mathscr{P}_2} + [(x_0-q)w,v]_{\mathscr{P}_2} = 0.$$

It follows that $w=0$ and that

$$(x_0-q)f(q) \equiv 0, \quad q\in\Omega,$$

and so $f\equiv 0$ in Ω since f is continuous at the point x_0.

We now show that R is isometric. Given (F_1,G_1) and (F_2,G_2) be two elements in the relation, corresponding to $q_1\in\Omega$, $u_1,v_1\in\mathscr{P}_2$ and to $q_2\in\Omega$, $u_2,v_2\in\mathscr{P}_2$ respectively, we have

$$[F_2,F_1]=[(x_0-q_1)K_S(q_1,q_2)(x_0-\overline{q_2})u_2,u_1]_{\mathscr{P}_2}+[(x_0-q_1)(x_0-\overline{q_2})v_2,v_1]_{\mathscr{P}_2}.$$

Furthermore

$$[G_2,G_1]=[g_2,g_1]+[h_2,h_1].$$

with $G_1=\begin{pmatrix}g_1\\ h_1\end{pmatrix}$ where

$$\begin{aligned}g_1(\cdot)&=K_S(\cdot,q_1)(x_0+\overline{q_1})u_1-2x_0K_S(\cdot,x_0)u_1+\sqrt{2x_0}K_S(\cdot,x_0)(x_0-\overline{q_1})v_1,\\ h_1&=\sqrt{2x_0}(S(q_1)^{[*]}-S(x_0)^{[*]})u_1+S(x_0)^{[*]}(x_0-\overline{q_1})v_1\end{aligned}$$

(and similarly for G_2). The fact that R is isometric is equivalent to

$$[F_2,F_1]=[g_2,g_1]+[h_2,h_1].\tag{8.59}$$

We distinguish terms which involve only u_1,u_2, terms which involve only v_1,v_2 and similarly for u_1,v_2 and v_1,u_2 in the computations. We now write the above inner terms separately:

Terms involving u_1,u_2. To show that these terms are the same on both sides of (8.59) we have to check that

$$\begin{aligned}[(x_0-q_1)K_S(q_1,q_2)(x_0-\overline{q_2})u_2,u_1]=&[(x_0+q_1)\big(K_S(q_1,q_2)(x_0+\overline{q_2})-\\ &-2x_0K_S(x_0,q_2)(x_0+\overline{q_2})-\\ &-2x_0(x_0+q_1)K_S(q_1,x_0)+4x_0^2K_S(x_0,x_0)\big)u_2,u_1]+\\ &+2x_0[(S(q_1)-S(x_0))(S(q_2)^{[*]}-S(x_0)^{[*]})u_2,u_1].\end{aligned}$$

From (8.56) this is equivalent to prove that

$$\begin{aligned}&(x_0-q_1)I_{\mathscr{P}_2}k(q_1,q_2)(x_0-\overline{q_2})-(x_0-q_1)S(q_1)k(q_1,q_2)S(q_2)^{[*]}(x_0-\overline{q_2})=\\ &=(x_0+q_1)I_{\mathscr{P}_2}k(q_1,q_2)(x_0+\overline{q_2})-(x_0+q_1)S(q_1)k(q_1,q_2)S(q_2)^{[*]}(x_0+\overline{q_2})-\\ &\quad-2x_0(I_{\mathscr{P}_2}-S(q_1)S(x_0)^{[*]})-2x_0(I_{\mathscr{P}_2}-S(x_0)S(q_2)^{[*]})+\\ &\quad+2x_0\left(I_{\mathscr{P}_2}-S(x_0)S(x_0)^{[*]}\right)+\\ &\quad+2x_0(S(q_1)-S(x_0)(S(q_2)^{[*]}-S(x_0)^{[*]}).\end{aligned}$$

But this amounts to check (8.58).

Terms involving v_1, v_2. We use the formula for $K_S(x_0,x_0)$, see (8.56), to verify that these terms are the same on both sides of (8.59), that is, to show that

$$[(x_0-\overline{q_2})v_2,(x_0-\overline{q_1})v_1] == [(x_0-q_1)S(x_0)S(x_0)^{[*]}(x_0-\overline{q_2})v_2,v_1]+ \\ +2x_0[(x_0-q_1)K_S(x_0,x_0)(x_0-\overline{q_2})v_2,v_1].$$

Terms involving u_2, v_1. There are no terms involving u_2 and v_1 on the left side of (8.59). We need to show that the sums of the terms on the right, that is,

$$\sqrt{2x_0}[(x_0-q_1)S(x_0)(S(q_2)^{[*]}-S(x_0)^{[*]})u_2,v_1]+ \\ +\sqrt{2x_0}[(x_0-q_1)\left(K_S(x_0,q_2)(x_0+\overline{q_2})-2x_0K_S(x_0,x_0)\right)u_2,v_1] = \\ =[Xu_2,v_1],$$

with

$$X=\sqrt{2x_0}(x_0-q_1)S(x_0)(S(q_2)^{[*]}-S(x_0)^{[*]})+\sqrt{2x_0}(x_0-q_1)\left(I_{\mathscr{P}_2}-S(x_0)S(q_2)^{[*]}\right)- \\ -\sqrt{2x_0}(x_0-q_1)\left(I_{\mathscr{P}_2}-S(x_0)S(x_0)^{[*]}\right),$$

add up to 0. Using (8.57) we see that $X=0$. Finally the terms involving u_1, v_2 form symmetric expression to the previous one.

The spaces $\mathscr{P}(S)\oplus\mathscr{P}_2$ and $\mathscr{P}(S)\oplus\mathscr{P}_1$ are Pontryagin spaces with the same index. By Theorem 5.7.10 a densely defined contractive relation defined on a pair of Pontryagin spaces with the same index extends to the graph of an everywhere defined contraction, and it follows that the relation R extends to the graph of an isometry, which we will denote by V. Set

$$V^{[*]}=\begin{pmatrix} B & F \\ G & H\end{pmatrix} \,:\, \begin{pmatrix}\mathscr{P}(S)\\ \mathscr{P}_2\end{pmatrix} \longrightarrow \begin{pmatrix}\mathscr{P}(S)\\ \mathscr{P}_1\end{pmatrix}. \tag{8.60}$$

In the following step we compute the adjoint of V. Recall that the operator R_{x_0} has been defined in (7.41).

STEP 2: *The following formulas hold:*

$$Bf=-(I+2x_0R_{x_0})f, \tag{8.61}$$

$$Fu=-\sqrt{2x_0}R_{x_0}Su, \tag{8.62}$$

$$Gf=\sqrt{2x_0}f(x_0), \tag{8.63}$$

$$H=S(x_0). \tag{8.64}$$

The formula for H is clear. In the verification of (8.61)-(8.63) note that we make use of property (5.4), which is assumed in force since the coeffiicent space $\mathscr{P}_2$ is two-sided. To check (8.61), take $f \in \mathscr{P}(S)$. Then, for $p \in \Omega$ and $u \in \mathscr{P}_2$ we have:

$$\begin{aligned}
[(x_0-p)Bf(p),u]_{\mathscr{P}_2} &= \underbrace{[Bf(p),(x_0-\overline{p})u]_{\mathscr{P}_2}}_{\text{where we use (5.4)}}\\
&= [Bf,K_S(\cdot,p)(x_0-\overline{p})u]_{\mathscr{P}(S)}\\
&= [f,B^{[*]}(K_S(\cdot,p)(x_0-\overline{p})u)]_{\mathscr{P}(S)}\\
&= [f,K_S(\cdot,p)(x_0+\overline{p})u-2x_0K_S(\cdot,x_0)u]_{\mathscr{P}(S)}\\
&= [f(p),(\overline{p}+x_0)u]-2x_0[f(x_0),u]_{\mathscr{P}_2}\\
&= \underbrace{[(p+x_0)f(p),u]_{\mathscr{P}_2}}_{\text{where we use (5.4)}} -2x_0f(x_0),u]_{\mathscr{P}_2}\\
&= [(p+x_0)f(p)-2x_0f(x_0),u]_{\mathscr{P}_2}.
\end{aligned}$$

Thus,

$$(x_0-p)(Bf(p)) = (p+x_0)f(p)-2x_0f(x_0), \quad p\in\Omega,$$

which can be rewritten as (8.61).

To compute (8.62) we take $u,v\in\mathscr{P}_2$. We have:

$$\begin{aligned}
[(x_0-p)((Fv)(p)),u]_{\mathscr{P}_2} &= \underbrace{[Fv,K_S(\cdot,p)(x_0-\overline{p})u]_{\mathscr{P}(S)}}_{\text{where (5.4) has been used}}\\
&= [v,\sqrt{2x_0}(S(p)^{[*]}-S(x_0)^{[*]})u]_{\mathscr{P}_2}\\
&= [\sqrt{2x_0}(S(p)-S(x_0))v,u]_{\mathscr{P}_2},
\end{aligned}$$

and so

$$(x_0-p)(Fv(p)) = \sqrt{2x_0}(S(p)-S(x_0))v, \quad p\in\Omega.$$

Finally, we have:

$$\begin{aligned}
[(x_0-p)Gf,v]_{\mathscr{P}_2} &= \underbrace{[Gf,(x_0-\overline{p})v]_{\mathscr{P}_2}}_{\text{using (5.4)}}\\
&= [f,G^{[*]}(x_0-\overline{p})v)]_{\mathscr{P}(S)}\\
&= [f,\sqrt{2x_0}K_S(\cdot,x_0)(x_0-\overline{p})v]_{\mathscr{P}(S)}\\
&= \sqrt{2x_0}[(x_0-p)f(x_0),v]_{\mathscr{P}_2}.
\end{aligned}$$

STEP 3: *Formula* (8.55) *holds for* p *near* x_0*:*

We check the formula for real p. The result follows then by slice hyperholomorphic extension. We first remark that the operator R_{x_0} is bounded since B is bounded. Let $f\in\mathscr{P}(S)$, with power series expansion

$$f(p)=\sum_{n=0}^{\infty}(p-x_0)^n f_n, \quad f_0,f_1,\ldots\in\mathscr{P}_2,$$

around x_0. We have

$$f_n = \frac{1}{\sqrt{2x_0}} G R_{x_0}^n f, \quad n = 0, 1, \ldots$$

and so, for real $p = x$ near x_0 we can write:

$$f(x) = \sum_{n=0}^{\infty} (x - x_0)^n f_n = \frac{1}{\sqrt{2x_0}} G(I - (x - x_0) R_{x_0})^{-1} f.$$

With $f = R_{x_0} S u = -\frac{1}{\sqrt{2x_0}} F u$ where $u \in \mathscr{P}_1$ we have

$$(R_{x_0} S u)(x) = -\frac{1}{2x_0} G(I - (x - x_0) R_{x_0})^{-1} F u = -G(2x_0 I - 2(x - x_0) x_0 R_{x_0})^{-1} F u$$

and so, since $B = -I - 2x_0 R_{x_0}$,

$$\begin{aligned} S(x)u &= S(x_0)u + (x - x_0)(R_{x_0} S u)(x) \\ &= S(x_0)u - (x - x_0) G(2x_0 I - 2(x - x_0) x_0 R_{x_0})^{-1} F u \\ &= S(x_0)u - (x - x_0) G(2x_0 I + (x - x_0)(B + I))^{-1} F u \\ &= S(x_0)u - (x - x_0) G((x + x_0) I + (x - x_0) B)^{-1} F u. \end{aligned}$$

STEP 4: *Assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ are Hilbert spaces. The function S admits a slice hypermeromorphic extension to $\mathbb{H}_+$, with at most a finite number of spheres of poles.*

We first show that the operator

$$(x_0 + x) I + (x - x_0) B$$

is invertible for all real x, with the possible exception of a finite set in $\mathbb{R}$. The operator $V^{[*]}$ is a contraction between Pontryagin spaces of same index, and so its adjoint V is a contraction; see Theorem 5.7.8. So it holds that

$$B^{[*]} B + G^{[*]} G \leq I.$$

But

$$\langle G^{[*]} G f, f \rangle = \langle G f, G f \rangle_{\mathscr{P}_2} \geq 0$$

since $\mathscr{P}_2$ is here assumed to be a Hilbert space, and so B is a contraction. It admits a maximal strictly negative invariant subspace, say $\mathscr{M}$ (see [165, Theorem 1.3.11] for the complex case and Theorem 5.7.9 for the quaternionic case). Writing

$$\mathscr{P}(S) = \mathscr{M} [+] \mathscr{M}^{[\perp]},$$

the operator matrix representation of B is upper triangular with respect to this decomposition where

$$B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}.$$

The operator B_{22} is a contraction from the Hilbert space $\mathscr{M}^{[\perp]}$ into itself, and so $I - \frac{x_0-x}{x_0+x}B_{22}$ is invertible for every $x > 0$. The operator B_{11} is a contraction from the finite dimensional anti-Hilbert space $\mathscr{M}$ onto itself, and so has right eigenvalues *outside the open unit ball.* So the operator $I - \frac{x_0-x}{x_0+x}B_{11}$, is invertible in $x > 0$, except the points $x \neq x_0$ such that $\frac{x+x_0}{x-x_0}$ is a real eigenvalue of B_{11} of modulus greater or equal to 1. There is a finite number of such points since a $n \times n$ quaternionic matrix has exactly n right eigenvalues (counting multiplicity) up to equivalence (in other words, it has exactly n spheres of eigenvalues); see Theorem 4.3.6. Thus

$$I - \frac{x_0-x}{x_0+x}B = \begin{pmatrix} I - \dfrac{x_0-x}{x_0+x}B_{11} & -\dfrac{x_0-x}{x_0+x}B_{12} \\ 0 & I - \dfrac{x_0-x}{x_0+x}B_{22} \end{pmatrix}$$

is invertible for all $x > 0$, with the possible exception of a finite number of points. We now use Proposition 7.4.2 to extend $S(x)$ computed in STEP 3 to a slice hypermeromorphic function in $\mathbb{H}_+$ via the formula

$$\begin{aligned} S(p)u &= S(x_0)u + \frac{x_0-p}{x_0+p}G \star \left(I - \frac{x_0-p}{x_0+p}B\right)^{-\star} F \\ &= S(x_0)u + \frac{p-x_0}{p+x_0} \star \left(G - \frac{x_0-\bar{p}}{x_0+\bar{p}}GB\right)\left(\frac{|x_0-p|^2}{|x_0+p|^2}B^2 - 2\mathrm{Re}\left(\frac{x_0-p}{x_0+p}\right)B + I\right)^{-1} F. \end{aligned}$$

Let $t = \dfrac{\mathrm{Re}\, q}{|q|^2}$ where $q = \dfrac{x_0-p}{x_0+p} \in \mathbb{B}$. We have

$$|q|^2B^2 - 2(\mathrm{Re}\, q)B + I = |q|^2 \begin{pmatrix} B_{11}^2 - 2tB_{11} + \frac{1}{|q|^2} & B_{11}B_{12} + B_{12}B_{22} - 2tB_{12} + \frac{1}{|q|^2} \\ 0 & B_{22}^2 - 2tB_{22} + \frac{1}{|q|^2} \end{pmatrix}.$$

The operators $B_{11}^2 - 2tB_{11} + \frac{1}{|q|^2}$ and $B_{22}^2 - 2tB_{22} + \frac{1}{|q|^2}$ are invertible for q such that $\frac{1}{|q|^2}$ is in the resolvent sets of B_{11} and B_{22} respectively, in both cases the complement of a compact nonempty set; see Theorem 7.2.3. For the Hilbert space contraction B_{22} this will happen when $|q| < 1$. The operator B_{11} is acting in a finite dimensional anti-Hilbert space, and thus has just point S-spectrum which is inside the closed unit ball. The point S-spectrum coincides with the set of right eigenvalues, and so it consists of a finite number of (possibly degenerate) spheres. Finally one uses the Potapov-Ginzburg transform to show that S has a slice hypermeromorphic extension when $\mathscr{P}_1$ and $\mathscr{P}_2$ are Pontryagin spaces with the same index.

STEP 5: *A function S with a realization of the form* (8.55) *is in a class* $\mathcal{S}_{\kappa'}(\mathscr{P}_1, \mathscr{P}_2, \mathbb{H}_+)$ *for some* $\kappa' \leq \kappa$. *Furthermore* $\kappa = \kappa'$ *when the realization is observable.*

One way to prove this assertion would be going via a direct computation, as in the proof of Theorem 8.3.6. We here present a slightly different approach, where the computations are made for real values of the variable, and then slice hyperholomorphic extension is used. Let $p = x$ and $q = y$ near x_0. A direct computation leads to

$$\frac{I_{\mathscr{P}_2} - S(x)S(y)^{[*]}}{x+y} = G(I(x_0+x) - (x+x_0)B)^{-1}(I(y+x_0) - (y-x_0)B)^{-[*]}G^{[*]},$$

where $B = -(I + x_0 A)$. Thus, with

$$K(x,y) = G(I(x_0+x) - (x+x_0)B)^{-1}(I(y+x_0) - (y-x_0)B)^{-[*]}G^{[*]},$$

we have

$$I_{\mathscr{P}_2} - S(x)S(y)^{[*]} = xK(x,y) + K(x,y)y,$$

and the result follows by observing that (8.55) is the hyperholomorphic extension of $S(x)$.

STEP 6: *An observable realization is unique up to a unitary transformation.*

As in the proof of Theorem 8.3.6 we have

$$G_1(I_{\mathscr{X}_1} - uB_1)^{-1}(I_{\mathscr{X}_1} - vB_1)^{-[*]}G_1^{[*]} = G_2(I_{\mathscr{X}_2} - uB_2)^{-1}(I_{\mathscr{X}_2} - vB_2)^{-[*]}G_2^{[*]},$$

where u, v are in a real neighborhood of the origin, and where the indices 1 and 2 correspond to two observable and coisometric realizations, with state spaces $\mathscr{X}_1$ and $\mathscr{X}_2$, respectively. Then the domain and range of the relation R spanned by the pairs

$$((I_{\mathscr{X}_1} - vB_1)^{-*}G_1^* h, (I_{\mathscr{X}_2} - vB_2)^{-*}G_2^* k), \quad h, k \in \mathscr{P}_2,$$

are dense. By Theorem 5.7.10) R is the graph of a unitary map, which provides the desired equivalence. □

Example. The case where $\dim \mathscr{P}(S) < \infty$ and moreover $\mathscr{P}_1$ and $\mathscr{P}_2$ are equal and also of finite dimension corresponds to the class of rational functions unitary with an indefinite metric on the imaginary plane, and will be considered in greater details in the next chapter. When $\mathscr{P}_1 = \mathscr{P}_2 = \mathbb{H}$ and $\mathscr{P}(S)$ has finite dimension then S is a finite Blaschke product, as defined in Section 11.8. When moreover $\dim \mathscr{P}(S) = 1$, S is (up to a right multiplicative unitary constant) a Blaschke factor based on a $a \in \mathbb{H}$, $a \neq \bar{a}$, that is

$$S(p) = b_a(p) = (p - \bar{a})^{-\star} \star (p - a).$$

For real $p = x$ we have $b_a(x) = (x + \bar{a})^{-1}(x - a)$ (see (6.74)). Furthermore, with $B = \frac{1-\bar{a}}{1+\bar{a}}$

we have:

$$\begin{aligned}
b_a(x) &= b_a(1) + b_a(x) - b_a(1)\\
&= \frac{1-a}{1+\bar{a}} + (x-1)\frac{2\mathrm{Re}(a)}{(x+\overline{a})(1+\overline{a})}\\
&= \frac{1-a}{1+\bar{a}} + (x-1)\frac{2\mathrm{Re}(a)(1+B)}{(x(1+B)+(1-B))(1+\overline{a})}\\
&= \frac{1-a}{1+\bar{a}} + (x-1)\frac{2\mathrm{Re}(a)}{(x(1+B)+(1-B))}\frac{2}{(1+\overline{a})^2}\\
&= \frac{1-a}{1+\bar{a}} + (x-1)\frac{2\mathrm{Re}(a)}{1+\overline{a}}((x+1)+(x-1)B)^{-1}\frac{2}{1+\overline{a}}
\end{aligned}$$

since

$$\frac{1+B}{1+\overline{a}} = \frac{2}{(1+\overline{a})^2}.$$

To conclude we have:

$$b_a(p) = H - (p-1)G\star((p+1)+(p-1)B)^{-\star}F$$

is slice hyperholomorphic, extends $b_a(x)$, and the matrix

$$\begin{pmatrix} B & F\\ G & H\end{pmatrix} = \begin{pmatrix} \dfrac{1-\overline{a}}{1+\overline{a}} & \dfrac{2\sqrt{\mathrm{Re}\,a}}{1+\overline{a}}\\[2ex] -\dfrac{2\sqrt{\mathrm{Re}\,a}}{1+\overline{a}} & \dfrac{1-a}{1+\bar{a}}\end{pmatrix}$$

is unitary. One can obtain from this formula the realization for a finite Blaschke product using formula for the product of realizations (for the variable $(p-1)(p+1)^{-1}$ rather than p). See Proposiiton 7.4.4 and Chapter 9, and in particular Theorem 9.1.6.

8.9 Herglotz and generalized Herglotz functions

In this section we prove a realization theorem for generalized Herglotz function. The form of the realization is first guessed from the realization given in Section 8.7 for generalized Carathéodory function. Then the main result, Theorem 8.9.1 is proved as in the previous cases using Shmulyan's theorem. More precisely, starting from (8.47) (with $\Phi_0 \in \mathcal{C}_\kappa(\mathscr{P},\mathbb{B})$)

$$\Phi_0(p) = \frac{1}{2}C\star(I-pU)^{-\star}\star(I+pU)\star C^{[*]} + \frac{\Phi_0(0)-\Phi_0(0)^{[*]}}{2},$$

we see that

$$\Phi_0(0) = \frac{1}{2}CC^{[*]} + \frac{\Phi_0(0)-\Phi_0(0)^{[*]}}{2}, \quad \text{and thus} \quad \frac{1}{2}CC^{[*]} = \frac{\Phi_0(0)+\Phi_0(0)^{[*]}}{2},$$

and write

$$\begin{aligned}\Phi_0(p) &= \frac{1}{2}C\star(I-pU)^{-\star}\star(I+pU)\star C^{[*]} - \frac{1}{2}CC^{[*]} + \frac{1}{2}CC^{[*]} + \frac{\Phi_0(0)-\Phi_0(0)^{[*]}}{2}\\ &= \frac{1}{2}C\star(I-pU)^{-\star}\star(I+pU)\star C^{[*]} - \frac{1}{2}CC^{[*]} + \frac{\Phi_0(0)+\Phi_0(0)^{[*]}}{2} +\\ &\quad + \frac{\Phi_0(0)-\Phi_0(0)^{[*]}}{2}\\ &= \Phi_0(0) + pC(I-pU)^{-\star}\star UC^{[*]}.\end{aligned}$$

This suggests that, with the change of variable $p \mapsto \dfrac{p-x_0}{p+x_0}$, which sends $\mathbb{H}_+$ onto $\mathbb{B}$ we have as realization for generalized Herglotz functions expressions of the form

$$\Phi(p) = \Phi(x_0) + (p-x_0)C\star((p+x_0)I-(p-x_0)U)^{-\star}\star UC^{[*]},$$

with U being coisometric, or, with $B=-U$ and $C=G$, namely of the form (8.65) given in the next theorem. We follow the arguments of [32] for the proof.

Theorem 8.9.1. *Let $\mathscr{P}$ be a two-sided Pontryagin space. A $\mathbf{B}(\mathscr{P})$-valued function Φ slice hyperholomorphic in an axially symmetric s-domain Ω containing the point $x_0>0$ is the restriction of a function in the class $\mathcal{H}_\kappa(\mathscr{P},\mathbb{H}_+)$ if and only if there exists a right quaternionic Pontryagin space $\mathscr{P}_\kappa$ of index κ and operators*

$$\begin{pmatrix} B & BG^{[*]}\\ G & H\end{pmatrix} : \begin{pmatrix}\mathscr{P}_\kappa\\ \mathscr{P}\end{pmatrix} \longrightarrow \begin{pmatrix}\mathscr{P}_\kappa\\ \mathscr{P}\end{pmatrix},$$

with B verifying

$$(I+2x_0B)(I+2x_0B)^{[*]} = I,$$

and such that Φ can be written as

$$\Phi(p) = H-(p-x_0)G\star((p+x_0)I+(p-x_0)B)^{-\star}BG^{[*]}. \tag{8.65}$$

Furthermore, Φ has a unique slice hypermeromorphic extension to $\mathbb{H}_+$. Finally, when the pair (G,B) is observable, the realization is unique up to a unitary isomorphism of Pontryagin right quaternionic spaces.

Proof. Given $\Phi\in\mathcal{H}_\kappa(\mathscr{P},\mathbb{H}_+)$, we denote by $\mathscr{L}(\Phi)$ the associated right reproducing kernel Pontryagin space of $\mathscr{P}$-valued functions with reproducing kernel K_Φ. We proceed in a number of steps to prove the theorem. The main step is to check that the backward shift operator R_{x_0} is bounded in $\mathscr{L}(\Phi)$. The rest of the proof is similar to the proofs of the realization theorems appearing in the previous sections. To do that R_{x_0} is bounded we define:

$$B = I+2x_0R_{x_0} \tag{8.66}$$

is coisometric.

STEP 1: *B is a (continuous) coisometry in $\mathscr{L}(\Phi)$.*

Let $\mathscr{R}_{x_0}$ be the linear relation on $\mathscr{L}(\Phi)\times\mathscr{L}(\Phi)$ generated by the linear span of the pairs

$$\mathscr{R}_{x_0}=(K_\Phi(\cdot,p)(\overline{p}-x_0)u,(K_\Phi(\cdot,p)-K_\Phi(\cdot,x_0))u),\quad p\in\Omega,\ u\in\mathscr{P}. \tag{8.67}$$

Let $h\in\mathscr{L}(\Phi)$ be such that

$$[h,K_\Phi(\cdot,p)(\overline{p}-x_0)u]=0,\ \ \forall p\in\Omega\quad\text{and}\quad u\in\mathscr{P}.$$

Then

$$(p-x_0)h(p)=0,\quad\forall p\in\Omega$$

and $h\equiv 0$ in Ω (recall that the elements of $\mathscr{L}(\Phi)$ are slice hyperholomorphic in Ω). Thus the domain of this relation is dense. We now show that

$$(f,g)\in\mathscr{R}_{x_0}\quad\Longrightarrow\quad[f,f]=[f+2x_0g,f+2x_0g], \tag{8.68}$$

and first prove that

$$[f,g]+[g,f]+2x_0[g,g]=0. \tag{8.69}$$

This is a lengthy calculation, which we take (as most of this section) from [32]. In the computations use is made of property (5.4) and of

$$K_\Phi(x_0,x_0)=\frac{1}{2x_0}\left(\Phi(x_0)+\Phi(x_0)^{[*]}\right). \tag{8.70}$$

An element in $\mathscr{R}_{x_0}$ can be written as (f,g) with

$$\begin{aligned} f(p)&=\sum_{j=1}^m K_\Phi(p,p_j)(\overline{p_j}-x_0)u_j,\quad\text{where}\quad u_1,\dots,u_m\in\mathscr{P},\\ g(p)&=\sum_{j=1}^m K_\Phi(p,p_j)u_j-K_\Phi(p,x_0)d,\quad\text{where}\quad d=\sum_{j=1}^m u_j.\end{aligned} \tag{8.71}$$

With f and g as in (8.71) we have:

$$[f,g]=\sum_{i,j=1}^m[K_\Phi(p_i,p_j)(\overline{p_j}-x_0)u_j,u_i]_{\mathscr{P}}-[\left(\sum_{j=1}^m K_\Phi(x_0,p_j)(\overline{p_j}-x_0)u_j\right),d]_{\mathscr{P}},$$

$$[g,f]=\sum_{i,j=1}^m[K_\Phi(p_i,p_j)u_j,(\overline{p_i}-x_0)u_i]_{\mathscr{P}}-[d,\left(\sum_{i=1}^m K_\Phi(x_0,p_i)(\overline{p_i}-x_0)u_i\right)]_{\mathscr{P}}.$$

Thus, using the assumed property (5.4) of the inner product in $\mathscr{P}$

$$
\begin{aligned}
[f,g]+[g,f] = {} & -2x_0\left(\sum_{i,j=1}^{m}[K_\Phi(p_i,p_j)u_j,u_i]_{\mathscr{P}}\right)+\\
& +\sum_{i,j=1}^{m}[\left\{p_iK_\Phi(p_i,p_j)+K_\Phi(p_i,p_j)\overline{p_j}\right\}u_j,u_i]_{\mathscr{P}}-[\left(\sum_{j=1}^{m}K_\Phi(x_0,p_j)\overline{p_j}u_j\right),d]_{\mathscr{P}}+\\
& +x_0[\left(\sum_{j=1}^{m}K_\Phi(x_0,p_j)u_j\right),d]_{\mathscr{P}}-[d,\left(\sum_{i=1}^{m}K_\Phi(x_0,p_i)\overline{p_i}u_i\right)]_{\mathscr{P}}+\\
& +x_0[d,\left(\sum_{j=1}^{m}K_\Phi(x_0,p_j)u_j\right)]_{\mathscr{P}}.
\end{aligned}
$$

Taking into account (8.4) we have

$$
\begin{aligned}
[f,g]+[g,f] = {} & -2x_0\left(\sum_{i,j=1}^{m}[K_\Phi(p_i,p_j)u_j,u_i]_{\mathscr{P}}\right)+\\
& +[d,\left(\sum_{i=1}^{m}\Phi(p_i)u_i\right)]_{\mathscr{P}}+[\left(\sum_{j=1}^{m}\Phi(p_j)^{[*]}u_j\right),d]_{\mathscr{P}}-\\
& -[\left(\sum_{j=1}^{m}K_\Phi(x_0,p_j)\overline{p_j}u_j\right),d]_{\mathscr{P}}+x_0[\left(\sum_{j=1}^{m}K_\Phi(x_0,p_j)u_j\right),d]_{\mathscr{P}}-\\
& -[d,\left(\sum_{i=1}^{m}uK_\Phi(x_0,p_i)\overline{p_i}u_i\right)]_{\mathscr{P}}+x_0[d,\left(\sum_{i=1}^{m}K_\Phi(x_0,p_i)u_i\right)]_{\mathscr{P}}.
\end{aligned}
$$

We now turn to $[g,g]$. We have:

$$
\begin{aligned}
[g,g] = {} & \left(\sum_{i,j=1}^{m}[K_\Phi(p_i,p_j)u_j,u_i]_{\mathscr{P}}\right)-[\left(\sum_{j=1}^{m}K_\Phi(x_0,p_j)u_j\right),d]_{\mathscr{P}}-\\
& -[d,\left(\sum_{i=1}^{m}K_\Phi(x_0,p_i)u_i\right)]_{\mathscr{P}}+[K_\Phi(x_0,x_0)d,d]_{\mathscr{P}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
[f,g]+[g,f]+2x_0[g,g] &= [d,\left(\sum_{i=1}^m \Phi(p_i)^{[*]}u_i\right)]_{\mathscr{P}} + [\left(\sum_{j=1}^m \Phi(p_j)^{[*]}u_j\right),d]_{\mathscr{P}} - \\
&\quad - [\left(\sum_{j=1}^m K_\Phi(x_0,p_j)\overline{p_j}u_j\right),d]_{\mathscr{P}} - x_0[\left(\sum_{j=1}^m K_\Phi(x_0,p_j)u_j\right),d]_{\mathscr{P}} - \\
&\quad - [d,\left(\sum_{i=1}^m K_\Phi(x_0,p_i)\overline{p_i}u_i\right)]_{\mathscr{P}} - x_0[d,\left(\sum_{i=1}^m K_\Phi(x_0,p_i)u_i\right)]_{\mathscr{P}} + \\
&\quad + 2x_0[K_\Phi(x_0,x_0)d,d]_{\mathscr{P}} \\
&= [d,\left(\sum_{i}^m J\Phi(p_i)^{[*]}u_i\right),d]_{\mathscr{P}} + [\left(\sum_{j=1}^m \Phi(p_j)^{[*]}u_j\right),d]_{\mathscr{P}} - \\
&\quad - [\left(\sum_{j=1}^m K_\Phi(x_0,p_j)(\overline{p_j}+x_0)u_j\right),d]_{\mathscr{P}} - \\
&\quad - [d,\left(\sum_{i=1}^m K_\Phi(x_0,p_i)(\overline{p_i}+x_0)u_i\right),d]_{\mathscr{P}} + 2x_0[K_\Phi(x_0,x_0)d,d]_{\mathscr{P}}.
\end{aligned}
$$

Using (8.70), we obtain

$$
\begin{aligned}
[f,g]+[g,f]+2x_0[g,g] &= [d,\left(\sum_{i=1}^m \Phi(p_i)^{[*]}u_i\right)]_{\mathscr{P}} + [\left(\sum_{j=1}^m \Phi(p_j)^{[*]}u_j\right),d]_{\mathscr{P}} - \\
&\quad - [\left(\sum_{j=1} \left(\Phi(x_0)+\Phi(p_j)^{[*]}\right)u_j\right),d]_{\mathscr{P}} - \\
&\quad - [d,\left(\sum_{i=1}^m \left(\Phi(p_i)^{[*]}+\Phi(x_0)\right)u_i\right)]_{\mathscr{P}} + 2x_0[K_\Phi(x_0,x_0)d,d]_{\mathscr{P}} \\
&= 0
\end{aligned}
$$

and so we have proved (8.69). Equation (8.68) follows since

$$
[f+2x_0g,f+2x_0g] = [f,f]+2x_0\left([f,g]+[g,f]+2x_0[g,g]\right).
$$

Equation (8.68) expresses that the linear space of functions $(f,f+2x_0g)$ with f,g as in (8.71) define an isometric relation $\mathscr{R}$ in $\mathscr{L}(\Phi)\times\mathscr{L}(\Phi)$. By Theorem 5.7.10, $\mathscr{R}$ extends to the graph of a (continuous) isometry, say B^*, on $\mathscr{L}(\Phi)$. We have for $h\in\mathscr{L}(\Phi)$

$$
\begin{aligned}
u^*(p-x_0)((Bh(p)) &= [Bh,K_\Phi(\cdot,p)(\overline{p}-x_0)u] \\
&= [h,B^*(K_\Phi(\cdot,p)(\overline{p}-x_0)u)] \\
&= [h,K_\Phi(\cdot,p)(\overline{p}-x_0)u+2x_0(K_\Phi(\cdot,p)-K_\Phi(\cdot,x_0))u] \\
&= u^*\left((p-x_0)h(p)+2x_0h(p)-2x_0h(x_0)\right) \\
&= u^*\left((p+x_0)h(p)-2x_0h(x_0)\right).
\end{aligned}
$$

Thus

$$(p-x_0)((Bh(p)) = ((p+x_0)h(p) - 2x_0h(x_0). \tag{8.72}$$

Taking real $p = x$ near x_0 we get

$$Bh(x) = ((I + 2x_0R_{x_0})h)(x).$$

It follows from slice hyperholomorphic extension that $B = I + 2x_0R_{x_0}$, and so the operator R_{x_0} is a bounded operator in $\mathscr{L}(\Phi)$ (in particular $\mathscr{R}_{x_0}$ extends to the graph of $R^*_{x_0}$).

STEP 2: *The function $p \mapsto (R_{x_0}\Phi\eta)(p)$ belongs to $\mathscr{L}(\Phi)$ for every $\eta \in \mathscr{P}$.*

Let G denote the point evaluation at the point x_0. We show that

$$R_{x_0}\Phi\eta = BG^*\eta. \tag{8.73}$$

By definition of the point evaluation we have $(G^*\eta)(p) = K_\Phi(p,x_0)\eta$ where $\eta \in \mathscr{P}$. See Lemma 5.10.3. From (8.72), and with $h = G^*\eta$ and taking $p = x$ real we have

$$\begin{aligned}(x-x_0)((BG^*)\eta(x)) &= ((p+x_0)(G^*\eta)(x) - 2x_0(G^*\eta)(x_0)\\ &= (x+x_0)\frac{\Phi(x)+\Phi(x_0)^{[*]}}{x+x_0}\eta - 2x_0\frac{\Phi(x_0)+\Phi(x_0)^{[*]}}{2x_0}\eta\\ &= (\Phi(x)-\Phi(x_0))\eta,\end{aligned}$$

and hence the result. Since we already know that the operator R_{x_0} is bounded, it follows (and the proof is the same as in STEP 6 of Theorem 8.8.1 for generalized Schur functions of the half-space) that realization formula (8.65) holds.

STEP 3: *The function Φ admits a slice hypermeromorphic extension to $\mathbb{H}_+$.*

Without loss of generality we assume that $x_0 = 1$ (this amounts to replace Φ by $\Phi(px_0)$; this transformation does not affect the number of negative squares of the kernel K_Φ neither the property of existence of a slice hypermeromorphic extension). Using Theorem 5.7.9 we know that $T = I + 2B$ has a strictly negative maximal invariant subspace, say $\mathscr{N}$, on which it is bijective. Consider now the operator matrix representation

$$T = \begin{pmatrix} T_{11} & T_{12}\\ 0 & T_{22}\end{pmatrix}$$

of T along the direct and orthogonal sum $\mathscr{L}(\Phi) = \mathscr{N}[\oplus]\mathscr{N}^{[\perp]}$. Thus T_{11} is a bijective contraction from a anti-Hilbert space onto itself, and T_{22} is a contraction from a Hilbert space into itself. Thus for $x > 0$ in a neighborhood of x_0,

$$(1+x)I + (x-1)B = (x+3)\begin{pmatrix} I + \frac{x-1}{x+3}T_{11} & \frac{x-1}{x+3}T_{12}\\ 0 & I + \frac{x-1}{x+3}T_{22}\end{pmatrix}.$$

Considering hyperholomorphic extension, the operator $(I - \frac{p-1}{p+3}T_{22})$ is invertible in $\mathbb{H}_+$ since the map $p \mapsto \frac{p-1}{p+3}$ sends $\mathbb{H}_+$ into $\mathbb{B}$. Furthermore, there are only a finite number of spheres where $(I - \frac{p-1}{p+3}T_{11})$ is not invertible.

To complete the proof of the theorem two facts remain to be proved. First, a function Φ admitting a realization of the form (8.65) is in a class $\mathcal{H}_\kappa(\mathscr{P}, \mathbb{H}_+)$. The proof is as in the case of the functions S and is based on the identity

$$\Phi(x) + \Phi(y)^{[*]} = (x+y)G(I(x_0+x) - (x_0-x)B)^{-1}(I(x_0+y) - (x_0-y)B)^{-[*]}G^{[*]},$$

where x, y are real and in a neighborhood of x_0.

Next we need to verify that observable realization of the form (8.65) is unique up to a isomorphism of quaternionic Pontryagin spaces. This is done as in the proofs of similar statements earlier in the chapter. $\square$

Chapter 9

Rational slice hyperholomorphic functions

Rational functions play an important role in linear system theory, and in the noncommutative case we mention in particular the works [222, 260, 261]. We also mention [66] which is set in the framework of noncommutative probability. There the approach uses a topological algebra of a very special structure (dual of a countably normed nuclear space, with a series of inequalities on the norms), and is also quite far from the quaternionic setting.

In this chapter we present various results pertaining to rational function of a *real variable*, with values in $\mathbb{H}^{n\times m}$. The various formulas presented in this section have slice hyperholomorphic extensions to suitable neighborhoods of the real axis. See Section 7.4.

The last four sections of the chapter are devoted to the study quaternionic rational functions with symmetries (J-unitary in an appropriate sense). Some of the results in these sections could be obtained from the infinite dimensional results presented in the previous chapter, but the approach here is different. We use the notion of minimal realization of a rational function, and the fact that the functions at hand are defined (besides a finite number of points) across the boundary.

9.1 Definition and first properties

Definition 9.1.1. A function $r(x)$ of the real variable x and with values in $\mathbb{H}^{n\times m}$ is rational if r is obtained from a finite number of addition, multiplication and (when invertible at the origin) division of matrix polynomials.

Remark 9.1.2. The above definition would make sense for any (possibly noncommutative field) which contains the real numbers as its center. Here, the particularity of the quaternions makes that the denominator can always be scalar. See equation (9.1) below.

We denote by $\mathbb{H}^{n\times m}(x)$ the skew field of $\mathbb{H}^{n\times m}$-valued rational function in the real variable x. In the following theorem we give a number of equivalent characterizations of rational functions. We first need a definition. For a convergent $\mathbb{H}^{n\times m}$-valued power series

$$f(x) = f_0 + f_1x + f_2x^2 + \cdots$$

of the real variable x we set (see also (7.41))

$$(R_0f)(x) = \begin{cases} \dfrac{f(x)-f(0)}{x}, & \text{if } x \neq 0, \\ f_1, & \text{if } x = 0. \end{cases}$$

The main result of the section is the characterization of rational functions in terms of realizations (see Theorem 9.1.8). We begin with some preliminaries.

Proposition 9.1.3. *Let $t \in \mathbb{H}[x]$. Then, $\bar{t}$ defined by $\bar{t}(x) = \overline{t(x)} \in \mathbb{H}[x]$ and $t\bar{t} \in \mathbb{R}[x]$. Moreover $r \in \mathbb{H}^{n\times m}(x)$ if and only if it can written as*

$$r(x) = \frac{M(x)}{m(x)}, \tag{9.1}$$

where M is a $\mathbb{H}^{n\times m}$-valued polynomial and $m \in \mathbb{R}[x]$. In particular, r is rational if and only if its entries are rational.

Proof. The first claim is clear. Moreover, any function of the form (9.1) is rational from the definition. We show the converse claim. It is enough to prove that the inverse of a square (say $\mathbb{H}^{n\times n}$-valued) rational function is still rational. For $n = 1$, this follows from the first claim since

$$t^{-1}(x) = \frac{\overline{t(x)}}{t(x)\overline{t(x)}} = \frac{\bar{t}(x)}{(t\bar{t})(x)}.$$

Now suppose that the induction holds for $n-1$ and let

$$r(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$$

be a $\mathbb{H}^{n\times n}$-valued rational function. Without loss of generality, we can assume that the $(1,1)$ entry $a(x) \in \mathbb{H}(x)$ is not identically equal to 0 (otherwise, multiply r on the left or on the right by a permutation matrix; this does not change the property of r or of r^{-1} being rational). We write (see for instance [169, (0.3), p. 3])

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c(x)a(x)^{-1} & I_{n-1} \end{pmatrix} \times \\ \times \begin{pmatrix} a(x) & 0 \\ 0 & d(x)-c(x)a(x)^{-1}b(x) \end{pmatrix} \begin{pmatrix} 1 & a(x)^{-1}b(x) \\ 0 & I_{n-1} \end{pmatrix}, \tag{9.2}$$

and so $d(x) - c(x)a(x)^{-1}b(x) \in \mathbb{H}^{(n-1)\times(n-1)}(x)$ is invertible for all, but at most a finite number of, values $x \in \mathbb{R}$ (to see this, it suffices to use the map χ (see (4.8)) and reduce the question to that of a matrix-valued rational function of a real variable, but with complex coefficients). We can thus use the induction for $n-1$. From (9.2) we get

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a(x)^{-1}b(x) \\ 0 & I_{n-1} \end{pmatrix} \times \\ \times \begin{pmatrix} (a(x))^{-1} & 0 \\ 0 & (d(x)-c(x)a(x)^{-1}b(x))^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c(x)a(x)^{-1} & I_{n-1} \end{pmatrix}, \tag{9.3}$$

and this proves the induction for n. □

We also note that the realization formulas (2.6), (2.9) and (2.10) still hold when the matrices have quaternionic entries, but the variable is real. More explicitly we have the following two theorems, whose proofs are omitted because they are as in the complex case.

Theorem 9.1.4. *Let r be a $\mathbb{H}^{n\times n}$-valued rational function with realization $r(x) = D + xC(I_N - xA)^{-1}B$, and assume D invertible. Then,*

$$r^{-1}(x) = D^{-1} - xD^{-1}C(I_N - x(A - BD^{-1}C))^{-1}BD^{-1} \tag{9.4}$$

is a realization of r^{-1}.

Remark 9.1.5. The operator

$$A^{\times} \overset{\text{def.}}{=} A - BD^{-1}C$$

plays a key role in the sequel.

Theorem 9.1.6. *If r_j, $j = 1,2$ are rational functions, respectively $\mathbb{H}^{n\times m}$ and $\mathbb{H}^{m\times t}$-valued, defined in neighborhood of the origin, and with realizations*

$$r_j(x) = D_j + x(I_{n_j} - xA_j)^{-1}B_j, \quad j = 1,2$$

then a realization of r_1r_2 is given by $D = D_1D_2$ and

$$A = \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix}, \ B = \begin{pmatrix} B_1D_2 \\ B_2 \end{pmatrix}, \ C = \begin{pmatrix} C_1 & D_1C_2 \end{pmatrix}, \tag{9.5}$$

and a realization of $r_1 + r_2$ is given by $D = D_1 + D_2$ and

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \ C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}. \tag{9.6}$$

Remark 9.1.7. With A,B,C,D given by (9.5), and assuming D_1 and D_2 invertible, we have

$$A^{\times} = \begin{pmatrix} A_1^{\times} & 0 \\ -B_2D_2^{-1}D_1^{-1}C_1 & A_2^{\times} \end{pmatrix}.$$

We can now state the main result of the section:

Theorem 9.1.8. *Let $r(x) = r_0 + r_1 x + \cdots$ be a $\mathbb{H}^{n\times m}$-valued power series of a real variable convergent in a neighborhood of the origin. Then the following are equivalent:*
(1) *r is rational.*
(2) *r can be written in the form*

$$r(x) = D + xC(I_N - xA)^{-1}B, \tag{9.7}$$

where $D = r(0)$ and $(C,A,B) \in \mathbb{H}^{n\times N} \times \mathbb{H}^{N\times N} \times \mathbb{H}^{N\times m}$ for some $N \in \mathbb{N}$.
(3) *The coefficients of the power series defining r can be written as*

$$r_t = CA^{t-1}B, \quad t = 1,2,\ldots$$

where $(C,A,B) \in \mathbb{H}^{n\times N} \times \mathbb{H}^{N\times N} \times \mathbb{H}^{N\times m}$ for some $N \in \mathbb{N}$.
(4) *The right linear span $\mathscr{M}(r)$ of the columns of the functions $R_0 r, R_0^2 r, \ldots$ is finite dimensional.*

Proof. First we prove (1) $\Longrightarrow$ (2). Assume that r is rational, that is, of the form (9.1), with

$$M(x) = \sum_{u=0}^{N_1} x^u M_u \quad \text{and} \quad m(x) = \sum_{u=0}^{N_2} x^u m_u.$$

Then

$$M(x) = D + xC(I - xA)^{-1}B,$$

where $D = M_0$, $N = N_1 m$, and

$$A = \begin{pmatrix} 0_m & I_m & 0_m & \cdots & \\ 0_m & 0_m & I_m & 0_m & \cdots \\ & \vdots & & & \vdots \\ 0_m & \cdots & \cdots & 0_m & I_m \\ 0_m & 0_m & \cdots & 0_m & 0_m \end{pmatrix},$$

$$B = \begin{pmatrix} 0_m \\ 0_m \\ \vdots \\ I_m \end{pmatrix}, \quad C = \begin{pmatrix} M_{N_1} & M_{N_1-1} & \cdots & M_1 \end{pmatrix}.$$

Similarly we obtain a realization $m(x) = d + xc(I - xa)^{-1}b$ for $m(x)$. By hypothesis $m_0 \neq 0$. We thus obtain a realization for $\frac{M}{m}$ using formulas (9.4) and(9.5).

Items (2) and (3) are clearly equivalent, in view of the formula

$$(I - xA)^{-1} = \sum_{u=0}^{\infty} x^u A^u,$$

which will hold at least in a neighborhood of the origin. To prove the implication $(2) \Longrightarrow (4)$ we note that

$$(R_0^t r)(x) = C(I_N - xA)^{-1}A^{t-1}B, \quad t = 1,2,\ldots$$

Thus $\mathscr{M}(r)$ is included in the linear span of the columns of the function $x \mapsto C(I_N - xA)^{-1}$, and is in particular finite dimensional.

Assume now that (4) is in force. Then there exists an integer $m_0 \in \mathbb{N}$ such that for every $v \in \mathbb{H}^q$, there exist vectors $u_0, \ldots, u_{m_0-1}$ such that

$$R_0^{m_0} r v = \sum_{m=0}^{m_0-1} R_0^m r u_m. \tag{9.8}$$

Of course, the u_j need not be unique. Let E denote the $\mathbb{H}^{p\times m_0 q}$-valued slice hyperholomorphic function

$$E = \begin{pmatrix} R_0 r & R_0^2 r & \ldots & R_0^{m_0} r \end{pmatrix}.$$

Then, in view of (9.8), there exists a matrix $A \in \mathbb{H}^{m_0 q \times m_0 q}$ such that

$$R_0 E = EA,$$

so that

$$E(x) - E(0) = xE(x)A,$$

and so

$$E(x) = E(0)(I - xA)^{-1}$$

and

$$(R_0 r)(x) = E(x) \begin{pmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus we have

$$r(x) - r(0) = xE(0)(I - xA)^{-1} \begin{pmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

It follows that r is of the form (9.7), and so (2) holds.

Finally the implication $(2) \Longrightarrow (1)$ follows from Proposition 9.1.3 applied to $(I - xA)^{-1}$. □

The following direct corollary will be used in Section 10.6 devoted to first order discrete linear systems.

Corollary 9.1.9. *Let* $(c,a,b) \in \mathbb{H}^{1\times N} \times \mathbb{H}^{N\times N} \times \mathbb{H}^{N\times 1}$*, and let*

$$s_{-\ell} = ca^{\ell}b, \quad \ell = 0,1,\dots. \tag{9.9}$$

Then, the function $\sum_{u=0}^{\infty} x^u ca^u b$ *is rational.*

Definition 9.1.10. Under the hypothesis and with the notations of Theorem 9.1.8, we define the linear operators

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathscr{M}(r) \times \mathbb{H}^m \longrightarrow \mathscr{M}(r) \times \mathbb{H}^n$$

by

$$\begin{aligned} Af &= R_0 f, && f \in \mathscr{M}(r),\\ Bb &= R_0 rb, && f \in \mathscr{M}(r),\\ Cf &= f(0), && f \in \mathscr{M}(r),\\ D &= r(0), && f \in \mathscr{M}(r). \end{aligned} \tag{9.10}$$

Equation (9.7) holds with this choice of (A,B,C,D), and so (2) holds, and (9.10) is then called the backward-shift realization of the function r.

In the setting of complex numbers the backward-shift realization plays a key role. See for instance [179, 180].

9.2 Minimal realizations

Realizations are useful in particular when they are, in some sense, unique. As in the complex case, one can introduce the notion of minimality, and the corresponding result on uniqueness of a minimal realization up to a similarity matrix. The interpretation of minimality in terms of local degrees of poles is still missing.

Definition 9.2.1. We say that the triple of matrices $(C,A,B) \in \mathbb{H}^{n\times N} \times \mathbb{H}^{N\times N} \times \mathbb{H}^{N\times m}$ is minimal if the following two conditions hold: The pair (C,A) is observable, meaning that

$$\cap_{t=0}^{\infty} \ker CA^t = \{0\} \tag{9.11}$$

and the pair (A,B) is controllable, meaning that

$$\cap_{t=0}^{\infty} \ker B^* A^{*t} = \{0\}\,. \tag{9.12}$$

The number N is called the degree of the minimal realization.

The proof of the following lemma is easy and will be omitted.

Lemma 9.2.2. *Conditions* (9.11) *and* (9.12) *are equivalent to the following two conditions, where $f \in \mathbb{H}^N$ and Ω denotes some real neighborhood of the origin;*

$$C(I_N - xA)^{-1} f \equiv 0,\ x \in \Omega \quad \Longleftrightarrow \quad f = 0 \tag{9.13}$$

and

$$B^*(I_N - xA^*)^{-1} f \equiv 0,\ x \in \Omega \quad \Longleftrightarrow \quad f = 0. \tag{9.14}$$

Theorem 9.2.3. *The backward-shift realization is minimal.*

Proof. The formula

$$A^t Bb = R_0^t R_0 rb = R_0^{t+1} b, \quad t = 0,1,2, \quad b \in \mathbb{H}^m,$$

shows that $\cup_{t=0}^{\infty} \text{ran} A^t B = \mathscr{M}(r)$. On the other hand, if

$$f(x) = \sum_{t=0}^{\infty} f_t x^t \in \mathscr{M}(r),$$

we have

$$CA^t f = f_t, \quad t = 0,1,\ldots$$

and so $\cap_{t=0}^{\infty} \ker CA^t = \{0\}$. $\square$

Theorem 9.2.4. *Let r be a $\mathbb{H}^{n\times m}$-valued function of a real variable defined in a neighborhood of the origin. Then r admits a minimal realization, and two minimal realizations have the same degree, and are equal up to an invertible similarity matrix.*

Proof. The first claim follows from Theorem 9.2.3. The proof of the second claim is divided in a number of steps. Let thus $r(x) = D_j + xC_j(I_{n_j} - xA_j)^{-1}B_j,\ j = 1,2$ be two minimal realizations of r. Then:

STEP 1: *It holds that*

$$\begin{aligned}\frac{r(x)-r(y)}{x-y} &= C_1(I_{n_1} - xA_1)^{-1}(I_{n_1} - yA_1)^{-1}B_1 \\ &= C_2(I_{n_2} - xA_2)^{-1}(I_{n_2} - yA_2)^{-1}B_2, \quad x,y \in (-\varepsilon,\varepsilon)\end{aligned} \tag{9.15}$$

for some $\varepsilon > 0$.

STEP 2: *One can define linear maps U and V from $\mathbb{H}^{n_1}$ into $\mathbb{H}^{n_2}$ via:*

$$U\left(\sum_{t=1}^{T}(I_{n_1} - y_t A_1)^{-1}B_1 a_t\right) = \sum_{t=1}^{T}(I_{n_2} - y_t A_2)^{-1}B_2 a_t, \tag{9.16}$$

and

$$V\left(\sum_{t=1}^{T}(I_{n_1} - y_t A_1^*)^{-1}C_1^* a_t\right) = \sum_{t=1}^{T}(I_{n_2} - y_t A_2^*)^{-1}C_2^* a_t.$$

A priori U and V are linear relations. They are everywhere defined in view of the conditions (9.13) and (9.14). In fact they are graphs of operators as we now prove. Assume $\sum_{t=1}^{T}(I_{n_1}-y_tA_1)^{-1}B_1a_t=0$; then

$$C_1(I_{n_1}-xA_1)^{-1}\left(\sum_{t=1}^{T}(I_{n_1}-y_tA_1)^{-1}B_1a_t\right)\equiv 0,$$

and so, by (9.15),

$$C_2(I_{n_2}-xA_2)^{-1}\left(\sum_{t=1}^{T}(I_{n_2}-y_tA_2)^{-1}B_2a_t\right)\equiv 0,$$

so that $\sum_{t=1}^{T}(I_{n_1}-y_tA_2)^{-1}B_2a_t=0$, and similarly for U.

STEP 3: *The map U^* and V^* satisfy*

$$U^*\left((I_{n_2}-yA_2^*)^{-1}C_2^*\right) = (I_{n_1}-yA_1^*)^{-1}C_1^*, \tag{9.17}$$

$$V^*\left((I_{n_2}-yA_2)^{-1}B_2\right) = (I_{n_1}-yA_1)^{-1}B_1. \tag{9.18}$$

Indeed, by definition of the adjoint we have

$$\begin{aligned}
\langle U^*\left((I_{n_2}-yA_2^*)^{-1}C_2^*b\right),(I_{n_1}-xA_1)^{-1}B_1c\rangle &=\\
&=\langle (I_{N_2}-yA_2^*)^{-1}C_2^*b,U\left((I_{n_1}-xA_1)^{-1}B_1c\right)\rangle\\
&=\langle (I_{n_2}-yA_2^*)^{-1}C_2^*b,(I_{n_2}-xA_2)^{-1}B_2c\rangle\\
&=\left(b^*\frac{R(x)-R(y)}{x-y}c\right)^*\\
&=\langle (I_{n_1}-yA_1^*)^{-1}C_1^*b,(I_{n_1}-xA_1)^{-1}B_1c\rangle,
\end{aligned}$$

and hence the result. Similarly,

$$\begin{aligned}
\langle V^*\left((I_{n_2}-yA_2)^{-1}B_2b\right),(I_{n_1}-xA_1^*)^{-1}C_1c\rangle &=\\
&=\langle (I_{n_2}-yA_2)^{-1}B_2b,V\left((I_{n_1}-xA_1^*)^{-1}C_1c\right)\rangle\\
&=\langle (I_{n_2}-yA_2)^{-1}B_2b,(I_{n_2}-xA_2^*)^{-1}C_2^*c\rangle\\
&=c^*\frac{R(x)-R(y)}{x-y}b\\
&=\langle (I_{n_1}-yA_1)^{-1}B_1b,(I_{n_1}-xA_1^*)^{-1}C_1^*c\rangle,
\end{aligned}$$

and hence the formula for V^*.

STEP 4: $n_1 = n_2 \overset{\text{def.}}{=} n$ *and* $V^*U = I_n$.

Indeed, by definition of U and V^* we get from (9.18)

$$V^*U\left(\sum_{t=1}^{T}(I_{n_1} - y_tA_1)^{-1}B_1a_t\right) = \sum_{t=1}^{T}(I_{n_1} - y_tA_1)^{-1}B_1a_t, \tag{9.19}$$

and similarly, from the definition of V and U^* we get from (9.17) that

$$VU^*\left(\sum_{t=1}^{T}(I_{n_1} - y_tA_2^*)^{-1}C_2^*a_t\right) = \sum_{t=1}^{T}(I_{n_1} - y_tA_2^*)^{-1}C_2^*a_t. \tag{9.20}$$

By the observability of the pair (C_1, A_1) and the controllability of the pair (A_1, B_1) we have $V^*U = I_{n_1}$ and $VU^* = I_{n_2}$, and so $n_1 = n_2$, and the maps U and V^* are inverse to each other.

STEP 5: *Two minimal realizations are similar.*

The operator U satisfies $UB_1 = B_2$ by construction and similarly V^* satisfies $C_1V^* = C_2$. From (9.15) we have for $u, v \in \mathbb{N}_0$

$$C_2A_2^uA_2A_2^vB_2 = C_1A_1^uA_1A_1^vB_1,$$

and similarly, by definition of U and V we have

$$C_1A_1^uV^*A_2UA_1^vB_1 = C_1A_1^uA_1A_1^vB_1,$$

and hence $U^{-1}A_2U = A_1$. □

The following two corollaries are proved as in the classical case.

Corollary 9.2.5. *Let r be a $\mathbb{H}^{n\times n}$-valued function of a real variable defined in a neighborhood of the origin, let*

$$r(x) = D + xC(I_N - xA)^{-1}B,$$

be a minimal realization of r, and assume D invertible. Then (9.4) *is a minimal realization of r^{-1}.*

Corollary 9.2.6. *Let r be a $\mathbb{H}^{n\times n}$-valued function of a real variable defined in a neighborhood of the origin, and let*

$$r(x) = D + xC(I_N - xA)^{-1}B,$$

be a minimal realization of r. Then,

$$r(x)^* = D^* + xB^*(I - xA^*)^{-1}C^* \tag{9.21}$$

is a minimal realization of the function $x \mapsto r(x)^*$.

To conclude this section we connect the real points where r is defined and the intersection of the spectrum of A with the real line.

Theorem 9.2.7. *Let r be a $\mathbb{H}^{n\times n}$-valued function of a real variable defined in a real neighborhood of the origin, and let*

$$r(x) = D + xC(I_N - xA)^{-1}B$$

be a minimal realization of r. Then, r is defined at the real point x if and only if $x = 0$ or $1/x \notin \sigma_S(A) \cap \mathbb{R}$.

Proof. Without loss of generality we can assume that A is in Jordan form,

$$A = \operatorname{diag}(J_{s_1}(\lambda_1), J_{s_2}(\lambda_2), \ldots, J_{s_m}(\lambda_m)),$$

where, for instance $J_{s_1}(\lambda_1)$, denotes the Jordan block of size s_1 associated to the eigenvalue λ_1. See Theorem 4.3.21. Then, with

$$C = \begin{pmatrix} C_1 & C_2 & \ldots & C_m \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}$$

the corresponding block matrix decompositions of C and B. Let $x_1, \ldots, x_v$ be the real eigenvalues of A, possibly repeated. We can write:

$$r(x) = D + \sum_{u=1}^{v} \left(\sum_{t=0}^{u} \frac{C_u N_u^t B_u}{(x - x_u)^{t+1}} \right) + s(x),$$

where $s(x)$ is the part corresponding to the standard nonreal eigenvalues. Assume that r is defined at a point x_u. This forces all corresponding matrices $C_u N_u^t B_u$ (possibly coming from different Jordan blocks associated to the given eigenvalue) to be 0, and this contradicts the minimality condition. $\square$

9.3 Realizations of unitary rational functions

In the complex setting, rational functions which take unitary values (with respect to a possibly indefinite metric) on the imaginary line or the unit circle play an important role in the theory of linear systems, in interpolation theory and related topics. We begin by considering the case of a real variable. The symmetries (9.22) (purely imaginary space) and (9.25) (unit ball) are motivated by the quaternionic setting.

In the complex case, the realizations of unitary rational functions were studied in particular in [184] and in [57]. As in these works, the strategy here is to use the uniqueness, up to similarity, of the minimal realization. We begin by the purely imaginary space case.

Theorem 9.3.1. *Let $J \in \mathbb{H}^{n\times n}$ be a signature matrix and let Θ be a rational $\mathbb{H}^{n\times n}$-valued function defined at the origin, and with minimal realization $\Theta(x) = D + xC(I - xA)^{-1}B$. Then, Θ satisfies*

$$\Theta(x)J\Theta(-x)^* = J \tag{9.22}$$

at the real points where it is defined, if and only if D is J unitary (that is, $DJD^* = J$) and there exists an Hermitian matrix H such that

$$\begin{aligned} HA + A^*H &= C^*JC, \\ C &= DJB^*H. \end{aligned} \tag{9.23}$$

When these conditions are in force it holds that

$$\Theta(x) = (I_n + xC(I_N - xA)^{-1}H^{-1}C^*J)D. \tag{9.24}$$

Proof. Let Ω be the open subset of the real line where Θ is defined. Setting $x = 0$ in (9.22) we obtain that $DJD^* = J$, and the quaternionic matrix D is invertible. Thus we can rewrite (9.22) as

$$\Theta(x) = J\Theta(-x)^{-*}J, \quad x \in \Omega,$$

and thus

$$D + xC(I - xA)^{-1}B = J(D^{-*} + xD^{-*}B^*(I + x(A - BD^{-1}C)^*)^{-*}C^*D^{-*})J, \quad x \in \Omega.$$

The two sides of the above equality are minimal realizations of the same rational $\mathbb{H}^{n\times n}$-valued function of the real variable x. They are thus similar and there exists a uniquely defined matrix H such that

$$\begin{pmatrix} H & 0 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -(A - BD^{-1}C)^* & C^*D^{-*}J \\ JD^{-*}B^* & JD^{-*}J \end{pmatrix}\begin{pmatrix} H & 0 \\ 0 & I_n \end{pmatrix}.$$

These equations can be rewritten as

$$\begin{aligned} HA &= -(A^* - C^*D^{-*}B^*)H, \\ HB &= C^*D^{-*}J, \\ C &= JD^{-*}B^*H, \\ D &= JD^{-*}J, \end{aligned}$$

or, equivalently,

$$\begin{aligned} HA + A^*H &= C^*JC, \\ C &= DJB^*H, \\ DJD^* &= J. \end{aligned}$$

These equations are also satisfied by H^* and so $H = H^*$ by uniqueness of the similarity matrix.

Conversely, if H is a solution of (9.23), one obtains (9.24) and a direct computation shows that

$$\frac{J-\Theta(x)J\Theta(y)^*}{x+y}=C(I_N-xA)^{-1}H^{-1}(I_N-yA)^{-*}C^*,\quad x,y\in\Omega,$$

and in particular (9.22) is in force. □

Now we turn to the unit ball case.

Theorem 9.3.2. *Let $J\in\mathbb{H}^{n\times n}$ be a signature matrix and let Θ be a rational $\mathbb{H}^{n\times n}$-valued function defined and invertible at the origin, and with minimal realization $\Theta(x)=D+xC(I-xA)^{-1}B$. Then, Θ satisfies*

$$\Theta(x)J\Theta(1/x)^*=J \tag{9.25}$$

(at the real points where it is defined) if and only if there exists an invertible Hermitian matrix H such that

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}^*\begin{pmatrix}H & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}H & 0\\ 0 & J\end{pmatrix}. \tag{9.26}$$

Proof. We rewrite (9.25) as

$$D^*+B^*(xI_N-A^*)^{-1}C^*=J(D^{-1}-D^{-1}C(xI_N-A^\times)^{-1}BD^{-1})J.$$

Since $\Theta(x)$ is defined and invertible at the origin condition (9.25) forces $\Theta(1/x)$ to be defined at the origin too. Since

$$\Theta(1/x)=D+C(xI_N-A)^{-1}B,$$

Theorem 9.2.7 implies that A is invertible. Hence

$$D^*+B^*(xI_N-A^*)^{-1}C^*=D^*-B^*A^{-*}C^*+xB^*A^{-*}(xI_N-A^*)^{-1}C^*,$$

and so

$$D^*-B^*A^{-*}C^*+xB^*A^{-*}(xI_N-A^*)^{-1}C^*=J(D^{-1}-D^{-1}C(xI_N-A^\times)^{-1}BD^{-1})J.$$

This in turn is an equality between two minimal realizations of a given rational matrix-valued function. The result then follows by proceeding as in [57, Theorem 3.1, p.197]. □

Remark 9.3.3. We note the formula (see also (8.18))

$$\frac{J-\Theta(x)J\Theta(y)^*}{1-xy}=C(I-xA)^{-1}H^{-1}(I-yA)^{-*}C^*.$$

Remark 9.3.4. We note that in fact the hypothesis of invertibility at the origin can be removed. To this end, one uses the results of Section 8.3, and note that as it follows from the formula

$$\frac{J-\Theta(x)J\Theta(y)^*}{1-xy}=-\frac{1}{y}R_{1/y}\Theta(x)(\Theta(1/y))^*,\quad y\neq 0,$$

the space $\mathscr{P}(\Theta)$ (or more precisely, the restrictions of its functions to the real line) is finite dimensional.

9.4 Rational slice hyperholomorphic functions

We now lift to the slice hyperholomorphic setting the results of the previous three sections.

Definition 9.4.1. The $\mathbb{H}^{n\times m}$-valued function r slice hyperholomorphic in an axially symmetric s-domain will be called rational if its restriction to the real line is rational in the sense of Definition 9.1.1.

Theorem 9.4.2. *Let r be a rational matrix-valued function defined at the origin, and with minimal realization $D+pC\star(I-pA)^{-\star}B$, and let $[p_1],\ldots,[p_N]$ be the spheres corresponding to the poles of r. Then*

$$\sigma_S(A)=\left\{\left[p_1^{-1}\right],\ldots,\left[p_N^{-1}\right]\right\}$$

Proof. We use as minimal realization the backward-shift realization. The elements of the state space $\mathscr{M}(r)$ are slice hyperholomorphic in $\mathbb{H}\setminus\{[p_1],\ldots,[p_N]\}$. The space is R_0-invariant, and so its eigenfunctions are of the form

$$f\star(1-pa)^{-\star}. \tag{9.27}$$

By the definition of $\mathscr{M}(r)$, (9.27) is a finite linear combinations of elements of the form $R_0^n rc$. It follows that, for $a\neq 0$, we have $\left[a^{-1}\right]\in\{[p_1],\ldots,[p_N]\}$.
Conversely, $\left\{\left[p_1^{-1}\right],\ldots,\left[p_N^{-1}\right]\right\}\subset\sigma_S(A)$ from the realization formula. □

Corollary 9.4.3. *In the setting of the previous theorem assume r slice hyperholomorphic in the closed ball. Then, $r_S(A)<1$ and the function $t\mapsto r(tp)$ is real analytic for all p on the unit sphere in a neighborhood of* 1.

We now turn to conditions (9.22) and (9.25).

Definition 9.4.4. Let $J\in\mathbb{H}^{n\times n}$ be a signature matrix. A rational $\mathbb{H}^{n\times n}$-valued function is called J-unitary on the purely imaginary quaternions if

$$\Theta(p)\star J\star\Theta^c(-p)=J. \tag{9.28}$$

Note that (9.28) is the slice hyperholomorphic extension of (9.22).

Definition 9.4.5. Let $J\in\mathbb{H}^{n\times n}$ be a signature matrix. A rational $\mathbb{H}^{n\times n}$-valued function is called J-unitary on the unit ball if

$$\Theta(p)\star J\star\Theta^c(1/p)=J. \tag{9.29}$$

Note that (9.29) is the slice hyperholomorphic extension of (9.25).
Let Θ be a slice hyperholomorphic $\mathbb{H}^{n\times n}$-valued rational function, with domain of definition $\Omega(\Theta)\subset\mathbb{H}$. The counterpart of the kernel K_Θ is now

$$K_\Theta(p,q)=\sum_{t=0}^{\infty}p^t(J-\Theta(p)J\Theta(q)^*)\overline{q}^t,$$

but the counterpart of (2.14) is not so clear, as illustrated by the following example (see [20] and [39]).

Example. The function

$$\Theta(p) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\star\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}p & i\\ pi & 1\end{pmatrix} \tag{9.30}$$

does not take unitary values on the unit sphere.
Indeed for p of modulus 1 we have:

$$\Theta(p)\Theta(p)^* = \begin{pmatrix}1 & \frac{i-pi\overline{p}}{2}\\ \frac{pi\overline{p}-i}{2} & 1\end{pmatrix},$$

and

$$\Theta(j)\Theta(j)^* = \begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$$

is not a contraction (its eigenvalues are 0 and 2), let alone a unitary matrix. On the other hand, the associated space $\mathscr{P}(\Theta)$ is finite dimensional since

$$K_\Theta(p,q) = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}.$$

We note that the positivity of the kernel $K_\Theta(p,q)$ does not imply the condition

$$J - \Theta(p)J\Theta(p)^* \geq 0 \quad \forall p \in \mathbb{B}_1 \quad \text{where } \Theta \text{ is defined.} \tag{9.31}$$

This is because the entries of the matrix $\Theta(p)J\Theta(p)^*$ need not be real and so will not commute with the factors p^t and $\overline{p}^t$. This is illustrated by the above example.
The scalar case, that is $J = 1$, is of special importance. Then $\Theta(p)J\Theta(p)^* = |\Theta(p)|^2 \in \mathbb{R}$ and the positivity of K_Θ and (9.31) are equivalent.

The next theorem characterizes slice-hyperholomorphic functions J-unitary (in an appropriate sense) on the unit sphere or on the real axis. In [57] a proof was given of this result (in the complex variable setting) under the hypothesis that Θ is invertible at infinity and at the origin. The operator A is then invertible because the chosen realization is minimal. Another proof was given in [52] without these hypothesis, using reproducing kernel Hilbert spaces. This approach is still applicable in our setting and is the one we take in the proof of the theorem. The standard computations follow [13, Exercise 7.7.16].

Theorem 9.4.6. *Let $(C,A) \in \mathbb{H}^{n\times N} \times \mathbb{H}^{N\times N}$ be an observable pair of matrices, and assume that $1 \in \rho_S(A)$. Let $J \in \mathbb{H}^{n\times n}$ be a signature matrix, and let P denote an invertible Hermitian matrix. Let $\mathscr{M}$ denote the linear space spanned by the columns of the matrix function $F(p) = C\star(I_N - pA)^{-\star}$, and endow $\mathscr{M}$ with the inner product*

$$[F(\cdot)\xi, F(\cdot)\eta]_P = \eta^* P\xi.$$

Then there exists a rational function Θ such that $\mathscr{M} = \mathscr{P}(\Theta)$ if and only if P is solution of the Stein equation (4.16).
The function Θ is then given by the formula (up to a right multiplicative J-unitary constant)

$$\Theta(p) = I_n - (1-p)\star C\star(I_N - pA)^{-\star}P^{-1}(I_N - A^*)^{-1}C^*J. \tag{9.32}$$

Proof. We consider real p and q. The result follows by slice hyperholomorphic extension. With $F(p) = C(I_N - pA)^{-\star}$ and $x, y \in \mathbb{R}$ such that $(I_N - xA)$ and$(I_N - yA)$ are invertible we can write

$$\begin{aligned} J - \Theta(x)J\Theta(y)^* &= (1-x)F(x)P^{-1}F(1)^* + (1-y)F(1)P^{-1}F(y)^* - \\ &\quad - (1-x)(1-y)F(x)P^{-1}F(1)^*JF(1)P^{-1}F(y)^* \\ &= F(x)P^{-1}(I_N - A)^{-*} \times \\ &\quad \times \{ \ (1-x)(I_N - yA)^*P(I_N - A) + \\ &\qquad + (1-y)(I_N - A)^*P(I_N - xA) - \\ &\qquad - (1-x)(1-y)C^*JC \ \} \times \\ &\quad \times (I_N - A)^{-1}P^{-1}F(y)^*. \end{aligned}$$

In view of the observability of the pair (C,A), the reproducing kernel formula

$$\frac{J - \Theta(x)J\Theta(y)^*}{(1-xy)} = F(x)P^{-1}F(y)^*$$

will hold if and only if we have

$$\begin{aligned} (1-xy)(I_N - A)^*P(I_N - A) = (1-x)&(I_N - yA)^*P(I_N - A) + \\ &+ (1-y)(I_N - A)^*P(I_N - xA) - \\ &- (1-x)(1-y)C^*JC, \end{aligned}$$

or equivalently

$$\begin{aligned} (1-xy)(I_N - A)^*P(I_N - A) = (1-x)&(I_N - yA)^*P(I_N - A) + \\ &+ (1-y)(I_N - A)^*P(I_N - xA) - \\ &+ (1-x)(1-y)(P + A^*PA) + \\ &+ (1-x)(1-y)(C^*JC - P - A^*PA). \end{aligned}$$

But it holds that:

$$\begin{aligned} (1-xy)(I_N - A)^*P(I_N - A) = (1-x)&(I_N - yA)^*P(I_N - A) + \\ &+ (1-y)(I_N - A)^*P(I_N - xA) - \\ &+ (1-x)(1-y)(P + A^*PA). \end{aligned}$$

Thus we get the formula

$$\begin{aligned} \frac{J - \Theta(x)J\Theta(y)^*}{(1-xy)} &= F(x)P^{-1}F(y)^* + \\ &+ \frac{(1-x)(1-y)}{1-xy}F(x)P^{-1}(I_N - A)^{-*}\Delta(I_N - A)^{-1}P^{-1}F(y)^*, \end{aligned}$$

where we have set $\Delta = C^*JC - P - A^*PA$. Thus formula (2.21) is in force if and only if

$$F(x)P^{-1}(I_N - A)^{-*}\Delta(I_N - A)^{-1}P^{-1}F(y)^* \equiv 0.$$

Since the pair (C,A) is observable, this will be the case if and only if $\Delta = 0$, that is, if and only if the assumed Stein equation holds.

□

Corollary 9.4.7. *Let (C,A) be an observable pair of matrices and assume that there exists an invertible Hermitian solution to the equation*

$$P - A^*PA = C^*JC. \tag{9.33}$$

Assume moreover that $1 \in \rho_S(A)$. Then, the function

$$\Theta(p) = I + (1-p)\star C\star(I - pA)^{-\star}P(I-A)^{-*}C^*J \tag{9.34}$$

is rational and satisfies (9.25).

For future use we need (see the proof of Theorem 10.1.8):

Proposition 9.4.8. *Assume $P > 0$. Then D^{-1} is invertible at the point $p = 0$.*

Proof. We have

$$\sum_{u=0}^{\infty} p^n \left(J - \Theta(p)J\Theta(p)^*\right)\overline{p}^n \geq 0.$$

For $p = 0$ we have

$$J \geq \Theta(0)J\Theta(0)^*$$

from which we get

$$I + C(0)C(0)^* \geq D(0)D(0)^*$$

and hence the result. □

An important example of such function, in the setting of discrete systems, is given in Theorem 10.7.5.

For a signature matrix $J \in \mathbb{H}^{n\times n}$ we first define $\mathrm{H}^2(J,\mathbb{B})$ to be the space $(\mathrm{H}^2(\mathbb{B}))^n$ endowed with the possibly indefinite inner product

$$[f,g] = \langle f, Jg\rangle_{(\mathrm{H}^2(\mathbb{B}))^n}.$$

Setting

$$k_J(p,q) = \sum_{u=0}^{\infty} p^u J\overline{q}^u,$$

we note that

$$[f, k_J(\cdot,q)\xi] = \xi^* f(q), \quad q \in \mathbb{B}_1, \xi \in \mathbb{H}^n.$$

When J has real entries, we futhermore have

$$k_J(p,q) = (1 - p\overline{q})^{-\star}J.$$

Proposition 9.4.9. *Let Θ be a rational function J-unitary on the unit sphere, and let (C,A) be an observable pair such that (the finite dimensional Pontryagin space) $\mathscr{P}(\Theta)$ is spanned by the columns of the matrix-function $C\star(I-pA)^{-\star}$. Then $\mathscr{P}(\Theta)$ is isometrically included in $\mathrm{H}^2(J,\mathbb{B})$ if and only if $\sigma_S(A)\subset\mathbb{B}$.*

Proof. Assume that $\mathscr{P}(\Theta)$ is included isometrically in $\mathrm{H}^2(J,\mathbb{B})$. In particular, it is included (as a set) inside $(\mathrm{H}^2(\mathbb{B}))^n$, where n is the size of J ($J\in\mathbb{R}^{n\times n}$). Since near the origin an element of $\mathscr{P}(\Theta)$ can be written as

$$F(p)\xi=\sum_{u=0}^{\infty}p^uCA^u\xi,\quad\text{with}\quad F(p)=C\star(I-pA)^{-\star},$$

for some $\xi\in\mathbb{H}^n$ it follows that

$$\sum_{u=0}^{\infty}\xi^*A^{*u}C^*CA^u\xi<\infty,\quad\forall\xi\in\mathbb{H}^n.\tag{9.35}$$

Let now $\lambda\in\sigma_S(A)$ and let $\xi\neq0$ be such that $A\xi=\xi\lambda$. We have $A^u\xi=\xi\lambda^u$ (with $u\in\mathbb{N}$) and so $C\xi\in\cap_{u=0}^{\infty}\ker CA^u$. Since the pair (C,A) is observable we have $C\xi\neq0$. Hence (9.35) can be rewritten as

$$\sum_{u=0}^{\infty}|\lambda|^{2u}\|C\xi\|^2<\infty$$

and so $\lambda\in\mathbb{B}$. Hence $\mathscr{P}(\Theta)\subset\mathrm{H}^2(J,\mathbb{B})$. But now the series

$$P=\sum_{u=0}^{\infty}A^{*u}C^*JCA^u$$

converges absolutely, and we have

$$\begin{aligned}[F(p)\xi,F(p)\eta]_{\mathscr{P}(\Theta)}&=\eta^*P\xi\\&=\sum_{u=0}^{\infty}\eta^*A^{*u}C^*JCA^u\xi\\&=[F(p)\xi,F(p)\eta]_{\mathrm{H}^2(J,\mathbb{B})}.\end{aligned}$$

The converse is clear and uses, for instance, (4.4.11). □

Theorem 9.4.10. *Let Θ be a rational function J-unitary on the unit sphere, and let (C,A) be an observable pair such that (the finite dimensional Pontryagin space) $\mathscr{P}(\Theta)$ is spanned by the columns of the matrix-function $C\star(I-pA)^{-\star}$. Then the operator M_Θ of $\star$ multiplication by Θ is an isometry from $\mathrm{H}^2(J,\mathbb{B})$ into itself if and only if $\sigma_S(A)\subset\mathbb{B}$. When this condition is in force we have*

$$\mathscr{P}(\Theta)=\operatorname{ran}(J-M_\Theta JM_\Theta^*)=\mathrm{H}^2(J,\mathbb{B})\ominus M_\Theta\mathrm{H}^2(J,\mathbb{B}).$$

Proof. Assume first that $\sigma_S(A) \subset \mathbb{B}$. By the previous theorem $\mathscr{P}(\Theta)$ is isometrically included in $\mathrm{H}^2(J,\mathbb{B})$. Moreovoer Θ (given by (9.34)) is bounded in norm in $\mathbb{B}$, and so the operator M_Θ is bounded from $(\mathrm{H}^2(\mathbb{B}))^n$ into itself (see Corollary 6.2.8). Thus, by Theorem 7.5.2,

$$\left(M_\Theta^* \left(\xi \star (1-\cdot\overline{q})^{-\star}\right)\right)(p) = \sum_{u=0}^{\infty} \Theta(q)^* \overline{q}^u \xi,$$

and

$$K_\Theta(p,q)\xi = \left((J - M_\Theta J M_\Theta^*)\left(\xi \star (1-\cdot\overline{q})^{-\star}\right)\right)(p)$$

and so $\mathscr{P}(\Theta) \subset \operatorname{ran}(J - M_\Theta J M_\Theta^*) \subset \mathrm{H}^2(J,\mathbb{B})$. We set

$$\Gamma = J - M_\Theta J M_\Theta^*.$$

We have

$$\begin{aligned}\eta^* K(p,q)\xi &= \langle K_\Theta(\cdot q)\xi, K_\Theta(\cdot,p)\eta\rangle_{\mathscr{P}(\Theta)} \\ &= [\Gamma\left(\xi \star (1-\cdot\overline{q})^{-\star}\right), \left(J\eta \star (1-\cdot\overline{p})^{-\star}\right)]_{\mathrm{H}^2(J,\mathbb{B})}\end{aligned}$$

(by Cauchy's formula)

$$= [\Gamma\left(\xi \star (1-\cdot\overline{q})^{-\star}\right), \Gamma\left(J\eta \star (1-\cdot\overline{p})^{-\star}\right)]_{\mathrm{H}^2(J,\mathbb{B})}$$

since $\mathscr{P}(\Theta)$ is isometrically included in $\mathrm{H}^2(J,\mathbb{B})$, and so we have $\Gamma^2 = \Gamma$ and every element in $\mathrm{H}^2(J,\mathbb{B})$ can be written as an orthogonal sum

$$f = \Gamma f + (I-\Gamma)f.$$

It follows that $\mathrm{H}^2(J,\mathbb{B}) = \mathscr{P}(\Theta) \oplus M_\Theta \mathrm{H}^2(J,\mathbb{B})$. ☐

As a consequence we have the following result:

Theorem 9.4.11. *Let $b \in \mathcal{S}$ be such that the kernel K_b is positive definite and assume that the associated reproducing kernel Hilbert space is finite dimensional. Then, b is a finite Blaschke product.*

Proof. Let n be the dimension of $\mathscr{H}(b)$ and let $(C,A) \in \mathbb{H}^{1\times n} \times \mathbb{H}^{n\times n}$ be a controllable pair such that the entries of $C \star (I_n - pA)^{-\star}$ form a basis of $\mathscr{H}(b)$. Then $\sigma_S(A) \subset \mathbb{B}$ since $\mathscr{H}(b) \subset \mathrm{H}^2(\mathbb{B})$ (a priori contractively). By the previous theorem we have

$$\mathscr{H}(b) = \mathrm{H}^2(\mathbb{B}) \ominus M_b \mathrm{H}^2(\mathbb{B}).$$

Let now $f(p) = (1-p\overline{a})^{-\star}$ be an eigenvector of R_0 with eigenvalue $a \in \mathbb{B}$. Using Theorem 6.3.7 we have that the span of f in $\mathrm{H}^2(\mathbb{B})$ is a $\mathscr{H}(b_a)$ space, where b_a is the Blaschke factor at a. Because of the isometric inclusion the function $b_a^{-\star} \star b$ is still a Schur function, and the associated reproducing kernel Hilbert space has dimension $n-1$. We get the result by iterating. ☐

9.5 Linear fractional transformation

Linear fractional transformations and interpolation problems originated with the analysis of the previous century and even earlier; we have in mind here the Schur algorithm and the associated solutions of the Carathéodory-Fejér interpolation problem (see Theorem 10.4.1) and the results of Stieltjes on moment problems (see []). The connection between the underlying reproducing kernel spaces seems to have been noticed at a much later stage in the work of de Branges and Rovnyak; see [105, Theorem 13, p.305], [156, §31, Theorem 31]. The following result is the quaternionic version of the result of de Branges and Rovnyak, when the linear fractional transformation coefficients are moreover assumed rational. For the next result it is useful to recall that $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Theorem 9.5.1. *Let Θ be a rational J_0-inner function, defined at $p = 1$ and such that $\Theta(1) = I_2$, and let s be a Schur function. Then, there is a Schur function e such that*

$$s(p) = (a(p)\star e(p) + b(p))\star(c(p)\star e(p) + d(p))^{-\star} \tag{9.36}$$

if and only if the map $M_{\begin{pmatrix} 1 & -s \end{pmatrix}}$ is a contraction from $\mathscr{H}(\Theta)$ into $\mathscr{H}(s)$.

Proof. Using the first item in Proposition 7.5.2 we obtain

$$\left(M^*_{\begin{pmatrix} 1 & -s \end{pmatrix}} K_s(\cdot, q)\right)(p) = \sum_{t=0}^{\infty} p^n \left(\begin{pmatrix} 1 \\ -s(q) \end{pmatrix} - \Theta(p)J\Theta(q)^* \star_r \begin{pmatrix} 1 \\ -s(q) \end{pmatrix} \right) \overline{q}^t,$$

and so

$$\begin{aligned}
&(M_{\begin{pmatrix} 1 & -s \end{pmatrix}} M^*_{\begin{pmatrix} 1 & -s \end{pmatrix}} K_s(\cdot, q))(p) \\
&\qquad = K_s(p,q) - \sum_{t=0}^{\infty} p^t \left(\begin{pmatrix} 1 & -s(p) \end{pmatrix} \star \Theta(p)J\Theta(q)^* \star_r \begin{pmatrix} 1 \\ -s(q) \end{pmatrix} \right) \overline{q}^t \\
&\qquad \le K_s(p,q),
\end{aligned}$$

and therefore the kernel

$$\sum_{t=0}^{\infty} p^t \left(\begin{pmatrix} 1 & -s(p) \end{pmatrix} \star \Theta(p)J\Theta(q)^* \star_r \begin{pmatrix} 1 \\ -s(q) \end{pmatrix} \right) \overline{q}^t = \sum_{t=0}^{\infty} p^t \left(A(p)\overline{A(q)} - B(p)\overline{B(q)} \right) \overline{q}^t$$

is positive definite in $\mathbb{B}$, with

$$A(p) = (a - s\star c)(p) \quad \text{and} \quad B(p) = (b - s\star d)(p).$$

The point $p = 1$ is not an interpolation node, and so Θ is well defined at $p = 1$. From (10.42) we have

$$\Theta(1) = I_2 \tag{9.37}$$

and so $(a^{-1}c)(1)=0$. Since s is bounded by 1 in modulus in $\mathbb{B}_1$ it follows that $(a-s\star c)\not\equiv 0$. Thus $e=-(a-s\star c)^{-\star}\star(b-s\star d)$ is defined in $\mathbb{B}$, with the possible exception of spheres of poles. Since

$$\sum_{t=0}^{\infty} p^t\left(A(p)\overline{A(q)}-B(p)\overline{B(q)}\right)\overline{q}^t = A(p)\star\left\{\sum_{t=0}^{\infty} p^t(1-e(p)\overline{e(q)})\overline{q}^t\right\}\star_r\overline{A(q)},$$

we have from Proposition 7.5.8 that the kernel

$$K_e(p,q)=\sum_{t=0}^{\infty} p^t(1-e(p)\overline{e(q)})\overline{q}^t,$$

where

$$e=-(a-s\star c)^{-\star}\star(b-s\star d), \tag{9.38}$$

is positive definite in its domain of definition. By Theorem 8.4.4 the function e extends to a Schur function. From (9.38) we get $s\star(c\star e+d)=a\star e+b$. To conclude we remark that (9.37) implies that

$$(d^{-1}c)(1)=0.$$

Thus, in a way similar to the proof that $(a-s\star c)\not\equiv 0$, we have $c\star e+d\not\equiv 0$ and we get that s is of the form (9.36).
Conversely, assume that s is of the form (9.36). Then the formula

$$\begin{aligned} K_s(p,q) = &\begin{pmatrix}1 & -s(p)\end{pmatrix}\star K_\Theta(p,q)\star_r\begin{pmatrix}1\\ -\overline{s(q)}\end{pmatrix}+\\ &+(a-s\star c)(p)\star K_e(p,q)\star_r\overline{(a-s\star c)(q)}\end{aligned} \tag{9.39}$$

implies that $M_{\begin{pmatrix}1 & -s\end{pmatrix}}$ is a contraction from $\mathscr{H}(\Theta)$ into $\mathscr{H}(s)$. □

9.6 Backward-shift operators

In this section we study finite dimensional backward operators for backward shifts centered at the origin or at $p=1$. We will not give the proof of the first lemma.

Lemma 9.6.1. *A finite dimensional subspace (say of dimension N) $\mathscr{M}$ of $\mathbb{H}^n$-valued functions slice hyperholomorphic in a neighborhood of the origin is R_0-invariant if and only if it can be written as*

$$\mathscr{M}=\left\{F(p)\xi\ ;\ \xi\in\mathbb{H}^N\right\}$$

where $F(p)=C\star(I_N-pA)^{-\star}$, where $(C,A)\in\mathbb{H}^{n\times N}\times\mathbb{H}^{N\times N}$ is an observable pair of matrices.

We begin with some preliminary results and definitions.

Definition 9.6.2. Let f be a slice hyperholomorphic function in a neighborhood Ω of $p=1$, and let $f(p)=\sum_{t=0}^{\infty}(p-1)^t f_t$ be its power series expansion at $p=1$. We define

$$R_1 f(p)=\sum_{t=1}^{\infty}(p-1)^t f_t. \tag{9.40}$$

Denoting by ext the slice hyperholomorphic extension we have

$$R_1 f(p)=\mathrm{ext}\left(R_1 f|_{p=x}\right). \tag{9.41}$$

The following proposition is the counterpart of Lemma 9.6.1 when R_0 is replaced by R_1.

Proposition 9.6.3. *Let $\mathscr{N}$ be a finite dimensional space, of dimension d, of $\mathbb{H}^m$-valued functions, slice hyperholomorphic in a neighborhood of the point $p=1$ and R_1-invariant. Then $\mathscr{N}$ is spanned by the columns of $G\star(I_d-pT)^{-\star}$, where the matrices $(G,T)\in\mathbb{H}^{m\times d}\times\mathbb{H}^{d\times d}$ such that $\xi\in\mathbb{H}^d$,*

$$G\star(I_d-pT)^{-\star}\xi\equiv 0 \quad\Longrightarrow\quad \xi=0.$$

Proof. Let $F(p)$ be built from the columns of a basis of $\mathscr{N}$ and note that there exists $B\in\mathbb{H}^{d\times d}$ such that $R_1F=FB$. Restricting to $p=x$, where x is real, we have

$$\frac{F(x)-F(1)}{x-1}=F(x)B,$$

and so

$$F(x)(I_d+B-xB)=F(1). \tag{9.42}$$

We claim that I_d+B is invertible. Let $\xi\in\mathbb{H}^d$ be such that $B\xi=-\xi$. Then, (9.42) implies that

$$xF(x)\xi=F(1)\xi,\quad x\in(-1,1).$$

Thus $F(1)\xi=0$ (by setting $x=0$) and so $F(x)\xi=0$ and so $\xi=0$. Hence

$$F(x)=F(1)(I_d+B)^{-1}(I_d-xB(I_d+B)^{-1})^{-1},$$

and the result follows. □

Lemma 9.6.4. *Let $f(p)=F(p)\xi$ where $F(p)=C\star(I_N-pA)^{-\star}$ and $\xi\in\mathbb{H}^N$. Then*

$$R_1 f(p)=F(p)A(I_N-A)^{-1}\xi. \tag{9.43}$$

Proof. First of all, recall that

$$F(p)=C\star(I_N-pA)^{-\star}=(C-\bar{p}CA)(I_n-2\mathrm{Re}pA+|p|^2A^2)^{-1},$$

so

$$F(1)=(C-CA)(I_N-2A+A^2)^{-1}=C(I_N-A)^{-1}.$$

Let us compute

$$\begin{aligned}R_1f(p) &= (p-1)^{-1}(f(p)-f(1)) = (p-1)^{-1}(C\star(I_N-pA)^{-\star}\xi - C(I_N-A)^{-1}\xi)\\ &= C\star(p-1)^{-1}((I_N-pA)^{-\star}-(I_N-A)^{-1})\xi\\ &= C\star(p-1)^{-1}\star(I_N-pA)^{-\star}\star((I_N-A)-(I_N-pA))(I_N-A)^{-1}\xi\\ &= C\star(p-1)^{-1}\star(I_N-pA)^{-\star}\star(p-1)A(I_N-A)^{-1}\xi\\ &= C\star(I_N-pA)^{-\star}A(I_N-A)^{-1}\xi\\ &= F(p)A(I_N-A)^{-1}\xi.\end{aligned}$$

□

Using the notation of the preceding lemma we also state:

Lemma 9.6.5. *Let $\mathscr{M}$ be endowed with the inner product defined by the Stein equation* (9.33)*,*

$$P - A^*PA = C^*JC,$$

and let $f,g \in \mathscr{M}$. Then

$$[f,g]+[R_1f,g]+[f,R_1g] = g(1)^*Jf(1). \tag{9.44}$$

Proof. Let $f(p) = F(p)\xi$ and $g(p) = F(p)\eta$ with $\xi,\eta \in \mathbb{H}^N$. We have

$$f(1) = C(I_N-A)^{-1}\xi \quad \text{and} \quad g(1) = C(I_N-A)^{-1}\eta.$$

These equations together with (9.43) show that (9.44) is equivalent to

$$P + P(I_N-A)^{-1}A + A^*(I_N-A)^{-*}P = (I_N-A)^{-*}C^*JC(I_N-A).$$

Multiplying this equation by $I_N - A^*$ on the left and by $I_N - A$ on the right we get the equivalent equation (10.26). □

Remark 9.6.6. Equation (9.44) corresponds to a special case of a structural identity which characterizes $\mathscr{H}(\Theta)$ spaces in the complex setting. A corresponding identity in the half space case was first introduced by de Branges, see [103], and improved by Rovnyak [251]. Ball introduced the corresponding identity in the setting of the open unit disk and proved the corresponding structure theorem (see [80]). In addition, see, e.g., [55, p. 17] for further discussions on this topic.

Proposition 9.6.7. *Let a and b be slice-hyperholomorphic functions defined in a neighborhood of the point $p=1$ such that the product $a\star b$ is well defined. Then,*

$$R_1(a\star b)(p) = (R_1a(p))\,b(1) + (a\star R_1b)(p). \tag{9.45}$$

Proof. By the Identity Principle, see [144, Theorem 4.2.4] the equality holds if and only if it holds for the restrictions to a complex plane $\mathbb{C}_{\mathrm{I}}$, $\mathrm{I} \in \mathbb{S}$, i.e., using the notation in Section 2, if and only if

$$(R_1(a \star b))_{\mathrm{I}}(z) = (R_1 a(z))_{\mathrm{I}}\, b(1) + (a \star R_1 b)_{\mathrm{I}}(z), \quad z \in \mathbb{C}_{\mathrm{I}}. \tag{9.46}$$

Let $\mathrm{J} \in \mathbb{S}$ be such that J is orthogonal to I and assume that

$$a_{\mathrm{I}}(z) = F(z) + G(z)\mathrm{J}, \qquad b_{\mathrm{I}}(z) = H(z) + L(z)\mathrm{J}.$$

Let us compute the left-hand side of (9.46), using the fact that $(R_1(a \star b))_{\mathrm{I}}(z) = R_1((a \star b)_{\mathrm{I}})$ and formula (6.13):

$$\begin{aligned}
R_1((a\star b)_{\mathrm{I}}) &= R_1\left(F(z)H(z) - G(z)\overline{L(\bar{z})} + (G(z)\overline{H(\bar{z})} + F(z)L(z))\mathrm{J}\right)\\
&= (z-1)^{-1}\Big(F(z)H(z) - G(z)\overline{L(\bar{z})} + (G(z)\overline{H(\bar{z})} + F(z)L(z))\mathrm{J}\\
&\quad -F(1)H(1) + G(1)\overline{L(1)} - (G(1)\overline{H(1)} + F(1)L(1))\mathrm{J}\Big).
\end{aligned}$$

On the right hand side of (9.46) we have $(R_1 a(z))_{\mathrm{I}}\, b(1) = (R_1 a_{\mathrm{I}}(z))\, b(1)$ which can be written as

$$\begin{aligned}
(R_1 a_{\mathrm{I}}(z))b(1) &= \left((z-1)^{-1}(F(z) + G(z)\mathrm{J} - F(1) - G(1)\mathrm{J})\right)(H(1) + L(1)\mathrm{J})\\
&= (z-1)^{-1}\Big(F(z)H(1) + F(z)L(1)\mathrm{J} + G(z)\overline{H(1)}\mathrm{J} - G(z)\overline{L(1)} - F(1)H(1)\\
&\quad -F(1)L(1)\mathrm{J} - G(1)\overline{H(1)}\mathrm{J} + G(1)\overline{L(1)}\Big),
\end{aligned}$$

and moreover,

$$\begin{aligned}
(a\star R_1 b)_{\mathrm{I}}(z) &= (F(z) + G(z)\mathrm{J}) \star \left((z-1)^{-1}(H(z) + L(z)\mathrm{J} - H(1) - L(1)\mathrm{J})\right)\\
&= (z-1)^{-1}(F(z) + G(z)\mathrm{J}) \star (H(z) + L(z)\mathrm{J} - H(1) - L(1)\mathrm{J})\\
&= (z-1)^{-1}(F(z)H(z) - G(z)\overline{L(\bar{z})} + (G(z)\overline{H(\bar{z})} + F(z)L(z))\mathrm{J})\\
&\quad - F(z)H(1) + G(z)\overline{L(1)} - (G(z)\overline{H(1)} + F(z)L(1))\mathrm{J}
\end{aligned}$$

from which the equality follows. ∎

Chapter 10

First applications: scalar interpolation and first order discrete systems

In the present chapter we discuss the Schur algorithm and some interpolation problems in the scalar case. We make use in particular of the theory of J-unitary rational functions presented in Chapter 9. We also discuss first order discrete systems. In the classical case, interpolation problems in the Schur class can be considered in a number of ways, of which we mention:

1. A recursive approach using the Schur algorithm (as in Schur's 1917 paper [257]) or its variant as in Nevanlinna's 1919 paper [240] in the scalar case.
2. The commutant lifting approach, in its various versions; see Sarason's seminal paper [254], the book [250] of Rosenblum and Rovnyak, and the book [176] of Foias and Frazho.
3. The state space method; see [86].
4. The fundamental matrix inequality method, due to Katsnelson, Kheifets and Yuditskii; see [223, 224].
5. The method based on extension of operators and Krein's formula; see the works of Krein and Langer [229, 228] and also [25].
6. The reproducing kernel method; see [26, 169].

This list is far from being exhaustive, and does not include in particular the papers and works which motivated Schur's work such the trigonometric moment problem and Herglotz's work. In most of these methods one constructs from the interpolation data a J-inner function which defines a linear fractional transformation describing the set of all solutions to the given problem.

In the present chapter we mainly use the reproducing kernel method. In the paper [20] as well as in part of the following section we use the fundamental matrix inequality method, suitably adapted to the present setting.

10.1 The Schur algorithm

In the setting of complex analysis, Theorem 10.1.1 below is the basis to the Schur algorithm and is an easy consequence of Schwarz' lemma and of the fact that if z_1 and z_2 are two complex numbers in the open unit disk, so is $\frac{z_1 - z_2}{1 - z_1\overline{z_2}}$. In particular, if $s(z)$ is a Schur function with $|s(0)| < 1$, then $\frac{s(z) - s(0)}{1 - s(z)\overline{s(0)}}$ is still a Schur function. In our setting, Schwarz' lemma still holds (see Lemma 6.1.11 in Chapter 6), but the pointwise product is replaced by the star product, and the fact that

$$(1 - s(p)\overline{s(0)})^{-\star} \star (s(p) - s(0))$$

is still a Schur function is not so clear.

Theorem 10.1.1. *Let s be a scalar Schur function which is not a unitary constant. Then the function defined by*

$$\sigma(p) = \begin{cases} p^{-1}(1 - s(p)\overline{s(0)})^{-\star} \star (s(p) - s(0)), & p \neq 0, \\ \dfrac{s'(0)}{1 - |s(0)|^2}, & p = 0, \end{cases} \tag{10.1}$$

is a Schur function.

Proof. Let $\Omega = (-1, 1) \setminus \{0\}$. The kernel

$$K_s(x, y) = \frac{1 - s(x)\overline{s(y)}}{1 - xy}$$

is positive definite for $x, y \in \Omega$. Let $\mathscr{H}(K_s)$ denote the associated reproducing kernel Hilbert space. The function

$$x \mapsto K_s(x, 0) = 1 - s(x)\overline{s(0)}$$

belongs to $\mathscr{H}(K_s)$ and has norm equal to $\sqrt{1 - |s(0)|^2}$. Thus the kernel

$$\frac{1 - s(x)\overline{s(y)}}{1 - xy} - \frac{(1 - s(x)\overline{s(0)})(1 - s(0)\overline{s(y)})}{1 - |s(0)|^2}$$

is also positive definite in Ω. Let

$$\theta_0(x) = \frac{1}{\sqrt{1 - |s(0)|^2}} \begin{pmatrix} 1 & s(0) \\ \overline{s(0)} & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, it follows that

$$\frac{J-\theta_0(x)J\theta_0(y)^*}{1-xy}=\frac{\begin{pmatrix}1\\ \overline{s(0)}\end{pmatrix}\begin{pmatrix}1\\ \overline{s(0)}\end{pmatrix}^*}{1-|s(0)|^2}.$$

Hence we have:

$$\begin{aligned}\frac{1-s(x)\overline{s(y)}}{1-xy}-\frac{(1-s(x)\overline{s(0)})(1-s(0)\overline{s(y)})}{1-|s(0)|^2}&=\\ &=\frac{1-s(x)\overline{s(y)}}{1-xy}-\frac{\begin{pmatrix}1 & -s(x)\end{pmatrix}(J-\theta_0(x)J\theta_0(y)^*)\begin{pmatrix}1 & -s(y)\end{pmatrix}^*}{1-xy}\\ &=\frac{\begin{pmatrix}1 & -s(x)\end{pmatrix}\theta_0(x)J\theta_0(y)^*\begin{pmatrix}1 & -s(y)\end{pmatrix}^*}{1-xy}\\ &=(1-s(x)\overline{s(0)})\frac{1-\sigma(x)\overline{\sigma(y)}}{1-xy}(1-s(0)\overline{s(y)}).\end{aligned}$$

Hence the kernel $\frac{1-\sigma(x)\overline{\sigma(y)}}{1-xy}$ is positive definite in Ω. But the function σ is slice hyperholomorphic in $\mathbb{B}$ and so by slice hyperholomorphic extension the kernel

$$\sum_{u=0}^{\infty}p^u(1-\sigma(p)\overline{\sigma(q)})\overline{q}^u$$

is positive definite in $\mathbb{B}$. Hence σ is a Schur function. $\square$

Remark 10.1.2. For an alternative proof, based on Schwarz lemma, see [39].

The Schur algorithm consists of iterating (10.1). It associates to s a sequence, finite or infinite, of Schur functions $s^{(n)}, n=0,1,2,\ldots$, with $s^{(0)}=s$, and

$$s^{(n+1)}(p)=\begin{cases}p^{-1}(1-s^{(n)}(p)\overline{s^{(n)}(0)})^{-\star}\star(s^{(n)}(p)-s^{(n)}(0)), & p\neq 0,\\ \dfrac{(s^{(n)})'(0)}{1-|s^{(n)}(0)|^2}, & p=0,\end{cases}\tag{10.2}$$

and of quaternions $\rho_n=s^{(n)}(0)$, $n=0,1,\ldots\in\mathbb{B}\cup\partial\mathbb{B}$, called Schur coefficients. If at some stage, $\rho_n\in\partial\mathbb{B}$, then the recursion stops. This happens if and only if s is a finite Blaschke product. This is based on Proposition 6.3.10.

The Schur algorithm can be translated on the level of the power series expansions. Indeed, let (whenever defined)

$$s^{(n)}(p)=\sum_{u=0}^{\infty}p^u s_{n,u}.$$

Then

$$\begin{aligned}(1-|\rho_n|^2)s_{n+1,0} &= s_{n,1}\\ (1-|\rho_n|^2)s_{n+1,1}-s_{n,1}\overline{\rho_n}s_{n+1,0} &= s_{n,2}\\ (1-|\rho_n|^2)s_{n+1,2}-s_{n,1}\overline{\rho_n}s_{n+1,1}-s_{n,2}\overline{\rho_n}s_{n+1,0} &= s_{n,3}\\ &\vdots\end{aligned}$$

We note that the above formulas are simple because we are evaluating at the origin; outside real points, the $\star$-multiplication is not so simply related to the coefficients.

Let us now denote by $L_T(h_0,\ldots,h_n)$ the $(n+1)\times(n+1)$ lower triangular Toeplitz matrix based on the quaternions $h_0,\ldots,h_n$.

Proposition 10.1.3. *Let s be a Schur function and assume that the function $s^{(j)}$ exists for $j=0,\ldots,n+1$. Assume that the Schur algorithm is well defined up to rank $n+1$. Then,*

$$\left(I_{j+1}-L_T(s_{n,0},\ldots,s_{n,j})\overline{\rho_n}\right)\begin{pmatrix}s_{n+1,0}\\ \vdots\\ s_{n+1,j}\end{pmatrix}=\begin{pmatrix}s_{n,1}\\ \vdots\\ s_{n,j+1}\end{pmatrix},\quad j=0,\ldots. \tag{10.3}$$

We note that (10.3) is a mere rewriting of the Schur algorithm. Using (10.3) we have the following result. We give at this stage only a formal explanation of the formula (10.5). A precise proof is differed to Section 10.4.

Theorem 10.1.4. *Let s be a Schur function with power series expansion $s(p)=\sum_{n=0}^{\infty}p^na_n$. There exist continuous functions $\varphi_{n,j}$ and ψ_n such that*

$$s_{n,j} = \varphi_{n,j}(a_0,\ldots,a_{n+j}),\quad j=0,1,\ldots \tag{10.4}$$

$$a_n = \psi_n(\rho_0,\ldots,\rho_n),\qquad n=0,1,\ldots. \tag{10.5}$$

Proof. We first prove (10.4) and proceed by induction on n. For $n=0$ we have $s_{0,j}=a_j$ for $j=0,1,\ldots$, and so the result trivially holds. Assume now the result true for n, and let $j\in\mathbb{N}_0$. It follows from (10.3) that $s_{n+1,j}$ is a continuous function of $s_{n,0},\ldots,s_{n,j+1}$. Applying the induction hypothesis for each of the $s_{n,0},\ldots,s_{n,j+1}$ we obtain the result at rank $n+1$.

□

Remark 10.1.5. Setting $j=0$ in (10.4) we have

$$\rho_n=\varphi_{n,0}(a_0,\ldots,a_n). \tag{10.6}$$

As a direct consequence of Theorem 10.1.1 we have:

Theorem 10.1.6. *Let $\rho \in \mathbb{B}$. Then a Schur function s satisfies $s(0) = \rho$ if and only if it can be written in the form*

$$s(p) = (\rho + p\sigma(p)) \star (1 + p\overline{\rho} \star \sigma(p))^{-\star} \tag{10.7}$$

for some Schur function σ.

Proof. Let s be a Schur function such that $s(0) = \rho$. The Schur algorithm asserts that the function σ defined by (10.1) is also a Schur function. Unwrapping s in function of σ we obtain (10.7). Conversely, a function of the form (10.7) is a Schur function. Indeed, as in the proof of the Schur algorithm we have

$$\frac{1 - s(x)\overline{s(y)}}{1 - xy} = \frac{(1 - s(x)\overline{\rho})(1 - \rho\overline{s(y)})}{1 - |\rho|^2} + (1 - s(x)\overline{\rho})\frac{1 - \sigma(x)\overline{\sigma(y)}}{1 - xy}(1 - \rho\overline{s(y)})$$

and so the kernel $\frac{1-s(x)\overline{s(y)}}{1-xy}$ is positive definite on $(-1,1)$, and hence s is a Schur function. It trivially satisfies $s(0) = \rho$. □

Let $\sigma(p) = \sigma_0 + p\sigma_1 + \cdots$. It follows from (10.7) that

$$\sigma_0 = \frac{a_1}{1 - |a_0|^2}$$

(with $s(p) = \sum_{n=0}^{\infty} p^n a_n$). This suggest that, as in the classical case, one can solve iteratively the following interpolation problem, called the Carathéodory-Fejér interpolation problem, in an iterative way. A (non-iterative) solution to this problem is given in Section 10.4.

Problem 10.1.7. *Given quaternions $a_0, \ldots, a_N$, find a necessary and sufficient condition for a Schur function s to exist whose power series expansion is*

$$s(p) = \underbrace{a_0 + \cdots + p^N a_N}_{\textit{fixed}} + p^{N+1} s_{N+1} + \cdots,$$

and describe the set of all solutions when this condition is in force.

To conclude we have the following two results:

Theorem 10.1.8. *Let $(\rho_n)_{n \in \mathbb{N}_0}$ be an infinite sequence of numbers in $\mathbb{B}$. Then there is a unique Schur function with Schur coefficients the ρ_n.*

Proof. As in Section 3.1 we note that the Schur algorithm can be rewritten as

$$p^{n+1} K_{n+1}(p) \star \begin{pmatrix} 1 & -s_{n+1}(p) \end{pmatrix} = \begin{pmatrix} 1 & -s(p) \end{pmatrix} \star \Theta_n(p), \quad n = 0, 1, \ldots \tag{10.8}$$

where K_{n+1} is a function slice hyperholomorphic in a neighborhood of the origin, and where

$$\Theta_n(p) = \theta_0(p) \star \cdots \star \theta_n(p), \tag{10.9}$$

with

$$\theta_k(p) = \frac{1}{\sqrt{1-|\rho_k|^2}} \begin{pmatrix} 1 & \rho_k \\ \overline{\rho_k} & 1 \end{pmatrix} \star \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{1-|\rho_k|^2}} \begin{pmatrix} p & \rho_k \\ p\overline{\rho_k} & 1 \end{pmatrix}, \quad k \in \mathbb{N}_0.$$

Let $\widehat{s}$ be another Schur function with the same Schur coefficients. It follows from (10.8) for $\widehat{s}$ that

$$\begin{pmatrix} 0 & s(p) - \widehat{s}(p) \end{pmatrix} \star \Theta_n(p) = p^{n+1} v_{n+1}(p)$$

for some function v_{n+1} slice hyperholomorphic at the origin. We note that Θ_n is J-inner and so its $(2,2)$ entry D_n is $\star$-invertible at the origin (see Proposition 9.4.8). Since

$$s(p) - \widehat{s}(p) = p^{n+1} \star v_{n+1}(p) \star D_n^{-\star}(p), \quad n = 0, 1, \dots$$

it follows that the power series of s and $\widehat{s}$ coincide, and so $s = \widehat{s}$. ☐

Theorem 10.1.9. *Let $(a_n)_{n\in\mathbb{N}_0}$ be a sequence of elements in $\mathbb{H}$. Then the series*

$$\sum_{u=0}^{\infty} p^u a_u$$

converges in $\mathbb{B}$ to a Schur function if and only if the corresponding ρ_n are all in $\mathbb{B}$ (or in case of a finite sequence, the last one is on $\partial\mathbb{B}$).

The proof is differed to the end of Section 10.4, after the solution of the Carathéodory-Fejér problem.

10.2 A particular case

In this section we give a short proof of the interpolation problem for Schur multipliers in the scalar case, and under two restrictive assumptions. We first recall some notation. Recall that

$$J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and recall that $\mathrm{H}^2(J_0, \mathbb{B})$ denotes the space of elements of the form $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, with $f_1, f_2 \in \mathrm{H}^2(\mathbb{B})$ and endowed with the form

$$[f, g]_{\mathrm{H}^2(J_0,\mathbb{B})} = \langle f, J_0 g \rangle_{(\mathrm{H}^2(\mathbb{B}))^2}, \quad f, g \in (\mathrm{H}^2(\mathbb{B}))^2. \tag{10.10}$$

Note that $\mathrm{H}^2(J_0, \mathbb{B})$ is a Krein space.

The interpolation data are given in terms of an observable pair of matrices $(C, A) \in \mathbb{H}^{2\times N} \times \mathbb{H}^{N\times N}$, and the first assumption is that the Stein equation (4.16)

$$P - A^* P A = C^* J C$$

has a solution which is strictly positive. Consider

$$\Theta(p) = I_2 - (1-p)\star C\star(I_N - pA)^{-\star}P^{-1}(I_N - A)^{-*}C^*J_0, \tag{10.11}$$

where we assume also that $I_N - A$ is invertible (see (9.32)). The second assumption is that the Hilbert space $\mathscr{H}(\Theta)$ with reproducing kernel

$$K_\Theta(p,q) = \sum_{u=0}^{\infty} p^u\,(J_0 - \Theta(p)J_0\Theta(q)^*)\,\overline{q}^u$$

is isometrically included in $\mathrm{H}^2(J_0,\mathbb{B})$ (or, equivalently, that the operator M_Θ of $\star$-multiplication by Θ is an isometry from $\mathrm{H}^2(J_0,\mathbb{B})$ into itself; see Theorem 9.4.10). These assumption covers in particular the Carathéodory-Fejér and Nevanlinna-Pick interpolation problems. They will not be met in the degenerate case (that is, when P is singular) or for boundary interpolation problems.

We set

$$\Theta(p) = \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \end{pmatrix},$$

where

$$a(p) = 1 - (1-p)\star C_1\star(I_N - pA)^{-\star}P^{-1}(I_N - A)^{-*}C_1^* \tag{10.12}$$

$$b(p) = -(1-p)\star C_1\star(I_N - pA)^{-\star}P^{-1}(I_N - A)^{-*}C_2^* \tag{10.13}$$

$$c(p) = -(1-p)\star C_2\star(I_N - pA)^{-\star}P^{-1}(I_N - A)^{-*}C_1^* \tag{10.14}$$

$$d(p) = 1 + (1-p)\star C_2\star(I_N - pA)^{-\star}P^{-1}(I_N - A)^{-*}C_2^*. \tag{10.15}$$

Theorem 10.2.1. *Assume that the function Θ in* (10.11) *is such that M_Θ is an isometry from $\mathrm{H}^2(J_0,\mathbb{B})$ into itself. Then the linear fractional transformation*

$$s(p) = (a(p)\star\sigma(p) + b(p))\star(ac(p)\star\sigma(p) + d(p))^{-\star}$$

describes all Schur functions $s(p) = \sum_{u=0}^{\infty} p^u s_u$ such that

$$\sum_{u=0}^{\infty} A^{*u}(C_1^* s_u - C_2^*) = 0 \tag{10.16}$$

when σ varies in the family of all Schur functions.

Proof. By hypothesis we have

$$\mathrm{H}^2(J_0,\mathbb{B}) = M_\Theta(\mathrm{H}^2(J_0,\mathbb{B})) \oplus \left(\mathrm{H}^2(J_0,\mathbb{B}) \ominus M_\Theta(\mathrm{H}^2(J_0,\mathbb{B}))\right). \tag{10.17}$$

We recall that the right quaternionic vector space $\mathrm{H}^2(J_0,\mathbb{B}) \ominus M_\Theta(\mathrm{H}^2(J_0,\mathbb{B}))$ is spanned by the columns of the function $C\star(I - pA)^{-\star}$, and that

$$\sum_{u=0}^{\infty} p^u\,(J_0 - \Theta(p)J_0\Theta(q)^*)\,\overline{q}^u \geq 0. \tag{10.18}$$

Since $\mathscr{H}(\Theta)$ is isometrically included in $\mathrm{H}^2(J_0,\mathbb{B})$ it follows from Proposition 9.4.9 that $\sigma_S(A)\subset\mathbb{B}$ and so the function Θ (and hence its entries a,b,c and d) are slice hyperholomorphic in a neighborhood of the closed unit ball.

STEP 1: $\max_{p\in\overline{\mathbb{B}}}|d^{-\star}(p)\star c(p)|<1$.

By $\star$-multiplying (10.18) by $\begin{pmatrix}0 & 1\end{pmatrix}$ on the left and $\star_r$-multiplying this same expression by $\begin{pmatrix}0\\1\end{pmatrix}$ on the right we get

$$\sum_{u=0}^{\infty}p^u\left(d(p)\overline{d(q)}-1-c(p)\overline{c(q)}\right)\overline{q}^u\geq 0. \tag{10.19}$$

Setting $p=q$ and since we are in the scalar case, we obtain

$$|d(p)|^2\geq 1+|c(p)|^2,\quad p\in\mathbb{B}, \tag{10.20}$$

and in particular $d(p)$ is invertible in the open unit ball and $|d^{-1}(p)c(p)|<1$ for such p's. The multiplication formula (6.16) implies then

$$\begin{aligned}d^{-\star}(p)\star c(p)&=(d(p)d(\widetilde{p}))^{-1}(d^c(p)c(\widetilde{p})\\&=d(\widetilde{p})^{-1}c(\widetilde{p})\end{aligned}$$

and so $|d^{-\star}(p)\star c(p)|<1$ in $\overline{\mathbb{B}}$.
Still putting $p=q$ in (10.19) and dividing by $d(p)$ (with $p\in\mathbb{B}$) we get

$$1-|d^{-1}(p)|^2-|d^{-1}(p)c(p)|^2\geq 0. \tag{10.21}$$

This inequality extends by continuity to $\overline{\mathbb{B}}$. By (10.15) we see that d is slice hyperholomorphic in a neighborhood of the closed unit ball $\overline{\mathbb{B}}$, and so (10.20) can be extended by continuity to $\overline{\mathbb{B}}$ as

$$|d(p)|\geq 1,\quad p\in\overline{\mathbb{B}}. \tag{10.22}$$

Hence $d^{-1}(p)\neq 0$ on the closed unit ball, and the claim follows from (10.21) and from the compactness of $\overline{\mathbb{B}}$.

STEP 2: *For every $s\in\mathbb{S}$ the function $(c\star\sigma+d)^{-\star}$ belongs to $\mathrm{H}^2(\mathbb{B})$ (and in fact is bounded in $\mathbb{B}$).*

This follows from step 1 and

$$c\star\sigma+d=d\star(d^{-\star}\star c\star\sigma+1)$$

since, by the product formula (6.1.21)

$$|(d(p)^{-\star}\star c(p)\star\sigma(p)|\leq\varepsilon$$

where $\varepsilon = \max_{p\in\overline{\mathbb{B}}} |d^{-\star}(p)\star c(p)| < 1$.

STEP 3: *Let $\sigma \in \mathcal{S}$. Then $s = T_\Theta(\sigma)$ satisfies* (10.16).

Indeed, $\sigma\star(c\star\sigma+d)^{-\star} \in \mathrm{H}^2(\mathbb{B})$ and so

$$\langle \Theta\star\begin{pmatrix}1\\ \sigma\star(c\star\sigma+d)^{-\star}\end{pmatrix}, C\star(I-pA)^{-\star}\xi\rangle = 0,$$

that is

$$\langle \begin{pmatrix}s\\ 1\end{pmatrix}, C\star(I-pA)^{-\star}\xi\rangle = 0,$$

and hence the result.

STEP 4: *Let s satisfies* (10.44). *Then it is of the form $s = T_\Theta(\sigma)$.*

Indeed, (10.16) means that

$$\begin{pmatrix}s\\ 1\end{pmatrix}$$

is orthogonal to $\left(\mathrm{H}^2(J_0,\mathbb{B})\ominus M_\Theta(\mathrm{H}^2(J_0,\mathbb{B}))\right)$ and hence, by (10.17), it is of the form $\Theta\star\begin{pmatrix}u\\ v\end{pmatrix}$ for some $u,v\in\mathrm{H}^2(\mathbb{B})$. Hence

$$s = a\star u + b\star v \tag{10.23}$$

$$1 = c\star u + d\star v. \tag{10.24}$$

Restricting to $p\in(-1,1)$ we note the following. If $v(x)\equiv 0$ for $x\in(-1,1)$ then (10.24) implies that both $u(x)\neq 0$ and $c(x)\neq 0$. Then, $s(x) = a(x)c(x)^{-1}$. But $a\star c^{-\star}$ is not a Schur function since

$$|a(1)|^2 - |c(1)|^2 = 1.$$

So we can divide by v and the kernel associated to $u\star v^{-\star}$ is positive definite on $(-1,1)$. By slice hyperholomorphic extension $u\star v^{-\star}$ is a Schur function. □

By Proposition 9.4.9, J_0-Blaschke products with singularities outside $\overline{\mathbb{B}}$ meet the condition of the previous theorem.

Remark 10.2.2. The cases

$$A = \begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots\\ & & & & \\ 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix}1 & 0 & \cdots & 0\\ \overline{a_0} & \overline{a_1} & \cdots & \overline{a_N}\end{pmatrix},$$

and

$$A=\operatorname{diag}(\overline{p_1},\ldots,\overline{p_N})\in\mathbb{H}^{N\times N},\quad C=\begin{pmatrix}1 & \cdots & 1\\ \overline{s_1} & \cdots & \overline{s_N}\end{pmatrix}\in\mathbb{H}^{2\times N},$$

correspond to the Carathódory-Fejér and Nevanlinna-Pick interpolation problem, respectively.

10.3 The reproducing kernel method

The method presented in Section 10.2 is *ad hoc*. For instance, it will not be applicable to study boundary interpolation problems, that is, when A has spectrum on the unit sphere. Similarly, it will not work (in the quaternionic case) in the matrix-valued case. Indeed (see example 9.4) for a (say $\mathbb{H}^{n\times n}$-valued) Schur function G the fact that

$$\sum_{u=0}^{\infty}p^u\,(I_n-G(p)G(p)^*)\,\overline{p}^u\geq 0$$

will not imply, in general, that $G(p)G(p)^*\leq I_n$ when $n>1$.

On the other hand, and as mentioned in the introduction to this chapter, there are numerous, and complementary, ways to attack the classical interpolation problems. The method we present here is based on the (quaternionic) theory of de Branges-Rovnyak spaces. It consists of five main steps, which are outlined in this section in the scalar case, and illustrated on two different examples in the sequel.

STEP 1: *One builds from the interpolation data a finite dimensional backward-shift invariant (possibly degenerate) inner product space $\mathscr{M}$ of rational functions slice hyperholomophic in a neighborhood of the origin. Thus (see Lemma 9.6.1) $\mathscr{M}$ is the right linear space spanned by the columns of a matrix-function of the form*

$$F(p)=C\star(I_N-pA)^{-\star},\tag{10.25}$$

where $(C,A)\in\mathbb{C}^{n\times N}$ is an observable pair of matrices. Assuming that the Stein equation

$$P-A^*PA=C^*JC\tag{10.26}$$

has a solution, one endows $\mathscr{M}$ with the (possibly degenerate and possibly indefinite) inner product

$$[F(\cdot)a,F(\cdot)b]=b^*Pa.\tag{10.27}$$

One then shows that $P\geq 0$ is a necessary condition for the problem to have a solution.

The space $\mathscr{M}$ bears various names; we will call it here the model space.

STEP 2: *One shows that for a given solution $s\in\mathcal{S}$ (if any) the map*

$$f\ \mapsto\ \begin{pmatrix}1 & -s(p)\end{pmatrix}\star f(p)$$

is an isometry from $\mathscr{M}$ into the reproducing kernel $\mathscr{H}(s)$ with reproducing kernel k_s associated to s.

STEP 3: *Assuming that $P > 0$ (that is, $\mathscr{M}$ is a Hilbert space), one shows it is a $\mathscr{H}(\Theta)$ space and that any solution (if any) can be expressed in terms of the linear fractional transformation associated to Θ.*

STEP 4: *One shows that the linear fractional transformation associated to Θ indeed describes all the solutions.*

STEP 5: *The case where $\mathscr{M}$ is degenerate is much more involved. One can show in this case, namely the scalar case, that there is then a unique solution, which is a finite Blaschke product.*

We illustrate this approach on two examples in the sequel. As in the setting of complex numbers, these are special cases of a much more general interpolation problem, called the bitangential interpolation problem, and of some of its variations. In the present book we do not consider this general problem, leaving it to future work.

10.4 Carathéodory-Fejér interpolation

In the case of Problem 10.1.7 the space $\mathscr{M}$ mentioned just above is the linear span of the columns $f_0, \ldots, f_N$ of the matrix polynomial function

$$F(p) = C \star (I_{N+1} - pA)^{-\star},$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ & & & & \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \overline{a_0} & \overline{a_1} & \cdots & \overline{a_N} \end{pmatrix}.$$

Equation (10.26) has then as unique solution

$$\begin{aligned} P &= C^*JC + A^*C^*JCA + \cdots + A^{(N-1)*}C^*JCA^{(N-1)} \\ &= I_{N+1} - S_{N+1}^* S_{N+1}, \end{aligned} \tag{10.28}$$

where S_{N+1} is the upper triangular Toeplitz matrix with diagonals $a_0, a_1, \ldots, a_N$.

We endow $\mathscr{M}$ with the inner product defined by P:

$$[F(p)a, F(p)b]_P = b^* P a, \quad a, b \in \mathbb{H}^{N+1}.$$

When P is invertible, we let

$$\Theta_N(p) = I_2 - (1-p)C \star (I_{N+1} - pA)^{-\star} \star P^{-1}(I_{N+1} - A)^{-*}C^*J.$$

Note that Θ_N is a matrix polynomial. By Theorem 9.4.6 we have:

$$F(p)P^{-1}F(q)^* = \sum_{u=0}^{\infty} p^u \left(J - \Theta_N(p)J\Theta(q)^*\right)\overline{q}^u.$$

Theorem 10.4.1. *A necessary and sufficient condition for Problem 10.1.7 to have a solution is that the matrix P defined by* (10.28) *is nonnegative. When* $P > 0$ *the set of all solutions is described by the linear fractional transformation*

$$s(p) = T_{\Theta(p)}(e(p)) \tag{10.29}$$

where e varies in $\mathcal{S}$. *When P is singular, the solution is unique and is a finite Blaschke product (or possibly, a unitary constant).*

In the proof below we consider steps 1-4 in the strategy mentioned above. The case of a singular Gram matrix is quite long and is given separately, after the proof of these steps.

Proof of Theorem 10.4.1 in the nonsingular case. We now prove the theorem following the strategy mentioned above. We have already built the space $\mathscr{M}$.

Proof of STEP 1: Assume that a solution s exist. Then,

$$M_s^*(p^j) = \sum_{v=0}^{j} p^v \overline{a_{j-v}}, \quad j = 0, \ldots, N, \tag{10.30}$$

and so

$$\langle (I - M_s M_s^*)p^u, p^v \rangle = \delta_{uv} - \langle M_s^* p^u, M_s^* p^v \rangle = P_{uv}, \tag{10.31}$$

which shows that $P \geq 0$ is a necessary condition for the problem to have a solution.

Proof of STEP 2: From (10.30) we have

$$\begin{pmatrix} I \\ M_s^* \end{pmatrix} p^j = f_j(p).$$

Thus, it is

$$\begin{pmatrix} 1 & -s(p) \end{pmatrix} \star F(p) = (I - M_s M_s^*)F.$$

This ends the proof since $\mathscr{H}(s) = \operatorname{ran}\sqrt{I - M_s M_s^*}$ with the range inner product (see (8.36)).

Proof of STEP 3: To see this, write

$$\begin{aligned} K_s(p,q) = &\begin{pmatrix} 1 & -s(p) \end{pmatrix} \star (I - p\overline{q})^{-\star} \star_r \begin{pmatrix} 1 & -s(q) \end{pmatrix}^* + \\ &+ \begin{pmatrix} 1 & -s(p) \end{pmatrix} \star \Theta_N(p) \star (I - p\overline{q})^{-\star} \star_r \Theta_N(q)^* \star_r \begin{pmatrix} 1 & -s(q) \end{pmatrix}^*, \end{aligned}$$

we see that the kernel

$$\begin{pmatrix}1 & -s(p)\end{pmatrix}\star\Theta_N(p)\star(I-p\overline{q})^{-\star}\star_r\Theta_N(q)^*\star_r\begin{pmatrix}1 & -s(q)\end{pmatrix}^* \tag{10.32}$$

is positive definite in $\mathbb{B}$. This kernel can be rewritten as

$$\begin{aligned}\sum_{u=0}^{\infty}p^u\,((a_N(p)-s(p)\star c_N(p))(a_N(q)-s(q)\star c_N(q))^*-\\ -(b_N(p)-s(p)\star d_Ne(p))(b_N(q)-s(q)\star d_N(q))^*)\,\overline{q}^u.\end{aligned} \tag{10.33}$$

Write

$$e(p)=-(a_N(p)-s(p)\star c_N(p))^{-\star}(b_N(p)-s(p)\star d_N(p)). \tag{10.34}$$

Then, (10.33) can be rewritten as

$$(a_N(p)-s(p)\star c_N(p))^{-\star}\star\left(\sum_{u=0}^{\infty}p^{u+1}\left(1-e(p)\overline{e(q)}\right)\overline{q}^{u+1}\right)\star_r((a_N(q)-s(q)\star c_N(p))^*)^{-\star_r}.$$

It follows (see Theorem 7.5.8) that the kernel $K_e(p,q)$ is positive definite in Ω, and hence is the restriction to Ω of a Schur function, which we still call e.

Proof of STEP 4: Let s be of the form (10.29). Then,

$$\begin{pmatrix}1 & -s(p)\end{pmatrix}\star\Theta_N(p)=p^{N+1}\star u(p)$$

where u is a $\mathbb{H}^{1\times 2}$-valued function slice hyperholomorphic at the origin. Indeed, let $c\in\mathbb{H}^2$. Then,

$$\begin{aligned}\langle M_{\begin{pmatrix}1 & -s\end{pmatrix}}\Theta_N c, z^j\rangle &= \langle\Theta_N c, M^*_{\begin{pmatrix}1 & -s\end{pmatrix}}z^j\rangle\\ &=\langle\Theta_N c, f_j\rangle\\ &=0\end{aligned}$$

because $\Theta_N c\in\Theta_N\mathrm{H}^2(\mathbb{B})$ and $\mathscr{M}=\mathrm{H}^2(\mathbb{B})\ominus\Theta_N\mathrm{H}^2(\mathbb{B})$. Then, as in the proof of [14, Theorem 6.4, p. 136], $\mathscr{H}(\Theta_N)$ is spanned by $g_0,\dots,g_N$, where $g_0,\dots,g_N$ are defined as $f_0,\dots,f_N$, from the coefficients of s. So s is a solution. □

Remark 10.4.2. The coefficient matrix-function Θ_N coincides, up to a multiplicative J_0-unitary constant on the right, with the function (10.9). In fact, in the nondegenerate case, the Carathéodory-Fejér problem can be solved iteratively using the Schur algorithm.

Proof of Theorem 10.4.1 in the singular case. If $|s_0|=1$, and so $s(z)\equiv a_0$ by the maximum modulus principle (see Theorem 6.1.10), and $a_1=\cdots=a_N=0$. There $P=0$ and there is a unique solution to the interpolation problem.

Assume now $|s_0| < 1$. Let $r \in \{1, \ldots, N\}$ be such that the main minor matrix $P_{[r]}$ is not singular but $P_{[r+1]}$ is singular. Note that $P_{[r]}$ is the Gram matrix of $f_0, \ldots, f_{r-1}$. By Theorem 9.4.6 there exists a J_0-unitary polynomial Θ_{r-1} such that the span of $f_0, \ldots, f_{r-1}$ is a $\mathscr{H}(\Theta_{r-1})$ space. We now define a space $\mathscr{N}_r$ by

$$\mathscr{M} = \mathscr{H}(\Theta_{r-1}) \oplus \Theta_{r-1} \star \mathscr{N}_r.$$

Note that the only possible singularity of elements of $\mathscr{N}_r$ is at the origin.

The strategy of the sequel of the proof is as follows:

(1) We prove that $\mathscr{N}_r$ is R_1-invariant, where the operator R_1 has been defined in (9.40).

(2) We prove that the elements of $\mathscr{N}_r$ are in fact polynomials.

(3) We show that $\mathscr{N}_r$ is neutral.

(4) We show that the interpolation problem has a unique solution.

(5) We show that the unique solution is a finite Blaschke product (or, possibly a unitary constant).

We follow the arguments in [53].

STEP 1: *The elements of $\mathscr{N}_r$ are slice hyperholomorphic in a neighborhood of $p = 1$ and $R_1\mathscr{N}_r \subset \mathscr{N}_r$.*

The first part is immediate since the elements of $\mathscr{N}_r$ have singularities possibly only at the origin. We now follow the argument in Step 1 in the proof of Theorem 3.1 in [53] (see p. 153). Let $n \in \mathscr{N}_r$. From (9.45) we have

$$(R_1(\Theta_{r-1} \star n))(p) = (R_1\Theta_{r-1})(p)n(1) + (\Theta_{r-1} \star R_1 n)(p). \tag{10.35}$$

To prove that $R_1 n \in \mathscr{N}_r$ we show that

$$[(R_1(\Theta_{r-1} \star n))(p) - (R_1\Theta_{r-1})(p)n(1), g]_{\mathscr{M}} = 0, \quad \forall g \in \mathscr{H}(\Theta_{r-1}). \tag{10.36}$$

Using (9.44) we have

$$\begin{aligned}[(R_1(\Theta_{r-1} \star n))(p), g]_{\mathscr{M}} &= g(1)^* J_0 (R_1(\Theta_{r-1} \star n))(1) - [\Theta_{r-1} \star n, g]_{\mathscr{M}} - [\Theta_{r-1} \star n, R_1 g]_{\mathscr{M}} \\ &= g(1)^* J_0 (R_1(\Theta_{r-1} \star n))(1)\end{aligned}$$

since

$$[\Theta_{r-1} \star n, g]_{\mathscr{M}} = 0 \quad \text{and} \quad [\Theta_{r-1} \star n, R_1 g]_{\mathscr{M}} = 0,$$

where the second equality follows from $R_1 g \in \mathscr{M}$. Moreover, for real $p = x$

$$(R_1\Theta_{r-1})(x) = -K_{\Theta_{r-1}}(x, 1) J_0 \Theta_{r-1}(1)^*,$$

and so, by slice hyperholomorphic extension,

$$(R_1\Theta_{r-1}(p) = -K_{\Theta_{r-1}}(p, 1) J_0 \Theta_{r-1}(1)^*.$$

Thus

$$\begin{aligned}[R_1\Theta_{r-1})(p)n(1),g]_{\mathscr{M}} &= -[K_{\Theta_{r-1}}(p,1)J_0\Theta_{r-1}(1)^*n(1),g]_{\mathscr{M}}\\ &= -(n(1)^*\Theta_{r-1}(1)^*g(1)^*)\\ &= -g(1)^*\Theta_{r-1}(1)J_0n(1),\end{aligned}$$

and so (10.36) is in force. This ends the proof of the second step.

Endow now $\mathscr{N}_r$ with the Hermitian form

$$[n_1,n_2]_{\mathscr{N}_r} = [\Theta_{r-1}\star n_1, \Theta_{r-1}\star n_2]_{\mathscr{M}}.$$

STEP 2: *The elements of $\mathscr{N}_r$ are polynomials.*

We first note that

$$(R_1+I)p^{-1} = 0,$$

and that, for $N>1$,

$$(R_1+I)p^{-N} = -p^{-(N-1)} + r_N(p) \tag{10.37}$$

where $r_N(p)$ is a linear span of the powers $p^{-1},\ldots,p^{-(N-2)}$. Iterating (10.37) we see that

$$(R_1+I)^{N-1}p^{-N} = (-1)^{N-1}p^{-1}.$$

Let $f\in\mathscr{N}_r$. Since it has at most a pole at the origin, we can write its Laurent expansion at the origin as

$$f(p) = p^{-N}b_N + \cdots + p^{-1}b_1 + a_0 + \cdots + p^M a_M \ \in\ \mathscr{N}_r, \quad \text{with } b_N\neq 0.$$

Then

$$(R_1+I)^{N-1}f(p) = p^{-1}b_N(-1)^{N-1} + \text{polynomial in } p.$$

Iterating R_1 enough times in the above will remove the polynomial part and we see that $\mathscr{N}_r$ contains an element of the form $p^{-1}b_N$ with $b_N\neq 0$. Thus the function

$$\Theta_{r-1}(p)\star p^{-1}b_N = p^{-1}\Theta_{r-1}(p)b_N \in \mathscr{M}.$$

Since this function is a polynomial we have $\Theta_{r-1}(0)b_N = 0$, and so the function

$$(\Theta_{r-1}(p)-\Theta_{r-1}(0))\star p^{-1}b_N \in \Theta_{r-1}\star\mathscr{N}_r.$$

On the other hand it belongs to $\mathscr{H}(\Theta_{r-1})$ and we obtain a contradiction unless $\Theta_{r-1}(p)\star p^{-1}b_N\equiv 0$. This last condition forces $b_N=0$ (since $\Theta_{r-1}(1)=I_2$).

STEP 3: *$\mathscr{N}_r$ is a neutral subspace.*

The space $\mathscr{N}_r$ is made of polynomials and is R_1-invariant. Thus a basis of $\mathscr{N}_r$ is of the form

$$\begin{aligned} g_0(z) &= \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \\ g_1(z) &= p \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} + \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \\ &\vdots \end{aligned}$$

We claim that $\|g_0\|_{\mathscr{N}_r} = 0$. By definition of the inner product in $\mathscr{N}_r$ we have

$$\|g_0\|^2_{\mathscr{N}_r} = |p_0|^2 - |q_0|^2.$$

If $\|g_0\|_{\mathscr{N}_r} > 0$, then Theorem 9.4.6 implies that the linear space of the function g_0 is a $\mathscr{H}(\theta)$ space for some J_0-inner function θ. This will contradict the minimality of r. We thus have

$$|p_0| = |q_0| \neq 0. \tag{10.38}$$

By definition of the inner product of $\mathscr{N}_r$, and since $[g_1, g_0] = 0$ by Cauchy-Schwartz inequality, we can write

$$\begin{aligned} 0 &= [g_0, g_1]_{\mathscr{N}_r} \\ &= [\Theta_{r-1} g_0, \Theta_{r-1} g_1]_{\mathscr{M}} \\ &= [\Theta_{r-1} g_0, \Theta_{r-1} g_1]_{\mathrm{H}^2(J_0,\mathbb{B})} \\ &= \overline{p_1} p_0 - \overline{q_1} q_0. \end{aligned}$$

In view of (10.38) this gives $|p_1| = |q_1|$ and so we get $[g_1, g_1]_{\mathscr{N}_r} = 0$. An easy induction will then show that $\mathscr{N}_r$ is neutral and that $C^* J_0 C = 0$.

STEP 4: *The interpolation problem has a unique solution.*

We first show that there is at most one solution. From the step 1 in the proof of Theorem 10.4.1, we have that any solution is such that the operator $M_{\begin{pmatrix} 1 & -s \end{pmatrix}}$ is an isometry from $\mathscr{M}$ into $\mathscr{H}(s)$. Since $\mathscr{N}_r$ is neutral we have in particular,

$$\begin{pmatrix} 1 & -s(p) \end{pmatrix} \star \Theta_{r-1}(p) \star n(p) = 0, \quad \forall n \in \mathscr{N}_r.$$

By the structure of C, this is equivalent to

$$\begin{pmatrix} 1 & -s(p) \end{pmatrix} \star \Theta_{r-1}(p) \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = 0,$$

and this defines s in a unique way.

We now show that a solution always exists. Replacing $a_0,\ldots,a_N$ by $ta_0,ta_1,\ldots,ta_N$, with $t\in(0,1)$ we see that the corresponding matrix $P(t)$, solution of the equation

$$P-A^*PA=C(t)^*J_0C(t)$$

with

$$C(t)=\begin{pmatrix}1&0\\0&t\end{pmatrix}C$$

is invertible for almost all $t\in(0,1)$. We then build a family of Schur functions solving the interpolation problem for the new data, and take a converging subsequence, using the normal family theorem. The resulting function is a solution.

STEP 5: *The solution is a finite Blaschke product (or a unitary constant).*

By the previous analysis the unique solution is of the form $s(p)=T_{\Theta_{r-1}(p)}(e)$ for some unitary constant e. This function is rational. The associated $\mathscr{H}(s)$ space is finite dimensional and so s is a finite Blaschke product by Theorem 9.4.11. □

We can now give a proof of Theorem 10.1.9.

Proof of Theorem 10.1.9: One direction is clear. If s is a Schur function, the Schur algorithm implies the claim. Conversely, assume first that the sequence of Schur coefficients is infinite, and build $a_0,a_1,\ldots$ via (10.5), and build Θ_n from the corresponding Carathéodory-Fejér interpolation problem. Let $s_n=T_{\Theta_n}(0)$. We know that s_n is a Schur function and that its first $n+1$ Taylor coefficients are $a_0,\ldots,a_n$. The result follows by taking the limit as $n\to\infty$. The case where there is only a finite number of Schur coefficients is clear. □

When one specializes the analysis of the previous section to $N=0$, the space $\mathscr{M}$ is a one dimensional space and it is spanned by the constant function

$$f_0(p)=\begin{pmatrix}1\\\overline{a_0}\end{pmatrix},$$

with norm

$$[f_0,f_0]=1-|a_0|^2.$$

$\mathscr{M}$ is the reproducing kernel Hilbert space with reproducing kernel

$$(J_0-\Theta_0(p)J_0\Theta_0(q)^*)\star(1-p\overline{q})^{-\star},\tag{10.39}$$

with

$$\Theta_0(p)=I_2-(1-p)\frac{1}{1-|\rho|^2}\begin{pmatrix}1\\\overline{\rho}\end{pmatrix}\begin{pmatrix}1\\\overline{\rho}\end{pmatrix}^*J.$$

Since Θ_0 is defined up to a right J_0-unitary matrix we can replace Θ_0 by

$$\begin{aligned}\Theta(p) &= \Theta_0(p)\begin{pmatrix}1 & \rho\\ \overline{\rho} & 1\end{pmatrix}\frac{1}{\sqrt{1-|\rho|^2}}\\ &= \frac{1}{\sqrt{1-|\rho|^2}}\left(\begin{pmatrix}1 & \rho\\ \overline{\rho} & 1\end{pmatrix} - (1-p)\begin{pmatrix}1\\ \overline{\rho}\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}\right) = \frac{1}{\sqrt{1-|\rho|^2}}\begin{pmatrix}p & \rho\\ p\overline{\rho} & 1\end{pmatrix},\end{aligned}$$

which is the coefficient function appearing in the Schur algorithm.

One can solve in a similar way the Nevanlinna-Pick interpolation problem:

Problem 10.4.3. *Given N pairs of points (p_u, s_u) in $\mathbb{H}_1 \times \overline{\mathbb{H}_1}$, find a necessary and sufficient condition for a Schur function s to exist such that*

$$s(p_u) = s_u, \quad u = 1, \dots, N$$

and describe the set of all solutions when this condition is in force.

See [20], where the fundamental inequality method is used.

10.5 Boundary interpolation

Already in the setting of a complex variable, boundary interpolation for Schur functions (that is, when the interpolation nodes are on the boundary of the unit ball) is much more involved than the case of inner points. The first difficulty is the statement of the problem, since boundary values exist only almost everywhere (and in the sense of nontangential limits) for a general Schur function. Another important difference is that the model space is not anymore included in the Hardy space. More precisely, recall first that we denote the boundary of the open unit ball by $\partial\mathbb{B}$. We will denote the interpolation nodes by $p_1, \dots, p_N$, and the interpolation values by $s_1, \dots, s_N$. We set

$$A = \mathrm{diag}\,(\overline{p_1}, \dots, \overline{p_N}) \in \mathbb{H}^{N\times N}, \quad C = \begin{pmatrix}1 & \cdots & 1\\ \overline{s_1} & \cdots & \overline{s_N}\end{pmatrix} \in \mathbb{H}^{2\times N}. \tag{10.40}$$

In the present setting, the matrix A in (10.25) is diagonal with unitary entries. The Stein equation (10.26)

$$P - A^*PA = C^*J_0C,$$

where the unknown is $P \in \mathbb{H}^{N\times N}$, is therefore not uniquely defined by the pair (C, A) (and may even fail to have a solution for certain choices of C). A necessary and sufficient condition for (10.26) to be solvable is that the diagonal entries of C^*J_0C vanish, that is, the interpolation values are also on the unit sphere. The off diagonal entries of (10.26) are uniquely determined by the equation

$$P_{uv} - p_u P_{uv}\overline{p_v} = 1 - s_u\overline{s_v}. \tag{10.41}$$

We denote by P the $N\times N$ Hermitian matrix with entries P_{uv} given by (10.41) for $u\neq v$ and with diagonal entries P_{uu} to be specified, $u,v=1,\ldots,N$. When P is invertible we define (as in Theorem 9.4.7; see (9.34))

$$\Theta(p)=I_2-(1-p)\star C\star(I_N-pA)^{-\star}P^{-1}(I_N-A)^{-*}C^*J_0=\begin{pmatrix}a(p) & b(p)\\ c(p) & d(p)\end{pmatrix}. \quad (10.42)$$

Note that I_N-A is invertible since the interpolation nodes p_u are all different from 1, and Θ is well defined in $\mathbb{B}$. Finally we denote by $\mathscr{M}$ the span of the columns of the function

$$F(p)=C\star(I_N-pA)^{-\star}=\sum_{t=0}^{\infty}p^tCA^t, \quad (10.43)$$

and endow $\mathscr{M}$ with the Hermitian form

$$[F(p)c,F(p)d]_{\mathscr{M}}=d^*Pc, \quad c,d\in\mathbb{H}^N.$$

When $P>0$ it follows from Theorem 9.4.6 that $\mathscr{M}$ endowed with this inner product is the space $\mathscr{H}(\Theta)$ with Θ as in (10.42).

Following the general idea that the linear fractional transformation based on Θ will describe the set of solutions of an underlying interpolation problem (which will depend on the values P_{uu}), and following Problem 3.2.4, it seems natural to set the following problem, see [1]. Note that in this section we follow that paper. Note the condition that not only the interpolation points are distinct, but also the spheres they determine. This hypothesis is needed because the S-spectrum of a matrix, or in general of an operator (see Definition 7.2.1), consists of spheres (which may reduce to real points). Recall also that the S-spectrum of a matrix T coincides with the set of right eigenvalues of T; see Proposition 4.3.15.

Problem 10.5.1. *(see [1]) Given $p_1,\ldots,p_N\in\partial\mathbb{B}\setminus\{1\}$ such that $[p_u]\cap[p_v]=\emptyset$ for $u\neq v$, $s_1,\ldots,s_N\in\partial\mathbb{B}$, and $\kappa_1,\ldots,\kappa_N\in[0,\infty)$, find a necessary and sufficient condition for a slice hyperholomorphic Schur function s to exist such that the conditions*

$$\lim_{\substack{r\to1\\ r\in(0,1)}} s(rp_u) = s_u, \quad (10.44)$$

$$\lim_{\substack{r\to1\\ r\in(0,1)}} \frac{1-s(rp_u)\overline{s_u}}{1-r} \leq \kappa_u \quad (10.45)$$

hold for $u=1,\ldots N$, and describe the set of all Schur functions satisfying (10.44)-(10.45) *when this condition is in force.*

As in the discussion following Theorem 3.2.3 we note that (10.44)-(10.45) imply that

$$\lim_{\substack{r\to1\\ r\in(0,1)}} \frac{1-|s(rp_u)|^2}{1-r^2}\leq\kappa_u, \quad u=1,\ldots,N, \quad (10.46)$$

since

$$\frac{1-|s(rp_u)|^2}{1-r^2} = \frac{1-s(rp_u)\overline{s_u}}{(1-r)(1+r)} + (s(rp_u)\overline{s_u})\frac{1-s_u\overline{s(rp_u)}}{(1-r)(1+r)}. \tag{10.47}$$

On the other hand, Carathéodory's theorem does not seem to have a direct counterpart here, and (10.46) is not equivalent to (10.44)-(10.45). Thus, because of the lack of commutativity, we will not present a solution of the above interpolation problem, but of some modification of it.

To make this point more precise, we now state a result, which can be seen as a counterpart of Carathéodory's theorem (see Theorem 3.2.3) in the setting of slice hyperholomorphic functions. We note that the condition (10.49) will hold in particular for rational functions s, as is proved using a realization of s; see Chapter 9 and in particular Theorem 9.4.2 and Corollary 9.4.3 there.

Theorem 10.5.2. *Let s be a slice hyperholomorphic Schur function, and assume that at some point p_u of modulus 1 we have*

$$\sup_{r\in(0,1)} \frac{1-|s(rp_u)|}{1-r} < \infty. \tag{10.48}$$

Assume moreover that the function $r \mapsto s(rp_u)$ has a development in series with respect to the real *variable r at $r = 1$:*

$$s(rp_u) = s_u + (r-1)a_u + O(r-1)^2. \tag{10.49}$$

Then

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \sum_{t=0}^{\infty} r^t p_u^t (1-s(r_n p_u)\overline{s_u})\overline{p_u}^t = (a_u\overline{s_u} - \overline{p_u}a_u\overline{s_u}\,\overline{p_u})(1-\overline{p_u}^2)^{-1} \geq 0.$$

Proof. From (10.48), we have

$$\sup_{r\in(0,1)} \frac{1-|s(rp_u)|^2}{1-r^2} < \infty. \tag{10.50}$$

But

$$\langle K_s(\cdot, rp_u), K_s(\cdot, rp_u)\rangle_{\mathscr{H}(s)} = \frac{1-|s(rp_u)|^2}{1-r^2},$$

and the family of functions $K_s(\cdot, rp_u)$ is uniformly bounded in norm in $\mathscr{H}(s)$; it has a weakly convergent subsequence. Since in a quaternionic reproducing kernel Hilbert space weak convergence implies pointwise convergence, the weak limit is equal to the function g_u defined by

$$g_u(p) = \sum_{t=0}^{\infty} p^t (1-s(p)\overline{s_u})\overline{p_u}^t. \tag{10.51}$$

Using (7.31) we have

$$\begin{aligned}
g_u(rp_u) &= \sum_{t=0}^{\infty} r^t p_u^t (1-s(rp_u)\overline{s_u})\overline{p_u}^t \\
&= ((1-s(rp_u)\overline{s_u}) - r\overline{p_u}(1-s(rp_u)\overline{s_u})\overline{p_u})\left(1-2\mathrm{Re}\,(rp_u)+r^2\overline{p_u}^2\right)^{-1} \\
&= ((1-s(rp_u)\overline{s_u}) - r\overline{p_u}(1-s(rp_u)\overline{s_u})\overline{p_u})\left((1-r)(1-r\overline{p_u}^2)\right)^{-1} \\
&= (a_u\overline{s_u} - \overline{p_u}a_u\overline{s_u}\,\overline{p_u})(1-\overline{p_u}^2)^{-1} + O(r-1),
\end{aligned} \tag{10.52}$$

where we have used (10.49) to get to the last line. From

$$0 \le \langle g_u, g_u\rangle_{\mathscr{H}(s)} = \lim_{n\to\infty} \langle g_u, K_s(\cdot, r_n p_u)\rangle_{\mathscr{H}(s)} = \lim_{n\to\infty} g_u(r_n p_u),$$

where $(r_n)_{n\in\mathbb{N}}$ is a sequence of numbers in $(0,1)$ with limit equal to 1 we obtain

$$\lim_{n\to\infty} \sum_{t=0}^{\infty} r_n^t p_u^t (1-s(r_n p_u)\overline{s_u})\overline{p_u}^t \ge 0,$$

and thus

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \sum_{t=0}^{\infty} r^t p_u^t (1-s(rp_u)\overline{s_u})\overline{p_u}^t \ge 0. \tag{10.53}$$

The result follows from (10.53) and (10.52). □

Taking for example $s(p) = \dfrac{1+pa}{2}$, (with $|a| \le 1$) we see that

$$(a_u\overline{s_u} - \overline{p_u}a_u\overline{s_u}\,\overline{p_u})(1-\overline{p_u}^2)^{-1} \ne a_u\overline{s_u}$$

unless $\overline{p_u}a_u\overline{s_u} = a_u\overline{s_u}\overline{p_u}$. This condition will hold in particular when $ap_u = p_u a$.

We prove the following theorem (see [1],which we follow here). Note that when $\ell_u > 0$ note that (8.23) becomes

$$\ell_u \le \kappa_u,$$

that is, condition (10.45). Thus, in opposition to the case of inner interpolation nodes, the statement is different from the complex setting in view of the noncommutativity. In that latter setting, with the same Θ, one gets a complete description of the solutions of Problem 10.5.1.

Theorem 10.5.3.
(1) *There always exists a Schur function so that* (10.44) *holds.*
(2) *Fix* $\kappa_1, \ldots, \kappa_N \ge 0$ *and assume* $P > 0$. *Any solution of Problem 10.5.1 is of the form* (9.36)*, that is,*

$$s(p) = (a(p)\star e(p) + b(p))\star(c(p)\star e(p)+d(p))^{-\star},$$

where a,b,c,d are as in (10.42) and e is a slice hyperholomorphic Schur function.
(3) *Conversely, any function of the form* (9.36) *satisfies* (10.44). *If the limits*

$$\ell_u = \lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1-s(rp_u)\overline{s_u}}{1-r}, \quad u=1,\ldots,N, \tag{10.54}$$

exist and are real, then s satisfies (10.45).
(4) *If e is a unitary constant, then the limits* (10.54) *exist (but are not necessarily real) and satisfy*

$$\frac{|\ell_u-\overline{p_u}\ell_u\overline{p_u}|^2}{|1-\overline{p_u}^2|} \le (\mathrm{Re}\,\ell_u)\kappa_u, \quad u=1,\ldots,N. \tag{10.55}$$

To prove this theorem we follow the strategy outlined in Section 10.3. The computations are different, and more complicated, than the case of inner points. In particular, use is made of the very useful formula (7.31). We now give the proofs of the steps listed in Section 10.3.

Proof of STEP 1: We use an approximation argument, and view the boundary interpolation problem as a limit of interpolation problems with inner interpolation nodes (of the kind considered in [20]). Corollary 8.3.11 applied to the points $rp_1,\ldots,rp_N$ with $r\in(0,1)$ shows that the $N\times N$ matrix $P(r)$ with (u,v) entry equal to

$$P_{uv}(r)=K_s(rp_u,rp_v), \quad u,v=1,\ldots N,$$

is positive and it is the unique solution of the matrix equation

$$P(r)-r^2A^*P(r)A=C(r)^*J_0C(r),$$

where

$$C(r)=\begin{pmatrix} 1 & \cdots & 1\\ \overline{s(rp_1)} & \cdots & \overline{s(rp_N)}\end{pmatrix},$$

and A is as in (10.40). Since we are in the scalar case, we have for any solution (if such a solution exists) of the interpolation problem, with associated reproducing kernel Hilbert space $\mathscr{H}(s)$.

$$P_{uu}(r)=K_s(rp_u,rp_u)=\frac{1-|s(rp_u)|^2}{1-r^2}, \quad u=1,\ldots N,$$

(in the matrix-valued case we would need to use (7.31) to compute $K_s(rp_u,rp_u)$) and so

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} K_s(rp_u,rp_u)=\lim_{\substack{r\to 1\\ r\in(0,1)}}\frac{1-|s(rp_u)|^2}{1-r^2}\le\kappa_u, \quad u=1,\ldots N,$$

and

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} C(r) = C,$$

since s is a solution of Problem 10.5.1.

By Lemma 4.2.3, $1-2\mathrm{Re}(p_u)\overline{p_v}+\overline{p_v}^2 \neq 0$ since $1-2\mathrm{Re}(p_u)x+x^2$ is the minimal (or companion) polynomial associated with the sphere $[p_u]$ and vanishes exactly at points on the sphere $[p_u]$, and $p_v \notin [p_u]$. It follows that $\lim_{\substack{r\to 1\\ r\in(0,1)}} P_{uv}(r)$ exists and is in fact equal to P_{uv} for $u \neq v$ by uniqueness of the solution of the equation

$$x - p_u x\overline{p_v} = 0. \tag{10.56}$$

Hence $P \geq 0$ since $P(r) \geq 0$ for all $r \in (0,1)$.

Proof of STEP 2: The formula

$$g_u(p) = \begin{pmatrix} 1 & -s(p) \end{pmatrix} \star f_u(p), \quad \forall p \in \mathbb{B},$$

where s is a solution (if any) of Problem 10.5.1 and where f_u is the u-th column of the matrix F, shows that the range of the $\star$ multiplication operator $M_{\begin{pmatrix} 1 & -s \end{pmatrix}}$ is inside $\mathscr{H}(s)$. That this map is a contraction follows from (10.51) since for $u \neq v$

$$\begin{aligned}
\langle g_v, g_u\rangle_{\mathscr{H}(s)} &= \lim_{n\longrightarrow\infty} \langle g_v, g_{u,r_n}\rangle_{\mathscr{H}(s)}\\
&= \lim_{n\longrightarrow\infty} g_v(r_n p_u)\\
&= \lim_{n\longrightarrow\infty} \sum_{t=0}^{\infty} r_n^t p_u^t (1-s(r_n p_u)\overline{s_v})\overline{p_v}^t\\
&= \lim_{n\longrightarrow\infty} \left((1-s(r_n p_u)\overline{s_v}) - r_n\overline{p_u}(1-s(r_n p_u)\overline{s_v})\overline{p_v}\right)(1-2r_n\mathrm{Re}(p_u)\overline{p_v}+r_n^2\overline{p_v}^2)^{-1}\\
&= \left((1-s_u\overline{s_v}) - \overline{p_u}(1-s_u\overline{s_v})\overline{p_v}\right)(1-2\mathrm{Re}(p_u)\overline{p_v}+\overline{p_v}^2)^{-1},
\end{aligned}$$

where we have used formula (7.31) and the fact that $[p_u]\cap[p_v] = \emptyset$ (recall that we assume here $u \neq v$; see (4.7)). It follows from Proposition 4.4.8 (see formula (4.19)) that

$$P_{uv} = \left((1-s_u\overline{s_v}) - \overline{p_u}(1-s_u\overline{s_v})\overline{p_v}\right)(1-2\mathrm{Re}(p_u)\overline{p_v}+\overline{p_v}^2)^{-1}. \tag{10.57}$$

Let now $c \in \mathbb{H}^N$. Then,

$$\left(M_{\begin{pmatrix} 1 & -s \end{pmatrix}} Fc\right)(p) = \sum_{u=1}^{N} g_u(p)c_u$$

and we have

$$
\begin{aligned}
\|(M_{\begin{pmatrix}1 & -s\end{pmatrix}}Fc\|^2_{\mathscr{H}(s)} &= \sum_{u,v=1}^{N} \overline{c_u}\left(\langle g_v, g_u\rangle_{\mathscr{H}(s)}\right) c_v \\
&= \sum_{u=1}^{N} |c_u|^2 \|g_u\|^2_{\mathscr{H}(s)} + \sum_{\substack{u,v=1\\ u\neq v}}^{N} \overline{c_u}\left(\langle g_v, g_u\rangle_{\mathscr{H}(s)}\right) c_v \\
&= \sum_{u=1}^{N} |c_u|^2 \|g_u\|^2_{\mathscr{H}(s)} + \sum_{\substack{u,v=1\\ u\neq v}}^{N} \overline{c_u} P_{uv} c_v \\
&\leq \sum_{u=1}^{N} |c_u|^2 \kappa_u + \sum_{\substack{u,v=1\\ u\neq v}}^{N} \overline{c_u} P_{uv} c_v \\
&= c^* P c \\
&= \|Fc\|^2_{\mathscr{M}},
\end{aligned}
$$

where we have used (10.57) and (10.46). Thus the $\star$-multiplication by $\begin{pmatrix}1 & -s(p)\end{pmatrix}$ is a contraction from $\mathscr{M}$ into $\mathscr{H}(s)$.

Proof of STEP 3: Let Θ be defined by (10.42). From Theorem 9.4.6 we have

$$
F(p)P^{-1}F(q)^* = K_\Theta(p,q), \tag{10.58}
$$

with

$$
K_\Theta(p,q) = \sum_{t=0}^{\infty} p^t \left(J_0 - \Theta(p) J_0 \Theta(q)^*\right) \overline{q}^t. \tag{10.59}
$$

Proof of STEP 4: We know that

$$
g_u(p) = \begin{pmatrix}1 & -s(p)\end{pmatrix} \star f_u(p) = \sum_{t=0}^{\infty} p^t (1 - s(p)\overline{s_u})\overline{p_u}^t \in \mathscr{H}(s) \tag{10.60}
$$

and

$$
\|g_u\|^2_{\mathscr{H}(s)} \leq \kappa_u.
$$

Hence,

$$
\begin{aligned}
|g_u(rp_u)|^2 &= |\langle g_u(\cdot), K_s(\cdot, rp_u)\rangle_{\mathscr{H}(s)}|^2 \\
&\leq \left(\|g_u\|^2_{\mathscr{H}(s)}\right) \cdot K_s(rp_u, rp_u) \\
&\leq \kappa_u \cdot \frac{1 - |s(rp_u)|^2}{1 - r^2} \\
&\leq \frac{2\kappa_u}{1 - r}.
\end{aligned} \tag{10.61}
$$

In view of Proposition 7.4.2 (see (7.31)), we can write

$$\begin{aligned} g_u(rp_u) &= \sum_{t=0}^{\infty} r^t p_u^t (1 - s(rp_u)\overline{s_u})\overline{p_u}^t \\ &= ((1 - s(rp_u)\overline{s_u}) - r\overline{p_u}(1 - s(rp_u)\overline{s_u})\overline{p_u})\,(1 - 2r\mathrm{Re}(p_u)\overline{p_u} + r^2\overline{p_u}^2)^{-1} \\ &= ((1 - s(rp_u)\overline{s_u}) - r\overline{p_u}(1 - s(rp_u)\overline{s_u})\overline{p_u})\,((1-r)(1 - r\overline{p_u}^2))^{-1}, \end{aligned} \tag{10.62}$$

and so we have

$$\frac{|(1 - s(rp_u)\overline{s_u}) - r\overline{p_u}(1 - s(rp_u)\overline{s_u})\overline{p_u}|}{|1 - r\overline{p_u}^2|} \le \sqrt{2\kappa_u} \cdot \sqrt{1-r}.$$

Set now

$$X_u = 1 - \sigma_u\overline{s_u},$$

where σ_u is a limit, possibly via a subsequence, of $s(rp_u)$ as $r \to 1$. The above inequality implies that $X_u = \overline{p_u}X_u\overline{p_u}$. By Lemma 4.1.6, $X_u = \alpha i + \beta j + \gamma k$, where $\alpha, \beta, \gamma \in \mathbb{R}$. From $\sigma_u\overline{s_u} = 1 - X_u$ we have

$$|\sigma_u\overline{s_u}|^2 = 1 + \alpha^2 + \beta^2 + \gamma^2.$$

Since $\sigma_u \in \mathbb{B}_1$ we have $|\sigma_u\overline{s_u}| \le 1$ and so $\alpha = \beta = \gamma = 0$. Thus, $X_u = 0$ and $\sigma_u\overline{s_u} = 1$. Hence $\sigma_u = s_u$ and the limit $\lim_{\substack{r\to 1\\ r\in(0,1)}} s(rp_u)$ exists and is equal to s_u, and hence (10.44) is satisfied.

To prove that (10.45) is met we proceed as follows. From (10.61) we have in particular

$$|g_u(rp_u)|^2 \le \kappa_u \cdot \frac{1 - |s(rp_u)|^2}{1 - r^2},$$

and using (10.62) we obtain:

$$\frac{|X(r) - r\overline{p_u}X(r)\overline{p_u}|^2}{(1-r)^2|1 - r\overline{p_u}^2|^2} \le \kappa_u \cdot \frac{1 - |s(rp_u)|^2}{1 - r^2}, \tag{10.63}$$

where we have set $X(r) = 1 - s(rp_u)\overline{s_u}$. Assume now that (10.54) is in force and let

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1 - s(rp_u)\overline{s_u}}{1-r} = \ell_u \in \mathbb{R}. \tag{10.64}$$

Then (10.63) together with (10.47) imply that

$$\ell_u^2 \le \ell_u\kappa_u,$$

from which we get that $\ell_u \ge 0$ and

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1 - s(rp_u)\overline{s_u}}{1-r} \le \kappa_u.$$

Proof of STEP 5: We first note that the limits (10.44) hold in view of the previous step. Since $|e|=1$, equality (9.39) implies that the multiplication operator $M_{\begin{pmatrix}1 & -s\end{pmatrix}}$ is unitary and so the space $\mathscr{H}(s)$ is finite dimensional. From Theorem 9.3.2 we see that s can be written in the form

$$s(p) = H + pG\star(I-pT)^{-\star}F, \tag{10.65}$$

where the block matrix

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix}$$

is such that

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix}\begin{pmatrix} P^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} T & F \\ G & H \end{pmatrix}^* = \begin{pmatrix} P^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

for a uniquely determined positive matrix $P \in \mathbb{H}^{v\times v}$, where $v = \dim \mathscr{H}(s)$. The formula (8.18) reads here:

$$\sum_{u=0}^{\infty} p^u(1-s(p)\overline{s(q)})\overline{q}^u = G\star(I-pT)^{-\star}P^{-1}(I-T^*\overline{q})^{-\star_r}\star_r G^*, \tag{10.66}$$

and implies that s is unitary on the unit sphere. Equation (10.65) implies that for every p on the unit sphere the function $r\mapsto s(rp)$ is real analytic for r in a neighborhood of the origin, and so $\lim_{\substack{r\to 1\\ r\in(0,1)}} s(rp)$ exists and is unitary. For $p=p_u$ it follows that the limits (10.45) exist. Then (10.47) leads to

$$\lim_{\substack{r\to 1\\ r\in(0,1)}} \frac{1-|s(rp_u)|^2}{1-r^2} = \mathrm{Re}\,\ell_u$$

and the conclusion follows then from (10.63).

We now consider the degenerate case. We denote by r the rank of P and assume that the main $r\times r$ minor of P is invertible. This can be done by rearranging the interpolation points. The arguments in the proof are different from their counterparts in the proof of Theorem 10.4.1. In the case of that theorem, the matrix A has one eigenvalue of multiplicity N, while here we have N simple eigenvalues. Of course, both results are particular cases of a much more involved result (yet to be stated and proven), concerning the case of general A (with spectrum in the closed unit ball). Even in the complex setting, the case of general A is quite involved. The main point is to write, if possible, the space $\mathscr{M}$ as a direct and orthogonal sum

$$\mathscr{M} = \mathscr{H}(\Theta)\oplus\Theta\mathscr{N}, \tag{10.67}$$

where $\mathscr{N}$ is neutral. The orthogonality is defined by the metric induced by the chosen solution P of the underlying Stein equation (10.26).

Theorem 10.5.4. *Assume that P is singular. Then Problem 10.5.1 has at most one solution, and the latter is then a finite Blaschke product. It has a unique solution satisfying* (10.55) *for* $u=1,\ldots,r$.

Proof. We proceed in a number of steps. Recall that $r = \operatorname{rank} P$. If $r = 0$, the Stein equation (10.26) implies that $C^* J_0 C = 0$, and so $1 - s_u \overline{s_v} = 0$ for $u \neq v \in \{1, \ldots, N\}$. Thus $s_1 = \cdots = s_N$ and the function $s \equiv s_1$ is a solution since

$$\frac{1 - s(rp_u)\overline{s_u}}{1 - r} \equiv 0,$$

and so the second condition in (10.45) is satisfied for any choice of κ_u, $u = 1, \ldots, N$.

Assume that s is another solution of Problem 10.5.1. The map $M_{\begin{pmatrix} 1 & -s \end{pmatrix}}$ of $\star$-multiplication by $\begin{pmatrix} 1 & -s(p) \end{pmatrix}$ is a contraction from $\mathscr{M}$ into $\mathscr{H}(s)$ (see the second step in the proof of Theorem 10.5.3). Thus

$$\begin{pmatrix} 1 & -s(p) \end{pmatrix} \star f_u(p) \equiv 0, \quad u = 1, \ldots, N,$$

that is $g_u \equiv 0$, where f_u is the u-th column of the matrix (10.43) and g_u has been defined in (10.60). From (7.31) we have (for $|p| < 1$)

$$g_u(p) = ((1 - s(p)\overline{s_u}) - \overline{p}(1 - s(p)\overline{s_u})\overline{p_u})\,(1 - 2\mathrm{Re}(p)\overline{p_u} + |p|^2 p_u^2)^{-1},$$

since

$$1 - 2\mathrm{Re}(p)\overline{p_u} + |p|^2 p_u^2 \neq 0$$

for $|p| < 1$. Hence

$$(1 - s(p)\overline{s_u}) = \overline{p}(1 - s(p)\overline{s_u})\overline{p_u}, \quad \forall p \in \partial\mathbb{B}.$$

Taking absolute values of both sides of this equality we get $(1 - s\overline{s_u}) \equiv 0$, and so $s(p) \equiv s_u$. Thus, there is only one solution when $r = 0$. In the rest of the proof we assume $r > 0$. By reindexing the interpolating nodes we assume that the principal minor of order r of the matrix P is invertible. Thus the corresponding space is a $\mathscr{H}(\Theta_r)$ space, and we can write

$$\mathscr{M} = \mathscr{H}(\Theta_r) \oplus \Theta_r \star \mathscr{N}_r,$$

since Θ_r is $\star$-invertible. As in the proof of Theorem 10.4.1 we see that the elements of $\mathscr{N}_r$ are slice hyperholomorphic in a neighborhood of $p = 1$ and

$$R_1 \mathscr{N}_r \subset \mathscr{N}_r. \tag{10.68}$$

It follows from Proposition 9.6.3 that $\mathscr{N}_r$ is spanned by the columns of a function $F_{\mathscr{N}_r}(p) = G \star (I_{N-r} - pT)^{-\star}$, where matrices $(G, T) \in \mathbb{H}^{2 \times (N-r)} \times \mathbb{H}^{(N-r) \times (N-r)}$ such that $\xi \in \mathbb{H}^{N-r}$,

$$F_{\mathscr{N}_r} \xi \equiv 0 \quad \Longrightarrow \quad \xi = 0.$$

We endow now $\mathscr{N}_r$ with the Hermitian fom

$$[n_1, n_2]_{\mathscr{N}_r} = [\Theta_r \star n_1, \Theta_r \star n_2]_{\mathscr{M}}.$$

The sequel of the proof is divided into a number of steps.

STEP 1: *The space $\mathscr{N}_r$ is neutral and $G^*J_0G = 0$.*

We follow [53, Step 2 of proof of Theorem 3.1, p. 154] and use (10.36). The space $\mathscr{N}_r$ is neutral by construction since $r = \operatorname{rank} P$. We first show that the inner product in $\mathscr{N}_r$ satisfies (9.44). We follow the arguments in [53, p. 154] and using (9.44) in $\mathscr{M}$ we have for $n_1, n_2 \in \mathscr{M}$:

$$\begin{aligned}
[R_1 n_1, n_2]_{\mathscr{N}_r} &= [\Theta_r \star R_1 n_1, \Theta_r \star n_2]_{\mathscr{M}} \\
&= [R_1(\Theta_r \star n_1), \Theta_r \star n_2]_{\mathscr{M}} - [(R_1\Theta_r)(n_1(1)), \Theta_r \star n_2]_{\mathscr{M}}
\end{aligned}$$

(where we used (10.35))

$$= [R_1(\Theta_r \star n_1), \Theta_r \star n_2]_{\mathscr{M}},$$

since $(R_1\Theta_r)(n_1(1)) \in \mathscr{H}(\Theta_r)$, and so $[(R_1\Theta_r)(n_1(1)), \Theta_r \star n_2]_{\mathscr{M}} = 0$.
Similarly,

$$\begin{aligned}
[n_1, R_1 n_2]_{\mathscr{N}_r} &= [\Theta_r \star n_1, \Theta_r \star R_1 n_2]_{\mathscr{M}} \\
&= [\Theta_r \star n_1, (R_1\Theta_r)(n_2(1))]_{\mathscr{M}} - [\Theta_r \star n_1, (R_1\Theta_r)(n_2(1))]_{\mathscr{M}} \\
&= [\Theta_r \star n_1, (R_1\Theta_r)(n_2(1))]_{\mathscr{M}}.
\end{aligned}$$

Thus, with $m_1 = \Theta_r \star n_1$ and $m_2 = \Theta_r \star n_2$,

$$\begin{aligned}
[n_1, n_2]_{\mathscr{N}_r} + [R_1 n_1, n_2]_{\mathscr{N}_r} + [n_1, R_1 n_2]_{\mathscr{N}_r} &= [m_1, m_2]_{\mathscr{M}} + [R_1 m_1, m_2]_{\mathscr{M}} + [m_1, R_1 m_2]_{\mathscr{M}} \\
&= m_2(1)^* J_0 m_1(1) \\
&= n_2(1) J_0 n_1(1)
\end{aligned}$$

since $m_v(1) = (\Theta_r \star n_v)(1) = \Theta_r(1) n_v(1)$ for $v = 1, 2$ and $\Theta_r(1)^* J_0 \Theta_r(1) = J_0$.

From Lemma 9.6.5 we get

$$P_{\mathscr{N}_r} - T^* P_{\mathscr{N}_r} T = G^* J_0 G,$$

and so $G^* J_0 G = 0$.

STEP 2: *Problem 10.5.1 has at most one solution.*

Let

$$\Theta_r(p) = \begin{pmatrix} a_r(p) & b_r(p) \\ c_r(p) & d_r(p) \end{pmatrix}.$$

Any solution to the interpolation problem satisfies in particular the nondegenerate problem built from the first r interpolation conditions. From the study of the nondegenerate case, we know that any solution is of the form

$$s(p) = (a_r(p) \star e(p) + b_r(p)) \star (c_r(p) \star e(p) + d_r(p))^{-\star}, \tag{10.69}$$

for some Schur function e (we show below that e is a uniquely determined unitary cosntant). Furthermore as in Step 1, for every $n \in \mathscr{N}_r$ we have

$$\begin{pmatrix} 1 & -s \end{pmatrix} \star \Theta_r \star n \equiv 0.$$

Thus

$$(a - sc) \star \begin{pmatrix} 1 & -e \end{pmatrix} \star n \equiv 0,$$

and so

$$\begin{pmatrix} 1 & -e \end{pmatrix} \star n \equiv 0.$$

Since $G^* J_0 G = 0$ we conclude in the way as in step 1. Indeed, let

$$G = \begin{pmatrix} h_1 & \dots & h_{N-r} \\ k_1 & \dots & k_{N-r} \end{pmatrix}.$$

At least one of the h_u or k_u is different from 0 and $G^* J_0 G = 0$ implies that

$$\overline{h_u} h_v = \overline{k_u} k_v, \quad \forall u, v = 1, \dots, N - r,$$

and so e is a unitary constant.

We now show that the solution, when it exists, is a finite Blaschke product.

STEP 3: *Let s be given by* (10.69). *Then the associated space $\mathscr{H}(s)$ is finite dimensional.*
This follows from

$$K_s(p,q) = \begin{pmatrix} 1 & -s(p) \end{pmatrix} \star K_{\Theta_r}(p,q) \star_r \begin{pmatrix} 1 \\ \overline{s(q)} \end{pmatrix} + \\ + \underbrace{\begin{pmatrix} 1 & -s(p) \end{pmatrix} \star \Theta_r(p) J_0 \Theta_r(q)^* \star_r \begin{pmatrix} 1 \\ \overline{s(q)} \end{pmatrix}}_{\text{is equal to 0 since } |e| = 1},$$

where K_{Θ_r} is defined as in (10.59) (with Θ_r in place of Θ). See the proof of step 3 in Theorem 10.5.3.

STEP 4: *The space $\mathscr{H}(s)$ is isometrically included in the Hardy space* $\mathrm{H}^2(\mathbb{B})$.
We know that the space $\mathscr{H}(s)$ is contractively included in $\mathrm{H}^2(\mathbb{B})$. We now recall that (see Corollary 8.3.9)

$$\|R_0 f\|^2_{\mathscr{H}(s)} \leq \|f\|^2_{\mathscr{H}(s)} - |f(0)|^2, \quad \forall f \in \mathscr{H}(s). \tag{10.70}$$

Here, the space $\mathscr{H}(s)$ is finite dimensional and R_0 invariant. Thus R_0 has a right eigenvector f of the form

$$f(p) = d \star (1 - p\overline{a})^{-\star}, \tag{10.71}$$

where $d \in \mathbb{H}$ and eigenvalue $a \in \mathbb{H}$.

Any eigenvector of R_0 is of the form (10.71), and $a \in \mathbb{B}$ since $\mathscr{H}(s)$ is inside the Hardy space. Thus, see Proposition 4.3.15, the S-spectrum of A is inside $\mathbb{B}$, and the claim now follows from Theorem 9.4.10.

STEP 5: *It holds that* $s(a) = 0$.

From the one dimensional scalar version of Theorem 9.4.6 follows that the span of f endowed with the norm

$$\|f\|^2 = \frac{|f(0)|^2}{1-|a|^2} \tag{10.72}$$

equals $\mathscr{H}(b_a)$, where b_a is a Blaschke factor, see (6.42). From (10.72) we get that $\mathscr{H}(b_a)$ is contractively included in $\mathscr{H}(s)$ and from Corollary 7.5.3 the kernel

$$K_s(p,q) - K_{b_a}(p,q) = \sum_{t=0}^{\infty} p^t (b_a(p)\overline{b_a(q)} - s(p)\overline{s(q)})\overline{q}^t \tag{10.73}$$

is positive definite in $\mathbb{B}$. But $b_a(a) = 0$. Thus, setting $p = q = a$ in (10.73) leads to $s(a) = 0$.

STEP 6: *We can write* $s = b_a \star \sigma_1$, *where* σ_1 *is a Schur function.*

Since a Schur function is bounded in modulus and thus belongs to the space $\mathrm{H}^2(\mathbb{B})$ (see [20]), the representation $s = b_a \star \sigma_1$ with $\sigma_1 \in \mathrm{H}^2(\mathbb{B})$, follows from [34, Proof of Theorem 6.2, p. 109]. To see that σ_1 is a Schur multiplier we note that

$$K_s(p,q) - K_{b_a}(p,q) = b_a(p) \star K_{\sigma_1}(p,q) \star_r \overline{b_a(q)} \tag{10.74}$$

implies that $b_a(p) \star K_{\sigma_1}(p,q) \star_r \overline{b_a(q)}$ is positive definite in $\mathbb{B}$ and hence $K_{\sigma_1}(p,q)$ is as well by [35, Proposition 5.3].

STEP 7: *It holds that* $\dim(\mathscr{H}(\sigma_1)) = \dim(\mathscr{H}(s)) - 1$.

The decomposition (10.74) gives the decomposition

$$K_s(p,q) = K_{b_a}(p,q) + b_a(p) \star K_{\sigma_1}(p,q) \star_r \overline{b_a(q)}.$$

The corresponding reproducing kernel spaces do not intersect. Indeed, all elements in the reproducing kernel Hilbert space with reproducing kernel $b_a(p) \star K_{\sigma_1}(p,q) \star_r \overline{b_a(q)}$ vanish at the point a while non zero elements in $\mathscr{H}(b_a)$ do not vanish anywhere. So the decomposition is orthogonal in $\mathscr{H}(s)$ by Theorem 5.9.4. The claim on the dimensions follows.

After a finite number of iterations, this procedure leads to a constant σ_ℓ, for some positive integer ℓ. This constant has to be unitary since the corresponding space $\mathscr{H}(\sigma_\ell)$ reduces to $\{0\}$.

STEP 8: *Problem 10.5.1 has a unique solution satisfying* (10.55).

To see this it suffices to use item (4) of Theorem 10.5.3 with Θ_r instead of Θ and $e = \sigma_\ell$ □

One can plug a unitary constant e also in the linear fractional transformation (9.36) and the same arguments lead to:

Corollary 10.5.5. *If Problem 10.5.1 has a solution, it is a Blaschke product of degree* $\operatorname{rank} P$.

Remark 10.5.6. The arguments in Steps 5-7 take only into account the fact that the space $\mathscr{H}(\Theta)$ is finite dimensional and that e is a unitary constant. In particular, they also apply in the setting of [20], and in that paper too, the solution of the interpolation problem is a Blaschke product of degree $\operatorname{rank} P$ when the Pick matrix is degenerate.

We conclude by observing that given a Blaschke factor the operator of multiplication by b_a is an isometry from $\mathrm{H}^2(\mathbb{B})$ into itself (see Proposition 9.4.9 or Theorem 8.4.9), and so is the operator of multiplication by a finite Blaschke product B. The degree of the Blaschke product equals the dimension of the space $\mathrm{H}^2(\mathbb{B}) \ominus B\mathrm{H}^2(\mathbb{B})$. Thus the previous argument shows in fact that $\mathscr{H}(s)$ is isometrically included inside $\mathrm{H}^2(\mathbb{B})$ and that $\mathscr{H}(s) = \mathrm{H}^2(\mathbb{B}) \ominus M_s\mathrm{H}^2(\mathbb{B})$.

10.6 First order discrete linear systems

The first order discrete systems (3.9), (3.10), (3.11), (3.12) discussed in Section 3.3 still make sense in the quaternionic setting. Let us recall that these systems arise in a natural way in Schur analysis in the study of the Toeplitz and Nehari extension problems. These problems also make sense in our present setting, and we discuss the corresponding first order discrete systems. We note that all the computations made on the level of the power series are still valid in the present setting (they amount to take a real variable x rather than a quaternionic variable). Since the (discrete) Wiener algebra has been defined (see [29] and Section 6.4) one can also define the corresponding characteristic spectral functions (such as the scattering function), first for real x, and then (at least in the case of rational functions) for a quaternionic variable by slice hyperholomorphic extension. The significance of these functions in quaternionic system theory has still to be explicited.

We consider the scalar quaternionic case. Interestingly enough, in the rational case, it is easier to consider systems of the form (3.11), (3.12), that is (and with the $\star$ product defined in (6.57))

$$B_{n+1}(p) = \begin{pmatrix} 1 & \rho_n \\ \nu_n & 1 \end{pmatrix}^* \star \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \star B_n(p), \quad n = 0, 1, \dots \tag{10.75}$$

which correspond to the matrix-valued case in the complex setting. We follow the analysis presented in [63], and which has been briefly reviewed in Section 3.3.

Definition 10.6.1. Let $(\alpha_n, \beta_n) \in \mathbb{H}^2$ $(n = 1, 2, \dots)$ and let $\Delta_n \in \mathbb{H}^{2\times 2}$ $(n = 0, 1, 2, \dots)$ be a sequence of strictly positive diagonal matrices. The sequence (α_n, β_n) $(n = 1, 2, \dots)$ is said to be Δ–admissible if

$$\begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix} J_0 \Delta_n \begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix}^* = J_0 \Delta_{n-1}, \quad n = 1, 2, \dots \tag{10.76}$$

where $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let us consider the sequence $(\alpha_n, \beta_n)_{n=1,2,\dots}$. The sequence $\Delta = (\Delta_n)_{n=0,1,\dots}$ is said to be an *associated sequence* to $(\alpha_n, \beta_n)_{n=1,2,\dots}$. It follows from Lemma 4.4.1 that the product

$$\alpha_n \beta_n \in [0, 1), \quad n = 1, 2, \dots.$$

The key result in the theory is the following theorem, which allows to deduce all properties of the underlying discrete system. See Theorem 3.3.2 for the complex valued setting.

Theorem 10.6.2. *Let (α_n, β_n) be a Δ–admissible sequence for some sequence of block diagonals matrices $\Delta = (\Delta_n)$ and assume that:*

$$\sum_{n=0}^{\infty} (|\alpha_n| + |\beta_n|) < \infty. \tag{10.77}$$

Then the canonical first order discrete system (10.75) *has a unique solution $X_n(p)$ with entries in the Wiener algebra and such that*

$$\lim_{n\to\infty} \begin{pmatrix} p^{-n} & 0 \\ 0 & 1 \end{pmatrix} \star X_n(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad |p| = 1. \tag{10.78}$$

Proof. Set

$$Z(p) = Z = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F_n = \begin{pmatrix} 0 & \overline{\beta_n} \\ \overline{\alpha_n} & 0 \end{pmatrix}.$$

An induction argument shows that for every positive integer n it holds that

$$Z^n \star (I_2 + Z^{-n} F_n Z^n) \star \cdots \star (I_2 + Z^{-1} F_1 Z) = \\ = \begin{pmatrix} 1 & \alpha_n \\ \beta_n & 1 \end{pmatrix}^* \star \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \star \cdots \star \begin{pmatrix} 1 & \alpha_1 \\ \beta_1 & 1 \end{pmatrix}^* \star \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}. \tag{10.79}$$

Hence, the solution $X_n(p)$ to the canonical discrete system (10.75) with initial condition $X_0(p)$ can be rewritten in two different ways as

$$X_n(p) = Z^n(I_2 + Z^{-n}F_nZ^n)\star\cdots\star(I_2 + Z^{-1}F_1Z)\star X_0(p)$$

and

$$\Delta_n^{1/2}X_n(p) = \left(C_n\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}\star\cdots\star C_1\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}\right)\star\Delta_n^{1/2}\star X_0(p),$$

where

$$C_n = \Delta_n^{1/2}\begin{pmatrix} 1 & \alpha_n\\ \beta_n & 1\end{pmatrix}^*\Delta_{n-1}^{-1/2},\quad n = 1,2,\ldots.$$

In view of (10.77) the infinite product

$$\overset{\curvearrowleft}{\prod_{\ell=1}^{\star\infty}}(I_{2p} + Z^{-\ell}F_\ell Z^\ell) = \lim_{n\to\infty}(I_{2p} + Z^{-n}F_nZ^n)\star\cdots\star(I_{2p} + Z^{-1}F_1Z) \tag{10.80}$$

converges both pointwise for $|p| = 1$ and in the norm of $\mathscr{W}^{2\times 2}$. For every n the matrix function

$$\begin{aligned} Q_n(p) &= \Delta_n^{1/2}\star(I_2 + Z^{-n}F_\ell Z^n)\star\cdots\star(I_2 + Z^{-1}F_1Z)\\ &= Z^{-n}C_n\star\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}\star\cdots\star C_1\star\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}\end{aligned}$$

satisfies (note that Q_n is a finite product, and so both $Q_n(x)$ and $Q_n(1/x)$ make sense)

$$Q_n(x)J_0Q_n(1/x)^* = J_0,\quad x\in(-1,1)\setminus(0).$$

It follows that

$$Q_n(p)\star J_0Q_n^c(1/p) = J_0. \tag{10.81}$$

Let $Y(p)$ be its limit and let $\Delta_\infty = \lim_{n\longrightarrow\infty}\Delta_n$. We claim that

$$Y(p)J_0\star Y^c(1/p) = \Delta_\infty^{-1}J_0. \tag{10.82}$$

In view of the continuity of the $\star$ product we obtain (10.82) by letting $n\to\infty$ in (10.81). To conclude the proof of the theorem it suffices to take $X_0(p) = Y(p)^{-\star}$, that is to chose $X_n(p)$ to be equal to:

$$\begin{aligned} &X_n(p) =\\ &= \begin{pmatrix} p^n & 0\\ 0 & 1\end{pmatrix}\left((I_2 + Z^{-n}F_nZ^n)\star\cdots\star(I_{2p} + Z^{-1}F_1Z)\right)\left(\overset{\curvearrowleft}{\star}{}_{\ell=1}^{\infty}(I_2 + Z^{-\ell}F_\ell Z^\ell)\right)^{-1}.\end{aligned} \tag{10.83}$$

□

The scattering function of the discrete system is defined in a similar way as in the complex case.

Theorem 10.6.3. *The system* (10.75) *has a unique* $\mathbb{H}^{2\times 2}$*–valued solution* $A_n(p)$ *with the following properties:*
(*a*) $\begin{pmatrix} 1 & -1 \end{pmatrix} A_0(p) = 0$, *and*
(*b*) $\begin{pmatrix} 0 & 1 \end{pmatrix} A_n(p) = 1 + o(n), \quad |p| = 1.$
It then holds that

$$\begin{pmatrix} 1 & 0 \end{pmatrix} A_n(p) = p^n S(p) + o(n),$$

where

$$S(p) = (Y_{11}(p) + Y_{12}(p)) \star (Y_{21}(p) + Y_{22}(p))^{-\star}. \tag{10.84}$$

Note that (10.84) is a spectral factorization and that the function S is unitary in the sense that

$$S(p) \star S^c(p) = 1.$$

Following the arguments of [63], it is possible to prove counterparts of Theorems 3.3.4, 3.3.5, 3.3.7 and 3.3.11 in the quaternionic setting. Here we chose a different avenue, and focus in the following section on the rational case.

10.7 Discrete systems: the rational case

We focus on the scalar rational case, that is, in the setting of Section 9.1, we consider sequences of numbers of the form (9.9)

$$s_{-\ell} = ca^\ell b, \quad \ell = 0, 1, \ldots$$

where $(c, a, b) \in \mathbb{H}^{1\times N} \times \mathbb{H}^{N\times N} \times \mathbb{H}^{N\times 1}$. We assume that the spectral radius $\rho(a) < 1$ and define

$$A = \begin{pmatrix} c \\ ca \\ ca^2 \\ \vdots \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b & ab & a^2b & \cdots \end{pmatrix}.$$

Then $A \in \mathbf{B}(\mathbb{H}^N, \ell_2(\mathbb{N}_0, \mathbb{H}^N))$ and $B \in \mathbf{B}(\ell_2(\mathbb{N}_0, \mathbb{H}^N), \mathbb{H}^N)$. We define operators Γ_n as in (1.13). Note that

$$A^*A = \Omega_0 \quad \text{and} \quad BB^* = \Delta,$$

where Δ and Ω_n are solutions of the Stein equations

$$\Delta - a\Delta a^* \quad = \quad bb^*, \tag{10.85}$$

$$\Omega_0 - a^*\Omega_0 a \quad = \quad c^*c. \tag{10.86}$$

Furthermore we have $\Gamma_0 = AB$, and so $\Gamma_0^*\Gamma_0 = B^*\Omega_0 B$. Thus $\|\Gamma_0\|^2 = \|\Omega_0\Delta\|$. Under the assumption that $\|\Gamma_0\| < 1$ we define sequences $a^{(n)}, b^{(n)}, c^{(n)}$ and $d^{(n)}$ as in (1.14).

Following (1.15) we set

$$\begin{aligned}
\alpha_n(p) &= a_{n0} + p^{-1}a_{n1} + \cdots && (10.87)\\
\beta_n(p) &= b_{n0} + p^{-1}b_{n1} + \cdots && (10.88)\\
\gamma_n(p) &= c_{n0} + pc_{n1} + \cdots && (10.89)\\
\delta_n(p) &= d_{n0} + pd_{n1} + \cdots . && (10.90)
\end{aligned}$$

We now compute these functions in the rational case (see (2.24)-(2.27)). The proof in the quaternionic case is the same as in the complex case for real p, and one uses then slice hyperholomorphic extension. We repeat below the arguments from [58].

Proposition 10.7.1. *Assume* $\|\Omega_0\Delta\| < 1$. *Then the following formulas hold:*

$$\begin{aligned}
\alpha_n(p) &= 1 + pca^n \star (pI_N - a)^{-\star}(I - \Delta\Omega_n)^{-1}\Delta a^{*n}c^*,\\
\beta_n(p) &= pca^n \star (pI_N - a)^{-\star}(I - \Delta\Omega_n)^{-1}b,\\
\gamma_n(p) &= b^* \star (I_N - pa^*)^{-\star}(I - \Omega_n\Delta)^{-1}a^{*n}c^*,\\
\delta_n(p) &= 1 + b^* \star (I - pa^*)^{-\star}(I - \Omega_n\Delta)^{-1}\Omega_n b.
\end{aligned}$$

Proof. Let $n \in \mathbb{N}_0$, and let $A_n = Aa^n$. Note that $\Omega_n = A_n^*A_n$ is the unique solution of the equation

$$\Omega_0 - a^*\Omega_0 a = a^{*n}c^*ca^n.$$

We have

$$I_{\ell_2} - \Gamma_n\Gamma_n^* = I_{\ell_2} - UV$$

with $U = A_n$ and $V = BB^*A_n^*$. Thus $I_{\ell_2} - \Gamma_n\Gamma_n^*$ is invertible if and only if the *matrix*

$$I_N - VU = I_N - BB^*A_n^*A_n = I_N - \Delta\Omega_n$$

is invertible. The formula

$$\begin{aligned}
(I_{\ell_2} - \Gamma_n\Gamma_n^*)^{-1} &= (I_{\ell_2} - A_nBB^*A_n^*)^{-1}\\
&= I_{\ell_2} + A_n(I_N - \Delta\Omega_n)^{-1}\Delta A_n^* && (10.91)\\
&= I_{\ell_2} + \begin{pmatrix} ca^n \\ ca^{n+1} \\ \vdots \end{pmatrix}(I_N - \Delta\Omega_n)^{-1}\Delta A_n^*, && (10.92)
\end{aligned}$$

gives

$$(I_{\ell_2} - \Gamma_n\Gamma_n^*)^{-1}e = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} ca^n \\ ca^{n+1} \\ \vdots \end{pmatrix}(I_N - \Delta\Omega_n)^{-1}\Delta a^{n*}c^*$$

and hence the formula for $\alpha_n(x)$ for real x. Similarly

$$\Gamma_n(I_{\ell_2}-\Gamma_n^*\Gamma_n)^{-1}e=\begin{pmatrix} ca^n \\ ca^{n+1} \\ \vdots \end{pmatrix}(I_N-\Delta\Omega_n)^{-1}b$$

and so

$$\begin{aligned}\Gamma_n(I_{\ell_2}-\Gamma_n^*\Gamma_n)^{-1} &= A_nB(I_N+B^*(I_N-\Omega_n\Delta)^{-1}\Omega_nB)\\ &= A_n\left(I_N+\Delta(I_N-\Omega_n\Delta)^{-1}\Omega_n\right)B\\ &= A_n(I_N-\Delta\Omega_n)^{-1}B \qquad (10.93)\\ &= \begin{pmatrix} ca^n \\ ca^{n+1} \\ \vdots \end{pmatrix}(I-\Delta\Omega_n)^{-1}B,\end{aligned}$$

from which follows the formula for $\beta_n(x)$. To compute $\gamma_n(x)$ and $\delta_n(x)$ we note that

$$\begin{aligned}(I_{\ell_2}-\Gamma_n^*\Gamma_n)^{-1} &= (I_{\ell_2}-B^*A_n^*A_nB)^{-1}\\ &= I_{\ell_2}+B^*(I_N-\Omega_n\Delta)^{-1}\Omega_nB\end{aligned}$$

and

$$\begin{aligned}\Gamma_n^*(I_{\ell_2}-\Gamma_n\Gamma_n^*)^{-1} &= B^*A_n^*\left(I_{\ell_2}+A_n(I_N-\Delta\Omega_n)^{-1}\Delta\right)A_n^*\\ &= B^*(I_N+\Omega_n(I_N-\Delta\Omega_n)^{-1}\Delta)A_n^*\\ &= B^*(I_N-\Omega_n\Delta)^{-1}A_n^*.\end{aligned}$$

So,

$$\begin{aligned}(I_{\ell_2}-\Gamma_n^*\Gamma_n)^{-1}e &= \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}+\begin{pmatrix} b^* \\ b^*a^* \\ \vdots \end{pmatrix}(I_N-\Omega_n\Delta)^{-1}\Omega_nb\\ \Gamma_n^*(I_{\ell_2}-\Gamma_n\Gamma_n^*)^{-1}e &= \begin{pmatrix} b^* \\ b^*a^* \\ \vdots \end{pmatrix}(I_N-\Omega_n\Delta)^{-1}a^{*n}c^*\end{aligned}$$

and the formulas for $\gamma_n(x)$ and $\delta_n(x)$ follow.

The required formula of a quaternionic variable p are then obtained by slice hyperholomorphic extension. □

Set (compare with (1.16))

$$H_n(p) = \begin{pmatrix} \alpha_n(p) & \beta_n(p) \\ \gamma_n(p) & \delta_n(p) \end{pmatrix}. \tag{10.94}$$

In view of Proposition 10.7.1 and since we have $\rho(a) < 1$ we have

$$\lim_{n\to\infty} H_n(p) = I_2, \quad \forall p \in \mathbb{B}. \tag{10.95}$$

The formulas show in fact that the limit is I_2 for every p where $H_n(p)$ is defined.

The following theorem is the rational counterpart of Theorem 10.6.2.

Theorem 10.7.2. *Let $H_n(p)$ be defined by* (10.94). *The matrix function*

$$X_n(p) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \star (H_n^c(1/p)) \star \begin{pmatrix} p^n I & 0 \\ 0 & 1/p \end{pmatrix}$$

is the unique solution to the canonical first order discrete system (10.75) *with*

$$\begin{aligned} \rho_n &= -ca^n(I - \Delta\Omega_{n+1})^{-1}b, \\ \nu_n &= -b^*(I - \Omega_n a\Delta a^*)^{-1}a^{*n}c^*, \end{aligned} \tag{10.96}$$

subject to the asymptotic condition

$$\lim_{n\to\infty} \begin{pmatrix} p^{-n} & 0 \\ 0 & 1 \end{pmatrix} \star X_n(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad |p| = 1. \tag{10.97}$$

The function

$$M_n(p) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} H_n^c(1/p) \star \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \star H_0^c(1/p)^{-\star} \begin{pmatrix} 1 & 0 \\ 0 & 1/p \end{pmatrix}$$

is the unique solution to the canonical first order discrete system (10.75) *subject to the initial condition $M_0(p) = I_2$.*

To prove Theorem 10.7.2 we first need the following result:

Theorem 10.7.3. *The coefficient matrix functions $H_n(p)$ satisfy the recurrence*

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} H_{n+1}(p) = H_n(p) \begin{pmatrix} 1 & \rho_n \\ \nu_n & 1 \end{pmatrix} \star \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \tag{10.98}$$

where ρ_n and ν_n are defined by (10.96).

Proof. We begin with a preliminary computation:

$$\begin{aligned} a(I_N - \Delta\Omega_{n+1})^{-1}\Delta a^* &= \\ &= a(I_N - \Delta a^*\Omega_n a)^{-1}\Delta a^* \\ &= a(I_N + \Delta a^*\Omega_n a + \Delta a^*\Omega_n a\Delta a^*\Omega_n a + \cdots)\Delta a^* \\ &= a\Delta a^*(I_N + \Omega_n a\Delta a^* + \Omega_n a\Delta a^*\Omega_n a\Delta a^* + \cdots) \\ &= a\Delta a^*(I_N - \Omega_n a\Delta a^*)^{-1}. \end{aligned} \tag{10.99}$$

Therefore

$$
\begin{aligned}
&\left(a(I_N-\Delta\Omega_{n+1})^{-1}\Delta a^*-(I_N-\Delta\Omega_n)^{-1}\Delta\right)a^{*n}c^* = \\
&\quad = (I_N-\Delta\Omega_n)^{-1}\left((I_N-\Delta\Omega_n)a\Delta a^*-\Delta(I_N-\Omega_n a\Delta a^*)\right)(I_N-\Omega_n a\Delta a^*)^{-1}a^{*n}c^* \\
&\quad = (I_N-\Delta\Omega_n)^{-1}(a\Delta a^*-\Delta)(I_N-\Omega_n a\Delta a^*)^{-1}a^{*n}c^* \\
&\quad = -(I_N-\Delta\Omega_n)^{-1}bb^*(I_N-\Omega_n a\Delta a^*)^{-1}a^{*n}c^*.
\end{aligned} \tag{10.100}
$$

To prove (10.98) we set $p=x$ real. We need to prove the recursions

$$\alpha_{n+1}(x) = \alpha_n(x)+\overline{\rho_n}\beta_n(x), \tag{10.101}$$

$$\beta_{n+1}(x) = x(\rho_n\alpha_n(x)+\beta_n(x)), \tag{10.102}$$

$$x\gamma_{n+1}(x) = \gamma_n(x)+\overline{\rho_n}\delta_n(x), \tag{10.103}$$

$$\delta_{n+1}(x) = \delta_n(x)+\rho_n\gamma_n(x). \tag{10.104}$$

To prove (10.101) we write:

$$
\begin{aligned}
&\alpha_{n+1}(x)-\alpha_n(x) = \\
&\quad = ca^n x(pI_N-a)^{-1}\left(a(I_N-\Delta\Omega_{n+1})^{-1}a^*-(I_N-\Delta\Omega_n)^{-1}\right)\Delta a^{*n}c^* \\
&\quad = -ca^n x(xI_N-a)^{-1}(I_N-\Delta\Omega_n)^{-1}bb^*(I_N-\Omega_n a\Delta a^*)^{-1}a^{*n}c^*\Delta a^{*n}c^* \\
&\quad = -\beta_n(x)b^*(I_N-\Omega_n a\Delta a^*)^{-1}a^{*n}c^*.
\end{aligned}
$$

So we obtain with ν_n as in (10.96):

$$\alpha_{n+1}(x)=\alpha_n(x)+\beta_n(x)\nu_n. \tag{10.105}$$

By slice hyperholomorphic extension we obtain

$$\alpha_{n+1}(p)=\alpha_n(p)+\beta_n(p)\nu_n. \tag{10.106}$$

To prove (10.102) we proceed similarly. Let $x\neq 0\in\mathbb{R}$. Then:

$$
\begin{aligned}
&\frac{\beta_{n+1}(x)}{x}-\beta_n(x) = \\
&= ca^{n+1}(xI_N-a)^{-1}(I_N-\Delta\Omega_{n+1})^{-1}b-ca^n x(xI_N-a)^{-1}(I_N-\Delta\Omega_n)^{-1}b \\
&= ca^n(a-xI_N+xI_N)(xI_N-a)^{-1}(I_N-\Delta\Omega_{n+1})^{-1}b- \\
&\quad -ca^n x(xI_N-a)^{-1}(I_N-\Delta\Omega_n)^{-1}b \\
&= \rho_n+ca^n x(xI_N-a)^{-1}(I_N-\Delta\Omega_{n+1})^{-1}b- \\
&\quad -ca^n x(xI_N-a)^{-1}(I_N-\Delta\Omega_n)^{-1}b \\
&= \rho_n+ca^n x(xI_N-a)^{-1}\left((I_N-\Delta\Omega_{n+1})^{-1}-(I_N-\Delta\Omega_n)^{-1}\right)b
\end{aligned} \tag{10.107}
$$

But

$$\begin{aligned}\left((I_N-\Delta\Omega_{n+1})^{-1}-(I_N-\Delta\Omega_{n+1})^{-1}\right)b &= (I_N-\Delta\Omega_n)^{-1}(\Delta\Omega_{n+1}-\Delta\Omega_n)(I_N-\Delta\Omega_{n+1})^{-1}b\\ &= -(I_N-\Delta\Omega_n)^{-1}(\Delta a^{*n}c^*ca^n(I_N-\Delta\Omega_{n+1})^{-1}b\\ &= -(I_N-\Delta\Omega_n)^{-1}\Delta a^{*n}c^*\rho_n.\end{aligned}$$

Hence, (10.107) may be rewritten as (10.102). We now prove (10.104). We have:

$$\begin{aligned}\delta_{n+1}(x)-\delta_n(x) &= b^*(I_N-xa^*)^{-1}\left((I_N-\Omega_n\Delta)^{-1}\Omega_n-(I_N-\Omega_{n+1}\Delta)^{-1}\Omega_{n+1}\right)b\\ &= b^*(I_N-xa^*)^{-1}(I_N-\Omega_n\Delta)^{-1}(\Omega_n-\Omega_{n+1})(I_N-\Omega_{n+1}\Delta)^{-1}\Omega_{n+1}b\\ &= -b^*(I_N-xa^*)^{-1}(I_N-\Omega_n\Delta)^{-1}a^{*n}c^*ca^n(I_N-\Omega_{n+1}\Delta)^{-1}\Omega_{n+1}b\\ &= \gamma_n(x)\rho_n.\end{aligned}$$

To prove (10.103) we first note that

$$(I_N-\Omega_{n+1}\Delta)^{-1}a^*=a^*(I_N-\Omega_n a\Delta a^*)^{-1}, \tag{10.108}$$

as is easily verified by cross–multiplying. Thus, using (10.108), we have

$$\begin{aligned}x\gamma_{n+1}(x)-\gamma_n(x) &= b^*(I_N-xa^*)^{-1}\left[(I_N-\Omega_{n+1}\Delta)^{-1}xa^*-(I_N-\Omega_n\Delta)^{-1}\right]a^{*n}c^*\\ &= b^*(I_N-xa^*)^{-1}\left[xa^*(I_N-\Omega_n a\Delta a^*)^{-1}-(I_N-\Omega_n\Delta)^{-1}\right]a^{*n}c^*\\ &= b^*(I_N-xa^*)^{-1}\left[(xa^*-I_N+I_N)(I_N-\Omega_n a\Delta a^*)^{-1}-(I_N-\Omega_n\Delta)^{-1}\right]\times\\ &\quad\times a^{*n}c^*\\ &= \nu_n+b^*(I_N-xa^*)^{-1}\left[(I_N-\Omega_n a\Delta a^*)^{-1}-(I_N-\Omega_n\Delta)^{-1}\right]a^{*n}c^*\\ &= \nu_n+b^*(I_N-xa^*)^{-1}(I_N-\Omega_n\Delta)^{-1}\left[I_N-\Omega_n\Delta-I_N+\Omega_n a\Delta a^*\right]\times\\ &\quad\times(I_N-\Omega_n a\Delta a^*)^{-1}a^{*n}c^*\\ &= \nu_n-b^*(I_N-xa^*)^{-1}(I_N-\Omega_n\Delta)^{-1}\Omega_n bb^*(I_N-\Omega_n a\Delta a^*)^{-1}a^{*n}c^*\\ &= \nu_n(1+b^*(I_N-xa^*)^{-1}(I_N-\Omega_n\Delta)^{-1}\Omega_n b)\\ &= \nu_n\delta_n(x).\end{aligned}$$

□

Proof of Theorem 10.7.2. Setting $p=\frac{1}{x}\in\mathbb{R}\setminus\{0\}$ in (10.98) and taking adjoints, we obtain:

$$(H_{n+1}(1/x))^*\begin{pmatrix}1&0\\0&1/x\end{pmatrix}=\begin{pmatrix}1&0\\0&1/x\end{pmatrix}\begin{pmatrix}1&\rho_n\\\nu_n&1\end{pmatrix}^*(H_n(1/x))^*.$$

Multiply both sides of this equality by the matrix $\begin{pmatrix}1&0\\0&x\end{pmatrix}$ on the left and by the matrix

$\begin{pmatrix} x^{n+1} & 0 \\ 0 & 1 \end{pmatrix}$ on the right. We obtain

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} (H_{n+1}(1/x))^* \begin{pmatrix} 1 & 0 \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} x^{n+1} & 0 \\ 0 & 1 \end{pmatrix} = \\
&\qquad = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} 1 & \rho_n \\ \nu_n & 1 \end{pmatrix}^* (H_n(1/x))^* \begin{pmatrix} x^{n+1} & 0 \\ 0 & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 1 & \rho_n \\ \nu_n & 1 \end{pmatrix}^* \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} (H_n(1/x))^* \begin{pmatrix} x^n & 0 \\ 0 & 1/x \end{pmatrix}.
\end{aligned}$$

By slice hyperholomorphic extension, and by definition of H_n^c (see Definition 6.1.24) we see that the functions

$$BX_n(p) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \star (H_n^c(1/p)) \star \begin{pmatrix} p^n I & 0 \\ 0 & 1/p \end{pmatrix}$$

satisfy (10.75). The second claim is then clear. The first claim follows from (10.95). □

Quaternionic counterparts of some of the theorems appearing in Section 3.3 (that is, Theorems 3.3.4, 3.3.5, 3.3.7 and 3.3.11) in the rational case can be obtained using the above theorem.

We now consider a minimal realization for H_n in the quaternionic setting. See Theorem 2.4.3 for the complex setting. We begin with a lemma:

Lemma 10.7.4. *The numbers*

$$\begin{aligned}
t_n &= 1 + ca^n(I - \Delta\Omega_n)^{-1}\Delta a^{*n}c^*, \\
u_n &= 1 + b^*(I - \Omega_n\Delta)^{-1}\Omega_n b.
\end{aligned}$$

are strictly positive.

Proof. This comes from the fact that the matrices $(I_N - \Delta\Omega_n)^{-1}\Delta$ and $(I - \Omega_n\Delta)^{-1}\Omega_n$ are non negative. □

Theorem 10.7.5. *A minimal realization of the matrix function $H_n(p)$ is given by $H_n(p) = D_n + C_n(pI - A)^{-\star}B_n$ where*

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix}, \qquad C_n = \begin{pmatrix} ca^n & 0 \\ 0 & b^* \end{pmatrix},$$

$$B_n = \begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix} \cdot \begin{pmatrix} (I - \Delta\Omega_n)^{-1}\Delta & (I - \Delta\Omega_n)^{-1} \\ -(I - \Omega_n\Delta)^{-1} & -(I - \Omega_n\Delta)^{-1}\Omega_n \end{pmatrix} \cdot \begin{pmatrix} a^{*n}c^* & 0 \\ 0 & b \end{pmatrix},$$

$$D_n = \begin{pmatrix} I_p + ca^n(I - \Delta\Omega_n)^{-1}\Delta a^{*n}c^* & ca^n(I - \Delta\Omega_n)^{-1}b \\ 0 & I_p \end{pmatrix},$$

and the function

$$H_n(p)\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix}$$

is J_0–unitary on $\partial\mathbb{B}$.

Proof. We use Theorem 9.3.2, and show that there is an Hermitian invertible matrix such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & -J_0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & -J_0 \end{pmatrix}. \tag{10.109}$$

Note that (10.109) can be rewritten as

$$\begin{aligned} H - A^*HA &= -C^*J_0C, && (10.110)\\ C^*J_0D &= A^*HB, && (10.111)\\ J_0 - D^*J_0D &= -B^*HB. && (10.112) \end{aligned}$$

We have

$$\begin{aligned} H_n(p)\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix} &= D_n \cdot \begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix} + \\ &\quad + C_n \star (pI - A)^{-\star} B_n \cdot \begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix}. \end{aligned}$$

We check (10.110)–(10.112) are satisfied for this realization, with associated Hermitian matrix given by

$$X_n = \begin{pmatrix} -\Omega_n & -I \\ -I & -a\Delta a^* \end{pmatrix}.$$

More precisely, we have,

$$\begin{aligned} X_n - A^*X_nA &= \begin{pmatrix} -\Omega_n & -I \\ -I & -a\Delta a^* \end{pmatrix} - \\ &\quad - \begin{pmatrix} a^* & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} -\Omega_n & -I \\ -I & -a\Delta a^* \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix} \\ &= \begin{pmatrix} -\Omega_n + a^*\Omega_n a & -I + I \\ -I + I & -a\Delta a^* + a^{-1}a\Delta a^* a^{-*} \end{pmatrix} \\ &= \begin{pmatrix} -a^{*n}c^*ca^n & 0 \\ 0 & bb^* \end{pmatrix} \\ &= -C_n^*J_0C_n, \end{aligned}$$

that is, (10.110) holds. We now check (10.111):

$$
\begin{aligned}
A^*X_nB_n &= \begin{pmatrix} a^* & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} -\Omega_n & -I \\ -I & -a\Delta a^* \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix}\times \\
&\quad \times\begin{pmatrix} (I-\Delta\Omega_n)^{-1}\Delta & (I-\Delta\Omega_n)^{-1} \\ -(I-\Omega_n\Delta)^{-1} & -(I-\Omega_n\Delta)^{-1}\Omega_n \end{pmatrix}\cdot\begin{pmatrix} a^{*n}c^* & 0 \\ 0 & b \end{pmatrix} \\
&= \begin{pmatrix} -a^*\Omega_n a & -I \\ -I & -\Delta \end{pmatrix}\begin{pmatrix} (I-\Delta\Omega_n)^{-1}\Delta & (I-\Delta\Omega_n)^{-1} \\ -(I-\Omega_n\Delta)^{-1} & -(I-\Omega_n\Delta)^{-1}\Omega_n \end{pmatrix}\times \\
&\quad \times\begin{pmatrix} a^{*n}c^* & 0 \\ 0 & b \end{pmatrix} \\
&= \begin{pmatrix} W_1 & W_2 \\ 0 & -I \end{pmatrix}\begin{pmatrix} a^{*n}c^* & 0 \\ 0 & b \end{pmatrix}
\end{aligned}
$$

where we have set

$$
\begin{aligned}
W_1 &= (I-a^*\Omega_n a\Delta)(I-\Omega_n\Delta)^{-1} = (I-\Omega_{n+1}\Delta)(I-\Omega_n\Delta)^{-1} \\
W_2 &= (\Omega-a^*\Omega a)(I-\Omega_n\Delta)^{-1} \quad = a^{*n}c^*ca^n(I-\Delta\Omega_n)^{-1}b.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
C_n^*J_0D_n &= \begin{pmatrix} a^{*n}c^* & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}\times \\
&\quad \times\begin{pmatrix} I_p+ca^n(I-\Delta\Omega_n)^{-1}\Delta a^{*n}c^* & ca^n(I-\Delta\Omega_n)^{-1}b \\ 0 & I_p \end{pmatrix} \\
&= \begin{pmatrix} a^{*n}c^*(I_p+ca^n(I-\Delta\Omega_n)^{-1}\Delta a^{*n}c^*) & a^{*n}c^*ca^n(I-\Delta\Omega_n)^{-1}b \\ 0 & -b \end{pmatrix} \\
&= \begin{pmatrix} (1+\Omega_n(I-\Delta\Omega_n)^{-1}\Delta)a^{*n}c^* & a^{*n}c^*ca^n(I-\Delta\Omega_n)^{-1}b \\ 0 & -b \end{pmatrix} \\
&= \begin{pmatrix} W_1 & W_2 \\ 0 & -I \end{pmatrix}\begin{pmatrix} a^{*n} & 0 \\ 0 & b \end{pmatrix} \\
&= A^*X_nB_n,
\end{aligned}
$$

and hence

$$
C_n^*J_0D_n\cdot\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix} = A^*X_nB_n\cdot\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix}.
$$

It remains to check (10.112): we have thus to check that

$$
\begin{aligned}
J_0-\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix}D_n^*J_0D_n\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix} &= \\
-\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix}B_n^*X_nB_n\begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix}.
\end{aligned}
$$

In view of (10.111),

$$D_n^* J_0 D_n - B_n^* X_n B_n = (D_n^* - B_n^* A^{-*} C_n^*) J_0 D_n,$$

and we first compute $D_n - C_n A^{-1} B_n$; we have:

$$C_n A^{-1} B_n =$$
$$= \begin{pmatrix} ca^n & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} (I_N - \Delta\Omega_n)^{-1}\Delta & (I - \Delta\Omega_n)^{-1} \\ -(I_N - \Omega_n\Delta)^{-1} & -(I_N - \Omega_n\Delta)^{-1}\Omega_n \end{pmatrix} \cdot \begin{pmatrix} a^{*n}c^* & 0 \\ 0 & b \end{pmatrix},$$

and thus

$$D_n - C_n A_n^{-1} B_n = \begin{pmatrix} 1 & 0 \\ b^*(I_N - \Omega_n\Delta)^{-1} a^{*n} c^* & 1 + b^*(I_N - \Omega_n\Delta)^{-1}\Omega_n b \end{pmatrix}.$$

We have

$$D_n - C_n A^{-1} B_n = \begin{pmatrix} 1 & 0 \\ d_{12}^* & u_n \end{pmatrix}$$

where we have denoted by d_{12} the $(1,2)$ entry of D_n. Thus, since t_n is the $(1,1)$ entry of D_n we obtain

$$(D_n - C_n A^{-1} B_n)^* J_0 D_n = \begin{pmatrix} t_n & 0 \\ 0 & -u_n \end{pmatrix},$$

from which we conclude that (10.112) holds.

□

We conclude with:

Lemma 10.7.6. *The sequence (α_n, β_n) is Δ–admissible.*

Proof. Indeed, let

$$U_n = \begin{pmatrix} t_n^{-1/2} & 0 \\ 0 & u_n^{-1/2} \end{pmatrix}.$$

From the recursion (10.98) with $z = 1$ we have

$$H_{n+1}(1) U_{n+1} = H_n(1) U_n U_n^{-1} \begin{pmatrix} 1 & \rho_n \\ \nu_n & 1 \end{pmatrix} U_{n+1}.$$

Writing that both sides are J_0–unitary matrices we obtain

$$U_{n+1} \begin{pmatrix} 1 & \rho_n \\ \nu_n & 1 \end{pmatrix}^* U_n^{-2} J_0 \begin{pmatrix} I_p & \rho_n \\ \nu_n & I_p \end{pmatrix} U_{n+1} = J_0,$$

and hence

$$\begin{pmatrix} 1 & \rho_n \\ \nu_n & 1 \end{pmatrix} U_{n+1}^2 J \begin{pmatrix} 1 & \rho_n \\ \nu_n & 1 \end{pmatrix}^* = U_n^2 J,$$

and hence the result. We note that $\lim_{n\longrightarrow\infty} \Delta_n = I_2$, but $\Delta_0 \neq I_2$.

□

Chapter 11

Interpolation: operator-valued case

In classical Schur analysis, operator-valued functions appear naturally in a number of different settings, of which we mention:

1. The characteristic operator function, when the imaginary part of the operator is not of finite rank (but possibly trace class, or of Hilbert-Schmidt class). See for instance [109, 110, 238].

2. The time-varying case, when the complex numbers are replaced by diagonal operators. See [42, 160, 161, 162].

3. Interpolation in the Hardy space of the polydisk. One can reduce the problem to an operator-valued problem in one variable. The values are then assumed to be Hilbert-Schmidt operators. See [23, 24].

In this chapter we consider left-interpolation problems in the Hardy space $\mathbf{H}^2_{\mathscr{H}}(\mathbb{B})$ and in the set of Schur multipliers $\mathcal{S}(\mathscr{H}_1, \mathscr{H}_2, \mathbb{B})$, where $\mathscr{H}, \mathscr{H}_1$ and $\mathscr{H}_2$ are two sided quaternionic Hilbert spaces. We note that much remains to be done in interpolation of slice hyperholomorphic functions. We mention in particular:

1. Two sided interpolation problems.

2. Interpolation problems in generalized Schur classes.

3. The case of several quaternionic variables.

11.1 Formulation of the interpolation problems

When considering a (say $\mathbb{H}^{m\times n}$-valued) function slice hyperholomorphic in a neighborhood of the origin with power series expansion

$$f(p)=\sum_{u=0}^{\infty}p^u f_u,$$

the (left) tangential values of f at a point p_0 in the direction $c\in\mathbb{H}^m$ is *not* defined by

$$c^*f(p_0) \tag{11.1}$$

but by

$$\sum_{u=0}^{\infty}p_0^u c^* f_u. \tag{11.2}$$

The motivation for such a definition comes from (at least) three related remarks:
(1) First, the space of functions corresponding to the homogeneous problem $c^*f(p_0)=0$ is not M_p-invariant. The space of functions satisfying $\sum_{u=0}^{\infty}p_0^u c^* f_u=0$ is M_p-invariant.
(2) The second remark pertains to the point evaluation in Hardy spaces of $\mathbb{H}^m$-valued function. Then, for $f\in(\mathrm{H}^2(\mathbb{B}))^m$, $c\in\mathbb{H}^m$ and $p_0\in\mathbb{B}$ we have

$$\langle f(p),c\star(1-p\overline{p_0})^{-\star}\rangle_{(\mathrm{H}^2(\mathbb{B}))^m}=\sum_{u=0}^{\infty}p_0^u c^* f_u,$$

that is, formula (11.2).
(3) There are nice formulas for the reproducing kernel of the space $f_0\star(1-p\overline{p_0})^{-\star}$ in the norm of the Hardy space. No such formulas hold for the space spanned by $(1-p\overline{p_0})^{-\star}f_0$.

More generally the definition of a (left) point evaluation in the vector-valued case is given as follows:

Definition 11.1.1. Let $\mathscr{X}$ and $\mathscr{H}$ be two-sided quaternionic Hilbert spaces. Given an output-stable pair (C,A) with $C\in\mathbf{B}(\mathscr{X},\mathscr{H})$ and $A\in\mathbf{B}(\mathscr{X})$ one can define a *left-tangential functional calculus* $f\to(C^*f)^{\wedge L}(A^*)$ on $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ by

$$(C^*f)^{\wedge L}(A^*)=\sum_{k=0}^{\infty}A^{*k}C^*f_k=\mathscr{O}^*_{C,A}f \quad \text{if} \quad f(p)=\sum_{k=0}^{\infty}p^k f_k\in\mathrm{H}^2_{\mathscr{H}}(\mathbb{B}), \tag{11.3}$$

and where $\mathscr{O}_{C,A}$ denotes the observability operator (see (7.44)).

The fact that the left-evaluation map amounts to the adjoint of the observability operator $\mathscr{O}^*_{C,A}$ was justified in Proposition 7.6.6. Since $S(p)u$ belongs to $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ for any $S\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ and $u\in\mathscr{H}_1$, the evaluation (11.3) also extends to the class $\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ by setting

$$(C^*S)^{\wedge L}(A^*)=\sum_{k=0}^{\infty}A^{*k}C^*S_k \quad \text{if} \quad S(p)=\sum_{k=0}^{\infty}p^kS_k. \tag{11.4}$$

With the left-tangential evaluation in hand we may now formulate the following left-tangential operator-argument interpolation problems.

Problem 11.1.2. $\mathbf{IP}(\mathrm{H}^2_{\mathscr{H}}(\mathbb{B}))$*:* Given the output-stable pair $(C,A) \in \mathbf{B}(\mathscr{X},\mathscr{H}_2) \times \mathbf{B}(\mathscr{X})$ and the vector $X \in \mathscr{X}$, find all $f \in \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ such that

$$(C^* f)^{\wedge L}(A^*) = X. \tag{11.5}$$

Problem 11.1.3. $\mathbf{IP}(\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B}))$*:* Given the output-stable pair $(C,A) \in \mathbf{B}(\mathscr{X},\mathscr{H}_2) \times \mathbf{B}(\mathscr{X})$ and the operator $N \in \mathbf{B}(\mathscr{H}_1,\mathscr{X})$, find all $S \in \mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ such that

$$(C^* S)^{\wedge L}(A^*) = N. \tag{11.6}$$

Several remarks are in order. We first notice that for certain special choices of data, interpolation conditions (11.5) and (11.6) amount to well-known conditions of Nevanlinna-Pick type.

Example. Let $n \in \mathbb{N}$ and let $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{H}$ and $\mathscr{X} = \mathbb{H}^n$, so that C, N and A take the form

$$C = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix}, \qquad N = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \qquad A = \begin{pmatrix} \bar{p}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{p}_n \end{pmatrix}, \tag{11.7}$$

with $c_1,\dots,c_n,b_1,\dots,b_n \in \mathbb{H}$ and $p_1,\dots,p_n \in \mathbb{B}$. Then

$$(C^* S)^{\wedge L}(A^*) = \sum_{k=0}^{\infty} \begin{pmatrix} p_1^k & & 0 \\ & \ddots & \\ 0 & & p_n^k \end{pmatrix} \begin{pmatrix} c_1^* S_k \\ \vdots \\ c_n^* S_k \end{pmatrix}$$

and condition (11.6) amounts to n left-tangential conditions

$$\sum_{k=0}^{\infty} p_j^k c_j^* S_k = b_j \quad \text{for} \quad j = 1,\dots,n. \tag{11.8}$$

More generally, for arbitrary spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, and $\mathscr{X} = \mathscr{H}_2^n$, set

$$C = \begin{pmatrix} I_{\mathscr{H}_2} & I_{\mathscr{H}_2} & \dots & I_{\mathscr{H}_2} \end{pmatrix}, \quad N = \begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix}, \quad A = \begin{pmatrix} \bar{p}_1 I_{\mathscr{H}_2} & & 0 \\ & \ddots & \\ 0 & & \bar{p}_n I_{\mathscr{H}_2} \end{pmatrix}, \tag{11.9}$$

where $B_1,\dots,B_n \in \mathbf{B}(\mathscr{H}_2,\mathscr{H}_1)$ and $p_1,\dots,p_n \in \mathbb{B}$. Conditions (11.8) become the operator-valued Nevanlinna-Pick interpolation conditions

$$S(p_j) = \sum_{k=0}^{\infty} p_j^k S_k = B_k^* \quad \text{for} \quad j = 1,\dots,n.$$

Remark 11.1.4. The requirement that the unknown interpolant S should belong to the Schur class $\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ (that is, the multiplication operator M_S is a contraction from $\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$ to $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$) makes the problem $\mathbf{IP}(\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B}))$ norm-constrained.

Remark 11.1.5. The space $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ can be interpreted as the de Branges-Rovnyak space associated to the zero function $S\equiv 0\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$. On could also consider the tangential interpolation problem in de Branges-Rovnyak spaces; see [84, 85] for the complex variable case. This will be considered in the sequel to the present book.

11.2 The problem $\mathbf{IP}(\mathrm{H}^2_{\mathscr{H}}(\mathbb{B}))$: the non-degenerate case

A general two-sided interpolation problem for matrix-valued function in the classical Hardy space was studied in [16, §5] by one of the authors and V. Bolotnikov. The computations done there for the left-sided problem (see [16, §3]), while the right-sided problem (see [16, §4]) depend only on the coefficients of power series, and are formally still valid here for left (resp. right) slice hyperholomorphic functions. Because of the noncommutativity of the variable with the coefficients, we here focus on the left-sided interpolation. With appropriate interpretations it is indeed possible to consider two-sided problems. These will be presented elsewhere. Furthermore, to make these formal computations precise, extra conditions need to be added to insure continuous invertibility of operators (which, in the above mentioned work, are just matrices). Here we assume that the given pair (C,A) is not only output-stable (which is needed in order to define the left-tangential evaluation (11.3)) but also *exactly observable*. The latter means that the observability Gramian $\mathscr{G}_{C,A}$ is strictly positive definite. See also [17, §2 and §3] for similar considerations for interpolation in the family of upper triangular Hilbert-Schmidt operators, which form a "time varying" version of the classical Hardy space.

Under the above hypothesis, one particular solution to the problem $\mathbf{IP}(\mathrm{H}^2_{\mathscr{H}}(\mathbb{B}))$ can be written explicitly as (compare with [16, (3.1), p. 42])

$$f_{\min}(p)=C\star(I_{\mathscr{X}}-pA)^{-\star}\mathscr{G}_{C,A}^{-1}X=\sum_{k=0}^{\infty}p^kCA^k\mathscr{G}_{C,A}^{-1}X. \tag{11.10}$$

Indeed, by (11.3), (11.10) and (7.45), we have

$$(C^*f_{\min})^{\wedge L}(A^*)=\sum_{k=0}^{\infty}A^{*k}C^*CA^k\mathscr{G}_{C,A}^{-1}X=\mathscr{G}_{C,A}\mathscr{G}_{C,A}^{-1}X=X.$$

On the other hand, all solutions to the problem $\mathbf{IP}(\mathrm{H}^2_{\mathscr{H}}(\mathbb{B}))$ can be written as $f=f_0+g$ where f_0 is a particular solution (the minimal norm solution, given in (11.10)) and where g is the general solution of the homogeneous problem

$$(C^*g)^{\wedge L}(A^*)=\mathscr{O}_{C,A}^*g=0. \tag{11.11}$$

The latter condition means that g belongs to the orthogonal (in the metric of $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$) complement of $\mathrm{ran}\,\mathscr{O}_{C,A}$, i.e., the solution set for the homogeneous problem (11.11) is

M_p-invariant. It turns out that this solution set is a closed subspace of $\mathrm{H}^2_{\mathscr{H}}(\mathbb{B})$ and then the Beurling-Lax theorem (see Theorem 8.4.12) will lead us to Theorem 11.2.1. First a remark: In view of Theorem 5.3.11, there exists an injective solution

$$\begin{pmatrix} B \\ D \end{pmatrix} : \mathscr{H}_1 \to \mathscr{X} \oplus \mathscr{H}_2$$

to the factorization problem

$$\begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} B^* & D^* \end{pmatrix} = \begin{pmatrix} \mathscr{G}_{C,A}^{-1} & 0 \\ 0 & I_{\mathscr{H}_2} \end{pmatrix} - \begin{pmatrix} A \\ C \end{pmatrix} \mathscr{G}_{C,A}^{-1} \begin{pmatrix} A^* & C^* \end{pmatrix}, \tag{11.12}$$

that is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathscr{G}_{C,A}^{-1} & 0 \\ 0 & I_{\mathscr{H}_2} \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} \mathscr{G}_{C,A}^{-1} & 0 \\ 0 & I_{\mathscr{H}_2} \end{pmatrix}.$$

We can thus define a function S as in (8.13):

Theorem 11.2.1. *Let $(C,A) \in \mathbf{B}(\mathscr{X}, \mathscr{H}_2) \times \mathbf{B}(\mathscr{X})$ be an exactly observable output-stable pair, Then:*

1. *All solutions f to the problem $\mathbf{IP}(\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B}))$ are parametrised by the formula*

$$f(p) = f_{\min}(p) + S(p) \star h(p) \tag{11.13}$$

   where $f_{\min}$ is defined in (11.10), S is defined by (8.13), and where h is a free parameter from $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$. Different parameters produce different solutions via formula (11.13).

2. *The representation (11.13) is orthogonal in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$; moreover, we have*

$$\|f\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} = \|f_{\min}\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} + \|S \star h\|^2_{\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})} = \langle \mathscr{G}_{C,A}^{-1} X, X \rangle_{\mathscr{H}_2} + \|h\|^2_{\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})}. \tag{11.14}$$

Proof. Since $\mathscr{G}_{C,A}$ is strictly positive definite, we can define the operators

$$\widetilde{C} = C \mathscr{G}_{C,A}^{-\frac{1}{2}} \quad \text{and} \quad \widetilde{A} = \mathscr{G}_{C,A}^{\frac{1}{2}} A \mathscr{G}_{C,A}^{-\frac{1}{2}} \tag{11.15}$$

and it follows from the Stein identity (7.46) that the pair $(\widetilde{C}, \widetilde{A})$ is isometric. Furthermore, we conclude from (7.47) that

$$\mathscr{O}^*_{\widetilde{C},\widetilde{A}} = \mathscr{G}_{C,A}^{-\frac{1}{2}} \mathscr{O}_{C,A} \tag{11.16}$$

from which we see that condition (11.10) is equivalent to $\mathscr{O}^*_{\widetilde{C},\widetilde{A}} g = 0$. By (11.11),

$$\mathscr{G}_{\widetilde{C},\widetilde{A}} = \mathscr{G}_{C,A}^{-\frac{1}{2}} \mathscr{O}^*_{C,A} \mathscr{O}_{C,A} \mathscr{G}_{C,A}^{-\frac{1}{2}} = \mathscr{G}_{C,A}^{-\frac{1}{2}} \mathscr{G}_{C,A} \mathscr{G}_{C,A}^{-\frac{1}{2}} = I_{\mathscr{X}}$$

and hence, for any $X \in \mathscr{X}$, we have

$$\begin{aligned}\|X\|^2 = \lim_{n\to\infty}\left\langle \sum_{k=0}^{n} \widetilde{A}^{*k}\widetilde{C}^{*}\widetilde{C}\widetilde{A}^{k}X,\, X\right\rangle &= \lim_{n\to\infty}\left\langle \sum_{k=0}^{n} \widetilde{A}^{*k}(I_{\mathscr{X}}-\widetilde{A}^{*}\widetilde{A})\widetilde{A}^{k}X,\, X\right\rangle\\ &= \lim_{n\to\infty}\left\langle \left(I_{\mathscr{X}}-\widetilde{A}^{*n+1}\widetilde{A}^{n+1}\right)X,\, X\right\rangle\\ &= \|X\|^2 - \lim_{n\to\infty}\|\widetilde{A}^{n+1}x\|^2.\end{aligned}$$

Therefore, $\|\widetilde{A}^{n}X\|$ tends to zero for every $X \in \mathscr{X}$ as $n\to\infty$ meaning that the operator $\widetilde{A}$ (and hence, also A) is strongly stable. By Theorem 7.6.12, $\operatorname{ran}\mathscr{O}_{\widetilde{C},\widetilde{A}}$ is a closed M_p^*-invariant subspace of $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$. Therefore its orthogonal complement $(\operatorname{ran}\mathscr{O}_{\widetilde{C},\widetilde{A}})^{\perp}$ is a closed M_p-invariant subspace of $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$. By Theorem 8.4.12, there exists a strongly inner multiplier $S\in\mathbf{IP}(\mathbb{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B}))$ such that $(\operatorname{ran}\mathscr{O}_{\widetilde{C},\widetilde{A}})^{\perp} = S\star\mathrm{H}^2_{\mathscr{C}_2}(\mathbb{B})$. The construction of S suggested in Corollary 8.4.11 applies to operators $\widetilde{C}$ and $\widetilde{A}$ and being translated to the original C and A leads to the function described in the formulation of the theorem. Since condition (11.10) is equivalent to $\mathscr{O}^*_{\widetilde{C},\widetilde{A}}g=0$, it follows that all solutions $g\in\mathrm{H}^2_{\mathscr{C}_2}(\mathbb{B})$ to the homogeneous interpolation problem (11.10) are described by the formula $g=S\star h$ where the parameter h runs through the space $\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$. Since $F_{\min}$ is a particular solution of the problem, the formula (11.13) indeed describes the solution set of the problem $\mathbf{IP}(\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B}))$. Since S is strongly inner, different parameters h lead via this formula to different solutions f. It is readily seen from formula (11.10) that $f_{\min}$ belongs to $\operatorname{ran}\mathscr{O}_{C,A}=\mathscr{H}(S)$ and therefore, is orthogonal to $S\star\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$. Therefore, the representation (11.13) is orthogonal and the first equality in (11.14) follows. The second equality follows from the isometric property of the operator M_S and the equality

$$\|f_{\min}\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} = \sum_{k=0}^{\infty}\|CA^k\mathscr{G}_{C,A}^{-1}X\|^2_{\mathscr{H}_2} = \left\langle \mathscr{G}_{C,A}^{-1}X, X\right\rangle_{\mathscr{X}}$$

which follows from the power series representation in (11.10). □

Remark 11.2.2. If the S-spectrum of A does not contain the point 1, one can choose $\mathscr{H}_1$ be equal to $\mathscr{H}_2$ and a fairly explicit formula for S in Theorem 11.2.1, motivated by the reproducing kernel formula (2.20), is the following (compare with [16, (2.19), p. 38]):

$$S(p) = I-(1-p)C\star(I-pA)^{-\star}\mathscr{G}_{C,A}^{-1}(I-A^*)^{-1}C^*. \tag{11.17}$$

Remark 11.2.3. It follows from (11.14) that $f_{\min}$ is the solution of the problem $\mathbf{IP}(\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B}))$ with the minimally possible norm. The latter formula also allows us to describe all solutions of the following norm-constrained problem $\mathbf{IP}_\gamma(\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B}))$: All functions $f\in\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ satisfying interpolation condition (11.5) and the norm constraint $\|f\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})}\le\gamma$ are given by the formula (11.13) where h is a function in $\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$ such that $\|f\|^2_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})}\le\gamma-\langle\mathscr{G}_{C,A}^{-1}X,X\rangle_{\mathscr{X}}$.

11.3 Left-tangential interpolation in $\mathbb{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$.

The left-tangential interpolation in $\mathbb{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ is solved in the following five sections, and we here summarize the stategy used. Let

$$P=\mathscr{G}_{C,A}-\mathscr{G}_{N,A}\in\mathbf{B}(\mathscr{X}). \tag{11.18}$$

The condition $X\geq 0$ is necessary for the problem to have a solution. In this section we prove that a necessary and sufficient condition for $S\in\mathbb{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ to have a solution is that the kernel

$$\begin{pmatrix}P & B(q)^*\\ B(p) & K_S(p,q)\end{pmatrix},\quad\text{where } B(p)=C-S(p)N)\star(I_{\mathscr{X}}-pA)^{-\star},$$

is positive definite in $\mathbb{B}$. When P is bounded invertible, one can go further. Defining a block operator-valued function Θ via the formula (11.39), we show that S is a solution to the interpolation problem if and only if the kernel

$$\sum_{k=0}^{\infty}p^k\begin{pmatrix}I_{\mathscr{H}_2} & -S(p)\end{pmatrix}\Theta(p)J\Theta(q)^*\begin{pmatrix}I_{\mathscr{H}_2} & -S(q)\end{pmatrix}^*\overline{q}^k \tag{11.19}$$

is positive definite in $\mathbb{B}$. Such a condition already appears in Sections 10.5 and 10.4. To translate this condition in terms of a linear fractional transformation is relatively easy in the scalar case, but requires extra care in the operator-valued setting. We first need, in Section 11.5 to solve the case where there is a finite number of interpolation points (this is still an infinite dimensional problem because the values are in a (possibly) infinite dimensional Hilbert space. Building on that section we prove a factorization theorem in Section 11.6 which allows to obtain the description of the set of solutions in terms of a linear fractional transformation in Section 11.7.

In this section we present necessary and sufficient conditions for $S\in\mathbb{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ to be a solution of Problem $\mathbf{IP}(\mathbb{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B}))$. The main result of the section is the following theorem, which characterizes all solutions S to the interpolation problem $\mathbf{IP}(\mathbb{S}(\mathscr{H}_1,\mathscr{Y},\mathbb{B}))$ in terms of positive definite kernels and in terms of the reproducing kernel Hilbert space $\mathscr{H}(S)$. We assume that the necessary conditions for the problem to have a solution are satisfied, that is, the pairs (C,A) and (N,A) are output stable and the operator P given by (11.18) is positive semidefinite.

Theorem 11.3.1. *Let S be an $\mathbf{B}(\mathscr{H}_1,\mathscr{H}_2)$-valued function slice hyperholomorphic in $\mathbb{B}$, and let*

$$B(p)=(C-S(p)N)\star(I_{\mathscr{X}}-pA)^{-\star}. \tag{11.20}$$

Then, the following conditions are equivalent:
(1) *The function $S\in\mathbb{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ satisfies* (11.6).
(2) *For every $X\in\mathscr{X}$, the function $BX\ :\ p\mapsto B(p)X$ belongs to $\mathscr{H}(S)$ and*

$$\|BX\|^2_{\mathscr{H}(S)}=\langle PX,X\rangle_{\mathscr{X}}. \tag{11.21}$$

(3) *The kernel*

$$\begin{pmatrix} P & B(q)^* \\ B(p) & K_S(p,q) \end{pmatrix} \tag{11.22}$$

is positive definite in $\mathbb{B}$.

(4) *The operator*

$$\mathbb{P} = \begin{pmatrix} P & M_B^* \\ M_B & I - M_S M_S^* \end{pmatrix} \quad : \quad \begin{pmatrix} \mathscr{X} \\ \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B}) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathscr{X} \\ \mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B}) \end{pmatrix} \tag{11.23}$$

is positive semi-definite.

The method used can be seen as a combination of the Fundamental Matrix Inequality method (FMI), see [223, 224], together with the reproducing kernel Hilbert space method. We refer to Chapter 10 and [20] for the scalar versions of some of these computations. For a sample of papers where the Fundamental Matrix Inequality method is applied to interpolation we mention [15, 18].

We begin with a preliminary result:

Proposition 11.3.2. *Assume that Problem* $\mathbf{IP}(\mathbb{S}(\mathscr{H}_1, \mathscr{H}_2, \mathbb{B}))$ *has a solution. Then:*
(a) *The pair* (N, A) *is output stable.*
(b) *The operator*

$$P = \mathscr{G}_{C,A} - \mathscr{G}_{N,A} \tag{11.24}$$

is positive semi-definite.

Proof.
(a) Let $S(p) = \sum_{n=0}^{\infty} p^n S_n$ (with $S_n \in \mathbf{B}(\mathscr{H}_1, \mathscr{H}_2)$ for $n \in \mathbb{N}_0$) and let $f_0, \ldots, f_M \in \mathscr{H}_1$. In view of (7.47) we have

$$(\mathscr{O}_{C,A}^* S)(f_n) = \lim_{M \to \infty} \sum_{k=0}^{M} A^{*k} C^* S_k f_n$$

in the topology of $\mathscr{X}$. In view of the interpolation condition (11.6) we can write:

$$\begin{aligned} \sum_{n=0}^{M} A^{*n} N^* f_n &= \sum_{n=0}^{M} A^{*n} (\mathscr{O}_{C,A}^* S)(f_n) \\ &= \lim_{T \to \infty} \sum_{n=0}^{M} A^{*n} \left(\sum_{t=0}^{T} A^{*t} C^* S_t f_m \right) \\ &= \lim_{R \longrightarrow \infty} \sum_{r=0}^{R} A^{*r} C^* \left(\sum_{t+m=r} S_t f_m \right) \\ &= \mathscr{O}_{C,A}^* M_S f, \end{aligned}$$

with $f(p) = \sum_{n=0}^{M} p^m f_m$. The operator $\mathscr{O}_{C,A}^* M_S$ is continuous and therefore the application which to f associates $\sum_{n=0}^{M} A^{*n} N^* f_n$ extends continuously to $\mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B})$ to a map which is by definition $\mathscr{O}_{N,A}$. Thus if S is a solution it holds that

$$\mathscr{O}_{C,A}^* M_S = \mathscr{O}_{N,A}^*. \tag{11.25}$$

In view of (b) (which is proved below) we note that this condition implies the interpolation condition. Indeed, let $u \in \mathscr{H}_1$ and $X \in \mathscr{X}$. Then,

$$\begin{aligned}\langle \mathscr{O}_{N,A}^* u, X\rangle_{\mathscr{X}} &= \langle u, \mathscr{O}_{N,A} X\rangle_{\mathbf{H}_2(\mathbb{B},\mathscr{H}_1)}\\ &= \langle u, NX\rangle_{\mathscr{H}_1}\end{aligned}$$

(since $(\mathscr{O}_{N,A}X)(p) = \sum_{n=0}^\infty p^n NA^n X$)

$$= \langle N^* u, X\rangle_{\mathscr{X}}.$$

Thus restricting (11.25) leads to

$$\mathscr{O}_{C,A}^* M_S\big|_{\mathscr{H}_1} = \mathscr{O}_{N,A}^*\big|_{\mathscr{H}_1} = N^*,$$

which is the interpolation condition.

(b) Let S be a solution of the interpolation problem. By the discussion above, (11.25) holds and thus:

$$\mathscr{G}_{C,A} - \mathscr{G}_{N,A} = \mathscr{O}_{C,A}^* \left(I_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} - M_S M_S^* \right) \mathscr{O}_{C,A} \geq 0$$

since M_S is a contraction. □

Proof of Theorem 11.3.1. We begin by showing that (1) $\Longrightarrow$ (2). By hypothesis, $I - M_S M_S^* \geq 0$, thus we have

$$\begin{aligned}\langle (I_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} - M_S M_S^*)\mathscr{O}_{C,A}X, \mathscr{O}_{C,A}X\rangle &= \|\mathscr{O}_{C,A}X\|^2 - \|M_S^*\mathscr{O}_{C,A}X\|^2\\ &= \|\mathscr{O}_{C,A}X\|^2 - \|\mathscr{O}_{N,A}X\|^2\\ &= \langle (\mathscr{G}_{C,A} - \mathscr{G}_{N,A})X, X\rangle\\ &= \langle PX, X\rangle_{\mathscr{X}}.\end{aligned} \tag{11.26}$$

From the definition of $B(p)$, see (11.20), we have

$$B(p)X = \mathscr{O}_{C,A}X - S(p)\mathscr{O}_{N,A}X = \mathscr{O}_{C,A}X - S(p)M_S^*\mathscr{O}_{C,A}X = (I - M_S M_S^*)\mathscr{O}_{C,A}x. \tag{11.27}$$

Therefore $B(p)x \in \mathscr{H}(S)$, and we deduce from (11.26) and (11.27):

$$\begin{aligned}\|B(p)x\|^2_{\mathscr{H}(S)} &= \|(I - M_S M_S^*)\mathscr{O}_{C,A}x\|^2_{\mathscr{H}(S)}\\ &= \langle (I - M_S M_S^*)\mathscr{O}_{C,A}x, \mathscr{O}_{C,A}x\rangle_{\mathbf{H}_2(\mathbb{B},\mathscr{Y})} = \langle Px, x\rangle_{\mathscr{X}}.\end{aligned}$$

We now show that (2) $\Longrightarrow$ (3). Equality (11.21) implies that the kernel

$$\langle Px,x\rangle_{\mathscr{X}} K_S(p,q) - B(p)x(B(q)x)^* \tag{11.28}$$

is positive. We now have two cases: If $Px \neq 0$ then the positivity of (11.28) implies

$$\begin{pmatrix} \langle Px,x\rangle_{\mathscr{X}} & (B(q)x)^* \\ B(p)x & K_S(p,q) \end{pmatrix} \geq 0. \tag{11.29}$$

On the other hand, if $Px = 0$ then (11.28) implies that $B(p)x$ is identically 0 and so (11.29) follows. Since (11.29) holds for every $x \in \mathscr{X}$, the kernel (11.22) is positive in $\mathbb{B}$.
We now prove that (3) $\Longrightarrow$ (4). Let

$$f(p) = \sum_{j=1}^{n} \begin{pmatrix} x_j \\ (1-p\bar{a}_j)^{-\star} y_j \end{pmatrix} \tag{11.30}$$

where $x_j \in \mathscr{X}$, $y_j \in \mathscr{Y}$, $a_j \in \mathbb{B}$. We will show that

$$\langle \mathbb{P}f, f\rangle_{\mathscr{X}\oplus \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})} \geq 0. \tag{11.31}$$

Since the set of vectors of the form (11.30) is dense in $\mathscr{X}\oplus \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$, assertion (4) follows from (11.31). To verify the validity of (11.31) we observe that

$$\begin{aligned}\langle \mathbb{P}x,x\rangle &= \sum_{i,j=1}^{n} \left\langle \begin{pmatrix} P & M_B^* \\ M_B & M_S M_S^* \end{pmatrix} \begin{pmatrix} x_j \\ (1-p\bar{a}_j)^{-\star} y_j \end{pmatrix}, \begin{pmatrix} x_i \\ (1-p\bar{a}_i)^{*} y_i \end{pmatrix} \right\rangle \\ &= \sum_{i,j=1}^{n} \left\langle \begin{pmatrix} P & B(a_j)^* \\ B(a_i) & K_S(a_i,a_j) \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix}, \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right\rangle \geq 0\end{aligned}$$

by the positivity of the kernel in (11.22).
Finally, we show that (4) $\Longrightarrow$ (1). Since $I - M_S M_S^* \geq 0$, it follows that $S \in \mathcal{S}(\mathscr{H}_1, \mathscr{H}_2, \mathbb{B})$. The condition $\mathbb{P} \geq 0$ can be written in an equivalent form as the positivity of the matrix

$$\begin{pmatrix} I & \mathscr{O}_{N,A} & M_S^* \\ \mathscr{O}_{N,A}^* & \mathscr{G}_{C,A} & \mathscr{O}_{C,A}^* \\ M_S & \mathscr{O}_{C,A} & I \end{pmatrix} : \begin{pmatrix} \mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B}) \\ \mathscr{X} \\ \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B}) \end{pmatrix} \rightarrow \begin{pmatrix} \mathrm{H}^2_{\mathscr{H}_1}(\mathbb{B}) \\ \mathscr{X} \\ \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B}) \end{pmatrix}, \tag{11.32}$$

in fact the Schur complement of the $(1,1)$ entry of this matrix is

$$\begin{pmatrix} \mathscr{G}_{C,A} & \mathscr{O}_{C,A}^* \\ \mathscr{O}_{C,A} & I \end{pmatrix} - \begin{pmatrix} \mathscr{O}_{N,A}^* \\ M_S \end{pmatrix} \begin{pmatrix} \mathscr{O}_{N,A} & M_S^* \end{pmatrix} = \begin{pmatrix} P & M_B^* \\ M_B & I - M_S M_S^* \end{pmatrix} = \mathbb{P}$$

and thus the positivity of (11.32) is equivalent to the positivity of (11.22). On the other hand, the positivity of (11.32) is also equivalent to the positivity of the Schur complement of the (3,3) entry, that is

$$\begin{pmatrix} I & \mathscr{O}_{N,A} \\ \mathscr{O}_{N,A}^* & \mathscr{G}_{C,A} \end{pmatrix} - \begin{pmatrix} M_S^* \\ \mathscr{O}_{C,A}^* \end{pmatrix} \begin{pmatrix} M_S & \mathscr{O}_{C,A} \end{pmatrix} = \begin{pmatrix} I - M_S^* M_S & \mathscr{O}_{N,A} - M_S^* \mathscr{O}_{C,A} \\ \mathscr{O}_{N,A}^* - \mathscr{O}_{C,A}^* M_S & 0 \end{pmatrix},$$

therefore we conclude that $M_S^* \mathscr{O}_{C,A} = \mathscr{O}_{N,A}$. $\square$

11.4 Interpolation in $\mathcal{S}(\mathcal{H}_1,\mathcal{H}_2,\mathbb{B})$. The non degenerate case

We now assume that the operator P is boundedly invertible, and wish to give a description of all solutions in terms of a linear fractional transformation. This is the content of the this section together with the next three ones. The coefficient operator matrix of this linear fractional transformation is given by the function Θ defined in (11.39).

Theorem 11.4.1. *A function $S \in \mathcal{S}(\mathcal{H}_1,\mathcal{H}_2,\mathbb{B})$ is a solution to the interpolation problem if and only if the kernel*

$$\sum_{k=0}^{\infty} p^k \begin{pmatrix} I_{\mathcal{H}_2} & -S(p) \end{pmatrix} \Theta(p) J \Theta(q)^* \begin{pmatrix} I_{\mathcal{H}_2} & -S(q) \end{pmatrix}^* \overline{q}^k \tag{11.33}$$

is positive definite in $\mathbb{B}$.

Before we prove this theorem we need some preliminary results. We set

$$J = \begin{pmatrix} I_{\mathcal{H}_2} & 0 \\ 0 & -I_{\mathcal{H}_1} \end{pmatrix}$$

and let $\mathcal{K} = \mathcal{X} \oplus \mathcal{H}_2 \oplus \mathcal{H}_1$ be endowed with metric

$$\widetilde{J} = \begin{pmatrix} P & 0 \\ 0 & J \end{pmatrix}.$$

Lemma 11.4.2. *The space*

$$\mathcal{K}_0 := \operatorname{ran} \begin{pmatrix} A \\ C \\ N \end{pmatrix}$$

is a uniformly positive subspace of $\mathcal{K}$.

Proof. It follows from the Stein equation

$$P - A^*PA = \begin{pmatrix} C^* & N^* \end{pmatrix} J \begin{pmatrix} C \\ N \end{pmatrix} \tag{11.34}$$

that $\langle Px, x\rangle_{\mathcal{X}} = [Tx, Tx]_{\mathcal{K}}$, where we have set

$$T = \begin{pmatrix} A \\ C \\ N \end{pmatrix}.$$

Since P is strictly positive definite, there exists $\varepsilon > 0$ such that

$$\langle Px, x\rangle_{\mathcal{X}} \geq \varepsilon \|x\|^2_{\mathcal{X}}, \quad \forall x \in \mathcal{X}.$$

Then, with

$$\|x\|_{\widetilde{J}} = [Tx, \tilde{J}Tx]_{\mathscr{K}},$$

we have

$$\begin{aligned}\|Tx,Tx\|^2_{\widetilde{J}} &= \langle Ax,Ax\rangle_{\mathscr{X}} + \langle Cx,Cx\rangle_{\mathscr{H}_2} + \langle Nx,Nx\rangle_{\mathscr{H}_1}\\ &\le (\|A\|^2+\|C\|^2+\|N\|^2)\|x\|^2_{\mathscr{X}}\\ &= \frac{\|A\|^2+\|C\|^2+\|N\|^2}{\varepsilon}\varepsilon\|x\|^2_{\mathscr{X}}\\ &\le \frac{\|A\|^2+\|C\|^2+\|N\|^2}{\varepsilon}[Tx,Tx]_{\mathscr{K}}.\end{aligned}$$

□

Therefore, see Theorem 5.8.11, the Krein-space orthogonal complement $\mathscr{K}_0^{[\perp]}$ of $\mathscr{G}_0$ is also a Krein space with the inner product inherited from $\mathscr{K}$ with inertia equal to the complement of the inertia of P with respect to the inertia of $\widetilde{J}$ on $\mathscr{K}$, that is, with inertia equal to that of J.

Lemma 11.4.3. *There is an isometry* $\begin{pmatrix} B\\ D_1\\ D_2\end{pmatrix}$ *from* $(\mathscr{H}_2\oplus\mathscr{H}_1, J)$ *to* $\mathscr{K}$ *so that*

$$\mathscr{K}_0^{[\perp]} = \operatorname{ran}\begin{pmatrix} B\\ D_1\\ D_2\end{pmatrix}.$$

Proof. $\mathscr{K}_0^{\perp}$ is a Krein space and so has a fundamental decomposition

$$\mathscr{K}_0^{\perp} = \mathscr{V}_+ + \mathscr{V}_-.$$

The Hilbert spaces $\mathscr{H}_2$ and $(\mathscr{V}_+, [\cdot,\cdot]_{\mathscr{K}})$ are isomorphic, and similarly for $\mathscr{H}_1$ and $(\mathscr{V}_-, -[\cdot,\cdot]_{\mathscr{K}})$. It suffices to take a unitary map U_+ from $\mathscr{H}_2$ to $\mathscr{V}_+$ and unitary map U_- from $\mathscr{H}_2$ to $\mathscr{V}_-$ (the latter being endowed with $-[\cdot,\cdot]_{\mathscr{K}}$). Then the map

$$H(y+v) = U_+y + U_-v$$

answers the question. □

For such an isometry, we have

$$B^*PB + D_1^*D_1 - D_2^*D_2 = J \quad\text{and}\quad B^*PT + D_1^*E - D_2^*N = 0$$

which together with (11.34) can be written in the matrix form as

$$\begin{pmatrix} T^* & E^* & N^*\\ B^* & D_1^* & D_2^*\end{pmatrix}\begin{pmatrix} P & 0\\ 0 & J\end{pmatrix}\begin{pmatrix} T & B\\ E & D_1\\ N & D_2\end{pmatrix} = \begin{pmatrix} P & 0 & 0\\ 0 & I_{\mathscr{H}_2} & 0\\ 0 & 0 & -I_{\mathscr{H}_1}\end{pmatrix}. \tag{11.35}$$

Furthermore, the Krein-space adjoint of the same isometry equals

$$\begin{pmatrix} B \\ D_1 \\ D_2 \end{pmatrix}^* = J\begin{pmatrix} B^* & D_1^* & D_2^* \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & J \end{pmatrix}.$$

Therefore, the orthogonal (Krein-space) projection $\mathscr{P}_{\mathscr{K}_0^{[\perp]}}$ of $\mathscr{K}$ onto $\mathscr{K}_0^{[\perp]}$ is given by

$$\mathscr{P}_{\mathscr{K}_0^{[\perp]}} = \begin{pmatrix} B \\ D_1 \\ D_2 \end{pmatrix}\begin{pmatrix} B \\ D_1 \\ D_2 \end{pmatrix}^* = \begin{pmatrix} B \\ D_1 \\ D_2 \end{pmatrix} J\begin{pmatrix} B^* & D_1^* & D_2^* \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & J \end{pmatrix}. \tag{11.36}$$

On the other hand the Krein-space orthogonal projection of $\mathscr{K}$ onto $\mathscr{K}_0$ is given by

$$\mathscr{P}_{\mathscr{K}_0} = \begin{pmatrix} T \\ E \\ N \end{pmatrix}\begin{pmatrix} T \\ E \\ N \end{pmatrix}^* = \begin{pmatrix} T \\ E \\ N \end{pmatrix} P^{-1}\begin{pmatrix} T^* & E^* & N^* \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & J \end{pmatrix}. \tag{11.37}$$

Substituting (11.36) and (11.37) into equality $\mathscr{P}_{\mathscr{K}_0} + \mathscr{P}_{\mathscr{K}_0^{[\perp]}} = I$ and multiplying the resulting equality by $\begin{pmatrix} P^{-1} & 0 \\ 0 & J \end{pmatrix}$ on the right we get

$$\begin{pmatrix} T & B \\ E & D_1 \\ N & D_2 \end{pmatrix}\begin{pmatrix} P^{-1} & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} T^* & E^* & N^* \\ B^* & D_1^* & D_2^* \end{pmatrix} = \begin{pmatrix} P^{-1} & 0 \\ 0 & J \end{pmatrix}. \tag{11.38}$$

We conclude that given T,E,N such that (E,T) and (N,T) are output stable and P is a strictly positive definite operator satisfying (11.34), one can always find operators B, D_1 and D_2 such that relations (11.35) and (11.38) hold.

We now introduce the operator-valued function Θ which will give a linear fractional representation of the set of all solutions of the interpolation problem in the nondegenerate case.

Proposition 11.4.4. *With the above notations, let*

$$\Theta(p) = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} + p\begin{pmatrix} E \\ N \end{pmatrix} \star (I - pA)^{-\star} B. \tag{11.39}$$

(1) *Then the following identities hold:*

$$\sum_{j=0}^{\infty} p^k \left(J - \Theta(p)J\Theta(q)^*\right)\bar{q}^k = \begin{pmatrix} E \\ N \end{pmatrix} \star (I - pA)^{-\star} P^{-1}\left(I - \bar{q}A^*\right)^{-\star_r} \star_r \begin{pmatrix} E^* & N^* \end{pmatrix}, \tag{11.40}$$

where $p,q\in\mathbb{B}$.
(2) *For* $x\in(-1,1)$ *we have*

$$J-\Theta(x)J\Theta(x)^* = (1-x^2)\begin{pmatrix}E\\N\end{pmatrix}(I-xA)^{-1}P^{-1}(I-xA^*)^{-1}\begin{pmatrix}E^* & N^*\end{pmatrix}, \tag{11.41}$$

$$J-\Theta(x)^*J\Theta(x) = (1-x^2)B^*(I-xA^*)^{-1}P(I-xA)^{-1}B. \tag{11.42}$$

(3) *The functions* Θ_{22}^{-1} *and* $\Theta_{22}^{-1}\Theta_{21}$ *extend to operator-valued Schur multipliers.*

Proof.
(1) and (2): Note that (11.40) holds due to (11.38), and (11.41) is the specialization of (11.40) for $p=q=x\in(-1,1)$. Furthermore, (11.42) holds due to (11.35).

(3) Take also $p=q=x\in(-1,1)$ in equality (11.40). Decomposing Θ as

$$\Theta=\begin{pmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{21} & \Theta_{22}\end{pmatrix}:\begin{pmatrix}\mathscr{H}_2\\ \mathscr{H}_1\end{pmatrix}\to\begin{pmatrix}\mathscr{H}_2\\ \mathscr{H}_1\end{pmatrix} \tag{11.43}$$

we see from the $(2,2)$-entries in the equalities (11.41) and (11.42) that

$$I_{\mathscr{H}_1}+\Theta_{21}(x)\Theta_{21}(x)^*\le\Theta_{22}(x)\Theta_{22}(x)^*$$
$$I_{\mathscr{H}_2}+\Theta_{12}(x)^*\Theta_{12}(x)\le\Theta_{22}(x)^*\Theta_{22}(x),$$

from which we conclude that $\Theta_{22}(x)$ is boundedly invertible and $\|\Theta_{22}(x)^{-1}\Theta_{21}(x)\|<1$ for each $x\in(-1,1)$. Using Proposition 7.4.1 we obtain a slice hyperholomorphic inverse of Θ_{22} in $|p|<r$. By (11.40) we have that

$$\sum_{n=0}^{\infty}p^n(I_{\mathscr{U}}-(\Theta_{22}(p))^{-\star}((\Theta_{22}(q))^{-\star})^*-((\Theta_{22}(p))^{-\star}\Theta_{21}(p))((\Theta_{22}(q)\Theta_{21}(q))^{-\star})^*)\overline{q}^n$$

is positive definite in $|p|<r$, and so, by Corollary 8.4.4 both Θ_{22}^{-1} and $\Theta_{22}^{-1}\Theta_{21}$ extend to operator-valued Schur multipliers. □

Remark 11.4.5. We note that in the complex-valued case we also have (rather than (11.42))

$$\sum_{j=0}^{\infty}\bar{w}^k(J-\Theta(w)^*J\Theta(z))z^k=B^*(I-\bar{w}A^*)^{-1}P(I-zA)^{-1}B. \tag{11.44}$$

Proof of Theorem 11.4.1. Since P is boundedly invertible, the positivity of the kernel (11.22) is equivalent to the positivity of the kernel

$$K_S(p,q)-B(p)\star P^{-1}\star_r B(q)^*.$$

In turn, this kernel can be rewritten in the equivalent form

$$
\begin{aligned}
K_S(p,q) - B(p)\star P^{-1}\star_r B(q)^* &= \sum_{k=0}^{\infty} p^k (I_{\mathcal{H}_2} - S(p)S(q)^*)\overline{q}^k - \\
&- \begin{pmatrix} I_{\mathcal{H}_2} & -S(p)\end{pmatrix}\star\begin{pmatrix} C\\ N\end{pmatrix}\star(I_{\mathcal{H}(K_S)} - pA)^{-\star}\star P^{-1}\star_r (I_{\mathcal{H}(K_S)} - A^*\overline{q})^{-\star_r}\star_r \\
&\qquad \star_r \begin{pmatrix} C^* & N^*\end{pmatrix}\star_r\begin{pmatrix} I_{\mathcal{H}_2}\\ S(q)^*\end{pmatrix}\\
&= \begin{pmatrix} I_{\mathcal{H}_2} & -S(p)\end{pmatrix}\star\Big(\sum_{k=0}^{\infty} p^k J\overline{q}^k - \begin{pmatrix} C\\ N\end{pmatrix}\star(I_{\mathcal{H}(K_S)} - pA)^{-\star}\star P^{-1}\star_r \\
&\qquad \star_r (I_{\mathcal{H}(K_S)} - A^*\overline{q})^{-\star_r}\star_r \begin{pmatrix} C^* & N^*\end{pmatrix}\Big)\star_r\begin{pmatrix} I_{\mathcal{H}_2}\\ S(q)^*\end{pmatrix}.
\end{aligned}
$$

and, using (11.40) this last expression can be rewritten as

$$
\begin{aligned}
K_S(p,q) - B(p)\star P^{-1}\star_r B(q)^* &= \begin{pmatrix} I_{\mathcal{H}_2} & -S(p)\end{pmatrix}\star\left(\sum_{k=0}^{\infty} p^k \left(J - (J - \Theta(p)J\Theta(q)^*)\right)\overline{q}^k\right)\star_r \\
&\qquad \star_r\begin{pmatrix} I_{\mathcal{H}_2}\\ -S(q)^*\end{pmatrix}.
\end{aligned}
$$

Thus we have

$$
K_S(p,q) - B(p)\star P^{-1}\star_r B(q)^* = \sum_{k=0}^{\infty} p^k \begin{pmatrix} I_{\mathcal{H}_2} & -S(p)\end{pmatrix}\star\Theta(p)J\Theta(q)^*\star_r\begin{pmatrix} I_{\mathcal{H}_2}\\ -S(q)^*\end{pmatrix}\overline{q}^k,
$$

which ends the proof. □

The kernel (11.33) can be rewritten as

$$
\sum_{k=0}^{\infty} p^k \left(U(p)U(q)^* - V(p)V(q)^*\right)\overline{q}^k
$$

where

$$
U(p) = \Theta_{11}(p) - S(p)\star\Theta_{21}(p) \quad \text{and} \quad V(p) = -\Theta_{12}(p) + S(p)\star\Theta_{22}(p).
$$

To conclude the description of all solutions of the interpolation $\mathcal{S}(\mathcal{H}_1,\mathcal{H}_2,\mathbb{B})$ we need to prove the existence of a Schur function $E\in\mathcal{S}(\mathcal{H}_1,\mathcal{H}_2,\mathbb{B})$ such that

$$
U(p)\star E(p) = V(p). \tag{11.45}
$$

This is easily done when one considers the interpolation with a finite number of interpolating points. See Section 11.5. This special result in turn allows to prove a version of Leech's theorem in the present setting, and this will give the factorization (11.45).

11.5 Interpolation: The case of a finite number of interpolating conditions

We now study the interpolation problem $\mathbf{IP}(\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B}))$ when only a finite number of interpolating points and directions are given. Note that we cannot use Corollary 8.4.4 (see also Remark 8.4.5) when the coefficient spaces are infinite dimensional since the value of a function at a point corresponds then to an infinite number of tangential interpolation conditions.

We consider the case where

$$A = \text{diag}\,(M_{\overline{p_1}}, M_{\overline{p_2}}, \cdots, M_{\overline{p_m}}), \quad C = \begin{pmatrix} M_{c_1} & M_{c_2} & \cdots & M_{c_m} \end{pmatrix}, \tag{11.46}$$

and

$$N = \begin{pmatrix} M_{n_1} & M_{n_2} & \cdots & M_{n_m} \end{pmatrix}, \tag{11.47}$$

with $(p_k, c_k, n_k) \in \mathbb{B} \times \mathscr{H}_2 \times \mathscr{H}_1$, $k = 1, \ldots, m$. We simplify the notation and set $M_\eta = \eta$ for the various vectors. Using (5.14) we note that

$$C^*C - N^*N =$$

$$= \begin{pmatrix} \langle c_1, c_1\rangle - \langle n_1, n_1\rangle & \langle c_2, c_1\rangle - \langle n_2, n_1\rangle & \cdots & \langle c_m, c_1\rangle - \langle n_m, n_1\rangle \\ \langle c_1, c_2\rangle - \langle n_1, n_2\rangle & \langle c_2, c_2\rangle - \langle n_2, n_2\rangle & \cdots & \langle c_m, c_2\rangle - \langle n_m, n_2\rangle \\ & & \langle c_k, c_\ell\rangle - \langle n_k, n_\ell\rangle & \\ \langle c_1, c_m\rangle - \langle n_1, n_m\rangle & \langle c_2, c_m\rangle - \langle n_2, n_m\rangle & \cdots & \langle c_m, c_m\rangle - \langle n_m, n_m\rangle \end{pmatrix}.$$

The Gram operator is now a $m \times m$ matrix and its (ℓ, k) entry is

$$\sum_{t=0}^{\infty} p_\ell^t \left(\langle c_k, c_\ell\rangle - \langle n_k, n_\ell\rangle\right) \overline{p_k}^t.$$

The interpolation condition

$$\sum_{t=0}^{\infty} A^{*t} C^* S_t = N^*$$

becomes

$$\sum_{t=0}^{\infty} A^{*t} C^* S_t \xi = N^* \xi,$$

i.e.

$$\sum_{t=0}^{\infty} p_\ell^t \langle S_t \xi, c_\ell\rangle = \langle \xi, n_\ell\rangle, \quad \ell = 1, \ldots, m, \quad \forall \xi \in \mathscr{H}_1.$$

Theorem 11.5.1. *Assume the $c_j \neq 0$ (but not necessarily linearly independent). Then, a necessary and sufficient condition for the interpolation problem corresponding to* (11.46)-(11.47) *to be solvable is that the Gram matrix is nonnegative.*

Proof. We proceed in a number of steps.

STEP 1: *The matrix G with (ℓ,k) entry*

$$\sum_{t=0}^{\infty} p_\ell^t \langle c_k, c_\ell\rangle \overline{p_k}^t$$

is invertible.

Indeed, let

$$f_k(p) = c_k \star (1-p\overline{p_k})^{-\star} = \sum_{t=0}^{\infty} p^t c_k \overline{p_k}^t, \quad k=1,\ldots,m.$$

Then the functions $f_k \in \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$. Moreover, let $\alpha_1,\ldots,\alpha_m \in \mathbb{H}$ be such that

$$\sum_{k=1}^{m} f_k(p)\alpha_k = 0, \quad \forall p \in \mathbb{B}.$$

Then, by the reproducing kernel property

$$\sum_{k=1}^{m} \overline{\alpha_k} \langle h(p_k), c_k\rangle_{\mathscr{H}_2} = 0, \quad \forall h \in \mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B}).$$

But, from the interpolation problem in the Hardy space, see Section 11.2, there exist functions in $\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})$ that

$$\langle h(p_1), c_1\rangle_{\mathscr{H}_2} \neq 0, \quad \text{and} \quad \langle h(p_k), c_k\rangle_{\mathscr{H}_2} = 0, \quad k=2,\ldots,m.$$

Thus $\alpha_1 = 0$, and similarly for the other indices. So the functions $f_1,\ldots,f_m$ are linearly independent, and $G > 0$ since

$$G_{\ell,k} = \sum_{t=0}^{\infty} p_\ell^t \langle c_k, c_\ell\rangle \overline{p_k}^t = \langle f_k, f_\ell\rangle_{\mathrm{H}^2_{\mathscr{H}_2}(\mathbb{B})}.$$

STEP 2: *The matrix $G(\varepsilon)$ with (ℓ,k) entry*

$$\sum_{t=0}^{\infty} p_\ell^t \left(\langle c_k, c_\ell\rangle - \varepsilon^2 \langle n_k, n_\ell\rangle\right) \overline{p_k}^t$$

is invertible for all value of $\varepsilon \in (0,1)$.

We cannot use the notion of determinant but will use Schur complements (and in particular Proposition 4.3.13) to prove the claim. Write

$$G(\varepsilon) = \begin{pmatrix} g_{11}(\varepsilon) & b(\varepsilon) \\ b(\varepsilon)^* & D(\varepsilon) \end{pmatrix}$$

with

$$g_{11}(\varepsilon)=\frac{\langle c_1,c_1\rangle-\varepsilon^2\langle n_1,n_1\rangle}{1-|p_1|^2}.$$

From Proposition 4.3.13 we get

$$G(\varepsilon)>0 \qquad \Longleftrightarrow \qquad \begin{cases} \begin{cases}\varepsilon\in[0,1) \text{ if } \langle c_1,c_1\rangle=\langle n_k,n_\ell\rangle\\ \varepsilon\in[0,1], \text{ otherwise}\end{cases}\\ \\ \text{and the } (1,1) \text{ entry of the Hermitian matrix}\\ \ D(\varepsilon)g_{11}(\varepsilon)-b(\varepsilon)^*b(\varepsilon)\\ \text{is strictly positive.}\end{cases}$$

We now reiterate the argument. But

$$(D(\varepsilon)g_{11}(\varepsilon)-b(\varepsilon)^*b(\varepsilon))_{11}$$

is a polynomial with real coefficients, and is a decreasing function of ε and strictly positive at the origin (these last two claims come from the definition of $G(\varepsilon)$ and from Step 1). Thus once more we get that at most the value $\varepsilon=1$ leads to a non -invertible matrix.

When $G(\varepsilon)>0$ we can apply the analysis in Section 11.4 and in particular Lemma 11.4.2 holds. So we can build the corresponding function defined by (11.39), which we denote by Θ^ε.

STEP 3: *Let $\varepsilon\in[0,1]$ be such that $G(\varepsilon)>0$. Then the block entry $\Theta^\varepsilon_{11}(x)$ is invertible and $\|\Theta^\varepsilon_{21}(x)(\Theta^\varepsilon_{11})^{-1}(x)\|<1$ in a real neighborhood of $x=1$.*

Indeed, since A is a finite matrix, of norm strictly less than 1 we can set $x=1$ in (11.41) and (11.42) and obtain:

$$J-\Theta^\varepsilon(1)J(\Theta^\varepsilon(1))^*=0 \quad\text{and}\quad J-(\Theta^\varepsilon(1))^*J\Theta^\varepsilon(1)=0.$$

Thus

$$\Theta^\varepsilon_{11}(1)(\Theta^\varepsilon_{11}(1))^* \;=\; I+\Theta^\varepsilon_{12}(1)(\Theta^\varepsilon_{12}(1))^*, \tag{11.48}$$

$$(\Theta^\varepsilon_{11}(1))^*\Theta^\varepsilon_{11}(1) \;=\; I+(\Theta^\varepsilon_{21}(1))^*\Theta^\varepsilon_{21}(1). \tag{11.49}$$

We get in particular that $\Theta^\varepsilon_{11}(1)$ is boundedly invertible. Furthermore, (11.49) implies that

$$I=(\Theta^\varepsilon_{11})^{-1}(1)^*((\Theta^{\varepsilon}{}^{-1}_{11})(1))+(\Theta^\varepsilon_{21}(1)(\Theta^\varepsilon_{11})^{-1}(1))^*(\Theta^\varepsilon_{21}(1)(\Theta^\varepsilon_{11})^{-1}(1)), \tag{11.50}$$

and in particular $\|\Theta^\varepsilon_{21}(1)(\Theta^\varepsilon_{11})^{-1}(1)\|<1$. By continuity (since Θ^ε is a rational function of x), $\Theta^\varepsilon_{11}(x)$ is boundedly invertible and $\|\Theta^\varepsilon_{21}(x)(\Theta^\varepsilon_{11})^{-1}(x)\|<1$ in a real neighborhood of $x=1$.

STEP 4: *Let S be any Schur multiplier. Then,* $(\Theta^{\varepsilon}_{11}(x) - S(x)\Theta^{\varepsilon}_{21}(x))$ *is invertible in an open set of the form* $(u,1)$.

For *real* $x \in (-1,1)$ we have that $\|S(x)\| \leq 1$ and the claim follows from the previous step since

$$\Theta^{\varepsilon}_{11}(x) - S(x)\Theta^{\varepsilon}_{21}(x) = \left(I - S(x)\Theta^{\varepsilon}_{21}(x)(\Theta^{\varepsilon}_{11})^{-1}(x)\right)\Theta_{11}(x)$$

in an open set of the form $(u,1)$. We note that both $(\Theta^{\varepsilon}_{11}(x) - S(x)\Theta^{\varepsilon}_{21}(x))$ and its inverse are restriction of slice hyperholomorphic functions in a real open interval.

STEP 5: *Let* $\varepsilon \in [0,1]$ *be such that* $G(\varepsilon) > 0$. *Then, the linear fractional transformation*

$$T_{\Theta^{\varepsilon}}(E) \tag{11.51}$$

describes the set of all solutions of the interpolation problem

$$\sum_{t=0}^{\infty} p^t_{\ell}\langle S_t\xi, c_{\ell}\rangle = \varepsilon\langle \xi, n_{\ell}\rangle, \quad \ell = 1,\dots,m, \quad \forall \xi \in \mathscr{H}_1,$$

when E varies in the class $\mathcal{S}(\mathscr{H}_1, \mathscr{H}_2, \mathbb{B})$.

From Theorem 11.4.1 we know that S is a solution of the interpolation problem if and only if the kernel (11.33)

$$\sum_{k=0}^{\infty} p^k\left(U^{\varepsilon}(p)U^{\varepsilon}(q)^* - V^{\varepsilon}(p)V^{\varepsilon}(q)^*\right)\overline{q}^k \tag{11.52}$$

is positive in $\mathbb{B}$, where

$$U^{\varepsilon}(p) = \Theta^{\varepsilon}_{11}(p) - S(p)\star\Theta^{\varepsilon}_{21}(p) \quad \text{and} \quad V^{\varepsilon}(p) = -\Theta^{\varepsilon}_{12}(p) + S(p)\star\Theta^{\varepsilon}_{22}(p).$$

In view of Step 4, $U^{\varepsilon}(x)$ is invertible in a set of the form $(u,1)$. Thus for $x,y \in (u,1)$ we can rewrite (11.52) as

$$U^{\varepsilon}(x)\frac{I_{\mathscr{H}_2} - E^{\varepsilon}(x)E^{\varepsilon}(y)^*}{1-xy}(U^{\varepsilon}(y))^*,$$

with $E^{\varepsilon} = (U^{\varepsilon})^{-1}V^{\varepsilon}$. It follows that the kernel

$$\frac{I_{\mathscr{H}_2} - E^{\varepsilon}(x)E^{\varepsilon}(y)^*}{1-xy}$$

is positive definite in $(u,1)$, and hence has an extension to a Schur multiplier, see Theorem 8.4.4. Still denoting by E^{ε} this extension we have

$$(\Theta^{\varepsilon}_{11} - S\star\Theta^{\varepsilon}_{21})\star E^{\varepsilon} = -\Theta^{\varepsilon}_{12} + S\star\Theta^{\varepsilon}_{22}.$$

Thus, we have

$$S\star(\Theta^{\varepsilon}_{21}\star E^{\varepsilon} + \Theta^{\varepsilon}_{22}) = \Theta^{\varepsilon}_{11}\star E^{\varepsilon} + \Theta^{\varepsilon}_{12}.$$

We now show that we can divide (with respect to the star product) by $(\Theta_{21}\star E+\Theta_{22})$, and thus obtain the linear fractional transformation (11.51) of S in terms of E^{ε}. Let $x\in(-1,1)$. By the proof of Proposition 11.4.4 we know that $\Theta_{22}(x)$ is boundedly invertible and that

$$\Theta_{22}^{\varepsilon}(x)^{-1}\Theta_{22}^{\varepsilon}(x)^{-*}+\Theta_{22}^{\varepsilon}(x)^{-1}\Theta_{21}^{\varepsilon}(x)(\Theta_{22}^{\varepsilon}(x)^{-1}\Theta_{21}^{\varepsilon}(x))^{*}\leq I_{\mathscr{H}_1}.$$

It follows that $\|\Theta_{22}^{\varepsilon}(x)^{-1}\Theta_{21}^{\varepsilon}(x)\|<1$ and therefore

$$\|\Theta_{22}^{\varepsilon}(x)^{-1}\Theta_{21}^{\varepsilon}(x)E^{\varepsilon}(x)\|<1,\quad x\in(-1,1).$$

By Proposition 11.4.4, the $\mathbf{B}(\mathscr{H}_1)$-valued function

$$I_{\mathscr{H}_1}+(\Theta_{22}^{\varepsilon})^{-\star}\star\Theta_{21}^{\varepsilon}$$

is invertible (with respect to the star product) in a neighborhood of the origin. So S is given by the linear fractional transformation in some open set, and in all of $\mathbb{B}$ by slice hyperholomorphic extension.

We conclude the proof of the theorem taking subsequences using Theorem 8.5.1. □

11.6 Leech's theorem

The following result is the quaternionic version of a factorization theorem originally due to Leech, and which has found numerous applications in operator theory. See [233] for the original paper (an unpublished manuscript, written in 1971-1972) and see [220], [250], [41], [22] for background on, and applications of, the theorem in the complex variable setting.

In the statement, the hypothesis on $\mathscr{H}_1$ is crucial.

Theorem 11.6.1. *Let $\mathscr{H}_1,\mathscr{H}_2$ and $\mathscr{H}_3$ be two sided quaternionic Hilbert spaces. We assume that the space $\mathscr{H}_1$ is separable and has a Hilbert space basis made of vectors which commute with the quaternions. Let A and B be slice hyperholomorphic in $\mathbb{B}$ and respectively $\mathbf{B}(\mathscr{H}_2,\mathscr{H}_1)$- and $\mathbf{B}(\mathscr{H}_3,\mathscr{H}_1)$-valued and assume that the $\mathbf{B}(\mathscr{H}_1)$-valued kernel*

$$K_{A,B}(p,q)=\sum_{n=0}^{\infty}p^n(A(p)A(q)^*-B(p)B(q)^*)\overline{q}^n \tag{11.53}$$

is positive definite in $\mathbb{B}$. Then there exists $S\in\mathcal{S}(\mathscr{H}_3,\mathscr{H}_2,\mathbb{B})$ such that $B=A\star S$.

Proof. The strategy is as follows: Use the positivity condition (11.53) to solve a countable family of finite dimensional interpolation problems and use Theorem 8.5.1 to find a solution S such that $B=A\star S$. We consider $p_1,p_2,\ldots$ a dense subset of $\mathbb{B}$, and denote by $u_1,u_2,\ldots$ a Hilbert space basis of $\mathscr{H}_1$ which commutes with the quaternions. We proceed in a number of steps.

STEP 1: *Let $N,L\in\mathbb{N}$. There is $S_{N,L}\in\mathcal{S}(\mathscr{H}_3,\mathscr{H}_2,\mathbb{B})$ such that*

$$(A\star S_{N,L})(p_j)u_\ell = B(p_j)u_\ell,\quad j=1,2,\ldots N,\quad \ell=1,\ldots,L.$$

Indeed, using property (5.4) to go from the second to the third line, and using the fact that the vectors u_ℓ commutes with the quaternions we have:

$$\begin{aligned}
\langle K_{A,B}(p_j,p_k)u_m,u_\ell\rangle_{\mathscr{H}_1} &= \sum_{t=0}^\infty \langle p_j^t\left(A(p_j)A(p_k)^*-B(p_j)B(p_k)^*\right)\underbrace{\overline{p_k}^t u_m}_{\text{commute}},u_\ell\rangle_{\mathscr{H}_1}\\
&= \sum_{t=0}^\infty \langle p_j^t\left(A(p_j)A(p_k)^*u_m-B(p_j)B(p_k)^*u_m\right)\overline{p_k}^t,u_\ell\rangle_{\mathscr{H}_1}\\
&= \sum_{t=0}^\infty \langle \left(A(p_j)A(p_k)^*u_m-B(p_j)B(p_k)^*u_m\right)\overline{p_k}^t,\underbrace{\overline{p_j}^t u_\ell}_{\text{commute}}\rangle_{\mathscr{H}_1}\\
&= \sum_{t=0}^\infty \langle A(p_k)^*u_m\overline{p_k}^t,A(p_j)^*u_\ell\overline{p_j}^t\rangle_{\mathscr{H}_2}-\langle B(p_k)^*u_m\overline{p_k}^t,B(p_j)^*u_\ell\overline{p_j}^t\rangle_{\mathscr{H}_3}\\
&= \sum_{t=0}^\infty p_j^t\left\{\langle A(p_k)^*u_m,A(p_j)^*u_\ell\rangle_{\mathscr{H}_2}-\langle B(p_k)^*u_m,B(p_j)^*u_\ell\rangle_{\mathscr{H}_3}\right\}\overline{p_k}^t\\
&= \sum_{t=0}^\infty p_j^t\left\{\langle c_{k,m},c_{j,\ell}\rangle_{\mathscr{H}_2}-\langle n_{k,m},n_{j,\ell}\rangle_{\mathscr{H}_3}\right\}\overline{p_k}^t,
\end{aligned}$$

where we have set $c_{j,\ell}=A(p_j)^*u_\ell$ and $n_{j,\ell}=B(p_j)^*u_\ell$. Since $K_{A,B}$ is a positive definite kernel we get that the Gram matrix is non-negative. By the result in Section 11.5 on the finite dimensional case, there exists $S_{N,L}\in\mathcal{S}(\mathscr{H}_3,\mathscr{H}_2,\mathbb{B})$ such that

$$\sum_{t=0}^\infty p_j^t\langle (S_{N,L})_t\xi,c_{j,\ell}\rangle_{\mathscr{H}_2}=\langle\xi,n_{j,\ell}\rangle_{\mathscr{H}_3},\quad j=1,\ldots,M,\quad \ell=1,\ldots L,\ \text{and}\ \forall\xi\in\mathscr{H}_3.$$

Still using the fact that u_ℓ commutes with the quaternions the left side of this last equation can be rewritten as

$$\begin{aligned}
\sum_{t=0}^{\infty} p_j^t \langle \xi, (S_{N,L})_t^* A(p_j)^* u_\ell \rangle_{\mathscr{H}_3} &= \sum_{t=0}^{\infty} p_j^t \langle \xi, (S_{N,L})_t^* \left(\sum_{v=0}^{\infty} p_j^v A_v \right)^* u_\ell \rangle_{\mathscr{H}_3} \\
&= \sum_{t=0}^{\infty} p_j^t \langle \xi, (S_{N,L})_t^* \left(\sum_{v=0}^{\infty} A_v^* \overline{p_j}^v \right) u_\ell \rangle_{\mathscr{H}_3} \\
&= \sum_{t=0}^{\infty} p_j^t \langle \xi, (S_{N,L})_t^* \left(\sum_{v=0}^{\infty} A_v^* \underbrace{\overline{p_j}^v u_\ell}_{\text{commute}} \right) \rangle_{\mathscr{H}_3} \\
&= \sum_{t=0}^{\infty} p_j^t \langle \xi, (S_{N,L})_t^* \left(\sum_{v=0}^{\infty} A_v^* u_\ell \overline{p_j}^v \right) \rangle_{\mathscr{H}_3} \\
&= \sum_{s=0}^{\infty} \langle \xi, \left(\sum_{t+v=s} (S_{N,L})_t^* A_v^* \right) \overline{p_j^s} u_\ell \rangle_{\mathscr{H}_3} \\
&= \langle \xi, ((A \star S_{N,L})(p_j))^* u_\ell \rangle_{\mathscr{H}_3}
\end{aligned}$$

and the interpolation condition becomes

$$\langle \xi, ((A \star S_{N,L})(p_j))^* u_\ell \rangle_{\mathscr{H}_3} = \langle \xi, (B(p_j))^* u_\ell \rangle_{\mathscr{H}_3}, \quad j = 1, \ldots, N, \quad \ell = 1, \ldots L.$$

Since ξ is arbitrary we get

$$(A \star S_{N,\ell})(p_j) u_\ell = B(p_j) u_\ell, \quad j = 1, \ldots, N, \quad \ell = 1, \ldots L,$$

and so, using once more that the u_j commute with the quaternions, we get

$$(A \star S_{N,\ell})(p_j) = B(p_j), \quad j = 1, \ldots, N, \quad \ell = 1, \ldots L.$$

To conclude we first fix N and let L go to infinity. By Theorem 8.5.1 there exists $S_N \in \mathcal{S}(\mathscr{H}_3, \mathscr{H}_2, \mathbb{B})$ such that

$$(A \star S_N)(p_j) = B(p_j), \quad j = 1, \ldots, N.$$

Another application of Theorem 8.5.1 implies the existence of $S \in \mathcal{S}(\mathscr{H}_3, \mathscr{H}_2, \mathbb{B})$ such that

$$(A \star S)(p_j) = B(p_j), \quad j = 1, \ldots,$$

and this last equality extends to all of $\mathbb{B}$ by continuity. □

11.7 Interpolation in $\mathcal{S}(\mathscr{H}_1, \mathscr{H}_2, \mathbb{B})$. Nondegenerate case: Sufficiency

We conclude this chapter with the description of all solutions to the interpolation problem **IP**($\mathcal{S}(\mathscr{H}_1, \mathscr{H}_2, \mathbb{B})$) in the nondegenerate case. The case where the operator (11.24) is not boundedly invertible will be considered in a sequel to the present book. Examples of degenerate cases in the finite dimensional case have been presented earlier in the book; see Sections 10.4 and 10.5.

Theorem 11.7.1. *Assume that the operator P given by* (11.24) *is boundedly invertible, and let Θ be defined by* (11.39). *Assume furthermore that the space $\mathscr{H}_2$ is separable and has a Hilbert space basis made of vectors which commute with the quaternions. Then the set of all solutions to the interpolation* $\mathbf{IP}(\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B}))$ *is given by the linear fractional transformation*

$$S=(\Theta_{11}\star E+\Theta_{12})\star(\Theta_{21}\star E+\Theta_{22})^{-\star} \tag{11.54}$$

when E runs through $\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$.

Proof. From Theorem 11.4.1 the function $S\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B})$ is a solution of the interpolation problem $\mathbf{IP}(\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2,\mathbb{B}))$ if and only if the kernel

$$\sum_{n=0}^{\infty}p^n(U(p)U(q)^*-V(p)V(q)^*)\overline{q}^n$$

is positive definite in $\mathbb{B}$, where

$$U(p)=\Theta_{11}(p)-S(p)\star\Theta_{21}(p)\quad\text{and}\quad V(p)=-\Theta_{12}(p)+S(p)\star\Theta_{22}(p).$$

Leech's theorem implies that there exists $E\in\mathcal{S}(\mathscr{H}_1,\mathscr{H}_2)$ such that $V=U\star E$, that is

$$(\Theta_{11}-S\star\Theta_{21})\star E=-\Theta_{12}+S\star\Theta_{22},$$

or, equivalently,

$$S\star(\Theta_{21}\star E+\Theta_{22})=\Theta_{11}\star E+\Theta_{12}.$$

As in the proof of Step 5 of Theorem 11.5.1 one shows that one can divide (with respect to the star product) by $(\Theta_{21}\star E+\Theta_{22})$ in some open subset of $\mathbb{B}$, and thus obtain the linear fractional transformation (11.54) of S in terms of E, first in the given open subset of the origin, and then in all of $\mathbb{B}$ by slice hyperholomorphic continuation. See Corollary 8.4.4 for the latter. □

Epilogue

Classical Schur analysis and its various applications (for example to operator theory or to the theory of linear systems) lead to new problems in function theory. For instance, the characteristic operator function lead to the study of the multiplicative structure of matrix-valued functions meromorphic in the open unit disk, and contractive there with respect to some indefinite metric. See the fundamental work of Potapov [245].

Classical Schur analysis contains whole sectors not touched here, and expands to new directions, still to be developed in the setting of slice hyperholomorphic functions. Among the first we mention:

(a) The study of the characteristic operator functions (the ”s” in functions is not a misprint; various classes of operators will have different corresponding characteristic operator functions) and operator models. We also mention the study of these functions from the pure function theory point of view. Indeed, in the classical case, the connections between the function theory approach and the operator theory side lead to new results in both theories.

(b) Still in function theory, the counterparts of integral representation formulas for functions analytic and with a real positive part in a disk or an half-plane remains to be done, and is related to moment problems.

(c) Interpolation problems for slice hyperholomorphic functions in the half-space. In the classical setting, and as we already have remarked, quite a number of different (but of course related) methods have been developed. The study of these methods in the quaternionic setting (for instance the band method) should be conducive to new problems and methods in quaternionic analysis.

(d) The degenerate cases in the interpolation problems.

(e) Interpolation problems for generalized Schur functions.

Among the second we mention:

(f) Applications to the theory of linear systems.

(g) The case of several noncommuting variables.

(h) Operator models for commuting and noncommuting operators. In the complex case, and for two commuting operators, this problem is related to function theory on compact Riemann surfaces. See [236].

In classical Schur analysis these questions are considered for operator-valued functions. Thus the study of operator-valued slice hyperholomorphic functions developed in Chapter 7 will provide the ground to pursue these lines of research.

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Index

Notation Index