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Catino, G.; Mastrolia, P.; Monticelli, D.D.; Rigoli, M.

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ON THE GLOBAL STRUCTURE OF CONFORMAL GRADIENT SOLITONS WITH NONNEGATIVE RICCI TENSOR

GIOVANNI CATINO
Dipartimento di Matematica, Politecnico di Milano
Piazza Leonardo da Vinci 32, Milano 20133, Italy
giovanni.catino@polimi.it

CARLO MANTEGAZZA* and LORENZO MAZZIERI†
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 7, Pisa 56126, Italy
**c.mantegazza@sns.it*
†*l.mazzieri@sns.it*

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In this paper we prove that any complete conformal gradient soliton with nonnegative Ricci tensor is either isometric to a direct product $\mathbb{R} \times N^{n-1}$, or globally conformally equivalent to the Euclidean space $\mathbb{R}^n$ or to the round sphere $\mathbb{S}^n$. In particular, we show that any complete, noncompact, gradient Yamabe-type soliton with positive Ricci tensor is rotationally symmetric, whenever the potential function is nonconstant.

Keywords: Conformal geometry; Yamabe solitons; warped products.

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1. Introduction

A connected, complete Riemannian manifold (M^n, g) is called a *conformal gradient soliton* if there exists a *nonconstant* smooth function f, called *potential* of the soliton, such that

$$\nabla^2 f = \varphi g,$$

for some function $\varphi : M^n \to \mathbb{R}$. Tracing this equation with the metric g, we see immediately that the function φ must coincide with $\Delta f/n$. Hence, an equivalent characterization of conformal gradient solitons is given by the equation

$$\nabla^2 f = \frac{\Delta f}{n} g. \tag{1.1}$$

In this note we are going to fully detail a remark of Petersen and Wylie [9, Remark A.3] about the classification of these solitons; moreover, we revisit a result of Tashiro [10], who first studied their global structure.

Complete Riemannian manifolds admitting a vector field ∇f satisfying Eq. (1.1) were studied by many authors in the late 1960s. Solutions to Eq. (1.1) have also been considered in a work by Cheeger and Colding [4], where the authors give a characterization of warped product manifolds. In particular, they observe that, in the complement of the critical points of f, any conformal gradient soliton is isometric to a warped product on some open interval. Taking advantage of this, we will be able to drastically simplify the proof of the classification result for conformal gradient solitons given by Tashiro. Moreover, we further characterize conformal gradient solitons with nonnegative Ricci tensor, in the spirit of some recent works about the classification of Einstein-like structures, such as gradient Ricci solitons and quasi-Einstein manifolds.

Our main result reads as follows.

Theorem 1.1. *Let (M^n, g) be a complete conformal gradient soliton and let f be a potential function for it. Then, any regular level set Σ of f admits a maximal open neighborhood $U \subset M^n$ on which f only depends on the signed distance r to the hypersurface Σ. In addition, the potential function f can be chosen in such a way that the metric g takes the form*

$$g = dr^2 + (f'(r))^2 g^\Sigma \quad \text{on } U,$$

where g^Σ is the metric induced by g on Σ. As a consequence, f has at most two critical points on M^n and we have the following cases:

(1) *If f has no critical points, then (M^n, g) is globally conformally equivalent to a direct product $I \times N^{n-1}$ of some interval $I = (t_*, t^*) \subseteq \mathbb{R}$ with an $(n-1)$-dimensional complete Riemannian manifold (N^{n-1}, g^N). More precisely, the metric takes the form*

$$g = u^2(t)(dt^2 + g^N),$$

where $u : (t_*, t^*) \to \mathbb{R}$ is some positive smooth function. In this case, if (M^n, g) is also locally conformally flat, it is well known that (N^{n-1}, g^N) must have constant curvature.

(1′) *If, in addition, the Ricci tensor of (M^n, g) is nonnegative, then (M^n, g) is* isometric *to a direct product $\mathbb{R} \times N^{n-1}$, where (N^{n-1}, g^N) has nonnegative Ricci tensor. In this case, if (M^n, g) is also locally conformally flat, then either (M^n, g) is flat or it is a direct product of $\mathbb{R}$ with a quotient of the round sphere $\mathbb{S}^{n-1}$.*

(2) *If f has only one critical point $O \in M^n$, then (M^n, g) is globally conformally equivalent to the interior of an Euclidean ball of radius $t^* \in (0, +\infty]$. More*

precisely, on $M^n \backslash \{O\}$, *the metric takes the form*

$$g = v^2(t)(dt^2 + t^2 g^{\mathbb{S}^{n-1}}),$$

where $v : (0, t^*) \to \mathbb{R}$ *is some positive smooth function. In particular* (M^n, g) *is complete, noncompact and rotationally symmetric.*

(2′) *If, in addition, the Ricci tensor of* (M^n, g) *is nonnegative, then* (M^n, g) *is globally conformally equivalent to* $\mathbb{R}^n$.

(3) *If the function* f *has two critical points* $N, S \in M^n$, *then* (M^n, g) *is globally conformally equivalent to* $\mathbb{S}^n$. *More precisely, on* $M^n \backslash \{N, S\}$, *the metric takes the form*

$$g = w^2(t)(dt^2 + \sin^2(t) g^{\mathbb{S}^{n-1}}),$$

where $w : (0, \pi) \to \mathbb{R}$ *is some smooth positive function. In particular* (M^n, g) *is compact and rotationally symmetric.*

In Sec. 2 we will prove Theorem 1.1, whereas in Sec. 3 we will focus our attention on the classification of gradient Yamabe solitons. These are conformal gradient solitons satisfying the equation

$$\nabla^2 f = (R - \lambda) g,$$

for some constant λ. We will show that any complete, noncompact, gradient Yamabe soliton with nonnegative Ricci tensor either has constant scalar curvature, or it splits isometrically as a direct product $\mathbb{R} \times N^{n-1}$, or it is rotationally symmetric and globally conformally equivalent to $\mathbb{R}^n$ (see Theorems 3.2 and 3.6 for the generalization to the case of gradient k-Yamabe solitons).

2. Proof of Theorem 1.1

Let Σ be a regular level set of the function $f : M^n \to \mathbb{R}$, i.e. $|\nabla f| \neq 0$ on Σ, which exists by Sard's Theorem and the fact that f is nonconstant in our definition. We have that $|\nabla f|$ has to be constant on Σ. Indeed, for all $X \in T_p\Sigma$

$$\nabla_X |\nabla f|^2 = 2\nabla^2 f(\nabla f, X) = \frac{2\,\Delta f}{n} g(\nabla f, X) = 0.$$

From this we deduce that, in a neighborhood U of Σ not containing any critical point of f, such potential function only depends on the signed distance r to the hypersurface Σ. In particular $df = f' dr$. Moreover, if $\theta = (\theta^1, \ldots, \theta^{n-1})$ are coordinates *adapted* to the hypersurface Σ, we get

$$\nabla^2 f = \nabla df = f'' dr \otimes dr + f' \nabla^2 r = f'' dr \otimes dr + \frac{f'}{2} \partial_r g_{ij} d\theta^i \otimes d\theta^j,$$

as

$$\Gamma^r_{rr} = \Gamma^k_{rr} = \Gamma^r_{ir} = 0, \quad \Gamma^r_{ij} = -\frac{1}{2}\,\partial_r g_{ij}, \quad \Gamma^k_{ir} = \frac{1}{2} g^{ks} \partial_r g_{is}.$$

On the other hand, using Eq. (1.1), we have

$$\nabla^2 f = \frac{\Delta f}{n} g = \frac{\Delta f}{n}(dr \otimes dr + g_{ij} d\theta^i \otimes d\theta^j),$$

thus, $\Delta f = n f''$ and $g_{ij}\Delta f = \frac{n}{2} f' \partial_r g_{ij}$. These equations imply the family of ODE's

$$f''(r) g_{ij}(r,\theta) = \frac{f'(r)}{2} \partial_r g_{ij}(r,\theta).$$

Since $f'(0) \neq 0$ (otherwise Σ is not a regular level set of f) we can integrate these equations obtaining

$$g_{ij}(r,\theta) = (f'(r)/f'(0))^2 g_{ij}(0,\theta).$$

Therefore, in U the metric takes the form

$$g = dr \otimes dr + (f'(r)/f'(0))^2 \sigma_{ij}(\theta) d\theta^i \otimes d\theta^j,$$

where $g^{\Sigma}_{ij}(\theta) = g_{ij}(0,\theta)$ is the metric induced by g on Σ. We notice that, since $f = f(r)$, then the width of the neighborhood U is uniform with respect to the points of Σ, namely we can assume $U = \{r_* < r < r^*\}$, for some maximal $r_* \in [-\infty, 0)$ and $r^* \in (0, \infty]$. Moreover, by the scalar invariance of Eq. (1.1), we can assume that $f'(0) = 1$, possibly changing the function f. Hence, in U, the metric can be written as

$$g = dr \otimes dr + (f')^2 g^{\Sigma}, \tag{2.1}$$

where g^{Σ} denotes the induced metric on the level set Σ. Moreover, the Ricci tensor and the scalar curvature of the metric g take the form (see [1, Proposition 9.106]):

$$\mathrm{Ric}_g = -(n-1)\frac{f'''}{f'} dr \otimes dr + \mathrm{Ric}^{\Sigma} - ((n-2)(f'')^2 + f' f''') g^{\Sigma}, \tag{2.2}$$

$$R_g = -2(n-1)\frac{f'''}{f'} + \frac{R^{\Sigma} - (n-1)(n-2)(f'')^2}{(f')^2}. \tag{2.3}$$

Case 1: *f has no critical points.* Since (M^n, g) is complete, the width of the maximal neighborhood U is unbounded in both the negative and the positive direction of the signed distance r (i.e. $r_* = -\infty$ and $r^* = +\infty$). To complete the proof, it is sufficient to set

$$t(r) = \int_0^r \frac{1}{f'(z)} dz.$$

For $r \in (-\infty, +\infty)$, we have $t \in (t_*, t^*)$, where $t_* = \lim_{r \to r_*} t(r) \in [-\infty, 0)$ and $t^* = \lim_{r \to r^*} t(r) \in (0, +\infty]$. Moreover, $t'(r) \neq 0$ and r can be viewed as a function of t by inverting the expression above. From (2.1), the metric takes the form

$$g = u(t)^2 (dt^2 + g^{\Sigma}),$$

where $u(t) = f'(r(t))$.

Case 1′: *f has no critical points and* $\mathrm{Ric} \geq 0$. From formula (2.2) and the fact that g has nonnegative Ricci tensor, one has

$$0 \leq \mathrm{Ric}_g(\partial_r, \partial_r) = -(n-1)\frac{f'''}{f'}.$$

Hence, f' is a concave function defined on the whole real line that can never be zero. Thus, it must be constant, that is $f' \equiv 1$, according to our choice of $f'(0)$. This implies that (M^n, g) is isometric to the direct product $\mathbb{R} \times N^{n-1}$ of the real line with an $(n-1)$-dimensional complete Riemannian manifold with nonnegative Ricci tensor.

Remark 2.1. We notice that, in this case, the Ricci tensor has a zero eigenvalue at every point. Hence, there are no examples of such manifolds for which the Ricci tensor is positive definite at some point.

Case 2: *f has only one critical point $O \in M^n$.* In this case, since (M^n, g) is complete, we can assume that the width of the neighborhood U is unbounded in the positive direction of the signed distance (i.e. $r_* > -\infty$ and $r^* = +\infty$) and $f' \to 0$, as $r \to r_*$. By formula (2.2) and the smoothness of the metric g, we have that f'''/f' is bounded, as $r \to r_*$. Hence, from (2.3), we deduce that

$$R^\Sigma - (n-1)(n-2)(f'')^2 \to 0, \quad \text{as } r \to r_*.$$

In particular R^Σ is nonnegative and constant along Σ. Moreover, it is easy to see that

$$\lim_{r \to r_*} \frac{f'(r)}{r - r_*} = \lim_{r \to r_*} f''(r) = (R^\Sigma/(n-1)(n-2))^{1/2}. \tag{2.4}$$

To conclude the proof of Case 2 of Theorem 1.1, it remains to show that the induced metric g^Σ on the level set Σ is proportional to the round metric $g^{\mathbb{S}^{n-1}}$ of the $(n-1)$-dimensional sphere. This follows from the elementary fact that, infinitesimally, the metric g is approximately Euclidean near O. Indeed, the standard expansion of the metric g around O, written in normal coordinates $(x^1, \ldots, x^n)$, gives

$$\begin{aligned} g &= (\delta_{ij} + \eta_{ij}(x))dx^i \otimes dx^j \\ &= g^{\mathbb{R}^n} + \eta_{ij} dx^i \otimes dx^j, \end{aligned}$$

where $\eta_{ij} = \mathcal{O}(|x|^2)$. Passing to Riemannian polar coordinates, we write $x^i = s\phi^i(\vartheta^1, \ldots, \vartheta^{n-1})$, with $s = r - r_* \in (0, +\infty)$ and $(\vartheta^1, \ldots, \vartheta^{n-1})$ being local coordinates on $\mathbb{S}^{n-1}$. Notice that $|\phi^1|^2 + \cdots + |\phi^n|^2 = 1$ and $|x| = s$. Thus, one has

$$g = ds \otimes ds + \left(s^2 g^{\mathbb{S}^{n-1}}_{\alpha\beta} + s^2 \eta_{ij} \frac{\partial \phi^i}{\partial \vartheta^\alpha} \frac{\partial \phi^j}{\partial \vartheta^\beta} \right) d\vartheta^\alpha \otimes d\vartheta^\beta,$$

with $\eta_{ij} = \mathcal{O}(s^2)$. Comparing with expression (2.1), we see that, for $s \in (0, +\infty)$, we have

$$f'(s + r_*)^2 g^\Sigma = s^2 g_{\mathbb{S}^{n-1}} + s^2 \eta_{ij} \frac{\partial \phi^i}{\partial \vartheta^\alpha} \frac{\partial \phi^j}{\partial \vartheta^\beta} d\vartheta^\alpha \otimes d\vartheta^\beta.$$

Now, combining the fact that $\eta_{ij} = \mathcal{O}(s^2)$ with formula (2.4), if we take the limit as $s \to 0$ (which means $r \to r_*$) we obtain $R^\Sigma > 0$ and

$$g^\Sigma = c^2\, g_{\mathbb{S}^{n-1}},$$

with $c^2 = (n-1)(n-2)/R^\Sigma$. Therefore, on $M^n \backslash \{O\}$, we have

$$g = ds^2 + (cf'(s+r_*))^2 g_{\mathbb{S}^{n-1}}.$$

This proves that g is rotationally symmetric. To complete the proof, we set

$$t(s) = \exp\left(\frac{1}{c}\int_{-r_*}^{s} \frac{1}{f'(z+r_*)} dz\right). \tag{2.5}$$

For $s \in (0, +\infty)$, we have $t \in (0, t^*)$, where $t^* = \lim_{s\to+\infty} t(s) \in (0, +\infty]$. Notice that $t'(s) \neq 0$, hence, the coordinate s can be viewed as a function of t by inverting the expression above. Moreover,

$$\frac{dt}{t} = \frac{ds}{c\, f'(s+r_*)}$$

and the metric g can be expressed as in the statement, namely

$$g = v(t)^2 (dt^2 + t^2 g^{\mathbb{S}^{n-1}}),$$

where $v(t) = c\, f'(s(t)+r_*)/t$.

Case 2′: *f has one critical point $O \in M^n$ and* $\mathrm{Ric} \geq 0$. Like in the Case 1′ above, as $\mathrm{Ric} \geq 0$, we have that $z \mapsto f'(z+r_*)$ is a concave function. In particular, it is definitely bounded above by some linear function, as $z \to +\infty$. Then, by the very definition of t^* in Eq. (2.5), it follows that $t^* = +\infty$. This clearly implies that (M^n, g) is globally conformally equivalent to $\mathbb{R}^n$ and rotationally symmetric.

Case 3: *f has two critical points $N, S \in M^n$.* We assume that the width of the neighborhood U is bounded in both the negative and the positive directions of the signed distance (i.e. $r_* > -\infty$ and $r^* < +\infty$). In particular (M^n, g) is compact (it "closes" at the points N and S) and there cannot be other critical points around. The same argument used in the proof of Case 2 implies at once the rotational symmetry of g. Namely, on $M^n \backslash \{N, S\}$, we have

$$g = ds^2 + (cf'(s+r_*))^2 g_{\mathbb{S}^{n-1}},$$

where $c^2 = (n-1)(n-2)/R^\Sigma$. To complete the proof of Case 3, we set

$$t(s) = 2 \arctan \exp\left(\frac{1}{c}\int_{-r_*}^{s} \frac{1}{f'(z+r_*)} dz\right).$$

For $s \in (0, r^* - r_*)$, we have $t \in (0, \pi)$ and $t'(s) \neq 0$, hence, s can be viewed as a function of t by inverting the expression above. Moreover,

$$\frac{dt}{\sin(t)} = \frac{ds}{cf'(s+r_*)}$$

and the metric g can be expressed as in the statement, namely

$$g = w(t)^2(dt^2 + \sin^2(t)g^{\mathbb{S}^{n-1}}),$$

where $w(t) = cf'(s(t) + r_*)/\sin(t)$.

This completes the proof of Theorem 1.1.

3. Classification of Yamabe-Type Solitons with Nonnegative Ricci Tensor

Let (M^n, g), $n \geq 3$, be a complete Riemannian manifold verifying

$$\nabla^2 f = \varphi g, \tag{3.1}$$

for some smooth functions f and φ on M^n. When the potential function f is nonconstant, then, according to our definition, (M^n, g) is a *conformal gradient soliton* and Theorem 1.1 applies.

We first notice that, by taking the divergence of this equation, we have

$$\nabla_i \varphi = \Delta \nabla_i f = \nabla_j \nabla_i \nabla^j f = \nabla_i \Delta f + R_{ij} \nabla^j f,$$

where we interchanged the covariant derivatives. Now, using the fact that $\Delta f = n\varphi$, we obtain the following identity:

$$(n-1)\nabla_i \varphi = -R_{ij} \nabla^j f. \tag{3.2}$$

We will discuss now some geometric applications of Theorem 1.1 to *gradient Yamabe solitons* and *gradient k-Yamabe solitons*.

3.1. *Gradient Yamabe solitons*

A Riemannian manifold (M^n, g) is called a *gradient Yamabe soliton* if it satisfies Eq. (3.1) with $\varphi = R - \lambda$ for some constant $\lambda \in \mathbb{R}$, i.e. there exists a smooth function f (notice that here we are not excluding the case of a constant f) such that

$$\nabla^2 f = (R - \lambda)g. \tag{3.3}$$

If $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$, then the soliton is called *steady*, *shrinking* or *expanding*, respectively. We recall that gradient Yamabe solitons are self-similar solutions to the Yamabe flow

$$\frac{\partial}{\partial t} g = -Rg.$$

This flow was first introduced by Hamilton and we refer the reader to [6] and the references therein for further details on this subject.

Notice that any Riemannian manifold with constant scalar curvature moves by the Yamabe flow only by dilations. Hence, it is trivially a self-similar solution and a gradient Yamabe soliton with $R = \lambda$ and f constant. Thus, according to our

definitions, only gradient Yamabe solitons with nonconstant potential function f can be viewed as conformal gradient solitons.

On the other hand, it is well known (see, for instance, [5, Proposition B.16]) that any compact gradient Yamabe soliton has constant scalar curvature $R = \lambda$. For the sake of completeness, we report here the proof.

Theorem 3.1. *Any compact gradient Yamabe soliton has constant scalar curvature $R = \lambda$. Moreover, the potential function f is constant.*

Proof. Contracting Eq. (3.3) with the Ricci tensor and integrating over M^n, we obtain

$$\int_{M^n} (R-\lambda) R dV_g = \int_{M^n} R_{ij}\nabla^{ij} f dV_g = -\int_{M^n} \nabla_i R_{ij} \nabla_j f dV_g$$
$$= -\frac{1}{2}\int_{M^n} \langle \nabla R, \nabla f\rangle dV_g,$$

where in the last equality we have used Schur's lemma $2\,\mathrm{div}(\mathrm{Ric}) = dR$. Moreover, from Eq. (3.3), one has that $\Delta f = n(R-\lambda)$. Hence, it follows that $\lambda = \frac{1}{\mathrm{Vol}(M^n)}\int_{M^n} R dV_g$ and, from the previous computation, we get

$$\int_{M^n} (R-\lambda)^2 dV_g = -\frac{1}{2}\int_{M^n} \langle \nabla R, \nabla f\rangle dV_g = \frac{n}{2}\int_{M^n} (R-\lambda)^2 dV_g.$$

Since $n \geq 3$, this implies that R coincides with the constant λ. As an immediate consequence, by the relation (3.3), we have that Δf is zero. Since M^n is compact, the function f is constant as well. □

When the function f is constant, obviously the scalar curvature of the gradient Yamabe soliton is constant as well. In such a case, Theorem 1.1 does not apply. In the sequel we will always assume that relation (3.3) is satisfied by some nonconstant function f, so we will deal only with noncompact gradient Yamabe solitons. In this case, as an immediate application of Theorem 1.1, we can prove the following global result.

Theorem 3.2. *Let (M^n, g) be a complete, noncompact, gradient Yamabe soliton with nonnegative Ricci tensor and nonconstant potential function f. Then, we have the following two cases*:

(1) *either (M^n, g) is a direct product $\mathbb{R}\times N^{n-1}$ where (N^{n-1}, g^N) is an $(n-1)$-dimensional complete Riemannian manifold with nonnegative Ricci tensor. If in addition (M^n, g) is locally conformally flat, then either it is flat or the manifold (N^{n-1}, g^N) is a quotient of the round sphere $\mathbb{S}^{n-1}$;*
(2) *or (M^n, g) is rotationally symmetric and globally conformally equivalent to $\mathbb{R}^n$. More precisely, there exists a point $O \in M^n$ such that on $M^n\backslash\{O\}$, the metric*

has the form

$$g = v^2(t)(dt^2 + t^2 g^{\mathbb{S}^{n-1}}),$$

where $v : \mathbb{R}^+ \to \mathbb{R}$ *is some positive smooth function.*

From Remark 2.1, it is now easy to deduce the following corollary.

Corollary 3.3. *Let* (M^n, g) *be a complete, noncompact, gradient Yamabe soliton with nonnegative Ricci tensor and nonconstant potential function* f*. If the Ricci tensor is positive definite at some point, then* (M^n, g) *is rotationally symmetric and globally conformally equivalent to* $\mathbb{R}^n$*, in particular, it is locally conformally flat.*

It was proved by Daskalopoulos and Sesum in [6] that any complete, noncompact, locally conformally flat, gradient Yamabe soliton with positive sectional curvature has to be globally conformally equivalent to $\mathbb{R}^n$. Corollary 3.3 shows that one can remove the assumption of local conformal flatness and relax the hypothesis on the sectional curvature. In [6] the authors also provide a complete classification of all rotationally symmetric gradient Yamabe solitons in the steady, shrinking and expanding cases.

3.2. *Gradient k-Yamabe solitons*

A Riemannian manifold (M^n, g) is called a *gradient k-Yamabe soliton* if it satisfies Eq. (3.1) with $\varphi = 2(n-1)(\sigma_k - \lambda)$ for some constant $\lambda \in \mathbb{R}$, where σ_k denotes the σ_k-curvature of g. We recall that, if we denote by $\mu_1, \ldots, \mu_n$ the eigenvalues of the symmetric endomorphism $g^{-1}A$, where A is the Schouten tensor defined by

$$A = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{1}{2(n-1)} Rg\right),$$

then the σ_k-curvature of g is defined as the kth symmetric elementary function of $\mu_1, \ldots, \mu_n$, namely

$$\sigma_k = \sigma_k(g^{-1}A) = \sum_{i_1 < \cdots < i_k} \mu_{i_i} \cdot \ldots \cdot \mu_{i_k} \quad \text{for } 1 \leq k \leq n.$$

Notice that $\sigma_1 = \frac{1}{2(n-1)} R$, so gradient 1-Yamabe solitons simply correspond to gradient Yamabe solitons. The structure equation takes the form

$$\nabla^2 f = 2(n-1)(\sigma_k - \lambda)g, \tag{3.4}$$

for some constant $\lambda \in \mathbb{R}$. As usual, if $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$, then g is called *steady*, *shrinking* or *expanding*, respectively.

Again, we observe that only gradient k-Yamabe solitons with nonconstant potential function f can be viewed as conformal gradient solitons.

We have seen that, for $k = 1$, compact, gradient Yamabe solitons have constant scalar curvature. By means of a generalized Kazdan–Warner identity, for any $k \geq 2$, we can prove the following analogue of Theorem 3.1.

Theorem 3.4. *Any compact, gradient k-Yamabe soliton with nonnegative Ricci tensor has constant σ_k-curvature $\sigma_k = \lambda$. Moreover, the potential function f is constant.*

Proof. Let us suppose, by contradiction, that σ_k is nonconstant. Then f cannot be constant, since $\Delta f = 2n(n-1)(\sigma_k - \lambda)$. Hence, we can apply Theorem 1.1, obtaining that (M^n, g) is globally conformally equivalent to $\mathbb{S}^n$, in particular, g is locally conformally flat. It was proved in the proof in [8] that, on a compact, locally conformally flat, Riemannian manifold, one has

$$\int_{M^n} \langle X, \nabla \sigma_k \rangle dV_g = 0,$$

for every conformal Killing vector field X on (M^n, g). For $k = 1$, this obstruction corresponds to the well-known Kazdan–Warner identity, which holds on any compact Riemannian manifold (i.e. without assuming the locally conformally flatness, see [2]). From the structure equation (3.4), we know that ∇f is a conformal Killing vector field, hence, it follows that

$$\int_{M^n} \langle \nabla f, \nabla \sigma_k \rangle dV_g = 0.$$

Now, contracting the identity (3.2) with ∇f, and integrating over M^n, we obtain

$$0 = \int_{M^n} \langle \nabla f, \nabla \sigma_k \rangle dV_g = -\frac{1}{2(n-1)^2} \int_{M^n} \mathrm{Ric}(\nabla f, \nabla f) dV_g.$$

From the fact that g has nonnegative Ricci tensor, we obtain that $\mathrm{Ric}(\nabla f, \nabla f) = 0$ everywhere. Then, by Eq. (3.2), we get $\langle \nabla f, \nabla \sigma_k \rangle = 0$. Since g is rotationally symmetric, we have that σ_k is constant on the regular level sets of f. Hence, the condition $\langle \nabla f, \nabla \sigma_k \rangle = 0$ is sufficient to conclude that σ_k is constant. This implies that Δf is constant. Since M^n is compact, the only possibility is that f is constant and $\sigma_k = \lambda$. □

Remark 3.5. The same result holds if one considers a generalized k-Yamabe soliton structure (3.1) with $\varphi = \psi(\sigma_k)$, for every $k \geq 1$ and every strictly monotone function $\psi : \mathbb{R} \to \mathbb{R}$. For instance, in [7] the authors consider a fully nonlinear conformal flow with velocity $\varphi = \log(\sigma_k) - \lambda$, with $\sigma_k > 0$.

Again, when the function f is constant, obviously the σ_k-curvature of the gradient k-Yamabe soliton is also constant. Hence, in such a case Theorem 1.1 does

not apply. In the complete, noncompact case, as an immediate application of Theorem 1.1, we can prove the following global result.

Theorem 3.6. *Let (M^n, g) be a complete, noncompact, gradient k-Yamabe soliton with nonnegative Ricci tensor and nonconstant potential function f. Then, we have the following two cases*:

(1) *either (M^n, g) is a direct product $\mathbb{R} \times N^{n-1}$ where (N^{n-1}, g^N) is an $(n-1)$-dimensional complete Riemannian manifold with nonnegative Ricci tensor. If in addition (M^n, g) is locally conformally flat, then either it is flat or the manifold (N^{n-1}, g^N) is a quotient of the round sphere $\mathbb{S}^{n-1}$;*
(2) *or (M^n, g) is rotationally symmetric and globally conformally equivalent to $\mathbb{R}^n$. More precisely, there exists a point $O \in M^n$ such that on $M^n \backslash \{O\}$ the metric has the form*

$$g = v^2(t)(dt^2 + t^2 g^{\mathbb{S}^{n-1}}),$$

where $v : \mathbb{R}^+ \to \mathbb{R}$ is some positive smooth function.

From Remark 2.1, it is now easy to deduce the following corollary.

Corollary 3.7. *Let (M^n, g) be a complete, noncompact, gradient k-Yamabe soliton with nonnegative Ricci tensor and nonconstant potential function f. If the Ricci tensor is positive at some point, then (M^n, g) is rotationally symmetric and globally conformally equivalent to $\mathbb{R}^n$.*

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Note added in proof

During the editing of this work, H.-D. Cao, X. Sun and Y. Zhang posted on the *ArXiv Preprint Server* the manuscript [3], where a classification result for gradient Yamabe solitons similar to the one discussed in Sec. 3 (in particular, Theorem 3.2) is obtained.

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