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A nonlinear Steklov problem arising in corrosion modeling

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Abstract

We investigate the existence of pairs (λ, u), with $\lambda > 0$ and u harmonic function in the unit ball $B \subset \mathbb{R}^3$, such that the nonlinear boundary condition $\partial_\nu u = \lambda \sinh u$ holds on ∂B. This type of exponential boundary condition arises in corrosion modeling (Butler-Volmer condition). We prove existence of global branches of nontrivial solutions in the framework of analytic bifurcation theory and investigate their properties both analytically and numerically.

1 Introduction

In a simple mathematical model of electrochemical corrosion, i.e. a deterioration of a metal by electrochemical reaction with its environment, a (suitably defined) galvanic potential is represented by a function u harmonic in a domain $\Omega \subset \mathbb{R}^N$ whose boundary is partly electrochemically active and partly inert. In the inactive boundary region the current density flow $J \cdot \nu$ (ν is the outward unit normal to $\partial\Omega$) is of course zero, but in the active part it is modeled (by interpolating experimental data) by a difference of two exponentials according to the so-called Butler-Volmer formula :

$$\big(J \cdot \nu\big)(x) = \lambda\mu(x)\big(e^{\beta u(x)} - e^{-(1-\beta)u(x)}\big) + g(x), \qquad x \in \partial\Omega \tag{1.1}$$

Here $\beta \in (0, 1)$ is a constant depending on the constituents of the electrochemical system, the function $\mu(x)$ distinguishes between the active and the inert boundary regions (typically μ is the characteristic function of some subset $\subseteq \partial\Omega$), λ is a real parameter which may take negative as well as positive values and g is an externally imposed current (see [1] and references therein for a detailed discussion). Assuming $\mu(x) \geq 0$ and not identically vanishing, the resulting mathematical problem is quite different in the two cases, λ negative or positive; for, the corresponding *linearized problem*

$$\begin{aligned} \Delta u(x) &= 0 \quad \text{in } \Omega \\ \partial_\nu u(x) &= \lambda\, \mu(x) u(x) + g(x) \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

is a classical elliptic problem with a Robin (or mixed Neumann-Robin) boundary condition if $\lambda < 0$, while for $\lambda > 0$ it is a Steklov problem. We stress that in the latter case, there are nontrivial solutions of the problem with $g = 0$ (Steklov eigenvalue problem).
Another quite sensible parameter of the problem is the dimension N of the space. In fact, if $N = 2$ the nonlinear problem is subcritical in the energy space $H^1(\Omega)$ (thanks to the Moser-Trudinger inequality); on the other hand, if $N \geq 3$ (and therefore in the physically relevant case $N = 3$) the problem is supercritical (see the discussion in [2]). The two-dimensional case has been considered by various authors [1], [3], [4], [5], [6], [7].
The literature concerning the supercritical case is much more lacking and seems to take into consideration mainly the case $\lambda < 0$ (that is, with the Robin boundary condition; see, e.g., [8]). A first attempt to investigate the three-dimensional problem with $\lambda > 0$ (and vanishing external current g) is in [2]. For the reader's convenience, let us summarize with few details the main results obtained in [2].

The authors discuss the following problem: find a (non identically vanishing) function u in a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, satisfying the system

$$\begin{aligned} \Delta u(x) &= 0 \quad \text{in } \Omega \\ \partial_\nu u(x) &= \lambda\, \mu(x) \sinh[u(x)] \quad \text{on } \partial\Omega \end{aligned} \tag{1.3}$$

where $\lambda > 0$ and μ is a non negative function in $L^\infty(\partial\Omega)$.
By observing that the above problem has the line of trivial solutions $\{(\lambda, 0) \,|\, \lambda \in \mathbb{R}\}$, they look for *bifurcation solutions*. By applying classical results of Bifurcation Theory [9], [10], the authors prove that, for every eigenvalue κ of the linearized problem

$$\begin{aligned} \Delta u(x) &= 0 \quad \text{in } \Omega \\ \partial_\nu u(x) &= \lambda\, \mu(x) u(x) \quad \text{on } \partial\Omega \end{aligned} \tag{1.4}$$

(which is a classical *Steklov eigenvalue problem* [11]) the pair $(\kappa, 0)$ is a bifurcation point for (1.3). Further results on global existence are proved by assuming specific symmetries of the domain. By restricting the study of the problem (1.3) in the unit ball of $\mathbb{R}^3$ and taking $\mu(x) \equiv 1$, they prove the existence of a branch of global solutions bifurcating from the first eigenvalue $\lambda = 1$ of the linearized problem.
In the present paper, after recalling some general results about existence of global solutions (section 2) the analysis of the branch bifurcating from the first eigenvalue is expanded (section 3) and some new properties (local analiticy, blow up of the solutions,..) as well as open problems are presented. In section 4 we describe some numerical results illustrating the properties of the previously investigated bifurcation branch.

2 Global existence of the bifurcation solutions

Hereafter, we consider the problem (1.3) with μ non negative and bounded. For more details and some proofs of the results of this section, see [2]. As we will see below, it is convenient to search three dimensional solutions in the *Hilbert* space $H^{3/2}(\Omega)$.
Let $f \in L^2(\partial\Omega)$ satisfy $\int_{\partial\Omega} f = 0$; define the *Neumann to Dirichlet map*

$$\mathcal{G} f = v_0|_{\partial\Omega} \tag{2.1}$$

where v_0 is the *unique* harmonic function in Ω with Neumann datum f and such that $\int_{\partial\Omega} \mu v_0 = 0$.
By known regularity results [12] we have $v_0 \in H^{3/2}(\Omega)$ and therefore $\mathcal{G} f \in H^1(\partial\Omega)$.
Let us define the subspace

$$\dot{H}^1(\partial\Omega) = \Big\{ \phi \in H^1(\partial\Omega), \quad \int_{\partial\Omega} \mu\phi = 0 \Big\} \tag{2.2}$$

and the operator

$$G(\lambda, \phi) = \lambda \mathcal{G}\Big(\mu \sinh[\phi + s(\phi)]\Big) \tag{2.3}$$

where

$$s(\phi) = -\tanh^{-1}\Big(\frac{\int_{\partial\Omega} \mu \sinh(\phi)}{\int_{\partial\Omega} \mu \cosh(\phi)}\Big) = \frac{1}{2} \log\Big(\frac{\int_{\partial\Omega} \mu e^{-\phi}}{\int_{\partial\Omega} \mu e^{\phi}}\Big) \tag{2.4}$$

By known estimates on *two dimensional* manifolds, the exponentials $e^{\pm\phi}$ lie in $L^p(\partial\Omega)$ for every $p \geq 1$; moreover, by the definition (2.4) the argument of $\mathcal{G}$ at the right hand side of (2.3) has vanishing integral on $\partial\Omega$. Then, by standard calculations one can show that the operator $G(\lambda, \cdot)$ is a $\mathcal{C}^1$ map from $\dot{H}^1(\partial\Omega)$ in itself. Assume now that ϕ solves the *functional equation*

$$\phi = G(\lambda, \phi) = \lambda \mathcal{G}\Big(\mu \sinh[\phi + s(\phi)]\Big) \tag{2.5}$$

Then, the unique harmonic function $u_0 \in H^1(\Omega)$ such that $u_0|_{\partial\Omega} = \phi$, satisfies the variational equation

$$\int_\Omega \nabla u_0 \nabla v = \lambda \int_{\partial\Omega} \mu \sinh[u_0|_{\partial\Omega} + s(u_0|_{\partial\Omega})]v \tag{2.6}$$

for every v such that $\int_{\partial\Omega} \mu v = 0$.
Finally, by standard regularity results the function

$$u(x) = u_0(x) + s(u_0|_{\partial\Omega})$$

satisfies the boundary value problem (1.3). Then, the following result holds [2] :

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary and let μ be a bounded non negative function on $\partial\Omega$. Moreover, let κ be an eigenvalue of multiplicity n_κ of the linear problem* (1.4)*. Then, there is an $r_0 > 0$ such that for each $r \in (0, r_0)$ the bifurcation equation* (2.5) *has at least n_κ distinct pairs of non trivial solutions $(\lambda_m(r), \pm\phi_m(r)) \subset \mathbb{R} \times \dot{H}^1(\partial\Omega)$, $m = 1, 2, ..., n_\kappa$; moreover, as $r \to 0$, $\lambda_m(r) \to \kappa$ and $\|\phi_m(r)\|_{H^1(\partial\Omega)} = O(r)$.*

Thus, by the previous discussion, the nonlinear boundary value problem (1.3) has at least n_κ distinct pairs of non trivial solutions $(\lambda_m(r), \pm u_m(r)) \subset \mathbb{R} \times H^{3/2}(\Omega)$, $m = 1, 2, ...n_\kappa$ for $r \in (0, r_0)$.
In the case of bifurcation from eigenvalues of *odd multiplicity*, a global result holds (see [10], Theorem 1.10). By denoting with $\mathcal{S} \subset \mathbb{R} \times \dot{H}^1(\partial\Omega)$ the closure of the set of the non trivial solutions (λ, ϕ) to (2.5), we have

Proposition 2.2. *Let κ be an eigenvalue of odd multiplicity of the linear problem* (1.4) *and let $\mathcal{C}$ be the component (i.e. a closed connected subset maximal with respect to inclusion) of $\mathcal{S}$ to which $(\kappa, 0)$ belongs. Then, either $\mathcal{C}$ is unbounded or contains $(\bar{\kappa}, 0)$, where $\bar{\kappa} \neq \kappa$.*

From now on, *we consider the problem* (1.3) *with $\Omega \equiv B$ the unit ball of $\mathbb{R}^3$ and $\mu \equiv 1$.* It is well known that the eigenfunctions of the corresponding linear Steklov problem are the *homogenous harmonic polynomials* of degree n and that the Steklov eigenvalues are precisely n, $n = 0, 1, 2, ...$ Moreover, the dimension of each eigenspace is $2n + 1$. Hence, Proposition 2.2 applies to the component of $\mathcal{S}$ containing $(n, 0)$ for every $n = 1, 2, ...$
In a spherical domain it is natural to look for solutions with an *axial symmetry* with respect to a diameter (note that there are no nontrivial *radially* symmetric solutions to (1.3) in the ball). By suitably choosing the coordinate system, we may consider solutions symmetric with respect to the z axis, i.e. solutions which are constant along the parallel lines of the sphere; in spherical coordinates, they will only depend on the distance $r = \sqrt{x^2 + y^2 + z^2}$ from the origin, and on the polar angle θ.
Let us denote by $H^{3/2}_{ax}(B)$ the subspace of the functions $v \in H^{3/2}(B)$ with the above axial symmetry; the boundary traces $v|_{\partial B}$ with vanishing integral on the sphere will belong to a subspace of (2.2) denoted by $\dot{H}^1_{ax}(\partial B)$. Now, by rotational invariance of the Laplacian, by the symmetry of the Neumann condition on the sphere and by uniqueness of the solution to the Neumann problem, one can check that *the operator $G(\lambda, \cdot)$ defined by* (2.3) *maps $\dot{H}^1_{ax}(\partial B)$ in itself.* Moreover, the (non constant) axially symmetric eigenfunctions of the Steklov problem in the ball are those harmonic polynomials which (in polar coordinates) are independent of the azimuthal angle, that is $r^n P_n(\cos\theta)$, $n = 1, 2, ...$ where

the P_n are the *Legendre polynomials*. The restrictions of these eigenfunctions to the spherical surface span the subspace of axially symmetric, zero mean functions of $L^2(\partial B)$.
We now define axially symmetric $\phi \in \dot{H}^1_{ax}(\partial B)$ and $u_0 \in H^{3/2}_{ax}(B)$ as in (2.2) and (2.6) respectively; then (see [2], section 4) we find nontrivial solutions (λ, u) of (1.3) bifurcating from $(n, 0)$, $n = 1, 2, ...$ and such that $u \in H^{3/2}_{ax}(B)$. We stress that there is a *unique* (normalized) axially symmetric eigenfunction for every eigenvalue n, so that all the eigenvalues of the linear problem in $H^{3/2}_{ax}(B)$ are *simple*. Thus, we get

Proposition 2.3. *Let B be the unit ball and let $\mu \equiv 1$. Then, for any $n = 1, 2, ...$ there is a component $\mathcal{C}_n \subset \mathbb{R} \times \dot{H}^1_{ax}(\partial B)$ of $\mathcal{S}$ which meets the point $(n, 0)$; each $\mathcal{C}_n$ is either unbounded or meets $(m, 0)$, with $m \neq n$.*

Remark 2.4. It is worthwhile to recall the following properties of the solutions bifurcating from a simple eigenvalue λ_0 (see [13], [10], [14]): the set of *nontrivial solutions* near to $(\lambda_0, 0)$ consists precisely of a smooth (even analytic in our case, see below) curve $(\lambda(s), \Phi(s))$, where $s \in \mathcal{I}$, an open neighborhood of the origin. Moreover, $\Phi(s) = sv_0 + o(s)$, where v_0 is an eigenfunction corresponding to λ_0.
Hence, by Theorem 2.1, it follows that near to $(n, 0)$ each component $\mathcal{C}_n$ defined in the above proposition is represented by a curve $(\lambda(s), \Phi(s))$ such that $\Phi(-s) = -\Phi(s)$ for s small.

Since $G : \mathbb{R} \times \dot{H}^1_{ax}(\partial B) \to \dot{H}^1_{ax}(\partial B)$ is real analytic, further properties of $\mathcal{S}$ can be deduced in the framework of the analytic bifurcation theory due to Dancer (see [15, 16, 17]).

Proposition 2.5. *Let B be the unit ball, $\mu \equiv 1$, and, for any $n = 1, 2, ...$, let $\mathcal{C}_n$ denote the component of $\mathcal{S}$ which meets the point $(n, 0)$, according to Proposition 2.3. Then there exists a curve $\mathfrak{C}_n$ with the following properties:*

1. *$\mathfrak{C}_n = \{(\Lambda(s), \Phi(s)) : s \in [0, \infty)\}$, where $(\Lambda, \Phi) : [0, \infty) \to \mathbb{R} \times \dot{H}^1_{ax}(\partial B)$ is continuous;*
2. *$(\Lambda(0), \Phi(0)) = (n, 0)$, $\mathfrak{C}_n \subset \mathcal{C}_n$;*
3. *the set $\Sigma_n = \{s \geq 0 : \ker(\mathrm{Id} - \partial_\phi G(\Lambda(s), \Phi(s))) \neq \{0\}\}$ has no accumulation point;*
4. *at each point, $\mathfrak{C}_n$ has a local analytic re-parameterization (this holds, in particular, at each point of Σ_n);*
5. *one of the following occurs:*
   - *(a) $\|(\Lambda(s), \Phi(s))\| \to \infty$ as $s \to \infty$ (which is much stronger than the claim that $\mathfrak{C}_n$ is unbounded in $\mathbb{R} \times \dot{H}^1_{ax}(\partial B)$);*
   - *(b) $\mathfrak{C}_n$ is a closed loop.*

In particular, we can assume without loss of generality that (Λ, Φ) is C^∞; furthermore, outside the singular set Σ_n, which is discrete, ϕ (and hence its harmonic extension u) can be smoothly parameterized with respect to λ along $\mathfrak{C}_n$.

The previous result is simply [18, Theorem 9.1.1] written in our context.

Remark 2.6. Since for every solution $(\Lambda(s), \Phi(s))$ there is another solution $(\Lambda(s), -\Phi(s))$, we can define the curves

$$\tilde{\mathfrak{C}}_n = \{(\Lambda(-s), -\Phi(-s)) : s \in (-\infty), 0]\}$$

By Remark 2.4 above, the union of $\tilde{\mathfrak{C}}_n$ with $\mathfrak{C}_n$ form a continuous, locally analytic curve, which in a neighborhood of the origin takes the form

$$\big(\Lambda(s), \Phi(s)\big) = \big(n + o(1), sv_n + o(s)\big)$$

where v_n is an eigenfunction corresponding to the eigenvalue n.

It would be interesting to establish which of the alternatives of the previous propositions actually holds. For the analogous two-dimensional problem in a disk, the results obtained by variational methods seem to indicate that, in the $(\lambda, \|\phi\|)$ plane, the branches of solutions outgoing from $(n, 0)$ become asymptotic to the $\lambda = 0$ axis. Actually in [3] an explicit family of solutions of problem (1.3) in the case of the unit disk and for $\mu = 1$ is constructed. These solutions bifurcate from the Steklov eigenfunctions of the disk and become asymptotic to the $\lambda = 0$ axis, blowing up at equidistant points on the boundary (for any smooth two-dimensional domain, it has been proved in [7] that there are at least two distinct families of solutions which for $\lambda \to 0$ exhibit the same qualitative behaviour of the explicit solutions in the disk).
The analysis of the three dimensional problem, even in the case of axially symmetric solutions in the unit ball (with $\mu \equiv 1$) is much more complicated; hence, we will study in detail the component of the set of nontrivial solutions bifurcating from $(1, 0)$.

3 Analysis of the first branch

We first prove that we can further restrict our problem to the subspace of the axially symmetric functions u (in the unit ball) which are *odd* with respect to z; such subspace only contains the components of $\mathcal{S}$ which meet the points $(2k + 1, 0)$, $k = 0, 1, 2, ...$
In spherical coordinates, we may represent an axially symmetric function u by $u = \hat{u}(r, \cos\theta)$; by putting $\cos\theta = t$, $-1 \leq t \leq 1$, we get $u = \hat{u}(r, t)$. Then, if u is odd with respect to z, we have $\hat{u}(r, -t) = -\hat{u}(r, t)$. We still denote by ϕ the traces $\phi = \hat{u}(1, \cdot)$.
Now, let V be the subspace of the functions $\phi \in \dot{H}^1_{ax}(\partial B$ such that $\phi(-t) = -\phi(t)$; by the invariance of the Laplace operator with respect to the reflection $z \mapsto -z$ and by the symmetry of the Neumann condition on the sphere, it follows that *any solution of the Neumann problem in the ball with boundary data in V is axially symmetric and odd with respect to z.*
Hence, we can further restrict the functional formulation of the nonlinear equation (2.5) to the subspace V. Note that $s(\phi) = 0$ for every $\phi \in V$ (see equation (2.4)) so that $u = u_0$ for every solution of (1.3) defined below (2.6). Then, we can rephrase Propositions 2.3 and 2.5 in this context.

Proposition 3.1. *Let B be the unit ball and let $\mu = 1$. Then, for any $k = 0, 1, 2, ...$ there exist a curve $\mathfrak{D}_k$, enjoying the properties of the curve $\mathfrak{C}_n$ described in Proposition 2.5, and a connected set $\mathcal{D}_k$, enjoying the properties of the set $\mathcal{C}_n$ described in Proposition 2.3, such that*

$$(2k + 1, 0) \in \mathfrak{D}_k \subset \mathcal{D}_k \subset \mathcal{S} \subset \mathbb{R} \times V.$$

The main advantage of this restriction is that now we can describe some finer properties of the first branch. In fact, we can state

Proposition 3.2. *Let $\lambda > 0$, $u \in V$ be such that $(\lambda, u|_{\partial B}) \in \mathcal{D}_0$ and $u \neq 0$; we may assume that $u > 0$ at some point of the upper half-sphere $\partial B \cap \{z > 0\}$ (otherwise, take $-u$). Then, $u|_{B\cap\{z>0\}} > 0$ and (by writing as before $u = \hat{u}(r, \cos\theta) = \hat{u}(r, t)$ with r, θ, spherical coordinates) the map $t \mapsto \hat{u}(r, t)$ is strictly increasing for every $r > 0$. Furthermore, $\lambda < 1$.*

Proof. By Theorem 4.1 of [2] we can assume that any solution to problem (1.3) in a ball (and with smooth μ) is smooth up to the boundary. In the following, we will denote by u an axially symmetric solution as a function of the *cylindrical coordinates* that is

$$u = u(\rho, z)$$

where $\rho = \sqrt{x^2 + y^2}$. We have

$$\hat{u}(r,t) = u\big(r\sqrt{1-t^2}, rt\big) \tag{3.1}$$

Let us now define

$$\hat{v}(r,t) = \frac{1}{r}\hat{u}_t(r,t) \tag{3.2}$$

By (3.1) we have

$$\hat{v} = u_z - \frac{t}{\sqrt{1-t^2}}u_\rho = u_z - \frac{z}{\rho}u_\rho \equiv v(\rho, z) \tag{3.3}$$

Then, by applying to v the Laplace operator in cylindrical coordinates

$$\Delta v = v_{\rho\rho} + \frac{1}{\rho}v_\rho + v_{zz}$$

we find after some calculations

$$\Delta v = -\Delta\Big(\frac{z}{\rho}u_\rho\Big) = -\frac{2}{\rho}\partial_\rho\Big(u_z - \frac{z}{\rho}u_\rho\Big) = -\frac{2}{\rho}v_\rho$$

Then, the function v solves the equation

$$v_{\rho\rho} + \frac{3}{\rho}v_\rho + v_{zz} = 0 \tag{3.4}$$

for $r = \sqrt{\rho^2 + z^2} < 1$. But the left hand side is the expression of the Laplace operator in cylindrical coordinates in 5 dimensions applied to an axially symmetric function. Hence, v is harmonic (and axially symmetric) in the unit ball $\tilde{B} \subset \mathbb{R}^5$.
Moreover, by definition (3.2),

$$\hat{v}_r = -\frac{1}{r^2}\hat{u}_t + \frac{1}{r}\hat{u}_{tr} = -\frac{1}{r}\hat{v} + \frac{1}{r}\partial_t\hat{u}_r$$

and by recalling (1.3) we find on the unit sphere

$$\hat{v}_r(1,t) = -\hat{v}(1,t) + \partial_t\big(\lambda \sinh \hat{u}(1,t)\big) = -\hat{v}(1,t) + \lambda \cosh \hat{u}(1,t)\, \hat{v}(1,t)$$

that is

$$\hat{v}_r(1,t) = \big(\lambda \cosh \hat{u}(1,t)\; - 1\big)\, \hat{v}(1,t) \tag{3.5}$$

Hence, v is an axially symmetric solution of the *linear* eigenvalue problem (1.4) in a ball $\tilde{\Omega} \subset \mathbb{R}^5$, with weight $\mu(x) = \lambda \cosh u(x) - 1$ (and eigenvalue 1).
By our assumptions on u, we can write in a neighborhood of $(1,0)$ (see Remark 2.4)

$$\big(\lambda, u\big) = \big(1 + \rho(\epsilon), \epsilon(z + w(\epsilon))\big)$$

where ϵ lies in some interval $[-\bar{\epsilon}, \bar{\epsilon}]$ and ρ, w are such that:

1. $\rho : [-\bar{\epsilon}, \bar{\epsilon}] \to \mathbb{R}$ is continuous and $\rho(0) = 0$

2. the map $\epsilon \mapsto w(\epsilon) \equiv w(\epsilon; x, y, z)$, $(x, y, z) \in \bar{B}$, is continuous from $[-\bar{\epsilon}, \bar{\epsilon}]$ to $\mathcal{C}^1(\bar{B})$ and $w(0) = 0$

Then we can write

$$v = \frac{1}{r} u_t = \epsilon(1 + \hat{w}_t(\epsilon)/r)$$

where as before we set $\hat{w}(\epsilon) \equiv \hat{w}(\epsilon; r, t) = w(\epsilon; \rho, z)$. The function $\hat{w}_t(\epsilon)/r$ is harmonic in the unit ball $\tilde{B} \subset \mathbb{R}^5$ and has the same normal derivative as v on the boundary ∂B; it follows by (3.5) that such normal derivative is uniformly vanishing for $\epsilon \to 0$. Then, $\lim_{\epsilon \to 0} \hat{w}_t(\epsilon)/r = c$; now, by choosing $r = 1$ and recalling that $\hat{w}(\epsilon) \to 0$ in $\mathcal{C}^1(\bar{B})$ for $\epsilon \to 0$, we conclude $c = 0$.

From the above result it follows that $v > 0$ for ϵ small enough. We claim that $v > 0$ all along $\mathcal{D}_0$; if not, by continuity there is a pair $(\lambda, u) \in \mathcal{D}_0$ such that $v \geq 0$ and $v(x) = 0$ for some $x \in \partial\tilde{B}$ (the boundary the unit ball of $\mathbb{R}^5$). Then, by (3.5) we get $v_r(x) = 0$, contradicting the Hopf principle. Since $\hat{u}_t = rv$, we find that $t \mapsto \hat{u}(r, t)$ is strictly increasing for every $r > 0$. But $\hat{u}(r, 0) = 0$, so that $\hat{u} > 0$ for $t > 0$, i.e. $u > 0$ on the upper half ball. Finally, by integration of both sides of (3.5) we get

$$\int_{\partial B} (\lambda \cosh u - 1)\, v = 0$$

which is possible for a positive v only if $\lambda < 1$. □

Theorem 3.3. *The set $\mathfrak{D}_0$ is unbounded; more precisely, $0 < \Lambda(s) \leq 1$ and $\|\Phi(s)\|_{L^\infty(\partial B)} \to \infty$ as $s \to \infty$, where $\Phi(s)$, $s \in [0, +\infty)$ are the solutions defined in Proposition 2.5.*

Proof. By Proposition 2.5 either $\mathfrak{D}_0$ is unbounded, or it is a closed loop; in the latter case, by Remark 2.4 there exist two solutions of opposite sign at the beginning and at the end of the loop near to $(1, 0)$; since the nontrivial solutions in $\mathfrak{D}_0$ only vanish at $z = 0$ (by Proposition 3.2) it is readily checked that $\mathfrak{D}_0$ must intersect the λ axis at some other point, which is necessarily $(2j + 1, 0)$ for some $j > 0$.

By recalling that $\Phi(s) = u|_{\partial B}$ with u harmonic function (axially symmetric and odd with respect to the reflection $z \mapsto -z$) it now follows by continuity (see [2]) that there exists a $(\Lambda, \Phi) \in \mathfrak{D}_0 \subset \mathcal{D}_0$, $\Phi \neq 0$ such that $\Phi = \hat{u}(1, t)$ (see (3.1)) and $\hat{u}_t(1, t) = 0$ at some point $t \in (-1, 1)$ contradicting the positivity of (3.2) on $\mathcal{D}_0$.

Thus, we conclude that $\mathfrak{D}_0$ is unbounded; but we know from Proposition 3.2 that λ is bounded along $\mathcal{D}_0 \supset \mathfrak{D}_0$; then, again by Proposition 2.5, we have $\|\Phi\|_{H^1(\partial\Omega)} \to \infty$ along $\mathfrak{D}_0$. Hence, as remarked in [2], we also have that the uniform norm $\|\Phi\|_{L^\infty(\partial\Omega)}$ becomes arbitrarily large; we stress that, due to Proposition 3.2, the sup norm of Φ is given by the value $u(0, 0, 1)$ where u is defined as above. □

Before investigating the limiting behaviour of the solutions with increasing supremum norm along $\mathfrak{D}_0$, we point out some further properties of such solutions. Let us introduce the energy

$$\mathcal{E}_\lambda(u) = \frac{1}{2} \int_B |\nabla u|^2\, dx - \lambda \int_{\partial B} (\cosh u - 1)\, d\sigma \tag{3.6}$$

and assume that $(\Lambda(s), u(s))$ is the solution to (1.3) corresponding to the point $(\Lambda(s), \Phi(s)) \in \mathfrak{D}_0$; by denoting with u', Λ'. the derivatives with respect to s, we can compute

$$\frac{d\mathcal{E}_\lambda(u)}{ds} = \int_B \nabla u \nabla u'\, dx - \Lambda \int_{\partial B} (\sinh u)\, u'\, d\sigma - \Lambda' \int_{\partial B} (\cosh u - 1)\, d\sigma = -\Lambda' \int_{\partial B} (\cosh u - 1)\, d\sigma \tag{3.7}$$

the last equality following by the weak form of (1.3). Since the last integral is nonnegative, it follows by (3.7) that the energy is decreasing for $\Lambda' \geq 0$; in particular, if $s \notin \Sigma_0$ (see Proposition 2.5), we can take Φ, and consequently u, smoothly dependent on λ in certain intervals contained in $(0, 1)$. Then, in every such interval $\mathcal{E}_\lambda(u)$ *is strictly decreasing with respect to* λ.

As it is suggested by numerical experiments (see below) in the 3 dimensional problem we have $\Sigma_0 \neq \emptyset$, and $\Lambda'(s)$ changes its sign along $\mathfrak{D}_0$. It is an open problem to find whether Σ_0 is a finite or infinite discrete set.

3.1 Blow-up analysis

The final part of this section is devoted to the asymptotic analysis of solutions with increasing supremum norm along $\mathfrak{D}_0$. Taking into account Proposition 3.2 and Theorem 3.3, we have that any unbounded subset of $\mathfrak{D}_0$ contains a subsequence $(\lambda_j, u_j|_{\partial B})$, with u_j harmonic in B, such that $u_j > 0$ on $B \cap \{x_3 > 0\}$ and

$$0 < \lambda_j < 1, \qquad \max_{\overline{B}} u_j(x) = u_j(k) = M_j \to \infty \tag{3.8}$$

as $j \to \infty$ (here $x = (x_1, x_2, x_3)$, and $k = (0, 0, 1)$ denotes the north pole of B).
We choose a sequence r_j such that $r_j \to 0$ for $j \to +\infty$ and define the transformation

$$y = \frac{k - x}{r_j} \tag{3.9}$$

which maps B onto a sphere B_j (of radius r_j^{-1}, center at $y = r_j^{-1}k$ and outer normal $\nu_j = r_j y - k$) in the upper half plane $y_3 \geq 0$. Note that the the point k is mapped to the origin $y = 0$ and that the sequence B_j exhausts $\mathbb{R}^3_+$.
Let us now define

$$v_j(y) = \frac{e^{-M_j}}{r_j}\Big[M_j - u_j(-r_j y + k)\Big] \tag{3.10}$$

The functions v_j are harmonic in B_j, positive and symmetric with respect to the y_3 axis, with minimum $v_j(0) = 0$. Moreover, they satisfy the following boundary conditions

$$\partial_{\nu_j} v_j(y) = (r_j y - k) \cdot \nabla_y v_j(y) = \frac{e^{-M_j}}{r_j}(k - r_j y) \cdot \nabla_y u_j(-r_j y + k)$$

$$= -e^{-M_j}\partial_\nu u_j(-r_j y + k) = -\lambda_j e^{-M_j} \sinh(u_j(-r_j y + k)), \qquad y \in \partial B_j \tag{3.11}$$

By the assumptions on u_j we have

$$-\frac{\lambda_j}{2}\left(1 - e^{-2M_j}\right) \leq \partial_{\nu_j} v_j(y) < 0$$

for every $y \in \partial B_j \cap \{y_3 < r_j^{-1}\}$ (the lower half of the spherical surface ∂B_j).
By the above estimate, one can infer that the sequence v_j converges uniformly in every bounded set of $\mathbb{R}^3_+$. Of course, the form of the limit problem depends on the choice of r_j. By taking $r_j = e^{-M_j}$ we get from (3.10) and (3.11),

$$\partial_{\nu_j} v_j(y) = -\lambda_j e^{-M_j} \sinh(M_j - v_j(y)) = -\frac{\lambda_j}{2}\Big[e^{-v_j(y)} - e^{-2M_j + v_j(y)}\Big]. \tag{3.12}$$

Then (via suitable projections of the lower half spheres $B_j \cap \{y_3 < r_j^{-1}\}$ to the upper half space $y_3 > 0$ and letting $j \to \infty$) we obtain the following limit problem

$$\begin{array}{rcl} \Delta v &=& 0 \quad \text{in } \mathbb{R}^3_+ \\ \partial_\nu v &=& -\dfrac{\lambda^*}{2}\, e^{-v} \quad \text{on } \mathbb{R}^2 \times \{y_3 = 0\} \\ v &=& v(\rho, z), \quad v \geq 0 \text{ in } \mathbb{R}^3_+, \quad v(0) = 0, \end{array} \tag{3.13}$$

where $\lambda^* \in [0, 1]$ is an accumulation point for the sequence (λ_j). In this way, the study of the asymptotic behaviour of $\mathfrak{D}_0$ is related to the classification of the entire solutions of problem (3.13).

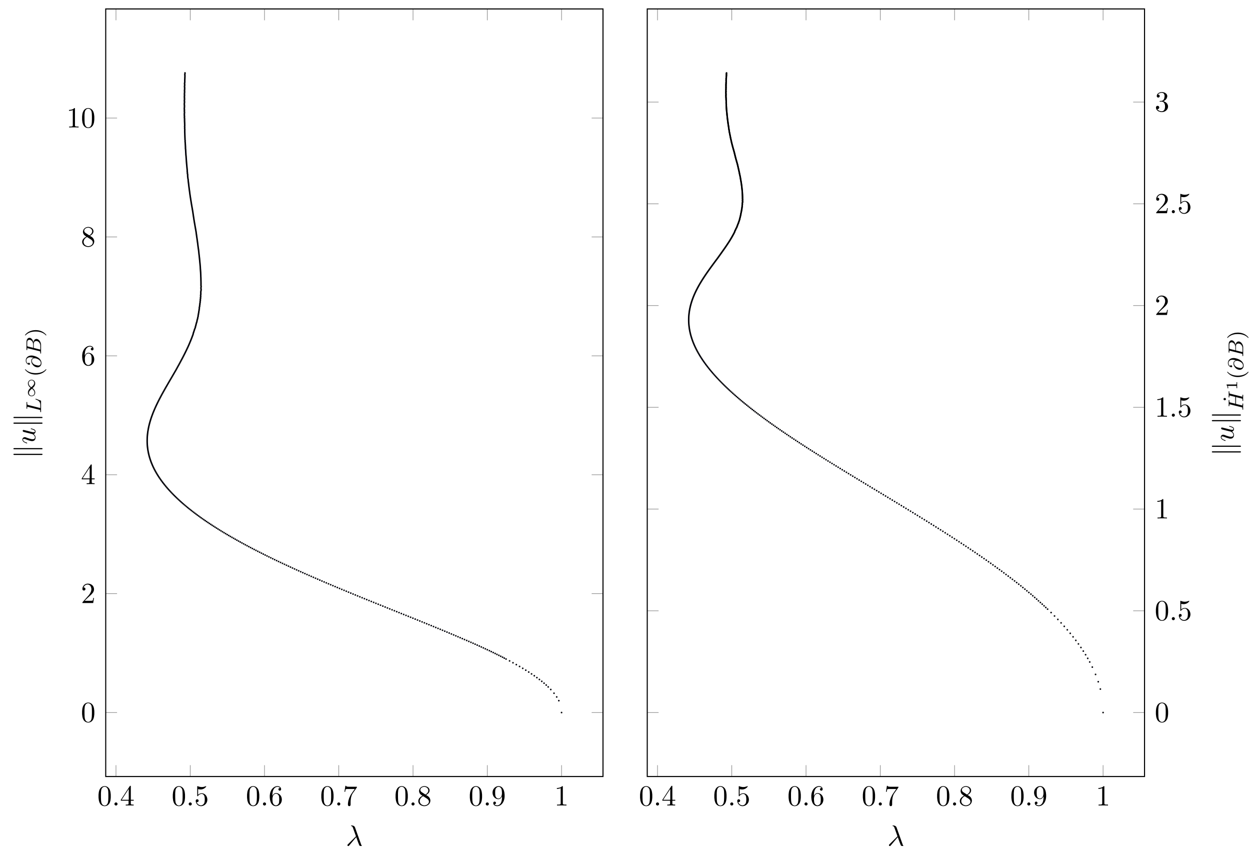


Figure 1: plot of the numerical simulations for the bifurcation branch $\mathfrak{D}_0$.

Remark 3.4. The function
$$v^*(y) = \frac{\lambda^*}{2} y_3$$
solves Problem (3.13). Moreover, if $\lambda^* > 0$ it has infinite Morse index.
Indeed, by writing the energy functional associated to the problem:
$$E(v) = \frac{1}{2}\int_{\mathbb{R}^3_+} |\nabla v|^2 - \frac{\lambda^*}{2}\int_{\mathbb{R}^2} e^{-v} \tag{3.14}$$
we have that v solves (3.13) if and only if
$$E'(v)[\phi] = \int_{\mathbb{R}^3_+} \nabla v \cdot \nabla \phi + \frac{\lambda^*}{2}\int_{\mathbb{R}^2} e^{-v}\phi = 0 \tag{3.15}$$
for every $\phi \in C^1_0(\mathbb{R}^3)$ (not necessarily vanishing on $\mathbb{R}^2 \times \{y_3 = 0\}$); therefore
$$E''(v)[\phi,\phi] = \int_{\mathbb{R}^3_+} |\nabla \phi|^2 - \frac{\lambda^*}{2}\int_{\mathbb{R}^2} e^{-v}\phi^2 \tag{3.16}$$
and
$$E''(v^*)[\phi,\phi] = \int_{\mathbb{R}^3_+} |\nabla \phi|^2 - \frac{\lambda^*}{2}\int_{\mathbb{R}^2} \phi^2. \tag{3.17}$$

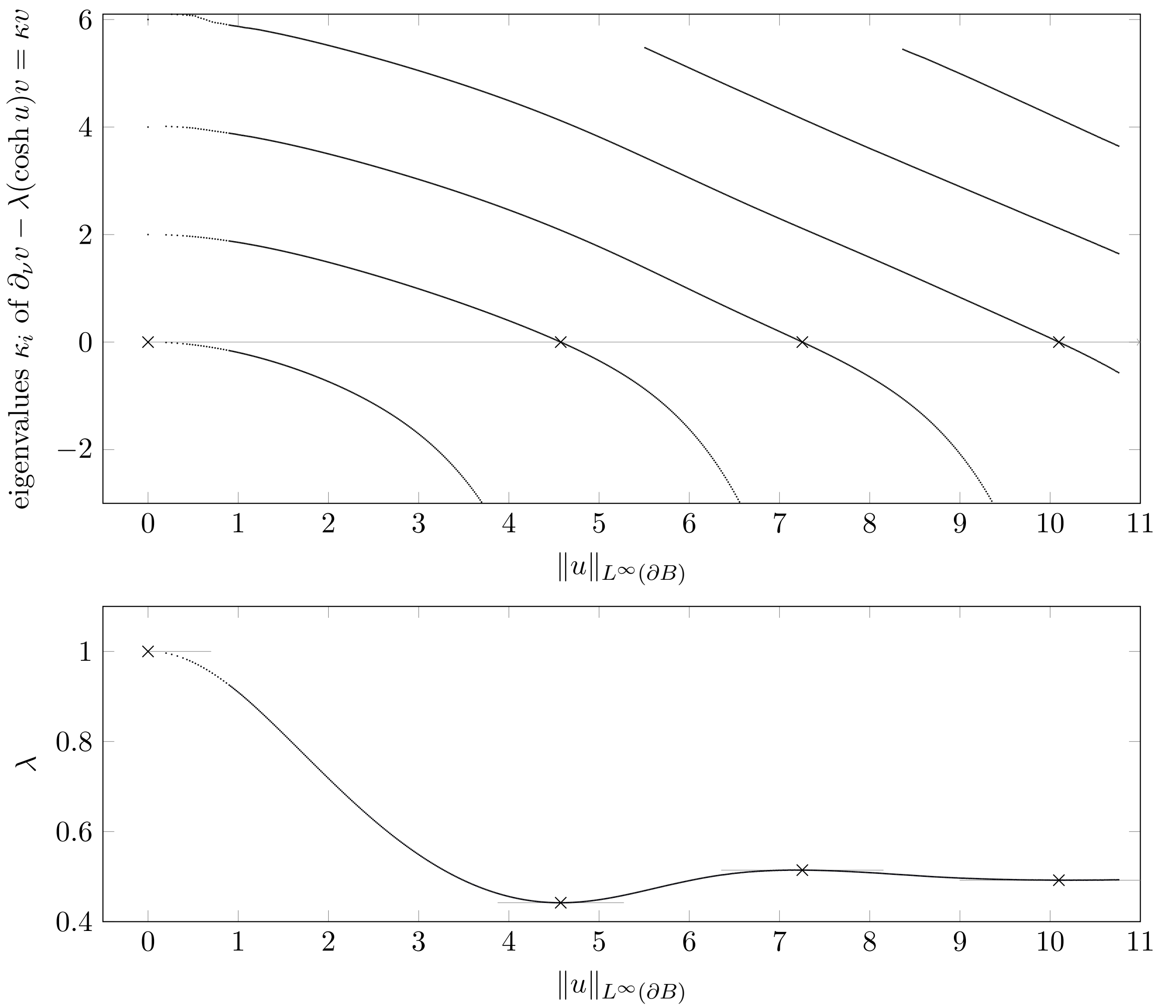


Figure 2: plot of the first 4 eigenvalues of the Steklov eigenvalues problem $\partial_\nu v - \lambda(\cosh u)v = \kappa v$ (above), and of the bifurcation parameter λ (below), as functions of $\|u\|_{L^\infty(\partial B)}$ along $\mathfrak{D}_0$. The turning points on the branch correspond to the increasing of the Morse index of the solution.

Now, for any fixed $\eta \in C_0^1(\mathbb{R}^3)$, $r > 0$ and $\xi \in \mathbb{R}^3$, we can choose $\phi_\xi(y) = r\eta(ry + \xi)$, so that

$$E''(v^*)[\phi_\xi, \phi_\xi] = r^2 \int_{\mathbb{R}^3_+} |\nabla \eta|^2 - \frac{\lambda^*}{2} \int_{\mathbb{R}^2} \eta^2 < 0 \qquad \text{for sufficiently small } r.$$

As a consequence, for any $m \in \mathbb{N}$, one can easily find $\xi_1, \dots, \xi_m \in \mathbb{R}^3$ in such a way that $W_m = \operatorname{span}\{\phi_{\xi_1}, \dots, \phi_{\xi_m}\}$ has dimension m and $E''(v^*)$ is negative defined on W_m.

It remains an open question whether Problem (3.13) admits other nontrivial solutions, apart from v^*, and in such a case whether solutions with finite Morse index may exist. In case $\lambda^* > 0$, the absence of finite Morse index solutions to (3.13) would indicate the presence of infinitely many secondary bifurcation points (turning points) along $\mathfrak{D}_0$. Such kind of behaviour is also suggested by the numerical simulations we discuss in the next section.

4 Numerical simulations

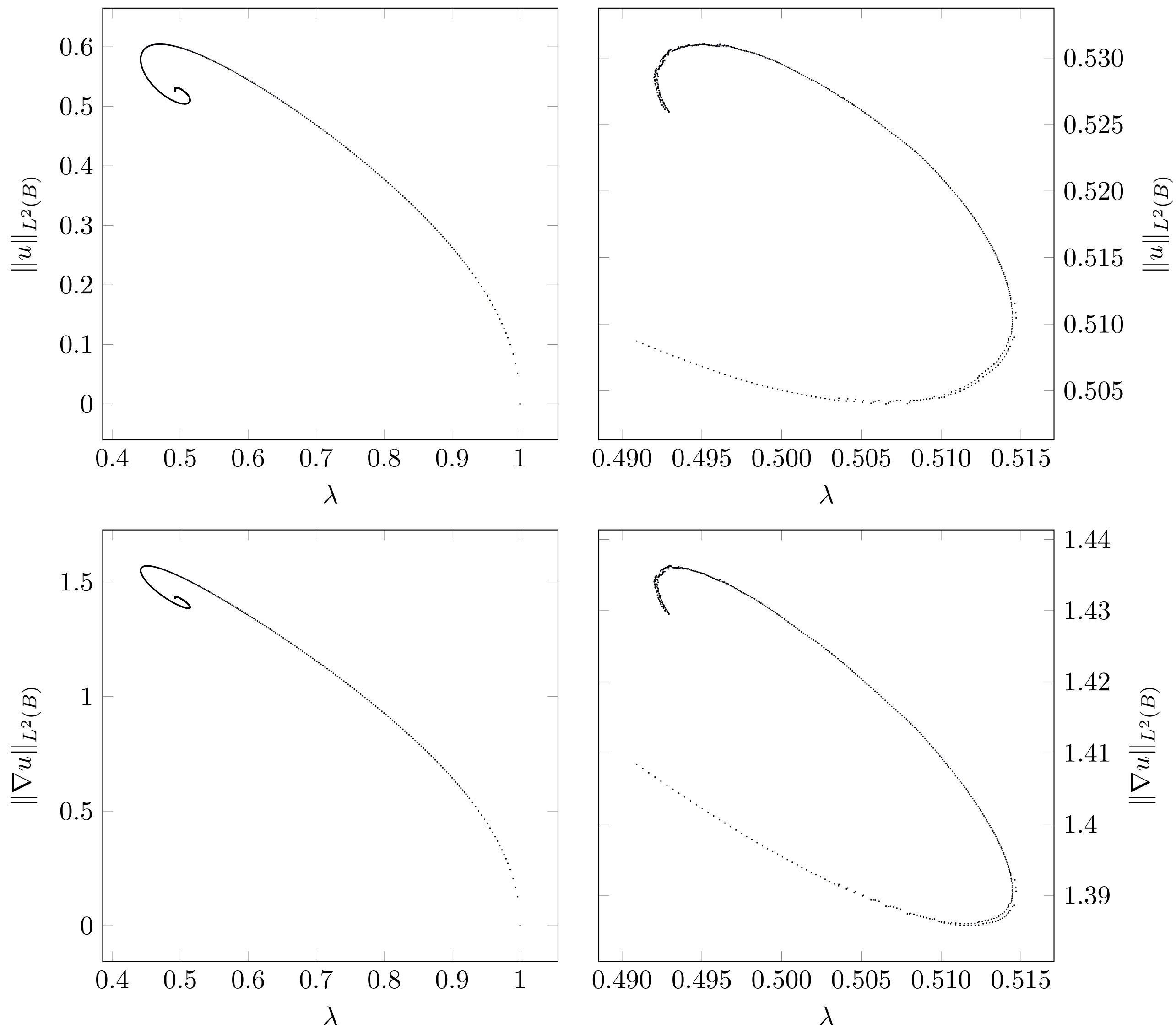
In this section we present and discuss the numerical scheme that was used in order to approximate the bifurcation branches of problem (1.3). We then give some comments on the numerical bifurcation diagram that we have obtained.
The mathematical literature concerning continuation methods is vast and it is not our aim to give here a complete list of the possible solutions already available, we just refer the interested reader to [19, 20] and to the references therein. As a matter of fact, the method we implemented can be traced back to the large class of predictor-corrector methods, where in our case the predictor step is obtained through a projection on a suitable space of solutions, while the corrector step consists in a time step of a suitable parabolic flow.
From a previous numerical investigation it is clear that λ can not be used to parametrize the curves of the bifurcation diagram, due to the non monotonicity of this quantity along these curves. Moreover the fact that for any $\lambda > 0$ the set of trivial solutions $(\lambda, 0)$ is also the stable branch discourages the direct use of more classical methods, such as the standard parabolic flow. As a concluding remark, we recall that even from the theoretical point of view, the branches of non trivial solutions are obtained in the functional space $\dot{H}^1(\partial B)$, which is somehow unnatural from a numerical point of view. For all these reasons, after preliminary numerical investigations we consider worthwhile to assume the $L^\infty(\partial B)$ norm of the solution to be a possible parameter to describe the curve, as it has already been shown that such norm is unbounded along any bifurcation branch.

Algorithm 1 (Continuation method)

1: initialize s as a small number and let α be a large positive constant
2: initialize $(\lambda, u) \leftarrow (1, sz)$
3: **repeat**
4:     set $s \leftarrow s + \epsilon$
5:     $(\gamma, v) \leftarrow (\lambda, u)$
6:     **repeat**
7:         $\tilde{v} \leftarrow \dfrac{v}{\|v\|_{L^\infty}} \cdot s$ ▷ Predictor step
8:         $\gamma \leftarrow \dfrac{\int_{B_1} |\nabla \tilde{v}|^2}{\int_{\partial B_1} \sinh(\tilde{v})\tilde{v}}$
9:         Solve $\begin{cases} \Delta v = 0 & \text{in } B_1(0) \\ \partial_\nu v + \alpha(v - \tilde{v}) = \gamma \sinh(\tilde{v}) & \text{on } \partial B_1(0) \end{cases}$ ▷ Corrector step
10:     **until** convergence with a prescribed tolerance
11:     $(\lambda, u) \leftarrow (\gamma, v)$
12: **until** blow-up

Remark 4.1. Similar results can be obtained also in the case of parameter the $\dot{H}^1(\partial B)$, even though the resulting method seems less efficient in terms of convergence rate.

Let us point out that both the $L^\infty(\partial B)$ and $\dot{H}^1(\partial B)$ norms constitute an unnatural choice as parameters from a point of view of the numerical method used in the corrector step (step 9 in the algorithm), which is discretized using its weak formulation in $H^1(B)$. As a particular consequence, this makes the predictor step a priori unfeasible, as the set $\{u \in H^1(B) : \|u\|_{L^\infty(\partial B)} = s\}$ (and $\{u \in H^1(B) : \|u\|_{\dot{H}^1(\partial B)} = s\}$) is not closed in the topology of $H^1(B)$: this complicates the convergence analysis, which is not carried out in the following.

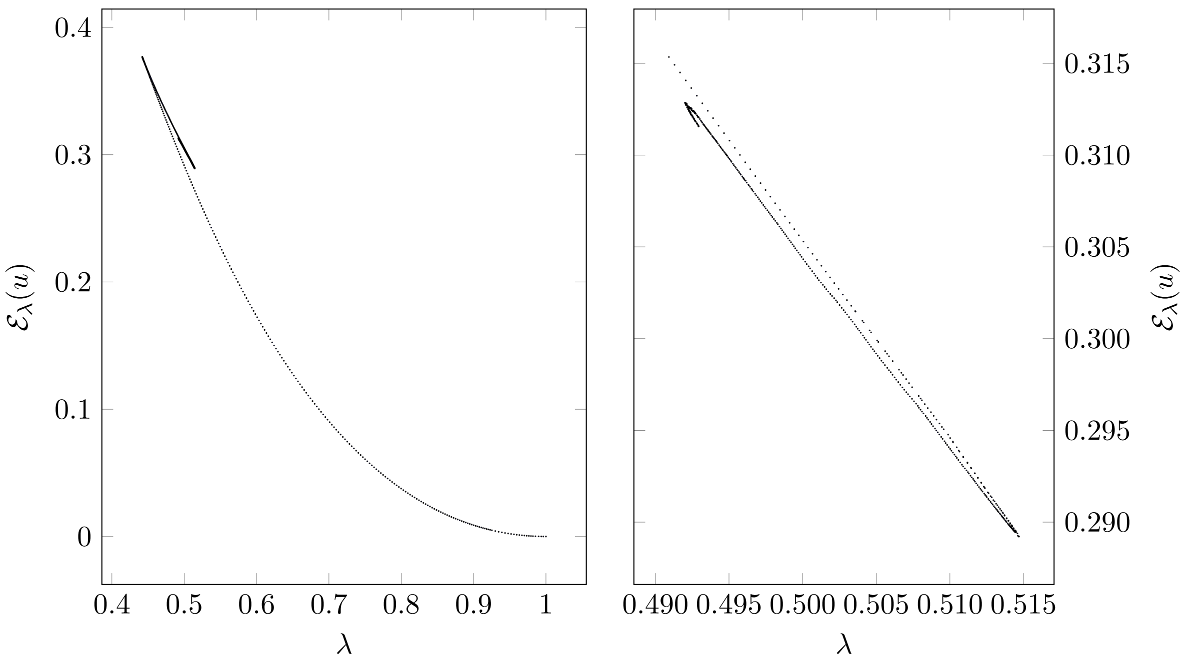


Remark 4.2. One may also try to use a more sophisticated algorithm, such as the Newton's method. As it turns out from numerical investigation, the Morse index of the solutions increases by one unit at each turning point, but the negative eigenvalues of the linearized operator diverge rather rapidly (see Figure 2). Any attempt we tried in stabilizing such algorithm led us to loose the convergence of the original method, and for this reason we chose to focus our attention on a more stable, even if less efficient, fixed point method.

Now we proceed with some comments on the numerical bifurcation diagrams. The plots are obtained from the simulation data using the LaTeX-graphics packages Ti*k*Z and PGFPLOTS. To start with, as we already mentioned, the $L^\infty(\partial B)$ norm of the solution is increasing along the branch essentially by construction. Also the $\dot{H}^1(\partial B)$ norm appears to increase (Figure 1). On the other hand, other norms are not monotone, and the simulations suggest that $\mathfrak{D}_0$ may be bounded in $H^1(B)$ (Figure 4).
The energy

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\int_B |\nabla u|^2\,dx - \lambda\int_{\partial B}(\cosh u - 1)\,d\sigma$$

exhibits an analogous behavior along the branch, but it looses smoothness: the turning points of $\mathfrak{D}_0$



become corner points for $\mathcal{E}_\lambda$. Moreover, according to (3.7), $\mathcal{E}_\lambda(u)$ is decreasing with respect to λ in every interval of smooth dependence (Figure 4).

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