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GLEASON'S PROBLEM AND SCHUR MULTIPLIERS IN THE MULTIVARIABLE QUATERNIONIC SETTING

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Abstract. We define and study the counterparts of Gleason's problem, of the Arveson's space and of Schur multipliers when the unit ball of $\mathbb{C}^N$ is replaced by the unit ball of $\mathbb{H}^N$. Schur multipliers are characterized in terms of coisometric operator matrices in quaternionic spaces. We define the counterpart of Blaschke factors in this setting.

Contents

1. Introduction and preliminaries

This paper is part of a series of works where classical Schur analysis is considered in the setting of slice-hyperholomoprhic functions; see for instance [2, 4, 3, 1]. In these papers the case of functions of one (as opposed to several) quaternionic variable was considered. Slice-hyperholomorphic functions of several quaternionic variables have been considered in [15, 17]. In the present work we consider the case of several quaternionic variables. A key player in the paper is a non-commutative version of Gleason's problem. To be more precise we first need some notation. We denote by $\mathbb{M}$ the free monoid generated by $p_1, \ldots, p_N$, and by $\widetilde{\ell}$ the set of finite ordered sequences a of pairs of integers $(n_1, \alpha_1), \ldots, (n_k, \alpha_k)$, with $k \in \mathbb{N}$ and $n_i \in \{1, \ldots, N\}$ and $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, k$ and moreover $n_1 \neq n_2 \neq n_3 \cdots$. We set

$$p^a = p_{n_1}^{\alpha_1} p_{n_2}^{\alpha_2} \cdots p_{n_k}^{\alpha_k} \in \mathbb{M} \tag{1.1}$$

where $n_1 \neq n_2 \neq n_3 \cdots$. When $\alpha_1 = \cdots = \alpha_k = 0$ we set $p^a = 1$. Furthermore $\mathbb{H}_1^N$ denotes the N dimensional unit ball in $\mathbb{H}^N$, that is the set of elements $(p_1, \ldots, , p_N)$ such that $\sum_{n=1}^N |p_n|^2 < 1$.

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With these definitions at hand, consider a right quaternionic Hilbert space $\mathcal{H}$ of converging $\mathcal{P}$-valued power series, defined in an open subset $\Omega \subset \mathbb{H}_1^N$, and where the coefficient space $\mathcal{P}$ is a two-sided Pontryagin space. Thus elements of $\mathcal{H}$ are converging series of the form

$$f(p) = \sum_{a \in \widetilde{\ell}} p^a q_a, \tag{1.2}$$

where the coefficients $q_a \in \mathcal{P}$.

We say that Gleason's is solvable in $\mathcal{H}$ if for every $f \in \mathcal{H}$ there exist $f_1, \ldots, f_N \in \mathcal{H}$ such that

$$f(p) - f(0) = \sum_{n=1}^{N} p_n f_n(p). \tag{1.3}$$

Note that such a choice is highly non-unique when $N > 1$. Furthermore, equation (1.3) *does not* imply that any of the functions

$$p \ \mapsto \ p_n f_n(p)$$

belongs to $\mathcal{H}$.

We refer to [20, 21] for more information on Gleason's problem. This problem was studied in the quaternionic setting in the framework of Fueter series in [11]. In the classical setting, condition for uniqueness of the decomposition (1.3) have been given in [7].

The purpose of this paper is to characterize a family of reproducing kernel Hilbert spaces of converging power series in several quaternionic variables, and in which Gleason's problem is solvable. It can be seen as the quaternionic version of [5].

We note that non-commutative interpolation with underlying field the complex number has a long history; see for instance [19, 16, 18, 13]. Although some formulas that we obtain have similarities with corresponding formulas in the aforementioned works, our context is different.

Given a two-sided quaternionic vector space $\mathcal{V}$ endowed with an Hermitian form $[\cdot, \cdot]$ we assume that, besides the usual condition

$$[va, wb] = \overline{b}[v, w]a, \quad \forall v, w \in \mathcal{V} \quad \text{and} \quad \forall a, b \in \mathbb{H},$$

the Hermitian form also satisfies

$$[v, aw] = [\overline{a}v, w], \forall v, w \in \mathcal{V} \quad \text{and} \quad \forall a \in \mathbb{H}. \tag{1.4}$$

This property plays a crucial role in the sequel; see for instance the proof of STEP 1 of Theorem 5.1.

We denote by * the adjoint with respect to the Hermitian forms.

Given a quaternionic right Pontryagin space $\mathcal{P}$ we denote by $\nu_-(\mathcal{P})$ its index of negativity, that is the dimension of a maximal strictly negative subspace of $\mathcal{P}$.

We now recall the following definitions (see [10] for the first). The definitions make sense since the spectral theorem holds for Hermitian quaternionic matrices; see [22].

Definition 1.1. *A $\mathbb{H}$-valued function $K(u,v)$ defined on a set B has κ negative squares if it is Hermitian:*

$$K(u,v)=\overline{K(v,u)},\quad \forall u,v\in A,$$

and if for every $n\in\mathbb{N}$ and every $u_1,\ldots,u_n\in B$, the $n\times n$ Hermitian matrix with (t,s) component $K(u_s,u_t)$ has at most κ strictly negative eigenvalues, and exactly κ strictly negative eigenvalues for some choice of $n,u_1,\ldots,u_n$.

Definition 1.2. *Let $A=A^*$ be an everywhere defined bounded self-adjoint operator in the Pontryagin space $\mathcal{P}$. We denote by $\nu_-(A)$ the number of negative squares of the function*

$$K(f,g)=[Af,g]_{\mathcal{P}},\quad f,g\in\mathcal{P}.$$

2. Schur multiplier

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two-sided quaternionic Pontryagin spaces with same index of negativity (we will call them the coefficient spaces), and let $\Omega\subset\mathbb{H}_1^N$ denote a neighborhood of the origin. Furthermore, let $s:\Omega\to\mathbf{L}(\mathcal{P}_1,\mathcal{P}_2)$. Consider the equation

$$k_s(p,q)-\sum_{j=1}^N p_jk_s(p,q)\bar{q}_j=I-s(p)s(q)^* \tag{2.1}$$

where the unknown is the function $k_s(p,q)$, and where

$$p=(p_1,....,p_N)\quad\text{and}\quad q=(q_1,....,q_N)\in\Omega\subset\mathbb{H}_1^N.$$

First note that (2.1) indeed makes sense since the coefficient spaces are assumed two-sided.

Remark 2.1. If one considers fundamental symmetries J_1 and J_2 such that the forms $[\cdot,J_1\cdot]_{\mathcal{P}_1}$ and $[\cdot,J_2\cdot]_{\mathcal{P}_2}$ are positive definite, then the right hand side of (2.1) takes the familar form

$$J_2-s(p)J_1s(q)^*,$$

where now the * denotes adjoint between Hilbert spaces.

Proposition 2.2. *Equation* (2.1) *has a unique solution, given by the power series expansion*

$$k_s(p,q)=\sum_{a\in\widetilde{\ell}}p^a(I-s(p)s(q)^*)\overline{q^a}. \tag{2.2}$$

Indeed, (2.2) follows from iterating (2.1). Furthermore, the difference of two solutions k_1 and k_2 will correspond to (2.2) with $I-s(p)s(q)^*$ replaced by 0, and hence $k_1-k_2\equiv 0$.

Definition 2.3. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two-sided quaternionic Pontryagin spaces with same index of negativity. A function $s:\Omega\subset(\mathbb{H}^N)_1\to\mathbf{L}(\mathcal{P}_1,\mathcal{P}_2)$ defined in a neighborhood of the origin is called Schur multiplier function if the kernel k_s (that is, the unique solution of equation* (2.1)*) is positive definite in Ω.*

3. The case when the coefficient spaces are Hilbert spaces

In the complex case, it is well known that the positivity of the analog of the kernel k_s in an open subset U of the unit disk (in the case of one complex variable), or more generally, in an open subset of the unit ball, implies that s is the restriction to U of an analytic function. We now study the counterpart of this result in our setting.

Definition 3.1. *The space $\mathcal{F}$ of power series of the form* (1.2) *and such that $\sum_{a\in\widetilde{\ell}}|q_a|^2<\infty$ is the full Fock space associated to the quaternionic variables $p_1,\dots,p_N$.*

In the complex setting, the space $\mathcal{F}$ with non commutative variables and complex coefficients is called the Fock space and used in [12, 14]. The corresponding reproducing kernel is called the non-commutative Szegö kernel.

It is readily seen that it is the reproducing kernel Hilbert space with reproducing kernel Hilbert space

$$k_0(p,q)=\sum_{a\in\widetilde{\ell}}p^a\overline{q^a}. \tag{3.1}$$

Theorem 3.2. *Let s be defined in $\mathbb{H}_1^N$ and assume that the kernel $k_s(p,q)$ is positive definite in Ω. Then s is a converging power series in $\mathbb{H}_1^N$.*

Proof. The method to prove the theorem is classical. We begin with a remark. Since $k_s(q,q)\geq 0$ we have

$$k_s(q,q)=\frac{1-|s(q)|^2}{1-\sum_{u=1}^N|q_u|^2}\geq 0,$$

and in particular $|s(q)|\leq 1$. Thus the function $p\mapsto\sum_{u=0}^\infty p^u\overline{s(q)}\overline{q}^uh\in\mathcal{F}$. Consider now the linear relation $Y\subset\mathcal{F}\times\mathcal{F}$ spanned by the elements of the form

$$(k(p,q)h,\sum_{u\in\widetilde{\ell}}p^u\overline{s(q)}\overline{q}^uh),\quad q\in\mathbb{H}_1\quad\text{and}\quad h\in\mathbb{H}. \tag{3.2}$$

The positivity of the kernel forces Y to be contractive. Since it is densely defined, it extends to the graph of a contraction T, whose adjoint is given by the formula

$$(T^*(k(\cdot,q)h))(p)=\sum_{a\in\widetilde{\ell}}p^as(p)\overline{q}^ah. \tag{3.3}$$

Setting $q=0$ we get that $s=T^*1\in\mathcal{F}$, and in particular s is a converging power series in $\mathbb{H}_1^N$. □

More generally it is readily seen that:

Proposition 3.3. *Let $f(p)=\sum_{u\in\widetilde{\ell}}p^uf_u$. Then*

$$(T^*f)(p)=\sum_{u\in\widetilde{\ell}}p^us(p)f_u \tag{3.4}$$

Definition 3.4. *The operator* (3.3) *will be denoted by M_s, and is called a multiplication operator, and a function s such that the kernel $k_s(p,q)$ is positive definite in $\mathbb{H}_1^N$ is called a Schur multiplier.*

When $N=1$, these operators have been defined and studied in [2]. We note that

$$((I-M_sM_s^*)(k(\cdot,q)u))\,(p)=k_s(p,q)u. \tag{3.5}$$

In particular we have the following result, whose proof is the same as in the Hilbert space setting. For the proof of the existence of a positive (Hermitian) squareroot, see for instance [8].

Theorem 3.5. *Let s be a $\mathbf{L}(\mathcal{H}_1,\mathcal{H}_2)$-valued Schur multiplier, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two-sided quaternionic Hilbert space. Then*

$$\mathcal{H}(s)=\text{ran }\sqrt{I-M_sM_s^*}$$

with the operator range norm defined by

$$\|(I-M_sM_s^*)f\|^2_{\mathcal{H}(s)}=\langle (I-M_sM_s^*)f,f\rangle_{\mathcal{F}},\quad f\in\mathcal{F}.$$

As a corollary of the previous results we have:

Proposition 3.6. *Let $U\subset\mathbb{H}_1$ be such that $k_s(p,q)$ is positive definite on U, and assume that $0\in U$. Then s is the restriction of a power series to U.*

Proof. The relation Y defined in (3.2) need not be densely defined anymore. Let $\mathcal{D}$ denote the closed linear span of the functions $k(\cdot,q)$, with $q\in U$. We extend Y to a densely defined linear relation by setting it equal to 0 on $\mathcal{F}\ominus\mathcal{D}$. Formula (3.3) is still valid, but only on U. The result follows by setting $q=0$ in (3.3). □

4. Realization theorem

Theorem 4.1. *A function $s:\Omega\subset\mathbb{H}_1^N\rightarrow\mathbf{L}(\mathcal{P}_1,\mathcal{P}_2)$ defined in a neighborhood of the origin is a Schur multiplier if and only if there exist a right quaternionic Hilbert space $\mathcal{H}$ and a coisometric operator*

$$V=\begin{pmatrix}T & F\\ G & H\end{pmatrix}\quad \mathcal{H}\oplus\mathcal{P}_1\ \longrightarrow\ \mathcal{H}^N\oplus\mathcal{P}_2$$

such that

$$(s(p)^*u)\ =\ H^*u+F^*\left(k_s(\cdot,p)p^*u\right) \tag{4.1}$$

$$k_s(\cdot,p)u-\sum_{n=1}^N T_n^*\left(k_s(\cdot,p)\overline{p_n}u\right)\ =\ G^*u. \tag{4.2}$$

Proof. Let s be Schur multiplier, and let $\mathcal{H}(s)$ denote the associated reproducing kernel Hilbert space of $\mathcal{P}_2$-valued functions defined on Ω and with reproducing kernel (2.2). As in [5], we set

$$\mathcal{H}_N(s)=(\mathcal{H}(s))^N\ominus\mathcal{N},$$

where $\mathcal{N}$ denotes the space of functions $f\in(\mathcal{H}(s))^N$ such that $pf(p)\equiv 0$. We define a linear relation

$$R\subset\begin{pmatrix}(\mathcal{H}_N(s))^N\\ \mathcal{P}_1\end{pmatrix}\times\begin{pmatrix}\mathcal{H}(s)\\ \mathcal{P}_2\end{pmatrix}$$

as the right linear span of elements of the form:

$$\left(\begin{pmatrix}k_s(.,q)q^*u\\ v\end{pmatrix},\begin{pmatrix}(k_s(.,q)-k_s(.,0))u+k_s(.,0)v\\ (s(q)^*-s(0)^*)u+s(0)^*v\end{pmatrix}\right).$$

where $u, v \in \mathcal{P}_2$ and q runs in Ω.
STEP 1: *The relation R is isometric and densely defined.*

Indeed,

$$\left\langle \begin{pmatrix} (k_s(.,q)-k_s(.,0))u_1+k_s(.,0)u_2 \\ (s(q)^*-s(0)^*)u_1+s(0)^*u_2 \end{pmatrix}, \begin{pmatrix} (k_s(.,p)-k_s(.,0))v_1+k_s(.,0)v_2 \\ (s(p)^*-s(0)^*)v_1+s(0)^*v_2 \end{pmatrix} \right\rangle =$$
$$\begin{aligned} &= \langle k_s(p,q)u_1, v_1\rangle - \langle k_s(p,0)u_1, v_1\rangle - \langle k_s(0,q)u_1, v_1\rangle + \langle k_s(0,0)u_1, v_1\rangle + \\ &\quad + \langle k_s(p,0)u_2, v_1\rangle - \langle k_s(0,0)u_2, v_1\rangle + \langle k_s(0,0)u_2, v_2\rangle + \langle k_s(0,q)u_1, v_2\rangle - \\ &\quad - \langle k_s(0,0)u_1, v_2\rangle + \langle (s(p)-s(0))(s(q)^*-s(0)^*)u_1, v_1\rangle + \\ &\quad + \langle s(0)s(0)^*u_2, v_2\rangle + \langle s(0)(s(q)^*-s(0)^*)u_1, v_2\rangle + \langle (s(p)-s(0))s(0)^*u_2, v_1\rangle \end{aligned} \tag{4.3}$$

Now we compute

$$\begin{aligned} &\langle (s(p)-s(0))(s(q)^*-s(0)^*)u_1, v_1\rangle = \\ &\qquad = \langle (s(p)s(q)^* - I + I - s(p)s(0)^* + I - s(0)s(q)^* + s(0)s(0)^* - I)u_1, v_1\rangle \\ &\qquad = \langle (k_s(p,0)+k_s(0,q)-k_s(0,0))u_1, v_1\rangle - (I - s(p)s(q)^*)\langle u_1, v_1\rangle \end{aligned}$$

where we have used the formula of the kernel $k_s(p,q)$. Taking into account this equality, we see that the right hand side of (4.3) reduces to

$$\langle pk_s(p,q)q^*u_1, v_1\rangle + \langle u_2, v_2\rangle = \left\langle \begin{pmatrix} k_s(.,q)q^*u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} k_s(.,p)p^*v_1 \\ v_2 \end{pmatrix} \right\rangle .$$

Hence, R is a densely defined isometry relation between right Pontryagin spaces of same index. By the quaternionic version of a theorem of Shmulyan (see [6, Theorem 1.4.2] for the complex case and [3, Theorem 7.2] for the quaternionic version), R can be extended in a unique way to the graph of an everywhere defined continuous isometric operator V, which we denote in the form

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix}^*$$

It follows then from the definition of the relation that:

$$T^*(k_s(.,q)q^*u) = (k_s(.,q)-k_s(.,0))u \tag{4.4}$$
$$F^*(k_s(.,q)q^*u) = (s(q)^*-s(0)^*)u \tag{4.5}$$
$$G^*u_2 = k_s(.,0)u_2 \tag{4.6}$$
$$H^*u_2 = s(0)^*u_2 \tag{4.7}$$

STEP 2: *s is a Schur multiplier.*

We write

$$T = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{pmatrix}, \quad \text{with} \quad T_n \in \mathbf{L}(\mathcal{H}(s), \mathcal{H}(s)),\ n = 1, \ldots, N.$$

Equation (4.4) can be rewritten as (with p instead of q)

$$k_s(\cdot,p)u-\sum_{n=1}^{N}T_n^*\left(k_s(\cdot,p)\overline{p_n}u\right)=G^*u \tag{4.8}$$

Using (4.5) we can write

$$\begin{aligned}\langle (I-s(p)s(q)^*)u,v\rangle &= \langle u,v\rangle-\langle s(q)^*u,s(p)^*v\rangle\\ &=\langle (HH^*+GG^*)u,v\rangle-\\ &\qquad -\langle (H^*u+F^*\left(k(\cdot,q)q^*u\right)),(H^*v+F^*\left(k(\cdot,p)p^*v\right))\rangle\\ &=\langle (HH^*+GG^*)u,v\rangle-\langle HH^*u,v\rangle-\\ &\qquad -\langle HF^*\left(k(\cdot,q)q^*u\right),v\rangle-\langle u,HF^*\left(k(\cdot,p)p^*v\right)\rangle-\\ &\qquad -\langle (k(\cdot,q)q^*u),FF^*\left(k(\cdot,p)p^*v\right)\rangle\\ &=\langle GG^*u,v\rangle-\langle (k(\cdot,q)q^*u),(I-TT^*)\left(k(\cdot,p)p^*v\right)\rangle\\ &\qquad +\langle GT^*\left(k(\cdot,q)q^*u\right),v\rangle+\langle u,GT^*\left(k(\cdot,p)p^*v\right)\rangle-\\ &=\langle GG^*u,v\rangle-\langle (k(\cdot,q)q^*u),(k(\cdot,p)p^*v)\rangle+\\ &\qquad +\langle T^*\left(k(\cdot,q)q^*u\right),T^*\left(k(\cdot,p)p^*v\right)\rangle+\\ &\qquad +\langle G\left(T^*\left(k(\cdot,q)q^*u\right)\right),v\rangle+\langle u,G\left(T^*\left(k(\cdot,p)p^*v\right)\right)\rangle.\end{aligned}$$

We now use (4.4) and write the above as

$$\begin{aligned}\langle (I-s(p)s(q)^*)u,v\rangle &= \langle GG^*u,v\rangle-\langle k(\cdot,q)q^*u,k(\cdot,p)p^*v\rangle+\\ &\qquad +\langle G^*u-k_s(\cdot,q)u\,,\,G^*v-k_s(\cdot,p)v\rangle+\\ &\qquad +\langle G\left(k_s(\cdot,q)u-G^*u\right),v\rangle+\langle u,G\left(k_s(\cdot,p)v-G^*v\right)\rangle.\end{aligned}$$

Hence,

$$\langle (I-s(p)s(q)^*)u,v\rangle=\langle k_s(\cdot,q)u\,,\,k_s(\cdot,p)v\rangle-\langle k_s(\cdot,q)q^*u,k_s(\cdot,p)p^*v\rangle,$$

which is (2.1).

These same computations show in fact that a function s satisfying (4.1) and (4.2) is a Schur multiplier. □

5. A structure theorem

We now present a characterization of spaces associated to Schur multipliers. The statement and proof are adapted from the commutative version. See [5, Theorem 3.2, p. 260] for the latter.

Note that equation (5.1) in the statement of Theorem 5.1, since the coefficient space is assumed two-sided. Equation (5.1) *does not* imply that any of the functions

$$p\ \mapsto\ p_n(T_nf)(p)$$

belongs to the space $\mathcal{H}$ mentioned in the theorem.

Theorem 5.1. *Let $\mathcal{P}$ be a two-sided quaternionic Pontraygin space, and let $\mathcal{H}$ be a right-sided quaternionic reproducing kernel Hilbert space of $\mathcal{P}$-valued functions defined in a neighborhood $\Omega\subset\mathbb{H}_1^N$ of the origin of $\mathbb{H}^N$. Then there exists a Pontryagin space*

$\mathcal{P}_1$ of same index as $\mathcal{P}$ and a $\mathbf{L}(\mathcal{P}_1,\mathcal{P})$-valued Schur multiplier s such that $\mathcal{H}=\mathcal{H}(s)$ if and only there exist linear bounded operators $T_1,\ldots,T_N$ such that

$$f(p)-f(0)=\sum_{n=1}^{N}p_n(T_nf)(p) \tag{5.1}$$

and

$$\sum_{n=1}^{N}\|T_nf\|^2\le\|f\|^2-[f(0),f(0)]_{\mathcal{P}} \tag{5.2}$$

Proof. We proceed in a number of steps.

STEP 1: *Equation* (5.1) *is equivalent to equation* (4.4).
Indeed, let $k(p,q)$ denote the reproducing kernel of $\mathcal{H}$. From (5.1) we have for every $f\in\mathcal{H}$ and $u\in\mathcal{P}$

$$\begin{aligned}\langle f,k(\cdot,p)u\rangle_{\mathcal{H}}-\langle f,k(\cdot,0)u\rangle_{\mathcal{H}}&=\sum_{n=1}^{N}[p_n(T_nf)(p),u]_{\mathcal{P}}\\&=\sum_{n=1}^{N}[(T_nf)(p),\overline{p_n}u]_{\mathcal{P}}\\&=\sum_{n=1}^{N}\langle T_nf,k(\cdot,p)\overline{p_n}u\rangle_{\mathcal{H}},\\&=\sum_{n=1}^{N}\langle f,T_n^*\left(k(\cdot,p)\overline{p_n}u\right)\rangle_{\mathcal{H}},\end{aligned}$$

where we have used (1.4) to go from the first to the second line. Equation (4.4) follows. The converse statement is proved by reading backwards the arguments.

We now define $\mathcal{H}_0^N=\mathcal{H}^N\ominus\mathcal{N}$, where $\mathcal{N}$ is the subspace of elements of $\mathcal{H}^N$ such that $pf(p)\equiv 0$.

STEP 2: *There exist a Pontryagin space $\mathcal{P}_1$ and operators $H\in\mathbf{L}(\mathcal{P}_1,\mathcal{P})$ and $F\in\mathbf{L}(\mathcal{P}_1,\mathcal{H}_0^N)$ such that*

$$I_{\mathcal{H}_0^N\oplus\mathcal{P}}-\begin{pmatrix}T\\C\end{pmatrix}\begin{pmatrix}T\\C\end{pmatrix}^*=\begin{pmatrix}F\\H\end{pmatrix}\begin{pmatrix}F\\H\end{pmatrix}^*. \tag{5.3}$$

We follow the arguments from [4, p. 862]. Let

$$E=\begin{pmatrix}T\\C\end{pmatrix},$$

where C denotes the evaluation at the origin. From the equality

$$\begin{aligned}&\begin{pmatrix}I_{\mathcal{H}}&0\\E&I_{\mathcal{H}_0^N\oplus\mathcal{P}}\end{pmatrix}\begin{pmatrix}I_{\mathcal{H}}&0\\0&I_{\mathcal{H}_0^N\oplus\mathcal{P}}-EE^*\end{pmatrix}\begin{pmatrix}I_{\mathcal{H}}&0\\E&I_{\mathcal{H}_0^N\oplus\mathcal{P}}\end{pmatrix}^*=\\&\qquad=\begin{pmatrix}I_{\mathcal{H}}&E^*\\0&I_{\mathcal{H}_0^N\oplus\mathcal{P}}\end{pmatrix}\begin{pmatrix}I_{\mathcal{H}}-E^*E&0\\0&I_{\mathcal{H}_0^N\oplus\mathcal{P}}\end{pmatrix}\begin{pmatrix}I_{\mathcal{H}}&E^*\\0&I_{\mathcal{H}_0^N\oplus\mathcal{P}}\end{pmatrix}^*,\end{aligned}$$

we get

$$\nu_-(I_{\mathcal{H}_0^N\oplus\mathcal{P}} - EE^*) = \nu_-(\mathcal{P}).$$

By [4, Theorem 6.7,p. 859], there exists a Pontryagin space $\mathcal{P}_1$ such that

$$\nu_-(\mathcal{P}_1) = \nu_-(\mathcal{P}),$$

and linear bounded operators

$$F \ : \ \mathcal{P}_1 \ \longrightarrow \ \mathcal{H}_0^N \quad \text{and} \quad H \ : \ \mathcal{P}_1 \ \longrightarrow \ \mathcal{P}$$

such that

$$I_{\mathcal{H}_0^N\oplus\mathcal{P}} - \begin{pmatrix} T \\ C \end{pmatrix}\begin{pmatrix} T \\ C \end{pmatrix}^* = \begin{pmatrix} F \\ H \end{pmatrix}\begin{pmatrix} F \\ H \end{pmatrix}^*.$$

STEP 3: *Let $p \in \Omega$. The formula*

$$s(p)^*u = H^*u + F^*\left(k(\cdot, p)p^*u\right)$$

defines a linear bounded operator $s(p) \in \mathbf{L}(\mathcal{P}_1, \mathcal{P})$ and s is a Schur multiplier.

The computations are the same as in the proof of STEP 2 of Theorem 4.1. □

6. Example: Blaschke factors and interpolation in the Fock space

As an example we characterize the one dimensional $\mathcal{H}(s)$ spaces which are isometrically included in $\mathcal{F}$, and connect this result with homogeneous interpolation in $\mathcal{F}$. The analysis is inspired by [9], but there is a difference. Apparently one cannot iterate the procedure and consider multipoint homogeneous interpolation problems.

Theorem 6.1. *$\mathcal{M}$ is a one dimensional vector space isometrically included in $\mathcal{F}$ of the form $\mathcal{H}(s)$ if and only if s is of the form*

$$s(p) = -a + p\sum_{u\in\widetilde{\ell}} p^u\overline{a^u}\left(I_N - a^*a\right)^{1/2}\sqrt{1-|a|^2} \tag{6.1}$$

for some $a \in \mathbb{H}_1^N$.

Proof. Let s be a (not necessarily scalar-valued) Schur multiplier such that the associated space $\mathcal{H}(s)$ is one dimensional, and let $f(p)$ be a basis of $\mathcal{H}(s)$. From (5.1) we have

$$f(p) - f(0) = \sum_{n=1}^N p_n f(p)\overline{a_n} \tag{6.2}$$

for some quaternions $a_1, \ldots, a_N$. Note that $T_n = a_n$ here, or, more precisely,

$$(T_n f q)(p) = f(p)a_n q.$$

In particular, we have that $f(0) \neq 0$. Otherwise, iterating (6.2) leads to $f \equiv 0$. With $a = (a_1, \ldots, a_N)$ we have $a \in \mathbb{H}_1^N$ and

$$f(p) = k_0(p, a),$$

and $\mathcal{H}(s)$ is included inside the Fock space. From (2.2) and the formula for a one dimensional reproducing kernel Hilbert space we have

$$k_s(a,a) = \frac{1-|s(a)|^2}{1-|a|^2} = \frac{k_0(0,a)k_0(a,0)}{\|k_0(\cdot,a)\|^2_{\mathcal{H}(s)}},$$

and so

$$\|k_0(\cdot,a)\|^2_{\mathcal{H}(s)}=\frac{1}{(1-|s(a)|^2)(1-|a|^2)}\le\frac{1}{1-|a|^2}$$

and so $\mathcal{H}(s)$ (if it exists, and with s still to be determined) will be isometrically included inside $\mathcal{F}$ if and only if $s(a)=0$. We claim is that a possible s is given by (6.1). To that purpose, we note that the matrix

$$\begin{pmatrix}T & F\\ G & H\end{pmatrix}=\begin{pmatrix}\dfrac{a^*}{\sqrt{1-|a|^2}} & (I_N-a^*a)^{1/2}\\ & -a\end{pmatrix}\tag{6.3}$$

is unitary. With s as in (6.1), and setting

$$Y(p)=\sum_{u\in\widetilde{\ell}}p^uT^u,$$

we have that

$$Y(p)-\sum_{j=1}^N p_jY(p)\overline{a_j}=I.\tag{6.4}$$

Taking into account that

$$(I_N-a^*a)^{1/2}\,a^*=a^*\sqrt{1-|a|^2}$$

we have:

$$\begin{split}s(p)s(q)^*=|a|^2-pY(p)T(1-|a|^2)-TY(q)^*q^*(1-|a|^2)+\\ +(1-|a|^2)pY(p)TT^*Y(q)^*q^*-\\ -(1-|a|^2)pY(p)Y(q)^*q^*,\end{split}$$

and hence

$$\begin{split}1-s(p)s(q)^*&=(1-|a|^2)\left\{(I+pY(p)T)(I+T^*Y(q)^*q^*)-pY(p)Y(q)^*q^*\right\}\\ &=(1-|a|^2)\left\{Y(p)Y(q)^*-pY(p)Y(q)^*q^*\right\}\end{split}$$

which shows that s is a Schur multiplier. Furthermore the reproducing kernel of $\mathcal{H}(s)$ is

$$(1-|a|^2)Y(p)Y(q)^*=(1-|a|^2)k_0(p,a)k_0(q,a)=k_0(p,q)$$

by the formula for the reproducing kernel, so that $\mathcal{H}(s)$ has for basis by $k_0(p,a)$. $\square$

Definition 6.2. *Let $a\in\mathbb{H}_1$. The Schur multiplier corresponding to* (6.3) *is called a Blaschke factor,and will be denoted by $b_a(p)$.*

Remark 6.3. If one removes the hypothesis of being isometrically included, then the example

$$s(p)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & b_a(p)\end{pmatrix}$$

shows that $s(a)$ need not be equal to 0.

Proposition 6.4. *Let $a\in\mathbb{H}_1$. An element $f\in\mathcal{F}$ is such that $f(a)=0$ if and only if it is of the form $M_{b_a}g$ for some $g\in\mathcal{F}^N$.*

Proof. We first note that

$$\mathcal{F} = \operatorname{ran} \sqrt{(I - M_{b_a} M_{b_a}^*)} + \operatorname{ran} \sqrt{M_{b_a} M_{b_a}^*}. \tag{6.5}$$

where

$$\operatorname{ran}(I - M_{b_a} M_{b_a}^*) = \text{span } \{k_0(\cdot, a)\}$$

is one dimensional. Since $k_0(\cdot, a) \notin \operatorname{ran} \sqrt{M_{b_a} M_{b_a}^*}$ (otherwise we would have $\mathcal{F} = \operatorname{ran} \sqrt{(I - M_{b_a} M_{b_a}^*)}$) the sum in (6.5) is direct and orthogonal and this ends the proof. □

In opposition to [9] it seems difficult to iterate this procedure to more than one point, because of the non-commutativity appearing in particular in formula (3.4). Indeed if a function $f \in \mathcal{F}$ satisfies conditions

$$f(a_1) = 0 \quad \text{and} \quad f(a_2) = 0$$

for some pre-assigned points a_1 and a_2 in $\mathbb{H}_1^N$, then

$$f(p) = (M_{b_{a_1}} g)(p) = \sum_{u=0}^{\infty} p^u b_{a_1}(p) g_u, \quad \text{where} \quad g(p) = \sum_{u=0}^{\infty} p^u g_u \in \mathcal{F}^N.$$

The second interpolation condition then becomes

$$\sum_{u=0}^{\infty} a_2^u b_{a_1}(a_2) g_u = 0,$$

which does not seem to be expressable in terms of $g(a_2)$ unless $b_{a_1}(a_2)$ and a_2 commute.

Remark 6.5. The solution of equation (6.4),

$$Y(p) - \sum_{j=1}^{N} p_j Y(p) T_j = I$$

can be seen as the non commutative version of the resolvent of the N-tuple of operators $(T_1, \ldots, T_N)$.

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