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On the Asymptotic Behaviour of Solutions to the Fractional Porous Medium Equation with Variable Density

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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO THE FRACTIONAL POROUS MEDIUM EQUATION WITH VARIABLE DENSITY

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ABSTRACT. We are concerned with the long time behaviour of solutions to the fractional porous medium equation with a variable density. We prove that if the density decays slowly at infinity, then the solution approaches the Barenblatt-type solution of a proper singular fractional problem. If, on the contrary, the density decays rapidly at infinity, we show that the minimal solution multiplied by a suitable power of the time variable converges to the minimal solution of a certain fractional sublinear elliptic equation.

CONTENTS

1. INTRODUCTION

We investigate the asymptotic behaviour, as $t \to \infty$, of nonnegative solutions to the following parabolic nonlinear, degenerate, *nonlocal* weighted problem:

$$\begin{cases} \rho(x)u_t + (-\Delta)^s (u^m) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{1.1}$$

where the initial datum u_0 is nonnegative and belongs to

$$L_\rho^1(\mathbb{R}^d) = \left\{ u : \|u\|_{1,\rho} = \int_{\mathbb{R}^d} |u(x)| \, \rho(x) \mathrm{d}x < \infty \right\},$$

and the weight ρ is assumed to be positive, locally essentially bounded away from zero (namely $\rho^{-1} \in L_{\mathrm{loc}}^\infty(\mathbb{R}^d)$) and to satisfy suitable decay conditions as at infinity, which we shall specify later. As for the parameters involved, we shall assume throughout the paper that $m > 1$ and $d > 2s$.

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Moreover, for all $s \in (0,1)$, the symbol $(-\Delta)^s$ denotes the fractional Laplacian operator, that is

$$(-\Delta)^s(\phi)(x) = p.v. \; C_{s,d} \int_{\mathbb{R}^d} \frac{\phi(x) - \phi(y)}{|x-y|^{d+2s}} \, \mathrm{d}y \quad \forall x \in \mathbb{R}^d, \; \forall \phi \in C_c^\infty(\mathbb{R}^d), \tag{1.2}$$

$C_{s,d}$ being a suitable positive constant depending only on s and d. For less regular functions, the fractional Laplacian is meant in the usual distributional sense.

For weights $\rho(x)$ that decay *slowly* as $|x| \to \infty$, we shall also be able to consider the more general problem

$$\begin{cases} \rho(x)u_t + (-\Delta)^s(u^m) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ \rho(x)u = \mu & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{1.3}$$

where μ is a positive finite measure. More precisely, here we shall assume that ρ complies with the following assumptions:

$$c_R \le \rho(x) \le c\,|x|^{-\gamma} \quad \forall x \in B_R, \; \forall R > 0, \quad \lim_{|x| \to \infty} \rho(x)|x|^\gamma = c_\infty$$

for $\gamma \in (0, 2s)$ and suitable *strictly positive* constants c_R, c and c_∞ (B_R denotes the ball of radius R centred at $x = 0$, while B_R^c denotes its complement). Note that in this case $\rho(x)$ is allowed to have a singularity as $|x| \to 0$.

The local version of problem (1.1), that is

$$\begin{cases} \rho(x)u_t - \Delta(u^m) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{1.4}$$

has been largely studied in the literature (see e.g. [13, 14, 16, 17, 19, 24, 21, 31]). In particular, for $d \ge 3$, it is shown that (1.4) admits a unique very weak solution if $\rho(x)$ decays *slowly* as $|x| \to \infty$, while nonuniqueness prevails when $\rho(x)$ decays *fast enough* as $|x| \to \infty$. In the latter case, uniqueness can be restored by imposing on the solutions proper extra conditions at infinity. Also note that, independently of the behaviour of $\rho(x)$ as $|x| \to \infty$, existence and uniqueness of the so-called *weak energy solutions* (namely solutions belonging to suitable functional spaces) hold true (see [16]). Furthermore, the long time behaviour of solutions to problem (1.4) has been addressed in [30, 32] and [20]. To be specific, in [32] it is proved that if $\|u_0\|_{1,\rho} = M$, $\rho > 0$ and $\rho(x) \sim |x|^{-\gamma}$ as $|x| \to \infty$, for some $\gamma \in [0, 2)$, then the solution u to problem (1.4) satisfies

$$\lim_{t \to \infty} \|u(t) - u_M^*(t)\|_{1,\rho} = 0$$

and

$$\lim_{t \to \infty} t^\alpha \|u(t) - u_M^*(t)\|_\infty = 0.$$

Here, u_M^* is the self-similar Barenblatt solution of mass $\int_{\mathbb{R}^d} u_M^* \rho = M$, that is

$$u_M^*(x,t) = t^{-\alpha} F\left(t^{-\kappa}|x|\right) \quad \forall (x,t) \in \mathbb{R}^d \times (0,\infty),$$

with

$$F(\xi) = \left(C - k\xi^{2-\gamma}\right)_+^{\frac{1}{m-1}} \quad \forall \xi \ge 0$$

for suitable positive constants C and k depending on M, m, d, γ. Moreover,

$$\alpha = (d - \gamma)\kappa, \quad \kappa = \frac{1}{d(m-1) + 2 - m\gamma}.$$

We stress that u_M^* solves the singular problem

$$\begin{cases} |x|^{-\gamma} u_t - \Delta(u^m) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ |x|^{-\gamma} u = M\delta & \text{on } \mathbb{R}^d \times \{0\}, \end{cases}$$

where $M = \|u_0\|_{1,\rho}$ and δ is the *Dirac delta* centred at $x = 0$. Note that, for $\rho \equiv 1$, and so $\gamma = 0$, the same asymptotic results have been shown in [15] and in [34].

On the contrary, in [20] it is proved that if $\rho > 0$ and $\rho(x) \sim |x|^{-\gamma}$ as $|x| \to \infty$, for some $\gamma > 2$, then the minimal solution to problem (1.4), which is unique in the class of solutions fulfilling

$$\frac{1}{R^{d-1}} \int_{\partial B_R} \int_0^t u^m(x, \tau) \, \mathrm{d}\tau \mathrm{d}S \to 0 \quad \text{as } R \to \infty$$

for all $t > 0$, satisfies

$$t^{\frac{1}{m-1}} u(x, t) \to (m-1)^{-\frac{1}{m-1}} W^{\frac{1}{m}}(x) \quad \text{as } t \to \infty, \quad \text{uniformly w.r.t. } x \in \mathbb{R}^d.$$

Here W is the unique (minimal) positive solution to the sublinear elliptic equation

$$-\Delta W = \rho \, W^{\frac{1}{m}} \quad \text{in } \mathbb{R}^d,$$

and it is such that

$$\lim_{|x| \to \infty} W(x) = 0.$$

Problem (1.1) with $\rho \equiv 1$, nonnegative initial datum u_0 in $L^1(\mathbb{R}^d)$ and $s \in (0, 1)$, namely

$$\begin{cases} u_t + (-\Delta)^s (u^m) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{1.5}$$

has been recently addressed in the breakthrough papers [9, 10]. In particular, existence, uniqueness and qualitative properties of solutions have been studied. Furthermore, the asymptotic behaviour, as $t \to \infty$, has been investigated in [36]. More precisely, it is first shown that, for any $M > 0$, there exists a unique solution u_M^* to the singular problem

$$\begin{cases} u_t + (-\Delta)^s (u^m) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u = M\delta & \text{on } \mathbb{R}^d \times \{0\}. \end{cases}$$

Furthermore, such u_M^* has the following self-similar form:

$$u_M^*(x, t) = t^{-\alpha} f(t^{-\kappa} |x|) \quad \forall (x, t) \in \mathbb{R}^d \times (0, \infty),$$

where

$$\alpha = \frac{d}{d(m-1) + 2s}, \quad \kappa = \frac{1}{d(m-1) + 2s}$$

and the profile $f : [0, \infty) \to (0, \infty)$ is a bounded, Hölder continuous decreasing function, with $f(r) \to 0$ as $r \to \infty$. In view of such properties, u_M^* is still called a *Barenblatt-type* solution. Then it is proved that the solution u to problem (1.5) satisfies

$$\lim_{t \to \infty} \|u(t) - u_M^*(t)\|_1 = 0$$

and

$$\lim_{t \to \infty} t^\alpha \|u(t) - u_M^*(t)\|_\infty = 0. \tag{1.6}$$

Existence and uniqueness of nonnegative bounded solutions to problem (1.1) for nonnegative initial data $u_0 \in L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and strictly positive weights have been investigated in [27, 26]. More precisely, it is proved that if $\gamma \in (0, 2s)$ and there exists $C_0 > 0$ such that if

$$\rho(x) \geq C_0 |x|^{-\gamma} \quad \text{a.e. in } B_1^c,$$

then problem (1.1) admits a unique bounded solution. Furthermore, when $\gamma \in (2s, \infty)$ and there exists $C_0 > 0$ such that

$$\rho(x) \leq C_0 |x|^{-\gamma} \quad \text{a.e. in } B_1^c, \tag{1.7}$$

we have existence of solutions satisfying a proper decaying condition at infinity. Within this class of solutions, uniqueness can be restored provided (1.7) holds true with $\gamma \in (d, \infty)$, basically as a consequence of the results of [26]. In addition, in the present paper we shall prove uniqueness under the weaker requirement that (1.7) holds true with $\gamma \in (4s \wedge d, \infty)$ (see Theorem 2.4 below). Actually, for generic positive densities $\rho \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ such that $\rho^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$, namely without assuming further conditions on $\rho(x)$ as $|x| \to \infty$, one can also prove existence and uniqueness of

weak energy solutions in the same spirit of [16] (see Proposition 2.3 below). The point is that the uniqueness results of Theorem 2.4 hold for a more general notion of solution, and we shall use them as such.

The main goal of this paper is to study the large time behaviour of solutions to problem (1.1). To this end, similarly to the results recalled above in the local case, we shall distinguish two situations:

i) $\rho(x) \to 0$ as $|x| \to \infty$ *slowly*, in the sense that for a suitable $\gamma \in (0, 2s)$ there holds

$$\lim_{|x| \to \infty} \rho(x)|x|^\gamma = c_\infty > 0 \, ; \tag{1.8}$$

ii) $\rho(x) \to 0$ as $|x| \to \infty$ *rapidly*, in the sense that for a suitable $\gamma \in (2s, \infty)$ (1.7) holds true.

In case i) we shall describe the asymptotic behaviour of solutions to problem (1.3), namely with initial data which can be positive finite measures. Such asymptotics is obtained in terms of a Barenblatt-type solution to a proper nonlocal singular problem, that is the unique solution $u_M^{c_\infty}$ to

$$\begin{cases} c_\infty |x|^{-\gamma} u_t + (-\Delta)^s (u^m) = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \, , \\ c_\infty |x|^{-\gamma} u = M\delta & \text{on } \mathbb{R}^d \times \{0\} \, , \end{cases} \tag{1.9}$$

where $M > 0$ is the (fixed) mass and c_∞ is as in (1.8). Existence and uniqueness of solutions to (1.9) actually follow from the results established in [18] for the more general problem (1.3). In particular, existence is ensured supposing that $\gamma \in (0, 2s \wedge (d - 2s))$, while uniqueness holds under the weaker condition $\gamma \in (0, 2s) \cap (0, d - 2s]$.

Coming back to the asymptotics of the solutions to the evolution equations considered, we shall show that

$$\lim_{t \to \infty} \|u(t) - u_M^{c_\infty}(t)\|_{1, |x|^{-\gamma}} = \lim_{t \to \infty} \int_{\mathbb{R}^d} |t^\alpha u(t^\kappa x, t) - u_M^{c_\infty}(x, 1)| \, |x|^{-\gamma} \mathrm{d}x = 0 \, , \tag{1.10}$$

where

$$\alpha = (d - \gamma)\kappa \, , \quad \kappa = \frac{1}{d(m-1) + 2s - m\gamma} \, .$$

In order to prove (1.10), we partially follow the general strategy used in the literature to prove similar convergence results (see e.g. [15, 34, 35, 31, 36]). However, here several technical difficulties arise, due to the simultaneous presence of the weight $\rho(x)$ and of the nonlocal operator $(-\Delta)^s$. To overcome them, we adapt to the present situation some ideas used in [18] to prove existence. Besides, the lack of known regularity results for the Barenblatt solutions considered, which hold true in the unweighted case because of the theory developed in [1], forces us to introduce a different argument in the final convergence step (which however does not allow to prove a stronger L^∞ convergence result of the type of (1.6)).

In case ii), the long time behaviour of the minimal solution to problem (1.1) is deeply linked with the minimal solution w to the following nonlocal sublinear elliptic equation:

$$(-\Delta)^s w = \rho \, w^\alpha \quad \text{in } \mathbb{R}^d \, , \tag{1.11}$$

where $\alpha = 1/m \in (0, 1)$. Note that the local case $s = 1$ has been thoroughly studied (see e.g. [5], [29] and references therein). For general $s \in (0, 1)$ it has been addressed in [25], following the same line of arguments of [5]. However, in [25] it is supposed that (1.7) holds true for $\gamma > d$ (with $d > 4s$) and $\rho \geq 0$ (with $\rho \not\equiv 0$). Furthermore, *energy solutions* have been dealt with. In the present work, existence of nontrivial *very weak solutions* is established whenever (1.7) holds for $\gamma > 2s$ (with $d > 2s$). In doing this, a central role will be played by the solution to the linear equation

$$(-\Delta)^s V = \rho \quad \text{in } \mathbb{R}^d \, .$$

We shall also prove uniqueness of very weak solutions to equation (1.11), satisfying proper decay conditions at infinity, assuming that (1.7) holds for $\gamma > 4s \wedge d$. We then show that, whenever (1.7) holds for $\gamma > 4s \wedge d$, there holds

$$\lim_{t \to \infty} t^{\frac{1}{m-1}} u(x, t) = (m-1)^{-\frac{1}{m-1}} w^{\frac{1}{m}}(x) \quad \text{for a.e. } x \in \mathbb{R}^d \, ,$$

where w is the minimal positive (very weak) solution to equation (1.11) with $\alpha = 1/m$.

Organization of the paper. In Section 2 we give the definitions of solution to problems (1.1) and (1.3); moreover, preliminary results concerning the well posedness of the problems are stated. As for long time behaviour of solutions, our results both for fast decaying densities (Theorem 3.1) and for slowly decaying densities (Theorem 3.3) are stated in Section 3. In Section 4 we consider the sublinear elliptic equation (1.11), and we show some new existence and uniqueness results for the corresponding solutions in Theorems 4.4 and 4.5, which have also an independent interest. We take advantage of such results in Section 5 in order to prove Theorem 3.1. Finally, in Section 6 we prove Theorem 3.3.

In Appendix A some useful properties of Riesz potential are discussed. In Appendix B the well posedness of problem (1.1) for rapidly decaying densities is proved: here we improve in various directions previous results in [26].

2. Preliminary results

We start this section by providing a suitable definition of weak solution to problem (1.1), which will be primarily interesting for the case of rapidly decaying densities. We shall always assume $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ and $\rho^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. Hereafter, by the symbol $\dot{H}^s(\mathbb{R}^d)$ we shall denote the completion of $C^\infty_c(\mathbb{R}^d)$ w.r.t. the norm

$$\|\phi\|_{\dot{H}^s} = \left\|(-\Delta)^{\frac{s}{2}}(\phi)\right\|_2 \quad \forall \phi \in C^\infty_c(\mathbb{R}^d).$$

Definition 2.1. *A nonnegative function u is a weak solution to problem (1.1) corresponding to the nonnegative initial datum $u_0 \in L^1_\rho(\mathbb{R}^d)$ if:*

- $u \in C([0,\infty); L^1_\rho(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (\tau, \infty))$ *for all $\tau > 0$;*
- $u^m \in L^2_{\text{loc}}((0,\infty); \dot{H}^s(\mathbb{R}^d))$;
- *for any $\varphi \in C^\infty_c(\mathbb{R}^d \times (0,\infty))$ there holds*

$$\int_0^\infty \int_{\mathbb{R}^d} u(x,t)\varphi_t(x,t)\,\rho(x)\mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x,t)(-\Delta)^{\frac{s}{2}}(\varphi)(x,t)\,\mathrm{d}x\mathrm{d}t = 0; \quad (2.1)$$

- $\lim_{t \to 0} u(t) = u_0$ *in $L^1_\rho(\mathbb{R}^d)$.*

A classical notion in the literature is the following (see e.g. [10, Section 8.1]).

Definition 2.2. *Let u be a weak solution to problem (1.1) (according to Definition 2.1). We say that u is a* strong solution *if, in addition, $u_t \in L^\infty((\tau,\infty); L^1_\rho(\mathbb{R}^d))$ for every $\tau > 0$.*

Existence and uniqueness of weak solutions to problem (1.1), by means of standard techniques (see e.g. [9, 10, 16, 26]), are discussed in Appendix B. The first result we provide reads as follows (for a sketch of proof see again Appendix B – Parts I and II).

Proposition 2.3. *Let $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ be positive and such that $\rho^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. Then there exists a* unique *weak solution u to problem (1.1), in the sense of Definition 2.1, which is also a strong solution in the sense of Definition 2.2.*

Let us introduce the Riesz kernel of the s-Laplacian:

$$I_{2s}(x) = \frac{k_{s,d}}{|x|^{d-2s}} \quad \forall x \in \mathbb{R}^d \setminus \{0\}, \quad (2.2)$$

where $k_{s,d}$ is a suitable positive constant that depends only on s and d. Recall that for a sufficiently regular function f there holds

$$(-\Delta)^s (I_{2s} * f) = f,$$

namely the convolution against I_{2s} represents the operator $(-\Delta)^{-s}$.

2.1. **Rapidly decaying density.** Given a weak solution u to (1.1) and any fixed $t_0 \geq 0$, let us set

$$U(x,t) = \int_{t_0}^t u^m(x,\tau)\,\mathrm{d}\tau \quad \forall (x,t) \in \mathbb{R}^d \times (t_0, \infty)\,.$$

Notice that U depends implicitly on t_0 as well.

When $\rho(x)$ is a density that decays *sufficiently fast* as $|x| \to \infty$, we shall often need to deal with solutions to (1.1) which are meant in a more general sense with respect to the one of Definition 2.1, namely what we call *local strong solutions*. The corresponding definition is technical, and we leave it to Appendix B (see Definition B.4). The result we present here below concerns existence and uniqueness of local strong solutions.

Theorem 2.4. *Let $\rho \in L^\infty(\mathbb{R}^d)$ be positive and such that $\rho^{-1} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Let $u_0 \in L^1_\rho(\mathbb{R}^d)$ be nonnegative. Assume in addition that $\rho(x) \leq C_0\,|x|^{-\gamma}$ a.e. in B_1^c for some $\gamma > 2s$ and $C_0 > 0$. Then the weak solution to problem* (1.1) *provided by Proposition 2.3 is the* minimal solution *in the class of local strong solutions (according to Definition B.4) and satisfies*

$$U(x,t) \to 0 \quad as \ |x| \to \infty \tag{2.3}$$

for all $t_0 > 0$ and $t \geq t_0$. More precisely, for any $t_0 > 0$ there holds

$$U(x,t) \leq C\,(I_{2s} * \rho)(x) \quad for \ a.e. \ (x,t) \in \mathbb{R}^d \times (t_0, \infty) \tag{2.4}$$

for some $C = C(t_0) > 0$, whence (2.3) *follows by Corollary A.2. Furthermore:*

 (i) under the more restrictive assumption that $\gamma > d$, the solution is unique *in the class of local strong solutions satisfying*

$$u^m \in L^1_{(1+|x|)^{-\alpha}}(\mathbb{R}^d \times (0,T)) \quad \forall T > 0\,, \tag{2.5}$$

 given any $\alpha < 2s$;

 (ii) if u_0 is also bounded, then under the more restrictive assumption that $\gamma > 4s \wedge d$ the solution is unique *in the class of bounded local strong solutions satisfying*

$$u^m \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (0,T)) \quad \forall T > 0\,. \tag{2.6}$$

For the proof of Theorem 2.4, we refer the reader to Appendix B – Part III.

Remark 2.5. Note that, in case (i), if the initial datum u_0 belongs to $L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ then the solution provided by Proposition 2.3 is bounded in the whole of $\mathbb{R}^d \times (0, \infty)$, so that one can actually pick $t_0 = 0$ in (2.4) (see [26, Theorems 5.5 and 5.6]).
Statement (i) can be proved proceeding as in the proof of [26, Theorem 6.10], where condition (2.4) is required instead of (2.5). However, in view of the results collected in Appendices A and B, one easily deduces that (2.4) is stronger than (2.5) but the latter is actually enough.
Finally notice that, as concerns uniqueness, when $d \leq 4s$ condition (2.5) is *weaker* than (2.6). Hence, in this case, the uniqueness result of (ii) is just a consequence of the uniqueness result of (i).

2.2. **Slowly decaying densities.** In this subsection we deal with weights $\rho(x)$ which decay slowly as $|x| \to \infty$. More precisely, we shall assume once for all that the following hypotheses are satisfied:

$$c \leq \rho(x) \leq C_2|x|^{-\gamma} \ \text{for a.e. } x \in B_1\,, \quad C_1|x|^{-\gamma} \leq \rho(x) \leq C_2|x|^{-\gamma} \ \text{for a.e. } x \in B_1^c \tag{2.7}$$

for some positive constants c, C_1, C_2 and $\gamma \in (0, 2s)$. Note that $\rho(x)$ might possibly be unbounded as $x \to 0$.

Below we recall the definition of weak solution to the more general problem (1.3) given in [18, Definition 3.1]. Before doing it, following the same notation as in [23], we need to introduce some notions of convergence in measure spaces. Let $\mathcal{M}(\mathbb{R}^d)$ be the cone of positive, finite measures on $\mathbb{R}^d$. A sequence $\{\mu_n\} \subset \mathcal{M}(\mathbb{R}^d)$ is said to converge to $\mu \in \mathcal{M}(\mathbb{R}^d)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ if

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}\mu_n = \int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}\mu \quad \forall \phi \in C_b(\mathbb{R}^d)\,,$$

where $C_b(\mathbb{R}^d)$ is the space of continuous, bounded functions in $\mathbb{R}^d$. An analogous definition holds for $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$.

Definition 2.6. *By a weak solution to problem* (1.3), *corresponding to the initial datum* $\mu \in \mathcal{M}(\mathbb{R}^d)$, *we mean a nonnegative function* u *such that:*

$$u \in L^\infty((0,\infty); L^1_\rho(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (\tau, \infty)) \quad \forall \tau > 0, \tag{2.8}$$

$$u^m \in L^2_{\mathrm{loc}}((0,\infty); \dot{H}^s(\mathbb{R}^d)), \tag{2.9}$$

$$\int_0^\infty \int_{\mathbb{R}^d} u(x,t)\varphi_t(x,t)\, \rho(x)\mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x,t)\,(-\Delta)^{\frac{s}{2}}(\varphi)(x,t)\,\mathrm{d}x\mathrm{d}t = 0 \tag{2.10}$$
$$\forall \varphi \in C_c^\infty(\mathbb{R}^d \times (0,\infty))$$

and

$$\lim_{t \to 0} \rho\, u(t) = \mu \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d)).$$

It is plain that, when $\mu = \rho\, u_0 \in L^1(\mathbb{R}^d)$, a solution to (1.1) with respect to Definition 2.1 is also a solution to (1.3) with respect to Definition 2.6. However, Definition 2.6 permits to handle more general initial data (positive, finite measures). In particular, we cannot ask $u \in C([0,\infty); L^1_\rho(\mathbb{R}^d))$. Nevertheless, thanks to the fundamental Theorem 2.7 which we state below, when $\mu = \rho\, u_0 \in L^1(\mathbb{R}^d)$ such two solutions do coincide (provided the parameters γ, s and d meet the corresponding assumptions).

We recall now some well posedness results proved in [18]. In fact, thanks to the theory developed therein, we can guarantee existence and uniqueness of weak solutions to (1.3) (according to Definition 2.6). Besides, Proposition 4.1 of [18] ensures that

$$\int_{\mathbb{R}^d} u(x,t)\, \rho(x)\mathrm{d}x = \mu(\mathbb{R}^d) \quad \forall t > 0, \tag{2.11}$$

namely there is *conservation of mass*. This is actually a sole consequence of Definition 2.6 and the hypothesis $\gamma \in (0, 2s)$.

The next result is a crucial one but its proof follows along known lines.

Theorem 2.7. *Let* $d > 2s$ *and* $\gamma \in (0, 2s \wedge (d - 2s))$. *Assume that* ρ *satisfies* (2.7). *Then there exists a weak solution* u *to problem* (1.3), *in the sense of Definition 2.6, which satisfies the smoothing estimate*

$$\|u(t)\|_\infty \leq K\, t^{-\alpha} \mu(\mathbb{R}^d)^\beta \quad \forall t > 0, \tag{2.12}$$

where K *is a suitable positive constant depending only on* m, γ, s, d *and*

$$\alpha = \frac{d - \gamma}{(m-1)(d-\gamma) + 2s - \gamma}, \quad \beta = \frac{2s - \gamma}{(m-1)(d-\gamma) + 2s - \gamma}. \tag{2.13}$$

In particular, $u(t) \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *for all* $t > 0$. *Moreover,* u *satisfies the energy estimates*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}}(u^m)(x,t) \right|^2 \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} u^{m+1}(x,t_2)\, \rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} u^{m+1}(x,t_1)\, \rho(x)\mathrm{d}x \tag{2.14}$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |z_t(x,t)|^2\, \rho(x)\mathrm{d}x\mathrm{d}t \leq C \tag{2.15}$$

for all $t_2 > t_1 > 0$, *where* $z = u^{\frac{m+1}{2}}$ *and* C *is a positive constant that depends only on* t_1, t_2, m *and on*

$$\int_{\mathbb{R}^d} u^{m+1}(x,t_*)\, \rho(x)\mathrm{d}x$$

for some $t_* \in (0, t_1)$.

Furthermore, such solution is unique.

Remark 2.8. (i) The smoothing effect (2.12) can be proved as in [18, Proposition 4.6]. In fact, such proof only relies on the validity of the fractional Sobolev inequality

$$\|v\|_{2\frac{d-\gamma}{d-2s},\rho} \le \widetilde{C} \, \|(-\Delta)^s(v)\|_2 \quad \forall v \in \dot{H}^s(\mathbb{R}^d) \,,$$

which, thanks to the assumptions on ρ, is a trivial consequence of

$$\|v\|_{2\frac{d-\gamma}{d-2s},-\gamma} \le C_{S,\gamma} \, \|(-\Delta)^s(v)\|_2 \quad \forall v \in \dot{H}^s(\mathbb{R}^d) \,. \tag{2.16}$$

For the validity of (2.16), we refer the reader to [18, Lemma 4.5] and references quoted.

(ii) Thanks to the results of [18, Section 3.1] (which in turn go back to [10, Section 8.1]), or to the discussion in Appendix B – Part I (which applies to slowly decaying densities as well), we have that the solutions provided by Theorem 2.7 are also strong. In particular, they belong to $C((0,\infty); L^1_\rho(\mathbb{R}^d))$.

(iii) Note that, for $d \ge 4s$, the hypotheses of Theorem 2.7 on γ reduce to $\gamma \in (0, 2s)$.

3. MAIN RESULTS: LARGE TIME BEHAVIOUR OF SOLUTIONS

In this section we state our main results for the asymptotics (as $t \to \infty$) of the solutions to problems (1.1) and (1.3) provided by Proposition 2.3 and Theorem 2.7, respectively.

3.1. **Rapidly decaying densities.** As concerns solutions to (1.1) when $\rho(x)$ is a density that decays sufficiently fast as $|x| \to \infty$, we have the following result.

Theorem 3.1. *Let $\rho \in C^\sigma_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\sigma > 0$, with $\rho > 0$. Let $u_0 \in L^1_\rho(\mathbb{R}^d)$ be nonnegative. Assume in addition that $\rho(x) \le C_0|x|^{-\gamma}$ in B_1^c for some $\gamma > 2s$ and $C_0 > 0$. Let u be the (minimal) weak solution to problem (1.1) provided by Proposition 2.3 and w be the very weak solution to the sublinear elliptic equation (1.11), with $\alpha = 1/m$, provided by Theorem 4.4 below (which is also minimal in the class of solutions specified by the corresponding statement). Then,*

$$\lim_{t\to\infty} t^{\frac{1}{m-1}} u(x,t) = (m-1)^{-\frac{1}{m-1}} w^{\frac{1}{m}}(x) \quad \text{for a.e. } x \in \mathbb{R}^d \,.$$

Remark 3.2. Let $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be nonnegative. Let $\rho \in C^\sigma_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\sigma > 0$, with $\rho > 0$, and assume that condition (1.7) is satisfied with $\gamma > 4s \wedge d$. Then, by Theorem 2.4, the minimal solution u to (1.1) provided by Proposition 2.3 is characterized to be the unique solution in the class of local strong solutions such that $u^m \in L^1_{(1+|x|)^{-\beta}}(\mathbb{R}^d \times (0,T))$ for all $T > 0$, where β is as in (4.5).

Moreover, the minimal solution w to the sublinear elliptic equation (1.11) provided by Theorem 4.4 is the unique solution in a certain class of solutions such that $w \in L^1_{(1+|x|)^{-\beta}}(\mathbb{R}^d)$ for a suitably chosen value of β (see the statement of Theorem 4.5 below).

3.2. **Slowly decaying densities.** In the analysis of the long time behaviour of solutions to (1.3) when $\rho(x)$ is density that decays *slowly* as $|x| \to \infty$, a major role is played by the solution to the same problem in the particular case $\rho(x) = c_\infty|x|^{-\gamma}$ and $\mu = M\delta$, for given positive constants c_∞ and M (namely, the solution to (1.9)). From now on we shall denote such solution as $u_M^{c_\infty}$.

Let us define the positive parameters α and κ as follows:

$$\alpha = (d-\gamma)\kappa, \quad \kappa = \frac{1}{(m-1)(d-\gamma) + 2s - \gamma} \,. \tag{3.1}$$

Notice that α is the same parameter appearing in (2.13). It is immediate to check that, for any given $\lambda > 0$, the function

$$u_{M,\lambda}^{c_\infty}(x,t) = \lambda^\alpha u_M^{c_\infty}(\lambda^\kappa x, \lambda t)$$

is still a solution to problem (1.9). Hence, as a consequence of the uniqueness result contained in Theorem 2.7, $u_{M,\lambda}^{c_\infty}$ and $u_M^{c_\infty}$ must necessarily coincide, that is

$$u_M^{c_\infty}(x,t) = \lambda^\alpha u_M^{c_\infty}(\lambda^\kappa x, \lambda t) \quad \forall t, \lambda > 0, \text{ for a.e. } x \in \mathbb{R}^d \,. \tag{3.2}$$

As already mentioned, the special solution $u_M^{c_\infty}$, thanks to the self-similarity identity (3.2) it satisfies, will be crucial in the study of the asymptotic behaviour of *any* solution to (1.3) (provided ρ complies with (3.3) as well). This is thoroughly analysed in Section 6.

Our main result concerning the asymptotics of solutions to (1.3) is the following.

Theorem 3.3. *Let $d > 2s$ and $\gamma \in (0, 2s \wedge (d - 2s))$. Suppose that ρ satisfies (2.7) and that, in addition,*

$$\lim_{|x| \to \infty} \rho(x)|x|^\gamma = c_\infty > 0. \tag{3.3}$$

Let u be the unique weak solution to problem (1.3), in the sense of Definition 2.6, provided by Theorem 2.7 and corresponding to $\mu \in \mathcal{M}(\mathbb{R}^d)$ as initial datum, with $\mu(\mathbb{R}^d) = M > 0$. Then,

$$\lim_{t \to \infty} \|u(t) - u_M^{c_\infty}(t)\|_{1, |x|^{-\gamma}} = 0 \tag{3.4}$$

or equivalently

$$\lim_{t \to \infty} \int_{\mathbb{R}^d} |t^\alpha u(t^\kappa x, t) - u_M^{c_\infty}(x, 1)| \, |x|^{-\gamma} \, \mathrm{d}x = 0, \tag{3.5}$$

where $u_M^{c_\infty}$ is the Barenblatt solution defined as the unique solution to problem (1.9), and the parameters α, κ are as in (3.1).

Notice once again that the range of γ for which the above theorem holds true simplifies to $(0, 2s)$ when $d \geq 4s$, which is, to some extent, the maximal one for which one can expect a similar result. Theorem 3.1 will be proved in Section 5, while Theorem 3.3 will be proved in Section 6.

4. A SUBLINEAR ELLIPTIC EQUATION

Prior to analysing the asymptotic behaviour of solutions to (1.1) when $\rho(x)$ is a density that decays fast as $|x| \to \infty$ (discussed in Section 5), we need to study the sublinear elliptic equation (1.11), which naturally arises from such asymptotic analysis.

Let us recall that if φ is a smooth and compactly supported function defined in $\mathbb{R}^d$, we can consider its s-harmonic extension $\mathrm{E}(\varphi)$ to the upper half-space $\mathbb{R}_+^{d+1} = \{(x, y) : x \in \mathbb{R}^d, \, y > 0\}$, namely the unique smooth and bounded solution to the problem

$$\begin{cases} \mathrm{div}\left(y^{1-2s}\nabla\mathrm{E}(\varphi)\right) = 0 & \text{in } \mathbb{R}_+^{d+1}, \\ \mathrm{E}(\varphi) = \varphi & \text{on } \partial\mathbb{R}_+^{d+1} = \mathbb{R}^d \times \{y = 0\}. \end{cases}$$

It has been proved (see e.g. [6], [7], [10]) that

$$-\mu_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial \mathrm{E}(\varphi)}{\partial y}(x, y) = (-\Delta)^s(\varphi)(x) \quad \forall x \in \mathbb{R}^d,$$

where $\mu_s = \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)}$. It is therefore convenient to define the operators

$$L_s = \mathrm{div}\left(y^{1-2s}\nabla\right),$$

$$\frac{\partial}{\partial y^{2s}} = -\mu_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial}{\partial y}.$$

We also denote by X^s the completion of $C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1})$ w.r.t. the norm

$$\|\psi\|_{X^s} = \left(\mu_s \int_{\mathbb{R}_+^{d+1}} y^{1-2s} |\nabla\psi(x, y)|^2 \, \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \quad \forall \psi \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1}).$$

Furthermore, by the symbol X_{loc}^s, we shall mean the space of all functions v such that $\psi v \in X^s$ for any $\psi \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1})$.

It is possible to prove that there exists a well defined notion of trace on $\partial \mathbb{R}^{d+1}_+$ for every function in X^s (see e.g. [4, Section 2], [6, Section 3.1] or [10, Section 3.2]). Moreover, for every $v \in \dot{H}^s(\mathbb{R}^d)$ there exists a unique extension $\mathrm{E}(v) \in X^s$ such that

$$\mathrm{E}(v)(x,0) = v(x) \quad \text{for a.e. } x \in \mathbb{R}^d$$

and

$$\mu_s \int_{\mathbb{R}^{d+1}_+} y^{1-2s} \langle \nabla \mathrm{E}(v), \nabla \psi \rangle (x,y) \, \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^d} (-\Delta)^s(v)(x) \, (-\Delta)^s(\psi)(x,0) \, \mathrm{d}x$$

for any $\psi \in C_c^\infty(\mathbb{R}^{d+1}_+ \cup \partial \mathbb{R}^{d+1}_+)$.

Having at our disposal the above tools, we can provide suitable weak formulations of problem (1.11) which deal with the harmonic extension. In fact, at a formal level, looking for a solution w to (1.11) is the same as looking for a pair of functions $(w, \tilde{w})$ solving the problem

$$\begin{cases} L_s \tilde{w} = 0 & \text{in } \mathbb{R}^{d+1}_+, \\ \tilde{w} = w & \text{on } \partial \mathbb{R}^{d+1}_+, \\ \dfrac{\partial \tilde{w}}{\partial y^{2s}} = \rho \, w^\alpha & \text{on } \partial \mathbb{R}^{d+1}_+, \end{cases} \tag{4.1}$$

with $0 < \alpha < 1$.

Definition 4.1. *A local weak solution to problem* (4.1) *is a bounded nonnegative function w such that, for some nonnegative $\tilde{w} \in X^s_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^{d+1}_+ \cup \partial \mathbb{R}^{d+1}_+)$ (what we call a* local extension *for w), there holds $\tilde{w}|_{\partial \mathbb{R}^{d+1}_+} = w$ and*

$$\int_{\mathbb{R}^d} w^\alpha(x)\, \psi(x,0) \, \rho(x)\mathrm{d}x = \mu_s \int_{\mathbb{R}^{d+1}_+} y^{1-2s} \langle \nabla \tilde{w}, \nabla \psi \rangle (x,y) \, \mathrm{d}x\mathrm{d}y$$

for any $\psi \in C_c^\infty(\mathbb{R}^{d+1}_+ \cup \partial \mathbb{R}^{d+1}_+)$.

Definition 4.2. *A bounded, nonnegative function w is a very weak solution to problem* (1.11) *if it satisfies*

$$\int_{\mathbb{R}^d} w^\alpha(x)\varphi(x) \, \rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} w(x)(-\Delta)^s(\varphi)(x) \, \mathrm{d}x$$

for any $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Definition 4.3. *A nonnegative function $w \in \dot{H}^s(\mathbb{R}^d)$ is a weak solution to problem* (1.11) *if it satisfies*

$$\int_{\mathbb{R}^d} w^\alpha(x)\psi(x,0) \, \rho(x)\mathrm{d}x = \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(w)(x)(-\Delta)^{\frac{s}{2}}(\psi)(x,0) \, \mathrm{d}x$$

$$= \mu_s \int_{\mathbb{R}^{d+1}_+} y^{1-2s} \langle \nabla \mathrm{E}(w), \nabla \psi \rangle (x,y) \, \mathrm{d}x\mathrm{d}y \tag{4.2}$$

for any $\psi \in C_c^\infty(\mathbb{R}^{d+1}_+ \cup \partial \mathbb{R}^{d+1}_+)$.

Note that a bounded weak solution is a solution to (1.11) in the sense of both Definition 4.1 and Definition 4.2.

What follows in this section aims at studying existence and uniqueness of solutions to (4.1) (and (1.11)), according to Definition 4.1 (and 4.2, 4.3). Our results are the following.

Theorem 4.4 (existence). *Let $\alpha \in (0,1)$. Let $\rho \in C_{\mathrm{loc}}^{0,\sigma}(\mathbb{R}^d)$ (for some $\sigma \in (0,1)$) be strictly positive and such that $\rho(x) \leq C_0|x|^{-\gamma}$ in B_1^c for some $\gamma > 2s$ and $C_0 > 0$. Then there exists a local weak solution w to problem* (4.1), *which is minimal in the class of nonidentically zero local weak solutions (according to Definition 4.1). Moreover, w is a very weak solution to* (1.11) *(in the sense of Definition 4.2) and satisfies the estimate*

$$w(x) \leq C(I_{2s} * \rho)(x) \quad \text{for a.e. } x \in \mathbb{R}^d \tag{4.3}$$

for some $C > 0$.

Finally, if γ complies with the more restrictive condition

$$\gamma > \frac{d + 2s(\alpha + 1)}{\alpha + 2}, \tag{4.4}$$

then w is also a weak solution to (1.11) (according to Definition 4.3).

Theorem 4.5 (uniqueness). *Let $\alpha \in (0, 1)$. Let $\rho \in C_{\text{loc}}^{0,\sigma}(\mathbb{R}^d)$ (for some $\sigma \in (0, 1)$) be strictly positive and such that $\rho(x) \leq C_0 |x|^{-\gamma}$ in B_1^c for some $\gamma > 4s \wedge d$ and $C_0 > 0$. Let $\underline{w}$ be the minimal solution to problem (4.1) provided by Theorem 4.4. Let w be any other local weak solution to problem (4.1) (according to Definition 4.1), which is also a very weak solution to problem (1.11) (according to Definition 4.2) and such that $w \not\equiv 0$ and $w \in L_{(1+|x|)^{-\beta}}^1(\mathbb{R}^d)$, where*

$$\beta = \begin{cases} d - 2s & \text{if } d \geq 4s, \\ 2s - \varepsilon & \text{if } d < 4s, \end{cases} \tag{4.5}$$

for some $\varepsilon > 0$. Then $w = \underline{w}$ a.e. in $\mathbb{R}^d$.

Remark 4.6. Observe that, thanks to Corollary A.2 and Remark A.3, when $\gamma > 4s \wedge d$ the minimal solution w provided by Theorem 4.4 does belong to $L_{(1+|x|)^{-\beta}}^1(\mathbb{R}^d)$ with β as in (4.5). That is, the class of solutions in Theorem 4.5 among which we claim uniqueness is nonempty.

4.1. **Existence for the sublinear elliptic equation.** Here we shall prove all the properties of w claimed in Theorem 4.4, *except* the fact that w is a very weak solution to problem (1.11) in the sense of Definition 4.2 for all $\gamma > 2s$. This will be in fact a consequence of the asymptotic analysis of Section 5.

Let us start off with some preliminaries. We consider first the following problem: find $(w_R, \tilde{w}_R)$ such that

$$\begin{cases} L_s \tilde{w}_R = 0 & \text{in } \Omega_R, \\ \tilde{w}_R = 0 & \text{on } \Sigma_R, \\ \tilde{w}_R = w_R & \text{on } \Gamma_R, \\ \dfrac{\partial \tilde{w}_R}{\partial y^{2s}} = \rho \, w_R^\alpha & \text{on } \Gamma_R, \end{cases} \tag{4.6}$$

where $\Omega_R = \{(x, y) \in \mathbb{R}_+^{d+1} : |(x, y)| < R\}$, $\Sigma_R = \partial \Omega_R \cap \{y > 0\}$ and $\Gamma_R = \partial \Omega_R \cap \{y = 0\}$. We denote by $X_0^s(\Omega_R)$ the completion of $C_c^\infty(\Omega_R \cup \Gamma_R)$ w.r.t. the norm

$$\|\psi\|_{X_0^s(\Omega_R)} = \left(\mu_s \int_{\Omega_R} y^{1-2s} |\nabla \psi(x, y)|^2 \, dxdy \right)^{\frac{1}{2}} \quad \forall \psi \in C_c^\infty(\Omega_R \cup \Gamma_R).$$

Definition 4.7. *A weak solution to problem (4.6) is a pair of nonnegative functions $(w_R, \tilde{w}_R)$ such that:*

- *$w_R^\alpha \in L^1(B_R)$, $\tilde{w}_R \in X_0^s(\Omega_R)$;*
- *$\tilde{w}_R|_{\Gamma_R} = w_R$;*
- *for any $\psi \in C_c^\infty(\Omega_R \cup \Gamma_R)$ there holds*

$$\int_{B_R} w_R^\alpha(x) \, \psi(x, 0) \, \rho(x) dx = \mu_s \int_{\Omega_R} y^{1-2s} \langle \nabla \tilde{w}_R, \nabla \psi \rangle (x, y) \, dxdy. \tag{4.7}$$

The next existence result concerning problem (4.6) can be proved by standard variational methods (see e.g. [4]).

Proposition 4.8. *Let $\alpha \in (0, 1)$. Let $\rho \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ be positive and such that $\rho^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$. Then there exists a nonidentically zero weak solution $(w_R, \tilde{w}_R)$ to problem (4.6), in the sense of Definition 4.7.*

The following regularity and comparison results for problem (4.6) will be crucial in the proof of Theorem 4.4 (specially as for minimality).

Proposition 4.9. *Let $\alpha \in (0,1)$. Let $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ be positive and such that $\rho^{-1} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Let $(w_R, \tilde{w}_R)$ be a solution to problem (4.6) (according to Definition 4.7), such that $w_R \in L^\infty(B_R)$ and $\tilde{w}_R \in L^\infty(\Omega_R)$. Then, for some $\beta \in (0,1)$,*

$$\|\tilde{w}_R\|_{C^{0,\beta}(\overline{\Omega}_R)} \leq C_1 \,, \tag{4.8}$$

where C_1 is a positive constant depending only on s, d, R and $\|w_R\|_\infty$, $\|\tilde{w}_R\|_\infty$.

If furthermore $\rho \in C^{0,\sigma}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\sigma \in (0,1)$, then, for some $\beta_1 \in (0,1)$,

$$\left\|\frac{\partial \tilde{w}_R}{\partial y^{2s}}\right\|_{C^{0,\beta_1}(\overline{\Omega}_R)} \leq C_2 \,, \tag{4.9}$$

where C_2 is a positive constant depending on $\|w_R\|_{C^{0,\beta}(\overline{B}_R)}$, $\|\rho\|_{C^{0,\sigma}(\overline{B}_R)}$ and on the same quantities as for C_1.

Proof. It is a direct consequence of [6, Lemma 4.5]. $\qquad\square$

Lemma 4.10. *Let $\alpha \in (0,1)$. Let $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ be positive and such that $\rho^{-1} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$.*

  (i) *Let $(w_R^{(1)}, \tilde{w}_R^{(1)})$ and $(w_R^{(2)}, \tilde{w}_R^{(2)})$ be a subsolution and a supersolution, respectively, to problem (4.6) (in a weak sense, in agreement with Definition 4.7). Assume that $\tilde{w}_R^{(1)}, \tilde{w}_R^{(2)} \geq 0$ a.e. in Ω_R, $w_R^{(1)} \geq 0$ a.e. in B_R, $w_R^{(2)} > 0$ a.e. in B_R and $\tilde{w}_R^{(1)}|_{\Sigma_R} \leq \tilde{w}_R^{(2)}|_{\Sigma_R}$ a.e. in Σ_R. Then $\tilde{w}_R^{(1)} \leq \tilde{w}_R^{(2)}$ a.e. in Ω_R and $w_R^{(1)} \leq w_R^{(2)}$ a.e. in B_R.*

  (ii) *Suppose in addition that $\rho \in C^{0,\sigma}_{\mathrm{loc}}(\mathbb{R}^d)$ (for some $\sigma \in (0,1)$). Let $(w_R, \tilde{w}_R)$ be a solution to problem (4.6) (according to Definition 4.7), such that $w_R \in L^\infty(B_R)$ and $\tilde{w}_R \in L^\infty(\Omega_R)$. Then (in particular) $\tilde{w}_R \in C(\overline{\Omega}_R)$ and either $(w_R, \tilde{w}_R) \equiv (0,0)$ or $w_R > 0$ in B_R and $\tilde{w}_R > 0$ in Ω_R.*

Proof. Statement (i) follows by performing minor modifications to the proof of [4, Lemma 5.3]. In order to prove (ii) just notice that, thanks to (4.8) and (4.9), we can exploit exactly the same arguments as in [6, Corollary 4.12] and get the assertion. $\qquad\square$

We are now in position to prove Theorem 4.4 as concerns the existence of a minimal local weak solution to (4.1). The fact that such solution is also a very weak solution to (1.11) (according to Definition 4.2) will be deduced in the end of the proof of Theorem 3.1 in Section 5.

Proof of Theorem 4.4 (first part). For any $R > 0$, by Proposition 4.8 we know that there exists a nontrivial solution $(w_R, \tilde{w}_R)$ to problem (4.6). Let now $(\chi_R, \tilde{\chi}_R)$ be the unique regular solution to the problem

$$\begin{cases} L_s \tilde{\chi}_R = 0 & \text{in } \mathcal{C}_R\,, \\ \tilde{\chi}_R = 0 & \text{on } \partial\mathcal{C}_R \cap \{y > 0\}\,, \\ \tilde{\chi}_R = \chi_R & \text{on } \Gamma_R\,, \\ \dfrac{\partial \tilde{\chi}_R}{\partial y^{2s}} = \rho & \text{on } \Gamma_R\,, \end{cases}$$

where $\mathcal{C}_R = B_R \times \{y > 0\}$. By standard results (see e.g. [8]), we have:

$$\tilde{\chi}_R(x,y) = \int_{B_R} G_R((x,y),z)\,\rho(z)\mathrm{d}z \quad \forall (x,y) \in \mathcal{C}_R\,, \tag{4.10}$$

where $G_R((x,y),z)$ (let $(x,y) \in \mathcal{C}_R$ and $z \in B_R$) is the Green function, namely the solution of

$$\begin{cases} L_s G_R(\cdot, z) = 0 & \text{in } \mathcal{C}_R\,, \\ G_R(\cdot, z) = 0 & \text{on } \partial\mathcal{C}_R \cap \{y > 0\}\,, \\ \dfrac{\partial G_R(\cdot, z)}{\partial y^{2s}} = \delta_z & \text{on } \Gamma_R\,, \end{cases}$$

for each $z \in B_R$. It is well known that the Green functions are positive and ordered w.r.t. R, that is, if $R_1 \leq R_2$ then

$$0 < G_{R_1} \leq G_{R_2} \quad \text{in } \overline{\mathcal{C}}_{R_1}. \tag{4.11}$$

Furthermore, they are all bounded from above by the Green function G_+ for the half-space:

$$G_R((x,y),z) \leq G_+((x,y),z) \quad \forall (x,y) \in \overline{\mathcal{C}}_R, \ \forall z \in B_R, \ \forall R > 0, \tag{4.12}$$

where

$$G_+((x,y),z) = \frac{k_{s,d}}{|((x-z),y)|^{d-2s}} \quad \forall (x,y) \in \mathbb{R}_+^{d+1}, \ \forall z \in \mathbb{R}^d$$

(for the same constant $k_{s,d}$ appearing in (2.2)). The function G_+ solves

$$\begin{cases} L_s G_+(\cdot, z) = 0 & \text{in } \mathbb{R}_+^{d+1}, \\ \dfrac{\partial G_+(\cdot, z)}{\partial y^{2s}} = \delta_z & \text{on } \partial \mathbb{R}_+^{d+1}, \end{cases}$$

for each $z \in \mathbb{R}^d$ (see again [8] and also [11]). From (2.2), (4.10) and (4.12) it clearly follows that, for any $R > 0$ and any $(x,y) \in \overline{\mathcal{C}}_R$,

$$\tilde{\chi}_R(x,y) \leq \int_{\mathbb{R}^d} G_+((x,y),z)\,\rho(z)\mathrm{d}z \leq \int_{\mathbb{R}^d} G_+((x,0),z)\,\rho(z)\mathrm{d}z = (I_{2s}*\rho)(x) \leq \|I_{2s}*\rho\|_\infty = \widehat{C} \tag{4.13}$$

(for the last inequality, see Corollary A.2). Now note that, for any test function ψ as in Definition 4.7, we have:

$$\mu_s \int_{\Omega_R} y^{1-2s}\langle \nabla \tilde{\chi}_R, \nabla \psi\rangle(x,y)\,\mathrm{d}x\mathrm{d}y = \mu_s \int_{\mathcal{C}_R} y^{1-2s}\langle \nabla \tilde{\chi}_R, \nabla \psi\rangle(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{B_R} \psi(x,0)\,\rho(x)\mathrm{d}x. \tag{4.14}$$

If we choose any $\overline{C} \geq \widehat{C}^{\frac{\alpha}{1-\alpha}}$, then the function $(\overline{C}\chi_R, \overline{C}\tilde{\chi}_R)$ is a supersolution to problem (4.6). In fact, thanks to (4.13) and (4.14), in this case there holds

$$\mu_s \int_{\Omega_R} y^{1-2s}\left\langle \nabla\left(\overline{C}\tilde{\chi}_R\right), \nabla \psi\right\rangle(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{B_R} \overline{C}\psi(x,0)\,\rho(x)\mathrm{d}x \geq \int_{B_R} \left[\overline{C}\chi_R(x)\right]^\alpha \psi(x,0)\,\rho(x)\mathrm{d}x$$

for all *nonnegative* ψ as above. Hence, thanks to (4.10) and (4.11), we are in position to apply the comparison principle provided by Lemma 4.10-(i) with the choices $(w_R^{(1)}, \tilde{w}_R^{(1)}) = (w_R, \tilde{w}_R)$ and $(w_R^{(2)}, \tilde{w}_R^{(2)}) = (\overline{C}\chi_R, \overline{C}\tilde{\chi}_R)$, to get:

$$\tilde{w}_R \leq \overline{C}\tilde{\chi}_R \quad \text{a.e. in } \Omega_R, \tag{4.15}$$

and

$$w_R \leq \overline{C}\chi_R \quad \text{a.e. in } B_R. \tag{4.16}$$

In particular, by (4.13), (4.15) and (4.16) we deduce that $w_R \in L^\infty(B_R)$ and $\tilde{w}_R \in L^\infty(\Omega_R)$. We can now exploit Lemma 4.10-(ii) and infer that

$$\tilde{w}_R > 0 \quad \text{in } \Omega_R \tag{4.17}$$

and

$$w_R > 0 \quad \text{in } B_R. \tag{4.18}$$

Let $0 < R_1 < R_2$. The strict positivity, for all $R > 0$, of $(w_R, \tilde{w}_R)$ given by (4.17) and (4.18) allows us to apply again Lemma 4.10-(i), this time with the choices $(w_R^{(1)}, \tilde{w}_R^{(1)}) = (w_{R_1}, \tilde{w}_{R_1})$ and $(w_R^{(2)}, \tilde{w}_R^{(2)}) = (w_{R_2}, \tilde{w}_{R_2})$, to get:

$$\tilde{w}_{R_1} \leq \tilde{w}_{R_2} \quad \text{in } \Omega_R, \quad w_{R_1} \leq w_{R_2} \quad \text{in } B_R \quad \forall R_2 > R_1 > 0. \tag{4.19}$$

We need to pass to the limit on $(w_R, \tilde{w}_R)$ as $R \to \infty$. Given any *fixed* $\eta \in C_c^\infty(\mathbb{R}_+^{d+1} \cup \partial \mathbb{R}_+^{d+1})$, for every $R > 0$ large enough we can pick (after approximation) $\psi = \tilde{w}_R \eta^2$ as a test function in Definition 4.7. So, it is easily seen that

$$
\begin{aligned}
&\mu_s \int_{\Omega_R} y^{1-2s} \left| \nabla \tilde{w}_R(x,y) \right|^2 \eta^2 \, \mathrm{d}x \mathrm{d}y \\
&\leq 2 \left\| w_R \right\|_\infty^{\alpha+1} \int_{B_R} \eta^2(x,0) \, \rho(x) \mathrm{d}x + 4 \mu_s \left\| \tilde{w}_R \right\|_\infty^2 \int_{\Omega_R} y^{1-2s} \left| \nabla \eta(x,y) \right|^2 \mathrm{d}x \mathrm{d}y \, .
\end{aligned}
\tag{4.20}
$$

From (4.13), (4.15), (4.16) and (4.20) we deduce that, for any $\Omega_0 \Subset \mathbb{R}_+^{d+1} \cup \partial \mathbb{R}_+^{d+1}$, there holds

$$
\int_{\Omega_0} y^{1-2s} \left| \nabla \tilde{w}_R(x,y) \right|^2 \, \mathrm{d}x \mathrm{d}y \leq K
\tag{4.21}
$$

for a suitable positive constant K independent of $R > 0$. By collecting (4.13), (4.15), (4.16) and (4.19), we infer that there exist the following (nontrivial) pointwise limits:

$$
\lim_{R \to \infty} \tilde{w}_R = \tilde{w} \in L^\infty(\mathbb{R}_+^{d+1}) , \quad \lim_{R \to \infty} w_R = w \in L^\infty(\mathbb{R}^d) .
\tag{4.22}
$$

Due to (4.21), by standard compactness arguments we can pass to the limit in the weak formulation (4.7) and infer that w is a local weak solution to (4.1) in the sense of Definition 4.1 (with local extension $\tilde{w}$).

Now we have to prove minimality. Hereafter, we shall denote by $\underline{w}$ the solution constructed above and by w any other nonidendically zero local weak solution to (4.1) (according to Definition 4.1). In particular, for R large enough $(w|_{B_R}, \tilde{w}|_{\Omega_R})$ is a nontrivial solution to problem (4.6), in the sense of Definition 4.7, except for the fact that $\tilde{w}|_{\Omega_R}$ is not necessarily zero on Σ_R (that is, $\tilde{w}$ has finite energy in Ω_R but does not belong to $X_0^s(\Omega_R)$). However, the regularity results of [6] still hold: namely, Lemma 4.10-(ii) is applicable in this case as well, ensuring that $w > 0$ in B_R. Because $(w_R, \tilde{w}_R)$ is also a weak solution to (4.6) and, trivially, $\tilde{w}_R|_{\Sigma_R} \leq \tilde{w}|_{\Sigma_R}$ on Σ_R, thanks to Lemma 4.10-(i) (with the choices $(w_R^{(1)}, \tilde{w}_R^{(1)}) = (w_R, \tilde{w}_R)$ and $(w_R^{(2)}, \tilde{w}_R^{(2)}) = (w|_{B_R}, \tilde{w}|_{\Omega_R})$) we deduce

$$
w_R \leq w|_{B_R} \quad \text{in } B_R ,
$$

whence $\underline{w} \leq w$ in Γ by letting $R \to \infty$, so that $\underline{w}$ is indeed minimal. The bound (4.3) is then just a consequence of (4.13), (4.16) and (4.22).

From the above method of proof one can check that, under the more restrictive condition (4.4), then $\underline{w}$ is also a *weak solution* to (1.11) in the sense of Definition 4.3. In fact, thanks to Remark A.3, the inequalities (4.13), (4.16) and condition (4.4) ensure that $\{\|w_R^{\alpha+1}\|_{1,\rho}\}$ is uniformly bounded with respect to R. As a consequence, it is easy to verify that estimate (4.21) holds with $\Omega_0 = \mathbb{R}_+^{d+1}$ (up to setting $\tilde{w}_R = 0$ in Ω_R^c). By passing to the limit as $R \to \infty$, this implies that $\underline{\tilde{w}} \in X^s$, $\underline{w} \in \dot{H}^s(\mathbb{R}^d)$, $\underline{\tilde{w}} = \mathrm{E}(\underline{w})$ and $\underline{w}$ satisfies (4.2).

As already remarked, the fact that $\underline{w}$ is a *very weak solution* to (1.11) in the sense of Definition 4.2 for all $\gamma > 2s$ will be deduced at the end of the asymptotic analysis of Section 5 (see the proof of Theorem 3.1). $\qquad \square$

4.2. Uniqueness for the sublinear elliptic equation.
In this section we prove our uniqueness result, stated in Theorem 4.5, for solutions to (1.11). The strategy of proof strongly relies on the uniqueness result provided by Theorem 2.4 for solutions to (1.1).

Proof of Theorem 4.5. Set $m = 1/\alpha$ and

$$
C_m = (m-1)^{-\frac{1}{m-1}} .
$$

For any $k \in \mathbb{N}$ let $\zeta_k \in C^\infty(\mathbb{R}^d)$ be such that $\zeta_k = 1$ in B_k, $\zeta_k = 0$ in B_{2k}^c and $0 \leq \zeta_k \leq 1$ in $\mathbb{R}^d$. Take $R > 2k$ and denote as $(v_{R,k}, \tilde{v}_{R,k})$ the unique strong solution to the following evolution problem (see

Appendix B – Part II):

$$\begin{cases} L_s\left(\tilde{v}_{R,k}^m\right) = 0 & \text{in } \Omega_R \times (0, \infty) \,, \\ \tilde{v}_{R,k} = 0 & \text{on } \Sigma_R \times (0, \infty) \,, \\ \dfrac{\partial\left(\tilde{v}_{R,k}^m\right)}{\partial y^{2s}} = \rho \dfrac{\partial v_{R,k}}{\partial t} & \text{on } \Gamma_R \times (0, \infty) \,, \\ v_{R,k} = C_m\, \zeta_k w^{\frac{1}{m}} & \text{on } B_R \times \{t = 0\} \,. \end{cases} \tag{4.23}$$

Let $(w_R, \tilde{w}_R)$ be defined as in the proof of Theorem 4.4. Since by hypothesis $w \in L^\infty(\mathbb{R}^d)$, thanks to (4.18) we can select a suitable $\tau_R > 0$ so that

$$\frac{w_{R+1}^{\frac{1}{m}}}{\tau_R^{\frac{1}{m-1}}} \geq w^{\frac{1}{m}} \quad \text{in } B_R \,. \tag{4.24}$$

We have:

$$\widetilde{U}_R = \frac{C_m \tilde{w}_{R+1}^{\frac{1}{m}}}{(t + \tau_R)^{\frac{1}{m-1}}} \leq \frac{C_m \tilde{w}_{R+1}^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} = \widetilde{U}_{0R} \quad \text{in } \overline{\Omega}_R \times (0, \infty) \,. \tag{4.25}$$

Set $U_R(\cdot, t) = \widetilde{U}_R(\cdot, 0, t)$ and $U_{0R}(\cdot, t) = \widetilde{U}_{0R}(\cdot, 0, t)$, for each $t > 0$. By definition of $(U_R, \widetilde{U}_R)$ and recalling (4.24), we get that $(U_R, \widetilde{U}_R)$ is a strong supersolution to (4.23). Hence, by the comparison principle stated in Proposition B.3 and (4.25), we deduce:

$$v_{R,k} \leq U_R \leq U_{0R} \quad \text{a.e. in } B_R \times (0, \infty) \,. \tag{4.26}$$

In addition to the above bounds we also have that, for any $k_2 > k_1$ and $R > 2k_1$, there holds

$$v_{R,k_1} \leq v_{R,k_2} \leq \frac{C_m w^{\frac{1}{m}}}{(t + 1)^{\frac{1}{m-1}}} = V \quad \text{a.e. in } B_R \times (0, \infty) \,. \tag{4.27}$$

Such inequalities follow by noticing that $(V, \widetilde{V})$ is a strong supersolution to (4.23) for all $R > 0$ and $k \in \mathbb{N}$, while $(v_{R,k_2}, \tilde{v}_{R,k_2})$ is a strong supersolution to (4.23) for $k = k_1$. One then applies again Proposition B.3.

Since for each $k \in \mathbb{N}$ we have $C_m \zeta_k w^{\frac{1}{m}} \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, by standard arguments (e.g. similar to the ones exploited in the proof of [26, Theorem 3.1], see also Appendix B – Part II) one sees that there exists the limit

$$v_{\infty,k} = \lim_{R \to \infty} v_{R,k} \quad \text{a.e. in } \mathbb{R}^d$$

and it is a solution of the problem

$$\begin{cases} \rho\,(v_{\infty,k})_t + (-\Delta)^s(v_{\infty,k}^m) = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \,, \\ v_{\infty,k} = C_m\, \zeta_k w^{\frac{1}{m}} & \text{on } \mathbb{R}^d \times \{0\} \,, \end{cases}$$

both in the sense of Definition 2.1 and in the sense of Definition B.6. Moreover, as a consequence of (4.27), such limit satisfies the bounds

$$v_{\infty,k_1} \leq v_{\infty,k_2} \leq V \quad \text{a.e. in } \mathbb{R}^d \times (0, \infty) \tag{4.28}$$

for all $k_2 > k_1$. Thanks to (4.28) we get the existence of the pointwise limit

$$v_\infty = \lim_{k \to \infty} v_{\infty,k} \leq V \quad \text{a.e. in } \mathbb{R}^d \times (0, \infty) \,; \tag{4.29}$$

by passing to the limit in the very weak formulation solved by $v_{\infty,k}$ for all $k \in \mathbb{N}$, we infer that v_∞ is a very weak solution, in the sense of Definition B.6, to the problem

$$\begin{cases} \rho\,(v_\infty)_t + (-\Delta)^s(v_\infty^m) = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \,, \\ v_\infty = C_m w^{\frac{1}{m}} & \text{on } \mathbb{R}^d \times \{0\} \,. \end{cases} \tag{4.30}$$

Now notice that V is also a very weak solution to (4.30). Because, by hypothesis, $w \in L^1_{(1+|x|)^{-\beta}}(\mathbb{R}^d)$, clearly $V^m \in L^1_{(1+|x|)^{-\beta}}(\mathbb{R}^d \times (0,T))$. Hence, thanks to (4.29), we deduce that also v_∞^m belongs to $L^1_{(1+|x|)^{-\beta}}(\mathbb{R}^d \times (0,T))$. We are therefore in position to apply Theorem 2.4 (after Remark B.7) and obtain

$$v_\infty = V \quad \text{a.e. in } \mathbb{R}^d \times (0,\infty) \,.$$

Passing to the limit in (4.26) (first as $R \to \infty$, then as $k \to \infty$) and using (4.22), we infer that

$$v_\infty \le \frac{C_m \underline{w}^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \quad \text{a.e. in } \mathbb{R}^d \times (0,\infty) \,,$$

Hence,

$$\frac{w^{\frac{1}{m}}}{\underline{w}^{\frac{1}{m}}} \le \frac{(t+1)^{\frac{1}{m-1}}}{t^{\frac{1}{m-1}}} \,, \tag{4.31}$$

and by letting $t \to \infty$ in (4.31) we deduce

$$w^{\frac{1}{m}} \le \underline{w}^{\frac{1}{m}} \quad \text{a.e. in } \mathbb{R}^d \,.$$

Since w is nontrivial and $\underline{w}$ is minimal, it follows that $w \equiv \underline{w}$. $\qquad\square$

5. Asymptotic behaviour for rapidly decaying densities: proofs

Before proving Theorem 3.1, we need the following intermediate result, which gives a crucial bound from above for the solution to problem (1.1) provided by Theorem 2.4.

Lemma 5.1. *Under the same assumptions and with the same notations as in Theorem 3.1, we have:*

$$u \le (m-1)^{-\frac{1}{m-1}} \, t^{-\frac{1}{m-1}} w^{\frac{1}{m}} \quad \text{a.e. in } \mathbb{R}^d \times (0,\infty) \,. \tag{5.1}$$

Proof. Suppose at first that $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Let C_m, $(w_R, \tilde{w}_R)$ and $(U_R, \widetilde{U}_R)$ (for a suitable $\tau_R > 0$ to be chosen later) be defined as in the proofs of Theorems 4.4 and 4.5. For any $R > 0$, let $(u_R, \tilde{u}_R)$ be the unique strong solution to the following evolution problem (see Appendix B – Part II):

$$\begin{cases} L_s \left(\tilde{u}_R^m \right) = 0 & \text{in } \Omega_R \times (0,\infty) \,, \\ \tilde{u}_R = 0 & \text{on } \Sigma_R \times (0,\infty) \,, \\ \dfrac{\partial \tilde{u}_R^m}{\partial y^{2s}} = \rho \dfrac{\partial u_R}{\partial t} & \text{on } \Gamma_R \times (0,\infty) \,, \\ u_R = u_0 & \text{on } B_R \times \{t=0\} \,. \end{cases} \tag{5.2}$$

By standard arguments (see again the proof of [26, Theorem 3.1] and Appendix B – Part II), we have that

$$\lim_{R\to\infty} u_R = u \quad \text{a.e. in } \mathbb{R}^d \times (0,\infty) \,, \quad \lim_{R\to\infty} \tilde{u}_R^m = \tilde{u}^m = \mathrm{E}(u^m) \quad \text{a.e. in } \mathbb{R}^{d+1}_+ \times (0,\infty) \,, \tag{5.3}$$

where u is the solution to (1.1) provided by Proposition 2.3. Note that, thanks to (4.18), for any $R > 0$ there holds

$$\min_{\overline{B}_R} w_{R+1} > 0 \,. \tag{5.4}$$

Hence, in view of (5.4) and recalling that we assumed $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we can pick $\tau_R > 0$ so that

$$\frac{C_m w_{R+1}^{\frac{1}{m}}}{\tau_R^{\frac{1}{m-1}}} \ge u_0 \quad \text{a.e. in } B_R \,. \tag{5.5}$$

Due to (5.5), $(U_R, \widetilde{U}_R)$ is a strong supersolution to problem (5.2). Therefore, by comparison principles (see Proposition B.3),

$$u_R \le U_R \quad \text{a.e. in } B_R \times (0,\infty) \,. \tag{5.6}$$

Because trivially $U_R \leq C_m t^{-\frac{1}{m-1}} w_{R+1}^{\frac{1}{m}}$, from (5.6) we deduce the fundamental estimate

$$u_R \leq C_m \, t^{-\frac{1}{m-1}} w_{R+1}^{\frac{1}{m}} \quad \text{a.e. in } B_R \times (0, \infty) \,. \tag{5.7}$$

By letting $R \to \infty$ in (5.7) and recalling (4.22) and (5.3), we finally get (5.1).
Consider now general data $u_0 \in L_\rho^1(\mathbb{R}^d)$. In this case, we have that

$$u = \lim_{n \to \infty} u_n \quad \text{a.e. in } \mathbb{R}^d \times (0, \infty) \,,$$

where for every $n \in \mathbb{N}$ we denote as u_n the solution to problem (1.1) corresponding to the initial datum $u_{0n} \in L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and the sequence $\{u_{0n}\}$ is such that $0 \leq u_{0n} \leq u_0$ in $\mathbb{R}^d$ for all $n \in \mathbb{N}$ and $u_{0n} \to u_0$ in $L_\rho^1(\mathbb{R}^d)$ as $n \to \infty$ (see [26, Section 6.2] and Appendix B – Parts I, II). In view of the first part of the proof, we know that for every $n \in \mathbb{N}$ there holds

$$u_n \leq C_m \, t^{-\frac{1}{m-1}} w^{\frac{1}{m}} \quad \text{a.e. in } \mathbb{R}^d \times (0, \infty) \,. \tag{5.8}$$

The assertion then follows by passing to the limit as $n \to \infty$ in (5.8). $\qquad \square$

Remark 5.2. As a consequence of the method of proof of Lemma 5.1 we also get the validity of the estimate

$$\mathrm{E}(u^m) \leq C_m^m \, t^{-\frac{m}{m-1}} \, \tilde{w} \quad \text{a.e. in } \mathbb{R}_+^{d+1} \times (0, \infty) \,, \tag{5.9}$$

where $\mathrm{E}(u^m)$ is the extension of u^m (see the beginning of Section 4) and $\tilde{w}$ is the local extension of w, in agreement with Definition 4.1, provided along the first part of the proof of Theorem 4.4. In fact it is enough to notice that, by standard comparison principles for sub- and supersolutions to the problem $L_s = 0$ in Ω_R, from (5.7) it follows that

$$\tilde{u}_R^m \leq C_m^m \, t^{-\frac{m}{m-1}} \, \tilde{w}_{R+1} \quad \text{a.e. in } \Omega_R \times (0, \infty) \,, \tag{5.10}$$

whence (5.9) upon letting $R \to \infty$ in (5.10).

Proof of Theorem 3.1 and end of proof of Theorem 4.4. Let us denote as $v(x, \tau)$ the following rescaling of $u(x, t)$:

$$u(x, t) = e^{-\beta\tau} v(x, \tau) \,, \quad t = e^\tau \,, \quad \beta = \frac{1}{m - 1} \,. \tag{5.11}$$

It is immediate to check that v is a (weak, and in particular very weak) solution to the equation

$$\rho v_\tau = -(-\Delta)^s(v^m) + \beta \rho v \quad \text{in } \mathbb{R}^d \times (0, \infty) \,,$$

in the sense that

$$- \int_0^\infty \int_{\mathbb{R}^d} v(x, \tau) \varphi_\tau(x, \tau) \, \rho(x) \mathrm{d}x \mathrm{d}\tau + \int_0^\infty \int_{\mathbb{R}^d} v^m(x, \tau) (-\Delta)^s(\varphi)(x, \tau) \, \mathrm{d}x \mathrm{d}\tau$$
$$= \beta \int_0^\infty \int_{\mathbb{R}^d} v(x, \tau) \varphi(x, \tau) \, \rho(x) \mathrm{d}x \mathrm{d}\tau + \int_{\mathbb{R}^d} u(x, 1) \varphi(x, 0) \, \rho(x) \mathrm{d}x \tag{5.12}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$. Moreover, $\mathrm{E}(v^m) \in L_{\mathrm{loc}}^2((0, \infty); X^s)$ and

$$- \int_0^T \int_{\mathbb{R}^d} v(x, \tau) \psi_\tau(x, 0, \tau) \, \rho(x) \mathrm{d}x \mathrm{d}\tau + \mu_s \int_0^T \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \left\langle \nabla \mathrm{E}(v^m), \nabla \psi \right\rangle (x, y, \tau) \, \mathrm{d}x \mathrm{d}y \mathrm{d}\tau$$
$$= \beta \int_0^T \int_{\mathbb{R}^d} v(x, \tau) \psi(x, 0, \tau) \, \rho(x) \mathrm{d}x \mathrm{d}\tau \tag{5.13}$$

for all $T > 0$ and $\psi \in C_c^\infty((\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1}) \times (0, T))$. Thanks to Lemma 5.1, we have:

$$v(x, \tau) \leq C_m w^{\frac{1}{m}}(x) \leq C_m \, \|w\|_\infty^{\frac{1}{m}} \quad \text{for a.e. } (x, \tau) \in \mathbb{R}^d \times (0, \infty) \,; \tag{5.14}$$

furthermore, recalling Remark 5.2,

$$\mathrm{E}(v^m)(x, y, \tau) \leq C_m^m \tilde{w}(x, y) \leq C_m^m \|\tilde{w}\|_\infty \quad \text{for a.e. } (x, y, \tau) \in \mathbb{R}_+^{d+1} \times (0, \infty) \,. \tag{5.15}$$

Now let us show that

$$v(x,\tau_2) \geq v(x,\tau_1) \quad \text{for a.e. } x \in \mathbb{R}^d, \quad \mathrm{E}(v^m)(x,y,\tau_2) \geq \mathrm{E}(v^m)(x,y,\tau_1) \quad \text{for a.e. } (x,y) \in \mathbb{R}^{d+1}_+$$
$$(5.16)$$

for all $\tau_2 \geq \tau_1 > 0$. To this purpose, first of all note that, similarly to [35, p. 182] (see also the original reference [2]), one can prove the fundamental Bénilan-Crandall inequality

$$u_t \geq -\frac{u}{(m-1)t} \quad \text{a.e. in } \mathbb{R}^d \times (0,\infty)$$

which, recalling (5.11), implies that

$$v_\tau \geq 0 \quad \text{a.e. in } \mathbb{R}^d \times (0,\infty).\tag{5.17}$$

Thanks to (5.17) we obtain the first inequality in (5.16), and therefore also the second one because the extension operator is order preserving. Hence, by (5.14), (5.15) and (5.16) we infer that there exist finite the limits

$$h(x) = \lim_{\tau\to\infty} v(x,\tau) \quad \text{for a.e. } x \in \mathbb{R}^d, \quad H(x,y) = \lim_{\tau\to\infty} \mathrm{E}(v^m)(x,y,\tau) \quad \text{for a.e. } (x,y) \in \mathbb{R}^{d+1}_+.$$
$$(5.18)$$

Moreover, since $u_0 \not\equiv 0$, (5.16) implies that $h \not\equiv 0$ and $H \not\equiv 0$, while (5.14) and (5.15) ensure that $h \in L^\infty(\mathbb{R}^d)$ and $H \in L^\infty(\mathbb{R}^{d+1}_+)$.

Let us set

$$g = C_m^{-m} h^m, \quad \tilde{g} = C_m^{-m} H.\tag{5.19}$$

First we want to prove that g (with the corresponding local extension $\tilde{g}$) is a solution to problem (4.1) (for $\alpha = 1/m$) in the sense of Definition 4.1. To this end, for any fixed $0 < \tau_1 < \tau_2$ and $0 < \epsilon < (\tau_2 - \tau_1)/2$, let $\zeta_\epsilon(\tau)$ be a smooth approximation of the function $\chi_{[\tau_1,\tau_2]}(\tau)$ such that

$$0 \leq \zeta_\epsilon(\tau) \leq 1 \quad \forall \tau \geq 0, \quad \zeta_\epsilon(\tau) = 0 \quad \forall \tau \notin [\tau_1,\tau_2], \quad \zeta_\epsilon(\tau) = 1 \quad \forall \tau \in [\tau_1+\epsilon, \tau_2-\epsilon].$$

Furthermore, we can and shall assume that

$$\zeta_\epsilon'(\tau) \to \delta(\tau - \tau_1) - \delta(\tau - \tau_2)$$

as $\epsilon \to 0$. Consider now a cut-off function η as in the first part of the proof of Theorem 4.4 and plug in the weak formulation (5.13) the test function $\psi = \zeta_\epsilon \eta^2 \mathrm{E}(v^m)$. Upon letting $\epsilon \to 0$, we get:

$$\frac{1}{m+1} \int_{\mathbb{R}^d} v^{m+1}(x,\tau_2)\, \eta^2(x,0)\, \rho(x)\mathrm{d}x + \mu_s \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^{d+1}_+} y^{1-2s} \left\langle \nabla \mathrm{E}(v^m), \nabla[\eta^2\, \mathrm{E}(v^m)] \right\rangle (x,y,\tau)\, \mathrm{d}x\mathrm{d}y\mathrm{d}\tau$$

$$= \frac{1}{m+1} \int_{\mathbb{R}^d} v^{m+1}(x,\tau_1)\, \eta^2(x,0)\, \rho(x)\mathrm{d}x + \beta \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^d} v^{m+1}(x,\tau)\, \eta^2(x,0)\, \rho(x)\mathrm{d}x\mathrm{d}\tau.$$

Thanks to (5.14) and (5.15), by setting $\tau_1 = \tilde{\tau}$, $\tau_2 = \tilde{\tau}+1$ and proceeding as in the proof of (4.21), we obtain the estimate

$$\int_{\tilde{\tau}}^{\tilde{\tau}+1} \int_{\Omega_0} y^{1-2s} \left| \nabla \mathrm{E}(v^m)(x,y,\tau) \right|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}\tau \leq K\tag{5.20}$$

for any $\Omega_0 \Subset \mathbb{R}^{d+1}_+ \cup \partial\mathbb{R}^{d+1}_+$ and a suitable constant $K > 0$ independent of $\tilde{\tau} > 0$. Take any function $\phi \in C_c^\infty(\mathbb{R}^{d+1}_+ \cup \partial\mathbb{R}^{d+1}_+)$. By plugging in (5.13) the test function $\psi(x,y,\tau) = \phi(x,y)\zeta_\epsilon(\tau)$, with $\tau_1 = \tilde{\tau}$ and $\tau_2 = \tilde{\tau}+1$, and letting $\epsilon \to 0$, we infer that

$$\int_{\mathbb{R}^d} [v(x,\tilde{\tau}+1) - v(x,\tilde{\tau})]\, \phi(x,0)\, \rho(x)\mathrm{d}x + \mu_s \int_{\tilde{\tau}}^{\tilde{\tau}+1} \int_{\mathbb{R}^{d+1}_+} y^{1-2s} \left\langle \nabla \mathrm{E}(v^m)(x,y,\tau), \nabla\phi(x,y) \right\rangle \mathrm{d}x\mathrm{d}y\mathrm{d}\tau$$

$$= \beta \int_{\tilde{\tau}}^{\tilde{\tau}+1} \int_{\mathbb{R}^d} v(x,\tau)\phi(x,0)\, \rho(x)\mathrm{d}x\mathrm{d}\tau.$$

$$(5.21)$$

Thanks to (5.14), (5.15), (5.18), (5.20) and standard local compactness arguments we can pass to the limit in (5.21) (along a suitable subsequence $\widetilde{\tau}_n \in [\widetilde{\tau}, \widetilde{\tau}+1]$, with $\widetilde{\tau}_n \to \infty$) to find that h and H satisfy

$$\mu_s \int_{\mathbb{R}^{d+1}_+} y^{1-2s} \langle \nabla H, \nabla \phi \rangle (x,y)\, \mathrm{d}x\mathrm{d}y = \beta \int_{\mathbb{R}^d} h(x)\phi(x,0)\, \rho(x)\mathrm{d}x$$

and $H(x,0) = h^m(x)$. That is, the function g (with $\tilde{g}$ as a local extension) defined in (5.19) is a local weak solution to (4.1) (for $\alpha = 1/m$) in the sense of Definition 4.1. Moreover, g is also a very weak solution to (1.11) (for $\alpha = 1/m$) in the sense of Definition 4.2. In order to prove this fact, we can proceed as above: for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ we plug in the weak formulation (5.12) the test function $\psi(x,\tau) = \zeta_\epsilon(\tau)\varphi(x)$, let $\epsilon \to 0$ and get

$$\int_{\mathbb{R}^d} [v(x,\widetilde{\tau}+1) - v(x,\widetilde{\tau})]\varphi(x)\rho(x)\mathrm{d}x = \int_{\widetilde{\tau}}^{\widetilde{\tau}+1} \int_{\mathbb{R}^d} [-v^m(x,\tau)(-\Delta)^s(\varphi)(x) + \beta v(x,\tau)\varphi(x)\rho(x)]\, \mathrm{d}x\mathrm{d}\tau\,.$$
$$(5.22)$$

By letting $\widetilde{\tau} \to \infty$ in (5.22) and recalling (5.19), we finally deduce that

$$0 = -\int_{\mathbb{R}^d} g(x)(-\Delta)^s(\varphi)(x)\, \mathrm{d}x + \int_{\mathbb{R}^d} g^{\frac{1}{m}}(x)\varphi(x)\rho(x)\mathrm{d}x\,. \tag{5.23}$$

Now note that, passing to the limit in (5.14) as $\tau \to \infty$, we obtain:

$$g(x) \le w(x) \quad \text{for a.e. } x \in \mathbb{R}^d\,. \tag{5.24}$$

Because $g \not\equiv 0$ and it is a solution to (4.1) in the sense of Definition 4.1, the minimality of w ensured by Theorem 4.4 and (5.24) necessarily imply that $g \equiv w$. Hence, thanks to (5.23), we can conclude the proof of Theorem 4.4 by inferring that the minimal solution provided by it is also a very weak solution to (1.11) (for $\alpha = 1/m$) in the sense of Definition 4.2.

$\square$

6. Asymptotic behaviour for slowly decaying densities: proofs

In order to prove Theorem 3.3, we first need some preliminary results. The following one has been proved in [18, Lemma 4.8].

Lemma 6.1. *Let $\gamma \in (0, 2s)$, $v \in L^1_{|x|^{-\gamma}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and U_γ^v be the Riesz potential of $|x|^{-\gamma}v$, that is*

$$U_\gamma^v = I_{2s} * \left(|x|^{-\gamma}v\right)\,.$$

The following properties hold true:

- *U_γ^v belongs to $C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for all p satisfying*

$$p \in \left(\frac{d}{d-2s}, \infty\right]\,. \tag{6.1}$$

- *Under the additional condition*

$$\gamma < d - 2s\,,$$

  for any $p > 1$ such that

$$p \in \left(\frac{d}{d-2s}, \frac{d}{\gamma}\right)$$

  there holds

$$U_\gamma^v \in W^{r,p}(\mathbb{R}^d)$$

  for all $r \in (0, 2s)$.

In all the above cases, the norms $\|U_\gamma^v\|_{L^p(\mathbb{R}^d)}$ and $\|U_\gamma^v\|_{W^{r,p}(\mathbb{R}^d)}$ can be estimated from above by a constant that depends on v only through $\|v\|_{1,|x|^{-\gamma}}$ and $\|v\|_\infty$.

Let u be a weak solution to problem (1.3), according to Definition 2.1. For any $\lambda > 0$, set

$$u_\lambda(x,t) = \lambda^\alpha u(\lambda^\kappa x, \lambda t) \quad \forall (x,t) \in \mathbb{R}^d \times (0,\infty) , \tag{6.2}$$

where α, κ are defined in (3.1). Notice that (6.2) is the same scaling under which $u_M^{c_\infty}$ is invariant (see Section 3.2).

Proposition 6.2. *Let the assumptions of Theorem 3.3 hold true. Then, for any sequence $\lambda_n \to \infty$, $\{u_{\lambda_n}\}$ converges to $u_M^{c_\infty}$ almost everywhere in $\mathbb{R}^d \times (0,\infty)$ along subsequences.*

Proof. For notational simplicity, we shall again put $c_\infty = 1$. We shall also assume, with no loss of generality, that $\mu = \rho u_0 \in L^1(\mathbb{R}^d)$ (recall e.g. the smoothing effect (2.12)).

Here we shall not give a fully detailed proof, since the procedure follows closely the one performed in the proof of [18, Theorem 3.2]. To begin with, note that u_λ solves the problem

$$\begin{cases} \rho_\lambda \, u_t + (-\Delta)^s \, (u_\lambda^m) = 0 & \text{in } \mathbb{R}^d \times (0,\infty) , \\ u_\lambda = u_{0\lambda} & \text{on } \mathbb{R}^d \times \{0\} , \end{cases} \tag{6.3}$$

where

$$\rho_\lambda(x) = \lambda^{\kappa\gamma} \rho(\lambda^\kappa x) , \quad u_{0\lambda}(x) = \lambda^\alpha u_0(\lambda^\kappa x) \quad \forall x \in \mathbb{R}^d . \tag{6.4}$$

It is easily seen that (recall the conservation of mass (2.11))

$$\|u_\lambda(t)\|_{1,\rho_\lambda} = \|u_{0\lambda}\|_{1,\rho_\lambda} = M \quad \forall t, \lambda > 0 . \tag{6.5}$$

Claim 1: *There exists a subsequence $\{u_{\lambda_m}\} \subset \{u_{\lambda_n}\}$ which converges pointwise a.e. in $\mathbb{R}^d \times (0,\infty)$ to some function u. Furthermore, the limit u satisfies (2.8), (2.9) and (2.10).*

Noticing that

$$\frac{\overline{c}_0}{1 + |x|^\gamma} \leq \rho_\lambda(x) \leq \frac{\overline{C}_0}{|x|^\gamma} \quad \text{for a.e. } x \in \mathbb{R}^d, \ \forall \lambda > 0 \tag{6.6}$$

for suitable positive constants $\overline{c}_0, \overline{C}_0$ independent of λ and combining the smoothing effect (2.12) with (6.5), we obtain:

$$\|u_\lambda(t)\|_\infty \leq K t^{-\alpha} M^\beta \quad \forall t, \lambda > 0 , \tag{6.7}$$

where $K > 0$ is a constant depending only on $\overline{C}_0$, m, γ, s and d. In particular,

$$\int_{\mathbb{R}^d} u_\lambda^{m+1}(x,t) \, \rho_\lambda(x) \mathrm{d}x \leq K^m t^{-\alpha m} M^{\beta m + 1} \quad \forall t, \lambda > 0 . \tag{6.8}$$

By (2.14) and (6.8) we infer that

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} (u_\lambda^m)(x,t) \right|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{m+1} \int_{\mathbb{R}^d} u_\lambda^{m+1}(x,t_2) \, \rho_\lambda(x) \mathrm{d}x \leq K^m t_1^{-\alpha m} M^{\beta m + 1} \tag{6.9}$$

for all $\lambda > 0$ and all $t_2 > t_1 > 0$. On the other hand, due to (2.15),

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |(z_\lambda)_t(x,t)|^2 \, \rho_\lambda(x) \mathrm{d}x\mathrm{d}t \leq C \quad \forall t_2 > t_1 > 0, \ \forall \lambda > 0 , \tag{6.10}$$

where $z_\lambda = u_\lambda^{\frac{m+1}{2}}$ and C is another positive constant depending on t_1 and t_2 but independent of λ. In view of (6.5), (6.7), (6.9) and (6.10), by standard compactness arguments (see again the proof of [18, Theorem 3.2]) the sequence $\{u_{\lambda_n}\}$ admits a subsequence $\{u_{\lambda_m}\}$ converging pointwise a.e. in $\mathbb{R}^d \times (0,\infty)$ to some function u which complies with (2.8) and (2.9). Moreover, because of the assumptions on ρ, (6.6) holds true and

$$\lim_{\lambda \to \infty} \rho_\lambda(x) = |x|^{-\gamma} \quad \text{for a.e. } x \in \mathbb{R}^d . \tag{6.11}$$

It is then immediate to pass to the limit in the weak formulation solved by u_{λ_m} and find that u satisfies also (2.10), and Claim 1 is shown. However, (2.10) does not provide any information about

the initial datum assumed by $u(t)$. To this end it is convenient to exploit some results in potential theory, following [23] or [36]. Hence, let us introduce the Riesz potential $U_\lambda(t)$ of $\rho_\lambda u_\lambda(t)$, that is

$$U_\lambda(t) = I_{2s} * (\rho_\lambda u_\lambda(t)) \quad \forall t, \lambda > 0 \,.$$

Claim 2: *For any $\lambda > 0$, the function U_λ satisfies the following differential equation:*

$$(U_\lambda)_t(t) = -u_\lambda^m(t) \quad \text{for a.e. } t > 0 \,. \tag{6.12}$$

In order to prove (6.12) rigorously, one proceeds exactly as in the proof of [18, Theorem 3.2]. Notice however that, formally, $(-\Delta)^s(U_\lambda)(t) = \rho_\lambda u_\lambda(t)$, so that (6.12), still at a formal level, just follows by applying the operator $(-\Delta)^{-s}$ to both sides of the differential equation in (6.3).

Claim 3: *Let $U_{0\lambda} = I_{2s} * (\rho_\lambda u_{0\lambda})$. For any fixed $\lambda > 0$, the following equality holds:*

$$\lim_{t \to 0} U_\lambda(x, t) = U_{0\lambda}(x) \quad \text{for a.e. } x \in \mathbb{R}^d \,. \tag{6.13}$$

In fact, by (6.12), we deduce that $U_\lambda(t)$ has an absolutely continuous version (for instance in $L^1_{\text{loc}}(\mathbb{R}^d)$) which is nonincreasing t. Consequently, $U_\lambda(t)$ admits a pointwise limit as $t \to 0$. Since we also know (Definition 2.1) that $\rho_\lambda u_\lambda(t)$ converges to $\rho_\lambda u_{0\lambda}$ in $L^1(\mathbb{R}^d)$ as $t \to 0$, Theorem 3.8 of [22] guarantees the identification (a.e. in $\mathbb{R}^d$) between the pointwise limit of $\{U_\lambda(t)\}$ and the Riesz potential of $\rho_\lambda u_{0\lambda}$, whence (6.13) and Claim 3 is proved.

Now we need to deal with the convergence of $\{U_\lambda\}$ as $\lambda \to \infty$.

Claim 4: *Up to subsequences,*

$$\lim_{m \to \infty} U_{\lambda_m}(y, t) = \left[I_{2s} * \left(|x|^{-\gamma} u(t) \right) \right](y) = U(y, t) \quad \text{for a.e. } (y, t) \in \mathbb{R}^d \times (0, \infty) \,. \tag{6.14}$$

Exploiting (6.5), (6.7) and Lemma 6.1 we deduce that

$$\sup_{\lambda \geq 1} \sup_{t \geq \tau} \|U_\lambda(t)\|_{W^{r,p}(\mathbb{R}^d)} < \infty \quad \forall \tau > 0$$

for any $r \in (0, 2s)$ and p complying with (6.1). From standard Hölder embeddings for fractional Sobolev spaces (see e.g. [12, Theorem 8.2]), this implies in turn that

$$\sup_{\lambda \geq 1} \sup_{t \geq \tau} \|U_\lambda(t)\|_{C^\beta(\Omega)} < \infty \quad \forall \Omega \Subset \mathbb{R}^d, \ \forall \tau > 0 \,, \tag{6.15}$$

provided r is sufficiently close to $2s$, p is sufficiently close to d/γ and $\beta = r - d/p$. But (6.7) and (6.12) ensure that $\{U_\lambda\}$ is uniformly Lipschitz in time. Combining this information with (6.15) yields

$$\sup_{\lambda \geq 1} \|U_\lambda\|_{C^\beta(\Omega \times (t_1, t_2))} < \infty \quad \forall \Omega \Subset \mathbb{R}^d, \ \forall t_2 > t_1 > 0 \,.$$

In particular, there exists a function $U \in C^\beta_{\text{loc}}(\mathbb{R}^d \times (0, \infty))$ such that, up to subsequences,

$$\lim_{m \to \infty} U_{\lambda_m}(x, t) = U(x, t) \quad \forall (x, t) \in \mathbb{R}^d \times (0, \infty) \,. \tag{6.16}$$

Thanks to (6.6), (6.7) and (6.11), by dominated convergence we infer that for a.e. $t > 0$

$$\lim_{m \to \infty} \rho_{\lambda_m} u_{\lambda_m}(t) = |x|^{-\gamma} u(t) \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d)) \,.$$

Recalling that $\|\rho_{\lambda_m} u_{\lambda_m}(t)\|_1 = M$, in view of (6.16) and [22, Theorem 3.8] we deduce (6.14).

Claim 5: *The following limit holds true:*

$$\lim_{t \to 0} U(x, t) = M I_{2s}(x) \quad \text{for a.e. } x \in \mathbb{R}^d \,. \tag{6.17}$$

Proceeding as in [36, Section 6], we multiply (6.12) by $\rho_\lambda(x)$, integrate in $\mathbb{R}^d \times (t_1, t_2)$ and use (6.5) and (6.7) on the r.h.s. to get

$$\int_{\mathbb{R}^d} |U_\lambda(x, t_2) - U_\lambda(x, t_1)| \, \rho_\lambda(x) \mathrm{d}x \leq K^{m-1} M^{1+\beta(m-1)} \frac{t_2^{1-\alpha(m-1)} - t_1^{1-\alpha(m-1)}}{1 - \alpha(m-1)} \,. \tag{6.18}$$

Letting $t_1 \to 0$ in (6.18), exploiting (6.13) and Fatou's Lemma yields

$$\int_{\mathbb{R}^d} |U_\lambda(x, t_2) - U_{0\lambda}(x)| \, \rho_\lambda(x) \mathrm{d}x \le K^{m-1} M^{1+\beta(m-1)} \frac{t_2^{1-\alpha(m-1)}}{1 - \alpha(m-1)} \,. \tag{6.19}$$

Now notice that $\{\rho_\lambda u_{0\lambda}\}$ tends to $M\delta$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ as $\lambda \to \infty$. In fact, $\|\rho_\lambda u_{0\lambda}\|_1 = M$ and for any $\phi \in C_c(\mathbb{R}^d)$ one has (recalling (3.1) and (6.4))

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}^d} \phi(x) \rho_\lambda(x) u_{0\lambda}(x) \, \mathrm{d}x$$

$$= \lim_{\lambda \to \infty} \lambda^{\alpha + \kappa\gamma} \int_{\mathbb{R}^d} \phi(x) \rho(\lambda^\kappa x) u_0(\lambda^\kappa x) \, \mathrm{d}x = \lim_{\lambda \to \infty} \int_{\mathbb{R}^d} \phi\left(\frac{y}{\lambda^\kappa}\right) u_0(y) \, \rho(y) \mathrm{d}y = M\phi(0) \,.$$

In particular, as a direct consequence of [22, Theorem 3.8],

$$\liminf_{\lambda \to \infty} U_{0\lambda}(x) = M I_{2s}(x) \quad \text{for a.e. } x \in \mathbb{R}^d \,. \tag{6.20}$$

Using (6.11), (6.14), (6.20), the fact that $U_\lambda(t)$ is nonincreasing w.r.t. t and applying Fatou's Lemma to (6.19) we obtain

$$\int_{\mathbb{R}^d} |U(x, t_2) - M I_{2s}(x)| \, |x|^{-\gamma} \, \mathrm{d}x \le C^{m-1} M^{1+\beta(m-1)} \frac{t_2^{1-\alpha(m-1)}}{1 - \alpha(m-1)} \,. \tag{6.21}$$

Letting $t_2 \to 0$ in (6.21) we deduce in particular the validity of (6.17).

We can finally prove the following result.

Claim 6: *There holds*

$$\operatorname*{ess\,lim}_{t \to 0} |x|^{-\gamma} u(t) = M\delta \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d)) \,. \tag{6.22}$$

Passing to the limit in (6.5) as $\lambda = \lambda_m \to \infty$ entails

$$\big\| |x|^{-\gamma} u(t) \big\|_1 \le M \quad \text{for a.e. } t > 0 \,. \tag{6.23}$$

Estimate (6.23) implies that $|x|^{-\gamma} u(t)$ converges, up to subsequences, to some positive, finite measure ν in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ as $t \to 0$. However, a priori such ν may depend on the particular subsequence. The fact that $\nu = M\delta$, and so that (6.22) holds at least in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$, follows thanks to (6.17) and [22, Theorems 1.12 and 3.8] (for the details, see the proof of [18, Theorem 3.2]). In order to get such convergence also in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ it is enough to show that

$$\operatorname*{ess\,lim}_{t \to 0} \big\| |x|^{-\gamma} u(t) \big\|_1 = M \,. \tag{6.24}$$

By the convergence of $|x|^{-\gamma} u(t)$ to $M\delta$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ we have

$$M \le \operatorname*{ess\,lim\,inf}_{t \to 0} \big\| |x|^{-\gamma} u(t) \big\|_1 \,. \tag{6.25}$$

But letting $t \to 0$ in (6.23) entails

$$\operatorname*{ess\,lim\,sup}_{t \to 0} \big\| |x|^{-\gamma} u(t) \big\|_1 \le M \,. \tag{6.26}$$

Combining (6.25) and (6.26), (6.24) clearly follows, so that Claim 6 is proved and we can conclude that u satisfies (1.9), that is $u = u_M^{c_\infty}$. □

We are now in position to prove Theorem 3.3.

Proof of Theorem 3.3. Take any sequence $\lambda_n \to \infty$. Our first aim is to prove that, along any of the subsequences $\{\lambda_m\} \subset \{\lambda_n\}$ given by Proposition 6.2, there holds

$$\lim_{m \to \infty} \int_{B_R} |u_{\lambda_m}(x, t) - u_M^{c_\infty}(x, t)| \, |x|^{-\gamma} \, \mathrm{d}x = 0 \quad \forall R > 0 \,, \, \forall t > 0 \,. \tag{6.27}$$

Thanks to the smoothing estimates (2.12), (6.7) and to the fact that for almost every $t > 0$ we know that $\{u_{\lambda_m}(t)\}$ converges pointwise almost everywhere to $u_M^{c\infty}(t)$, by dominated convergence

$$\lim_{m\to\infty} \int_{B_R} |u_{\lambda_m}(x,t) - u_M^{c\infty}(x,t)| \, \mathrm{d}x = 0 \quad \forall R > 0, \text{ for a.e. } t > 0. \tag{6.28}$$

Moreover, estimate (B.7) for u_λ reads

$$\|(u_\lambda)_t(t)\|_{1,\rho_\lambda} \leq \frac{2}{(m-1)\,t} M \quad \text{for a.e. } t > 0. \tag{6.29}$$

Gathering (6.29) and (6.6), we can assert that for every $R, \tau > 0$ there exists a positive constant $C(R,\tau)$ (independent of λ) such that

$$\|(u_\lambda)_t(t)\|_{L^1(B_R)} \leq C(R,\tau) \quad \text{for a.e. } t \geq \tau. \tag{6.30}$$

Of course (6.30) also holds for $u_M^{c\infty}$. It is now possible to infer that (6.28) actually holds for *every* $t > 0$:

$$\lim_{m\to\infty} \int_{B_R} |u_{\lambda_m}(x,t) - u_M^{c\infty}(x,t)| \, \mathrm{d}x = 0 \quad \forall R > 0, \; \forall t > 0. \tag{6.31}$$

In fact, for any given $t_0, \varepsilon > 0$, there exists $t > t_0$ such that (6.28) holds and $|t - t_0| \leq \varepsilon$. Exploiting (6.30), we get:

$$\int_{B_R} |u_{\lambda_m}(x,t_0) - u_M^{c\infty}(x,t_0)| \, \mathrm{d}x$$
$$\leq \int_{B_R} |u_{\lambda_m}(x,t_0) - u_{\lambda_n}(x,t)| \, \mathrm{d}x + \int_{B_R} |u_{\lambda_m}(x,t) - u_M^{c\infty}(x,t)| \, \mathrm{d}x + \int_{B_R} |u_M^{c\infty}(x,t) - u_M^{c\infty}(x,t_0)| \, \mathrm{d}x$$
$$\leq 2\,C(R,t_0)\,\varepsilon + \int_{B_R} |u_{\lambda_m}(x,t) - u_M^{c\infty}(x,t)| \, \mathrm{d}x. \tag{6.32}$$

Letting $m \to \infty$ in (6.32) yields

$$\limsup_{m\to\infty} \int_{B_R} |u_{\lambda_m}(x,t_0) - u_M^{c\infty}(x,t_0)| \, \mathrm{d}x \leq 2\,C(R,t_0)\varepsilon. \tag{6.33}$$

Letting now $\varepsilon \to 0$ in (6.33) shows that (6.28) holds for $t = t_0$ as well. The validity of (6.27) is then just a consequence of (6.31), the local integrability of $|x|^{-\gamma}$ and the uniform bound over $\|u_{\lambda_m}(t) - u_M^{c\infty}(t)\|_\infty$ ensured by the smoothing estimates (2.12) and (6.7).

The consequence of Proposition 6.2 and what we proved above is that *any* sequence $\lambda_n \to \infty$ satisfies (6.27) along subsequences. We can thus infer that

$$\lim_{\lambda\to\infty} \int_{B_R} |u_\lambda(x,t) - u_M^{c\infty}(x,t)| \, |x|^{-\gamma} \, \mathrm{d}x = 0 \quad \forall R > 0, \; \forall t > 0. \tag{6.34}$$

Upon fixing $t = 1$, relabelling λ as t and recalling the definition of u_λ, note that (6.34) reads

$$\lim_{t\to\infty} \int_{B_R} |t^\alpha u(t^\kappa x, t) - u_M^{c\infty}(x,1)| \, |x|^{-\gamma} \, \mathrm{d}x = 0 \quad \forall R > 0.$$

Performing the change of variable $y = t^\kappa x$ and using the fact that $\alpha + \kappa(\gamma - d) = 0$, we obtain:

$$\lim_{t\to\infty} \int_{B_{Rt^\kappa}} \left|u(y,t) - t^{-\alpha} u_M^{c\infty}(t^{-\kappa}y, 1)\right| |y|^{-\gamma} \, \mathrm{d}y = \lim_{t\to\infty} \int_{B_{Rt^\kappa}} |u(y,t) - u_M^{c\infty}(y,t)| \, |y|^{-\gamma} \, \mathrm{d}y = 0 \tag{6.35}$$

for all $R > 0$, where we used (3.2) with $\lambda = t^{-1}$.

From now on we shall denote as ε_R any function of the spatial variable (possibly constant) which is independent of t and vanishes uniformly as $R \to \infty$. Going back to the original variable $x = t^{-\kappa}y$ we find that

$$\int_{B_{Rt^\kappa}^c} u_M^{c\infty}(y,t) \, |y|^{-\gamma} \, \mathrm{d}y = \int_{B_R^c} u_M^{c\infty}(x,1) \, |x|^{-\gamma} \, \mathrm{d}x = \varepsilon_R \quad \forall R > 0. \tag{6.36}$$

Hence, the conservation of mass for $u_M^{c_\infty}$, (6.35) and (6.36) imply that

$$\lim_{t\to\infty} \int_{B_{Rt^\kappa}} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y = Mc_\infty^{-1} - \varepsilon_R \quad \forall R > 0\,. \tag{6.37}$$

Next we show that

$$\lim_{t\to\infty} \int_{\mathbb{R}^d} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y = Mc_\infty^{-1}\,. \tag{6.38}$$

To this end first notice that, thanks to (2.7) and (3.3), there holds

$$|y|^{-\gamma} = \frac{\rho(y)}{c_\infty + \varepsilon_R(y)} \quad \forall y \in B_R^c\,,$$

whence

$$\int_{\mathbb{R}^d} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y = \int_{B_R} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y + \int_{B_R^c} u(y,t)\,\frac{\rho(y)}{c_\infty + \varepsilon_R(y)}\,\mathrm{d}y\,. \tag{6.39}$$

Thanks to (6.39) and the conservation of mass (2.11) for u, we get:

$$\left| \int_{\mathbb{R}^d} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y - Mc_\infty^{-1} \right| = \left| \int_{\mathbb{R}^d} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y - \int_{\mathbb{R}^d} u(y,t)\,\frac{\rho(y)}{c_\infty}\,\mathrm{d}y \right|$$

$$\leq \int_{B_R} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y$$

$$+ \int_{B_R} u(y,t)\,\frac{\rho(y)}{c_\infty}\,\mathrm{d}y + \frac{\|\varepsilon_R\|_\infty}{c_\infty(c_\infty - \|\varepsilon_R\|_\infty)} \int_{B_R^c} u(y,t)\,\rho(y)\,\mathrm{d}y\,. \tag{6.40}$$

Letting $t \to \infty$ in (6.40), using the smoothing effect (2.12) (as a decay estimate) and the fact that both $\rho(y)$ and $|y|^{-\gamma}$ are locally integrable, we obtain:

$$\limsup_{t\to\infty} \left| \int_{\mathbb{R}^d} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y - Mc_\infty^{-1} \right| \leq \frac{M\,\|\varepsilon_R\|_\infty}{c_\infty(c_\infty - \|\varepsilon_R\|_\infty)}\,. \tag{6.41}$$

By letting $R \to \infty$ in (6.41) we get (6.38). Now notice that

$$\int_{\mathbb{R}^d} |u(y,t) - u_M^{c_\infty}(y,t)|\,|y|^{-\gamma}\,\mathrm{d}y \leq \int_{B_{Rt^\kappa}} |u(y,t) - u_M^{c_\infty}(y,t)|\,|y|^{-\gamma}\,\mathrm{d}y$$

$$+ \int_{B_{Rt^\kappa}^c} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y + \int_{B_{Rt^\kappa}^c} u_M^{c_\infty}(y,t)\,|y|^{-\gamma}\,\mathrm{d}y\,. \tag{6.42}$$

Moreover, (6.37) and (6.38) imply that

$$\lim_{t\to\infty} \int_{B_{Rt^\kappa}^c} u(y,t)\,|y|^{-\gamma}\,\mathrm{d}y = \varepsilon_R\,. \tag{6.43}$$

Collecting (6.35), (6.36), (6.42) and (6.43) we finally get

$$\limsup_{t\to\infty} \int_{\mathbb{R}^d} |u(y,t) - u_M^{c_\infty}(y,t)|\,|y|^{-\gamma}\,\mathrm{d}y \leq 2\varepsilon_R\,,$$

whence (3.4) follows by letting $R \to \infty$. The validity of (3.5) is just a consequence of (3.4) and the change of variable $y = t^\kappa x$ (one exploits again the scaling property (3.2) of $u_M^{c_\infty}$). $\square$

Appendix A. Some technical results concerning Riesz potentials

We discuss here some properties of the Riesz potential $I_{2s} * f$ of a function f. To begin with, note that it is straightforward to show that, if $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ is such that

$$\int_{\mathbb{R}^d} \frac{|f(y)|}{1 + |y|^{d-2s}}\,\mathrm{d}y < \infty\,, \tag{A.1}$$

then $I_{2s} * f \in C(\mathbb{R}^d)$. From [33, Theorem 2] (see also [26, Proposition 5.1]) and [26, Remark 5.3] we get the next result.

Proposition A.1. *Let $d \geq 1$ and $r > 1$, with $\frac{2}{r} < 2s < d$. Let*

$$\nu > 2s - \frac{d}{r}. \tag{A.2}$$

Suppose that $f = f(|x|) \in L^\infty_{loc}(\mathbb{R}^d)$, with $f(|x|)|x|^\nu \in L^r(\mathbb{R}^d)$. Then there exists a constant $C > 0$ such that

$$|(I_{2s} * f)(x)| \leq C \, \||x|^\nu f\|_{L^r(\mathbb{R}^d)} \, |x|^{2s-\nu-\frac{d}{r}} \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Corollary A.2. *Let $d > 2s$. Let $\rho \in L^\infty(\mathbb{R}^d)$ be such that $\rho^{-1} \in L^\infty_{loc}(\mathbb{R}^d)$. Suppose moreover that $\rho(x) \leq C_0|x|^{-\gamma}$ in B_1^c for some $\gamma > 2s$ and $C_0 > 0$. Then, $I_{2s} * \rho \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and*

$$(I_{2s} * \rho)(x) \to 0 \quad \text{as } |x| \to \infty.$$

More precisely, for some $C > 0$ we have

$$(I_{2s} * \rho)(x) \leq C|x|^{2s-\nu-\frac{d}{r}} \quad \forall x \in \mathbb{R}^d, \tag{A.3}$$

provided $2s < \nu < \gamma$ and

$$r > \max\left\{\frac{1}{s}, \frac{d}{\gamma - \nu}\right\}. \tag{A.4}$$

Proof. In view of the hypotheses on ρ, we can choose $\widetilde{\rho}(x) = \widetilde{\rho}(|x|) \in C(\mathbb{R}^d)$ such that

$$\rho(x) \leq \widetilde{\rho}(x) \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{A.5}$$

Furthermore, we can and shall assume that $\widetilde{\rho}(x) \leq C_1|x|^{-\gamma}$ in B_1^c for some $\gamma > 2s$ and $C_1 > 0$. Note that $|x|^\nu \widetilde{\rho} \in L^r(\mathbb{R}^d)$ whenever $(\gamma - \nu)r > d$. It is plain that $0 \leq I_{2s} * \rho \leq I_{2s} * \widetilde{\rho}$. In order to apply Proposition A.1 (with $f \equiv \widetilde{\rho}$), we need to find $r > 1$ and $\nu > 0$ such that (A.2) and (A.4) are fulfilled. Since $\gamma > 2s$, there certainly exists ν satisfying $2s < \nu < \gamma$, whence $r > 1$ such that (A.2) and (A.4) hold true. The thesis then follows thanks to (A.5) and the discussion before Proposition A.1 (in view of the assumptions on ρ, the integral (A.1) is clearly finite for $f \equiv \rho$). $\square$

Remark A.3. Since in (A.3) and (A.4) we can choose ν arbitrarily close to γ, in fact we have that, under the same assumptions of Corollary A.2, for all positive ε there exists $C > 0$ such that

$$(I_{2s} * \rho)(x) \leq C|x|^{2s-\gamma+\varepsilon} \quad \forall x \in \mathbb{R}^d.$$

A direct calculation shows however that the above formula also holds for $\varepsilon = 0$. In particular, it is immediate to see that if $\gamma > 4s$ then $I_{2s} * \rho \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d)$. More in general, $I_{2s} * \rho \in L^1_{(1+|x|)^{-\alpha}}(\mathbb{R}^d)$ for all $\alpha > d + 2s - \gamma$.

APPENDIX B. WELL POSEDNESS OF THE PARABOLIC PROBLEM FOR RAPIDLY DECAYING DENSITIES

Throughout this section, we shall use of the same notations as in Section 4.

Part I. If $\rho \in L^\infty_{loc}(\mathbb{R}^d)$ is positive and such that $\rho^{-1} \in L^\infty_{loc}(\mathbb{R}^d)$, u_0 is nonnegative and such that $u_0 \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then we can argue as in the proof of [10, Theorem 7.3 (first construction)] in order to get the existence of a weak solution to problem (1.1), in the sense of Definition 2.1, which is bounded in the whole of $\mathbb{R}^d \times (0, \infty)$. Furthermore, the following L^1_ρ comparison principle holds true:

$$\int_{\mathbb{R}^d} [u_1(x,t) - u_2(x,t)]_+ \, \rho(x)\mathrm{d}x \leq \int_{\mathbb{R}^d} [u_{01} - u_{02}]_+ \, \rho(x)\mathrm{d}x \quad \forall t > 0, \tag{B.1}$$

where u_1 and u_2 are the solutions to problem (1.1), constructed as above, corresponding to the initial data $u_{01} \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $u_{02} \in L^1_\rho(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, respectively.

As for uniqueness, a quite standard result for (suitable) weak solutions to problem (1.1) is the following.

Proposition B.1. *Let $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ be positive and such that $\rho^{-1} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Let u and v be two nonnegative weak solutions to* (1.1), *corresponding to the same nonnegative $u_0 \in L^1_\rho(\mathbb{R}^d)$, in the sense that:*

$$u, v \in L^{m+1}_\rho(\mathbb{R}^d \times (0, \infty)), \tag{B.2}$$

$$u^m, v^m \in L^2_{\mathrm{loc}}([0, \infty); \dot{H}^s(\mathbb{R}^d)) \tag{B.3}$$

and

$$-\int_0^\infty \int_{\mathbb{R}^d} u(x,t)\varphi_t(x,t)\,\rho(x)\mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x,t)(-\Delta)^{\frac{s}{2}}(\varphi)(x,t)\,\mathrm{d}x\mathrm{d}t$$

$$= -\int_0^\infty \int_{\mathbb{R}^d} v(x,t)\varphi_t(x,t)\,\rho(x)\mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(v^m)(x,t)(-\Delta)^{\frac{s}{2}}(\varphi)(x,t)\,\mathrm{d}x\mathrm{d}t \tag{B.4}$$

$$= \int_{\mathbb{R}^d} u_0(x)\varphi(x,0)\,\rho(x)\mathrm{d}x$$

holds true for any $\varphi \in C^\infty_c(\mathbb{R}^d \times [0, \infty))$. Then, $u = v$ a.e. in $\mathbb{R}^d \times (0, \infty)$.

Proof. In view of the hypotheses on u and v, using a standard approximation argument one can show that the so-called *Oleĭnik's test function*

$$\varphi(x,t) = \int_t^T [u^m(x,\tau) - v^m(x,\tau)]\,\mathrm{d}\tau \ \text{ in } \mathbb{R}^d \times (0,T], \quad \varphi = 0 \ \text{ in } \mathbb{R}^d \in (T, \infty),$$

is in fact an admissible test function in the weak formulations (B.4) (for each $T > 0$). The conclusion then follows by arguing exactly as in [10, Theorem 6.1] (see also the subsequent remark). $\qquad\square$

Let us discuss some further properties of the solutions we constructed, which can be proved by means of standard tools. To begin with note that, by proceeding exactly as in [35, Lemma 8.5], one can show that $\rho\, u_t(t)$ is a Radon measure on $\mathbb{R}^d$ satisfying the inequality

$$\|\rho\, u_t(t)\|_{\mathcal{M}(\mathbb{R}^d)} \le \frac{2}{(m-1)\,t}\,\|u_0\|_{1,\rho} \quad \text{for a.e. } t > 0, \tag{B.5}$$

where here, as opposed to Subsection 2.2, with a slight abuse of notation we indicate by $\mathcal{M}(\mathbb{R}^d)$ the Banach space of Radon measures on $\mathbb{R}^d$ endowed with the usual norm of the total variation. Letting $z = u^{\frac{m+1}{2}}$ and following [10, Lemma 8.1] we also get the validity of the estimate

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |z_t(x,t)|^2\,\rho(x)\mathrm{d}x\mathrm{d}t \le C \quad \forall t_2 > t_1 > 0 \tag{B.6}$$

for some positive constant C depending on t_1, t_2 and m. In view of (B.6) and the general result provided by [3, Theorem 1.1] one infers that $u_t \in L^1_{\mathrm{loc}}((0, \infty); L^1_\rho(\mathbb{R}^d))$. Moreover, the inequality

$$\|u_t(t)\|_{1,\rho} \le \frac{2}{(m-1)\,t}\,\|u_0\|_{1,\rho} \quad \text{for a.e. } t > 0 \tag{B.7}$$

holds true as a direct consequence of (B.5). In particular, our solution u is also a *strong solution* to problem (1.1) in the sense of Definition 2.2. The fact that solutions are strong permits to assert that they also solve the differential equation in (1.1), for a.e. $t > 0$, in the L^1 sense. This allows to get the following energy estimate (for the details, see e.g. [18, Sections 4.1 and 4.2]):

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}}(u^m)(x,t)\right|^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{m+1}\int_{\mathbb{R}^d} u^{m+1}(x,t_2)\,\rho(x)\mathrm{d}x = \frac{1}{m+1}\int_{\mathbb{R}^d} u^{m+1}(x,t_1)\,\rho(x)\mathrm{d}x,$$

$$\tag{B.8}$$

for all $t_2 > t_1 > 0$. Furthermore, by suitably exploiting the celebrated Stroock-Varopoulos inequality (see [10, Proposition 8.5] or [18, Section 4.2]), one can show that for any $p \in [1, \infty]$ the L^p_ρ norm of $u(t)$ does not increase in time.

Now suppose that, in addition to the above hypotheses, $\rho \in L^\infty(\mathbb{R}^d)$. Thanks to the latter assumption, from the classical fractional Sobolev embedding (we refer the reader e.g. to the survey paper

[12] and references quoted therein) one immediately deduces the validity of the following weighted, fractional Sobolev inequality:

$$\|v\|_{\frac{2d}{d-2s},\rho} \le \widetilde{C}_S \left\|(-\Delta)^{\frac{s}{2}}(v)\right\|_2 \quad \forall v \in \dot{H}^s(\mathbb{R}^d)\,, \tag{B.9}$$

where $\widetilde{C}_S = \widetilde{C}_S(\|\rho\|_\infty, s, d)$ is a suitable positive constant. By interpolation it is straightforward to check that, as a consequence of (B.9), also the weighted, fractional Nash-Gagliardo-Nirenberg inequality

$$\|v\|_{q,\rho} \le \widetilde{C}_{GN} \left\|(-\Delta)^{\frac{s}{2}}(v)\right\|_2^{\frac{1}{a+1}} \|v\|_{p,\rho}^{\frac{a}{a+1}} \quad \forall v \in L_\rho^p(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d) \tag{B.10}$$

holds true for any $a \ge 0$, $p \ge 1$ and

$$q = \frac{2d(a+1)}{d\frac{a}{p} + d - 2s}\,,$$

where $\widetilde{C}_{GN} = \widetilde{C}_{GN}(\|\rho\|_\infty, a, p, s, d)$ is another suitable positive constant. Taking advantage of (B.10), by means of the same techniques as in [10, Section 8.2] or [18, proof of Proposition 4.6], one can prove the smoothing estimate

$$\|u(t)\|_\infty \le K\, t^{-\alpha_p} \|u_0\|_{p,\rho}^{\beta_p} \quad \forall t > 0\,,\ \forall p \ge 1\,, \tag{B.11}$$

where

$$\alpha_p = \frac{d}{d(m-1)+2sp}\,, \quad \beta_p = \frac{2sp\alpha_p}{d}$$

and $K = K(\|\rho\|_\infty, m, s, d) > 0$.

Still under the additional assumption $\rho \in L^\infty(\mathbb{R}^d)$, it is possible to construct solutions to (1.1) corresponding to any nonnegative data $u_0 \in L_\rho^1(\mathbb{R}^d)$. One proceeds picking a sequence of nonnegative data $u_{0n} \in L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that $u_{0n} \to u_0$ in $L_\rho^1(\mathbb{R}^d)$ and pass to the limit in (2.1) as $n \to \infty$ by exploiting (B.1), (B.8) and (B.11) for $p = 1$ (see also [26, Theorem 6.5 and Remark 6.11]). Such solutions are still strong because the L_ρ^1 comparison principle (B.1) is preserved (which is in fact one of the main tools to prove that solutions are strong – see again [10, Section 8.1] and references quoted). We have therefore proved the existence result contained in Proposition 2.3. As concerns uniqueness, one can reason as follows. Proposition B.1, in particular, ensures that if $u_0 \in L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ then the solution to (1.1) that we constructed above is unique in the class of weak solutions satisfying (B.2), (B.3) and (B.4). Moreover, any weak solution $u(x,t)$ to (1.1), in the sense of Definition 2.1, is such that $u(x, t + \varepsilon)$ is a weak solution to (1.1), corresponding to the initial datum $u_0(x, \varepsilon) \in L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, satisfying (B.2), (B.3) and (B.4), for any $\varepsilon > 0$. Thanks to these properties, one can then proceed exactly as in the proof of [26, Theorem 6.7].

Part II. We describe here another method for constructing weak solutions to problem (1.1). Take again nonnegative initial data $u_0 \in L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and consider the following problem (see also the discussion at the beginning of Section 4):

$$\begin{cases} L_s\left(\tilde{u}_R^m\right) = 0 & \text{in } \Omega_R \times (0, \infty)\,, \\ \tilde{u}_R = 0 & \text{on } \Sigma_R \times (0, \infty)\,, \\ \tilde{u}_R = u_R & \text{on } \Gamma_R \times (0, \infty)\,, \\ \dfrac{\partial\left(\tilde{u}_R^m\right)}{\partial y^{2s}} = \rho\, \dfrac{\partial u_R}{\partial t} & \text{on } \Gamma_R \times (0, \infty)\,, \\ u_R = u_0 & \text{on } B_R \times \{t = 0\}\,. \end{cases} \tag{B.12}$$

Definition B.2. *A weak solution to problem* (B.12) *is a pair of nonnegative functions* $(u_R, \tilde{u}_R)$ *such that:*

- $u_R \in C([0, \infty); L_\rho^1(B_R)) \cap L^\infty(B_R \times (\tau, \infty))$ *for all* $\tau > 0$;
- $\tilde{u}_R^m \in L_{\text{loc}}^2((0, \infty); X_0^s(\Omega_R))$;
- $\tilde{u}_R|_{\Gamma_R \times (0, \infty)} = u_R$;

- *for any $\psi \in C_c^\infty((\Omega_R \cup \Gamma_R) \times (0, \infty))$ there holds*

$$-\int_0^\infty \int_{B_R} u_R(x,t)\, \psi_t(x,0,t)\, \rho(x)\mathrm{d}x\mathrm{d}t + \mu_s \int_0^\infty \int_{\Omega_R} y^{1-2s} \langle \nabla(\tilde{u}_R^m), \nabla\psi \rangle (x,y,t)\, \mathrm{d}x\mathrm{d}y\mathrm{d}t = 0\,;$$

- $\lim_{t\to 0} u_R(t) = u_0|_{B_R}$ *in* $L_\rho^1(B_R)$.

Weak sub– and supersolutions to (B.12) are defined in agreement with Definition B.2. In addition, we say that $(u_R, \tilde{u}_R)$ is a *strong solution* if $(u_R)_t \in L^\infty((\tau, \infty); L_\rho^1(B_R))$ for every $\tau > 0$. By means of the same arguments used in the proof of [10, Theorem 6.2], it is direct to deduce the next comparison principle.

Proposition B.3. *Let $\rho \in L_{\mathrm{loc}}^\infty(\mathbb{R}^d)$ be positive and such that $\rho^{-1} \in L_{\mathrm{loc}}^\infty(\mathbb{R}^d)$. Let $(u_R^{(1)}, \tilde{u}_R^{(1)})$ and $(u_R^{(2)}, \tilde{u}_R^{(2)})$ be a strong subsolution and a strong supersolution, respectively, to problem (B.12). Suppose that $u_R^{(1)} \le u_R^{(2)}$ on $B_R \times \{t = 0\}$ and $\tilde{u}_R^{(1)} \le \tilde{u}_R^{(2)}$ on $\Sigma_R \times (0, \infty)$. Then $u_R^{(1)} \le u_R^{(2)}$ in $B_R \times (0, \infty)$ and $\tilde{u}_R^{(1)} \le \tilde{u}_R^{(2)}$ in $\Omega_R \times (0, \infty)$.*

Making use of quite standard tools (see e.g. [10, 26]), one can prove that for any $R > 0$ and $u_0 \in L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique strong solution $(u_R, \tilde{u}_R)$ to problem (B.12) (in the sense of Definition B.2). Moreover, the limit function $u = \lim_{R\to\infty} u_R$ (note that the family $\{u_R\}$ is monotone in R thanks to Proposition B.3) is nonnegative, bounded in $\mathbb{R}^d \times (0, \infty)$ and such that (B.2), (B.3) and (B.4) hold true. Hence, in view of Proposition B.1, such a u necessarily coincides with the solution constructed in Part I: this in particular ensures that $u \in C([0, \infty), L_\rho^1(\mathbb{R}^d))$. Again, for general data $u_0 \in L_\rho^1(\mathbb{R}^d)$, we can select a sequence $\{u_{0n}\} \subset L_\rho^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that $0 \le u_{0n} \le u_0$ and $u_{0n} \to u_0$ in $L_\rho^1(\mathbb{R}^d)$ and pass to the limit in (2.1) as $n \to \infty$ to get a solution to (1.1) in the sense of Definition 2.1 (which still coincides with the one obtained in Part I).

Finally, we should note that in [10] and [26] the approximating problems are a little different from (B.12) (namely, cylinders in the upper plane are used instead of half-balls). However, this change does not affect the construction of the solution u. Indeed, the present idea of using problem (B.12) is taken from [9, Section 2], where the case $s = 1/2$ and $\rho \equiv 1$ is studied.

Part III. Let us now address the following problem, which is the analogue of (B.12) in the whole upper plane:

$$\begin{cases} L_s(\tilde{u}^m) = 0 & \text{in } \mathbb{R}_+^{d+1} \times (0, \infty)\,, \\ \tilde{u} = u & \text{on } \partial\mathbb{R}_+^{d+1} \times (0, \infty)\,, \\ \dfrac{\partial(\tilde{u}^m)}{\partial y^{2s}} = \rho \dfrac{\partial u}{\partial t} & \text{on } \partial\mathbb{R}_+^{d+1} \times (0, \infty)\,, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{t = 0\}\,. \end{cases} \qquad \text{(B.13)}$$

Definition B.4. *A nonnegative function u is a local weak solution to problem (B.13) corresponding to the nonnegative initial datum $u_0 \in L_\rho^1(\mathbb{R}^d)$ if, for some nonnegative function $\tilde{u}$ such that*

$$\tilde{u}^m \in L_{\mathrm{loc}}^2((0, \infty); X_{\mathrm{loc}}^s) \cap L^\infty(\mathbb{R}_+^{d+1} \times (\tau, \infty)) \quad \forall \tau > 0\,,$$

there hold:

- $u \in C([0, \infty); L_\rho^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (\tau, \infty))$ *for all $\tau > 0$;*
- $\tilde{u}|_{\partial\mathbb{R}_+^{d+1} \times (0, \infty)} = u$;
- *for any $\psi \in C_c^\infty((\mathbb{R}_+^{d+1} \cup \partial\mathbb{R}_+^{d+1}) \times (0, \infty))$,*

$$-\int_0^\infty \int_{\mathbb{R}^d} u(x,t)\, \psi_t(x,0,t)\, \rho(x)\mathrm{d}x\mathrm{d}t + \mu_s \int_0^\infty \int_{\mathbb{R}_+^{d+1}} y^{1-2s} \langle \nabla(\tilde{u}^m), \nabla\psi \rangle (x,y,t)\, \mathrm{d}x\mathrm{d}y\mathrm{d}t \qquad \text{(B.14)}$$

  (in fact $\tilde{u}^m$ is a local extension for u^m);
- *for any $\varphi \in C_c^\infty(\mathbb{R}^d \times (0, \infty))$,*

$$-\int_0^\infty \int_{\mathbb{R}^d} u(x,t)\varphi_t(x,t)\, \rho(x)\mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^d} u^m(x,t)(-\Delta)^s(\varphi)(x,t)\, \mathrm{d}x\mathrm{d}t = 0\,; \qquad \text{(B.15)}$$

- $\lim_{t \to 0} u(t) = u_0$ *in* $L^1_\rho(\mathbb{R}^d)$.

Moreover, we say that u *is a* local strong solution *if, in addition,* $u_t \in L^\infty((\tau, \infty); L^1_{\rho, \text{loc}}(\mathbb{R}^d))$ *for every* $\tau > 0$.

Notice that (B.15) is related to the so-called *very weak* formulation of problem (1.1) (see also Definition B.6 below). For local weak solutions, in general $u^m \notin L^2_{\text{loc}}((0, \infty); \dot{H}^s(\mathbb{R}^d))$. Hence, equivalence between (B.14) and (B.15) cannot be established.

The criterion of Proposition B.1 here is not applicable in order to prove uniqueness. However, it is possible to restore the latter by imposing extra integrability conditions, as stated in Theorem 2.4. In order to prove it, we need some preliminaries. Given a nonnegative $f \in C^\infty_c(\mathbb{R}^d)$, let $h = I_{2s} * f$, so that

$$(-\Delta)^s(h) = f \quad \text{in } \mathbb{R}^d. \tag{B.16}$$

Exploiting the properties of I_{2s} and of the convolution operation, it is not difficult to show that $h \in C^\infty(\mathbb{R}^d)$, $h \geq 0$ and

$$h(x) + |\nabla h(x)| \leq |x|^{-d+2s} \quad \forall x \in \mathbb{R}^d.$$

Now take a cut-off function $\xi \in C^\infty_c(\mathbb{R}^d)$ such that $0 \leq \xi \leq 1$ in $\mathbb{R}^d$, $\xi = 1$ in $B_{1/2}$ and $\xi = 0$ in B^c_1. For any $R > 0$, let

$$\xi_R(x) = \xi\left(\frac{x}{R}\right) \quad \forall x \in \mathbb{R}^d. \tag{B.17}$$

After straightforward computations, we obtain:

$$(-\Delta)^s(h\xi_R)(x) = h(x)(-\Delta)^s(\xi_R)(x) + (-\Delta)^s(h)(x)\,\xi_R(x) + \mathcal{B}(h, \xi_R)(x) \quad \forall x \in \mathbb{R}^d,$$

where $\mathcal{B}(\phi_1, \phi_2)(x)$ is the bilinear form defined as

$$\mathcal{B}(\phi_1, \phi_2)(x) = 2\,C_{s,d} \int_{\mathbb{R}^d} \frac{(\phi_1(x) - \phi_1(y))(\phi_2(x) - \phi_2(y))}{|x - y|^{d+2s}}\,\mathrm{d}y \quad \forall x \in \mathbb{R}^d$$

and $C_{s,d}$ is the positive constant appearing in (1.2). By means of the same techniques used in the proof of [28, Lemma 3.1], we get the following result.

Lemma B.5. *Let* $f \in C^\infty_c(\mathbb{R}^d)$, *with* $f \geq 0$, $h = I_{2s} * f$ *and* ξ_R *be as in* (B.17). *Then, for any* $T > 0$ *and* $v \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (0, T))$, *there holds*

$$\lim_{R \to \infty} \int_0^T \int_{\mathbb{R}^d} |v(x,t)\,h(x)\,(-\Delta)^s(\xi_R)(x)|\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} |v(x,t)\,\mathcal{B}(h, \xi_R)(x)|\,\mathrm{d}x\mathrm{d}t = 0.$$

Having at our disposal Lemma B.5, we are now in position to prove Theorem 2.4.

Proof of Theorem 2.4. Let $\underline{u}$ be the weak solution to problem (1.1) provided by Proposition 2.3. First of all, note that its minimality in the class of solutions described in Definition B.4 (namely *local strong solutions*) is a consequence of the construction outlined in Part II and the comparison principle given in Proposition B.3. Moreover, estimates (2.3) and (2.4) can be obtained using the same arguments as in [26, Theorem 5.5], combined with the smoothing effect (B.11) and [26, Remark 6.11].

(i) The thesis is a consequence of estimate (2.4), Remark A.3 and the method of proof of [26, Theorems 6.9 and 6.10], which here can be exploited with inessential modifications in view of (B.11) and [26, Remark 6.11]. For further references, let us just mention that it is appropriate to take as a test function in (B.15) $\varphi(x,t) = \xi_R(x)\eta_\epsilon(t)$, where $\eta_\epsilon(t)$ properly tends to $\chi_{[t_1,t_2]}(t)$ as $\epsilon \to 0$ (let $t_2 > t_1 > 0$ be fixed).

(ii) We claim that $\underline{u}^m \in L^1_{(1+|x|)^{-d+2s}}(\mathbb{R}^d \times (0, T))$ for all $T > 0$: in fact, this follows immediately from estimate (2.4) and Remark A.3. Let u be another bounded local strong solution to (1.1). Since, by definition, both u and $\underline{u}$ belong to

$$C([0, \infty), L^1_{\rho, \text{loc}}(\mathbb{R}^d)) \cap L^\infty((0, \infty) \times \mathbb{R}^d),$$

it is direct to see that for any $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$ there holds

$$
\begin{aligned}
&- \int_0^\infty \int_{\mathbb{R}^d} u(x,t)\varphi_t(x,t)\,\rho(x)\mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^d} u^m(x,t)(-\Delta)^s(\varphi)(x,t)\,\mathrm{d}x\mathrm{d}t \\
&= - \int_0^\infty \int_{\mathbb{R}^d} \underline{u}(x,t)\varphi_t(x,t)\,\rho(x)\mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^d} \underline{u}^m(x,t)(-\Delta)^s(\varphi)(x,t)\,\mathrm{d}x\mathrm{d}t \qquad (\text{B.18}) \\
&= \int_{\mathbb{R}^d} u_0(x)\varphi(x,0)\,\rho(x)\mathrm{d}x\,.
\end{aligned}
$$

Let $\eta \in C^\infty(\mathbb{R})$ be such that

$$
\eta = 1 \ \text{ in } \ (-\infty, 0]\,, \ \eta = 0 \ \text{ in } \ [1, \infty)\,, \ 0 \le \eta \le 1\,, \ \eta' \le 0 \ \text{ in } \ \mathbb{R}\,.
$$

For any $\epsilon \in (0, T)$ and $\tau \in (0, T - \epsilon)$, set

$$
\eta_{\epsilon\tau}(t) = \eta\left(\frac{t - \tau}{T - \epsilon - \tau}\right) \quad \forall t > 0\,.
$$

Now take the test function

$$
\varphi(x,t) = h(x)\xi_R(x)\eta_{\epsilon\tau}(t) \quad \forall (x,t) \in \mathbb{R}^d \times (0, \infty)
$$

(h is given by (B.16)) and plug it into the weak formulation solved by $u - \underline{u}$ (according to (B.18)). We get:

$$
\begin{aligned}
&\int_0^{T-\epsilon} \int_{\mathbb{R}^d} f(x)\xi_R(x)\eta_{\epsilon\tau}(t)[u^m(x,t) - \underline{u}^m(x,t)]\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^{T-\epsilon} \int_{\mathbb{R}^d} h(x)\xi_R(x)\eta'_{\epsilon\tau}(t)[u(x,t) - \underline{u}(x,t)]\,\rho(x)\mathrm{d}x\mathrm{d}t \qquad (\text{B.19}) \\
&\quad - \int_0^{T-\epsilon} \int_{\mathbb{R}^d} [h(x)(-\Delta)^s(\xi_R)(x) + \mathcal{B}(h, \xi_R)(x)]\,\eta_{\epsilon\tau}(t)\,[u^m(x,t) - \underline{u}^m(x,t)]\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}
$$

Because $\underline{u} \le u$ and $\eta'_{\epsilon\tau} \le 0$, from (B.19) we deduce that

$$
\begin{aligned}
0 &\le \int_0^{T-\epsilon} \int_{\mathbb{R}^d} f(x)\xi_R(x)\eta_{\epsilon\tau}(t)[u^m(x,t) - \underline{u}^m(x,t)]\,\mathrm{d}x\mathrm{d}t \\
&\le \int_0^{T-\epsilon} \int_{\mathbb{R}^d} |h(x)(-\Delta)^s(\xi_R)(x) + \mathcal{B}(h, \xi_R)(x)|\,[u^m(x,t) + \underline{u}^m(x,t)]\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}
\qquad (\text{B.20})
$$

Letting $R \to \infty$ in (B.20) and applying Lemma B.5 to the r.h.s., with the choice $v = u^m + \underline{u}^m$ (recall that by hypothesis (2.6) holds both for u and $\underline{u}$), we infer that $u = \underline{u}$ in the region $\mathrm{supp}\, f \times (0, T - \epsilon)$. Thanks to the arbitrariness of f, T and ϵ we finally obtain that $u = \underline{u}$ in the whole of $\mathbb{R}^d \times (0, \infty)$. $\quad\square$

Let us consider the next definition of *very weak* solution to problem (1.1).

Definition B.6. *A nonnegative function $u \in L^\infty(\mathbb{R}^d \times (0, \infty))$ is a very weak solution to problem (1.1) corresponding to the nonnegative initial datum $u_0 \in L^\infty(\mathbb{R}^d)$ if, for any $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$, (B.18) holds true.*

Clearly, any *bounded* weak solution to (1.1) (according to Definition 2.1) is also a very weak solution in the sense of Definition B.6.

Remark B.7. From the proof of the uniqueness results in Theorem 2.4, it follows that if u_1 and u_2 are *very weak* solutions to problem to (1.1) (in the sense of Definition B.6), having the same integrability properties required in the hypotheses of Theorem 2.4 and such that $u_1 \le u_2$ a.e. in $\mathbb{R}^d \times (0, \infty)$, then $u_1 \equiv u_2$. We should note that, in case (i), a minor change in the proof is needed since Definition B.6 does not imply that $\lim_{t \to 0} u(t) = u_0$ in $L^1_{\rho, \mathrm{loc}}(\mathbb{R}^d)$. However, the initial condition is taken in the sense of (B.18): this is enough in order to repeat the proof of Theorem 2.4-(i) (to be specific, it suffices to choose $t_1 = 0$ when we define the test function η_ε).

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