DIPARTIMENTO DI MATEMATICA
"Francesco Brioschi"
POLITECNICO DI MILANO

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Collezione dei *Quaderni di Dipartimento*, numero **QDD 171**
Inserito negli *Archivi Digitali di Dipartimento* in data 17-1-2014

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Torsional instability in a fish-bone model for suspension bridges

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Abstract

We consider a mathematical model for the study of the dynamical behavior of suspension bridges. We show that internal resonances, which depend on the bridge structure only, are the origin of torsional instability. We obtain both theoretical and numerical estimates of the thresholds of instability. Our method is based on a finite dimensional projection of the phase space which reduces the stability analysis of the model to the stability of suitable nonlinear Hill equations. This gives an answer to a long-standing question about the origin of torsional instability.
Keywords: suspension bridges, torsional stability, Hill equation.
Mathematics Subject Classification: 37C75, 35G31, 34C15.

1 Introduction

The collapse of the Tacoma Narrows Bridge, which occurred in 1940, raised many questions about the stability of suspension bridges. In particular, since the Federal Report [1] considers *the crucial event in the collapse to be the sudden change from a vertical to a torsional mode of oscillation*, see also [31], a natural question appears to be:

why do torsional oscillations appear suddenly in suspension bridges? (**Q**)

The main purpose of the present paper is to try to give an answer to **(Q)** by analyzing a suitable mathematical model. We are here concerned with the main span, namely the part of the roadway between the towers, which has a rectangular shape with two long edges (of the order of 1km) and two shorter edges (of the order of 20m) fixed and hinged between the towers. Due to the large discrepancy between these measures we model the roadway as a degenerate plate, that is, a beam representing the midline of the roadway with cross sections which are free to rotate around the beam. We call this model a *fish-bone*, see Figure 1. The grey part is the roadway, the two black cross sections are

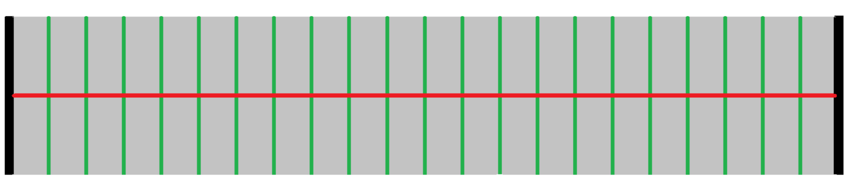

Figure 1: The model of a fish-bone plate.

between the towers, they are fixed and the plate is hinged there. The red line contains the barycenters of the cross sections and is the line where the downwards vertical displacement y is computed. The green orthogonal lines are virtual cross sections seen as rods that can rotate around their barycenter, the angle of rotation with respect to the horizontal position being denoted by θ. We assume that the roadway has length L and width 2ℓ with $2\ell \ll L$. The kinetic energy of a rotating object is $\frac{1}{2}J\dot{\theta}^2$, where J is the moment of inertia and $\dot{\theta}$ is the angular velocity. The moment of inertia of a rod of length 2ℓ about the perpendicular axis through its center is given by $\frac{1}{3}M\ell^2$ where M is the mass of the rod. Hence, the kinetic energy of a rod having mass M and half-length ℓ, rotating about its center with angular velocity $\dot{\theta}$, is given by $\frac{M}{6}\ell^2\dot{\theta}^2$. On the other hand, the bending energy of the beam depends on its curvature and this leads to a fourth order equation, see [5]. Note that M is also the mass per unit length in the longitudinal direction. The hangers are prestressed and the equilibrium position of the midline is $y = 0$, recall that $y > 0$ corresponds to a downwards displacement of the midline. The equations for this system read

$$\begin{cases} My_{tt} + EIy_{xxxx} + f(y + \ell \sin\theta) + f(y - \ell \sin\theta) = 0 & 0 < x < L \quad t > 0 \\ \frac{M\ell^2}{3}\theta_{tt} - \mu\ell^2\theta_{xx} + \ell\cos\theta\,(f(y + \ell\sin\theta) - f(y - \ell\sin\theta)) = 0 & 0 < x < L \quad t > 0, \end{cases} \tag{1}$$

where $\mu > 0$ is a constant depending on the shear modulus and the moment of inertia of the pure torsion, $EI > 0$ is the flexural rigidity of the beam, f represents the restoring action of the prestressed hangers and therefore also includes the action of gravity. We have not yet simplified by ℓ the second equation in (1) in order to emphasize all the terms.

To (1) we associate the following boundary-initial conditions:

$$y(0,t) = y_{xx}(0,t) = y(L,t) = y_{xx}(L,t) = \theta(0,t) = \theta(L,t) = 0 \qquad t \geq 0 \tag{2}$$

$$y(x,0) = \eta_0(x)\,, \quad y_t(x,0) = \eta_1(x)\,, \quad \theta(x,0) = \theta_0(x)\,, \quad \theta_t(x,0) = \theta_1(x) \qquad 0 < x < L\,. \tag{3}$$

The first four boundary conditions in (2) model a beam hinged at its endpoints whereas the last two boundary conditions model the fixed cross sections between towers.

In a slightly different setting, involving mixed space-time fourth order derivatives, a *linear* version of (1) was first suggested by Pittel-Yakubovich [27], see also [35, Chapter VI]; this model, with the addition of an external forcing representing the wind, was studied with a parametric resonance approach and an instability was found for a sufficiently large action of the wind. This approach received severe criticisms from engineers [29, p.841], see also [4, 18] for the physical point of view. The reason is that "too much importance is attributed to the action of the wind" as if some kind of forced resonance would be involved. And it is clear that, in a windstorm, a precise phenomenon such as forced resonance is quite unlikely to be seen [20, Section 1]. More recently, Moore [25] considered (1) with

$$f(s) = k\left[\left(s + \frac{Mg}{2k}\right)^+ - \frac{Mg}{2k}\right],$$

a nonlinearity which models hangers behaving as linear springs of elastic constant $k > 0$ if stretched but exert no restoring force if compressed; here g is gravity. This nonlinearity, first suggested by McKenna-Walter [24], describes the possible slackening of the hangers (occurring for $s \leq -\frac{Mg}{2k}$) which was observed during the Tacoma Bridge collapse, see [1, V-12]. But Moore considers the case where the hangers do not slacken: then f becomes *linear*, $f(s) = ks$, and the two equations in (1) decouple. In this situation there is obviously no interaction between vertical and torsional oscillations and, consequently, no possibility to give an answer to **(Q)**.

The demand for reliable nonlinear models from the engineers dates of about half a century ago. For instance, Robinson-West [34, p.15] write: *Actually, some linear theories serve a simple introduction to some of the essential problems of the stiffened suspension bridge. Nevertheless, all major modern bridges are such that linear theories are unacceptable.* And it is nowadays established that suspension bridges behave nonlinearly, see [7, 8, 13, 19] and references therein. Whence, nonlinear restoring forces f in (1) appear unavoidable if one wishes to have a realistic model. A nonlinear f was introduced in (1) by Holubová-Matas [16] who were able to prove well-posedness for a forced-damped version of (1).

For a slightly different model, numerical results obtained by McKenna [21] were able to show a sudden development of large torsional oscillations as soon as the hangers lose tension, that is, as soon as the restoring force becomes nonlinear. Further numerical results by Doole-Hogan [10] and McKenna-Tuama [23] show that a purely vertical periodic forcing may create a torsional response. An answer to **(Q)** was recently given in [2] by using suitable Poincaré maps for a suspension bridge modeled by several coupled (second order) nonlinear oscillators. When enough energy is present within the structure a resonance may occur, leading to an energy transfer between oscillators. The results in [2] are, again, purely numerical. Whence, so far no theoretical explanation of the origin of torsional oscillations has been given, nor any effective way to estimate the conditions which may create torsional instability. This naturally leads to the following question (see [22, Problem 7.4]): *can one employ the tools of nonlinear analysis to say anything further in terms of stability*?

In this paper we consider the fish-bone model and we display the same phenomenon of sudden transition from purely vertical to torsional oscillations. Let us mention that a somehow related behavior of self-excited oscillations is visible in nonlinear beam equations, see [3, 14]. Here we provide a detailed theoretical explanation of how internal resonances occur in (1), yielding instability.

In Theorem 1 we prove well-posedness of (1)-(2)-(3) for a wide class of nonlinearities f. The proof is based on a Galerkin method which enables us to project (1) on a finite dimensional subspace of the phase space and to study the instability of the vertical oscillating modes in terms of suitable Hill equations [15]. This procedure is motivated by classical engineering literature. Bleich-McCullough-Rosecrans-Vincent [6, p.23] write that *out of the infinite number of possible modes of motion in which a suspension bridge might vibrate, we are interested only in a few, to wit: the ones having the smaller numbers of loops or half waves.* The physical reason why only low modes should be considered is that higher modes require large bending energy; this is well explained by Smith-Vincent [32, p.11] who write that *the higher modes with their shorter waves involve sharper curvature in the truss and, therefore, greater bending moment at a given amplitude and accordingly reflect the influence of the truss stiffness to a greater degree than do the lower modes.* The suggestion to restrict attention to lower modes, mathematically corresponds to project an infinite dimensional phase space on a finite dimensional subspace, a technique which should be attributed to Galerkin [12]. This projection enables us to determine both theoretical and numerical bounds for stability, see Sections 3 and 7, and to explain the origin of torsional instability.

The obtained results yield the following answer to question **(Q)**. The onset of large torsional oscillations is due to a resonance which generates an energy transfer between different oscillation modes. When the bridge is oscillating vertically with sufficiently large amplitude, part of the energy is suddenly transferred to a torsional mode giving rise to wide torsional oscillations. Estimates of “large amplitudes” may be obtained theoretically.

2 Simplification of the model and well-posedness

It is not our purpose to give the precise quantitative behavior of the model under consideration. Therefore, in this section we make several simplifications which do not modify the qualitative behavior of the nonlinear system (1). First of all, up to scaling we may assume that $L = \pi$; this will simplify the Fourier series expansion. Then we take $EI = 3\mu = 1$ although these parameters may be fairly different in actual bridges. Moreover, we are not interested in describing accurately the behavior of the bridge under large torsional oscillations. Instead, we are willing to describe how small torsional oscillations may suddenly become larger ones. And for small θ the following approximations are legitimate

$$\cos\theta \cong 1 \quad \text{and} \quad \sin\theta \cong \theta\,.$$

Then we set $z := \ell\theta$ and this cancels the dependence of (1) on the width ℓ; to recover the dependence on ℓ, note that $\theta = \frac{z}{\ell}$ so that smaller ℓ yield larger θ, that is, less stability. Finally, note that the change of variable $t \mapsto \sqrt{M}t$ results in a positive or negative delay in the occurrence of any (possibly catastrophic) phenomenon; whence, we may take $M = 1$. After all these changes, (1) becomes

$$\begin{cases} y_{tt} + y_{xxxx} + f(y+z) + f(y-z) = 0 & (0 < x < \pi,\ t \geq 0) \\ z_{tt} - z_{xx} + 3f(y+z) - 3f(y-z) = 0 & (0 < x < \pi,\ t \geq 0)\,. \end{cases} \tag{4}$$

The boundary-initial conditions (2)-(3) may be rewritten as

$$y(0,t) = y_{xx}(0,t) = y(\pi,t) = y_{xx}(\pi,t) = z(0,t) = z(\pi,t) = 0 \qquad (t \geq 0)\,, \tag{5}$$

$$y(x,0) = \eta_0(x)\,, \quad y_t(x,0) = \eta_1(x)\,, \quad z(x,0) = \zeta_0(x)\,, \quad z_t(x,0) = \zeta_1(x) \qquad (0 < x < \pi)\,, \tag{6}$$

where $\zeta_0(x) := \ell\theta_0(x)$ and $\zeta_1(x) := \ell\theta_1(x)$. If f is nondecreasing, as in the physical situation, then

$$F(s) := \int_0^s f(\tau)\,d\tau > 0 \tag{7}$$

is a convex function. Therefore, the convex and coercive functional (here $' = \frac{d}{dx}$)

$$J(y,z) = \frac{\|y''\|_2^2}{2} + \frac{\|z'\|_2^2}{6} + \int_0^\pi [F(y+z) + F(y-z)]\,dx \qquad \big(y \in H^2 \cap H^1_0(0,\pi)\,,\ z \in H^1_0(0,\pi)\big)$$

admits a unique critical point, which is the absolute minimum and coincides with $(y,z) = (0,0)$; here and in the sequel $\|\cdot\|_2$ denotes the $L^2(0,\pi)$-norm. Hence, (4) admits a unique stationary solution (equilibrium) given by $y = z = 0$ and corresponding to the initial conditions $\eta_0 = \eta_1 = \zeta_0 = \zeta_1 = 0$.

We say that the functions

$$y \in C^0(\mathbb{R}_+; H^2 \cap H^1_0(0,\pi)) \cap C^1(\mathbb{R}_+; L^2(0,\pi)) \cap C^2(\mathbb{R}_+; H^*(0,\pi))$$

$$z \in C^0(\mathbb{R}_+; H^1_0(0,\pi)) \cap C^1(\mathbb{R}_+; L^2(0,\pi)) \cap C^2(\mathbb{R}_+; H^{-1}(0,\pi))$$

are solutions to (4)-(5)-(6) if they satisfy the initial conditions (6) and if

$$\langle y_{tt}, \varphi\rangle_{H^*} + (y_{xx}, \varphi'') + (f(y-z) + f(y+z), \varphi) = 0 \quad \forall\,\varphi \in H^2 \cap H^1_0(0,\pi)\,, \forall t > 0\,,$$

$$\langle z_{tt}, \psi\rangle_{H^{-1}} + (z_x, \psi') + 3(f(y+z) - f(y-z), \psi) = 0 \quad \forall\,\psi \in H^1_0(0,\pi)\,, \forall t > 0\,,$$

where $\langle\cdot,\cdot\rangle_{H^{-1}}$ and $\langle\cdot,\cdot\rangle_{H^*}$ are the duality pairings in $H^{-1} = (H^1_0(0,\pi))'$ and $H^* = (H^2 \cap H^1_0(0,\pi))'$ while $(\cdot,\cdot)$ denotes the scalar product in $L^2(0,\pi)$. We have

Theorem 1. *Let $\eta_0 \in H^2 \cap H^1_0(0,\pi)$, $\zeta_0 \in H^1_0(0,\pi)$, $\eta_1, \zeta_1 \in L^2(0,\pi)$. Assume that $f \in \mathrm{Lip}_{loc}(\mathbb{R})$ is nondecreasing, with $f(0) = 0$, and $|f(s)| \leq C(1 + |s|^p)$ for every $s \in \mathbb{R} \setminus \{0\}$ and for some $p \geq 1$. Then there exists a unique solution (y,z) to* (4)-(5)-(6).

The proof of Theorem 1 is given in Section 4. It is based on a Galerkin procedure which suggests to approximate (4) with a finite dimensional system. In the next section, we study in some detail these approximate systems.

3 Finite dimensional torsional stability

3.1 Approximated n-modes solutions

Consider the solution (y,z) to (4)-(5)-(6), as given by Theorem 1, and let us expand it in Fourier series with respect to x:

$$y(x,t)=\sum_{j=1}^{\infty}y_j(t)\sin(jx)\ ,\quad z(x,t)=\sum_{j=1}^{\infty}z_j(t)\sin(jx)\ ,\tag{8}$$

where the functions y_j and z_j are the unknowns. We focus the attention to the lowest n oscillatory modes, $j\le n$. As already mentioned in the introduction, this finite dimensional approximation is physically justified. We make a finite dimensional projection of (4), that is, we fix an integer n and consider the space $X_n:=\mathrm{span}\{\sin(jx)\}_{j=1}^{n}$. Let P_n denote the orthogonal projection from $H^2\cap H^1_0(0,\pi)$ onto X_n. We say that the functions $y^n,z^n\in C^2(\mathbb{R}_+;X_n)$ are **approximated n-mode solutions** to (4)-(5)-(6) if $y^n(\cdot,t),z^n(\cdot,t)\in X_n$ for all $t\ge 0$ and if they satisfy the initial conditions

$$y^n(x,0)=P_n\eta_0(x),\ y^n_t(x,0)=P_n\eta_1(x),\ z^n(x,0)=P_n\zeta_0(x),\ z^n_t(x,0)=P_n\zeta_1(x)\quad(0<x<\pi).\tag{9}$$

Fix $n\ge 1$. Approximated n-mode solutions take the form

$$y^n(x,t)=\sum_{j=1}^{n}y_j(t)\sin(jx)\ ,\quad z^n(x,t)=\sum_{j=1}^{n}z_j(t)\sin(jx)\tag{10}$$

where the unknowns y_j and z_j solve a system of $2n$ second order ODE's and also depend on n.

Put $(Y,Z):=(y_1,...,y_n,z_1,...,z_n)\in\mathbb{R}^{2n}$ and

$$U(Y,Z):=\frac{2}{\pi}\int_0^{\pi}\Big(F(y^n+z^n)+F(y^n-z^n)\Big)\,dx\,,$$

with $F(s)=\int_0^s f(\tau)\,d\tau$. In order to project (4) onto X_n we multiply the equations there by $\sin(jx)$ $(j=1,...,n)$ and we integrate over $(0,\pi)$. After some integrations by parts we obtain the system

$$\begin{cases}\ddot y_j(t)+j^4y_j(t)+U_{y_j}(Y,Z)=0\\ \ddot z_j(t)+j^2z_j(t)+3\,U_{z_j}(Y,Z)=0\end{cases}\qquad(j=1,...,n)\,.\tag{11}$$

Due to the integration, the dependence on x is lost and the boundary conditions have disappeared. To (11) we associate the initial conditions

$$Y(0)=Y_0\,,\quad \dot Y(0)=Y_1\,,\quad Z(0)=Z_0\,,\quad \dot Z(0)=Z_1\,,\tag{12}$$

where the components of the n-vector Y_0 are the n Fourier coefficients of $P_n\eta_0$; similarly for Y_1, Z_0, Z_1. The conserved total energy of (11) is given by

$$E:=\frac{|\dot Y|^2}{2}+\frac{|\dot Z|^2}{6}+\frac12\sum_{j=1}^{n}j^4y_j^2+\frac16\sum_{j=1}^{n}j^2z_j^2+U(Y,Z)\,.\tag{13}$$

Note that (13), together with (7), yields the boundedness of the L^∞-norm of each of the y_j, $\dot y_j$, z_j and $\dot z_j$. In order to attempt an answer to the main question **(Q)**, we choose arbitrary data Y_0 and Y_1 and "small" data Z_0 and Z_1. The initial (and constant) energy (13) is then approximately given by

$$E\approx\frac{|Y_1|^2}{2}+\frac12\sum_{j=1}^{n}j^4y_j(0)^2+U(Y_0,0)\,.$$

In the next subsections we will show that:

if the energy E in (13) *is small enough then small initial torsional oscillations remain small for all time $t > 0$, whereas if E is large (that is, the vertical oscillations are initially large) then small torsional oscillations suddenly become wider.*

Therefore, a crucial role is played by the amount of energy inside the system (4). This will be analyzed within (11), that is, in the finite dimensional projection of (4). To obtain precise results, we consider a specific nonlinearity f satisfying the assumptions of Theorem 1. Since our purpose is merely to describe the qualitative phenomenon, the choice of the nonlinearity is not important; it is shown in [2] that several different nonlinearities yield the same qualitative behavior for the solutions. We take

$$f(s) = s + \gamma s^3 \quad \text{for } \gamma > 0\,, \tag{14}$$

which allows to simplify several computations. Let us also mention that Plaut-Davis [28, Section 3.5] make the same choice and that this nonlinearity appears in several elastic contexts, see e.g. [17, (1)].

The parameter γ measures how far is f from a linear function. In Sections 3.2-3.3 (and 7) we consider the case $\gamma = 1$, that is,

$$f(s) = s + s^3 \tag{15}$$

while we postpone the discussion for general $\gamma > 0$ until Section 3.4. When f is as in (15) we have

$$f(y+z) + f(y-z) = 2y(1+y^2+3z^2) \quad \text{and} \quad f(y+z) - f(y-z) = 2z(1+3y^2+z^2) \tag{16}$$

so that (4) reduces to

$$\begin{cases} y_{tt} + y_{xxxx} + 2y(1+y^2+3z^2) = 0 & (0 < x < \pi,\ t \geq 0) \\ z_{tt} - z_{xx} + 6z(1+3y^2+z^2) = 0 & (0 < x < \pi,\ t \geq 0)\,. \end{cases} \tag{17}$$

In order to determine the projected system, we need to multiply by $\sin(jx)$ ($j = 1, ..., n$) the equations in (17) and then integrate over $(0, \pi)$. Therefore, we will intensively exploit the following

Lemma 2. *Let $a, b, c \in \mathbb{N}$ be such that $a + b + c = 4$ and let*

$$I(a,b,c) = \frac{1}{\pi}\int_0^\pi \sin^a(x)\sin^b(2x)\sin^c(3x)\,dx\,.$$

Then $I(3,1,0) = I(2,1,1) = I(1,3,0) = I(1,1,2) = I(1,0,3) = I(0,3,1) = I(0,1,3) = 0$ *and*

$$I(4,0,0) = I(0,4,0) = I(0,0,4) = \tfrac{3}{8},\ I(2,2,0) = I(2,0,2) = I(0,2,2) = \tfrac{1}{4},\ I(1,2,1) = -I(3,0,1) = \tfrac{1}{8}.$$

3.2 The 1-mode system

By (10), the approximated 1-mode solutions have the form

$$y(x,t) = y_1(t)\sin x\ , \qquad z(x,t) = z_1(t)\sin x\ .$$

Then, by (16) and Lemma 2, (11) reads

$$\begin{cases} \ddot{y}_1 + 3y_1 + \frac{3}{2}y_1^3 + \frac{9}{2}y_1 z_1^2 = 0 \\ \ddot{z}_1 + 7z_1 + \frac{9}{2}z_1^3 + \frac{27}{2}z_1 y_1^2 = 0\ , \end{cases} \tag{18}$$

with some initial conditions

$$y_1(0) = \eta_0\,,\ \dot{y}_1(0) = \eta_1\,,\ z_1(0) = \zeta_0\,,\ \dot{z}_1(0) = \zeta_1\ . \tag{19}$$

We are interested in determining the stability of the solution $z_1 \equiv 0$ corresponding to $\zeta_0 = \zeta_1 = 0$. Let us make precise what is meant by this.

Fix η_0 and η_1. If we take $\zeta_0 = \zeta_1 = 0$, then the unique solution to (18)-(19) is $(y_1, z_1) = (y^p, 0)$ with $y^p = y^p(\eta_0, \eta_1)$ being the unique (periodic) solution of the autonomous equation

$$\ddot{y} + 3y + \frac{3}{2}y^3 = 0 \,, \qquad y(0) = \eta_0 \,,\ \dot{y}(0) = \eta_1 \,, \tag{20}$$

which admits the conserved quantity

$$E = \frac{\dot{y}^2}{2} + \frac{3}{2}y^2 + \frac{3}{8}y^4 \equiv \frac{\eta_1^2}{2} + \frac{3}{2}\eta_0^2 + \frac{3}{8}\eta_0^4 \,. \tag{21}$$

The standard procedure to deduce the stability of $(y^p, 0)$ consists in studying the behavior of the perturbed vector $(y_1 - y^p, z_1)$ where (y_1, z_1) solves (18), see [33, Chapter 5]. This leads to linearize the system (18) around $(y^p, 0)$ and, subsequently, to apply the Floquet theory for differential equations with periodic coefficients. The torsional component z of the linearization of system (18) around $(y^p, 0)$ satisfies the following Hill equation [15]:

$$\ddot{z} + a(t)z = 0 \quad \text{with} \quad a(t) = 7 + \frac{27}{2}y^p(t)^2 \,. \tag{22}$$

Then we state

Definition 3. *We say that the periodic solution $(y^p, 0)$ to (18) at energy E is* **torsionally stable** *if the trivial solution $z \equiv 0$ of (22) is stable.*

By exploiting a stability criterion given in [36] for equation (22), in Section 5 we prove

Theorem 4. *The solution $(y^p, 0)$ to (18)-(19) is torsionally stable provided that*

$$\|y^p\|_\infty \le \sqrt{\frac{10}{21}} \approx 0.69$$

or, equivalently, provided that the conserved energy E in (21) satisfies

$$E \le \frac{235}{294} \approx 0.799 \,.$$

As already remarked, Definition 3 is the usual one. Nevertheless, the stability results obtained in [26] for suitable *nonlinear* Hill equations suggest that different equivalent definitions can be stated, possibly not involving a linearization process. In particular, by [26] we know that the stability of the trivial solution to (22) implies the stability of the trivial solution to

$$\ddot{z} + a(t)z + \frac{9}{2}z^3 = 0 \quad \text{with} \quad a(t) = 7 + \frac{27}{2}y^p(t)^2 \,.$$

We also refer to [9] and references therein for stability results for nonlinear first order planar systems. As far as we are aware, there is no general theory for nonlinear systems of any number of equations but it is reasonable to expect that similar results might hold. This is why, in our numerical experiments, we consider system (18) without any linearization. The below numerical results suggest that the threshold of instability is larger than the one in Theorem 4. Clearly, they only give a "local stability" information (for finite time), but the observed phenomenon is very precise and the thresholds of torsional instability are determined with high accuracy. The pictures in Figure 2 display the plots of the solutions to (18) with initial data

$$y_1(0) = \|y_1\|_\infty = 10^4 z_1(0) \,, \quad \dot{y}_1(0) = \dot{z}_1(0) = 0 \tag{23}$$

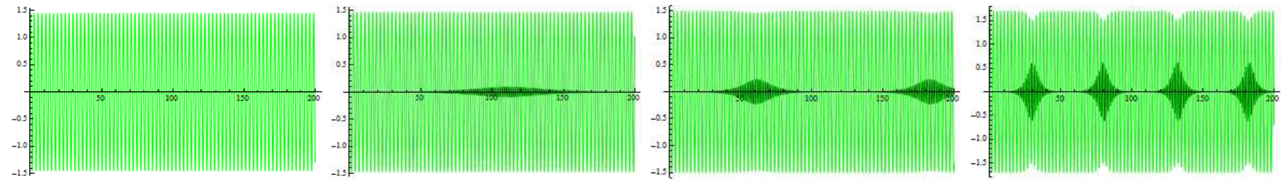

Figure 2: On the interval $t \in [0, 200]$, plot of the solutions y_1 (green) and z_1 (black) to (18)-(23) for $\|y_1\|_\infty = 1.45,\ 1.47,\ 1.5,\ 1.7$ (from left to right).

for different values of $\|y_1\|_\infty$. The green plot is y_1 and the black plot is z_1. For $\|y_1\|_\infty = 1.45$ no wide torsion appears, which means that the solution $(y_1, 0)$ is torsionally stable. For $\|y_1\|_\infty = 1.47$ we see a sudden increase of the torsional oscillation around $t \approx 50$. Therefore, the stability threshold for the vertical amplitude of oscillation lies in the interval $[1.45, 1.47]$. Finer experiments show that the threshold is $\|y_1\|_\infty \approx 1.46$, corresponding to a critical energy of about $E \approx 4.9$: these values should be compared with the statement of Theorem 4. When the amplitude is increased further, for $\|y_1\|_\infty = 1.5$ and $\|y_1\|_\infty = 1.7$, the appearance of wide torsional oscillations is anticipated (earlier in time) and amplified (larger in magnitude). This phenomenon continues to increase for increasing $\|y_1\|_\infty$. We then tried different initial data with $\dot{y}_1(0) \neq 0$; as expected, the sudden appearance of torsional oscillations always occurs at the energy level $E \approx 4.9$, no matter of how it is initially distributed between kinetic and potential energy of y_1. Summarizing, we have seen that the "true" thresholds are larger than the ones obtained in Theorem 4.

Let us try to give a different point of view of this phenomenon. If we slightly modify the parameters involved we can prove that the nonlinear frequency of z_1 is larger than the frequency of y_1, which shows that the mutual position of z_1 and y_1 varies and may create the spark for an energy transfer. Instead of $EI = 3\mu = 1$, we take $EI = \mu = 1$ and (18) should then be replaced by

$$\begin{cases} \ddot{y}_1 + 3y_1 + \frac{3}{2}y_1^3 + \frac{9}{2}y_1 z_1^2 = 0 \\ \ddot{z}_1 + 9z_1 + \frac{9}{2}z_1^3 + \frac{27}{2}z_1 y_1^2 = 0\ . \end{cases} \tag{24}$$

Then we prove

Proposition 5. *Let (y_1, z_1) be a nontrivial solution of (24). Let $t_1 < t_2$ be two consecutive critical points of $y_1(t)$. Then there exists $\tau \in (t_1, t_2)$ such that $z_1(\tau) = 0$.*

Proof. For any solution (y_1, z_1) of (24) we have

$$\frac{d}{dt}\Big[\dot{y}_1^3 \dot{z}_1\Big] + 9\frac{d}{dt}\left[y_1 z_1 + \frac{y_1 z_1^3}{2} + \frac{z_1 y_1^3}{2}\right]\dot{y}_1^2 = 0\,. \tag{25}$$

By integrating (25) by parts over (t_1, t_2) we obtain

$$\begin{aligned} 0 &= 9\int_{t_1}^{t_2} \frac{d}{dt}\left[y_1(t)z_1(t) + \frac{y_1(t)z_1(t)^3}{2} + \frac{z_1(t)y_1(t)^3}{2}\right]\dot{y}_1(t)^2\,dt \\ &= -18\int_{t_1}^{t_2} y_1(t)z_1(t)\left[1 + \frac{z_1(t)^2}{2} + \frac{y_1(t)^2}{2}\right]\ddot{y}_1(t)\dot{y}_1(t)\,dt \\ &= 54\int_{t_1}^{t_2} y_1(t)^2 z_1(t)\left[1 + \frac{z_1(t)^2}{2} + \frac{y_1(t)^2}{2}\right]\left[1 + \frac{y_1(t)^2}{2} + \frac{3z_1(t)^2}{2}\right]\dot{y}_1(t)\,dt \end{aligned}$$

where, in the last step, we used $(24)_1$. In the integrand, $y_1^2[1 + \frac{z_1^2}{2} + \frac{y_1^2}{2}][1 + \frac{y_1^2}{2} + \frac{3z_1^2}{2}] \geq 0$ and also $\dot{y}_1$ has fixed sign so that the integral may vanish only if $z_1(t)$ changes sign in (t_1, t_2). $\square$

Proposition 5 shows that the nonlinear frequency of z_1 is always larger than the frequency of y_1. If the frequency of z_1 reaches a multiple of the frequency of y_1 then an internal resonance is created and this yields a possible transfer of energy from y_1 to z_1.

3.3 The 2-modes system

When $n \geq 2$ different vertical oscillation modes are visible and a precise characterization is needed.

Definition 6. *Let $n \geq 2$ and $1 \leq j \leq n$; we call $Y_j = (0, ..., y_j, ..., 0)$ the j-th **vertical oscillating mode** if $(Y_j, 0) \in \mathbb{R}^{2n}$ is a solution to* (11)*.*

Whence, we may identify a j-th vertical oscillating mode with the unique nontrivial component y_j of the vector $(Y, Z) \in \mathbb{R}^{2n}$. For every j there exist infinitely many j-th vertical oscillating modes, each one being identified by the values of $y_j(0)$ and $\dot{y}_j(0)$. Any j-th vertical oscillating mode is the (periodic) solution $y_j^p = y_j^p(t)$ of the second order autonomous nonlinear equation

$$\ddot{y}(t) + (j^4 + 2)y(t) + \frac{3}{2}y(t)^3 = 0\,, \tag{26}$$

which admits the conserved energy

$$E_j = \frac{\dot{y}(t)^2}{2} + (j^4 + 2)\frac{y(t)^2}{2} + \frac{3y(t)^4}{8} = \frac{\dot{y}(0)^2}{2} + (j^4 + 2)\frac{y(0)^2}{2} + \frac{3}{8}y(0)^4 \geq 0\,. \tag{27}$$

The energy E_j of such solution is constant in time and merely depends on the initial conditions, $E_j = E_j(y_j(0), \dot{y}_j(0))$. Two j-th vertical oscillating modes are equal up to a time translation if and only if they have the same energy: the energy E_j determines univocally the maximum amplitude of oscillation of y_j^p, see (50) below.

Our purpose is to study the torsional stability of the Y_j's. To this end, as in the 1-mode case, we consider the linearized system around $(Y_j, 0)$ and we give the following definition.

Definition 7. *We say that the j-th vertical oscillating mode $Y_j = (0, ..., y_j, ..., 0)$ at energy E_j is* **torsionally stable** *if the torsional components of the linear system, obtained by linearizing* (11) *around $(Y_j, 0)$, are stable.*

Let us now fix $n = 2$ in (10) and put

$$y(x,t) = y_1(t)\sin x + y_2(t)\sin(2x)\,, \qquad z(x,t) = z_1(t)\sin x + z_2(t)\sin(2x)\,.$$

Then, after integration over $(0, \pi)$, we see that y_j and z_j satisfy the system

$$\begin{cases} \ddot{y}_1(t) + y_1(t) = -\dfrac{2}{\pi}\displaystyle\int_0^\pi [f(y(x,t) - z(x,t)) + f(y(x,t) + z(x,t))] \sin x \, dx \\ \ddot{y}_2(t) + 16y_2(t) = -\dfrac{2}{\pi}\displaystyle\int_0^\pi [f(y(x,t) - z(x,t)) + f(y(x,t) + z(x,t))] \sin(2x) \, dx \\ \ddot{z}_1(t) + z_1(t) = \dfrac{6}{\pi}\displaystyle\int_0^\pi [f(y(x,t) - z(x,t)) - f(y(x,t) + z(x,t))] \sin x \, dx \\ \ddot{z}_2(t) + 4z_2(t) = \dfrac{6}{\pi}\displaystyle\int_0^\pi [f(y(x,t) - z(x,t)) - f(y(x,t) + z(x,t))] \sin(2x) \, dx\,. \end{cases}$$

By (16) and Lemma 2, this system becomes

$$\begin{cases} \ddot{y}_1 + 3y_1 + \dfrac{9}{2}y_1 z_1^2 + 3y_1 z_2^2 + 3y_1 y_2^2 + \dfrac{3}{2}y_1^3 + 6z_1 z_2 y_2 = 0 \\ \ddot{y}_2 + 18y_2 + \dfrac{9}{2}y_2 z_2^2 + 3y_2 z_1^2 + 3y_2 y_1^2 + \dfrac{3}{2}y_2^3 + 6z_1 z_2 y_1 = 0 \\ \ddot{z}_1 + 7z_1 + \dfrac{27}{2}z_1 y_1^2 + 9z_1 y_2^2 + 9z_1 z_2^2 + \dfrac{9}{2}z_1^3 + 18y_1 y_2 z_2 = 0 \\ \ddot{z}_2 + 10z_2 + \dfrac{27}{2}z_2 y_2^2 + 9z_2 y_1^2 + 9z_2 z_1^2 + \dfrac{9}{2}z_2^3 + 18y_1 y_2 z_1 = 0\,, \end{cases} \tag{28}$$

while the energy becomes

$$\begin{aligned} E &= \frac{1}{2}(\dot{y}_1^2 + \dot{y}_2^2) + \frac{1}{6}(\dot{z}_1^2 + \dot{z}_2^2) + \frac{3}{2}y_1^2 + 9y_2^2 + \frac{7}{6}z_1^2 + \frac{5}{3}z_2^2 + 6y_1 y_2 z_1 z_2 \\ &\quad + \frac{9}{4}(y_1^2 z_1^2 + y_2^2 z_2^2) + \frac{3}{2}(y_1^2 y_2^2 + y_1^2 z_2^2 + y_2^2 z_1^2 + z_1^2 z_2^2) + \frac{3}{8}(y_1^4 + y_2^4 + z_1^4 + z_2^4)\,. \end{aligned}$$

Since our purpose is to emphasize perturbations of linear equations, it is more convenient to rewrite the two last equations in (28) as

$$\begin{cases} \ddot{z}_1 + \left(7 + \dfrac{27}{2}y_1^2 + 9y_2^2 + 9z_2^2\right) z_1 + \dfrac{9}{2}z_1^3 = -18y_1 y_2 z_2 \\ \ddot{z}_2 + \left(10 + \dfrac{27}{2}y_2^2 + 9y_1^2 + 9z_1^2\right) z_2 + \dfrac{9}{2}z_2^3 = -18y_1 y_2 z_1\,. \end{cases} \tag{29}$$

In view of Definition 7, to deduce the stability of $Y_1 = (y_1^p, 0)$ and $Y_2 = (0, y_2^p)$, we linearize system (29) around $(Y_j, 0)$ and we consider its torsional part; we are so led to study the stability of the following systems of uncoupled Hill equations:

$$\begin{cases} \ddot{z}_1(t) + a_{1,j}(t) z_1(t) = 0 \\ \ddot{z}_2(t) + a_{2,j}(t) z_2(t) = 0 \end{cases} \qquad (j = 1, 2) \tag{30}$$

where $a_{i,j}(t) = i^2 + 6 + 9\alpha_{i,j} y_j^p(t)^2$ and $\alpha_{i,j} = 1$ if $i \neq j$, $\alpha_{i,i} = \frac{3}{2}$. By Definition 7, y_j^p is torsionally stable if the trivial solutions $(z_1, z_2) = (0, 0)$ of (30) are stable. Then we prove

Theorem 8. *The first vertical oscillating mode y_1^p of (28) is torsionally stable provided that*

$$\|y_1^p\|_\infty \le \frac{1}{\sqrt{3}} \approx 0.577 \iff E \le \frac{13}{24} \approx 0.542\,. \tag{31}$$

The second vertical oscillating mode y_2^p of (28) is torsionally stable provided that

$$\|y_2^p\|_\infty \le \sqrt{\frac{32}{51}} \approx 0.792 \iff E \le \frac{5024}{867} \approx 5.795\,.$$

Again, Theorem 8 merely gives a sufficient condition for the torsional stability and, numerically, the thresholds seem to be larger. Once more, numerics only shows local stability but the observed phenomena are very precise and hence they appear reliable. Here the situation is slightly more complicated because two modes (4 equations) are involved. Therefore, we proceed differently.

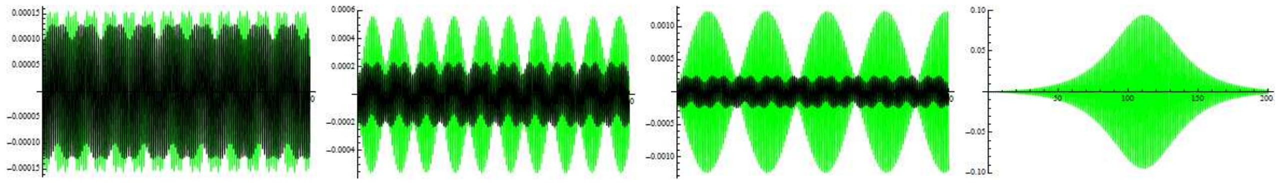

Figure 3: On the interval $t \in [0, 200]$, plot of the torsional components z_1 (green) and z_2 (black) to (28)-(32) for $\|y_1\|_\infty = 1,\ 1.4,\ 1.45,\ 1.47$ (from left to right).

We start by studying the stability of the first vertical oscillating mode. The pictures in Figure 3 display the plots of the torsional components (z_1, z_2) of the solutions to (28) with initial data

$$y_1(0) = \|y_1\|_\infty = 10^4 y_2(0) = 10^4 z_1(0) = 10^4 z_2(0)\,, \quad \dot{y}_1(0) = \dot{y}_2(0) = \dot{z}_1(0) = \dot{z}_2(0) = 0 \tag{32}$$

for different values of $\|y_1\|_\infty$. The green plot is z_1 and the black plot is z_2. Recalling that the initial torsional amplitudes are of the order of 10^{-4} we can see that, for $\|y_1\|_\infty = 1$, both torsional components remain small, although z_1 is slightly larger than z_2. By increasing the y_1 amplitude, $\|y_1\|_\infty = 1.4$ and $\|y_1\|_\infty = 1.45$, we see that z_1 and z_2 still remain small but now z_1 is significantly larger than z_2 and displays bumps. When $\|y_1\|_\infty = 1.47$, z_1 has become so large that z_2, which is still of the order of 10^{-4}, is no longer visible in the fourth plot of Figure 3. The threshold for the appearance of $z_1 \gg z_2$ is again $\|y_1\|_\infty \approx 1.46$, see Section 3.2. Therefore, it seems that the stability of the first vertical oscillating mode does not transfer energy on the second modes; but, as we now show, this is not true.

We increased further the initial datum up to $\|y_1\|_\infty = 3$. In Figure 4 we display the plot of all the components (y_1, y_2, z_1, z_2) of the corresponding solution to (28)-(32). One can see that some energy is

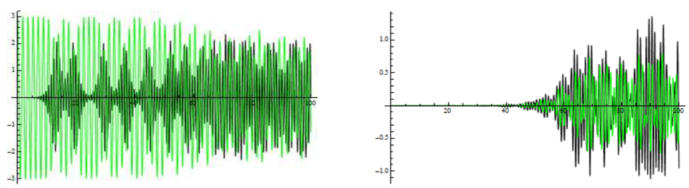

Figure 4: On the interval $t \in [0, 100]$, plot of the solution to (28)-(32) for $\|y_1\|_\infty = 3$. Left picture: green=y_1, black=z_1. Right picture: green=y_2, black=z_2.

also transferred to both the vertical and torsional second modes, although this occurs with some delay (in the second picture, the green oscillation is hidden but it is almost as wide as the black oscillation).

Concerning the stability of the second vertical oscillating mode, we just quickly describe our numerical results. The loss of stability appeared for $\|y_2\|_\infty \approx 0.945$ corresponding to $E \approx 8.33$; in this case, Theorem 8 gives a fairly good sufficient condition. For $\|y_2\|_\infty \leq 0.94$, both z_1 and z_2 (and also y_1) remain small and of the same magnitude, with the amplitude of oscillations of z_1 being almost constant while the amplitude of oscillations of z_2 being variable. For $\|y_2\|_\infty \geq 0.945$, z_2 suddenly displays the bumps seen in the above pictures. Finally, for $\|y_2\|_\infty \geq 1.08$, also y_1 and z_1 display sudden wide oscillations which, however, appear delayed in time when compared with z_2.

3.4 The impact of nonlinearity

Let $\gamma > 0$. When f is as in (14), the system (4) becomes

$$\begin{cases} y_{tt} + y_{xxxx} + 2y(1+\gamma y^2 + 3\gamma z^2) = 0 & (0 < x < \pi,\ t \geq 0) \\ z_{tt} - z_{xx} + 6z(1 + 3\gamma y^2 + \gamma z^2) = 0 & (0 < x < \pi,\ t \geq 0)\,. \end{cases}$$

By multiplying these equations by $\sin(jx)$ $(j = 1, ..., n)$ and integrating by parts over $(0, \pi)$, (11) takes the form

$$\begin{cases} \ddot{y}_j(t) + \mu_j y_j(t) + \gamma P_j(Y,Z) = 0 \\ \ddot{z}_j(t) + \nu_j z_j(t) + \gamma Q_j(Y,Z) = 0 \end{cases} \qquad (j = 1, ..., n) \tag{33}$$

where, for all $j = 1, ..., n$, $\mu_j, \nu_j > 0$ and P_j, Q_j are third order homogeneous polynomials in the $2n$ variables $y_1, ..., y_n, z_1, ..., z_n$. To (33) we associate some initial conditions; then we denote by (Y_γ, Z_γ) the corresponding solution to (33) and by E_γ the conserved energy. If we put $(\overline{Y}, \overline{Z}) = \sqrt{\gamma}\,(Y_\gamma, Z_\gamma)$, then $(\overline{Y}, \overline{Z})$ solves system (33) when $\gamma = 1$. Since each vertical oscillating mode of this system has its own thresholds for stability, we infer the thresholds for (33):

Proposition 9. *Let $\gamma > 0$. The energy threshold E_γ for the torsional stability of a vertical oscillating mode $(Y_\gamma, 0)$ of (33) satisfies $E_\gamma = E/\gamma$, where E is the energy threshold for the vertical oscillating mode $(\overline{Y}, 0)$ of (33) when $\gamma = 1$. Moreover, the widest amplitude threshold $\|Y_\gamma\|_\infty$ for the torsional stability satisfies $\|Y_\gamma\|_\infty = \|\overline{Y}\|_\infty/\sqrt{\gamma}$.*

The proof of Proposition 9 follows by noticing that the energy E of the solution $(\overline{Y}, \overline{Z})$ satisfies $E = \gamma E_\gamma$. From Proposition 9 we see that

$$\gamma \mapsto E_\gamma \qquad \text{and} \qquad \gamma \mapsto \|Y_\gamma\|_\infty$$

are decreasing with respect to γ and both tend to 0 if $\gamma \to \infty$, whereas they tend to ∞ if $\gamma \to 0$. This shows that the nonlinearity plays against stability:

more nonlinearity yields more instability and almost linear elastic behaviors are extremely stable.

3.5 The impact of aerodynamic forces

Even in absence of wind, an aerodynamic force is exerted on the bridge by the surrounding air in which the structure is immersed, and is due to the relative motion between the bridge and the air. Scanlan-Tomko [30] assume that the aerodynamic forces depend linearly on the derivatives. For the 1-mode system (18), this leads to the following modified system:

$$\begin{cases} \ddot{y}_1 + 3y_1 + \frac{3}{2}y_1^3 + \frac{9}{2}y_1 z_1^2 + \delta \dot{z}_1 = 0 \\ \ddot{z}_1 + 7z_1 + \frac{9}{2}z_1^3 + \frac{27}{2}z_1 y_1^2 + \delta \dot{y}_1 = 0 \end{cases} \tag{34}$$

with $\delta > 0$. As in (23), we take the initial conditions

$$y_1(0) = \sigma = 10^4 z_1(0)\,, \quad \dot{y}_1(0) = \dot{z}_1(0) = 0 \tag{35}$$

for different values of σ and we wish to highlight the differences, if any, between (18) and (34). For (34) we have no energy conservation; however, let us consider the (variable) energy function

$$E(t) = \frac{\dot{y}_1^2}{2} + \frac{\dot{z}_1^2}{6} + \frac{3}{2}y_1^2 + \frac{7}{6}z_1^2 + \frac{9}{4}y_1^2 z_1^2 + \frac{3}{8}(y_1^4 + z_1^4)\,. \tag{36}$$

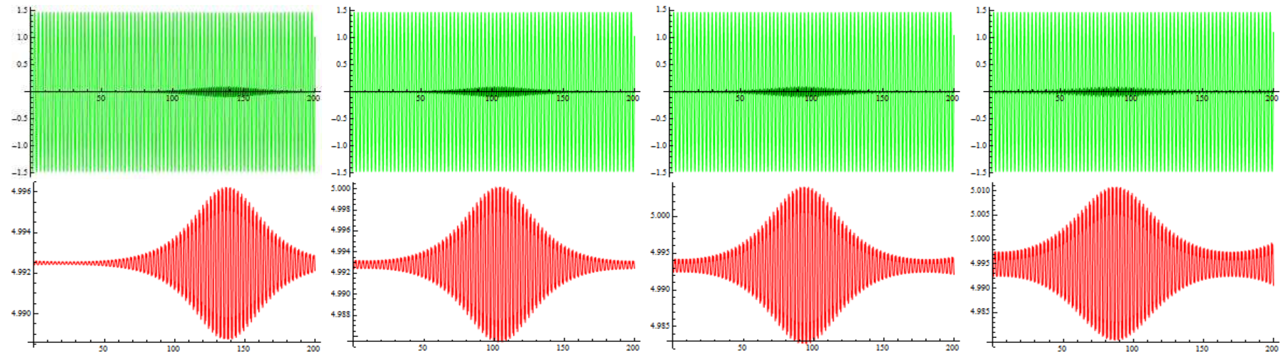

Figure 5: On the interval $t \in [0, 200]$, plot of the solutions y_1 (green) and z_1 (black) to (34)-(35) for $\sigma = 1.47$ and $\delta = 0.01,\ 0.02,\ 0.03,\ 0.05$ (from left to right). On the second line (red), the energy $E = E(t)$ defined in (36).

We first take $\sigma = 1.47$ and we modify the aerodynamic parameter δ. In Figure 5 we plot both the behavior of the solutions (first line) and the behavior of the energy $E(t)$ (second line), for increasing values of δ. The first line should be compared with the second picture in Figure 2 (case $\delta = 0$). We note that, as the aerodynamic parameter increases, the transfer of energy is anticipated but it is not amplified. Quite surprisingly, on the second line we see that the energy $E(t)$ remains almost constant except in the interval of time where the transfer of energy occurs: for increasing aerodynamic parameters δ we observe increasing variations in the energy behavior.

Then we maintain fixed $\delta = 0.01$ and we increase the initial energy, that is, the initial amplitude of oscillation. In Figure 6 we plot both the behavior of the solutions (first line) and the behavior of the energy $E(t)$ (second line), for increasing values of σ. It turns out that all the phenomena are

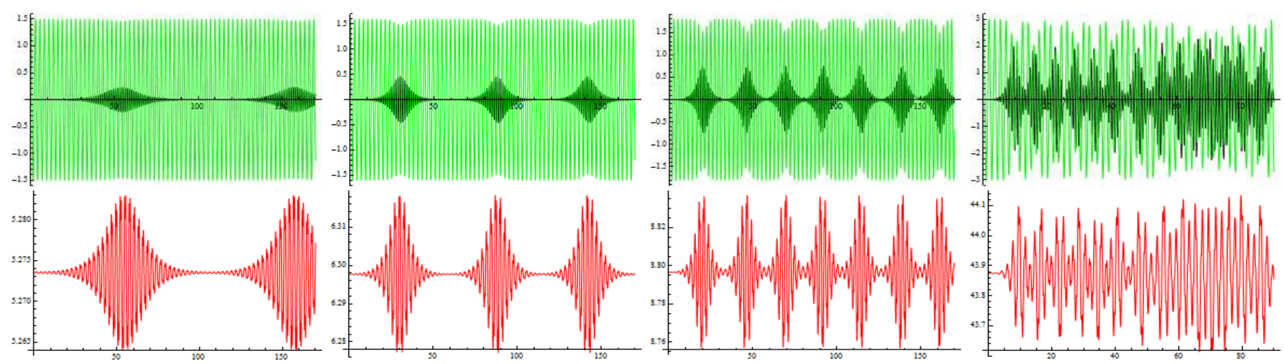

Figure 6: On the interval $t \in [0, 170]$, plot of the solutions y_1 (green) and z_1 (black) to (34)-(35) for $\delta = 0.01$ and $\sigma = 1.5,\ 1.6,\ 1.8,\ 3$ (from left to right). On the second line (red), the energy $E = E(t)$ defined in (36).

anticipated (in time) and amplified (in width) and reach a quite chaotic behavior for $\sigma = 3$ where we had to stop the integration time at $t = 90$.

These experiments seem to show that the onset of instability does not depend on aerodynamic forces. The amount of initial energy still seems to be the decisive parameter. The aerodynamic forces modify the internal energy only when the exchange of energy occurs.

4 Proof of Theorem 1

The existence and uniqueness issues are inspired to [16, Theorems 8 and 11] while the regularity statement is achieved by arguing as in [11, Lemma 8.1].

For the existence part we perform a Galerkin procedure. The sequence $\{\sin(jx)\}_{j\geq 1}$ is an orthogonal basis of the spaces $L^2(0,\pi), H^1_0(0,\pi)$ and $H^2\cap H^1_0(0,\pi)$. Then, for a given $n\in\mathbb{N}$, we set

$$y^n(x,t)=\sum_{j=1}^{n} y_j(t)\sin(jx)\ ,\quad z^n(x,t)=\sum_{j=1}^{n} z_j(t)\sin(jx)\,, \tag{37}$$

where y_j and z_j satisfy the system of ODE's

$$\begin{cases} \ddot{y}_j(t)+j^4y_j(t)+\dfrac{2}{\pi}\displaystyle\int_0^\pi [f(y^n(x,t)+z^n(x,t))+f(y^n(x,t)-z^n(x,t))]\,\sin(jx)\,dx=0 \\ \\ \ddot{z}_j(t)+j^2z_j(t)+\dfrac{6}{\pi}\displaystyle\int_0^\pi [f(y^n(x,t)+z^n(x,t))-f(y^n(x,t)-z^n(x,t))]\,\sin(jx)\,dx=0 \end{cases} \tag{38}$$

for $t>0$ and $j=1,...,n$. Moreover, writing the Fourier expansion of the initial data (6) as

$$\eta_0(x)=\sum_{j=1}^{\infty}\eta_0^j\sin(jx)\quad\text{in } H^2\cap H^1_0(0,\pi)\qquad y_1(x)=\sum_{j=1}^{\infty}\eta_1^j\sin(jx)\quad\text{in } L^2(0,\pi)$$

$$\zeta_0(x)=\sum_{j=1}^{\infty}\zeta_0^j\sin(jx)\quad\text{in } H^1_0(0,\pi)\qquad \zeta_1(x)=\sum_{j=1}^{\infty}\zeta_1^j\sin(jx)\quad\text{in } L^2(0,\pi)\,,$$

we assume that, for every $1\leq j\leq n$, the y_j's and the z_j's satisfy

$$y_j(0)=\eta_0^j\,,\quad \dot{y}_j(0)=\eta_1^j\,,\quad z_j(0)=\zeta_0^j\,,\quad \dot{z}_j(0)=\zeta_1^j\,. \tag{39}$$

The existence of a unique local solution to (38)-(39) in some maximal interval of continuation $[0,\tau_n)$, $\tau_n>0$, follows from standard theory of ODE's. Then, we multiply the first equation in (38) by $6\dot{y}_j(t)$ and the second equation by $2\dot{z}_j(t)$, then we add the so obtained $2n$ equations for $j=1$ to n, finally we integrate over $(0,t)$ to obtain

$$3\|\dot{y}^n(t)\|_2^2+3\|y^n_{xx}(t)\|_2^2+\|\dot{z}^n(t)\|_2^2+\|z^n_x(t)\|_2^2+12\int_0^\pi\left(F(y^n(t)+z^n(t))+F(y^n(t)-z^n(t))\right)dx\leq$$

$$3\|\dot{y}^n(0)\|_2^2+3\|y^n_{xx}(0)\|_2^2+\|\dot{z}^n(0)\|_2^2+\|z^n_x(0)\|_2^2+12\int_0^\pi\left(F(y^n(0)+z^n(0))+F(y^n(0)-z^n(0))\right)dx$$

where $y^n(t)=y^n(x,t)$, $z^n(t)=z^n(x,t)$ and $F(s)=\int_0^s f(\tau)\,d\tau$. Since $F\geq 0$, this yields

$$3\|\dot{y}^n(t)\|_2^2+3\|y^n_{xx}(t)\|_2^2+\|\dot{z}^n(t)\|_2^2+\|z^n_x(t)\|_2^2\leq C\quad\text{for any } t\in[0,\tau_n) \text{ and } n\geq 1 \tag{40}$$

for some constant C independent of n and t. Hence, $\{y^n\}$ and $\{z^n\}$ are globally defined in $\mathbb{R}_+$ and uniformly bounded, respectively, in the spaces $C^0([0,T];H^2\cap H^1_0(0,\pi))\cap C^1([0,T];L^2(0,\pi))$ and $C^0([0,T];H^1_0(0,\pi))\cap C^1([0,T];L^2(0,\pi))$ for all finite $T>0$. We show that they both admit a strongly convergent subsequence in the same spaces.

The estimate (40) shows that $\{y^n\}$ and $\{z^n\}$ are bounded and equicontinuous in $C^0([0,T];L^2(0,\pi))$. By the Ascoli-Arzelà Theorem we then conclude that, up to a subsequence, $y^n\to y$ and $z^n\to z$ strongly in $C^0([0,T];L^2(0,\pi))$. By (40) and the embedding $H^1_0(0,\pi)\subset L^\infty(0,\pi)$ we also infer that y^n and z^n are uniformly bounded in $[0,\pi]\times[0,T]$. Whence,

$$\left|\int_0^\pi F(y^n(t)+z^n(t))\,dx-\int_0^\pi F(y(t)+z(t))\,dx\right|$$

$$\leq \int_0^\pi |f(\tau(y^n(t)+z^n(t))+(1-\tau)(y(t)+z(t)))|\,(|y^n(t)-y(t)|+|z^n(t)-z(t)|)\ dx$$

for some $\tau=\tau(x,t,n)\in[0,1]$. Since y^n and z^n are uniformly bounded, so is $f(\tau(y^n+z^n)+(1-\tau)(y+z))$ and the latter inequality yields

$$\left|\int_0^\pi F(y^n(t)+z^n(t))\,dx-\int_0^\pi F(y(t)+z(t))\,dx\right|\leq C\Big(\|y^n(t)-y(t)\|_2+\|z^n(t)-z(t)\|_2\Big)\to 0\,. \quad (41)$$

We may argue similarly for $F(y^n-z^n)$.

Next, for every $n>m\geq 1$, we set $y^{n,m}:=y^n-y^m$ and $z^{n,m}:=z^n-z^m$. Repeating the computations which yield (40), for all $t\in[0,T]$ one gets

$$3\|\dot y^{n,m}(t)\|_2^2+3\|y^{n,m}_{xx}(t)\|_2^2+\|\dot z^{n,m}(t)\|_2^2+\|z^{n,m}_x(t)\|_2^2$$

$$=3\|\dot y^{n,m}(0)\|_2^2+3\|y^{n,m}_{xx}(0)\|_2^2+\|\dot z^{n,m}(0)\|_2^2+\|z^{n,m}_x(0)\|_2^2$$

$$-12\int_0^\pi [F(y^n(t)+z^n(t))-F(y^m(t)+z^m(t))+F(y^n(t)-z^n(t))-F(y^m(t)-z^m(t))]\ dx$$

$$+12\int_0^\pi [F(y^n(0)+z^n(0))-F(y^m(0)+z^m(0))+F(y^n(0)-z^n(0))-F(y^m(0)-z^m(0))]\ dx\,.$$

Therefore, by using (41), we infer that

$$\sup_{t\in[0,T]}\ \Big(3\|\dot y^{n,m}(t)\|_2^2+3\|y^{n,m}_{xx}(t)\|_2^2+\|\dot z^{n,m}(t)\|_2^2+\|z^{n,m}_x(t)\|_2^2\Big)\to 0 \qquad \text{as } n,m\to\infty$$

so that $\{y^n\}$ and $\{z^n\}$ are Cauchy sequences in the spaces $C^0([0,T];H^2\cap H^1_0(0,\pi))\cap C^1([0,T];L^2(0,\pi))$ and $C^0([0,T];H^1_0(0,\pi))\cap C^1([0,T];L^2(0,\pi))$, respectively. In turn this yields

$$y^n\to y \text{ in } C^0([0,T];H^2\cap H^1_0(0,\pi))\cap C^1([0,T];L^2(0,\pi)) \quad \text{as } n\to+\infty$$

and

$$z^n\to z \text{ in } C^0([0,T];H^1_0(0,\pi))\cap C^1([0,T];L^2(0,\pi)) \quad \text{as } n\to+\infty\,.$$

Let $\Upsilon\in C^\infty_c(0,T)$, $\varphi\in H^2\cap H^1_0(0,\pi)$ and $\psi\in H^1_0(0,\pi)$. We denote by φ^n and ψ^n the orthogonal projections of φ and ψ onto $X_n:=\text{span}\{\sin(jx)\}_{j=1}^n$ from, respectively, the spaces $H^2\cap H^1_0(0,\pi)$ and $H^1_0(0,\pi)$. Then (38) yields

$$\begin{cases} \displaystyle\int_0^T (\dot y^n(t),\varphi^n)\dot\Upsilon(t)\,dt=\int_0^T\left[(y^n_{xx}(t),(\varphi^n)'')+\frac{2}{\pi}(f(y^n(t)-z^n(t))+f(y^n(t)+z^n(t)),\varphi^n)\right]\Upsilon(t)\,dt \\ \\ \displaystyle\int_0^T (\dot z^n(t),\psi^n)\dot\Upsilon(t)\,dt=-\int_0^T\left[(z^n_x(t),(\psi^n)')+\frac{6}{\pi}(f(y^n(t)-z^n(t))-f(y^n(t)+z^n(t)),\psi^n)\right]\Upsilon(t)\,dt. \end{cases}$$

Since, by compactness,

$$f(y^n\pm z^n)\to f(y\pm z)\quad \text{in } C^0([0,T],L^2(0,\pi))\,,$$

by letting $n \to +\infty$ in the above system we conclude that $y_{tt} \in C^0([0,T];H^*)$ and $z_{tt} \in C^0([0,T];H^{-1})$. The verification of the initial conditions follows by noting that $y^n(0) \to y(0)$ in $H^2 \cap H^1_0(0,\pi)$, $\dot{y}^n(0) \to \dot{y}(0)$ in $L^2(0,\pi)$, $z^n(0) \to z(0)$ in $H^1_0(0,\pi)$ and $\dot{z}^n(0) \to \dot{z}(0)$ in $L^2(0,\pi)$. The proof of the existence part is complete, once we observe that all the above results hold for any $T > 0$.

Next we turn to the uniqueness issue. Since it follows by repeating the proof of [16, Theorem 11] with some minor changes we only give a sketch of it. Assume problem (4)-(5)-(6) admits two couples of solutions (y^1, z^1) and (y^2, z^2) and denote $(\bar{y}, \bar{z}) := (y^1 - y^2, z^1 - z^2)$. Next, we put $\mu_s(t) = -\int_t^s \bar{y}(\tau)\,d\tau$, $\eta_s(t) = -\int_t^s \bar{z}(\tau)\,d\tau$, $Y(t) = \int_0^t \bar{y}(\tau)\,d\tau$ and $Z(t) = \int_0^t \bar{z}(\tau)\,d\tau$ with $0 < t \le s$. Note that $\mu_s'(t) = \bar{y}(t)$, $\eta_s'(t) = \bar{z}(t)$, $\mu_s(t) = Y(t) - Y(s)$ and $\eta_s(t) = Z(t) - Z(s)$. Multiply the equation satisfied by $\bar{y}$ times μ_s and the one satisfied by $\bar{z}$ times η_s. By integrating, one deduces

$$\begin{cases} \frac{1}{2}\|\bar{y}(s)\|_2^2 + \frac{1}{2}\|Y_{xx}(s)\|_2^2 = \int_0^s \left(\left[f(y^1-z^1) + f(y^1+z^1) - f(y^2-z^2) - f(y^2+z^2)\right], \mu_s\right)\,dt \\ \\ \frac{1}{2}\|\bar{z}(s)\|_2^2 + \frac{1}{2}\|Z_{x}(s)\|_2^2 = 3\int_0^s \left(\left[f(y^1-z^1) - f(y^1+z^1) - f(y^2-z^2) + f(y^2+z^2)\right], \eta_s\right)\,dt. \end{cases}$$

Exploiting the fact that $f \in \mathrm{Lip}_{loc}(\mathbb{R})$ and (40), one infers that

$$\|\bar{y}(s)\|_2^2 + \|\bar{z}(s)\|_2^2 + \|Y_{xx}(s)\|_2^2 + \|Z_x(s)\|_2^2 \le C\int_0^s \left(\|\bar{y}(t)\|_2^2 + \|\bar{z}(t)\|_2^2 + \|Y_{xx}(t)\|_2^2 + \|Z_x(t)\|_2^2\right)\,dt$$

where both the Young and the Poincaré inequalities have been exploited. Hence, by the Gronwall Lemma, $\|\bar{y}(s)\|_2 = \|\bar{z}(s)\|_2 = 0$ and uniqueness follows.

5 Proof of Theorem 4

For any $E > 0$ we put

$$\Lambda_\pm(E) := 2\sqrt{1 + \frac{2}{3}E} \pm 2.$$

Then we prove

Lemma 10. *For any $\eta_0, \eta_1 \in \mathbb{R}$ problem (20) admits a unique solution $y = y^p$ which is periodic of period*

$$T(E) = \frac{8}{\sqrt{3}}\int_0^1 \frac{ds}{\sqrt{(\Lambda_+(E) + \Lambda_-(E)s^2)(1-s^2)}}\,. \tag{42}$$

In particular, the map $E \mapsto T(E)$ is strictly decreasing and $\lim_{E\to 0} T(E) = 2\pi/\sqrt{3}$.

Proof. The existence, uniqueness and periodicity of the solution y^p is a known fact from the theory of ODE's. For a given $E > 0$, we may rewrite (21) as

$$\dot{y}^2 = 2E - 3y^2 - \frac{3}{4}y^4\,. \tag{43}$$

Hence,

$$\|y^p\|_\infty = \sqrt{\Lambda_-(E)}\,. \tag{44}$$

Since (20) merely consists of odd terms, the period $T(E)$ of y^p is the double of the width of an interval of monotonicity for y^p. Since the problem is autonomous, we may assume that $y^p(0) = -\|y^p\|_\infty$ and

$\dot{y^p}(0) = 0$; then, by symmetry and periodicity, we have that $y^p(T/2) = \|y^p\|_\infty$ and $\dot{y^p}(T/2) = 0$. By rewriting (43) as

$$\dot{y} = \frac{\sqrt{3}}{2}\sqrt{(\Lambda_+(E) + y^2)(\Lambda_-(E) - y^2)} \qquad \forall t \in \left(0, \frac{T}{2}\right),$$

by separating variables, and upon integration over the time interval $(0, T/2)$ we obtain

$$\frac{T(E)}{2} = \frac{2}{\sqrt{3}} \int_{-\|y^p\|_\infty}^{\|y^p\|_\infty} \frac{dy}{\sqrt{(\Lambda_+(E) + y^2)(\Lambda_-(E) - y^2)}}.$$

Then, using the fact that the integrand is even with respect to y and through a change of variable,

$$T(E) = \frac{8}{\sqrt{3}} \int_0^{\|y^p\|_\infty} \frac{dy}{\sqrt{(\Lambda_+(E) + y^2)(\|y^p\|_\infty^2 - y^2)}} = \frac{8}{\sqrt{3}} \int_0^1 \frac{ds}{\sqrt{(\Lambda_+(E) + \Lambda_-(E)s^2)(1 - s^2)}},$$

which proves (42). Both the maps $E \mapsto \Lambda_\pm(E)$ are continuous and increasing for $E \in [0, \infty)$ and $\Lambda_-(0) = 0$, $\Lambda_+(0) = 4$. Whence, $E \mapsto T(E)$ is strictly decreasing and

$$\lim_{E \to 0} T(E) = T(0) = \frac{4}{\sqrt{3}} \int_0^1 \frac{ds}{\sqrt{1 - s^2}} = \frac{2\pi}{\sqrt{3}},$$

a result that could have also been obtained by noticing that, as $E \to 0$, the equation (20) tends to $\ddot{y} + 3y = 0$. $\square$

In the sequel, we need bounds for $T(E)$. From (42) we see that, by taking $s = 0$ in the first polynomial under square root,

$$T(E) \le \frac{8}{\sqrt{3\Lambda_+(E)}} \int_0^1 \frac{ds}{\sqrt{1 - s^2}} = \frac{4\pi}{\sqrt{3\Lambda_+(E)}} \implies \frac{16\pi^2}{T(E)^2} \ge 3\Lambda_+(E). \tag{45}$$

Moreover, by taking $s = 1$ in the first polynomial under square root, we infer

$$T(E) \ge \frac{8}{\sqrt{3}\sqrt{\Lambda_+(E) + \Lambda_-(E)}} \int_0^1 \frac{ds}{\sqrt{1 - s^2}} = \frac{2\pi}{\sqrt[4]{9 + 6E}} \implies \frac{4\pi^2}{T(E)^2} \le \sqrt{9 + 6E}. \tag{46}$$

Let us now consider (22). With the initial conditions $z(0) = \dot{z}(0) = 0$, the unique solution to (22) is $z \equiv 0$. We are interested in determining whether the trivial solution is stable in the Lyapunov sense, namely if the solutions to (22) with small initial data $|z(0)|$ and $|\dot{z}(0)|$ remain small for all $t \ge 0$. By Lemma 10, the function $y^p(t)^2$ is $T/2$-periodic. Then a is a positive $T/2$-periodic function and a stability criterion for the Hill equation due to Zhukovskii [36], see also [35, Chapter VIII], states that the trivial solution to (22) is stable provided

$$\frac{4\pi^2}{T(E)^2} \le a(t) \le \frac{16\pi^2}{T(E)^2}. \tag{47}$$

Let us translate this condition in terms of $\|y^p\|_\infty$. By the definition of a in (22) and by (44), we have

$$7 \le a(t) \le 7 + \frac{27}{2}\|y^p\|_\infty^2 = -20 + 9\sqrt{9 + 6E}.$$

Whence, (47) holds if both

$$\frac{4\pi^2}{T(E)^2} \le 7 \quad \text{and} \quad -20 + 9\sqrt{9 + 6E} \le \frac{16\pi^2}{T(E)^2}. \tag{48}$$

In turn, by (45)-(46), the inequalities in (48) certainly hold if

$$\sqrt{9+6E} \leq 7 \quad \text{and} \quad -20+9\sqrt{9+6E} \leq 3\Lambda_+(E)\ .$$

The first of such inequalities is fulfilled provided that $E \leq \frac{20}{3}$; the second inequality is satisfied if

$$E \leq \frac{235}{294} \approx 0.799\ , \qquad \|y^p\|_\infty \leq \sqrt{\frac{10}{21}} \approx 0.69\ , \tag{49}$$

which is more stringent and, therefore, yields a sufficient condition for (47) to hold. This proves Theorem 4.

Remark 11. *The sufficient condition* (47) *is fulfilled as long as both* (48) *hold. Numerically, we see that the former is satisfied for* $E \lessapprox 10.445$ *whereas the latter is satisfied for* $E \lessapprox 0.944$. *The most stringent is the second one which corresponds to* $\|y^p\|_\infty \lessapprox 0.74$, *not significantly better than* (49).

6 Proof of Theorem 8

For any $E > 0$, we put

$$\Lambda^j_\pm(E) = 2\sqrt{\frac{(j^4+2)^2}{9} + \frac{2}{3}E} \pm \frac{2}{3}(j^4+2) \qquad (j=1,2)\,.$$

Then (27), with $E_j = E$, reads

$$\dot{y_j}^2 = \frac{3}{4}(\Lambda^j_+(E) + y_j^2)(\Lambda^j_-(E) - y_j^2) \qquad (j=1,2)\,.$$

By this, since any j-th oscillating mode y_j^p satisfies (27), we deduce

$$\|y_j^p\|_\infty = \sqrt{\Lambda^j_-(E)} \qquad (j=1,2)\,. \tag{50}$$

Then, the same analysis performed in Section 5 yields that the y_j^p are periodic functions of period

$$T_j(E) = \frac{8}{\sqrt{3}} \int_0^1 \frac{ds}{\sqrt{(\Lambda^j_+(E) + \Lambda^j_-(E)s^2)(1-s^2)}}\,.$$

In particular, the map $E \mapsto T_j(E)$ is strictly decreasing and $\lim_{E\to 0} T_j(E) = 2\pi/\sqrt{j^4+2}$. Furthermore, the following estimates hold

$$T_j(E) \leq \frac{8}{\sqrt{3\Lambda^j_+(E)}} \int_0^1 \frac{ds}{\sqrt{1-s^2}} = \frac{4\pi}{\sqrt{3\Lambda^j_+(E)}} \implies \frac{16\pi^2}{T_j(E)^2} \geq 3\Lambda^j_+(E) \tag{51}$$

and

$$T_j(E) \geq \frac{8}{\sqrt{3}\sqrt{\Lambda^j_+(E) + \Lambda^j_-(E)}} \int_0^1 \frac{ds}{\sqrt{1-s^2}} = \frac{2\pi}{\sqrt[4]{(j^4+2)^2+6E}} \implies \frac{4\pi^2}{T_j(E)^2} \leq \sqrt{(j^4+2)^2+6E}\,. \tag{52}$$

Consider the first mode $(Y_1, 0) = (y_1^p, 0, 0, 0)$. For $j=1$ system (30) reads

$$\begin{cases} \ddot{z}_1(t) + (7 + \frac{27}{2}y_1^p(t)^2)z_1(t) = 0 \\ \ddot{z}_2(t) + (10 + 9y_1^p(t)^2)z_2(t) = 0\,. \end{cases} \tag{53}$$

If the trivial solution of both the equations in (53) is stable then system (53) itself is stable and Definition 7 is satisfied, see [35, Theorem II-Chapter III-vol 1]. Since the first equation in (53) coincides with (22), the proof of Theorem 4 yields the torsional stability provided that (49) holds. For the second equation in (53), by applying again the Zhukovskii stability criterion (47), we see that the trivial solution is stable provided that

$$\frac{4\pi^2}{T(E)^2} \le 10 + 9y_1^p(t)^2 \le \frac{16\pi^2}{T(E)^2}\,.$$

By arguing as in the proof of Theorem 4, see (48), we reach the bounds (31) which are more stringent than (49). Whence, these are the bounds for the torsional stability of the first vertical oscillating mode.

For the second vertical mode $(Y_2, 0) = (0, y_2^p, 0, 0)$ we proceed similarly, but now system (30) reads

$$\begin{cases} \ddot{z}_1(t) + (7 + 9y_2^p(t)^2)z_1(t) = 0 \\ \ddot{z}_2(t) + (10 + \frac{27}{2}y_2^p(t)^2)z_2(t) = 0\,. \end{cases} \tag{54}$$

Concerning the first equation, a different stability criterion for the Hill equation due to Zhukovskii [36], see also [35, Chapter VIII], states that the trivial solution is stable provided that

$$0 \le 7 + 9y_2^p(t)^2 \le \frac{4\pi^2}{T_2(E)^2}\,. \tag{55}$$

The left inequality is always satisfied while the second inequality is satisfied if $7 + 9\|y_2^p\|_\infty^2 \le \frac{4\pi^2}{T_2(E)^2}$. Whence, by (51) a sufficient condition for the stability is

$$7 + 9\|y_2^p\|_\infty^2 \le \frac{3}{4}\Lambda_+^2(E) \iff E \le \frac{38}{3}\,,\ \|y_2^p\|_\infty \le \frac{2}{\sqrt{3}}\,. \tag{56}$$

Next we focus on the second equation in (54). The stability of the trivial solution is ensured if

$$0 \le 10 + \frac{27}{2}y_2^p(t)^2 \le \frac{4\pi^2}{T_2(E)^2}\,,$$

that is, if $10 + \frac{27}{2}\|y_2^p\|_\infty^2 \le \frac{4\pi^2}{T_2(E)^2}$. Whence, by (51) with $j = 2$, a sufficient condition for the stability is

$$10 + \frac{27}{2}\|y_2^p\|_\infty^2 \le \frac{3}{4}\Lambda_+^2(E) \iff E \le \frac{5024}{867}\,,\ \|y_2^p\|_\infty \le \sqrt{\frac{32}{51}}\,. \tag{57}$$

This is more restrictive than (56) and is therefore a sufficient condition for the stability of the second vertical oscillating mode y_2^p.

7 Appendix: general n-modes systems

Definitions 6 and 7 hold for general n-modes systems. The main difference is that, for $n \ge 3$, the equations of the linearized system may not decouple. For instance, when $n = 3$, we put

$$y(x,t) = y_1(t)\sin x + y_2(t)\sin(2x) + y_3(t)\sin(3x)\,, \quad z(x,t) = z_1(t)\sin x + z_2(t)\sin(2x) + z_3(t)\sin(3x)\,.$$

By (16), Lemma 2, and by arguing as for (28), we find that y_j and z_j satisfy the system

$$\left\{\begin{array}{rcl}
\ddot{y}_1+3y_1+\frac{9}{2}y_1z_1^2+3y_1z_2^2+3y_1z_3^2+3y_1y_2^2+3y_1y_3^2-3y_1z_1z_3-\frac{3}{2}y_1^2y_3 & & \\
+\frac{3}{2}y_1^3+6y_2z_1z_2+3y_2z_2z_3+\frac{3}{2}y_2^2y_3-\frac{3}{2}y_3z_1^2+\frac{3}{2}y_3z_2^2+6y_3z_1z_3 & = & 0\\
\ddot{y}_2+18y_2+3y_2z_1^2+\frac{9}{2}y_2z_2^2+3y_2z_3^2+3y_2y_1^2+3y_2y_3^2+3y_2z_1z_3+3y_2y_1y_3 & & \\
+\frac{3}{2}y_2^3+6y_1z_1z_2+3y_1z_2z_3+3y_3z_1z_2+6y_3z_2z_3 & = & 0\\
\ddot{y}_3+83y_3+3y_3z_1^2+3y_3z_2^2+\frac{9}{2}y_3z_3^2+3y_3y_1^2+3y_3y_2^2+\frac{3}{2}y_3^3 & & \\
-\frac{3}{2}y_1z_1^2+\frac{3}{2}y_1z_2^2+\frac{3}{2}y_1y_2^2+6y_1z_1z_3-\frac{1}{2}y_1^3+3y_2z_1z_2+6y_2z_2z_3 & = & 0\\
\ddot{z}_1+7z_1+\frac{27}{2}z_1y_1^2+9z_1y_2^2+9z_1y_3^2-9z_1y_1y_3+9z_1z_2^2+9z_1z_3^2-\frac{9}{2}z_1^2z_3 & & \\
+\frac{9}{2}z_1^3+9z_2y_2y_3+18z_2y_1y_2+\frac{9}{2}z_3z_2^2+18z_3y_1y_3-\frac{9}{2}z_3y_1^2+\frac{9}{2}z_3y_2^2 & = & 0\\
\ddot{z}_2+10z_2+9z_2y_1^2+\frac{27}{2}z_2y_2^2+9z_2y_3^2+9z_2y_1y_3+9z_2z_1^2+9z_2z_3^2+9z_1z_2z_3 & & \\
+\frac{9}{2}z_2^3+18z_1y_1y_2+9z_1y_2y_3+9z_3y_1y_2+18z_3y_2y_3 & = & 0\\
\ddot{z}_3+15z_3+9z_3y_1^2+9z_3y_2^2+\frac{27}{2}z_3y_3^2+9z_3z_1^2+9z_3z_2^2+\frac{9}{2}z_3^3-\frac{9}{2}z_1y_1^2 & & \\
+\frac{9}{2}z_1y_2^2+\frac{9}{2}z_1z_2^2+18z_1y_1y_3-\frac{3}{2}z_1^3+9z_2y_1y_2+18z_2y_2y_3 & = & 0\,,
\end{array}\right. \tag{58}$$

while the energy becomes

$$\begin{gathered}
E=\tfrac{1}{2}(\dot{y}_1^2+\dot{y}_2^2+\dot{y}_3^2)+\tfrac{1}{6}(\dot{z}_1^2+\dot{z}_2^2+\dot{z}_3^2)+\tfrac{3}{2}y_1^2+9y_2^2+\tfrac{83}{2}y_3^2+\tfrac{7}{6}z_1^2+\tfrac{5}{3}z_2^2+\tfrac{5}{2}z_3^2\\
+\tfrac{3}{2}\left(y_1^2y_2^2+y_1^2y_3^2+y_2^2y_3^2+y_1^2z_2^2+y_1^2z_3^2+y_2^2z_1^2+y_2^2z_3^2+y_3^2z_1^2+y_3^2z_2^2+z_1^2z_2^2+z_1^2z_3^2+z_2^2z_3^2\right)\\
+\tfrac{3}{2}\left(y_1y_3y_2^2+y_1y_3z_2^2+z_1z_3y_2^2+z_1z_3z_2^2-z_1z_3y_1^2-y_1y_3z_1^2\right)\\
+\tfrac{9}{4}\left(y_1^2z_1^2+y_2^2z_2^2+y_3^2z_3^2\right)+\tfrac{1}{2}\left(y_3y_1^3+z_3z_1^3\right)+6\left(y_1y_2z_1z_2+y_1y_3z_1z_3+y_2y_3z_2z_3\right)\\
+3\left(y_1y_2z_2z_3+y_2y_3z_1z_2\right)+\tfrac{3}{8}\left(y_1^4+y_2^4+y_3^4+z_1^4+z_2^4+z_3^4\right)\,.
\end{gathered}$$

By considering the torsional components of the linearized system around $(Y_j,0)$, with Y_j having just the j-th nontrivial component y_j^p, we reduce to the following system of differential equations with periodic coefficients:

$$\left\{\begin{array}{l}
\ddot{z}_1(t)+a_{1,j}(t)z_1(t)+b_j(t)z_3(t)=0\\
\ddot{z}_2(t)+a_{2,j}(t)z_2(t)=0\\
\ddot{z}_3(t)+a_{3,j}(t)z_3(t)+b_j(t)z_1(t)=0\,,
\end{array}\right. \tag{59}$$

where $a_{i,j}(t)=i^2+6+9\alpha_{i,j}y_j^p(t)^2$, with $\alpha_{i,j}=1$ if $i\neq j$ and $\alpha_{i,i}=\frac{3}{2}$, and $b_j(t)=(-1)^j\frac{9}{2}y_j^p(t)^2$ if $j=1,2$ and $b_3(t)=0$. Differently from the 2-mode case, the above equations are coupled. The stability of such systems can be studied as in [35, Th.II vol.1]. Here, to avoid too many distinctions, we fix j and for every i we consider (59) subject the conditions $z_k(0)=0$, $\dot{z}_k(0)=0$ for $k\neq i$; this procedure decouples the equations and, once more, we reduce to Hill equations of the form (22).

Proposition 12. *For $j=1,2,3$ fixed, let $Z_i^j=(0,...,z_i^j,...,0)$ be the solution to system* (59) *subject the conditions $z_k(0)=\dot{z}_k(0)=0$ for $k\neq i$.*

If the first vertical oscillating mode y_1^p of (58) *satisfies*

$$\|y_1^p\|_\infty\leq\frac{1}{\sqrt{3}}\approx 0.577\iff E\leq\frac{13}{24}\approx 0.542\,, \tag{60}$$

then Z_i^1 is stable for every $i = 1, 2, 3$.

If the second vertical oscillating mode y_2^p of (58) *satisfies*

$$\|y_2^p\|_\infty \leq \frac{2}{\sqrt{11}} \approx 0.603 \iff E \leq \frac{402}{121} \approx 3.322\,,$$

then Z_i^2 is stable for every $i = 1, 2, 3$.

If the third vertical oscillating mode y_3^p of (58) *satisfies*

$$\|y_3^p\|_\infty \leq \frac{4}{\sqrt{3}} \approx 2.309 \iff E \leq 232\,,$$

then Z_i^3 is stable for every $i = 1, 2, 3$.

Proof. The three vertical oscillating modes y_j^p satisfy (26) with $j = 1, 2, 3$ and the analysis performed at the beginning of Section 6 is valid. Then, for $j = 1, 2, 3$ fixed, the nontrivial component of Z_i^j satisfies the Hill equation

$$\ddot{z}(t) + a_{i,j}(t)z(t) = 0 \tag{61}$$

for $i = 1, 2, 3$, with $a_{i,j}$ as in (59). Since $a_{i,j}$ is a positive $T_j/2$-periodic function, we exploit a stability criterion for the Hill equation due to Zhukovskii [36] which states that the trivial solution to (61) is stable provided that

$$\frac{4m^2\pi^2}{T_j(E)^2} \leq a_{i,j}(t) \leq \frac{4(m+1)^2\pi^2}{T_j(E)^2} \tag{62}$$

for some integer $m \geq 0$.

Fix $j = 1$, from the proofs of Theorems 4 and 8 we know that (62), with $m = 1$ and $i = 1$, namely (47), holds if (49) is satisfied. While (62) with $m = 1$ and $i = 2$ holds when (31) is satisfied. If $i = 3$, we test (62) with $m = 2$. By (52), the left inequality in (62) is satisfied if $4\sqrt{9 + 6E} \leq 15$, namely $E \leq \frac{27}{32}$. By (50) and (51), the right inequality in (62) follows if

$$15 + 9\Lambda_-^1(E) \leq \frac{27}{4}\Lambda_+^1(E)\,,$$

namely if $E \leq \frac{56}{3}$. The more restrictive is (31) and yields (60).

Fix $j = 2$. We test (62) with $m = 0$ for $i = 1, 2, 3$. When $i = 1$ and $i = 2$, (62) holds provided (56), respectively (57), is satisfied. When $i = 3$, by (50) and (51), (62) follows if

$$15 + 9\Lambda_-^2(E) \leq \frac{3}{4}\Lambda_+^2(E)\,,$$

namely if $E \leq \frac{402}{121}$ and $\|y_2^p\|_\infty \leq \sqrt{\Lambda_-^2(E)} \leq \frac{2}{\sqrt{11}}$.

Fix $j = 3$. By (50) and (51), (62) with $m = 0$ and $i = 1, 2, 3$ follows if

$$i^2 + 6 + 9\alpha_{i,3}\Lambda_-^3(E) \leq \frac{3}{4}\Lambda_+^3(E)\,.$$

Some computations yield $E \leq \frac{150328}{363}$ when $i = 1$, $E \leq \frac{143956}{363}$ if $i = 2$ and $E \leq 232$ if $i = 3$. The latter is the more restrictive and the conclusion follows. □

The same proof of Proposition 12 can be generalized to the case of n-modes. Indeed, by considering the torsional part of the linearization of system (11) around $(Y_j, 0)$ subject the conditions $z_k(0) = \dot{z}_k(0) = 0$ for $k \neq i$ we still obtain (61) with $a_{i,j}$ as in (59). Then, we may exploit the stability criterion (62). As $E \to 0$, (62) reads

$$m^2(j^4 + 2) \leq i^2 + 6 \leq (m+1)^2(j^4 + 2)\,. \tag{63}$$

For $1 \leq j \leq n$ fixed and for every $i = 1, ..., n$, we denote by $m_i = m_i(j)$ the least integer such that the right inequality holds in (63). Furthermore, let $L_i = L_i(j) > 0$ be such that the inequalities

$$m_i^2 \sqrt{(j^4+2)^2 + 6E} \leq i^2 + 6\,, \qquad i^2 + 6 + 9\alpha_{i,1}\Lambda_-^j(E) \leq \frac{3}{4}(m_i+1)^2 \Lambda_+^j(E)$$

hold for every $E \leq L_i$. From (50), (51) and (52), the above inequalities yield (62) with $m = m_i(j)$ and $E \leq \min_{1\leq i\leq n}\{L_i(j)\}$ and, in turn, the stability of (61) follows. Summarizing, we have proved

Proposition 13. *Let $\Lambda_-^j(E)$ and $L_i(j)$ be as defined above. Furthermore, let $Z_i^j = (0, ..., z_i^j, ..., 0)$ be the solution to the torsional part of to the linearized system (11) around $(Y_j, 0)$ and subject to the conditions $z_k(0) = \dot{z}_k(0) = 0$ for $k \neq i$.*

For $1 \leq j \leq n$ fixed, if the j-th vertical oscillating mode $Y_j = (0, ..., y_j^p, ...0)$ of system (11) at energy E satisfies

$$\|y_j^p\|_\infty \leq \sqrt{\Lambda_-^j(E)} \iff E \leq \min_{1\leq i\leq n}\{L_i(j)\}\,.$$

then Z_i^j is stable for every $i = 1, 2, 3$.

Remark 14. *When $j = n$, from (63), $m_i(n) = 0$ for every $i = 1, ..., n$. Then, if $\alpha_{i,j}$ is as in (59), (62) (with $m = 0$) follows if*

$$i^2 + 6 + 9\alpha_{i,n}\Lambda_-^n(E) \leq \frac{3}{4}\Lambda_+^n(E)\,,$$

that is,

$$L_i(n) := \frac{1}{6}\left[\frac{(n^4+2)(12\alpha_{i,n}+1) - 2(i^2+6)}{12\alpha_{i,n}-1}\right]^2 - \frac{(n^4+2)^2}{6}\,.$$

On the other hand, if $j = 1$, it is not difficult to verify that $m_i(1) > 0$ for every $i = 1, ..., n$.

We conclude by describing (with no plots) the numerical results obtained when $n = 3$. We found that the first vertical oscillating mode is stable up to $\|y_1\|_\infty \approx 1.07 \Leftrightarrow E \approx 2.21$: at this level, the instability of the z_1 and z_3 torsional oscillations appears while z_2 remains small. We then increased the energy: when we reached $\|y_1\|_\infty \approx 2.1 \Leftrightarrow E \approx 13.91$ also the z_2 oscillations lost stability. By increasing further the energy, the amplitude of the z_2 oscillations increased while the amplitudes of the z_1 and z_3 oscillations decreased: at $\|y_1\|_\infty \approx 3 \Leftrightarrow E \approx 43.87$ all the energy is transferred onto z_2 while z_1 and z_3 appear stable. A further increase of the energy lead to "chaos", that is, a disordered distribution of the energy from y_1 to all the other components y_2, y_3, z_1, $_2$, z_3.

We then found that the second vertical oscillating mode y_2 was stable up to $\|y_2\|_\infty \approx 0.69 \Leftrightarrow E \approx 4.37$, showing that Proposition 12 gives a fairly good condition: at this energy level, an instability of the z_1 and z_3 torsional oscillations appears while z_2 remains small. At $\|y_2\|_\infty \approx 0.8 \Leftrightarrow E \approx 5.91$ also the z_2 oscillations appear unstable, although its oscillations are quite small when compared with z_1 and z_3. Then unpredictable behaviors start. For instance, when we reach $\|y_2\|_\infty \approx 3 \Leftrightarrow E \approx 111.38$, we are back in the situation where z_1 and z_3 are (equally) unstable while z_2 appears stable. More generally, a quite chaotic behavior was apparent.

The numerical results suggest that the third vertical oscillating mode y_3 is stable up to $\|y_3\|_\infty \approx 2.7 \Leftrightarrow E \approx 322$, showing that Proposition 12 gives a fairly good condition: at this energy level, an instability of the z_3 torsional oscillation appears while z_1 and z_2 remain small. At $\|y_3\|_\infty \approx 3 \Leftrightarrow E \approx 404$ the behavior appears disordered and we could not detect any precise behavior; the only fact was that the z_1 and z_2 instability manifested delayed in time.

We tried many other experiments but, yet, we could not detect a general rule. The only evident thing is that the instability was lost for values slightly larger than the ones in Proposition 12.

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