

Classical dilations à la Hudson-Parthasarathy of Markov semigroups

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Abstract

We study the Classical Probability analogue of the dilations of a quantum dynamical semigroup defined in Quantum Probability via quantum stochastic differential equations. Given a homogeneous Markov chain in continuous time in a finite state space E , we introduce a second system, an environment, and a deterministic invertible time-homogeneous global evolution of the system E with this environment such that the original Markov evolution of E can be realized by a proper choice of the initial random state of the environment. We also compare this dilations with the dilations of a quantum dynamical semigroup in Quantum Probability: given a classical Markov semigroup, we extend it to a proper quantum dynamical semigroup for which we can find a Hudson-Parthasarathy dilation which is itself an extension of our classical dilation.

AMS Subject Classification: 60J27, 81S25

1 Introduction

We study the analogue in Classical Probability of the dilations in Quantum Probability of a quantum dynamical semigroup (QDS) in continuous time. A QDS T_t describes the evolution of a quantum system, possibly open, but “Markovian”, and homogeneous in time. If a QDS is uniformly continuous, then it is always possible to introduce a Hudson-Parthasarathy quantum stochastic differential equation and to employ its solution to dilate T_t by a quantum stochastic flow j_t . Such a dilation allows to represent the QDS by the conditional expectation of a quantum Markov process, analogously to the representation of a classical Markov semigroup (CMS) by a classical Markov process. Anyway, such a dilation in Quantum Probability enjoys a richer structure which allows to dilate the semigroup T_t at the same time also by a strongly continuous unitary group U_t , thus showing that the system evolution T_t and the flow j_t do not contradict the axioms of Quantum Mechanics, i.e. that they can arise from a Hamiltonian evolution U_t of an isolated bigger system, consisting of the given system and its environment [1, 6, 7, 11, 15–19]. In particular this implies that the Hamiltonian operator generating U_t gives an infinitesimal description of the dilation which is alternative, but equivalent, to that given by the Hudson-Parthasarathy equation [3, 4, 8, 9]. This feature differentiates the representation of a QDS by means of a quantum stochastic flow from the representation of a CMS by means of a Markov process, as usually there is no “Hamiltonian evolution” associated to this latter.

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Dilations analogous to the quantum ones have been recently introduced in Classical Probability for CMS in discrete time [10]. The first aim of this paper is to introduce them also in the continuous time context, choosing a self-contained approach in a completely classical framework. These classical dilations are interesting, not only to better understand the relationship between the two probabilistic theories, but also from a simply classical point of view, to better understand the relationship between Markov processes and deterministic invertible homogeneous dynamics. Indeed, we show that every homogeneous Markov chain in continuous time in a finite state space can always be realized as a deterministic invertible homogeneous evolution of the system coupled with a second system. The existence of such representations is theoretically relevant if Markov chains are applied to phenomena, like physical phenomena for example, for which an underlying theory postulates deterministic invertible homogeneous evolutions in absence of noise and external disturbances. For these phenomena the second system introduced by the dilation models the surrounding world, the environment, the source of the noise, which is given now a dynamical explanation. Of course, here the characterizing property is not simply that the global evolution is deterministic, as in innovation theory [20, 21], but that it is also invertible and homogeneous in time.

More precisely, we consider a system with finite state space E , undergoing a continuous time evolution given by a homogeneous Markov chain. Then, we introduce an environment with its state space (Γ, \mathcal{G}) , a measurable space, together with a group of measurable maps $\alpha_t : E \times \Gamma \rightarrow E \times \Gamma$, $t \in \mathbb{R}$, describing a global evolution. Thus, if (i, γ) is the state of the compound system at time 0, then $\alpha_t(i, \gamma)$ is its state at time t , where hence α_t gives a deterministic invertible homogeneous global evolution. Nevertheless, if the environment state is never observed and if initially it is randomly distributed with some law \mathbb{Q} on (Γ, \mathcal{G}) , then the evolution of the observed system turns out to be stochastic and, if Γ , \mathcal{G} , α_t and \mathbb{Q} are properly built, it is given by the original Markov chain. In this case, we say that $(\Gamma, \mathcal{G}, \alpha_t, \mathbb{Q})$ is a dilation of the Markov evolution in E .

Actually, as in the discrete time context, given only the state space E (arbitrary but finite), we build a *universal* dilation $(\Gamma, \mathcal{G}, \alpha_t, \{\mathbb{Q}\})$, where $\{\mathbb{Q}\}$ is an entire family of distributions which can produce any Markov chain in E : every Markov chain can be dilated by taking always the same model $(\Gamma, \mathcal{G}, \alpha_t)$ for the environment and the global evolution, and by choosing every time the proper distribution \mathbb{Q} for the initial state of the environment. Moreover, not only our construction allows to interpret each Markov chain as the stochastic dynamics resulting from the coupling with an environment, but at the same time it also represents the chain via an innovation process, where the innovation now is dynamically provided by the environment.

Our aim is similar to the aim of Lewis and Maassen [14] when they consider classical mechanics and, taken a linear Hamiltonian system modelling a particle and its environment, they describe how Gibbs states of the whole system lead to stationary Gaussian stochastic processes for the observables pertaining to the particle under consideration. However, we do not look for good global states, but for good states \mathbb{Q} of the environment alone which lead to Markov evolutions of the system E , our particle, for every independent choice of its initial state.

The second aim of the paper is to show that such dilations in Classical Probability are really analogous to the quantum dilations which inspire them. We prove that every CMS in E , considered on any finite time interval, admits an extension to a QDS for which we can find a quantum dilation which is itself an extension of the classical dilation $(\Gamma, \mathcal{G}, \alpha_t, \mathbb{Q})$ that we build for the CMS.

Thus these dilations are explicit constructions of that classical structures which would appear by studying the abelian algebras left invariant by quantum stochastic flows (§4.27 in [1]).

However, we shall not embed a whole universal dilation $(\Gamma, \mathcal{G}, \alpha_t, \{\mathbb{Q}\})$ in the quantum world,

as quantum dilations do not exhibit the same universality and they strictly depend on the QDS under consideration, so that it is not enough to change the environment state to get another QDS.

In the sequel, given a complex function f on a domain E , we shall denote with the same symbol f also its extension on a domain $E \times \Gamma$, $f(i, \gamma) = f(i)$. Similarly, given a map $\phi : E \rightarrow E$, we shall denote with the same symbol ϕ also its extension, by tensorizing with the identity, on a domain $E \times \Gamma$ to $E \times \Gamma$, $\phi(i, \gamma) = (\phi(i), \gamma)$.

2 Preliminaries

We consider a system with finite state space $E = \{1, \dots, N\}$ and power σ -algebra \mathcal{E} , fixed for the whole paper. We denote by $P = (P_{ij})_{i,j \in E}$ a stochastic matrix in E , so that $P_{ij} \geq 0$ and $\sum_j P_{ij} = 1$ for every i . As usual, we identify the elements of the complex abelian $*$ -algebra $\mathcal{L}^\infty(\mathcal{E})$, the system random variables $f : E \rightarrow \mathbb{C}$, with the column vectors in \mathbb{C}^N , so that every P defines an operator in $\mathcal{L}^\infty(\mathcal{E})$, which describes the one-step evolution $f \mapsto Pf$, $f \in \mathcal{L}^\infty(\mathcal{E})$. We denote by D a deterministic matrix in E , that is a stochastic matrix with a 1 in each row. Every D describes with matrix terminology a deterministic evolution β , where

$$D = (D_{ij})_{i,j \in E}, \quad \beta : E \rightarrow E, \quad D_{ij} = \delta_{\beta(i),j}, \quad (1)$$

so that $Df = f \circ \beta$. The invertible maps in E correspond to the special cases of permutation matrices. Labelled all deterministic matrices and the corresponding maps with indexes ℓ belonging to $L = \{1, \dots, N^N\}$, every stochastic matrix P determines the weights

$$p_\ell = P_{1\beta_\ell(1)} \cdots P_{N\beta_\ell(N)}, \quad \ell \in L, \quad (2)$$

which give a probability on the power σ -algebra of L and provide the representation

$$P = \sum_{\ell \in L} p_\ell D_\ell, \quad p_\ell \geq 0, \quad \sum_{\ell \in L} p_\ell = 1. \quad (3)$$

We denote by $R = (R_{ij})_{i,j \in E}$ a transition rate matrix in E , so that $R_{ij} \geq 0$ for every $i \neq j$ and $\sum_j R_{ij} = 0$ for every i . Every R generates a CMS e^{Rt} , $t \geq 0$, which consists of stochastic matrices. It describes the continuous-time homogeneous evolution

$$f \mapsto e^{Rt} f, \quad f \in \mathcal{L}^\infty(\mathcal{E}), \quad t \geq 0. \quad (4)$$

For every CMS e^{Rt} , there exists a Markov chain $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_k)_{k \in E})$ with transition probability functions given by e^{Rt} , i.e. a continuous-time stochastic process of random variables $X_t : \Omega \rightarrow E$, adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$, and a family of probability measures \mathbb{P}_k , $k \in E$, such that the starting distribution of the process depends on k , X_0 has Dirac distribution δ_k under \mathbb{P}_k , but the process always enjoys the Markov property with transition matrices e^{Rt} :

$$\mathbb{P}_k(X_{t+s} = j | \mathcal{F}_t) = \mathbb{P}_k(X_{t+s} = j | X_t) = (e^{Rs})_{X_t j}, \quad \forall k, j \in E, \quad t, s \geq 0.$$

Thus, a system random variable $f \in \mathcal{L}^\infty(\mathcal{E})$ has now a stochastic evolution given by the $*$ -unital homomorphism

$$j_t : \mathcal{L}^\infty(\mathcal{E}) \rightarrow \mathcal{L}^\infty(\mathcal{F}_t), \quad f \mapsto j_t(f) := f(X_t), \quad t \geq 0, \quad (5)$$

and the evolution (4) admits the representation

$$\left(e^{Rt}f\right)(k) = \mathbb{E}_k[f(X_t)], \quad \forall f \in \mathcal{L}^\infty(\mathcal{E}). \quad (6)$$

By means of the uniformization technique, a continuous-time Markov chain can always be realized as a discrete-time Markov chain moved by an independent Poisson process. Given the transition rate matrix R , taken a rate $\lambda > 0$ and a stochastic matrix P such that

$$R = \lambda(P - \mathbb{1}), \quad (7)$$

taken a discrete-time Markov chain X_n^* with transition matrix P and an independent Poisson process $N(t)$ with rate λ , then the process $X_t = X_{N(t)}^*$ is a continuous-time Markov chain with transition rate matrix R (e.g. [5]). For example, one can take

$$\lambda = \max_{i \in E} \{-R_{ii}\}, \quad P = \mathbb{1} + \frac{1}{\lambda} R. \quad (8)$$

Moreover, by means of representation (3) for P , it is always possible to realize the Markov chain via an innovation process. Let $\Gamma_{\mathbb{R}_+}^L$ be the canonical space of a marked simple point process on \mathbb{R}_+ with mark space L , that is the set of sequences $\gamma = (\ell_n, t_n)_{n \in \mathbb{N}}$ where $\ell_n \in L$, $t_n \in [0, +\infty]$, $0 < t_1 \leq t_2 \leq \dots \leq +\infty$, and $t_n < t_{n+1}$ for all $t_n < +\infty$ [2]. Denoting by $\Gamma_{\mathbb{R}_+}$ the canonical space of a simple point process on \mathbb{R}_+ , we have $\Gamma_{\mathbb{R}_+}^L = L^{\mathbb{N}} \times \Gamma_{\mathbb{R}_+}$. Then one can take

$$\begin{aligned} \Omega &= E \times L^{\mathbb{N}} \times \Gamma_{\mathbb{R}_+}, & \omega &= (i, (\ell_n)_{n \in \mathbb{N}}, (t_n)_{n \in \mathbb{N}}), \\ X_0(\omega) &= X_0^*(\omega) = i, & Y_n(\omega) &= \ell_n, & T_n(\omega) &= t_n, & X_n^* &= \beta_{Y_n}(X_{n-1}^*), & n \in \mathbb{N} \\ N_\ell(t) &= \sum_{n \in \mathbb{N}} I_{(Y_n = \ell, T_n \leq t)}, & N(t) &= \sum_{\ell \in L} N_\ell(t), & X_t &= X_{N(t)}^*, & t \geq 0, & (9) \\ \mathcal{F} &= \sigma(X_0) \otimes \sigma(Y_n; n \in \mathbb{N}) \otimes \sigma(T_n; n \in \mathbb{N}), & \mathcal{F}_t &= \sigma(X_0, N_\ell(s); \ell \in L, 0 \leq s \leq t), \\ \mathbb{P}_k &= \delta_k \otimes p^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}_+}^\lambda, \end{aligned}$$

where we denote by I_A the indicator of an event A , by p the probability on L associated to P , and by $Q_{\mathbb{R}_+}^\lambda$ the probability on $\sigma(T_n; n \in \mathbb{N})$ such that the random variables T_n are the arrival times of a Poisson process with rate λ . Then the random variables Y_n are i.i.d. with distribution p , the counting processes $N_\ell(t)$ are independent Poisson processes with rates $p_\ell \lambda$, the counting process $N(t)$ is a Poisson process with rate λ , the process X_n^* is a discrete-time Markov chain with transition matrix P and X_t is a continuous-time Markov chain with transition rate matrix R . The random variables Y_n are called marks. For every $t \geq 0$, the chain state X_t is a deterministic function of X_0 and of $N_\ell(s)$, $\ell \in L$, $0 \leq s \leq t$, and the random process $(N_\ell(t))_{\ell \in L}$ is an innovation process for X_t .

3 Dilations and universal dilations

Dilation of a classical Markov semigroup. We call dilation of the CMS e^{Rt} in $\mathcal{L}^\infty(\mathcal{E})$ a term

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (Z_t)_{t \geq 0}, (\mathbb{P}_k)_{k \in E}\right)$$

where

- every $Z_t = (X_t, \Upsilon_t)$ is a random variable on (Ω, \mathcal{F}) with values in $(E \times \Gamma, \mathcal{E} \otimes \mathcal{G})$, being (Γ, \mathcal{G}) a fixed measurable space,

- the term $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_k)_{k \in E})$ is a Markov chain with transition rates R ,
- the random variable (X_0, Υ_0) has distribution $\delta_k \otimes \mathbb{Q}$ under \mathbb{P}_k , being \mathbb{Q} a fixed distribution on \mathcal{G} ,
- there exists one-parameter group $(\alpha_t)_{t \in \mathbb{R}}$ of measurable maps $\alpha_t : E \times \Gamma \rightarrow E \times \Gamma$ such that $\alpha_0 = \text{Id}$ and $Z_t = \alpha_t(Z_0)$ for every $t \geq 0$.

Thus, besides the system E , a second system is introduced, an environment with state space (Γ, \mathcal{G}) . Their states X_t and Υ_t are asked to be random variables on a same measurable space (Ω, \mathcal{F}) such that the global state $Z_t = (X_t, \Upsilon_t)$ undergoes a deterministic invertible homogeneous evolution α_t . Therefore all the X_t and Υ_t are determined by Z_0 , so that X_t and Υ_t are measurable with respect to $\sigma(Z_0) = \sigma(X_0, \Upsilon_0) \subseteq \mathcal{F}$ and, depending on the probability chosen on \mathcal{F} , they are deterministic if and only if Z_0 is. Nevertheless, a probability \mathbb{P}_k typically fixes only the value of X_0 . The space (Ω, \mathcal{F}) is also endowed with a filtration \mathcal{F}_t . Note that only the X_t are asked to be adapted to \mathcal{F}_t so that, in particular, Υ_0 does not have to be \mathcal{F}_0 -measurable. Therefore the X_t are not trivially \mathcal{F}_0 -measurable, even if their values are completely determined by the values of X_0 and Υ_0 , and, neglecting the environment, each $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \mathbb{P}_k)$ can be a non trivial stochastic process. What we ask is that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \mathbb{P}_k)$ actually is a Markov chain starting from k with transition rates R . At the same time however, this Markov chain is compatible with a deterministic, invertible and homogeneous model for the evolution of E coupled with an environment Γ . In particular, as $X_0 = k$, the whole stochasticity of the process is due only to the randomness of the unobserved initial state Υ_0 of the environment.

A dilation gives another interpretation of every evolution (4), compatible with (6):

$$\left(e^{Rt} f \right)(k) = \mathbb{E}_k[f(X_t)] = \mathbb{E}_k[f(Z_t)] = \mathbb{E}_k[f(\alpha_t(k, \Upsilon_0))], \quad \forall f \in \mathcal{L}^\infty(\mathcal{E}).$$

Indeed, the stochastic evolution (5) of a system variable $f \in \mathcal{L}^\infty(\mathcal{E})$ is now described by the $*$ -unital homomorphism

$$j_t : \mathcal{L}^\infty(\mathcal{E}) \rightarrow \mathcal{L}^\infty(\mathcal{F}_t), \quad f \mapsto j_t(f) := f(X_t) = f(Z_t) = f \circ \alpha_t(Z_0), \quad (10)$$

which is injective as α_t is invertible. And now we could also consider global random variables $F : E \times \Gamma \rightarrow \mathbb{C}$ and their evolution $F \mapsto F(Z_t) = F \circ \alpha_t(Z_0)$.

Universal dilation. Let us denote by \mathcal{R} the set of transition rate matrices R in E . We call universal dilation of the CMS's in $\mathcal{L}^\infty(\mathcal{E})$ a term

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (Z_t)_{t \geq 0}, (\mathbb{P}_{k,R})_{k \in E, R \in \mathcal{R}} \right)$$

such that every $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (Z_t)_{t \geq 0}, (\mathbb{P}_{k,R})_{k \in E})$ is a dilation of the corresponding semigroup e^{Rt} . We call universal such a dilation because we ask that the same $\Omega, \mathcal{F}, \mathcal{F}_t$ and Z_t allow to represent all the CMS's in $\mathcal{L}^\infty(\mathcal{E})$, with the change of the probabilities $\mathbb{P}_{k,R}$ alone. Therefore, both the environment state space (Γ, \mathcal{G}) and the global evolution α_t depend only on the state space E , not on the particular CMS to be dilated.

Poisson dilation and Poisson universal dilation. In order to show that every state space E admits a universal dilation, now we consider a particular classes of dilations and of universal dilations. Let us describe all the special requirements we are interested in for the dilation of a CMS e^{Rt} .

First of all we want the sample space Ω to be just $E \times \Gamma$, the state space of the global system. As we want it to describe all the possible initial global states, we ask the random variable Z_0 to be the identity function and X_0 and Y_0 to be the coordinate variables: if $\omega = (i, \gamma)$, then $Z_0(\omega) = \omega$, $X_0(\omega) = i$ and $Y_0(\omega) = \gamma$. Thus, for all $t \geq 0$, $Z_t = (X_t, Y_t) = \alpha_t \circ Z_0 = Z_0 \circ \alpha_t$, $X_t = X_0 \circ \alpha_t$ and $Y_t = Y_0 \circ \alpha_t$.

We are interested in an environment with state space Γ equal to $\Gamma_{\mathbb{R}}^G$, the canonical space of a marked simple point process on \mathbb{R} with finite mark space G , that is the set of sequences $\gamma = (g_n, t_n)_{n \in \mathbb{Z}}$ where $g_n \in G$, $t_n \in [-\infty, +\infty]$, $-\infty \leq \dots \leq t_{-1} \leq t_0 \leq 0 < t_1 \leq t_2 \leq \dots \leq +\infty$, and $t_n < t_{n+1}$ for all $|t_n| < \infty$. Denoting by $\Gamma_{\mathbb{R}}$ the canonical space of a simple point process on \mathbb{R} , we have $\Gamma_{\mathbb{R}}^G = G^{\mathbb{Z}} \times \Gamma_{\mathbb{R}}$. Moreover, with clear meaning of symbols, $\Gamma_{\mathbb{R}}^G = \Gamma_{(-\infty, 0]}^G \times \Gamma_{\mathbb{R}_+}^G$. Thus the environment state γ is a whole trajectory of a marked simple point process. Later, the introduction of the global evolution α_t will allow the following rough interpretation of γ and of its time-parameter. If the environment state at time 0 is $\gamma = (g_n, t_n)_{n \in \mathbb{Z}}$, then at every instant $t_n > 0$ it will provide a sudden shock of type g_n to the system, thus causing an instantaneous transition which will be determined by the system state and by g_n .

We explicitly introduce the marks $Y_n(\omega) = g_n$, the arrival times $T_n(\omega) = t_n$, the processes $N_g(t) = \sum_{n \in \mathbb{N}} I_{(Y_n=g, 0 < T_n \leq t)}$, counting the arrivals of type g , and the process $N(t) = \sum_{g \in G} N_g(t)$, counting all the arrivals. Then, endowed G with its power σ -algebra, we want all the functions so far introduced to be measurable and so we ask \mathcal{G} to be the natural σ -algebra $\mathcal{G}_{\mathbb{R}} := \sigma(Y_0) = \sigma(Y_n; n \in \mathbb{Z}) \otimes \sigma(T_n; n \in \mathbb{Z})$ on $\Gamma_{\mathbb{R}}^G = G^{\mathbb{Z}} \times \Gamma_{\mathbb{R}}$, and \mathcal{F} to be $\mathcal{E} \otimes \mathcal{G}_{\mathbb{R}} = \sigma(X_0, Y_0)$ on Ω . Supposing that at time 0 only X_0 is observed and that later only the information carried by the processes $N_g(t)$ is acquired, we want the filtration $\mathcal{F}_t = \sigma(X_0, N_g(s); g \in G, 0 \leq s \leq t)$.

In order to get consistence between these definitions and the global evolution α_t , we ask that

$$\alpha_t = \begin{cases} \vartheta_t \circ \psi_t, & \text{if } t \geq 0, \\ \psi_{|t|}^{-1} \circ \vartheta_t, & \text{if } t \leq 0, \end{cases} \quad (11)$$

where ϑ_t and ψ_t are as follows. The family of maps ϑ_t is the group of the left shifts

$$\vartheta_t : \Gamma_{\mathbb{R}}^G \rightarrow \Gamma_{\mathbb{R}}^G, \quad \vartheta_t((g_n, t_n)_{n \in \mathbb{Z}}) = (g_n, t_n - t)_{n \in \mathbb{Z}}, \quad t \in \mathbb{R}, \quad (12)$$

where a renumbering is understood if $t_1 - t \leq 0$. Every ϑ_t is extended on Ω by tensorizing with the identity. The family of maps ψ_t has to be a right cocycle w.r.t. ϑ_t giving a global evolution which, up to ϑ_t , simply couples the system with every mark provided by the environment, always via a same invertible interaction. More precisely, first we ask an invertible map

$$\phi : E \times G \rightarrow E \times G, \quad (13)$$

which gives the instantaneous coupling between the system and a single mark. Thus the coupling between the system and m subsequent marks is given by the invertible map

$$\begin{aligned} \varphi_m : E \times \left(\prod_{n=1}^m G_n \right) &\rightarrow E \times \left(\prod_{n=1}^m G_n \right), & G_n &\equiv G, \\ \varphi_m &= \phi_m \circ \dots \circ \phi_1, & \phi_n &= \phi : E \times G_n \rightarrow E \times G_n. \end{aligned} \quad (14)$$

Then we ask that, given ϕ and φ_m , for every $t \geq 0$ the coupling between the system and the marks provided in the time-interval $(0, t]$ is given by

$$\psi_t : E \times \Gamma_{\mathbb{R}_+}^G \rightarrow E \times \Gamma_{\mathbb{R}_+}^G, \quad \psi_t(i, (g_n, t_n)_{n \in \mathbb{N}}) = (i', (g'_n, t'_n)_{n \in \mathbb{N}}), \quad (15)$$

where

$$t_m \leq t < t_{m+1} \quad \Rightarrow \quad \begin{cases} (i', (g'_n)_{n=1}^m) = \varphi_m(i, (g_n)_{n=1}^m), \\ g'_n = g_n, \\ t'_n = t_n, \end{cases} \quad \begin{array}{l} \text{if } n > m, \\ \text{if } n \in \mathbb{N}. \end{array} \quad (16)$$

Extended also ψ_t on Ω , for every invertible ϕ definitions (13)-(16) automatically give a family of bimeasurable maps satisfying the cocycle property

$$\psi_{t+s} = \vartheta_{-t} \circ \psi_s \circ \vartheta_t \circ \psi_t, \quad \forall t, s \geq 0,$$

and so definition (11) automatically give a group of measurable maps in $E \times \Gamma_{\mathbb{R}}^G$. Moreover, denoting by φ_m^E the projection of φ_m on E , the state of the system at a positive time t is $X_t = X_0 \circ \alpha_t = X_0 \circ \psi_t = \varphi_{N(t)}^E(X_0, Y_1, \dots, Y_{N(t)})$, which is automatically adapted to \mathcal{F}_t . Roughly speaking, when the system and the environment evolve from time 0 to time $t > 0$, first the map ψ_t couples the initial state X_0 of the system with the marks $Y_1, \dots, Y_{N(t)}$, giving the system state $X_t = \varphi_{N(t)}^E(X_0, Y_1, \dots, Y_{N(t)})$, and then the shift ϑ_t prepares the subsequent marks for the future couplings with the system. Thus ϑ_t could be interpreted as a free evolution of the environment.

At long last, we consider the probabilities \mathbb{P}_k . Of course, they have to be factorized as $\delta_k \otimes \mathbb{Q}$ on $\mathcal{F} = \mathcal{E} \otimes \mathcal{G}_{\mathbb{R}}$. We also require the distribution \mathbb{Q} to be factorized as $q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda}$ on $\mathcal{G}_{\mathbb{R}} = \sigma(Y_n; n \in \mathbb{Z}) \otimes \sigma(T_n; n \in \mathbb{Z})$, where q is a probability on G , and $Q_{\mathbb{R}}^{\lambda}$ is the probability on $\sigma(T_n; n \in \mathbb{Z})$ such that the random variables T_n are the arrival times of a Poisson process with some rate λ . Then the marks Y_n are i.i.d. with distribution q , the counting processes $N_g(t)$ are independent Poisson processes with rates $q_g \lambda$, and $N(t)$ is a Poisson process with rate λ . As we shall verify in the following Proposition, this guarantees the Markov property for the process X_t with respect to \mathcal{F}_t , so that the only point is to check if the resulting rates are the desired ones.

A dilation like this will be called Poisson in the following. Summarizing, a dilation is Poisson if

- $(\Gamma, \mathcal{G}) = (\Gamma_{\mathbb{R}}^G, \mathcal{G}_{\mathbb{R}})$, the canonical space of a marked simple point process on \mathbb{R} with finite mark space G ,
- $\Omega = E \times \Gamma_{\mathbb{R}}^G$, $\omega = (i, \gamma) = (i, (g_n, t_n)_{n \in \mathbb{Z}}) \in \Omega$, $i \in E$, $\gamma \in \Gamma_{\mathbb{R}}^G$, $g_n \in G$, $t_n \in \mathbb{R}$,
- $X_0(\omega) = i$, $\Upsilon_0 = (Y_n, T_n)_{n \in \mathbb{Z}}$, $\Upsilon_0(\omega) = \gamma$, $Y_n(\omega) = g_n$, $T_n(\omega) = t_n$, $Z_0(\omega) = \omega$,
- $N_g(t) = \sum_{n \in \mathbb{N}} I_{(Y_n = g, 0 < T_n \leq t)}$, $N(t) = \sum_{g \in G} N_g(t)$,
- $\mathcal{F} = \mathcal{E} \otimes \mathcal{G}_{\mathbb{R}} = \sigma(X_0) \otimes \sigma(Y_n; n \in \mathbb{Z}) \otimes \sigma(T_n; n \in \mathbb{Z})$,
- $\mathcal{F}_t = \sigma(X_0, N_g(s); g \in G, 0 \leq s \leq t)$, $t \geq 0$,
- $\mathbb{P}_k = \delta_k \otimes \mathbb{Q} = \delta_k \otimes q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda}$,
- α_t given by (11), with the shift ϑ_t given by (12) and with a cocycle ψ_t given by (13)-(16).

Then, for every $t \geq 0$ we have

- $Z_t = \alpha_t \circ Z_0 = Z_0 \circ \alpha_t = (X_t, \Upsilon_t)$,
- $\Upsilon_t = \Upsilon_0 \circ \alpha_t$,
- $X_t = X_0 \circ \alpha_t = X_0 \circ \psi_t = \varphi_{N(t)}^E(X_0, Y_1, \dots, Y_{N(t)})$.

A Poisson dilation is therefore specified by the term (G, ϕ, \mathbb{Q}) .

With a Poisson dilation the evolution of every global random variable is described by the group of $*$ -automorphisms

$$J_t : \mathcal{L}^\infty(\mathcal{F}) \rightarrow \mathcal{L}^\infty(\mathcal{F}), \quad J_t(F) = F \circ \alpha_t, \quad t \in \mathbb{R}.$$

Note that a Poisson dilation realizes a Markov chain via an innovation process, with the innovation dynamically provided by the environment.

Let ϕ^E denote the projection of ϕ on E .

Proposition 1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_k)_{k \in E})$ be a stochastic process given by a Poisson dilation $(G, \phi, q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^\lambda)$ for an arbitrary choice of G, ϕ, q and λ . Then the process is a Markov chain with transition rate matrix*

$$R = \lambda(P - \mathbb{1}), \quad P_{ij} = \sum_{g \in G} q_g \delta_{\phi^E(i,g),j}. \quad (17)$$

Proof. Firstly, we set

$$X_0^* = X_0, \quad X_n^* = \phi^E(X_{n-1}^*, Y_n), \quad n \in \mathbb{N}, \quad \mathcal{F}_m^* = \sigma(X_0^*, Y_n; 1 \leq n \leq m), \quad m = 0, 1, \dots$$

Then $X_t = X_{N(t)}^*$ and $(\Omega, \mathcal{F}, (\mathcal{F}_n^*)_{n \geq 0}, (X_n^*)_{n \geq 0}, (\mathbb{P}_k)_{k \in E})$ is a discrete-time Markov chain with transition matrix P . Moreover, for every $k \in E$, the independence of $(Y_n)_{n \in \mathbb{N}}$ and $(T_n)_{n \in \mathbb{N}}$ under \mathbb{P}_k implies the independence of the processes X_n^* and $N(t)$.

Secondly, by the latter independence and the Markov property of X_n^* , we show that for $f \in \mathcal{L}^\infty(\mathcal{E})$, $t \geq 0$ and $m = 0, 1, \dots$

$$\mathbb{E}_k [f(X_{N(t)+m}^*) | \mathcal{F}_t] = (P^m f)(X_{N(t)}^*). \quad (18)$$

To see this, let $A \in \sigma(N(s); 0 \leq s \leq t)$, $B \in \mathcal{F}_n^*$, $n \in \mathbb{N}$. Then

$$\begin{aligned} \int_{A \cap B \cap (N(t)=n)} f(X_{N(t)+m}^*) d\mathbb{P}_k &= \int_{A \cap B \cap (N(t)=n)} f(X_{n+m}^*) d\mathbb{P}_k \\ &= \mathbb{P}_k(A, N(t) = n) \int_B f(X_{n+m}^*) d\mathbb{P}_k = \mathbb{P}_k(A, N(t) = n) \int_B (P^m f)(X_n^*) d\mathbb{P}_k \\ &= \int_{A \cap B \cap (N(t)=n)} (P^m f)(X_n^*) d\mathbb{P}_k. \end{aligned}$$

Since $\{A \cap B \cap (N(t) = n); A \in \sigma(N(s); 0 \leq s \leq t), B \in \mathcal{F}_n^*, n \in \mathbb{N}\}$ is closed under finite intersections and generates \mathcal{F}_t , Eq.(18) follows by the Dynkin class theorem. Finally, since the Poisson process $N(t)$ has independent increments and rate λ ,

$$\begin{aligned} \mathbb{E}_k [f(X_{t+s}) | \mathcal{F}_t] &= \mathbb{E}_k [f(X_{N(t)+N(t+s)-N(t)}^*) | \mathcal{F}_t] \\ &= \sum_{m=0}^{\infty} \mathbb{E}_k [f(X_{N(t)+m}^*) | \mathcal{F}_t] \mathbb{P}_k(N(t+s) - N(t) = m) = \sum_{m=0}^{\infty} (P^m f)(X_{N(t)}^*) e^{-\lambda s} \frac{(\lambda s)^m}{m!} \\ &= (e^{\lambda(P-\mathbb{1})s} f)(X_t) \end{aligned}$$

for all $t, s \geq 0$. Hence $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_k)_{k \in E})$ is a Markov chain in E with transition rate matrix $R = \lambda(P - \mathbb{1})$. \square

A universal dilation will be called Poisson if it is given by a family of Poisson dilations (G, ϕ, \mathbb{Q}_R) , $R \in \mathcal{R}$, all of them with the same G and ϕ . Thus we fix the model for the environment and for the global evolution with G and ϕ , and then we require the existence of a family of initial distributions \mathbb{Q}_R for the environment state, each one giving rise to a different CMS for the system. A Poisson universal dilation is therefore specified by the term $(G, \phi, (\mathbb{Q}_R)_{R \in \mathcal{R}})$. A Poisson universal dilation is not uniquely determined by the state space E , but it always exists.

Theorem 2. *For every finite state space E , there exists a Poisson universal dilation $(G, \phi, (\mathbb{Q}_R)_{R \in \mathcal{R}})$ of the classical Markov semigroups in $\mathcal{L}^\infty(\mathcal{E})$.*

Proof. We only have to exhibit a proper mark space G , together with the coupling ϕ and the probability measures \mathbb{Q}_R on $\mathcal{G}_{\mathbb{R}}$.

We can take the same space G and coupling ϕ used in [10] for the analogous result in discrete-time. Given $E = \{1, \dots, N\}$ and the set L labelling the all possible maps $\beta : E \rightarrow E$, we set

$$G = E \times L, \quad i, j, k \in E, \quad \ell \in L, \quad g = (j, \ell) \in G.$$

Arbitrarily fixed $j = 1$, we focus on points $(1, \ell)$ in G . Thus, taken two points $(i, (1, \ell)) \neq (i', (1, \ell'))$ in $E \times G$, we get $(\beta_\ell(i), (i, \ell)) \neq (\beta_{\ell'}(i'), (i', \ell'))$ and so we can find an invertible map

$$\phi : E \times G \rightarrow E \times G, \quad \phi(i, (j, \ell)) = \begin{cases} (\beta_\ell(i), (i, \ell)), & \text{if } j = 1, \\ \dots, & \text{if } j \neq 1. \end{cases} \quad (19)$$

We choose an arbitrary ϕ satisfying (19).

Given a transition rate matrix R , we consider the rate λ and the stochastic matrix P associated to R by Eq.(8). Via P and Eq.(2), we obtain a probability p on L , that we use to define on $G = E \times L$ the probability

$$q = \delta_1 \otimes p.$$

Thus $P_{ij} = \sum_{\ell \in L} p_\ell \delta_{\beta_\ell(i), j} = \sum_{g \in G} q_g \delta_{\phi^E(i, g), j}$ and, if we define $\mathbb{Q}_R = q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^\lambda$, then every stochastic process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_{k, R})_{k \in E})$ is a Markov chain with rates R , independently of the definition of $\phi(i, (j, \ell))$ for $j \neq 1$. Therefore $(G, \phi, (\mathbb{Q}_R)_{R \in \mathcal{R}})$ is a Poisson universal dilation of the classical Markov semigroups in $\mathcal{L}^\infty(\mathcal{E})$. \square

Similarly to the corresponding construction in discrete-time, even if each $g \in G$ has two components, $g = (j, \ell)$, the probability constructed in the proof is always concentrated only on those g of the kind $g = (1, \ell)$, but we need the first component j to define an invertible ϕ . Analogously, we are considering the evolution only for positive times so that all the (g_n, t_n) , $n \leq 0$, are never involved in the interaction with the system, but they are needed to define an invertible shift ϑ_t .

We could find also other probabilities on \mathcal{F} , different from $\mathbb{P}_{k, R}$, but inducing the same law for the process X_t . Indeed, not only $(Y_n, T_n)_{n \leq 0}$ does not effect the evolution of $X_t = X_0 \circ \psi_t$, but the representation (7) usually holds for other λ and P different from (8), just as the representation (3) usually holds also for other probabilities p different from (2), and all these different choices could be as well employed in the construction.

The cocycle approach to Poisson dilations. Given a CMS e^{Rt} with a Poisson dilation $(G, \phi, q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda})$, where G, ϕ, λ, q can be defined as in the proof of Theorem 2 or not, we can rewrite the $*$ -unital injective homomorphism (10) as

$$j_t : \mathcal{L}^{\infty}(\mathcal{E}) \rightarrow \mathcal{L}^{\infty}(\mathcal{F}_t), \quad f \mapsto j_t(f) := f(X_t) = f \circ \psi_t, \quad t \geq 0. \quad (20)$$

If we denote by $\mathbb{E}_g[f \circ \phi]$ the system random variable in $\mathcal{L}^{\infty}(\mathcal{E})$ defined by $i \mapsto f \circ \phi(i, g)$, then the stochastic evolution (20) satisfies

$$j_0(f) = f(X_0), \quad j_t(f) = \sum_{g \in G} j_t \left(\mathbb{E}_g[f \circ \phi] - f \right) dN_g(t), \quad \forall f \in \mathcal{L}^{\infty}(\mathcal{E}), t \geq 0. \quad (21)$$

For every $f \in \mathcal{L}^{\infty}(\mathcal{E})$, this is a stochastic differential equation for the \mathcal{F}_t -adapted process $j_t(f)$ with respect to the noises $N_g(t)$.

More generally, as long as we consider only system random variables neglecting the environment, we can define a Markov chain with rates R avoiding α_t and ϑ_t and employing only the deterministic, invertible, but inhomogeneous global evolution ψ_t , which never involves the environment components (g_n, t_n) for $n \leq 0$. That means to reduce the sample space Ω from $E \times \Gamma_{\mathbb{R}}^G$ to $E \times \Gamma_{\mathbb{R}^+}^G$, to restrict here $\mathcal{F}, \mathcal{F}_t$ and \mathbb{P}_k , and to define the cocycle ψ_t , either by (13)-(16) or by (13) and (21). Anyway, thanks to the cocycle properties of ψ_t , it is always possible to introduce later $\Gamma_{(-\infty, 0]}^G$ and the shift ϑ_t , in order to recover the whole environment state space $(\Gamma_{\mathbb{R}}^G, \mathcal{G}_{\mathbb{R}})$, the evolution α_t and the initial environment distribution $q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda}$, so that the two constructions are equivalent and can be considered different descriptions of the same situation.

Choosing the cocycle approach, a Poisson dilation of a CMS e^{Rt} gives a Markov chain $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_k)_{k \in E})$, where

$$\begin{aligned} \Omega &= E \times \Gamma_{\mathbb{R}^+}^G, & \omega &= (i, (g_n, t_n)_{n \in \mathbb{N}}), \\ X_0(\omega) &= X_0^*(\omega) = i, & Y_n(\omega) &= g_n, & T_n(\omega) &= t_n, & X_n^* &= \phi^E(X_{n-1}^*, Y_n), & n &\in \mathbb{N} \\ N_g(t) &= \sum_{n \in \mathbb{N}} I_{(Y_n=g, T_n \leq t)}, & N(t) &= \sum_{g \in G} N_g(t), & X_t &= X_{N(t)}^*, & t &\geq 0, & (22) \\ \mathcal{F} &= \sigma(X_0) \otimes \sigma(Y_n; n \in \mathbb{N}) \otimes \sigma(T_n; n \in \mathbb{N}), & \mathcal{F}_t &= \sigma(X_0, N_g(s); g \in G, 0 \leq s \leq t), \\ \mathbb{P}_k &= \delta_k \otimes q^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}^+}^{\lambda}, \end{aligned}$$

and where, again, R is related to ϕ, q and λ by Eq. (17). This chain is specified by the term $(G, \phi, q^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}^+}^{\lambda})$.

Let us underline that the Markov chain (22) is similar to (9), as also this one is represented via a discrete-time Markov chain and an innovation process. Nevertheless, this chain is endowed with a reacher structure because the cocycle ψ_t implicitly introduces also the deterministic invertible homogeneous evolution α_t .

4 Dilations of classical Markov semigroups and of quantum dynamical semigroups

We want to compare a Poisson dilation with the dilation of a QDS in Quantum Probability.

Given a Hilbert space \mathcal{H} , always complex separable in the paper, let us denote its vectors by h , or $|h\rangle$ using Dirac's notation, so that $\langle h'|h\rangle$ denotes the scalar product (linear in h) and $|h'\rangle\langle h|$ denotes the operator $h'' \mapsto \langle h|h''\rangle h'$. Let $\mathcal{B}(\mathcal{H})$ be the complex $*$ -algebra of bounded operators

in \mathcal{H} . Let $\Gamma[\mathcal{H}]$ denote the symmetric Fock space over \mathcal{H} and, for every $h \in \mathcal{H}$, let $e(h) \in \Gamma[\mathcal{H}]$ denote the corresponding normalized exponential vector. Given a Hilbert space $L^2(\mu)$, with a probability measure μ on some measurable space, and given a measurable complex function f on the same measurable space, let m_f denote the multiplication operator

$$m_f : \text{Dom}(m_f) \rightarrow L^2(\mu), \quad \text{Dom}(m_f) = \{h \in L^2(\mu) : fh \in L^2(\mu)\}, \quad m_f h = fh,$$

which is bounded if and only if $f \in L^\infty(\mu)$. Given two Hilbert spaces \mathcal{H} and \mathcal{K} and a vector $\kappa \in \mathcal{K}$, let $\mathbb{E}_\kappa : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$ denote the conditional expectation with respect to $|\kappa\rangle\langle\kappa|$. In the sequel, given an operator a in \mathcal{H} , we shall identify it with its extension $a \otimes \mathbb{1}_\mathcal{K}$ in $\mathcal{H} \otimes \mathcal{K}$.

Quantum extension of a CMS. We denote by \mathcal{L} a Lindblad operator in $\mathcal{B}(\mathcal{H})$, that is an operator $\mathcal{L} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ admitting the representation

$$\mathcal{L}a = i[H, a] + \sum_{z \in Z} \left(R_z^* a R_z - \frac{1}{2} \{R_z^* R_z, a\} \right), \quad a \in \mathcal{B}(\mathcal{H}), \quad (23)$$

where Z is a discrete index set, H and R_z belong to $\mathcal{B}(\mathcal{H})$, $H^* = H$, $\sum_z R_z^* R_z$ strongly converges in $\mathcal{B}(\mathcal{H})$, and where $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ denote the commutator and the anticommutator respectively. Let us recall that representation (23) is not unique, and that every \mathcal{L} generates a uniformly continuous QDS $T_t = e^{\mathcal{L}t}$, $t \geq 0$, which consists of bounded, completely positive, normal and identity preserving operators $T_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$. Actually Eq.(23) gives the complete characterization of the generator of a uniformly continuous QDS.

In order to extend a CMS in $\mathcal{L}^\infty(\mathcal{E})$ by a QDS in some $\mathcal{B}(\mathcal{H})$, we take $\mathcal{H} = L^2(\mu_E)$, where μ_E is the uniform probability on (E, \mathcal{E}) , and, denoted by $\{|i\rangle\}_{i \in E}$ its canonical basis, we embed $\mathcal{L}^\infty(\mathcal{E}) = L^\infty(\mu_E)$ in $\mathcal{B}(\mathcal{H})$ by the $*$ -isomorphism $f \mapsto m_f = \sum_{i \in E} f(i) |i\rangle\langle i|$ between $L^\infty(\mu_E)$ and the subalgebra of the multiplication operators in $L^2(\mathcal{H})$.

Since \mathcal{H} is finite dimensional, every QDS in $\mathcal{B}(\mathcal{H})$ is uniformly continuous. We say that a QDS $e^{\mathcal{L}t}$ in $\mathcal{B}(\mathcal{H})$ extends a CMS e^{Rt} in $L^\infty(\mu_E)$ if

$$e^{\mathcal{L}t} m_f = m_{e^{Rt} f}, \quad \forall f \in L^\infty(\mu_E), t \geq 0.$$

It is enough to check that $\mathcal{L}m_f = m_{Rf}$ for all $f \in L^\infty(\mu_E)$. Such extension always exists, but it is not unique at all. For example, given the transition rate matrix R , considered a rate λ and a stochastic matrix P associated to R via (7), taken a probability p associated to P via (3), using notation (1), the CMS e^{Rt} is extended by the QDS $e^{\mathcal{L}t}$ generated by

$$\mathcal{L}a = \lambda \left(\sum_{\substack{\ell \in L \\ i \in E}} p_\ell |i\rangle\langle\beta_\ell(i)| a |\beta_\ell(i)\rangle\langle i| - a \right), \quad a \in \mathcal{B}(\mathcal{H}), \quad (24)$$

which admits representation (23) with $H = 0$, $Z = G = E \times L$, $R_z = R_{(i,\ell)} = \sqrt{p_\ell \lambda} |\beta_\ell(i)\rangle\langle i|$.

HP-dilation of a QDS. Given a uniformly continuous QDS $e^{\mathcal{L}t}$ in $\mathcal{B}(\mathcal{H})$, a typical Quantum Probability construction employs a Hudson-Parthasarathy equation to dilate $e^{\mathcal{L}t}$ at the same time by a quantum stochastic flow and by a group of $*$ -automorphisms [1, 6, 7, 11, 15–19].

Chosen a representation (23) for \mathcal{L} , taken the Hilbert space \mathfrak{H} generated by the basis $\{|z\rangle\}_{z \in Z}$, let $d\Lambda_{zz'}(t)$, $dA_z(t)$ and $dA_z^\dagger(t)$ be the corresponding canonical quantum noises in $\Gamma[L^2(\mathbb{R}_+; \mathfrak{H})]$.

Fix an arbitrary vector $\nu = \sum_z \nu_z |z\rangle \in \mathfrak{H}$ and an arbitrary unitary operator $S = \sum_{zz'} S_{zz'} \otimes |z\rangle\langle z'|$ in $\mathcal{H} \otimes \mathfrak{H}$. Define the bounded operators in \mathcal{H}

$$\begin{aligned} L_z &= R_z - \sum_{z'} S_{zz'} \nu_{z'}, \\ H_0 &= H + \frac{i}{2} \sum_{zz'} \left(R_z^* S_{zz'} \nu_{z'} - \bar{\nu}_{z'} S_{zz'}^* R_z \right) = H_0^*. \end{aligned} \quad (25)$$

Then the quantum stochastic differential equation for adapted processes in $\mathcal{H} \otimes \Gamma[L^2(\mathbb{R}_+; \mathfrak{H})]$

$$\begin{aligned} dV_t &= \left[\sum_{zz'} \left(S_{zz'} - \delta_{zz'} \right) d\Lambda_{zz'}(t) - \sum_{zz'} L_z^* S_{zz'} dA_{z'}(t) + \sum_z L_z dA_z^\dagger(t) - \left(iH_0 + \frac{1}{2} \sum_z L_z^* L_z \right) dt \right] V_t, \\ V_0 &= \mathbb{1}, \end{aligned}$$

is a Hudson-Parthasarathy equation. The properties of the coefficients guarantee that it admits a unique solution V_t , which is a strongly continuous unitary cocycle. The HP-dilation of $e^{\mathcal{L}t}$ is identified by the triple (\mathfrak{H}, ν, V_t) as follows.

The quantum stochastic flow is

$$j_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma[L^2(\mathbb{R}_+; \mathfrak{H})]), \quad j_t(a) = V_t^* a V_t, \quad t \geq 0, \quad (26)$$

which satisfies the quantum stochastic differential equation

$$\begin{aligned} dj_t(a) &= \sum_{zz'} j_t \left(\sum_{z''} S_{z''z}^* a S_{z''z'} - \delta_{zz'} a \right) d\Lambda_{zz'}(t) + \sum_z j_t \left(\sum_{z'} S_{z'z}^* [a, L_{z'}] \right) dA_z^\dagger(t) \\ &\quad + \sum_z j_t \left(\sum_{z'} [L_{z'}^*, a] S_{z'z} \right) dA_z(t) + j_t \left(i[H_0, a] + \sum_z (L_z^* a L_z - \frac{1}{2} \{L_z^* L_z, a\}) \right) dt. \end{aligned}$$

Denoted by Θ_t the second quantization of the left shift in $L^2(\mathbb{R}; \mathfrak{H})$, $v(r) \mapsto v(r+t)$, embedded $\mathcal{H} \otimes \Gamma[L^2(\mathbb{R}_+; \mathfrak{H})]$ in $\mathcal{H} \otimes \Gamma[L^2(\mathbb{R}; \mathfrak{H})]$ and extended here every operator by tensorizing with the identities, consider the strongly continuous unitary group in $\mathcal{H} \otimes \Gamma[L^2(\mathbb{R}; \mathfrak{H})]$

$$U_t = \begin{cases} \Theta_t V_t, & \text{if } t \geq 0, \\ V_{|t|}^* \Theta_t, & \text{if } t \leq 0. \end{cases} \quad (27)$$

The group of $*$ -automorphisms is

$$J_t : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma[L^2(\mathbb{R}; \mathfrak{H})]) \rightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma[L^2(\mathbb{R}; \mathfrak{H})]), \quad J_t(A) = U_t^* A U_t, \quad t \in \mathbb{R}. \quad (28)$$

Of course $J_t(a) = j_t(a)$ for every $a \in \mathcal{B}(\mathcal{H})$ and $t \geq 0$.

Then, for every $T > 0$, taken $v \in L^2(\mathbb{R}_+; \mathfrak{H})$ and $u \in L^2(\mathbb{R}; \mathfrak{H})$ such that $v(t) = u(t) = \nu$ for every $0 \leq t \leq T$, both $(j_t, e(v))$ and $(J_t, e(u))$ dilate $e^{\mathcal{L}t}$ in the time interval $[0, T]$, that is

$$e^{\mathcal{L}t} a = \mathbb{E}_{e(v)} [j_t(a)] = \mathbb{E}_{e(u)} [J_t(a)] \quad \forall a \in \mathcal{B}(\mathcal{H}), \quad 0 \leq t \leq T. \quad (29)$$

In particular, if $\nu = 0$, then the choices $v = 0$ and $u = 0$ give the usual dilations which hold for every $t \geq 0$.

Let us call such a construction a HP-dilation (\mathfrak{H}, ν, V_t) . This is highly non-unique, as representation (23) and the choices of S and ν are not.

Thanks to the adaptedness of V_t , when $0 \leq t \leq T < \infty$, we can consider the first equality of Eq. (29) with V_t adapted process in $\mathcal{H} \otimes \Gamma[L^2((0, T); \mathfrak{H})]$, with the quantum stochastic flow $j_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma[L^2((0, T); \mathfrak{H})])$ and with the state $e(v) \in \Gamma[L^2((0, T); \mathfrak{H})]$.

Quantum extension of a Poisson dilation. Given a CMS e^{Rt} , we can either represent it in Classical Probability with a Poisson dilation $(G, \phi, q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda})$, either extend it to a QDS $e^{\mathcal{L}t}$ and then represent this latter in Quantum Probability with a HP-dilation (\mathfrak{Z}, ν, V_t) . Now we want to prove that, if the quantum extension \mathcal{L} and its dilation (\mathfrak{Z}, ν, V_t) are properly chosen, then (\mathfrak{Z}, ν, V_t) is a quantum extension of $(G, \phi, q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda})$. This shows that a Poisson dilation is a classical analogue of a HP-dilation.

The first step to study this relationship is to embed a Poisson dilation $(G, \phi, q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda})$ in the quantum world. We want this embedding at the Hilbert space level, as this is the level where usual quantum stochastic calculus is defined. Therefore we should introduce first a proper measure on $(\Gamma_{\mathbb{R}}^G, \mathcal{G}_{\mathbb{R}})$. Taken μ_E on E , the proper measure on $(\Gamma_{\mathbb{R}}^G, \mathcal{G}_{\mathbb{R}})$ should give a product measure on $(E \times \Gamma_{\mathbb{R}}^G, \mathcal{E} \otimes \mathcal{G}_{\mathbb{R}})$ invariant for the deterministic invertible evolutions $\alpha_t, \vartheta_t, \psi_t$. Thus the natural choice would be the probability measure $\mu_G^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda}$, where μ_G denotes the uniform probability on G . Nevertheless, it would be singular with respect to the initial environment distribution $q^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda}$, so that this latter could not be obtained from a state in $L^2(\mu_G^{\otimes \mathbb{Z}} \otimes Q_{\mathbb{R}}^{\lambda})$. Therefore now we fix a finite time horizon $T > 0$ and we focus only for $0 \leq t \leq T$ on the Markov chain $(G, \phi, q^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}_+}^{\lambda})$.

Using notations (22), we introduce the σ -algebras on $\Gamma_{\mathbb{R}_+}^G$

$$\begin{aligned}\mathcal{G}_{\mathbb{R}_+} &= \sigma(Y_n; n \in \mathbb{N}) \otimes \sigma(T_n; n \in \mathbb{N}), \\ \mathcal{G}_t &= \sigma(N_g(s); g \in G, 0 \leq s \leq t) \subseteq \mathcal{G}_{\mathbb{R}_+}.\end{aligned}$$

Then we consider the probability measures $\mathbb{Q} = \mu_G^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}_+}^{\lambda}$ on $\mathcal{G}_{\mathbb{R}_+}$ and $\mathbb{P} = \mu_E \otimes \mathbb{Q}$ on $\mathcal{F} = \mathcal{E} \otimes \mathcal{G}_{\mathbb{R}_+}$. Given $T > 0$, we introduce also their restrictions \mathbb{Q}_T and \mathbb{P}_T to \mathcal{G}_T and \mathcal{F}_T . Thus the processes $N_g(t)$, $0 \leq t \leq T$, are i.i.d. Poisson processes with rates $\lambda/|G|$. Then we consider the Hilbert space $L^2(\mathbb{P}_T) = L^2(\mu_E) \otimes L^2(\mathbb{Q}_T)$ and we embed $L^\infty(\mathbb{P}_T) = L^\infty(\mu_E) \otimes L^\infty(\mathbb{Q}_T)$ in $\mathcal{B}(L^2(\mathbb{P}_T))$ by the $*$ -isomorphism $F \mapsto m_F$ between $L^\infty(\mathbb{P}_T)$ and the bounded multiplication operators in $L^2(\mathbb{P}_T)$.

The cocycle ψ_t generates the family of operators

$$\Psi_t : L^2(\mathbb{P}_T) \rightarrow L^2(\mathbb{P}_T), \quad \Psi_t \xi = \xi \circ \psi_t^{-1}, \quad 0 \leq t \leq T.$$

Proposition 3. *Every Ψ_t is a well defined unitary operator in $L^2(\mathbb{P}_T)$.*

Proof. Every Ψ_t is well defined and unitary because, thanks to the invertibility of ϕ , the maps ψ_t and ψ_t^{-1} preserve the probability measure \mathbb{P}_T .

To see this, let $A \in \sigma(N(s); 0 \leq s \leq T)$, $B = (X_0 = i, Y_1 = g_1, \dots, Y_m = g_m)$, $0 \leq n \leq m$. Then

$$\begin{aligned}\psi_t(A \cap (N(t) = n) \cap (N(T) = m) \cap B) &= A \cap (N(t) = n) \cap (N(T) = m) \cap \psi_t(B) = \\ A \cap (N(t) = n) \cap (N(T) = m) \cap &((X_0, Y_1, \dots, Y_n) = \varphi_n(i, g_1, \dots, g_n), Y_{n+1} = g_{n+1}, \dots, Y_m = g_m),\end{aligned}$$

and so, thanks to the invertibility of φ_n ,

$$\begin{aligned}\mathbb{P}_T(\psi_t(A \cap (N(t) = n) \cap (N(T) = m) \cap B)) &= Q_{\mathbb{R}_+}^{\lambda}(A, N(t) = n, N(T) = m) \frac{1}{|E|} \frac{1}{|G|^m} \\ &= \mathbb{P}_T(A, N(t) = n, N(T) = m, B).\end{aligned}$$

Since these events form a set closed under finite intersections and generate \mathcal{F}_t , the equality $\mathbb{P}_T(\psi_t(A)) = \mathbb{P}_T(A)$ follows for all $A \in \mathcal{F}_T$. Analogously, $\mathbb{P}_T(\psi_t^{-1}(A)) = \mathbb{P}_T(A)$ for all $A \in \mathcal{F}_T$.

□

The unitary family Ψ_t is a quantum extension of the deterministic invertible evolution ψ_t , as

$$\Psi_t^* m_F \Psi_t = m_{F \circ \psi_t}, \quad \forall F \in L^\infty(\mathbb{P}_T), \quad 0 \leq t \leq T.$$

In order to study the quantum stochastic calculus properties of the family of operators Ψ_t , we need to introduce a good isomorphism between $L^2(\mathbb{Q}_T)$ and a Fock space $\Gamma[L^2((0, T); \mathfrak{Z})]$. We take $\mathfrak{Z} = L^2(\mu_G)$ with its canonical basis $\{|g\rangle\}_{g \in G}$ and we choose the isomorphism which diagonalizes the number processes $\Lambda_{gg}(t)$ and which maps the state $e(v_0)$, with $v_0(t) = \sum_g \sqrt{\lambda/|G|} |g\rangle$ for all $0 \leq t \leq T$, to the constant function $1 \in L^2(\mathbb{Q}_T)$. Denoted by μ_T the Lebesgue measure on $(0, T)$, let us make the identification $L^2((0, T); \mathfrak{Z}) = L^2(\mu_G \otimes \mu_T)$ so that every $v : (0, T) \mapsto \mathfrak{Z}$, $v(t) = \sum_g v_g(t) |g\rangle$, corresponds to $v : G \times (0, T) \rightarrow \mathbb{C}$, $v(g, t) = \sqrt{|G|} v_g(t)$. Thus we introduce the isomorphism

$$I_T : \Gamma[L^2((0, T); \mathfrak{Z})] \rightarrow L^2(\mathbb{Q}_T),$$

$$I_T[e(v)] = \exp\left(-\frac{1}{2}\|v\|^2 + \frac{1}{2}\lambda T\right) \prod_{n=1}^{N(T)} \frac{v(Y_n, T_n)}{\sqrt{\lambda}}, \quad v \in L^2(\mu_G \otimes \mu_T) \cap \mathcal{C}([0, T]; \mathfrak{Z}).$$

The operator I_T is an isometry which turns out to be unitary thanks to the chaotic representation property of the Poisson process. Then

$$I_T \Lambda_{gg}(t) I_T^{-1} = m_{N_g(t)}, \quad \forall g \in G, \quad 0 \leq t \leq T,$$

and $I_T[e(v_0)] = 1$. Any choice of a state $\eta \in \Gamma[L^2((0, T); \mathfrak{Z})]$ with $\eta \neq e(v_0)$ corresponds to a change of probability on \mathcal{G}_T , that is the choice of a probability with Radon Nikodym derivative $|I_T[\eta]|^2$ with respect to \mathbb{Q}_T .

Extended I_T to an isomorphism between $\mathcal{H} \otimes \Gamma[L^2((0, T); \mathfrak{Z})]$ and $L^2(\mathbb{P}_T)$, we define the family of unitary operators

$$V_t := I_T^{-1} \Psi_t I_T : \mathcal{H} \otimes \Gamma[L^2((0, T); \mathfrak{Z})] \rightarrow \mathcal{H} \otimes \Gamma[L^2((0, T); \mathfrak{Z})], \quad 0 \leq t \leq T. \quad (30)$$

Thus the unitary quantum evolution in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma[L^2((0, T); \mathfrak{Z})])$, $A \mapsto V_t^* A V_t$, $0 \leq t \leq T$, admits an invariant abelian subalgebra where it gives just the classical evolution $\circ \psi_t$ in $L^\infty(\mathbb{P}_T)$. Indeed, the algebra of the multiplication operators $\{m_F, F \in L^\infty(\mathbb{P}_T)\} \subseteq \mathcal{B}(L^2(\mathbb{P}_T))$ is *-isomorphic to the algebra $\{M_F := I_T^{-1} m_F I_T, F \in L^\infty(\mathbb{P}_T)\} \subseteq \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma[L^2((0, T); \mathfrak{Z})])$, where we have

$$V_t^* M_F V_t = M_{F \circ \psi_t}, \quad \forall F \in L^\infty(\mathbb{P}_T), \quad 0 \leq t \leq T.$$

In particular, the flow j_t associated to V_t via (26) is a quantum extension of the classical homomorphism j_t associated to ψ_t via (20).

Theorem 4. *Let e^{Rt} be a classical Markov semigroup in a finite state space E and let (22) be the Markov chain provided by a Poisson dilation $(G, \phi, q^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}_+}^\lambda)$. Let $\mathcal{H} = L^2(\mu_E)$, let $\mathfrak{Z} = L^2(\mu_G)$, and let V_t be the family of unitary operators (30) in $\mathcal{H} \otimes \Gamma[L^2((0, T); \mathfrak{Z})]$. Then*

(1) V_t is a strongly continuous adapted process satisfying the Hudson-Parthasarathy equation

$$dV_t = \sum_{g, g' \in G} (S_{gg'} - \delta_{gg'}) V_t d\Lambda_{gg'}(t), \quad 0 \leq t \leq T, \quad (31)$$

$$V_0 = \mathbb{1},$$

where S is the unitary operator $\sum_{\substack{i \in E \\ g \in G}} |\phi(i, g)\rangle \langle i, g|$ in $\mathcal{H} \otimes \mathfrak{Z}$;

(2) the quantum stochastic flow $j_t(a) = V_t^* a V_t$ satisfies, on the algebra of the multiplication operators, the quantum stochastic differential equation

$$dj_t(m_f) = \sum_{g \in G} j_t \left(\sum_{i \in E} |i\rangle \langle \phi^E(i, g)| m_f |\phi^E(i, g)\rangle \langle i| - m_f \right) d\Lambda_{gg}(t), \quad \forall f \in L^\infty(\mu_E); \quad (32)$$

(3) taken $v(t) = \nu$ for all $0 \leq t \leq T$, with $\nu = \sum_{g \in G} \sqrt{\lambda q_g} |g\rangle \in \mathfrak{H}$,

$$\begin{aligned} & - |I_T[e(v)]|^2 = \frac{dq^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}_+}^\lambda |g_T}{dQ_T}, \\ & - (j_t, e(v)) \text{ dilates in } 0 \leq t \leq T \text{ the quantum dynamical semigroup } e^{\mathcal{L}t}, \text{ extending } e^{Rt}, \\ & \text{which is generated by} \end{aligned}$$

$$\mathcal{L}a = \lambda \left(\sum_{g, g' \in G} \sqrt{q_{g'} q_g} S_{gg'}^* a S_{gg'} - a \right), \quad a \in \mathcal{B}(\mathcal{H}). \quad (33)$$

Before of the proof, let us write explicitly the relation between ϕ and coefficients in (31), that is

$$S_{gg'} = \sum_{i, j \in E} |i\rangle \langle i, g| \phi(j, g') \langle j|.$$

Proof. (1) The family of unitary operators V_t is an adapted process in $\mathcal{H} \otimes \Gamma[L^2((0, T); \mathfrak{H})]$ because every V_t belongs to $\mathcal{B}(\mathcal{H} \otimes \Gamma[L^2((0, t); \mathfrak{H})]) \otimes \mathbb{1}_{\Gamma[L^2((t, T); \mathfrak{H})]}$. Indeed, for every $h, h' \in L^2(\mu_E)$, $v, v' \in L^2((0, T); \mathfrak{H})$, $0 \leq t \leq T$,

$$\begin{aligned} & \langle h \otimes e(v) | V_t h' \otimes e(v') \rangle \\ &= \exp\left(\lambda T - \frac{1}{2}\|v\|^2 - \frac{1}{2}\|v'\|^2\right) \\ & \quad \cdot \mathbb{E}_T \left[\bar{h}(X_0) \left(\prod_{n=1}^{N(T)} \frac{\bar{v}(Y_n, T_n)}{\sqrt{\lambda}} \right) \left(\left\{ h'(X_0) \prod_{n=1}^{N(t)} \frac{v'(Y_n, T_n)}{\sqrt{\lambda}} \right\} \circ \psi_t^{-1} \right) \prod_{n=N(t)+1}^{N(T)} \frac{v'(Y_n, T_n)}{\sqrt{\lambda}} \right] \\ &= \exp\left(\lambda T - \frac{1}{2}\|v\|^2 - \frac{1}{2}\|v'\|^2\right) \\ & \quad \cdot \mathbb{E}_T \left[\bar{h}(X_0) \left(\prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n, T_n)}{\sqrt{\lambda}} \right) \left(\left\{ h'(X_0) \prod_{n=1}^{N(t)} \frac{v'(Y_n, T_n)}{\sqrt{\lambda}} \right\} \circ \psi_t^{-1} \right) \right] \\ & \quad \cdot \mathbb{E}_T \left[\prod_{n=N(t)+1}^{N(T)} \frac{\bar{v}(Y_n, T_n) v'(Y_n, T_n)}{\lambda} \right] \\ &= \langle h \otimes e(v)|_{(0,t)} | V_t h' \otimes e(v')|_{(0,t)} \rangle \langle e(v)|_{(t,T)} | e(v')|_{(t,T)} \rangle, \end{aligned}$$

where the last V_t is the operator in $\mathcal{H} \otimes \Gamma[L^2((0, t); \mathfrak{H})]$ defined by Eq. (30) with $T = t$, which a posteriori is identified with its extension in $\mathcal{H} \otimes \Gamma[L^2((0, T); \mathfrak{H})]$ for $T \geq t$.

The adapted process V_t is strongly continuous. Since V_t is unitary for every t , it is enough to prove weak continuity. To see this, let $\xi \in L^2(\mathbb{P}_T)$, $\xi' \in L^\infty(\mathbb{P}_T)$, $0 \leq t \leq T$, $0 \leq t + s \leq T$, $\Delta N = |N(t + s) - N(t)|$. Then

$$\begin{aligned} & \left| \mathbb{E}_T [\bar{\xi} \Psi_{t+s} \xi'] - \mathbb{E}_T [\bar{\xi} \Psi_t \xi'] \right| \leq \int_{\Omega} |\xi| \cdot \left| \xi' \circ \psi_{t+s}^{-1} - \xi' \circ \psi_t^{-1} \right| d\mathbb{P}_T \\ & = \int_{\Delta N \geq 1} |\xi| \cdot \left| \xi' \circ \psi_{t+s}^{-1} - \xi' \circ \psi_t^{-1} \right| d\mathbb{P}_T \leq 2 \|\xi'\|_\infty \int_{\Delta N \geq 1} |\xi| d\mathbb{P}_T \xrightarrow{s \rightarrow 0} 0, \end{aligned}$$

because $\psi_{t+s}^{-1}(\omega) = \psi_t^{-1}(\omega)$ for all $\omega \in (\Delta N = 0)$ and $\mathbb{P}_T(\Delta N \geq 1) = 1 - e^{-\lambda|s|} \rightarrow 0$ as $s \rightarrow 0$. Thus $\mathbb{E}_T[\bar{\xi} \Psi_t \xi']$ is continuous for $0 \leq t \leq T$ if $\xi \in L^2(\mathbb{P}_T)$ and $\xi' \in L^\infty(\mathbb{P}_T)$, and the same is true for $\xi \in L^\infty(\mathbb{P}_T)$ and $\xi' \in L^2(\mathbb{P}_T)$, as the same argument works for $\Psi_t^* \xi = \xi \circ \psi_t$. If both ξ and ξ' belong to $L^2(\mathbb{P}_T)$, then the continuity of $\mathbb{E}_T[\bar{\xi} \Psi_t \xi']$ follows by standard arguments taking a sequence $\xi_n \in L^\infty(\mathbb{P}_T)$ such that $\xi_n \rightarrow \xi$ in $L^2(\mathbb{P}_T)$. Therefore Ψ_t and V_t are strongly continuous.

In order to show that the unitary strongly continuous adapted process V_t satisfies the Hudson-Parthasarathy equation (31), it is enough to show that

$$\begin{aligned} \mathfrak{f}(t) &:= \langle h \otimes e(v) | V_t h' \otimes e(v') \rangle \\ &= \mathfrak{f}(0) + \int_0^t \langle h \otimes e(v) | \sum_{g,g'} (S_{gg'} - \delta_{gg'}) V_s \bar{v}_{g'}(s) v'_g(s) h' \otimes e(v') \rangle ds, \end{aligned} \quad (34)$$

for all $h, h' \in L^2(\mu_E)$, $v, v' \in L^2((0, T); \mathfrak{H})$, $0 \leq t \leq T$. To prove this, we compute the right derivative of $\mathfrak{f}(t)$ in the special case $v, v' \in \mathcal{C}([0, T]; \mathfrak{H})$. Let $\xi = I_T[h \otimes e(v)]$, $\xi' = I_T[h' \otimes e(v')]$, $0 \leq t < t+s \leq T$, $\Delta N = N(t+s) - N(t)$, $\Delta N_g = N_g(t+s) - N_g(t)$. Then

$$\begin{aligned} &\mathfrak{f}(t+s) - \mathfrak{f}(t) \\ &= \mathbb{E}_T \left[\left(\bar{\xi} \circ \psi_{t+s} - \bar{\xi} \circ \psi_t \right) \xi' \right] \\ &= \sum_g \mathbb{E}_T \left[\left(\bar{\xi} \circ \psi_{t+s} - \bar{\xi} \circ \psi_t \right) \xi' \mid \Delta N = \Delta N_g = 1 \right] \mathbb{P}_T(\Delta N = \Delta N_g = 1) \\ &\quad + \mathbb{E}_T \left[\left(\bar{\xi} \circ \psi_{t+s} - \bar{\xi} \circ \psi_t \right) \xi' \mid \Delta N \geq 2 \right] \mathbb{P}_T(\Delta N \geq 2) \\ &= A + B. \end{aligned}$$

Setting

$$C_1 = \exp\left(\lambda T - \frac{1}{2}\|v\|^2 - \frac{1}{2}\|v'\|^2\right) \|h\|_\infty \|h'\|_\infty, \quad C_2 = \frac{\|v\|_\infty \|v'\|_\infty}{\lambda},$$

the second addendum B is bounded as follows:

$$\begin{aligned} |B| &\leq \mathbb{E}_T \left[\left(|\bar{\xi} \circ \psi_{t+s}| + |\bar{\xi} \circ \psi_t| \right) |\xi'| \mid \Delta N \geq 2 \right] \mathbb{P}_T(\Delta N \geq 2) \\ &\leq 2C_1 \mathbb{E}_T \left[C_2^{N(T)} \mid \Delta N \geq 2 \right] \mathbb{P}_T(\Delta N \geq 2) \\ &= 2C_1 e^{(C_2-1)\lambda T} \left(1 - e^{-C_2\lambda s} - C_2 \lambda s e^{-C_2\lambda s} \right) = o(s), \quad \text{as } s \rightarrow 0^+. \end{aligned}$$

In order to deal with the first addendum A , for $n \geq 1$ let us explicitly introduce the random variables $Y_n^{(t)} = Y_n \circ \psi_t$, the n -th marks at time t . Denoting by $\varphi_n^{G_n}$ the projection of the map φ_n on G_n (see Eq. (14)), we have

$$Y_n^{(t)} = Y_n \circ \psi_t = \begin{cases} \varphi_n^{G_n}(X_0, Y_1, \dots, Y_n), & \text{if } T_n \leq t, \\ Y_n, & \text{if } T_n > t, \end{cases}$$

so that $Y_n^{(t+s)} = Y_n^{(t)}$ if $T_n \leq t < t+s$, and $Y_n^{(t+s)} = Y_n^{(t)} = Y_n$ if $t < t+s < T_n$. Denoting by ϕ^G the projection of the map ϕ on G , we also have $Y_n^{(t+s)} = \phi^G(X_t, Y_n)$ if $T_{n-1} \leq t < T_n \leq t+s$.

Then, for every $g \in G$,

$$\begin{aligned}
& \mathbb{E}_T \left[\left(\bar{\xi} \circ \psi_{t+s} - \bar{\xi} \circ \psi_t \right) \xi' \mid \Delta N = \Delta N_g = 1 \right] \\
&= \exp \left(\lambda T - \frac{1}{2} \|v\|^2 - \frac{1}{2} \|v'\|^2 \right) \mathbb{E}_T \left[\left(\prod_{n=N(t+s)+1}^{N(T)} \frac{\bar{v}(Y_n, T_n) v'(Y_n, T_n)}{\lambda} \right) \right. \\
&\quad \cdot \left(\bar{h}(\phi^E(X_t, g)) \frac{\bar{v}(\phi^G(X_t, g), T_{N(t+s)})}{\sqrt{\lambda}} - \bar{h}(X_t) \frac{\bar{v}(g, T_{N(t+s)})}{\sqrt{\lambda}} \right) h'(X_0) \frac{v'(g, T_{N(t+s)})}{\sqrt{\lambda}} \\
&\quad \left. \cdot \prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n^{(t)}, T_n) v'(Y_n, T_n)}{\lambda} \mid \Delta N = \Delta N_g = 1 \right] \\
&= \exp \left(\lambda T - \frac{1}{2} \|v\|^2 - \frac{1}{2} \|v'\|^2 \right) \mathbb{E}_T \left[\prod_{n=N(t+s)+1}^{N(T)} \frac{\bar{v}(Y_n, T_n) v'(Y_n, T_n)}{\lambda} \right] \\
&\quad \cdot \mathbb{E}_T \left[\left(\bar{h}(\phi^E(X_t, g)) \frac{\bar{v}(\phi^G(X_t, g), T_{N(t+s)})}{\sqrt{\lambda}} - \bar{h}(X_t) \frac{\bar{v}(g, T_{N(t+s)})}{\sqrt{\lambda}} \right) h'(X_0) \frac{v'(g, T_{N(t+s)})}{\sqrt{\lambda}} \right. \\
&\quad \left. \cdot \prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n^{(t)}, T_n) v'(Y_n, T_n)}{\lambda} \mid \Delta N = \Delta N_g = 1 \right].
\end{aligned}$$

Thus

$$\begin{aligned}
A &= \exp \left(\lambda T - \frac{1}{2} \|v\|^2 - \frac{1}{2} \|v'\|^2 \right) \mathbb{E}_T \left[\prod_{n=N(t+s)+1}^{N(T)} \frac{\bar{v}(Y_n, T_n) v'(Y_n, T_n)}{\lambda} \right] \\
&\cdot \sum_g \left\{ \mathbb{E}_T \left[\left(\bar{h}(\phi^E(X_t, g)) \frac{\bar{v}(\phi^G(X_t, g), t)}{\sqrt{\lambda}} - \bar{h}(X_t) \frac{\bar{v}(g, t)}{\sqrt{\lambda}} \right) h'(X_0) \frac{v'(g, t)}{\sqrt{\lambda}} \right. \right. \\
&\quad \left. \left. \cdot \prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n^{(t)}, T_n) v'(Y_n, T_n)}{\lambda} \mid \Delta N = \Delta N_g = 1 \right] \right. \\
&+ \mathbb{E}_T \left[\left(\frac{\bar{v}(\phi^G(X_t, g), T_{N(t+s)}) v'(g, T_{N(t+s)})}{\lambda} - \frac{\bar{v}(\phi^G(X_t, g), t) v'(g, t)}{\lambda} \right) \bar{h}(\phi^E(X_t, g)) h'(X_0) \right. \\
&\quad \left. \cdot \prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n^{(t)}, T_n) v'(Y_n, T_n)}{\lambda} \mid \Delta N = \Delta N_g = 1 \right] \\
&+ \mathbb{E}_T \left[\left(\frac{\bar{v}(g, t) v'(g, t)}{\lambda} - \frac{\bar{v}(g, T_{N(t+s)}) v'(g, T_{N(t+s)})}{\lambda} \right) \bar{h}(X_t) h'(X_0) \right. \\
&\quad \left. \cdot \prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n^{(t)}, T_n) v'(Y_n, T_n)}{\lambda} \mid \Delta N = \Delta N_g = 1 \right] \left. \right\} \frac{\lambda s}{|G|} e^{-\lambda s} \\
&= A_1 + A_2 + A_3,
\end{aligned}$$

where the three addenda A_l correspond to the three expectations enclosed in curly brackets. Thanks to the continuity of v and v' , every function $\bar{v}(g, r) v'(g', r)/\lambda$ is continuous with respect to r , so that there exists a constant $C_3(s)$ such that

$$\left| \frac{\bar{v}(g, r) v'(g', r)}{\lambda} - \frac{\bar{v}(g, t) v'(g', t)}{\lambda} \right| \leq C_3(s), \quad \forall t \leq r \leq t+s, \quad g, g' \in G,$$

with $C_3(s) \rightarrow 0$ as $s \rightarrow 0^+$. Since $t < T_{N(t+s)} \leq t + s$, both A_2 and A_3 are bounded as follows:

$$\begin{aligned} |A_1| &\leq C_1 \mathbb{E}_T \left[C_2^{N(T)-N(t+s)} \right] C_3(s) \mathbb{E}_T \left[C_2^{N(t)} \right] \lambda s e^{-\lambda s} \\ &= C_1 e^{(C_2-1)\lambda(T-s)} C_3(s) \lambda s e^{-\lambda s} = o(s), \quad \text{as } s \rightarrow 0^+. \end{aligned}$$

Finally, let us show that A_1/s converges to the right limit. The following conditional expectation has an \mathcal{F}_t -measurable argument and hence it satisfies

$$\begin{aligned} &\mathbb{E}_T \left[\left(\bar{h}(\phi^E(X_t, g)) \frac{\bar{v}(\phi^G(X_t, g), t)}{\sqrt{\lambda}} - \bar{h}(X_t) \frac{\bar{v}(g, t)}{\sqrt{\lambda}} \right) h'(X_0) \frac{v'(g, t)}{\sqrt{\lambda}} \right. \\ &\quad \left. \cdot \prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n^{(t)}, T_n) v'(Y_n, T_n)}{\lambda} \middle| \Delta N = \Delta N_g = 1 \right] \\ &= \mathbb{E}_T \left[\left(\bar{h}(\phi^E(X_t, g)) \frac{\bar{v}(\phi^G(X_t, g), t)}{\sqrt{\lambda}} - \bar{h}(X_t) \frac{\bar{v}(g, t)}{\sqrt{\lambda}} \right) h'(X_0) \frac{v'(g, t)}{\sqrt{\lambda}} \right. \\ &\quad \left. \cdot \prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n^{(t)}, T_n) v'(Y_n, T_n)}{\lambda} \right] \\ &= \mathbb{E}_T \left[\left(\bar{h}(\phi^E(X_0, g)) \frac{\bar{v}(\phi^G(X_0, g), t)}{\sqrt{\lambda}} - \bar{h}(X_0) \frac{\bar{v}(g, t)}{\sqrt{\lambda}} \right) \left(\prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n, T_n)}{\sqrt{\lambda}} \right) \right. \\ &\quad \left. \cdot \frac{v'(g, t)}{\sqrt{\lambda}} \left\{ h'(X_0) \prod_{n=1}^{N(t)} \frac{v'(Y_n, T_n)}{\sqrt{\lambda}} \right\} \circ \psi_t^{-1} \right] \\ &= \mathbb{E}_T \left[\left(\sum_{j, g'} \langle j, g' | \phi(X_0, g) \rangle \bar{h}(j) \frac{\bar{v}(g', t)}{\sqrt{\lambda}} - \bar{h}(X_0) \frac{\bar{v}(g, t)}{\sqrt{\lambda}} \right) \left(\prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n, T_n)}{\sqrt{\lambda}} \right) \right. \\ &\quad \left. \cdot \frac{v'(g, t)}{\sqrt{\lambda}} \left\{ h'(X_0) \prod_{n=1}^{N(t)} \frac{v'(Y_n, T_n)}{\sqrt{\lambda}} \right\} \circ \psi_t^{-1} \right] \\ &= \frac{1}{\lambda} \left\langle \left(\sum_{g'} S_{g'g}^* v(g', t) - v(g, t) \right) h \otimes e(v|_{(0,t)}) \middle| v'(g, t) V_t h' \otimes e(v'|_{(0,t)}) \right\rangle \\ &\quad \cdot \exp\left(-\lambda t + \frac{1}{2} \|v|_{(0,t)}\|^2 + \frac{1}{2} \|v'|_{(0,t)}\|^2\right). \end{aligned}$$

Then

$$\begin{aligned}
\frac{A_1}{s} &= \exp\left(\lambda T - \frac{1}{2}\|v\|^2 - \frac{1}{2}\|v'\|^2\right) \mathbb{E}_T \left[\prod_{n=N(t+s)+1}^{N(T)} \frac{\bar{v}(Y_n, T_n) v'(Y_n, T_n)}{\lambda} \right] \\
&\quad \cdot \sum_g \mathbb{E}_T \left[\left(\bar{h}(\phi^E(X_t, g)) \frac{\bar{v}(\phi^G(X_t, g), t)}{\sqrt{\lambda}} - \bar{h}(X_t) \frac{\bar{v}(g, t)}{\sqrt{\lambda}} \right) h'(X_0) \frac{v'(g, t)}{\sqrt{\lambda}} \right. \\
&\quad \quad \left. \cdot \prod_{n=1}^{N(t)} \frac{\bar{v}(Y_n^{(t)}, T_n) v'(Y_n, T_n)}{\lambda} \Big| \Delta N = \Delta N_g = 1 \right] \frac{\lambda}{|G|} e^{-\lambda s} \\
&= \langle h \otimes e(v)|_{(0,t)} \rangle \sum_{gg'} (S_{gg'} - \delta_{gg'}) \bar{v}_g(t) v'_{g'}(t) V_t h' \otimes e(v'|_{(0,t)}) \langle e(v)|_{(t+s,T)} \rangle e(v'|_{(t+s,T)}) \\
&\quad \quad \quad \cdot \exp\left(-\frac{1}{2}\|v|_{(t,t+s)}\|^2 - \frac{1}{2}\|v'|_{(t,t+s)}\|^2\right) \\
&\xrightarrow{s \rightarrow 0^+} \langle h \otimes e(v) | \sum_{gg'} (S_{gg'} - \delta_{gg'}) \bar{v}_g(t) v'_{g'}(t) V_t h' \otimes e(v') \rangle.
\end{aligned}$$

Therefore, for every $h, h' \in \mathcal{H}$ and every $v, v' \in \mathcal{C}([0, T], \mathfrak{Z})$, the function \mathfrak{f} is a continuous function with continuous right derivative $D_+\mathfrak{f}(t) = \langle h \otimes e(v) | \sum_{g,g'} (S_{gg'} - \delta_{gg'}) \bar{v}_g(t) v'_{g'}(t) V_t h' \otimes e(v') \rangle$. Then \mathfrak{f} is continuously differentiable with $d\mathfrak{f}/dt = D_+\mathfrak{f}$. Finally, the density of $\mathcal{C}([0, T], \mathfrak{Z})$ in $L^2([0, T], \mathfrak{Z})$ gives Eq. (34) on the whole exponential domain.

(2) The Hudson-Parthasarathy equation (31) for the process V_t determines the quantum stochastic differential equation $dj_t(a) = \sum_{g,g'} j_t \left(\sum_{g''} S_{g''g}^* a S_{g''g'} - \delta_{gg'} a \right) d\Lambda_{gg'}(t)$ for the quantum stochastic flow $j_t(a) = V_t^* a V_t$. If $a = m_f$ with $f \in L^\infty(\mu_E)$, then

$$\begin{aligned}
\sum_{g''} S_{g''g}^* m_f S_{g''g'} &= \sum_{g''} \sum_{i,j,i',j'} |i\rangle \langle \phi(i, g) | j, g'' \rangle \langle j | m_f | i' \rangle \langle i', g'' | \phi(j', g') \rangle \langle j' | \\
&= \delta_{gg'} \sum_i |i\rangle \langle \phi^E(i, g) | m_f | \phi^E(i, g) \rangle \langle i |,
\end{aligned}$$

so that Eq.(32) follows for every multiplication operator in $L^2(\mu_E)$.

(3) If we take $\nu = \sum_g \sqrt{\lambda q_g} |g\rangle \in \mathfrak{Z}$ and $v(t) = \nu$ for every $0 \leq t \leq T$, then

$$|I_T[e(v)]|^2 = \prod_{n=1}^{N(T)} |G| q_{Y_n} = \frac{dq^{\otimes N} \otimes Q_\lambda |_{\mathcal{G}_T}}{dQ_T}.$$

Moreover, since V_t satisfies a Hudson-Parthasarathy equation with only $d\Lambda_{gg'}$ terms, the couple $(j_t, e(v))$ defines a QDS $e^{\mathcal{L}t}$ in $\mathcal{B}(\mathcal{H})$ with Lindblad generator \mathcal{L} admitting representation (23) with coefficients R_g and H given by (25) in the case $L_g = 0$ and $H_0 = 0$. Thus

$$R_g = \sum_{g'} S_{gg'} \nu_{g'}, \quad H = 0,$$

which give, together with our choice of ν , the Lindblad operator (33). Then

$$\mathcal{L}m_f = \lambda \left(\sum_{\substack{g \in G \\ i \in E}} q_g |i\rangle \langle \phi^E(i, g) | m_f | \phi^E(i, g) \rangle \langle i | - m_f \right) = m_{Rf}, \quad \forall f \in L^\infty(\mu_E).$$

□

Theorem 4 allows to state that the cocycle ψ_t of a Poisson dilation is a classical analogue of a Hudson-Parthasarathy cocycle, as it admits a quantum extension V_t satisfying the Hudson-Parthasarathy equation (31). In particular the associated quantum stochastic flow j_t satisfies, on the abelian algebra of the multiplication operators in $L^2(\mathbb{P}_T)$, the quantum stochastic differential equation (32), which is just a reformulation in operator terminology of the stochastic differential equation (21) satisfied by the corresponding classical homomorphism j_t .

Moreover, Theorem 4 allows to state that the whole Poisson dilation $(G, \phi, q^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}_+}^\lambda)$ of a CMS e^{Rt} is a classical analogue of a HP-dilation (\mathfrak{J}, ν, V_t) of a QDS $e^{\mathcal{L}t}$, as there is also a quantum environment state $e(v)$ which gives on side the right initial distribution of the classical environment to dilate e^{Rt} , and on the other side the right QDS to extend e^{Rt} .

Let us note also that if $(G, \phi, q^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}_+}^\lambda)$ is built as in Theorem 2, then

$$S_{(i,\ell)(1,\ell')} = \delta_{\ell\ell'} |\beta_\ell(i)\rangle\langle i|,$$

and the Lindblad operator (33) becomes just the Lindblad operator (24).

All of these results are obtained for an arbitrary but finite time horizon $T > 0$. They are consistent with respect to T , but it is not possible to set $T = +\infty$, as there is no isomorphism I_∞ , and $v(t) = \nu$ for all $t > 0$ does not belong to $L^2(\mathbb{R}_+; \mathfrak{J})$. Thus, starting from ψ_t and V_t , we can introduce separately the groups ϑ_t and α_t and the groups Θ_t and U_t , but we do not have a Hilbert space isomorphism to show that the group of $*$ -automorphisms (28) gives the group of $*$ -automorphisms $\circ\alpha_t$ on an invariant abelian subalgebra of multiplication operators.

In order to avoid the finite time horizon and to find a correspondence between the two dilations holding for all times, one can look at a Poisson dilation under other isomorphisms different from I_T . For example, one can consider the usual isomorphism

$$\begin{aligned} \hat{I} : \Gamma[L^2(\mathbb{R}_+; \mathfrak{J})] &\rightarrow L^2(\mathbb{Q}), \\ \hat{I}[e(v)] &= \exp\left(-\frac{1}{2}\|v\|^2 - \sqrt{\frac{\lambda}{|G|}} \sum_{g \in G} \int_{\mathbb{R}_+} v_g(s) ds\right) \prod_{n=1}^{\infty} \left(1 + \frac{v(Y_n, T_n)}{\sqrt{\lambda}}\right), \quad v \in \mathcal{C}_c([0, +\infty); \mathfrak{J}), \end{aligned}$$

such that

$$\hat{I}^{-1} m_{N_g(t)} \hat{I} = \Lambda_{gg}(t) + \sqrt{\frac{\lambda}{|G|}} (A_g^\dagger(t) + A_g(t)) + \frac{\lambda}{|G|} t, \quad \forall g \in G, 0 \leq t < \infty,$$

and $\hat{I}[e(0)] = 1$. This choice leads now to a strongly continuous adapted process of unitary operators $\hat{V}_t = \hat{I}^{-1} \Psi_t \hat{I}$ in $\mathcal{H} \otimes \Gamma[L^2(\mathbb{R}_+; \mathfrak{J})]$ defined for all $t > 0$. Nevertheless the new Hudson-Parthasarathy equation is not as simple as (31) and it is anyhow related to (31) by a Weyl transformation. Moreover, still no state $\eta \in \Gamma[L^2(\mathbb{R}_+; \mathfrak{J})]$ can give $|\hat{I}[\eta]\rangle^2 = dq^{\otimes \mathbb{N}} \otimes Q_{\mathbb{R}_+}^\lambda / d\mathbb{Q}$ on $\mathcal{G}_{\mathbb{R}_+}$, and again we can recover the right semigroups e^{Rt} and $e^{\mathcal{L}t}$ only up to a finite time T , namely by choosing an environment state $e(v)$,

$$v(t) = \begin{cases} \sum_{g \in G} \left(\sqrt{q_g \lambda} - \sqrt{\frac{\lambda}{|G|}}\right) |g\rangle, & \text{if } 0 \leq t \leq T, \\ u(t), & \text{if } t > T, \end{cases}$$

with an arbitrary $u \in L^2((T, +\infty); \mathfrak{J})$.

In order to eliminate the finite time horizon one can also leave the Hilbert space approach and study this correspondence on some C^* -algebras of bounded operators, but then the connection with quantum stochastic calculus is less direct. Anyway the basic result remains Theorem 4, which can be employed to find the preferred isomorphism.

Let us conclude by remarking that, if Theorem 4 allows to interpret a Poisson dilation as a classical analogue of a HP-dilation, at the same time Eq. (31) is only a particular case of a Hudson-Parthasarathy equation, so that it also suggests that other classical analogues could be found by coupling the system E with processes different from counting processes.

References

- [1] Accardi, L.; Lu, Y.G.; Volovich, I.: *Quantum theory and its stochastic limit*. Springer-Verlag, Berlin, 2002
- [2] Baccelli, F.; Brémaud, P.: *Palm probabilities and stationary queues*. Lecture Notes in Statistics, **41**. Springer-Verlag, Berlin, 1987
- [3] Chebotarev, A.M.: The quantum stochastic equation is unitarily equivalent to a symmetric boundary value problem for the Schrödinger equation. *Math. Notes* **61**, 510-518 (1997)
- [4] Chebotarev, A.M.: Quantum stochastic differential equation is unitary equivalent to a symmetric boundary value problem in Fock space. *Inf. Dimens. Anal. Quantum Probab. Relat. Top.* **1**, 175-199 (1998)
- [5] Ethier, S.N.; Kurtz, T.G.: *Markov processes. Characterization and convergence*. John Wiley & Sons, Inc., New York, 1986
- [6] Frigerio, A.: Covariant Markov dilations of quantum dynamical semigroups. *Publ. RIMS Kyoto Univ.* **21**, 657-675 (1985)
- [7] Frigerio, A.: Construction of stationary quantum Markov processes through quantum stochastic calculus. In *Quantum Probability and Applications II*. *Lect. Not. Math.* **1136**, 207-222. Berlin: Springer-Verlag, 1985
- [8] Gregoratti, M.: On the Hamiltonian operator associated to some quantum stochastic differential equations. *Inf. Dimens. Anal. Quantum Probab. Relat. Top.* **3** No.4, 483-503 (2000)
- [9] Gregoratti, M.: The Hamiltonian Operator Associated with Some Quantum Stochastic Evolutions. *Commun. Math. Phys.* **222**, 181-200 (2001)
- [10] Gregoratti, M.: Classical dilations à la Quantum Probability of Markov evolutions in discrete time. *Quaderno di Dipartimento QDD* **14**, math.PR/0702690v1
- [11] Hudson, R.L., Parthasarathy, K.R.: Quantum Itô's formula and stochastic evolutions. *Commun. Math. Phys.* **93**, 301-323 (1984)
- [12] Karr, A.F.: *Point processes and their statistical inference*. Marcel Dekker, Inc., New York, 1991
- [13] Kadison, R.V.; Ringrose, J.R.: *Fundamentals of the theory of operator algebras. Vol. II. Advanced theory*. Pure and Applied Mathematics, 100. Academic Press, Inc., Orlando, FL, 1986
- [14] Lewis, J.T.; Maassen, H.: Hamiltonian models of classical and quantum stochastic processes. *Quantum probability and applications to the quantum theory of irreversible processes (Villa Mondragone, 1982)*, 245-276, *Lecture Notes in Math.*, **1055**, Springer, Berlin, 1984

- [15] Maassen, H.: The construction of continuous dilations by solving quantum stochastic differential equations. Semesterbericht Funktionalanalysis Tübingen Sommersemester 1984, 183-204 (1984)
- [16] Maassen, H.: Quantum Markov processes on Fock space described by integral kernels. In *Quantum Probability and Applications II*. Lect. Not. Math. **1136**, 361-374. Berlin: Springer-Verlag, 1985
- [17] Meyer, P.A.: *Quantum probability for probabilists*. Lecture Notes in Mathematics, **1538**. Springer-Verlag, Berlin, 1993
- [18] Mohary, A.; Sinha, K.B.: Quantum stochastic flows with infinite degrees of freedom and countable state Markov processes. *Sankhyā* **52** A pt.1, 43-57 (1990)
- [19] Parthasarathy, K.R.: *An Introduction to Quantum Stochastic Calculus*. Birkhäuser, Basel-Boston-Berlin, 1992
- [20] Rozanov, Y.A.: *Innovation processes*. V.H. Winston & Sons, Washington, D.C.; John Wiley & Sons, New York, 1977
- [21] Tsirelson, B.: Within and beyond the reach of Brownian innovation. Proceedings of the International Congress of Mathematicians, Vol.III (Berlin, 1998). Doc. Math. Extra Vol.III, 311-320 (1998)