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Hartshorne, R.; Lella, P.; Schlesinger, E:

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SMOOTH CURVES SPECIALIZE TO EXTREMAL CURVES

ROBIN HARTSHORNE, PAOLO LELLA, AND ENRICO SCHLESINGER

Abstract. Let $H_{d,g}$ denote the Hilbert scheme of locally Cohen-Macaulay curves of degree d and genus g in projective three space. We show that, given a smooth irreducible curve C of degree d and genus g, there is a rational curve $\{[C_t] : t \in \mathbb{A}^1\}$ in $H_{d,g}$ such that C_t for $t \neq 0$ is projectively equivalent to C, while the special fibre C_0 is an extremal curve. It follows that smooth curves lie in a unique connected component of $H_{d,g}$. We also determine necessary and sufficient conditions for a locally Cohen-Macaulay curve to admit such a specialization to an extremal curve.

1. Introduction

In this paper we study curves in projective three space $\mathbb{P}^3$ over an algebraically closed field k. While smooth curves are often the first to be studied, the development of liaison theory in recent years [18] has shown that these are best understood in the context of the larger class of locally Cohen-Macaulay curves, that is, one-dimensional closed subschemes of $\mathbb{P}^3$ with no isolated points and no embedded points. For each d and (arithmetic) genus g, the locally Cohen-Macaulay curves form an open subset $H_{d,g}$ of the full Hilbert scheme $\mathrm{Hilb}_{d,g}$ of all closed subschemes of $\mathbb{P}^3$ with Hilbert polynomial $dn + 1 - g$. The smooth curves form an open subset $H^0_{d,g}$ of $H_{d,g}$.

It is known from classical examples that $H^0_{d,g}$ may not be connected. On the other hand, the thesis of the first author [8] showed that for any d, g the full Hilbert scheme $\mathrm{Hilb}_{d,g}$ is connected whenever it is nonempty. So it is natural to ask whether the Hilbert scheme $H_{d,g}$ of locally Cohen-Macaulay curves is connected, a question first raised by Martin-Deschamps and Perrin in their paper [20]. Up to now, the connectedness of $H_{d,g}$ has been established only for very small degree d, or for very large genus g; more precisely, for $d \leq 4$ [22, 25] and $g \geq \frac{1}{2}(d-3)(d-4) - 1$ [2, 12, 28]). The method is to identify all the irreducible components of $H_{d,g}$ and then to verify that they intersect each other in such a way as to make the whole set connected. It is the problem of finding all irreducible components of $H_{d,g}$ that seems to block any further progress in this direction.

Another approach makes use of the so-called *extremal curves*. Martin-Deschamps and Perrin showed in two papers [19, 20] that for any curve C of degree d and genus g the dimensions of the cohomology modules $H^1(\mathbb{P}^3, \mathcal{I}_C(n))$ are bounded by explicit functions of d, g (see exact statement below). Those curves that attain the maximum are called *extremal curves*, and the family of extremal curves forms a non-empty closed irreducible component of $H_{d,g}$ for each d, g. Thus one can ask which curves can be connected, through a chain of irreducible components of $H_{d,g}$ to an extremal curve. Various methods have shown for example that this is possible for smooth rational or elliptic curves, for ACM curves, for any curve whose Rao module is a complete intersection, or for any curve in the biliaison class of an extremal curve [11, 30, 27].

The purpose of this paper is to give a necessary and sufficient condition for an irreducible component V of $H_{d,g}$ to contain an extremal curve. We say that a curve C in $\mathbb{P}^3$ satisfies condition (*) if there is a point p in $\mathbb{P}^3$ such that:

1) for every line N containing p the degree of $C \cap N$ is at most two; furthermore, $\deg(C \cap N) = 2$ for at most finitely many such lines;
2) if N is a line through p meeting C in a scheme of degree 2, then there is a plane K through N such that the divisor $2N$ in K meets C in a scheme of degree ≤ 3.

Note that smooth irreducible curves satisfy condition (*). Our main result shows that an irreducible component V of $H_{d,g}$ contains an extremal curve if and only if a general curve $C \in V$ satisfies (*). We prove also the

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stronger result that if C satisfies (*), then C *specializes* to an extremal curve, meaning that there is a flat family $\{C_t\}_{t\in\mathbb{A}^1}$ such that $C = C_1$, for all $t \neq 0$, C_t is obtained from C by an automorphism of $\mathbb{P}^3$, and C_0 is an extremal curve. In the last section of the paper we give a geometric characterization of curves satisfying condition (*). From this it follows that in characteristic zero a reduced curve that has embedding dimension ≤ 2 at every point satisfies condition (*).

As a corollary of our theorem, we recover many earlier results, and in particular we show that any smooth irreducible curve specializes to an extremal curve. So for example we can conclude that the two irreducible components of $H^0_{9,10}$, neither of which meets the closure of the other [9, IV,6.4.3], can be connected via extremal curves in $H_{9,10}$.

The method we use, which appears already in the earlier paper [17] of the second and third authors, is to place the curve on a certain surface of the kind called a monoid by Cayley in his early study of space curves [4]. The limit of the curve under a weighted automorphism of $\mathbb{P}^3$ is then a multiple line on the limit surface, which one shows to be an extremal curve. The essential point, of course, is to show that the limit has no embedded points, and this is accomplished by a careful control of the degree and genus of the curves involved.

We should mention also that there are curves to which our results do not apply. For $d = 4$ and genus ≤ -3, there is a closed irreducible component of $H_{d,g}$, made up of the so-called *thick* 4-lines, that does not meet the component of extremal curves [25].

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2. Construction of specializations to extremal curves

In this section we establish notation and terminology and review some known results that we will need later. We work over an algebraically closed field k of arbitrary characteristic. We denote by the symbol $\mathcal{I}_X$ the ideal sheaf of a subscheme $X \subset \mathbb{P}^3$. Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3$, we define $h^i(\mathcal{F}) = \dim H^i(\mathbb{P}^3, \mathcal{F})$ and $H^i_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P}^3, \mathcal{F}(n))$. We write $R = k[x, y, z, w]$ for the homogeneous coordinate ring $H^0_*(\mathcal{O}_{\mathbb{P}^3})$ of $\mathbb{P}^3$.

Definition 2.1. A *curve* in $\mathbb{P}^3$ (or more precisely a locally Cohen-Macaulay curve) is a one dimensional subscheme $C \subset \mathbb{P}^3$ without zero dimensional associated points; this means that all irreducible components of C have dimension 1, and that C has no embedded points.

We denote by $H_{d,g}$ the Hilbert scheme parametrizing curves of degree d and arithmetic genus g in $\mathbb{P}^3$. This is an open subscheme of the full Hilbert scheme parametrizing all one dimensional subschemes of $\mathbb{P}^3$ with Hilbert polynomial $dn+1-g$. Martin-Deschamps and Perrin [19] have found sharp bounds for the Rao function $h^1(\mathcal{I}_C(n))$ of a curve in $H_{d,g}$. To state these bounds we need to introduce some notation. Let $a = \frac{1}{2}(d–2)(d–3)-g$ and $l = d - 2$. Then for $d \geq 2$ and $g \leq \frac{1}{2}(d-2)(d-3)$ define

$$\rho_{d,g}(n) = \begin{cases} 0 & \text{if} \quad n \leq -a \\ n + a & \text{if} \quad -a \leq n \leq 0 \\ a & \text{if} \quad 0 \leq n \leq l \\ a + l - n & \text{if} \quad l \leq n \leq a + l \\ 0 & \text{if} \quad n \geq a + l \end{cases}$$

Theorem 2.2 ([19, 23]). *Let $C \subset \mathbb{P}^3$ be a curve of degree d and arithmetic genus g. Then*

(1) C is a plane curve if and only if $g = \frac{1}{2}(d-1)(d-2)$; in this case $h^1(\mathcal{I}_C(n)) = 0$ for every $n \in \mathbb{Z}$.

(2) If C is not a plane curve, then $d \geq 2$, $g \leq \frac{1}{2}(d-2)(d-3)$ and

$$h^1(\mathcal{I}_C(n)) \leq \rho_{d,g}(n) \quad \text{for all } n \in \mathbb{Z}.$$

Remark 2.3. The set of pairs for which $H_{d,g}$ is non-empty was determined by Hartshorne in [10], see also [26]. The bound in the theorem is proven in [19] in characteristic zero; Nollet in [23] has shown that the same result is valid in characteristic p as well.

Definition 2.4. A curve $E \subset \mathbb{P}^3$ of degree d and genus g is called *extremal* if either it is a plane curve or it satisfies $h^1(\mathcal{I}_E(n)) = \rho_{d,g}(n)$ for every $n \in \mathbb{Z}$.

Remark 2.5. Our definition of extremal curves is equivalent to the one given by Hartshorne [11]. Martin-Deschamps and Perrin required extremal curves not to be ACM, that is, $g < \frac{1}{2}(d-2)(d-3)$.

It is a remarkable discovery of Martin-Deschamps and Perrin that extremal curves exist for every d, g for which $H_{d,g}$ is non-empty:

Theorem 2.6 ([10, 19, 20]). *If $H_{d,g}$ is non-empty, then there exist extremal curves of degree d and genus g, and they form a closed irreducible component $E_{d,g}$ of $H_{d,g}$.*

Remark 2.7. In fact in [19] it is shown that, for given d and g, extremal curves in $H_{d,g}$ have the same Hilbert function, that is, the functions $h^0(\mathcal{I}_E(n))$ and $h^2(\mathcal{I}_E(n))$ are constant on $E_{d,g}$; furthermore,

$$h^i(\mathcal{I}_C(n)) \leqslant h^i(\mathbb{P}^3, \mathcal{I}_E(n)) \qquad \text{for every } n \in \mathbb{Z} \text{ and for } i = 0,1,2 \tag{1}$$

for every curve C in $H_{d,g}$ and every extremal curve $E \in E_{d,g}$. Thus extremal curves have the largest possible cohomology.

Remark 2.8. Suppose $d \geq 2$ and $g < \frac{1}{2}(d-2)(d-3)$, that is, $a > 0$ and $l \geq 0$. Martin-Deschamps and Perrin show that the extremal curves are precisely the minimal curves associated to *extremal Koszul modules*. These are the modules of the form $R/(l_1, l_2, F, G)$ where l_1 and l_2 are linear forms, F and G are forms of degree a and $a + l$ respectively, and l_1, l_2, F, G have no common zeros. A curve is extremal if and only if its Rao module is an extremal module shifted to the left by $a-1$.

The following proposition describes extremal curves supported on the line $x = y = 0$.

Proposition 2.9 ([17] Proposition 2.5). *Let (d, g) be a pair of integers satisfying $d \geq 2$ and $g < \frac{1}{2}(d-2)(d-3)$. Let F and G be two forms in $k[z, w]$ of degrees a and $a + l$, respectively, with no common zeros. The surface S of equation $xG - y^{d-1}F = 0$ is irreducible and generically smooth along the line L of equations $x = y = 0$. It therefore contains a unique curve E of degree d supported on L. The curve E is extremal of degree d and genus g, and its Rao module is*

$$H^1_*(\mathcal{I}_E) \cong R/(x, y, F, G)(a-1) \cong k[z, w]/(F, G)(a-1)$$

The homogeneous ideal of E is generated by x^2, xy, y^d and $xG - y^{d-1}F$.

Proof. The surface S is irreducible because F and G have no common zeros, and it is smooth at points of L where G is different from zero. Therefore the ideal of L in the local ring $\mathcal{O}_{S,\xi}$ of the generic point ξ of L is generated by one function t , and the ideal of a curve of degree d supported on L must be $t^d\mathcal{O}_{S,\xi}$. Since a locally Cohen-Macaulay curve supported on L is determined by its ideal at the generic point of L, we see that there is a unique curve $D_m \subset S$ supported on L of degree m for every $m \geqslant 1$. For $m = d-1$, the curve $P = D_{d-1}$ is the planar multiple structure of equations $x = y^{d-1} = 0$. We note that $\mathcal{I}_P \otimes \mathcal{O}_L \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1-d)$ where the two generators are the images of x and y^{d-1}. The two forms F and G define a surjective map $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(1-d) \to \mathcal{O}_L(a-1)$; composing this with the natural map $\mathcal{I}_P \to \mathcal{I}_P \otimes \mathcal{O}_L$ we obtain a surjection $\phi : \mathcal{I}_P \to \mathcal{O}_L(a-1)$. We let E be the subscheme of $\mathbb{P}^3$ whose ideal sheaf is the kernel of ϕ. By construction we have an exact sequence

$$0 \to \mathcal{I}_E \to \mathcal{I}_P \to \mathcal{O}_L(a-1) \to 0 \tag{2}$$

This sequence shows that E is a (locally Cohen-Macaulay) curve of degree d and genus g, and that its homogeneous ideal is generated by x^2,xy,y^d and $xG - y^{d-1}F$. Therefore $E = D_d$ is the unique curve of degree d contained in S and supported on L. Finally, the long exact cohomology sequence associated to (2) shows that the Rao module of E is

$$M_E = k[z, w]/(F, G)(a-1) = R/(x, y, F, G)(a-1).$$

Hence E is an extremal curve. □

Given a weight vector $\omega \in \mathbb{N}^{N+1}$, the ω-degree of a monomial $\boldsymbol{x}^{\mathbf{n}} = x_0^{n_0} \cdots x_N^{n_N}$ is defined as $\deg_\omega \boldsymbol{x}^{\mathbf{n}} = \mathbf{n} \cdot \omega = n_0\omega_0 + \ldots + n_N\omega_N$, and the ω-degree of an arbitrary polynomial $P(\boldsymbol{x}) = \sum_{\mathbf{n}} c_{\mathbf{n}}\boldsymbol{x}^{\mathbf{n}}$ is the maximum degree of a monomial appearing in $P(\boldsymbol{x})$. In analogy with the notion of leading term of a polynomial with respect to a term ordering, one defines the *initial form* of $P(\boldsymbol{x})$ with respect to ω as

$$\mathrm{in}_\omega\big(P(\boldsymbol{x})\big) = \sum_{\substack{\mathbf{n} \\ \deg_\omega \boldsymbol{x}^{\mathbf{n}} = \deg_\omega P}} c_{\mathbf{n}}\boldsymbol{x}^{\mathbf{n}}.$$

For any ideal $I \subset k[x_0, \ldots, x_N]$ one defines the *initial ideal* $\mathrm{in}_\omega(I)$ of I with respect to ω as the ideal generated by all initial forms $\mathrm{in}_\omega(P)$ as P varies in I. We will use the following well known fact (see for example [3], [5, Theorem 15.17] or [16, Theorem 4.3.22]).

Proposition 2.10. *Given a subscheme $X \subset \mathbb{P}^N$ and a weight vector $\omega \in \mathbb{N}^{N+1}$, there is a flat family $X_t \subset \mathbb{P}^N \times \mathbb{A}^1$, whose fibres X_t for $t \neq 0$ are isomorphic to $X = X_1$ via an automorphism of $\mathbb{P}^N$ and whose special fibre X_0 is the subscheme of $\mathbb{P}^N$ defined by the initial ideal of X with respect to ω.*

Our basic remark is that the homogeneous polynomial $xG - y^{d-1}F$, if F and G are forms in z and w, is also homogeneous with respect to the weight vector $\omega = (d, 2, 1, 1)$. It follows that the ideal of the extremal curve E of Proposition 2.9 is homogeneous with respect to the grading defined by ω, and so it coincides with its initial ideal with respect to ω; we can then construct flat families having E as a special fibre by taking the initial ideal with respect to this particular weight vector.

Proposition 2.11. *Let (d, g) be a pair of integers satisfying $d \geq 2$ and $g < \frac{1}{2}(d-2)(d-3)$. Let $\omega = (d, 2, 1, 1)$. Let S be a surface whose defining equation f satisfies*

$$\mathrm{in}_\omega(f) = xG - y^{d-1}F$$

where F and G be two forms in $k[z, w]$ of degrees a and $a + l$, respectively, with no common zeros. If C is a curve of degree d and genus g on S that does not meet the line $z = w = 0$, then the saturation of the initial ideal of C with respect to ω is is the ideal generated by x^2, xy, y^d and $xG - y^{d-1}F$ and therefore defines an extremal curve.

Proof. Suppose C is a curve that does not meet the line M of equation $z = w = 0$. We can find a complete intersection curve D that contains C and does not meet the line M. Let g and h be equations for D, and let g_1 and h_1 their reduction modulo z and w in $k[x, y] \cong k[x, y, z, w]/(z, w)$. Since $D \cap M = \emptyset$, the radical of the ideal generated by g_1 and h_1 is the maximal ideal (x, y). Therefore we can find positive integers l and m such that x^l and y^m belong to (g_1, h_1). We can lift these to polynomials that belong to the ideal of D, and a fortiori to the ideal of C. This shows that x^l and y^m belong to $\mathrm{in}_\omega(I_C)$, therefore the curve C_0 defined by $\mathrm{in}_\omega(I_C)$ is supported on the line L of equation $x = y = 0$.

Now we add the hypothesis that C is contained in the surface S. Then, since $\mathrm{in}_\omega(I_S) = I_S$, the scheme C_0 is contained in S. By flatness, the Hilbert polynomial of C_0 coincides with that of C, so C_0 is a one dimensional subscheme of $\mathbb{P}^3$ of degree d and genus g. Let E be the largest Cohen-Macaulay curve contained in C_0: it is the curve of degree d obtained from C_0 throwing away its embedded points. By Proposition 2.9 E is the unique curve of degree d contained in S and supported on the line L; it is the extremal curve whose ideal is generated by x^2, xy, y^d and $xG - y^{d-1}F$. Since $E \subset C_0$ and the two schemes have the same Hilbert polynomial, we conclude $E = C_0$, and the statement is proven. □

3. Specialization of curves satisfying condition (*)

Now we come to our main result. We denote by $X \cap Y$ the scheme theoretic intersection of two closed subschemes of an ambient scheme V. If Z is a zero dimensional scheme of finite type over k and p is a closed point in the support of Z, we denote by $\mathrm{mult}_p(Z) = \mathrm{length}(\mathcal{O}_{Z,p})$ the multiplicity of Z at p, and by $\deg(Z) = \sum_{p \in \mathrm{Supp}(Z)} \mathrm{mult}_p(Z)$ the degree of Z. For a curve C in $\mathbb{P}^3$ we consider a condition (*) that expresses the idea that C behaves in some ways like a smooth curve:

Condition (*)

We say that a curve C in $\mathbb{P}^3$ satisfies condition (*) if there is a point p in $\mathbb{P}^3$ such that:

1) for every line N containing p the degree of $C \cap N$ is at most two; furthermore, $\deg(C \cap N) = 2$ for at most finitely many such lines;
2) if N is a line through p meeting C in a scheme of degree 2, then there is a plane K through N such that the divisor $2N$ in K meets C in a scheme of degree ≤ 3.

Remark 3.1. The set of points p with respect to which C satisfies condition (*) is open in $\mathbb{P}^3$. If 2) holds, then, for a general plane K in the pencil of planes through N, the divisor $2N$ in K meets C in a scheme of degree ≤ 3, and therefore, for at most finitely many planes K in the pencil, the divisor $2N$ in K meets C in a scheme of degree ≥ 4.

Proposition 3.2. *Condition (*) is an open condition on the Hilbert scheme.*

Proof. Suppose that C_0 is a curve satisfying (*). It will be sufficient to show that for any flat family $\mathcal{C} = \{C_t\}_{t \in T}$ of curves, parametrized by an irreducible curve T, with $C_{t_0} = C_0$, a general curve C_t in the family also satisfies (*). Choose $p \in \mathbb{P}^3$ that satisfies condition (*) for C_0. If a general curve in the family $\mathcal{C}$ does not satisfy (*), then either a) for a general C_t all lines from p to a point of C_t meet C_t at least twice, or b) for a general C_t there is a line N through p meeting C_t at least 3 times, or c) there is a line N through p such that all planar double lines $2N$ containing N intersect C_t with multiplicity at least 4. In each case, then, one can choose families of lines N_t for each C_t (possibly after a base extension $T' \to T$ of the family) whose limits N_0 by semicontinuity will contradict the property (*) that holds for C_0. Hence a general C_t in $\mathcal{C}$ must satisfy (*). □

Theorem 3.3.

a) An irreducible component V of $H_{d,g}$ contains an extremal curve if and only if a general curve C in V satisfies condition (*).

b) If C is a curve satisfying (*), then there is a flat family $\mathcal{C} = \{C_t\}_{t \in \mathbb{A}^1}$ such that $C_1 = C$, for $t \neq 0$, C_t is obtained from C by an automorphism of $\mathbb{P}^3$, and C_0 is an extremal curve (In this case we say C specializes *to an extremal curve).*

Proof of Theorem 3.3 (first part).

Step 1. First we deal with two special cases. If $g = \frac{1}{2}(d-1)(d-2)$, then $H_{d,g}$ is irreducible and consists of the plane curves of degree d. These obviously satisfy (*). If $g = \frac{1}{2}(d-2)(d-3)$, again $H_{d,g}$ is irreducible, and consists of ACM extremal curves. These have been described in [10, 3.5]. The general such curve is a nonsingular twisted cubic for $d = 3$, a nonsingular elliptic quartic for $d = 4$, and for $d \geq 5$ consists of a plane curve of degree $d-1$ with a line attached at one point. These general curves clearly satisfy (*), so part a) of the theorem is true in these two cases. For part b) we can take the trivial family $C_t = C$ for all t. Thus for the remainder of this proof we may assume $g < \frac{1}{2}(d-2)(d-3)$ since these are the only remaining values where $H_{d,g}$ is non-empty. (For $g = \frac{1}{2}(d-2)(d-3)$ it is not true that every extremal curve satisfies (*): consider for example the curve defined by the ideal (x^2, xy, y^{d-1}) for $d \geq 3$).

Step 2. We will show next that every extremal curve C with $g < \frac{1}{2}(d-2)(d-3)$ satisfies (*). We will use the classification of extremal curves of [20, Proposition 0.6]. If the curve C is the disjoint union of a line and a plane curve, then C obviously satisfies (*). The ideal of any other extremal curve can be brought with a change of variables into the form

$$I_C = (x^2, xy, Py^{b+2}, xG + y^{b+1}FP)$$

where F, G are forms in z, w of degrees a, $a+l$ respectively, with no common zeros, while P is a form in y, z, w of degree $l - b$ that is not a multiple of y. Here $a = \frac{1}{2}(d-2)(d-3) - g \geq 1$, $l = d - 2 \geq 1$, and $b \geq 0$. The curve C is contained in the double plane $2H = V(x^2)$, and, except at points of the line $L = V(x, y)$, the curve is contained in the reduced plane $H = V(x)$ [20, loc. cit.].

We claim that C satisfies (*) with respect to the point $p = (1, 0, 0, 0)$. Since C is contained in the double plane $2H$, the intersection of C with any line N through p has at most of degree 2 in any case. If N is a line joining p with a point q of $C \setminus L$, then $C \cap N$ has degree one because it is contained in $C \cap H$. Thus we need to worry only about lines $N = pq$ where $q = (0, 0, z_0, w_0)$ is a point of L. The ideal of the intersection $C \cap N$ is the saturation of

$$I_C + I_N = (x^2, y, xG, w_0 z - z_0 w).$$

Thus, if $G(z_0, w_0) \neq 0$, then $C \cap N$ consists of the reduced point q. Since G has finitely many zeros, corresponding to finitely many points q_i of L, there are finitely lines $N_i = pq_i$ that meets C in a scheme of degree 2. To finish, it suffices to show that for each of these lines N_i there is a plane K such that the intersection of C with the divisor $2N_i$ on K has degree at most three. We claim that, if $q_i = (0, 0, z_0, w_0)$, the plane $K = V(w_0 z - z_0 w)$ will do. To show this, observe that $w_0 z - z_0 w$ divides G, hence the ideal of $C \cap (2N_i)_K$ is the saturation of

$$(x^2, xy, y^2, w_0 z - z_0 w, y^{b+1} F P).$$

This shows $C \cap (2N_i)_K$ is a fat point of degree at most 3, and completes the proof that the extremal curves with $g < \frac{1}{2}(d-2)(d-3)$ all satisfy (*).

This together with Proposition 3.2 proves half of statemenent *a)* of the theorem: if an irreducible component V of $H_{d,g}$ contains an extremal curve then by the openness property, a general curve C in V must also satisfy (*). Thus it remains only to prove *b)*, since that statement implies the other half of *a)*.

Step 3. We will choose coordinates in $\mathbb{P}^3$ adapted to C. Let p be a point of $\mathbb{P}^3$ for which C satisfies (*). Then for every line N containing p, the degree of $C \cap N$ is at most two, and $\deg C \cap N = 2$ for at most finitely many lines N_i through p. For each N_i, let K_{ij} be the (at most) finitely many planes for which the divisor $2N_i$ in K_{ij} meets C in a scheme of degree ≥ 4. Finally, choose a line M through p such that a) M does not meet C, b) M is not contained in any plane containing two or more of the N_i, and c) M is not contained in any of the planes K_{ij}. Choose coordinates x, y, z, w in $\mathbb{P}^3$ so that $p = (1, 0, 0, 0)$ and M is the line $z = w = 0$.

Let H be the plane $x = 0$ and consider the projection π of the curve from p to H. The image will be a plane curve $C_0 \subset H$ of degree d. On the other hand, if we consider the flat family obtained by projecting away from p, as in [9, III, 9.8.3], or by applying Proposition 2.10 with weight vector $(1, 0, 0, 0)$, the limit is a scheme C' supported on C_0, and having nilpotent elements at points where the line N from p meets C twice. Note that the scheme C' is contained in the double plane $2H$ defined by $x^2 = 0$, since no line N meets C more than twice.

In general, when a scheme has embedded points there is no natural scheme structure on the set of embedded points. In our case, however, we can define a scheme Z representing the embedded points by taking the residual of the intersection C_0 of C' with H [13]. Since C' is contained in $2H$, the scheme Z will be contained in H and will have degree equal to the difference of the genus of C (which equals that of C') and of C_0, namely $\deg Z = \frac{1}{2}(d-1)(d-2) - g$. One can also obtain the structure sheaf $\mathcal{O}_Z$ as the quotient of $\mathcal{O}_{C'}$ by $\mathcal{O}_{C_0}$ or the quotient of $\pi_* \mathcal{O}_C$ by $\mathcal{O}_{C_0}$.

Note that lines N through p meeting C twice correspond to points of Z, also in the scheme sense: if z_0 is a point of Z of multiplicity r, then the cone from p over z_0 is an r-fold line N meeting C in $2r$ points. ☐

Before proceeding, we need a lemma.

Lemma 3.4. *Suppose $g \leq \frac{1}{2}(d-2)(d-3)$ and let $\nu = \frac{1}{2}(d-1)(d-2) - g$. For any choice of coordinates x, y, z, w on $\mathbb{P}^3$ and for any curve C in $\mathbb{P}^3$, the ideal of C contains a nonzero form f of degree $\nu + 1$ of type*

$$f(x, y, z, w) = x G_\nu(z, w) - \sum_{j=0}^{d-1} y^j F_{\nu+1-j}(z, w). \tag{3}$$

Here subscripts denote degree.

Proof. The proof is an easy dimension count. Inside $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(\nu + 1))$, the subspace T of forms as in (3) has dimension

$$\dim T = \nu + 1 + \sum_{j=0}^{d-1} (\nu + 2 - j) = \nu + 1 + d(\nu + 2) - \binom{d}{2} = (\nu + 1)(d + 1) + 1 - \frac{1}{2}(d-1)(d-2)$$

We consider the natural map from $H^0\big(\mathcal{O}_{\mathbb{P}^3}(\nu + 1)\big)$ to $H^0\big(\mathcal{O}_C(\nu + 1)\big)$. It is well known – and easy to show – that $h^1(\mathcal{O}_C(n)) = 0$ if $n \geq d - 2$: this is proven in characteristic zero in [19, Corollaire 2.4.(1)], and in any characteristic for example in [29, Proposition 3.2]. Now $\nu \geq d-3$ because $g \leq \frac{1}{2}(d-2)(d-3)$, hence we can conclude that $h^1(\mathcal{O}_C(\nu+1)) = 0$. Therefore by Riemann-Roch

$$h^0\big(\mathcal{O}_C(\nu+1)\big) = (\nu+1)d + 1 - g.$$

Now we observe that the dimension of T is greater than this number:

$$\dim T - h^0\big(\mathcal{O}_C(\nu+1)\big) = \nu + 1 - \frac{1}{2}(d-1)(d-2) + g = 1.$$

Hence there is a non zero form $f \in T$ whose image is zero in $H^0\big(\mathcal{O}_C(\nu+1)\big)$, that is, which is contained in $H^0\big(\mathcal{I}_C(\nu+1)\big)$. □

Proof of Theorem 3.3 (second part). **Step 4.** Using the coordinates chosen for C in **Step 3**, we apply Lemma 3.4 and conclude that C is contained in a surface S whose equation has the form (3) above. We will be using the surface S for the remainder of the proof. First we will show that G_ν is not identically zero.

Suppose by way of contradiction $G_\nu = 0$. We can find a plane K with the following properties: it contains the line M, it does not contain any of the lines N_i and it is not a component of the surface S. We may assume K is the plane of equation $w = 0$. Then the curve $S \cap K$ has equation

$$\sum_{j=0}^{d-1} y^j F_{\nu+1-j}(z,0) = z^{\nu+2-d} \sum_{j=0}^{d-1} c_j\, y^j z^{d-1-j} = 0.$$

Thus, as a divisor in K, the intersection $S \cap K$ is a cone consisting of M with multiplicity at least $\nu + 2 - d$, plus a divisor D of degree at most $d-1$. Since C is contained in S, the scheme $C \cap K$ is contained in $S \cap K$, and since C does not meet M, this scheme $C \cap K$, which is of degree d, must be contained in D. Since K contains none of the N_i, every line through p in K meets $C \cap K$ at most once, so the divisor D must have degree at least d, which is a contradiction.

Step 5. Now we relate the equation of S to the scheme of embedded points Z from **Step 3**. Let q be the point $M \cap H$ and let Z' be the projection of Z from q to the line L defined by $x = y = 0$. Since M was chosen not to belong to any of the planes K_{ij}, it follows that each line through q in H meets Z at most once. Hence the projection from Z to Z' is a closed immersion and Z' is a subscheme of L of degree ν.

On the other hand, the sheaf $\pi_*\mathcal{O}_C$ is generated over $\mathcal{O}_H$ by the local images of the variable x, because we projected away from $p = (1,0,0,0)$. The equation of S shows that

$$xG_\nu \equiv \sum_{j=0}^{d-1} y^j F_{\nu+1-j}(z,w) \qquad \text{mod } I_S.$$

Thus G_ν maps the local generator of $\pi_*\mathcal{O}_C$ into $\mathcal{O}_{C_0}$ and we conclude that G_ν annihilates Z. Since G_ν and Z' both have the same degree ν, we conclude that $G_\nu(z,w)$ is precisely the annihilator of the scheme Z' in L.

Step 6. Now we will show that G_ν and $F_{\nu+d-2}$ are forms in z, w with no common zeros. Let (z_0, w_0) be a zero of G_ν. By a linear change of variables we may assume that $(z_0, w_0) = (1, 0)$. Let K be the plane $w = 0$. Since K contains the line M, we conclude from the choices above that K contains at most one of the lines N_i, and that the divisor $2N_i$ in K meets C in a scheme of degree ≤ 3. The plane K is not a component of S: otherwise, we would have $S = K + S_1$ where S_1 has an equation of the form

$$f_1(x,y,z,w) = xG'(z,w) - \sum_{j=0}^{d-1} y^j F'_{\nu-j}(z,w) \tag{4}$$

where G' is not identically zero, and $\deg G' = \deg G_\nu - 1 = \nu - 1$. As the curve C meets properly every plane through M, it must be contained in S_1. Then, as in **Step 5**, we prove that G' kills Z' which is impossible since Z' has degree ν.

Consider the curve $S \cap K$ which has equation

$$\overline{f} = f(x,y,z,0) = -\sum_{j=0}^{d-1} y^j F_{\nu+1-j}(z,w) = -z^{\nu+2-d} \sum_{j=0}^{d-1} c_j y^j z^{d-1-j} \tag{5}$$

since $G_\nu(z,0)=0$. Now x does not appear in this equation so $S \cap K$ is a divisor in K supported on lines through p. In particular, $\overline{f}$ has a factor $z^{\nu+2-d}$, so the line M appears in $S\cap K$ with multiplicity $\geq \nu+2-d$. We will show that this multiplicity is exactly $\nu+2-d$, in other words that $c_{d-1}\neq 0$, hence $F_{\nu+2-d}(z,0)$ is not zero, hence that $(z,0)$ is not a common zero of G_ν and $F_{\nu+2-d}$, as desired.

Since C does not meet the line M, the plane K does not contain any component of C. Let W be the zero dimensional scheme $C\cap K$. Since $C\subset S$, we find $W\subset S\cap K$. So the curve $S\cap K$ contains the least divisor D of (possibly multiple) lines through p containing W. By construction, most of the reduced lines N through p meet W with degree 1; at most one N_i meets W in degree 2, and for that one, the divisor $2N_i$ in K meets W with degree ≤ 3. We will show that this divisor D has degree $\geq d-1$, hence M can appear in $S\cap K$ with multiplicity at most $(\nu+1)-(d-1)=\nu+2-d$ as required.

To verify the assertion that $\deg D\geq d-1$ we argue as follows. If $q\in W$ is a point for which the line $N=pq$ meets W with multiplicity one, then the scheme W is curvilinear at that point, transverse to N, so that if its multiplicity at q is r, we need N to appear with multiplicity r in D for D to contain W. If N is a line through p meeting W with multiplicity 2, and if N meets W in two distinct points q_1,q_2, then, since $2N$ meets W with multiplicity ≤ 3, one of these points must be a reduced point of W, the other is curvilinear transversal to N as above, so that if the multiplicity of the second point is r, we need N to appear with multiplicity r in D for D to contain W at the points q_1, q_2. Finally, if N meets W with multiplicity 2 at a single point q, and $2N$ meets it with multiplicity 2 also, then $\deg W=2$ at q, with tangent direction N. The last case is if N meets W at q with multiplicity 2, and $2N$ meets W at q with multiplicity 3. This is the only case that requires some calculation, and Lemma 3.5 below shows in this case that if the multiplicity of W at q is r, then we need N to appear with multiplicity $r-1$ in D for D to contain W at q. The conclusion is that since $\deg W=d$, we need a divisor D of degree degree $d-1$ or d to contain W, as claimed. □

Lemma 3.5. *Let $A=k[[x,y]]$ with maximal ideal $\mathfrak{m}=(x,y)$. Let I be an $\mathfrak{m}$-primary ideal such that* length $A/I+(y)=2$ *and* length $A/I+(y^2)=3$. *Then, replacing if necessary x with another element of $\mathfrak{m}\setminus\mathfrak{m}^2$, linearly independent of y modulo $\mathfrak{m}^2$, we can put the ideal I in one of the following forms:*

- *$I=(y-x^2,x^3)$, of colength 3;*
- *$I=(x^2,xy,y^s)$, with $s\geq 2$, of colength $s+1$*
- *$I=(x^2-y^s,xy,y^{s+1})$ with $s\geq 2$, of colength $s+2$*

In all three cases, the least power of y contained in I is one less than the colength.

Proof. Suppose I contains an element beginning with a linear form. Since $A/I+(y)$ has length 2 (not 1), that form must be y. Thus $I+(y)=(y,x^2)$, and I contains elements $y+f$, x^2+gy with $f\in\mathfrak{m}^2$ and $g\in A$. Hence I contains also $x^2-fg\in\mathfrak{m}^2$. If this element is non-zero, then $I+(y^2)=(y,x^2)$ would have colength only 2 (not 3). Hence g is a unit and f is a unit times x^2. We can absorb the unit into x, and obtain the first case $I=(y-x^2,x^3)$ above.

Suppose now $I\subseteq\mathfrak{m}^2$. Then length $A/I+(y^2)=3$ implies $\mathfrak{m}^2=I+(y^2)$. Then $xy-by^2\in I$ and replacing x with $x-by$ we may assume $xy\in I$. Since $x^2\in I+(y^2)$, we can find $a\in A$ such that x^2-ay^2 belong to I. As $xy\in I$, we can choose such an a in $k[[y]]$. If $a=0$, then $I=(x^2,xy,y^s)$ for some $s\geq 2$. This is the second case above.

If $a\neq 0$, then $ay^2=uy^s$ with u a unit and $s\geq 2$. Then

$$y^{s+1}=-u^{-1}y(x-uy^s)+u^{-1}xy\in I$$

In particular we can assume u is a nonzero element in k, and rescaling we can reduce to the case $u=1$. This gives the third case above (or the second case again if $y^t\in I$ for some $t\leq s$). □

Proof of Theorem 3.3 (third part). **Step 7.** We can now complete the proof of part *b)* of the theorem. Let C be a curve satisfying (*). Choose coordinates as in **Step 3** and a surface S containing C as in **Step 4** above. We have shown in **Step 6** that G_ν and $F_{\nu+d-2}$ have no common zeros. Thus C and S satisfy the hypotheses of Proposition 2.11, and we conclude that the flat family constructed there has for its limit an extremal curve. Thus C specializes to an extremal curve, as required. □

Corollary 3.6. *Every smooth irreducible curve can be specialized to an extremal curve.*

Proof. Any smooth irreducible curve satisfies (*). Just choose a point $p \in \mathbb{P}^3$ that lies on no trisecant line and no tangent line [9, IV 3.10]. □

Corollary 3.7. *Suppose C_0 is a curve satisfying condition (*) that is minimal in its biliaison class. Then every curve of degree d and genus g in the biliaison class of C_0 is in the connected component of $H_{d,g}$ that contains the extremal curves of degree d and genus g.*

Proof. The statement follows from Theorem 3.3.b and [30, Theorem 2.3]. □

4. Geometric characterization of curves satisfying condition (*)

The main result of this section is Theorem 4.5 that, at least in characteristic zero, gives a clear geometric characterization of curves that satisfy condition (*).

Lemma 4.1. *Suppose C satisfies condition (*). Then C has embedding dimension ≤ 2 at all but finitely many points.*

Proof. If C has embedding dimension 3 at a point q, then every line through q meets C with multiplicity at least 2. Since C satisfies (*), this can happen only at finitely many points. □

Lemma 4.2. *Suppose C is a curve that satisfies condition (*) and has embedding dimension 3 at a point q. Let T denote the tangent cone to C at q: it is naturally embedded in the affine tangent space $\mathbb{A}^3$ of C at q. Then one of the following possibilities occurs:*

1. *the cone T is the union of two curves T_1 and T_2, where T_1 is contained in a plane $H \subset \mathbb{A}^3$, and T_2 is a line transversal to the tangent plane to C_1;*
2. *there is a plane H containing the support of T, and the residual curve to $T \cap H$ in T is a reduced line;*
3. *the curve T is contained in the complete intersection of two quadric cones in $\mathbb{A}^3$, and therefore has multiplicity at most 4 at q.*

Proof. Let p be a point with respect to which C satisfies condition (*). Since C has embedding dimension 3 at q, the line $N = pq$ meets C at q with multiplicity 2, and for a general plane K through N the divisor $2N$ in K meets C at q with multiplicity 3. Fix such a plane K. Then near q the plane section $C \cap K$ has one of the forms listed in Lemma 3.5. In the first case $C \cap K$ has embedding dimension 1 at q, so C has at most embedding dimension 2 at q, a contradiction. So either the second case or the third case occur. In both cases, the ideal of $C \cap K$ in $\mathcal{O}_{K,q}$ is contained in $m^2_{K,q}$, and its image in m^2/m^3 has dimension at least 2. Thus the ideal of C at q also has image of dimension at least 2 in $m^2_{\mathbb{P}^3,q}/m^3_{\mathbb{P}^3,q}$, and so the tangent cone is contained in two distinct quadric cones. The statement then follows immediately. □

Lemma 4.3. *Suppose C satisfies condition (*). Then all non-reduced components of C have support in a single plane.*

Proof. **Step 1.** Suppose C is a non-reduced irreducible curve such that C_{red} is not a plane curve. We claim that C does not satisfy (*). Since C_{red} is not a plane curve, the secant variety of C_{red}, that is the closure of the union of lines meeting C_{red} in at least two distinct points, is the whole of $\mathbb{P}^3$. Thus a general point p of $\mathbb{P}^3$ lies on a line $N = \overline{qr}$ where q and r are distint points of C_{red}. Since C is non-reduced, any plane H through N will intersect C at q in a scheme of length at least two, and therefore the divisor $2N$ on H will intersect C with multiplicity at least 2 at q. For the same reason, $2N$ will intersect C with multiplicity at least 2 at r. Thus $2N$ will meet C at least in a scheme of degree 4, and C does not satisfy (*).
Step 2. Suppose C and D are non-reduced irreducible curves that $C_{red} \cup D_{red}$ is not a plane curve. The $C \cup D$ does not satisfy (*). Consider the join J of C_{red} and D_{red}, that is, the closure of the set of points lying on lines joining a point of $C_{red} \setminus D_{red}$ with a point of $D_{red} \setminus C_{red}$. Since $C_{red} \cup D_{red}$ is not a plane curve, the join J is the whole space $\mathbb{P}^3$. Thus a general point p of $\mathbb{P}^3$ lies on a line $N = \overline{qr}$ with $q \in C_{red}$ and $r \in D_{red} \setminus C_{red}$. We may also assume p lies on no plane containing either C_{red} or D_{red}. Then an argument similar to the one above shows that $C \cup D$ does not satisfy (*).
Step 3. Now suppose C is any curve, and that C_1, C_2, $\ldots C_r$ are the non-reduced components of C. If C satisfies condition (*), then the support of each C_i is contained in a plane H_i by Step 1. If one of the C_i, say C_1, is not a line, then H_1 is the unique plane containing the support of C_1, and H_1 contains the support of the

other non-reduced components by Step 2. Suppose finally that for every i the support of C_i is a line L_i. By Step 2 there is a plane H containing L_1 and L_2, and L_1 and L_2 meet in a point q. Suppose by way of contradiction that there is an i, say $i = 3$, such that L_i is not contained in H. By Step 2, the line L_3 meets both L_1 and L_2, and this can happen only at q. In particular, at q the curve C has embedding dimension 3. The tangent cone to C at q then contains at least three (planar) double lines supported on L_1, L_2 and L_3; but this contradicts Lemma 4.3. Therefore the support of C is contained in the plane H and the statement is proven. □

Lemma 4.4. *Suppose C is a non-reduced, irreducible curve that satisfies (*). Then either C has multiplicity ≤ 3 along its length, or else its underlying multiplicity 2 structure is planar.*

Proof. Let $D = C_{red}$. The conormal sheaf $\mathcal{I}_D/\mathcal{I}_D^2$ is generically locally free of rank 2. Since C has generically embedding dimension 2, it does not contain the first infinitesimal neighborhood $D^{(1)}$ of D, that is, $\mathcal{I}_C$ is not contained in $\mathcal{I}_D^2$. Therefore $(\mathcal{I}_D^2 + \mathcal{I}_C)/\mathcal{I}_D^2$ is generically a rank one locally free sheaf on D, and there is a unique locally Cohen-Macaulay curve $D_2 \subset D^{(1)}$ whose ideal in $D^{(1)}$ coincides with $(\mathcal{I}_D^2 + \mathcal{I}_C)/\mathcal{I}_D^2$ at the generic point. This curve D_2 is the multiplicity 2 structure underlying C. At most points q of D, the embedding dimension of D_2 is 2, and there is a well defined tangent plane H_q to D_2. This gives a rational map from D to the dual projective space. If this map is not constant, then a general point p of $\mathbb{P}^3$ is contained is some plane H_q. If $N = pq$, the divisor $2N$ in any plane K containing N meets C with multiplicity μ at q, where μ is the multiplicity of C along D. If $\mu \geq 4$, we thus get a contradiction to condition (*); thus, if $\mu \geq 4$, H_q is a fixed plane H for every q, and D_2 is contained in H. □

Theorem 4.5. *A curve $C \subset \mathbb{P}^3$ satisfies condition (*) if and only if it has the following properties:*

i) there are at most finitely points q at which the embedding dimension of C is 3; at each of these points the embedded tangent cone is contained in a pencil of two dimensional quadratic cones;
ii) all non-reduced components of C have support in a single plane;
iii) if C_0 is a non-reduced component of C, then either C_0 has multiplicity ≤ 3 along its length, or else its underlying multiplicity 2 structure is planar;
iv) suppose C_0 is a non-reduced component of C that has multiplicity 3 along its length and whose embedded tangent space at a general point $q \in C_0$ is a plane H_q; if H_q varies with q, then we require that all but finitely many lines N in H_q meet C_0 at q with multiplicity ≤ 2;
v) let $Z \subset \mathbb{P}^3$ denote the union of the lines that meet C in three or more distinct points; we require the closure of Z not to be the whole of $\mathbb{P}^3$.

Remark 4.6. If char $k = 0$, conditions *iv)* and *v)* are automatically satisfied; indeed, [6, Corollary 4.6.17] implies condition *v)*, and *iv)* follows from [10, Theorem 2.1]. On the other hand, if k has positive characteristic, condition *iv)* may fail for curves that satisfy *i)-iii)*. The paper [10] by the first author contains an example. Take any prime p, and let q be either p if $p \leq 3$ or 4 if p = 2. Then consider the surface X of equation $xw^q + y^3w^{q-2} + yz^q = 0$. Let L be the line $x = y = 0$, which is contained in the smooth locus of X. Take $C = 3L$ on X. Then one verifies that the tangent plane of C varies along the line L, and at every point any line in the tangent plane meets the curve 3 times. As for condition *v)*, we asked Joel Roberts about strange curves, and he gave us an example of a curve in characteristic $p > 2$ that is integral, but whose trisecants fill up the whole space. For $p \geq 3$, take the curve given by affine parametric equation (t, t^p, t^{p^2}). Any secant line meets the curve in p distinct points; thus the p-secant lines fill up the whole space. This curve does not satisfy condition *i)* because it has a bad singularity at infinity. We do not know if there is a curve that does satisfy *i)-iii)* but not *v)*.

Proof. Suppose C satisfies condition (*) with respect to a general point p. Then there are at most finitely lines N through p such that $\deg(C \cap N) > 1$. Together with the previous lemmas this property implies *i)-v)*.

If C satisfies *i)-v)*, then the set of points p that satisfy the following conditions is open and non-empty:

(1) p does not lie on any line that meets C in three or more distinct points;
(2) p lies on at most finitely many lines meeting C in two distinct points;
(3) if q is a singular point of C_{red}, then the line pq meets C only at q (there are at most finitely many such points q); furthermore, if C has embedding dimension two at $q \in C_{red}^{sing}$, then p does not belong to the tangent plane to C at q;

(4) p does not lie on any tangent line to C at a smooth point;
(5) if q is one of the at most finitely many points at which C has embedding dimension three, then (a) the line pq meets C only at q, (b) p does not belong to the embedded tangent cone $\mathcal{C}_q$ to C at q, (c)if the pencil of quadratic cones through $\mathcal{C}_q$ has a plane H as a common component, then $p \notin H$, and (d) if $\mathcal{C}_q$ is the complete intersection of two quadratic cones, there is a plane K containing $N = pq$ such that $\deg(C \cap (2N)_K) \leq 3$ (in case (d), if K is a plane meeting $\mathcal{C}_q$ properly at q, then for all but finitely lines $N \subset K$ through q one has $\deg(C \cap (2N)_K) \leq 3$ because $\mathcal{C}_q \cap K$ is a complete intersection of two conics);
(6) if C has a non-reduced component C_0, there are at most finitely many lines through p meeting C at a point $q_1 \in C_0$ and at a distinct point $q_2 \in C$; C has embedding dimension 2 at q_1, and the line pq_1 does not belong to the tangent plane to C at q_1; q_2 is a smooth point of C (for the support of the non-reduced components of C is contained in a single plane);
(7) if q is a point at which C has embedding dimension two and the tangent plane H_q to C at q contains a non-reduced sub-curve of C, then p does not belong H_q (usually H_q will be the single plane H containing the support of the non-reduced components of C; let C_0 be a non-reduced component of C, and let $D \subset H$ be the support of C_0; if D is not a line, then we must have $H_q = H$; on the other hand, if D is a line, then it is possible that H_q be different from H, but it will be the only plane containing the planar multiplicity two structure underlying C_0, and so it is uniquely determined by C_0, and there are at most finitely many such planes);
(8) suppose C_0 is a non-reduced component of C that has multiplicity ≤ 3 along its length; let U be as in *iv)*; then, if the tangent plane H_q is independent of $q \in U$, $p \notin T_q\, C_0$; otherwise, for every $q \in U$ the point p does not belong to any line N through q in H_q meeting C_0 at q with multiplicity ≥ 3.

Choose a point p that satisfies all of the conditions above. We claim that C satisfies condition (*) with respect to p. In the following we denote by N a line through p. We have to show: (a) no such line N meets C in a scheme of degree ≥ 3; (b) there are at most finitely such lines N that meet C in a scheme of degree 2, and for each of these lines N there is a plane K containing N such that $\deg(C \cap (2N)_K) \leq 3$.

By (1) any line N through p meets C in at most two distinct points, and by (2) at most finitely many of such lines N meet C in two distinct points. Suppose N meets C at two distinct points q_1 and q_2. If q_1 and q_2 are smooth points of C, then N is not tangent to C at these points, and so $\deg(C \cap N) = 2$; furthermore, if K is a plane through N that does not contain the tangent line to C neither at q_1 nor at q_2, then $\deg(C \cap (2N)_K) = 2$. Otherwise, we are in the situation of (6): q_1 is a point of a non-reduced component C_0, C has embedding dimension 2 at q_1, the line $N = pq_1$ does not belong to the tangent plane to C at q_1, and q_2 is a smooth point of C. Then $\deg(C \cap N) = 2$, and, if K is a plane through N that does not contain the tangent line to C at q_2, the degree of $C \cap (2N)_K$ is at most 3. Thus we have checked the required properties for lines meeting C in two distinct points.

Suppose now N meets C at a unique point q. If q is a smooth point of C or a singular point of C at which C has embedding dimension 2 and N is not contained in the tangent plane H_q to C at q, then $\deg(C \cap N) = 1$. Thus we are left to examine lines $N = pq$ where q is either a point such that C has embedding dimension 2 at q and p belongs to the tangent plane H_q, or a point at which C has embedding dimension 3. In the first case, by (3) q is a smooth point of C_{red}, and by (7) it is contained in a unique non-reduced component C_0 whose underlying multiplicity 2 structure is not planar, so that C_0 has multiplicity ≤ 3 along its length by *iii)*; by (8) the tangent plane H_q varies with q, so there are finitely many such $q \in C_0$ for which $p \in H_q$; furthermore, by (8) the line $N = pq$ meets C with multiplicity 2 at q, and for every plane K through N, $C \cap (2N)_K$ has degree at most 3 because has multiplicity ≤ 3 along its length. Finally, suppose q is a point at which C has embedding dimension 3. If the tangent cone $\mathcal{C}_q$ is contained in $H \cup L$, where H is a plane and L is a line transversal to N, then $p \notin H$ by (5), the line $N = pq$ meets C with multiplicity 2 at q, and, if K is a plane through N that does not contain L, then $(2N)_K$ meets $\mathcal{C}_q$ with multiplicity at most 3 at q, hence $\deg(C \cap (2N)_K) \leq 3$. If the tangent cone $\mathcal{C}_q$ is contained in a plane H plus an embedded line L , that is, the ideal of $\mathcal{C}_q$ contains x^2 and xy up to a choice of coordinates, then $p \notin H$ by (5), the line $N = pq$ meets C with multiplicity 2 at q, and, if K is a plane through N that does not contain L, $(2N)_K$ meets C with multiplicity at most 3 at q. Finally, the case in which the tangent cone $\mathcal{C}_q$ is the complete intersection of two quadratic cones is taken care by (5.d)

□

Example 4.7. Every smooth irreducible curve satisfies condition (*) cf. 3.6. The same holds for a smooth non connected curve whose trisecants do not fill the whole space: this latter condition is always satisfied if the curve lies on a quadric surface, or if $\operatorname{char} k = 0$. In particular, we obtain a stronger version of the main result of [17]: every smooth divisor on a smooth quadric surface specializes in a flat family to an extremal curve.

Example 4.8. Assume $\operatorname{char} k = 0$. Let C be a reduced curve that has embedding dimension at most 2 at every point. Then C satisfies condition (*). This applies for example to general ACM curves: see [14, Theorem 7.21].

Example 4.9. Assume $\operatorname{char} k = 0$. Suppose we are given curves C_i for $i = 0, \ldots, r$ that are disjoint, satisfy condition (*) and are reduced except possibly for C_0. Then their disjoint union C is a curve that satisfies condition (*).

Example 4.10. Assume $\operatorname{char} k = 0$. Suppose C satisfies condition (*) and D is a smooth irreducible curve meeting C at a single point p in such a way that the tangent line to D at p is not contained in the tangent space to C at p. Then $C \cup D$ satisfies condition (*).

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Department of Mathematics, University of California, Berkeley, California 94720–3840
E-mail address: robin@math.berkeley.edu

Dipartimento di Matematica dell'Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
E-mail address: paolo.lella@unito.it
URL: http://www.dm.unito.it/dottorato/dottorandi/lella/

Dipartimento di Matematica del Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
E-mail address: enrico.schlesinger@polimi.it
URL: http://www1.mate.polimi.it/~enrsch/