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A family of Hardy-Rellich type inequalities involving the L^2-norm of the Hessian matrices

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Abstract

We derive a family of Hardy-Rellich type inequalities in $H^2(\Omega)\cap H^1_0(\Omega)$ involving the scalar product between Hessian matrices. The constants found are optimal and the existence of a boundary remainder term is discussed.
Mathematics Subject Classification: $26D10$; $46E35$; $35J40$
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1 Introduction

Let $\Omega\subset\mathbb{R}^N(N\geq 2)$ be a bounded domain (open and connected) with Lipschitz boundary. By combining interpolation inequalities (see [1, Corollary 4.16]) with the classical Poincaré inequality, the Sobolev space $H^2(\Omega)\cap H^1_0(\Omega)$ becomes a Hilbert space when endowed with the scalar product

$$(u,v):=\int_\Omega D^2u\cdot D^2v\,dx=\sum_{i,j=1}^N\int_\Omega \partial^2_{ij}u\,\partial^2_{ij}v\,dx\quad\text{for all } u\,,v\in H^2(\Omega)\cap H^1_0(\Omega)\,,\tag{1}$$

which induces the norm $\|D^2u\|_2:=\left(\int_\Omega D^2u\cdot D^2u\,dx\right)^{1/2}=\left(\int_\Omega |D^2u|^2\,dx\right)^{1/2}$.
If, furthermore, Ω satisfies a uniform outer ball condition, see [3, Definition 1.2], some of the derivatives in (1) may be dropped. Then, the bilinear form

$$\langle u,v\rangle:=\int_\Omega \Delta u\,\Delta v\,dx\quad\text{for all } u\,,v\in H^2(\Omega)\cap H^1_0(\Omega)\tag{2}$$

defines a scalar product on $H^2(\Omega)\cap H^1_0(\Omega)$ with corresponding norm $\|\Delta u\|_2:=\left(\int_\Omega |\Delta u|^2\,dx\right)^{1/2}$. Easily, $\|D^2u\|_2^2\geq 1/N\|\Delta u\|_2^2$, for every $u\in H^2(\Omega)\cap H^1_0(\Omega)$. The converse inequality follows from [3, Theorem 2.2].

A well-known generalization of the first order Hardy inequality [15, 16] to the second order is the so-called Hardy-Rellich inequality [19] which reads

$$\int_\Omega |\Delta u|^2\,dx\geq\frac{N^2(N-4)^2}{16}\int_\Omega\frac{u^2}{|x|^4}\,dx\qquad\text{for all } u\in H^2_0(\Omega)\,.\tag{3}$$

Here $\Omega\subset\mathbb{R}^N$ $(N\geq 5)$ is a bounded domain such that $0\in\Omega$ and the constant $\frac{N^2(N-4)^2}{16}$ is optimal, in the sense that it is the largest possible. Further generalizations to (3) have appeared in [9] and in [17]. In [11] the validity of (3) was extended to the space $H^2\cap H^1_0(\Omega)$, see also [12]. One may wonder

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what happens in (3), if we replace the L^2-norm of the laplacian with $\|D^2u\|_2^2$. In $H_0^2(\Omega)$, a density argument and two integrations by parts yield that $\frac{N^2(N-4)^2}{16}$ is still the "best" constant. In $H^2\cap H_0^1(\Omega)$ the answer is less obvious and, to our knowledge, the corresponding inequality is not known, not even when Ω is smooth. This regard motivates the present paper.

Let ν be the exterior unit normal at $\partial\Omega$, we set

$$c_0 = c_0(\Omega) := \inf_{H^2\cap H_0^1(\Omega)\setminus H_0^2(\Omega)} \frac{\int_\Omega |D^2u|^2\,dx}{\int_{\partial\Omega} u_\nu^2\,d\sigma}\,. \tag{4}$$

The above definition makes sense as soon as Ω has Lipschitz boundary. Indeed, the normal derivative to a Lipschitz domain is defined almost everywhere on $\partial\Omega$ so that $u_\nu \in L^2(\partial\Omega)$ for any $u \in H^2\cap H_0^1(\Omega)$. By the compactness of the embedding $H^2(\Omega)\subset H^1(\partial\Omega)$ (see [18, Chapter 2 - Theorem 6.2]), the infimum in (4) is attained and $c_0>0$.
For $c>-c_0$, we aim to determine the largest $h(c)>0$ such that

$$\int_\Omega |D^2u|^2\,dx + c\int_{\partial\Omega} u_\nu^2\,d\sigma \ge h(c)\int_\Omega \frac{u^2}{|x|^4}\,dx \qquad \text{for all } u\in H^2\cap H_0^1(\Omega)\,. \tag{5}$$

In Section 3, for $\partial\Omega\in C^2$, we prove that there exists $C_N = C_N(\Omega)\in(-c_0,+\infty)$ such that:

- $h(c) < \frac{N^2(N-4)^2}{16}$, for $c\in(-c_0, C_N)$ and the equality is achieved in (5).
- $h(c) = \frac{N^2(N-4)^2}{16}$, for $c\in[C_N,+\infty)$ and, if $c>C_N$ ($u\not\equiv 0$), the inequality is strict in (5).

When Ω satisfies a suitable geometrical condition (see (25) in the following) and $C=C_N$, we show that the equality cannot be achieved in (5). At last, we derive lower and upper bounds for C_N and we discuss its sign, see Theorem 1 and Remark 3.
If $\Omega = B$, the unit ball in $\mathbb{R}^N (N\ge 5)$, several computations can be done explicitly. In Section 5, we show that $c_0(B)=1$, $C_N(B) = N-3-\frac{\sqrt{2(N^2-4N+8)}}{2}$ and we determine the (radial) functions for which the equality holds in (5) (when $c<C_N$). In particular, for all $u\in H^2\cap H_0^1(B)\setminus\{0\}$, we show that

$$\int_B |D^2u|^2\,dx + \left(N-3-\frac{\sqrt{2(N^2-4N+8)}}{2}\right)\int_{\partial B} u_\nu^2\,d\sigma > \frac{N^2(N-4)^2}{16}\int_B \frac{u^2}{|x|^4}\,dx \tag{6}$$

and the constants are optimal.
It's worth noting that $C_N(B)$ is positive when $N\ge 7$, negative when $N=5,6$, see Figure 1. Hence, in lower dimensions, the following Hardy-Rellich inequality (with a boundary remainder term) holds

$$\int_B |D^2u|^2\,dx > \frac{N^2(N-4)^2}{16}\int_B \frac{u^2}{|x|^4}\,dx \left(+|C_N|\int_{\partial B} u_\nu^2\,d\sigma\right) \qquad \text{for all } u\in H^2\cap H_0^1(B)\setminus\{0\}\,,$$

where $|C_5| = \sqrt{13/2}-2$ and $|C_6| = \sqrt{10}-3$. While, if $N\ge 7$, the "best" constant $h(0)$ is no longer the classical Hardy-Rellich one and we prove

$$\int_B |D^2u|^2\,dx \ge \frac{(N-1)(N-5)(2N-5)}{4}\int_B \frac{u^2}{|x|^4}\,dx \qquad \text{for all } u\in H^2\cap H_0^1(B)\,. \tag{7}$$

Here, $\frac{(N-1)(N-5)(2N-5)}{4} < \frac{N^2(N-4)^2}{16}$ and the equality in (7) is achieved by a unique positive radial function, see Theorem 2 in Section 5.

The plan of the paper is the following: in Section 2 we prove existence and positivity of solutions to a suitable biharmonic linear problem. The boundary conditions considered arise from (5). In Section 3 we state our statement about the family of inequalities (5) while, in Section 4, we put its proof. At last, in Section 5, we focus on the case $\Omega = B$ and we prove (6) and (7). The Appendix contains the proof of some estimates we need in Section 3.

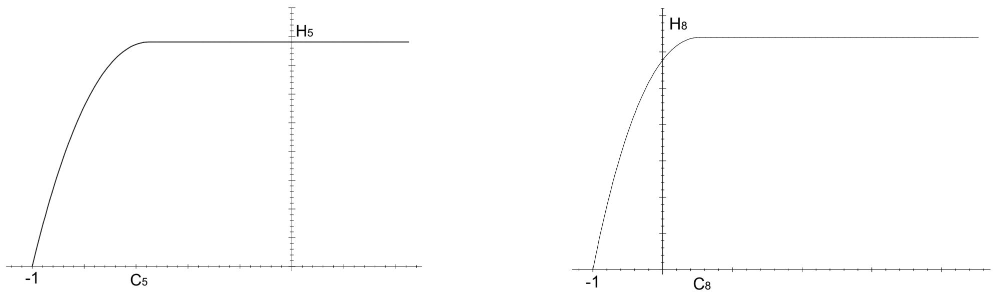

Figure 1: The plot of the map $(-c_0, +\infty) \in c \mapsto h(c)$ when $\Omega = B$, $N = 5$ or $N = 8$ (right). H_5 and H_8 denote the Hardy-Rellich constants, $c_0(B) = 1$.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a Lipschitz bounded domain which satisfies a uniform outer ball condition. We recall the definition of the first *Steklov* eigenvalue

$$d_0 = d_0(\Omega) := \inf_{H^2 \cap H^1_0(\Omega) \setminus H^2_0(\Omega)} \frac{\int_\Omega |\Delta u|^2 \, dx}{\int_{\partial\Omega} u_\nu^2 \, d\sigma} \,. \tag{8}$$

From the compactness of the embedding $H^2(\Omega) \subset H^1(\partial\Omega)$, the infimum in (8) is attained. Furthermore, due to [6], we know that the corresponding minimizer is unique, positive in Ω and solves the equation $\Delta^2 u = 0$ in Ω, subject the conditions $u = 0 = \Delta u - d_0 u_\nu$ on $\partial\Omega$.
Next, we assume that $\partial\Omega \in C^2$ and we denote with $|\Omega|$ and $|\partial\Omega|$ the Lebesgue measures of Ω and $\partial\Omega$. There holds

$$d_0(\Omega) \leq \frac{|\partial\Omega|}{|\Omega|} \,,$$

see, for instance, [10, Theorem 1.8]. Let $K(x)$ denote the mean curvature of $\partial\Omega$ at x,

$$\underline{K} := \min_{\partial\Omega} K(x) \quad \text{and} \quad \overline{K} := \max_{\partial\Omega} K(x) \,. \tag{9}$$

If Ω is convex, it was proved in [10, Theorem 1.7] that

$$d_0(\Omega) \geq N\underline{K} \,. \tag{10}$$

Notice that we adopt the convention that K is positive where the domain is convex.
Finally, from [14, Theorem 3.1.1.1] we recall

$$\int_\Omega |\Delta u|^2 \, dx = \int_\Omega |D^2 u|^2 \, dx + (N-1) \int_{\partial\Omega} K(x) \, u_\nu^2 \, d\sigma \qquad \text{for all } u \in H^2 \cap H^1_0(\Omega) \,. \tag{11}$$

Identity (11) is the basic ingredient to prove

Proposition 1. *Let Ω be a bounded domain with C^2 boundary, let c_0 and d_0 be as in* (4) *and* (8)*, $\overline{K}$ and $\underline{K}$ as in* (9)*. There holds*

$$\max\left\{ d_0(\Omega) - (N-1)\overline{K}; \frac{d_0(\Omega)}{N} \right\} \leq c_0(\Omega) \leq d_0(\Omega) - (N-1)\underline{K} \,. \tag{12}$$

Furthermore, if Ω is convex, then (*i*) $c_0 \geq \underline{K}$ *and the equality holds if and only if* Ω *is a ball;*

(*ii*) *the minimizer* u_0 *of* (4) *is unique (up to a multiplicative constant) and, if* $u_0(x_0) > 0$ *for some* $x_0 \in \Omega$, *then* $u_0 > 0$, $-\Delta u_0 \geq 0$ *in* Ω *and* $(u_0)_\nu < 0$ *on* $\partial\Omega$.

If $\Omega = B$, the unit ball in $\mathbb{R}^N$, since $K(x) \equiv 1$, Proposition 1-(*i*) yields $c_0(B) = 1$.

Proof. The estimates in (12) follow by combining (11) with (4) and (8). For the lower bound $d_0(\Omega)/N$, we exploit the fact that $\|D^2u\|_2^2 \geq 1/N\|\Delta u\|_2^2$, for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$.
Let Ω be convex, by (10) and (12), $c_0 \geq \underline{K}$. If $c_0 = \underline{K}$, by (10) and (12), we deduce that $d_0 = N\underline{K}$ and, by [10, Theorem 1.7], Ω must be a ball. On the other hand, if Ω is a ball, then $\underline{K} = \overline{K}$ and, by (12), we get $c_0 = d_0 - (N-1)\underline{K}$. Since, from [10], $d_0 = N\underline{K}$, statement (*i*) follows at once.
To prove statement (*ii*), by (11), we write (12) as

$$c_0 = \inf_{H^2\cap H_0^1(\Omega)\setminus H_0^2(\Omega)} \frac{\int_\Omega |\Delta u|^2\,dx - (N-1)\int_{\partial\Omega} K(x)\,u_\nu^2\,d\sigma}{\int_{\partial\Omega} u_\nu^2\,d\sigma}\,. \tag{13}$$

Let u_0 be a minimizer to c_0. As in [6], we define $\bar{u}_0 \in H^2 \cap H_0^1(\Omega)$ as the unique (weak) solution to

$$\begin{cases} -\Delta \bar{u}_0 = |\Delta u_0| & \text{in } \Omega \\ \bar{u}_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle for superharmonic functions,

$$|u_0| \leq \bar{u}_0 \quad \text{in } \Omega \qquad \text{and} \qquad |(u_0)_\nu| \leq |(\bar{u}_0)_\nu| \quad \text{on } \partial\Omega\,.$$

If Δu_0 changes sign, then the above inequalities are strict and, since K is positive, by (13), we infer

$$c_0 = \frac{\int_\Omega |\Delta u_0|^2\,dx - (N-1)\int_{\partial\Omega} K(x)\,(u_0)_\nu^2\,d\sigma}{\int_{\partial\Omega} (u_0)_\nu^2\,d\sigma} > \frac{\int_\Omega |\Delta \bar{u}_0|^2\,dx - (N-1)\int_{\partial\Omega} K(x)\,(\bar{u}_0)_\nu^2\,d\sigma}{\int_{\partial\Omega} (\bar{u}_0)_\nu^2\,d\sigma}\,,$$

a contradiction. This noticed, a further application of the maximum principle yields the positivity issue. Uniqueness follows by standard arguments. That is, by exploiting the fact that a (positive) minimizer to (4) solves the linear problem (15), here below, for $f \equiv 0$ and $c = -c_0$. □

Remark 1. *The problem of dealing with domains having a nonsmooth boundary goes beyond the purposes of the present paper. We limit ourselves to make a couple of remarks on the topic.*
If we drop the regularity assumption on $\partial\Omega$, *identity* (11) *is, in general, no longer true. Hence, the previous proof cannot be carried out. Assume that* $\Omega \subset \mathbb{R}^N (N \geq 2)$ *is a bounded domain with Lipschitz boundary which satisfies an outer ball condition. Due to* [3], *we know that there exist a sequence of smooth domains* $\Omega_m \nearrow \Omega$, *with* $\partial\Omega_m \in C^\infty$, *and a real constant* C *such that the mean curvatures satisfy* $K_m(x) \geq C$, *for every* $x \in \partial\Omega$ *and* $m \geq 1$. *Next, for* $u \in H^2 \cap H_0^1(\Omega)$ *fixed, define the sequence of functions* $\{u_m\}_{m\geq 1}$ *such that* $u_m \in H^2 \cap H_0^1(\Omega)$ *solves*

$$\begin{cases} -\Delta u_m = -\Delta u & \text{in } \Omega_m \\ u_m = 0 & \text{on } \partial\Omega_m. \end{cases}$$

When $C \geq 0$, *from* (11), *it is readily deduced that*

$$\int_{\Omega_m} |D^2 u_m|^2\,dx \leq \int_\Omega |\Delta u|^2\,dx$$

while, if $C<0$*, we get*

$$\int_{\Omega_m}|D^2u_m|^2\,dx \le \int_\Omega |\Delta u|^2\,dx-(N-1)C\int_{\partial\Omega_m}(u_m)_\nu^2\,d\sigma \le \left(1+\frac{(N-1)|C|}{d_0(\Omega)}\right)\int_\Omega|\Delta u|^2\,dx\,,$$

where d_0 *is as in* (8)*. Then, by a standard weak convergence argument, see* [14, *Theorem* 3.2.1.2]*, one concludes that*

$$\int_\Omega |D^2u|^2\,dx\le (1+\gamma(\Omega))\int_\Omega|\Delta u|^2\,dx \qquad \text{for all } u\in H^2\cap H^1_0(\Omega)\,, \tag{14}$$

where $\gamma(\Omega)=0$*, if* $C\ge 0$*, and* $\gamma(\Omega)=((N-1)|C|)/d_0(\Omega)$*, otherwise.*
Obviously, (14) *does not replace* (11)*. However, it can be exploited to obtain the first part of Proposition* 1 *for domains satisfying the above mentioned (weaker) regularity assumptions.*

For every $c>-c_0$ and for $f\in L^2(\Omega)$, we will consider the linear problem

$$\begin{cases}\Delta^2u=f(x) & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\\ u_{\nu\nu}+cu_\nu=0 & \text{on } \partial\Omega.\end{cases} \tag{15}$$

This choice of boundary conditions will be convenient in the next Section.
By solutions to (15) we mean weak solutions, that is functions $u\in H^2\cap H^1_0(\Omega)$ such that

$$\int_\Omega D^2u\cdot D^2v\,dx+c\int_{\partial\Omega}u_\nu\,v_\nu\,d\sigma=\int_\Omega f\,v\,dx \qquad \text{for all } v\in H^2\cap H^1_0(\Omega). \tag{16}$$

Indeed, formally, two integrations by parts give

$$\int_\Omega D^2u\cdot D^2v\,dx=\int_\Omega \Delta^2u\,v\,dx+\int_{\partial\Omega}u_{\nu\nu}\,v_\nu\,d\sigma \qquad \text{for all } v\in H^2\cap H^1_0(\Omega)\,, \tag{17}$$

see [7, formula (36)]. Then, plugging (17) into (16), by standard density arguments, we infer that u solves (15) pointwise. Since the boundary conditions in (15) have the same principal part of Navier boundary conditions ($u=0=\Delta u$ on $\partial\Omega$), they must satisfy the so-called *complementing conditions* [4]. See also [13, formula (2.22)]. Hence, standard elliptic regularity theory applies. Therefore, if $\partial\Omega\in C^4$ and $f\in L^2(\Omega)$, then $u\in H^4(\Omega)$ and (17) makes sense.
Solutions to (16) correspond to critical points of the functional

$$I_c(u):=\frac12\left(\int_\Omega|D^2u|^2\,dx+c\int_{\partial\Omega}u_\nu^2\,d\sigma\right)-\int_\Omega fu\,dx \qquad \text{for } u\in H^2\cap H^1_0(\Omega)\,.$$

For $c>-c_0$, I_c turns to be coercive. Since it is also strictly convex, there exists a unique critical point u_c which is the global minimum of I_c. When $\partial\Omega\in C^2$, thanks to (11), I_c writes

$$I_c(u)=\frac12\left(\int_\Omega|\Delta u|^2\,dx-\int_{\partial\Omega}\alpha_c(x)\,u_\nu^2\,d\sigma\right)-\int_\Omega fu\,dx \qquad \text{for } u\in H^2\cap H^1_0(\Omega)\,,$$

where $\alpha_c(x):=(N-1)K(x)-c$, for every $x\in\partial\Omega$. Then, the minimizer u_c to I_c also satisfies

$$\int_\Omega \Delta u_c\,\Delta v\,dx-\int_{\partial\Omega}\alpha_c(x)\,(u_c)_\nu\,v_\nu\,d\sigma=\int_\Omega f\,v\,dx \qquad \text{for all } v\in H^2\cap H^1_0(\Omega)\,. \tag{18}$$

From [13, Definition 5.21], we know that (18) is the definition of *weak* solutions to the equation $\Delta^2 u = f$ in Ω, subject to *Steklov* boundary conditions (with nonconstant parameter α_c). Namely, $u = 0 = \Delta u - \alpha_c(x)\, u_\nu$ on $\partial\Omega$. Arguing as in the proof of [13, Theorem 5.22], if $\alpha_c \geq 0$ and $0 \neq f \geq 0$, we infer that the minimizer u_c to I_c is positive. Furthermore, $-\Delta u_c \geq 0$ in Ω and $(u_c)_\nu < 0$ on $\partial\Omega$. We conclude that Δ^2, subject to the boundary conditions in (15), satisfies the *positivity preserving property* (p.p.p. in the following) if

$$-c_0 < c \leq (N-1)K(x) \quad \text{ for every } x \in \partial\Omega\,.$$

Notice that, if only the positivity of u is concerned, the lower bound for p.p.p. ($\alpha_c \geq 0$) can be weakened, see [13, Theorem 5.22].
We collect the conclusions so far drawn in the following

Proposition 2. *Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a Lipschitz bounded domain and c_0 be as in (4). For every $c > -c_0$, we have*

(*i*) *for every $f \in L^2(\Omega)$, problem (15) admits a unique solution $u \in H^2 \cap H^1_0(\Omega)$. Moreover, if $f \in H^k(\Omega)$ and $\partial\Omega \in C^{k+4}$ for some $k \geq 0$, then $u \in H^{k+4}(\Omega)$.*

(*ii*) *Assume, furthermore, that Ω is convex, $\partial\Omega \in C^2$ and $\underline{K}$ is as in (9). Then, for every $c \in (-c_0, (N-1)\underline{K}]$, if $f \geq 0$ ($f \not\equiv 0$) in Ω, the solution u of (15) satisfies $u > 0$, $-\Delta u \geq 0$ in Ω and $u_\nu < 0$ on $\partial\Omega$.*

Remark 2. *The convexity assumption in Proposition 2-(ii) is only needed to assure the non-emptiness of the interval $(-c_0, (N-1)\underline{K}]$ in which p.p.p. holds. If Ω is not convex, by (12), the same goal can be achieved by assuming that Ω satisfies one of the following inequalities*

$$N(N-1)|\underline{K}| < d_0(\Omega) \quad \text{or} \quad (N-1)(\overline{K} + |\underline{K}|) < d_0(\Omega)\,. \tag{19}$$

Compare with Proposition 3 in the Appendix.

3 Hardy-Rellich type inequalities with a boundary term

Before stating our results, we recall some facts from [5]. Set $H_N := \frac{N^2(N-4)^2}{16}$. For every bounded domain Ω such that $0 \in \Omega$ and for every $h \in [0, H_N]$, we know that

$$\int_\Omega |\Delta u|^2\, dx \geq h \int_\Omega \frac{u^2}{|x|^4}\, dx + d_1(h) \int_{\partial\Omega} u_\nu^2\, d\sigma \qquad \text{for all } u \in H^2 \cap H^1_0(\Omega)\,. \tag{20}$$

The optimal constant $d_1(h)$ is achieved, if and only if $h < H_N$, by a unique positive function $u_h \in H^2 \cap H^1_0(\Omega)$. Furthermore, $0 \leq d_1(h) < d_1(0) = d_0$, with d_0 as in (8). When $d_1(H_N) > 0$ (this was established only for strictly starshaped domains, namely such that $\min_{\partial\Omega}(x \cdot \nu) > 0$), (20) readily gives the Hardy-Rellich inequality (3) (for $u \in H^2 \cap H^1_0(\Omega)$) plus a boundary remainder term. See also the Appendix.

Let c_0 be as in (4). To obtain (5), for $c > -c_0$, we consider the minimization problem

$$h(c) := \inf_{H^2 \cap H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |D^2 u|^2\, dx + c \int_{\partial\Omega} (u_\nu)^2\, d\sigma}{\int_\Omega \frac{u^2}{|x|^4}\, dx}\,. \tag{21}$$

Clearly, $h(c) \geq 0$ and $h(-c_0) = 0$. On the other hand, since $\int_\Omega |D^2 u|^2\, dx = \int_\Omega |\Delta u|^2\, dx$, for all $u \in H^2_0(\Omega)$, (3) yields $h(c) \leq H_N$.

Formally, for every $c > -c_0$ fixed, the Euler equation corresponding to (21) is the eigenvalue problem

$$\begin{cases} \Delta^2 u = h \dfrac{u}{|x|^4} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u_{\nu\nu} + c u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Indeed, by solutions to (22) we mean functions $u \in H^2 \cap H^1_0(\Omega)$ such that

$$\int_\Omega D^2u\, D^2v\, dx + c\int_{\partial\Omega} u_\nu\, v_\nu\, d\sigma = h \int_\Omega \frac{uv}{|x|^4}\, dx \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega)\,, \tag{23}$$

see Section 2. By elliptic regularity, any solution to (22) belongs to $C^\infty(\Omega \setminus \{0\})$, whereas, up to the boundary, the solution is smooth as the boundary, see again Section 2. We prove

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ be a bounded domain such that $0 \in \Omega$ and $\partial\Omega \in C^2$. Let c_0 be as in (4) and $h(c)$ be as in (21). If $c > -c_0$, then $h(c) > 0$ and*

$$\int_\Omega \left|D^2u\right|^2\, dx + c\int_{\partial\Omega} u_\nu^2\, dS \geq h(c) \int_\Omega \frac{u^2}{|x|^4}\, dx \qquad \textit{for all } u \in H^2 \cap H^1_0(\Omega)\,. \tag{24}$$

Furthermore, there exists $C_N = C_N(\Omega) \in (-c_0, (N-1)\overline{K} - d_1(H_N)]$, where $\overline{K}$ is as in (9) and $d_1(h)$ is as in (20), such that

(i) $h(c)$ is increasing, concave and continuous with respect to $c \in (-c_0, C_N]$;

(ii) $h(c) = H_N$ for every $c \geq C_N$.

Moreover, the infimum in (21) is not achieved if $c > C_N$, achieved if $-c_0 < c < C_N$ and the minimizer $u_c \in H^2 \cap H^1_0(\Omega)$ solves (22) with $h = h(c)$.
Let now Ω be such that the following inequality is satisfied

$$(N-1)(\overline{K} - \underline{K}) \leq d_1(H_N) \quad \textit{for every } N \geq 5\,, \tag{25}$$

where $\underline{K}$ is as in (9). Then, $h(C_N)(= H_N)$ is not achieved. Furthermore, for every $-c_0 < c < C_N$, the minimizer u_c of $h(c)$ is unique, strictly positive, superharmonic in Ω and $(u_c)_\nu < 0$ on $\partial\Omega$.

Condition (25) excludes domains for which the curvature of the boundary has wide oscillations. This requirement is trivially satisfied if Ω is a ball ($\overline{K} = \underline{K}$). On the other hand, if Ω is not a ball, (25) yields $d_1(H_N) > 0$. To our knowledge, this issue has only been proved for strictly starshaped domains, see [5]. In the Appendix, by slightly modifying the proof of [5, Theorem 1], we provide an explicit constant $D_N = D_N(\Omega) > 0$ such that $d_1(H_N) \geq m\, D_N$, where $m := \min_{\partial\Omega}(x \cdot \nu) > 0$. Hence, when Ω is strictly starshaped, in stead of (25), one may check that

$$(N-1)(\overline{K} - \underline{K}) \leq m\, D_N \quad \text{for every } N \geq 5\,,$$

where D_N comes from (42) with $h = H_N$.
Theorem 1 as the following

Corollary 1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ be a bounded domain such that $0 \in \Omega$ and $\partial\Omega \in C^2$. There exists an optimal constant $C_N \in (-c_0, (N-1)\overline{K} - d_1(H_N)]$ such that*

$$\int_\Omega \left|D^2u\right|^2\, dx + C_N\int_{\partial\Omega} u_\nu^2\, dS \geq \frac{N^2(N-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4}\, dx \quad \forall\, u \in H^2 \cap H^1_0(\Omega)\,. \tag{26}$$

Furthermore, if Ω satisfies (25), the inequality in (26) is strict (for $u \not\equiv 0$).

Remark 3. *When $\Omega = B$, the unit ball in $\mathbb{R}^N (N \geq 5)$, C_N can be computed explicitly and we get*

$$C_N(B) = N - 1 - d_1(H_N) = N - 3 - \frac{\sqrt{2(N^2 - 4N + 8)}}{2},$$

see Section 5 for the details. Hence, in this case, the upper bound for C_N (given in Corollary 1) is sharp. As already remarked in the Introduction, $C_N(B) > 0$ if and only if $N \geq 7$. In the next Section (see, Lemma 2) we show that, if Ω is such that the following inequality is satisfied

$$(N-1)(\overline{K} - \underline{K}) < d_0 - d_1(H_N - \delta) \quad \text{for every } N \geq 5 \text{ and for some } \delta > 0, \tag{27}$$

then $C_N \geq (N-1)\underline{K} - d_1(H_N)$. When Ω is convex, this estimate supports the conjecture

$$\text{there exists } \overline{N} = \overline{N}(\Omega) \geq 5 : C_N(\Omega) > 0, \quad \text{for } N \geq \overline{N}.$$

This issue could be proved by providing a suitable upper bound for $d_1(H_N)$. Notice that, in view of (10), the estimate $d_1(H_N) < d_0(\Omega)$ does not suffices to deduce the sign of C_N.
On the other hand, if (25) holds and $\underline{K} < 0$ (Ω is not convex), the upper bound for C_N in Corollary 1 yields $C_N < 0$, for every $N \geq 5$.

4 Proof of Theorem 1 and Corollary 1

We use the same notations of the previous section. First we prove

Lemma 1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ be a Lipschitz bounded domain which satisfies a uniform outer ball condition and such that $0 \in \Omega$. If $h(c) < H_N$ for some $c > -c_0$, then the infimum in (21) is attained. Moreover, a minimizer weakly solves problem (22) for $h = h(c)$.*

Proof. Let $\{u_m\} \subset H^2 \cap H^1_0(\Omega)$ be a minimizing sequence for $h(c)$ such that

$$\int_\Omega \frac{u_m^2}{|x|^4}\, dx = 1. \tag{28}$$

Then,

$$\int_\Omega |D^2 u_m|^2\, dx + c \int_{\partial\Omega} (u_m)^2_\nu\, d\sigma = h(c) + o(1) \quad \text{as } m \to +\infty. \tag{29}$$

For $c > -c_0$, this shows that $\{u_m\}$ is bounded in $H^2 \cap H^1_0(\Omega)$. Exploiting the compactness of the trace map $H^2(\Omega) \to H^1(\partial\Omega)$, we conclude that there exists $u \in H^2 \cap H^1_0(\Omega)$ such that

$$u_m \rightharpoonup u \quad \text{in } H^2 \cap H^1_0(\Omega), \qquad (u_m)_\nu \to u_\nu \quad \text{in } L^2(\partial\Omega), \qquad \frac{u_m}{|x|^2} \to \frac{u}{|x|^2} \quad \text{in } L^2(\Omega), \tag{30}$$

up to a subsequence.
Now, from [10] we know that the space $H^2 \cap H^1_0(\Omega)$, endowed with (2), admits the following orthogonal decomposition

$$H^2 \cap H^1_0(\Omega) = W \oplus H^2_0(\Omega), \tag{31}$$

where W is the completion of

$$V = \left\{ v \in C^\infty(\overline{\Omega}) : \Delta^2 v = 0,\ v = 0 \text{ on } \partial\Omega \right\}$$

with respect to the norm induced by (2). Furthermore, if $u \in H^2 \cap H^1_0(\Omega)$ and if $u = w + z$ is the corresponding orthogonal decomposition with $w \in W$ and $z \in H^2_0(\Omega)$, then w and z are weak solutions to

$$\begin{cases} \Delta^2 w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \\ (w)_\nu = u_\nu & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta^2 z = \Delta^2 u & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \\ (z)_\nu = 0 & \text{on } \partial\Omega\,. \end{cases}$$

By this, the functions u_m, as given at the beginning, may be written as $u_m = w_m + z_m$, where $w_m \in W$ and $z_m \in H^2_0(\Omega)$. Assume now that (30) holds with $u \equiv 0$. By the first of the above Dirichlet problems, we deduce that $w_m \to 0$ in $H^2 \cap H^1_0(\Omega)$ and, in particular, that $\frac{w_m}{|x|^2} \to 0$ in $L^2(\Omega)$. This yields

$$\int_\Omega |D^2 u_m|^2\,dx = \int_\Omega |D^2 z_m|^2\,dx + o(1) = \int_\Omega |\Delta z_m|^2\,dx + o(1) \tag{32}$$

and

$$\int_\Omega \frac{u_m^2}{|x|^4}\,dx = \int_\Omega \frac{z_m^2}{|x|^4}\,dx + o(1)\,.$$

Then, by (3), (28)-(29)-(30) and the fact that $h(c) < H_N$, we infer that

$$H_N > h(c) + o(1) = \int_\Omega |D^2 u_m|^2\,dx + o(1) = \int_\Omega |\Delta z_m|^2\,dx + o(1) \geq H_N + o(1)\,,$$

a contradiction. Hence, $u \not\equiv 0$. If we set $v_m := u_m - u$, from (30) we obtain

$$v_m \rightharpoonup 0 \quad \text{in } H^2 \cap H^1_0(\Omega), \qquad (v_m)_\nu \to 0 \quad \text{in } L^2(\partial\Omega), \qquad \frac{v_m}{|x|^2} \to 0 \quad \text{in } L^2(\Omega), \tag{33}$$

In view of (33), we may rewrite (29) as

$$\int_\Omega |D^2 u|^2\,dx + \int_\Omega |D^2 v_m|^2\,dx + c\int_{\partial\Omega} u_\nu^2\,d\sigma = h(c) + o(1). \tag{34}$$

Moreover, by (28), (33) and the Brezis-Lieb Lemma [8], we have

$$\begin{aligned} 1 = \int_\Omega \frac{u_m^2}{|x|^4}\,dx &= \int_\Omega \frac{u^2}{|x|^4}\,dx + \int_\Omega \frac{v_m^2}{|x|^4}\,dx + o(1) \leq \int_\Omega \frac{u^2}{|x|^4}\,dx + \frac{1}{H_N}\int_\Omega |\Delta v_m|^2\,dx + o(1) \\ &= \int_\Omega \frac{u^2}{|x|^4}\,dx + \frac{1}{H_N}\int_\Omega |D^2 v_m|^2\,dx + o(1)\,, \end{aligned}$$

where the last equality is achieved by exploiting the decomposition (31), as explained above. Since $h(c) \geq 0$, the just proved inequality gives

$$h(c) \leq h(c)\int_\Omega \frac{u^2}{|x|^4}\,dx + \frac{h(c)}{H_N}\int_\Omega |D^2 v_m|^2\,dx + o(1).$$

By combining this with (34), we obtain

$$\begin{aligned} &\int_\Omega |D^2 u|^2\,dx + c\int_{\partial\Omega} u_\nu^2\,d\sigma \\ &\leq h(c)\int_\Omega \frac{u^2}{|x|^4}\,dx + \left(\frac{h(c)}{H_N} - 1\right)\int_\Omega |D^2 v_m|^2\,dx + o(1) \leq h(c)\int_\Omega \frac{u^2}{|x|^4}\,dx + o(1) \end{aligned}$$

which shows that $u \not\equiv 0$ is a minimizer.

□

Remark 4. *If $\partial\Omega \in C^2$, to deduce* (32)*, one may exploit* (11) *instead of the decomposition* (31)*. We leave here this (longer) proof since it highlights that the regularity assumption on $\partial\Omega$ (in the statement of Theorem* 1*) is not due to the existence issue.*

Next, we show

Lemma 2. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 5$) be a bounded domain, with $\partial\Omega \in C^2$ and such that $0 \in \Omega$. The map $(-c_0, +\infty) \ni c \mapsto h(c)$ is nondecreasing (increasing when achieved), concave, hence, continuous and*

$$h(c) = H_N \quad \textit{for every} \quad c \geq (N-1)\overline{K} - d_1(H_N)\,.$$

Moreover, if Ω satisfies (27) *and $H_N - \delta < h < H_N$, then*

$$h(c) \leq h \quad \textit{for every} \quad -c_0 < c \leq (N-1)\underline{K} - d_1(h)\,.$$

Proof. The properties of $h(c)$ follow from its definition, we only need to prove the estimates. By (11), the infimum in (21) may be rewritten as

$$h(c) = \inf_{u \in H^2 \cap H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^2\,dx - \int_{\partial\Omega} \alpha_c(x)(u_\nu)^2\,d\sigma}{\int_\Omega \frac{u^2}{|x|^4}\,dx}\,, \tag{35}$$

where $\alpha_c(x) = (N-1)K(x) - c$, as defined in Section 2. Then, if $\alpha_c(x) \leq d_1(H_N)$ for every $x \in \partial\Omega$, by (20), $h(c) \equiv H_N$ and the first estimate follows. Similarly, if $\alpha_c(x) \geq d_1(h)$ for every $x \in \partial\Omega$, by (20), we get the second estimate. Notice that assumption (27), suitably combined with (12), ensures that $(N-1)\underline{K} - d_1(h) > -c_0$, for every $H_N - \delta < h < H_N$. □

By Lemma 2, the number

$$C_N := \inf\{c > -c_0 : h(c) = H_N\} \tag{36}$$

is well-defined. Furthermore, we have

$$(N-1)\underline{K} - d_1(H_N) \leq C_N \leq (N-1)\overline{K} - d_1(H_N)\,, \tag{37}$$

where the lower bound has been proved for Ω satisfying (27). Then, we show

Lemma 3. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 5$) be a bounded domain such that $0 \in \Omega$ and $\partial\Omega \in C^2$. Let C_N be as in* (36)*, then the infimum in* (21) *is not achieved if $c > C_N$, achieved if $-c_0 < c < C_N$ and the minimizer (weakly) solves problem* (22) *for $h = h(c)$.*
Assume, furthermore, that Ω satisfies (25)*. Then, for every $-c_0 < c < C_N$, $h(c)$ is achieved by a unique positive function u_c which satisfies $-\Delta u_c \geq 0$ in Ω and $(u_c)_\nu < 0$ on $\partial\Omega$ while, $h(C_N)$ is not achieved.*

Proof. The first part of the statement comes from the definition of C_N combined with the previous lemmata. To prove the second part, we write (21) as in (35). From (25), combined with (37), we have that $C_N \leq (N-1)\underline{K}$. Then, $\alpha_c(x) \geq 0$ for every $x \in \partial\Omega$ and for every $-c_0 < c < C_N$. Hence, we may argue as in the proof of Proposition 1-(ii), to deduce the positivity of a minimizer u_c, together with the fact that $-\Delta u_c \geq 0$ in Ω and $(u_c)_\nu < 0$ on $\partial\Omega$. Since problem (22) is linear, once the positivity of a minimizer is known, the proof of its uniqueness is standard.
It remains to show that $h(c)$ is not achieved for $c = C_N$. If a minimizer of $h(C_N)$ exists, it would be a positive and superharmonic solution, vanishing on $\partial\Omega$, to the equation in (22) with $h = H_N$. Then, the same argument of [2, Theorem 2.2- (ii)] gives a contradiction. □

The proofs of Theorem 1 and Corollary 1 follow by combining the statements of the above lemmata.

5 Radial setting

When $\Omega = B$, the unit ball in $\mathbb{R}^N (N \geq 5)$, the mean curvature $K \equiv 1$. Then, for what remarked in Section 2, problems (20) and (21) become almost equivalent. Indeed, let u_h be the function achieving the equality in (20), for some $0 \leq h < H_N$. Then, by (35), u_h is also the minimizer of $h(c)$ for $c = c_h = N - 1 - d_1(h)$ and $h(c_h) = h$ (or, equivalently, u_h achieves the equality in (5)). Furthermore, the map $[0, H_N) \ni h \mapsto c_h$ is increasing, $c_0 = -1$ and $c_{H_N} = C_N$, where C_N is as in (37).
We briefly sketch the computations to determine (explicitly) the minimizer of $h(c)$. As in [5, Section 5], we introduce an auxiliary parameter $0 \leq \alpha \leq N - 4$ and we set

$$H(\alpha) := \frac{\alpha(\alpha + 4)(\alpha + 4 - 2N)(\alpha + 8 - 2N)}{16} . \tag{38}$$

The map $\alpha \mapsto H(\alpha)$ is increasing, $H(0) = 0$ and $H(N - 4) = H_N$ so that $0 \leq H(\alpha) \leq H_N$ for all $\alpha \in [0, N - 4]$. For $\alpha < N - 4$, let $\gamma_N(\alpha) := \sqrt{N^2 - \alpha^2 + 2\alpha(N - 4)}$ and

$$\overline{u}_\alpha(x) := |x|^{-\frac{\alpha}{2}} - |x|^{\frac{4-N+\gamma_N(\alpha)}{2}} \in H^2 \cap H^1_0(B) .$$

The function $\overline{u}_\alpha$ is a *positive* solution to problem (22) with $h = H(\alpha) < H_N$ and $c = c(\alpha)$, where

$$c(\alpha) := \frac{\alpha^2 - \alpha(N - 5) - N^2 + 3N - 4 + (N - 3)\gamma_N(\alpha)}{\alpha + 4 - N + \gamma_N(\alpha)} . \tag{39}$$

The map $[0, N - 4] \ni \alpha \mapsto c(\alpha)$ is increasing, $c(0) = -1$ and

$$C_N = c(N - 4) = N - 3 - \frac{\sqrt{2(N^2 - 4N + 8)}}{2} .$$

Since the first eigenfunction $u_{h(c)}$ of problem (22) is unique (by Lemma 3), when $\Omega = B$, it must be a radial function. Furthermore, $u_{h(c)}$ turns to be the only positive eigenfunction. To see this, let $v_{\bar{h}(c)}$ be another positive eigenfunction, corresponding to some $\bar{h}(c) > h(c)$. Write (23), first with $u_{h(c)}$ and test with $v_{\bar{h}(c)}$, then with $v_{\bar{h}(c)}$ and test with $u_{h(c)}$. Subtracting, we get

$$h(c) \int_B \frac{u_{h(c)} v_{\bar{h}(c)}}{|x|^4} \, dx = \bar{h}(c) \int_B \frac{u_{h(c)} v_{\bar{h}(c)}}{|x|^4} \, dx ,$$

a contradiction. By this, we conclude that $u_{h(c)} = \overline{u}_\alpha$, where $c = c(\alpha)$. Namely, $\overline{u}_\alpha$ is the minimizer of $h(c(\alpha)) = H(\alpha)$ for every $\alpha \in [0, N - 4)$. In turn, this shows

Theorem 2. *For every $0 \leq \alpha \leq N - 4$, there holds*

$$\int_B |D^2 u|^2 \, dx + c(\alpha) \int_{\partial B} u_\nu^2 \, d\sigma \geq H(\alpha) \int_B \frac{u^2}{|x|^4} \, dx \quad \text{for all } u \in H^2 \cap H^1_0(B) ,$$

where $H(\alpha)$ and $c(\alpha)$ are defined in (38) and (39). Furthermore, the best constant $H(\alpha)$ is attained if and only if $0 \leq \alpha < N - 4$, by multiples of the function

$$\overline{u}_\alpha(x) = |x|^{-\frac{\alpha}{2}} - |x|^{\frac{4-N+\sqrt{N^2-\alpha^2+2\alpha(N-4)}}{2}} .$$

As a Corollary of Theorem 2, we readily get (6) and (7). We just remark that, to get (7), one has to determine the unique solution α_N to the equation

$$c(\alpha) = 0 \quad \text{for} \quad \alpha \in (0, N - 4) \quad \text{and} \quad N \geq 7 .$$

By (39), we have that

$$c(\alpha)=0 \quad \Leftrightarrow \quad \alpha^4-2(N-5)\alpha^3-2(5N-13)\alpha^2+4(N^2-7N+8)+8(N^2-3N+2)=0$$

and the above polynomial can be factorized as follows

$$(\alpha+1-\sqrt{2N-1})(\alpha+1+\sqrt{2N-1})(\alpha-N+4-\sqrt{N^2-4N+8})(\alpha-N+4+\sqrt{N^2-4N+8})\,.$$

Then, since $\alpha\in(0,N-4)$ and $N\geq 7$, we obtain the unique solution $\alpha_N=\sqrt{2N-1}-1$. Finally, $H(\alpha_N)$, with $H(\alpha)$ as in (38), is the optimal constant in (7). See also Figure 1 for the trace of the curve $(0,N-4)\ni\alpha\mapsto(c(\alpha),H(\alpha))$ (or, equivalently, the plot of the map $(-c_0,+\infty)\ni c\mapsto h(c)$), when $N=5$ and $N=8$.

Appendix

Let $\Omega\subset\mathbb{R}^N (N\geq 5)$ be a bounded domain such that $0\in\Omega$ and $\partial\Omega\in C^2$. Denote by $|\Omega|$ its $N-$dimensional Lebesgue measure and by $\omega_N=|B|$, where B is the unit ball. Finally, set $\gamma=j_0^2\approx 2.4^2$, where j_0 is the first positive zero of the Bessel function J_0, and

$$A_N=A_N(\Omega):=\frac{N(N-4)}{2}\gamma\left(\frac{\omega_N}{|\Omega|}\right)^{2/N}. \tag{40}$$

Let $H_N:=\frac{N^2(N-4)^2}{16}$. From [11, Theorem 2], we know that

$$\int_\Omega|\Delta u|^2\,dx\geq H_N\int_\Omega\frac{u^2}{|x|^4}\,dx+A_N\int_\Omega\frac{u^2}{|x|^2}\,dx \qquad \text{for all } u\in H^2\cap H_0^1(\Omega)\,. \tag{41}$$

Next we prove

Proposition 3. *Let $0<h\leq H_N$ and $d_1(h)$ be the optimal constant in (20). If Ω is strictly starshaped with respect to the origin, then*

$$d_0>d_1(h)\geq\frac{2A_N\,m}{MA_N+h+4}\,, \tag{42}$$

where d_0 is as in (8), A_N is as in (40), $M:=\max_{\partial\Omega}|x|^2$ and $m:=\min_{\partial\Omega}(x\cdot\nu)$.

Proof. For $0<h<H_N$, let $u_h\in H^2\cap H_0^1(\Omega)$ be the (positive and superharmonic) function which achieves the equality in (20). Notice that u_h solves the equation in (22) subject the conditions $u_h=0=\Delta u_h=d_1(h)(u_h)_\nu$ on $\partial\Omega$. By (41), we get

$$d_1(h)\int_{\partial\Omega}(u_h)_\nu^2\,d\sigma=\int_\Omega|\Delta u_h|^2\,dx-h\int_\Omega\frac{u_h^2}{|x|^4}\,dx\geq(H_N-h)\int_\Omega\frac{u_h^2}{|x|^4}\,dx+A_N\int_\Omega\frac{u_h^2}{|x|^2}\,dx\,. \tag{43}$$

Next, in the spirit of the computations performed in [5, Theorem 1], we deduce

$$\begin{aligned}\int_\Omega\frac{u_h^2}{|x|^2}\,dx &= \int_\Omega(|x|^2u_h)\,\frac{u_h}{|x|^4}\,dx=\frac{1}{h}\int_\Omega(|x|^2u_h)\,\Delta^2u_h\,dx\\ &= \frac{1}{h}\int_\Omega\Delta(|x|^2u_h)\,\Delta u_h\,dx-\frac{1}{h}\int_{\partial\Omega}|x|^2\Delta u_h(u_h)_\nu\,d\sigma\\ &= \frac{1}{h}\int_\Omega\Delta u_h\left(2Nu_h+4x\cdot\nabla u_h+|x|^2\Delta u_h\right)dx-\frac{d_1(h)}{h}\int_{\partial\Omega}|x|^2(u_h)_\nu^2\,d\sigma\,.\end{aligned}$$

From [17, formula (1.3)], we have

$$\begin{aligned}\int_\Omega \Delta u_h\,(x\cdot\nabla u_h)\,dx &= \frac{N-2}{2}\int_\Omega |\nabla u_h|^2\,dx + \frac12\int_{\partial\Omega}(x\cdot\nu)(u_h)_\nu^2\,d\sigma \\ &= -\frac{N-2}{2}\int_\Omega u_h\Delta u_h\,dx + \frac12\int_{\partial\Omega}(x\cdot\nu)(u_h)_\nu^2\,d\sigma\end{aligned}$$

and we conclude

$$\int_\Omega \frac{u_h^2}{|x|^2}\,dx = \frac1h\int_\Omega(4\,u_h\Delta u_h\,+|x|^2|\Delta u_h|^2)\,dx + \frac1h\int_{\partial\Omega}(2\,(x\cdot\nu)-d_1(h)\,|x|^2)\,(u_h)_\nu^2\,d\sigma.$$

Finally, by exploiting the Young's inequality

$$\left|\int_\Omega u_h\Delta u_h\,dx\right| \le \frac14\int_\Omega |x|^2|\Delta u_h|^2\,dx + \int_\Omega \frac{u_h^2}{|x|^2}\,dx\,,$$

we deduce

$$\left(1+\frac4h\right)\int_\Omega \frac{u_h^2}{|x|^2}\,dx \ge \frac{2m-Md_1(h)}{h}\int_{\partial\Omega}(u_h)_\nu^2\,d\sigma\,,$$

where m and M are defined in the statement. Plugging this into (43), (42) follows for $h<H_N$.

The estimate for $d_1(H_N)$ comes by letting $h\to H_N$ in (42). Indeed, by definition of $d_1(H_N)$, we know that for all $\varepsilon>0$ there exists $u_\varepsilon\in H^2\cap H_0^1(\Omega)\setminus H_0^2(\Omega)$ such that

$$\frac{\int_\Omega |\Delta u_\varepsilon|^2\,dx - H_N\int_\Omega \frac{u_\varepsilon^2}{|x|^4}\,dx}{\int_{\partial\Omega}(u_\varepsilon)_\nu^2\,dS} \;<\; d_1(H_N)+\varepsilon.$$

Then, for all $h<H_N$ we have

$$\begin{aligned}d_1(H_N)\le d_1(h) &\le \frac{\int_\Omega |\Delta u_\varepsilon|^2\,dx - H_N\int_\Omega \frac{u_\varepsilon^2}{|x|^4}\,dx}{\int_{\partial\Omega}(u_\varepsilon)_\nu^2\,d\sigma} + (H_N-h)\frac{\int_\Omega \frac{u_\varepsilon^2}{|x|^4}\,dx}{\int_{\partial\Omega}(u_\varepsilon)_\nu^2\,d\sigma} \\ &< d_1(H_N)+\varepsilon+C_\varepsilon(H_N-h).\end{aligned}$$

Hence,

$$\lim_{h\to H_N} d_1(h) = d_1(H_N)$$

and we conclude. □

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