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STRONG LOCAL SURVIVAL OF BRANCHING RANDOM WALKS IS NOT MONOTONE

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Abstract. The aim of this paper is the study of the strong local survival property for discrete-time and continuous-time branching random walks. We study this property by means of an infinite dimensional generating function G and a maximum principle which, we prove, is satisfied by every fixed point of G. We give results about the existence of a strong local survival regime and we prove that, unlike local and global survival, in continuous time, strong local survival is not a monotone property in the general case (though it is monotone if the branching random walk is quasi transitive). We provide an example of an irreducible branching random walk where the strong local property depends on the starting site of the process. By means of other counterexamples we show that the existence of a pure global phase is not equivalent to nonamenability of the process, and that even a branching random walk with the same branching law at each site may not exhibit strong local survival.

Keywords: branching random walk, branching process, strong local survival, recurrence, generating function, maximum principle.
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1. Introduction

A branching process is a very simple population model (introduced in [10]) where particles breed and die (independently of each other) according to some random law. At any time, this process is completely characterized by the total number of particles alive. Branching random walks (in short, BRWs) add space to this picture: particles live in a spatially structured environment and the reproduction law, which may depend on the location, not only tells how many children the particle has, but also where it places them. The state of the process, at any time, is thus described by the collection of the numbers of particles alive at x, where x varies among the possible sites. In the literature one can find BRWs both in continuous and discrete time. The continuous-time setting has been studied by many authors (see [13, 14, 15, 16, 18] just to name a few). The discrete-time case has been initially considered as a natural generalization of branching processes (see [1, 6, 7, 8, 9, 12]). The definition of discrete-time BRW that we give in Section 2.1 is sufficiently general to include the discrete-time counterpart that every continuous-time BRW admits. Since every continuous-time BRW and its discrete-time counterpart have the same asymptotic behavior, it suffices to provide results for the discrete-time case. On the other hand, continuous-time examples naturally yield discrete-time ones.

The basic question which arises studying the BRW is whether it survives with positive probability and, in this case, if it visits a site infinitely many times. The first question asks whether there is global survival, that is, with positive probability at any time there is someone alive *somewhere*); while the second question deals with local survival, that is, whether with positive probability the process returns infinitely many times to some fixed site. It is clear that the probability of global survival is larger or equal to the probability of local survival. If the probability of global survival is strictly larger than the one of local survival, then the latter may be positive or null. In the first case, we say that there is non-strong local survival, in the second case there is pure global survival. When on the contrary, the probabilities of global and local survival are equal and strictly positive, we say that the BRW has strong local survival. Hence, strong local survival means that the events of local and global survival coincide (but for a null probability set) and have positive probability.

As in the case of branching processes, the probabilities of extinction are fixed points of an infinite-dimensional generating function G (Sections 2.3 and 3.1). In the particular case where there is no branching, one gets a random walk and the role of G and its fixed points is played by the transition matrix and the harmonic functions, respectively. It is thus natural to look for a maximum principle in the context of branching random walks as well (see Proposition 2.4). By means of the generating function and its properties we obtain conditions for strong local survival (Theorems 3.1, 3.2, Corollary 3.3) and we provide various counterexamples.

The speed of reproduction of a continuous-time BRW is proportional to a positive parameter λ (see Section 2.2). It is easily seen that the probability of local and global survival are nondecreasing functions with respect to λ; thus strong and local survival are monotone properties (meaning that if one of them holds for some λ_0 then it holds for all $\lambda \geq \lambda_0$). We show that this is not true for strong local survival (see Example 3.6). We also show that, unlike local and global survival, even in the irreducible case, one may have strong local survival or non-strong local survival depending on the starting vertex.

Here is the outline of the paper. In Section 2 we give the necessary definitions and some basic facts about discrete-time BRWs (Section 2.1), continuous-time BRWs (Section 2.2), the infinite-dimensional generating function G associated to a BRW (Section 2.3) and the special class of $\mathcal{F}$-BRWs (Section 2.4). This class contains properly the class of BRWs on quasi-transitive graphs (which were studied in [21]). In particular in Section 2.3 a maximum principle for the solutions of the equation $G(v) \geq v$, including all fixed points of G, is stated (Proposition 2.4).

Section 3 is devoted to the study of all the types of survival. In particular in Section 3.1 extinction probabilities are seen as fixed points of the generating function G. Theorem 3.1 gives equivalent conditions for strong local survival which are useful to prove that strong local survival is not monotone (Example 3.6). Some known results on local and global survival are informally discussed in Section 3.2, while in Section 3.3 we deal with pure global and strong local survival. From the maximum principle (Proposition 2.4) we derive Theorem 3.2 which describes some properties of fixed points of G for $\mathcal{F}$-BRWs. Corollary 3.3 shows that for an irreducible, quasi-transitive BRW, there are only three possible behaviours (independently of the starting vertex): global extinction, pure global survival or strong local survival (Examples 3.8 and 3.9 show that this does not hold for a generic irreducible $\mathcal{F}$-BRW). Thus for a quasi-transitive, irreducible, continuous-time BRW, strong local survival is monotone and the critical parameter is λ_s (defined in Section 2.2). In our construction of a continuous-time BRW where the strong local survival is not monotone there is a deep relation with the existence of a pure global phase. This last property, in the case of an irreducible $\mathcal{F}$-BRW is equivalent to nonamenability. Nevertheless in general nonamenability neither implies nor is implied by the existence of a pure global phase (Example 3.5). Example 3.7 shows that even for an irreducible BRW with null probability of having no children for a particle living at x (for some $x \in X$), it is possible to have strong local survival starting from some vertices and non-strong local survival starting from others. Finally we show (see Examples 3.8 and 3.9) that even fairly simple BRWs (such as BRWs with independent diffusion and with offspring distribution independent of the site) may have non-strong local survival.

2. BASIC DEFINITIONS AND PRELIMINARIES

2.1. **Discrete-time Branching Random Walks.** We start with the construction of a generic discrete-time BRW $\{\eta_n\}_{n\in\mathbb{N}}$ (see also [3] where it is called *infinite-type branching process*) on a set X which is at most countable; $\eta_n(x)$ represents the number of particles alive at x at time n. To this aim we consider a family $\mu = \{\mu_x\}_{x\in X}$ of probability measures on the (countable) measurable space $(S_X, 2^{S_X})$ where $S_X := \{f : X \to \mathbb{N} : \sum_y f(y) < \infty\}$. To obtain generation $n+1$ from generation n we proceed as follows: a particle at site $x \in X$ lives one unit of time, then a function $f \in S_X$ is chosen at random according to the law μ_x and the original particle is replaced by $f(y)$ particles at y, for all $y \in X$; this is done independently for all particles of generation n (a similar construction in random environment can be found in [11]). Note that the choice of f assigns simultaneously the total number of children and the location where they will live. We denote the BRW by the couple (X, μ).

Equivalently we could introduce the BRW by choosing first the number of children and afterwards their location. Indeed define $\mathcal{H} : S_X \to \mathbb{N}$ as $\mathcal{H}(f) := \sum_{y \in X} f(y)$ which represents the total number of children associated to f. Denote by ρ_x the measure on $\mathbb{N}$ defined by $\rho_x(\cdot) := \mu_x(\mathcal{H}^{-1}(\cdot))$; this is the law of the random number of children of a particle living at x. For each particle, independently, we pick a number n at random, according to the law ρ_x, and then we choose a function $f \in \mathcal{H}^{-1}(n)$ with probability $\mu_x(f)/\rho_x(n) \equiv \mu_x(f)/\sum_{g \in \mathcal{H}^{-1}(n)} \mu_x(g)$ and we replace the particle at x with $f(y)$ particles at y (for all $y \in X$).

In BRW theory a fundamental role is played by the *first-moment matrix* $M = (m_{xy})_{x,y \in X}$, where $m_{xy} := \sum_{f \in S_X} f(y)\mu_x(f)$ is the expected number of particles from x to y (that is, the expected number of children that a particle living at x sends to y). We suppose that $\sup_{x \in X} \sum_{y \in X} m_{xy} < +\infty$; most of the results of this paper still hold without this hypothesis, nevertheless it allows us to avoid dealing with an infinite expected number of offsprings. Note that the expected number of children generated by a particle living at x is $\sum_{y \in X} m_{xy} = \sum_{n \geq 0} n\rho_x(n) =: \bar{\rho}_x$. Given a function f defined on X we denote by Mf the function $Mf(x) := \sum_{y \in X} m_{xy} f(y)$ whenever the RHS converges absolutely for all x. We denote by $m_{xy}^{(n)}$ the entries of the nth power matrix M^n and we define

$$M_s(x, y) := \limsup_{n \to \infty} \sqrt[n]{m_{xy}^{(n)}}, \quad M_w(x) := \liminf_{n \to \infty} \sqrt[n]{\sum_{y \in X} m_{xy}^{(n)}}, \qquad \forall x, y \in X.$$

The BRW (X, μ) is called *non-oriented* or *symmetric* if $m_{xy} = m_{yx}$ for every $x, y \in X$. (X, μ) is called *nonamenable* if and only if

$$\inf \left\{ \frac{\sum_{x \in S, y \in S^\complement} m_{xy}}{|S|} : S \subseteq X, |S| < \infty \right\} =: \iota_{(X,\mu)} > 0,$$

and it is called *amenable* otherwise.

Given a generic discrete-time BRW we say that (x, y) is and edge if and only if a particle living at x can send a child at y with positive probability (from now on *wpp*). Let E_μ be the set of edges. Moreover there is a path from x to y, and we write $x \to y$, if it is possible to find a finite sequence $\{x_i\}_{i=0}^n$ (where $n \in \mathbb{N}$) such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E_\mu$ for all $i = 0, \ldots, n-1$. If $x \to y$ and $y \to x$ we write $x \rightleftharpoons y$. Observe that there is always a path of length 0 from x to itself.

We call the matrix $M = (m_{xy})_{x,y \in X}$ *irreducible* if and only if the graph (X, E_μ) is *connected*, otherwise we call it *reducible* (irreducibility of M means that the progeny of any particle can spread to any site of the graph). We denote by $\deg(x)$ the degree of a vertex x, that is, the cardinality of the set $\mathcal{N}_x := \{y \in X : (x, y) \in E_\mu\}$. Note that if (X, μ) is non-oriented then the graph (X, E_μ) is non-oriented (that is, $(x, y) \in E_\mu$ if and only if $(y, x) \in E_\mu$).

Definition 2.1. *The colony can survive in different ways: we say that the colony survives locally wpp in $A \subseteq X$ starting from $x \in X$ if*

$$q(x, A) := 1 - \mathbb{P}^{\delta_x}(\limsup_{n \to \infty} \sum_{y \in A} \eta_n(y) > 0) < 1;$$

we say that it survives globally wpp starting from x if

$$\bar{q}(x) := q(x, X) < 1.$$

Following [11], *we say that the there is strong local survival wpp in $A \subseteq X$ starting from $x \in X$ if*

$$q(x, A) = \bar{q}(x) < 1$$

and non-strong local survival wpp in A if $q(x, A) < \bar{q}(x) < 1$. Finally we say that the BRW is in a pure global survival phase starting from x if

$$\bar{q}(x) < q(x, x) = 1$$

where we write $q(x,y)$ instead of $q(x,\{y\})$ for all $x,y \in X$. From now on when we talk about survival, "wpp" will be tacitly understood. Often we will say simply that local survival occurs "starting from x" or "at x": in this case we mean that $x=y$.

Note that $q(x,A)$ are the probabilities of extinction in A starting from x (see Section 3.1). Roughly speaking, there is strong survival at y starting from x if and only if the probability of local survival at y starting from x conditioned on global survival starting from x is 1. One can show that strong local survival implies that for almost all realizations the process either survives locally (hence globally) or it goes globally extinct. There are many relations between $\bar q(x)$ and $q(x,y)$ and between $q(w,x)$ and $q(w,y)$ where $x,y,w \in X$ (see for instance Section 3.1 or [5, 22]); we observe that $\bar q(x) \le q(x,A)$ for all $x \in X$, $A \subseteq X$.

In order to avoid trivial situations where particles have a.s. one offspring we assume henceforth the following.

Assumption 2.2. *For all $x \in X$ there is a vertex $y \rightleftharpoons x$ such that $\mu_y(f : \sum_{w: w \rightleftharpoons y} f(w) = 1) < 1$, that is, in every equivalence class (with respect to $\rightleftharpoons$) there is at least one vertex where a particle can have inside the class a number of children different from one wpp.*

For a generic BRW, we call *diffusion matrix* the matrix P with entries $p(x,y) = m_{xy}/\bar\rho_x$. Note that, in the general case, the locations of the offsprings are not chosen independently (they are assigned by the chosen $f \in S_X$). In the particular case when the offsprings are dispersed independently according to P we call the process a *BRWs with independent diffusion*: in this case

$$\mu_x(f) = \rho_x\left(\sum_y f(y)\right) \frac{(\sum_y f(y))!}{\prod_y f(y)!} \prod_y p(x,y)^{f(y)}, \quad \forall f \in S_X. \tag{2.1}$$

Clearly in this case the expected number of children at y of a particle living at x is $m_{xy} = p(x,y)\bar\rho_x$.

2.2. **Continuous-time Branching Random Walks.** In continuous time each particle has an exponentially distributed random lifetime with parameter 1. The breeding mechanisms can be regulated by means of a nonnegative matrix $K = (k_{xy})_{x,y \in X}$ in such a way that for each particle at x, there is a clock with $Exp(\lambda k_{xy})$-distributed intervals (where $\lambda > 0$), each time the clock rings the particle breeds in y. We say that the BRW has a death rate 1 and a reproduction rate λk_{xy} from x to y. We observe (see Remark 2.3) that the assumption of a nonconstant death rate does not represent a significative generalization. We denote by (X,K) a family of continuous-time BRWs (depending on the parameter $\lambda > 0$), while we use the notation (X,μ) for a discrete-time BRW.

To a continuous-time BRW one can associate a discrete-time counterpart (see for instance [22]); in this sense the theory of continuous-time BRWs, as long as we are interested in the probability of survival (local, strong local and global), is a particular case of the theory of discrete-time BRWs.

Elementary calculations show that each particle living at x, before dying, has a random number of offsprings given by equation (2.1) where

$$\rho_x(i) = \frac{1}{1+\lambda k(x)}\left(\frac{\lambda k(x)}{1+\lambda k(x)}\right)^i, \qquad p(x,y) = \frac{k_{xy}}{k(x)}, \tag{2.2}$$

and this is the law of the discrete-time counterpart; note that the discrete-time counterpart of a continuous-time BRW is a BRW with independent diffusion and that ρ_x depends only on $\lambda k(x)$.

Remark 2.3. *The same construction applies to continuous-time BRWs with a death rate $d(x) > 0$ dependent on $x \in X$. In this case the discrete-time counterpart satisfies equation* (2.1) *where*

$$\rho_x(i) = \frac{d(x)}{d(x)+\lambda k(x)}\left(\frac{\lambda k(x)}{d(x)+\lambda k(x)}\right)^i, \qquad p(x,y) = \frac{k_{xy}}{k(x)}.$$

Hence, from the point of view of local and global survival, this process is equivalent to a continuous-time BRW with death rate 1 *and reproduction rate $\lambda k_{xy}/d(x)$ from x to y.*

Given $x_0 \in X$, two critical parameters are associated to the continuous-time BRW: the *global survival critical parameter* $\lambda_w(x_0)$ and the *local survival critical parameter* $\lambda_s(x_0)$. They are defined as

$$\lambda_w(x_0) := \inf\Big\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}\Big(\sum_{w\in X} \eta_t(w) > 0, \forall t\Big) > 0\Big\},$$
$$\lambda_s(x_0) := \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}\big(\limsup_{t\to\infty} \eta_t(x_0) > 0\big) > 0\},$$

$\mathbb{P}^{\delta_{x_0}}$ is the law of the process which starts with one individual in x_0. The process is called *globally supercritical*, *critical* or *subcritical* if $\lambda > \lambda_w$, $\lambda = \lambda_w$ or $\lambda < \lambda_w$; an analogous definition is given for the local behavior using λ_s instead of λ_w. We prove in this paper that it is not possible to define a *strong local survival critical parameter* analogously (see Example 3.6).

If the graph (X, E_μ) is connected (that is, the BRW is irreducible) then these values do not depend on the initial configuration, provided that this configuration is finite (that is, it has only a finite number of individuals), nor on the choice of x_0. In particular we say that there exists a *pure global survival phase* starting from x if the interval $(\lambda_w(x), \lambda_s(x))$ is not empty; clearly, if $\lambda \in (\lambda_w(x), \lambda_s(x))$ then the BRW is in a pure global survival phase according to Definition 2.1.

Given a continuous-time BRW (X, K) we define the analogs of $M_s(x,y)$ and $M_w(x)$

$$K_s(x,y) := \frac{M_s(x,y)}{\lambda} \equiv \limsup_{n\to\infty} \sqrt[n]{k^{(n)}_{xy}}, \quad K_w(x) := \frac{M_w(x)}{\lambda} \equiv \liminf_{n\to\infty} \sqrt[n]{\sum_{y\in X} k^{(n)}_{xy}}, \qquad \forall x, y \in X.$$

Note that K_s and K_w were introduced in [2, 3] where they were called M_s and M_w.

We say that a BRW is *site-breeding* if $k(x)$ does not depend on $x \in X$. We say that a BRW is *edge-breeding* if X has a multigraph structure (see [2, Section 2.1] for a formal definition) and k_{xy} is the number of edges from x to y; in this case to each edge there corresponds a constant reproduction rate λ.

2.3. Infinite-dimensional generating function.

We associate a generating function $G : [0,1]^X \to [0,1]^X$ to the family $\{\mu_x\}_{x\in X}$ which can be considered as an infinite dimensional power series (see also [3, Section 3]). More precisely, for all $z \in [0,1]^X$ the function $G(z) \in [0,1]^X$ is defined as the following weighted sum of (finite) products

$$G(z|x) := \sum_{f\in S_X} \mu_x(f) \prod_{y\in X} z(y)^{f(y)}.$$

Note that G is continuous with respect to the *pointwise convergence topology* of $[0,1]^X$ and nondecreasing with respect to the usual partial order of $[0,1]^X$ (see [3, Sections 2 and 3] for further details). Moreover, G represents the 1-step reproductions; we denote by $G^{(n)}$ the generating function associated to the n-step reproductions, which is inductively defined as $G^{(n+1)}(z) = G^{(n)}(G(z))$. It is worth mentioning that the function G arises naturally computing the extinction probabilities of the process, as shown in Section 3.1 (see also [12] for the use of generating functions in the study of branching processes); indeed extinction probabilities are fixed points of G.

When one is interested in the questions whether a global surviving BRW survives strong locally, it may be useful to condition the process on global survival. Given a generic discrete-time BRW, conditioning on global survival, we associate a BRW with no death with generating function

$$\widehat{G}(z|x) = \frac{G(v(z)|x) - \bar{q}(x)}{1 - \bar{q}(x)}, \tag{2.3}$$

where G is the generating function of the original BRW and $v : [0,1]^X \to [0,1]^X$ is defined as $v(z|x) := \bar{q}(x) + z(x)(1 - \bar{q}(x))$. In a more compact way equation (2.3) can be written as $\widehat{G} = T_{\bar{q}}^{-1} \circ G \circ T_{\bar{q}}$ where $T_w : [0,1]^X \to \{z \in [0,1]^X : w \le z\}$ is defined as $T_w z(x) := z(x)(1-w(x)) + w(x)$; note that T_w is nondecreasing and, if $w(x) < 1$ for all $x \in X$, bijective. In particular if $\bar{q} < \mathbf{1}$ then $T_{\bar{q}}$ is a bijective map from the set of fixed points of $\widehat{G}$ to the set of fixed points of G.

We have immediately that, for all $A \subseteq X$, the probability of local survival in A of the associated no-death BRW starting from x is $1-(T_{\bar q}^{-1} q(\cdot, A))(x) = (1-q(x,A))/(1-\bar q(x))$. This is clearly the probability of local survival in A of the original BRW conditioned on global survival (starting from x).

The following proposition is a sort of *maximum principle* for the function $(z-\bar q)/(\mathbf{1}-\bar q)$ where z is such that $G(z) \ge z$.

Proposition 2.4. *Let $z \in [0,1]^X$, $z \ge \bar q$ be a solution of the inequality $G(z) \ge z$. We define $\widehat z := (z-\bar q)/(\mathbf{1}-\bar q)$ if $\bar q < \mathbf{1}$ and $\widehat z(x) := 1$ for all x such that $\bar q(x)=1$. Then for all $x \in X$ such that the set $\mathcal{N}_x = \{y : (x,y) \in E_\mu\}$ is not empty, either $\widehat z(y) = \widehat z(x)$ for all $y \in \mathcal{N}_x$ or there exists $y \in \mathcal{N}_x$ such that $\widehat z(y) > \widehat z(x)$. In particular if $\widehat z(x) = 1$ then for all $y \in \mathcal{N}_x$ we have $\widehat z(y) = 1$. The same results hold if we take the set $\{y \in X : x \to y\}$ instead of $\mathcal{N}_x$.*

We observe that in a finite, final irreducible class (for instance if the BRW is irreducible and the set X is finite) then $\widehat z$ is constant if $z \ge \bar q$ is a solution of $G(z) \ge z$. Since the probabilities of extinction $\{q(\cdot, A)\}_{A \subseteq X}$ are solutions of $G(z) = z$, Proposition 2.4 applies with $z(\cdot) = q(\cdot, A)$ for all $A \subseteq X$. In this case $\widehat z(x)$ can be interpreted as the probability of local extinction in A conditioned on global survival (starting from x). Thus, if the BRW is irreducible, then this conditional probability is one everywhere, provided it is one somewhere.

2.4. **$\mathcal{F}$-BRWs.** Some results can be achieved if the BRW has some regularity; to this aim we introduce the concept of $\mathcal{F}$-BRW (see also [22, Definition 4.2]), which extends the concept of quasi-transitivity.

Definition 2.5. *We say that a BRW (X,μ) is locally isomorphic to a BRW (Y,ν) if there exists a surjective map $g : X \to Y$ such that*

$$\nu_{g(x)}(\cdot) = \mu_x\left(\pi_g^{-1}(\cdot)\right)$$

where $\pi_g : S_X \to S_Y$ is defined as $\pi_g(f)(y) = \sum_{z \in g^{-1}(y)} f(z)$ for all $f \in S_X$, $y \in Y$. We say that (X,μ) is a $\mathcal{F}$-BRW if it is locally isomorphic to some BRW (Y,ν) on a finite set Y.

Clearly, if (X,μ) is locally isomorphic to (Y,ν) then

$$G_X(z \circ g | x) = G_Y(z | g(x)) \tag{2.4}$$

for all $z \in [0,1]^Y$ and $x \in X$. We note that, since μ is uniquely determined by G, equation (2.4) holds if and only if (X,μ) is locally isomorphic to (Y,ν) and g is the map in Definition 2.5.

Using equation (2.4) and the fact that $\bar q = \lim_{n\to\infty} G^{(n)}(\mathbf{0})$ (see equation (3.5) with $A = X$), it is possible to prove that there is global survival for (X,μ) starting from x if and only if there is global survival for (Y,ν) starting from $g(x)$ (see [22, Theorem 4.3]).

In continuous time (see [3]) one can prove that (X,K) is *locally isomorphic* to $(Y,\widetilde K)$ if and only if there exists a surjective map $g : X \to Y$ such that $\sum_{z\in g^{-1}(y)} k_{xz} = \widetilde k_{g(x)y}$ for all $x \in X$ and $y \in Y$.

Let $\gamma : X \to X$ be an injective map. We say that $\mu = \{\mu_x\}_{x\in X}$ is γ-invariant if for all $x, y \in X$ and $f \in S_X$ we have $\mu_x(f) = \mu_{\gamma(x)}(f \circ \gamma^{-1})$. Moreover (X,μ) is *quasi transitive* if and only if there exists a finite subset $X_0 \subseteq X$ such that for all $x \in X$ there exists a bijective map $\gamma : X \to X$ and $x_0 \in X_0$ satisfying $\gamma(x_0) = x$ and μ is γ-invariant.

We note that every quasi-transitive BRW (see [22, Section 6.2]) is an $\mathcal{F}$-BRW. The class of $\mathcal{F}$-BRWs is strictly larger than the class of quasi-transitive BRWs, see Example 3.8 (other examples are [2, Examples 3.1 and 3.2]).

It is worth mentioning a particular subclass of $\mathcal{F}$-BRWs: a BRW is *locally isomorphic to a branching process* if and only if the laws of the offspring number $\rho_x = \rho$ is independent of $x \in X$ (see Definition 2.5). In particular a continuous-time BRW is locally isomorphic to a branching process if and only if $k(x)$ does not depend on $x \in X$.

3. SURVIVAL

3.1. **Probabilities of extinction.** Define $q_n(x,A)$ as the probability of extinction before generation $n+1$ in A starting with one particle at x, namely $q_n(x,A)=\mathbb{P}(\eta_k(x)=0,\,\forall k\geq n+1,\,\forall x\in A)$. It is clear that $\{q_n(x,A)\}_{n\in\mathbb{N}}$ is a nondecreasing sequence satisfying

$$\begin{cases} q_n(\cdot,A)=G(q_{n-1}(\cdot,A)), & \forall n\geq 1 \\ q_0(x,A)=0, & \forall x\in A, \end{cases} \tag{3.5}$$

hence there is a limit $q(x,A)=\lim_{n\to\infty}q_n(x,A)\in[0,1]^X$ which is the probability of local extinction in A starting with one particle at x (see Definition 2.1). Note that equation (3.5) defines completely the sequence $\{q_n(\cdot,A)\}_{n\in\mathbb{N}}$ only when $A=X$ (otherwise one needs the values $q_0(x,A)$ for $x\notin A$). Since G is continuous we have that $q(\cdot,A)=G(q(\cdot,A))$, hence these extinction probabilities are fixed points of G (and Proposition 2.4 applies). Note that $q(\cdot,\emptyset)=\mathbf{1}$ and $q(\cdot,X)=\bar{q}(\cdot)$. It can be shown (see [3, Corollary 2.2]) that $\bar{q}$ is the smallest fixed point of $G(z)$ in $[0,1]^X$, since it is $\bar{q}=\lim_{n\to\infty}G^{(n)}(\mathbf{0})$. Using the same arguments, one can prove that $\bar{q}$ is the smallest fixed point of $G^{(m)}$ for all $m\in\mathbb{N}$.

Note that $A\subseteq B$ implies $q(\cdot,A)\geq q(\cdot,B)$. In particular, $q(\cdot,y)\geq\bar{q}$ for all $y\in X$. Since for all finite $A\subseteq X$ we have $q(x,A)\geq 1-\sum_{y\in A}(1-q(x,y))$ then, for any given finite $A\subseteq X$, $q(x,A)=1$ if and only if $q(x,y)=1$ for all $y\in A$.

If $x\to x'$ and $A\subseteq X$ then $q(x',A)<1$ implies $q(x,A)<1$; as a consequence, if $x\rightleftharpoons x'$ and $y\rightleftharpoons y'$ then $q(x,A)<1$ if and only if $q(x',A)<1$ and $q(x,y)=q(x,y')$.

In the irreducible case, if $\rho_x(0)>0$ for all $x\in X$, we have that $\bar{q}(x)=q(x,A)$ for some $x\in X$ and a finite subset $A\subseteq X$ if and only if $\bar{q}(y)=q(y,B)$ for all $y\in X$ and all finite subsets $B\subseteq X$. On he other hand, if we drop the assumption $\rho_x(0)>0$ for all $x\in X$, we might actually have $\bar{q}(x)=q(x,A)<1$ and $\bar{q}(y)<q(y,A)$ for some $x,y\in X$ and a finite $A\subseteq X$ (see Example 3.7).

The following theorem is the main engine for proving that strong local survival is not monotone (Example 3.6).

Theorem 3.1. *We observe that the following assertions are equivalent for every nonempty subset $A\subseteq X$.*

(1) $q(x,A)=\bar{q}(x)$, for all $x\in X$;
(2) $q_0(x,A)\leq\bar{q}(x)$, for all $x\in X$;
(3) the probability of visiting A at least once starting from x is larger than the probability of global survival starting from x, for all $x\in X$:
(4) for all $x\in X$, either $\bar{q}(x)=1$ or the probability of visiting A at least once starting from x conditioned on global survival starting from x is 1;
(5) for all $x\in X$, either $\bar{q}(x)=1$ or the probability of local survival in A starting from x conditioned on global survival starting from x is 1 (strong local survival in A starting from x).

From this theorem we have that if there exists $x\in X$ such that $q(x,A)>\bar{q}(x)$ (that is, there is a positive probability of global survival and nonlocal survival in A starting from x) then there exists $y\in X$ such that $q_0(y,A)>\bar{q}(y)$ (that is, there is a positive probability that the colony survives globally starting from y without ever visiting A). Of course, $q_0(x,A)>\bar{q}(x)$ implies $q(x,A)>\bar{q}(x)$ but the converse is not true. In particular for a BRW with no death there is strong local survival in A starting from x for all $x\in X$ if and only if the probability of visiting A is 1 starting from every vertex.

We note that, *a priori*, there is no order relation between the events "visiting A at least once starting from x" and "global survival starting from x". Nevertheless if, for all $x\in X$, the probability of "visiting A at least once starting from x" is larger or equal to the probability of "global survival starting from x" then, using the equivalence (1) $\iff$ (2) we have that the probability of "global survival starting from x never visiting A" is 0 and this implies, whenever $\bar{q}(x)<1$, that there is strong local survival in A starting from x.

If equation (2.1) holds and $\rho(n) = \frac{1}{1+\bar\rho_x}(\frac{\bar\rho_x}{1+\bar\rho_x})^n$, we have that the survival probability in A, $v_A := \mathbf{1} - q(\cdot, A)$, satisfies the equality $Mv_A = v_A/(\mathbf{1} - v_A)$. In particular in the continuous-time case we have $\lambda K v_A = v_A/(\mathbf{1} - v_A)$.

3.2. **Local and global survival.** The fact that there is local survival or not, depends only on the first-moment matrix M (see for instance [22, Theorem 4.1]); indeed there is local survival starting from x if and only if $M_s(x,x) > 1$. In particular the BRW survives locally at x if and only if it does so when restricted to the irreducibility class of x. It is worth noting that if $[x]$, the irreducible class of $x \in X$, is finite, then $M_s(x,x)$ is the Perron-Frobenius eigenvalue of the submatrix $M' := (m_{yz})_{y,z\in[x]}$. In this case there is local survival at x if and only if $\max\{t > 0 : \exists v \neq \mathbf{0},\, M'v = tv\} > 1$. Given a continuous-time BRW (X,K), $\lambda_s(x) = 1/K_s(x,x) = 1/\limsup_{n\to\infty} \sqrt[n]{k^{(n)}_{xx}}$. If $\lambda = \lambda_s(x)$ then there is local extinction at x ([3, Theorems 4.1 and 4.7]).

The main equivalent condition for global survival starting from x is the existence of $z \in [0,1]^X$, $z(x) < 1$ such that $G(z|y) \le z(y)$, for all $y \in X$ (equivalently, such that $G(z|y) = z(y)$, for all $y \in X$); see for instance [22, Theorem 4.1].

Note that, in general, the global behavior does not depend only on M (see [22, Example 4.4]) unless there is a one-to-one correspondence between first moment matrices and processes. This is true for instance in the class of BRWs with independent diffusion satisfying equation (2.2) (hence for a continuous-time BRW). Indeed (see [22, 3]) in that case an equivalent condition for global survival starting from $x \in X$ is the existence of $v \in [0,1]^X$, $v(x) > 0$ such that

$$Mv \ge v/(\mathbf{1} - v), \qquad (\text{equivalently, } Mv = v/(\mathbf{1} - v))$$

(where the ratio is taken coordinatewise). A characterization of global survival by means of $M_w(x)$ holds for the class of $\mathcal{F}$-BRWs as well ([22, Theorem 4.3]): there is global survival starting from x if and only if $M_w(x) > 1$.. In particular for a continuous-time $\mathcal{F}$-BRWs the global critical value is known: $\lambda_w(x) = 1/K_w(x)$ and if $\lambda = \lambda_w(x)$ there is global extinction starting from x ([3, Theorem 4.8, Proposition 4.5]); note that for a generic BRW when $\lambda = \lambda_w(x)$ there might be global survival (see [3, Example 3]).

Another characterization of $\lambda_w(x)$ has been given in [3, Theorem 4.2] by means of the so-called *lower Collatz-Wielandt number.*

3.3. **Strong local and pure global survival.** The interest on the strong local behavior is fairly recent (see for instance [11, 19]). We start with some results which deal with strong and non-strong local survival in the general case and then we prove that it is not monotone, that is, given a generic continuous-time BRW it is not possible to find a strong local critical parameter. Indeed if we are dealing with a continuous-time BRW, it might happen that if λ is small enough or large enough there is strong local survival but in a intermediate interval for λ there might be global and local survival with different probabilities. You can find this behavior in the BRW of Example 3.6 (which relies on Theorem 3.1).

The first result describes some properties of fixed-points of G for $\mathcal{F}$-BRWs.

Theorem 3.2. *Let (X,μ) be an $\mathcal{F}$-BRW. Then, there exists at most one fixed point z for G such that $\sup_{x\in X} z(x) < 1$, namely $z = \bar q$. Moreover for all $x \in X$, either $q(\cdot,x) = \bar q(\cdot)$ or $\sup_{w\in X} q(w,x) = 1$. In particular when (X,μ) is irreducible then it is either $q(x,x) = \bar q(x)$ for all $x \in X$ or $\sup_{x\in X} q(x,x) = 1$.*

In particular we can describe the case when X is finite (not necessarily irreducible). Clearly in this case $\bar q(w) = \min_{x\in X: w\to x} q(w,x)$, hence for all w such that $\bar q(w) < 1$ there exists x such that $q(w,x) = \bar q(w)$. Moreover, using Theorem 3.2, for all $x \in X$ we have that it is either $q(\cdot,x) = \bar q(\cdot)$ or there exists $w \in X$ such that $q(w,x) = 1$. If the BRW is irreducible (and X is finite) then it is $\bar q(w) = q(w,w)$ for all $w \in X$ or $q(w,x) = 1$ for all $w, x \in X$.

Corollary 3.3. *Let (X, μ) be an irreducible and quasi-transitive BRW. Then the existence of $x \in X$ such that there is local survival at x (i.e. $q(x, x) < 1$) implies that there is strong local survival at y starting from w for every $w, y \in X$ (i.e $q(w, y) = \bar{q}(w)$).*

Hence for a quasi-transitive, irreducible BRW, whenever there is local survival, it is a strong local survival; in continuous-time this implies that there is global and local extinction if $\lambda \in [0, \lambda_w]$, pure global survival if $\lambda \in (\lambda_w, \lambda_s]$ and strong local survival if $\lambda \in (\lambda_s, +\infty)$ (the weak and strong critical behaviors have been proved in [2, 3]).

In the particular case of a quasi-transitive, irreducible BRW with no death and with independent diffusion, Corollary 3.3 was proved in [19, Theorem 3.7]. The proof we give in Section 4 is of a different nature.

Unlike Theorem 3.2, Corollary 3.3 does not hold for every $\mathcal{F}$-BRW; indeed, as Examples 3.8 and 3.9 show, for an irreducible $\mathcal{F}$-BRW there might be non-strong local survival.

Recall that, in the irreducible case, if $\rho_x(0) > 0$ for all $x \in X$, then strong local survival is a common property of all vertices as local and global survival are. This is clearly false in the reducible case but it might be false as well in the irreducible case if we drop the assumption $\rho_x(0) > 0$ for all $x \in X$ as Example 3.7 shows.

The following result is a natural generalization of [17, Theorem 3.1]; it follows by applying [17, Theorem 3.1] to the no-death BRW associated to a generic BRW as described in Section 2.3 (hence we omit the proof).

Theorem 3.4. *Let (X, μ) be an irreducible, globally surviving BRW. Then there is no strong local survival if and only if there exists a finite, nonempty set $A \subseteq X$ and a function $v \in [0, 1]^X$ such that $\bar{q} \le v$ and*

$$\begin{cases} G(v|x) \ge v(x), & \forall x \in A^{\mathsf{C}}, \\ (T_{\bar{q}}^{-1} v)(x_0) > \max_{x \in A} (T_{\bar{q}}^{-1} v)(x) & \textit{for some } x_0 \in A^{\mathsf{C}}, \end{cases}$$

where $T_{\bar{q}}^{-1} v = (v - \bar{q})/(\mathbf{1} - \bar{q})$ (and the ratio is taken coordinatewise).

Our construction of an example of a continuous-time BRW where there is no monotonicity for the strong local behavior relies on the existence of a *pure global phase*. The idea of *pure global survival* has been introduced in continuous-time BRW theory (and, more generally, in interacting particle theory) to define the situation where $\lambda_s(x) > \lambda_w(x)$. In this case for every $\lambda \in (\lambda_w(x), \lambda_s(x)]$ there is a positive probability of global survival starting from x but the colony dies out locally at x almost surely. A necessary condition for the existence of a pure global survival phase starting from x is clearly that $K_s(x, x) < K_w(x)$ (see [3, Theorem 4.3]). In some cases this condition is also sufficient (see [3, Proposition 4.5] and [2, Theorem 3.3]).

Clearly for an irreducible BRW, the existence of pure global survival does not depend on the starting vertex. In this case $M_s(x, y) = M_s$ and $M_w(x) = M_w$ for all $x, y \in X$. Analogously $\lambda_w(x) = \lambda_w$ and $\lambda_s(x) = \lambda_s$ for all $x \in X$ in the case of an irreducible continuous-time BRW.

A straightforward generalization of [2, Theorem 3.6] (we omit the proof) implies that an irreducible, non-oriented $\mathcal{F}$-BRW is nonamenable if and only if $M_s < M_w$. In particular an irreducible, continuous-time $\mathcal{F}$-BRW has a pure global phase if and only if it is nonamenable.

The following example shows that there are irreducible amenable BRWs with pure global survival and irreducible nonamenable BRWs with no pure global survival (see also [20]). Recall that, for an edge-breeding BRW on a graph (or a multigraph), nonamenability is equivalent to the usual nonamenability of the graph.

Example 3.5. *Consider an irreducible, edge-breeding continuous-time BRW on the (non-oriented) graph X obtained by attaching to a copy of $\mathbb{N}$ one branch T of the homogeneous tree $\mathbb{T}_3$. The BRW is amenable by the presence of the copy of $\mathbb{N}$. We claim that $\lambda_s^X = \lambda_s^{\mathbb{T}_3}$ and $\lambda_w^X = \lambda_w^{\mathbb{T}_3}$. Indeed $T \subset X \subset \mathbb{T}_3$, hence $\lambda_s^T \ge \lambda_s^X \ge \lambda_s^{\mathbb{T}_3}$ and $\lambda_w^T \ge \lambda_w^X \ge \lambda_w^{\mathbb{T}_3}$. But by approximation, $\lambda_s^T = \lambda_s^{\mathbb{T}_3}$. Indeed $\lambda_s^T \ge \lambda_s^{\mathbb{T}_3}$ and does not depend on the starting vertex; moreover T contains arbitrarily large* *balls isomorphic to balls of* $\mathbb{T}_3$*, hence by* [22, Theorem 5.2][1] *or* [4, Theorem 3.1] *their critical local parameters coincide. Note that by* [2, Remark 3.2] *since* $\mathbb{T}$ *is a disjoint union of three copies of* T*, then* $\lambda_w^T = \lambda_w^{\mathbb{T}_3}$*. Then we have* $\lambda_w^X = \lambda_w^T \leq \lambda_w^{\mathbb{T}_3}$

On the other hand, consider a nonamenable graph X' *such that the corresponding edge-breeding continuous-time BRW has a pure global survival phase (take for instance* $X' := \mathbb{T}_3$ *the homogeneous tree with degree* 3*). Following* [2, Remark 3.2]*, attach to a vertex of* X' *a complete graph with degree* $k > 1/\lambda_w^{X'}$ *by an edge. It is easy to show that the resulting graph* X *is still nonamenable, nevertheless, according to* [2, Remark 3.2]*, there is no pure global survival for the corresponding edge-breeding BRW. Roughly speaking, since* $\lambda_w^X \leq 1/k < 1/3$*, then for every* $\lambda \in (\lambda_w^X, 1/3)$ *the process cannot survive globally in* $X' := \mathbb{T}_3$ *hence it hits infinitely often with positive probability the complete graph, hence* $\lambda_s^X = \lambda_w^X$*.*

Pure global survival is a fragile property of a BRW. Finite modifications, such as for an edge-breeding BRW attaching a complete finite graph to a vertex or removing a set of vertices and/or edges, can create it or destroy it as it is shown in [2, Remark 3.2].

We construct an example of a continuous-time BRW, where if λ is small enough or large enough there is strong local survival but in a intermediate interval for λ there is global and local survival with different probabilities. This is obtained by modifying the edge-breeding BRW on a particular graph, namely the homogeneous tree $\mathbb{T}_d$. The crucial property that we need here is the existence of a pure global survival phase, thus the procedure applies to every BRW with such a phase.

Example 3.6. *Consider the edge-breeding continuous-time BRW on the homogeneous tree* $\mathbb{T}_d$ *with degree* $d \geq 3$*. It is easy to see that if* $\lambda \leq 1/d$ *the probabilities of survival are* 0*, if* $\lambda > 1/2\sqrt{d-1}$ *there is strong local survival (according to Corollary 3.3) and if* $\lambda \in (1/d, 1/2\sqrt{d-1}]$ *the probability of global survival is positive and independent of the starting point and the probability of local survival at any finite* $A \subseteq X$ *is* 0*.*

Fix $\lambda \in (1/d, 1/2\sqrt{d-1}]$*. According to Theorem 3.1, there exists* $x \in X$ *such that there is a positive probability of global survival starting from* x *without ever visiting* A *(clearly* $x \notin A$*). In this case, any modification of the rates in the subset* A *provides a new BRW such that there is still a positive probability of global survival starting from* x *without ever visiting* A *(since, the original BRW and the new one coincide until the first hitting time on* A*). On the other hand, if there is* $y \in A$ *such that* $x \to y$ *and we add a loop in* y *and a rate* $k_{yy} > 1/\lambda$ *then* $\bar{q}(x) < q(x,y) < 1$*; the first inequality holds by the discussion above on local modifications and the second one holds since* $\lambda k_{yy} > 1$ *implies local survival at* y *(then irreducibility implies local survival at* y *starting from* x*). This means that, for this fixed value of* λ*, we obtained a locally and globally (but not strong-locally) surviving BRW at* y *starting from* x*.*

Suppose now that $k_{yy} > d$*; then (see* [2, Remark 3.2]*) we have a new BRW such that* $\lambda'_w = \lambda'_s \leq 1/k_{yy}$*. In this case, when* $\lambda \leq \lambda'_w$ *there is global extinction. When* $\lambda > 1/2\sqrt{d-1}$ *there is strong local survival for the original BRW (by Corollary 3.3) which implies strong local survival for the new one (the probability of hitting* x *conditioned on global survival is* 1 *for both processes and Theorem 3.1 applies). If* $\lambda \in (\lambda'_w, 1/d]$ *there is local and global survival with the same probability since in order to survive globally, the process must visit* x *infinitely many times (it cannot survive globally in the branches of* $\mathbb{T}_d$*). If* $\lambda \in (1/d, 1/2\sqrt{d-1}]$ *then, according to the previous discussion, there is non-strong local survival for the new BRW.*

We show that even in the irreducible case, if $\rho_x(0) = 0$ for some $x \in X$, we might have strong local survival starting from some vertices and not from others.

Example 3.7. *Let us consider a modification of the discrete-time counterpart of the edge-breeding BRW on* $\mathbb{T}_d$ *with degree* $d \geq 3$ *and* $\lambda \in (1/d, 1/2\sqrt{d-1}]$ *. Let us fix a vertex* y*; in this modified*

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[1] We observe that in [22, Section 5.1] the hypotheses that M is a nonnegative matrix is missing, even though it is implicitly used. Moreover [22, Theorems 5.1 and 5.2] hold without the irreducibility hypothesis (since it is easy to prove that for all $x_0 \in X$ we have ${}_nR(x_0, x_0) \to R(x_0, x_0)$).

version we add, with probability one, one child at y for every particle at y. In this case $\bar{q}(y) = q(y, A) = 0$ for all $A \subseteq X$. On the other hand according to the discussion in Example 3.6, there is a vertex y such that $\bar{q}(x) < q(x, y)$.

In the last few examples we make use of the subclass of BRWs which are locally isomorphic to a branching process (which are particular $\mathcal{F}$-BRWs, see Section 2.4).

It is easy to show that for such a process: (1) there is global survival if and only if $\bar{\rho} > 1$; (2) there is local survival at x if and only if $\bar{\rho} > 1/\limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,x)} =: r(x,x)$. Hence, given a continuous-time BRW which is locally isomorphic to a branching process, $\lambda_w = 1/k$ and $\lambda_s(x) = r(x,x)/k$ (where $k = k(x)$ for all $x \in X$). It is clear that, in the irreducible case, there is pure global survival if and only if $1 < \bar{\rho} \le r$ (where $r = r(x,x)$ in this case does not depend on $x \in X$ due to irreducibility). This is possible if and only if $r > 1$ which is equivalent to nonamenability since in this case $M_s(x,y) = \bar{\rho}/r$ and $M_w(x) = \bar{\rho}$.

In general there may be non-strong local survival, even if the BRW is locally isomorphic to a branching process and it has independent diffusion as Examples 3.8 and 3.9 show.

Example 3.8. *Fix $X := \mathbb{N}$ and consider a BRW with the following reproduction probabilities. Every particle has two children with probability $3/4$ and no children with probability $1/4$. Each newborn particle is dispersed independently according to a nearest neighbor matrix P on $\mathbb{N}$. More precisely*

$$p(i,j) := \begin{cases} p_i & \text{if } j = i+1 \\ 1 - p_i & \text{if } j = i-1, \end{cases}$$

and $p_0 = 1$. The process described above is an irreducible $\mathcal{F}$-BRW for every choice of the set $\{p_i\}_{i \in \mathbb{N}\setminus\{0\}}$. The generating function of the total number of children is $z \mapsto 3z^2/4 + 1/4$ and its minimal fixed point is $1/3 = \bar{q}(x)$ (for all $x \in \mathbb{N}$).

Choose $p_1 < 5/9$; it is easy to show that the process confined to $\{0, 1\}$ survives (since the expected number of children at 0 every two generations (starting from 0) is $(3/2)^2(1 - p_1) > 1$). By irreducibility this implies that $q(x, y) < 1$ and $\bar{q}(x) < 1$ for all $x, y \in \mathbb{N}$.

Choose the p_is such that $\prod_{i=1}^\infty p_i^{2^i} > 0$ (or, equivalently, $\sum_{i=1}^\infty 2^i(1 - p_i) < +\infty$). Consider the branching process N_n representing the total number of particles alive at time n: for all n, $N_n \le 2^n$ almost surely. The probability, conditioned on global survival, that every particle places its children (if any) to its right, is the conditioned expected value of $\prod_{i=1}^\infty p_i^{N_i}$. But $\prod_{i=1}^\infty p_i^{N_i} \ge \prod_{i=1}^\infty p_i^{2^i} > 0$ almost surely. Hence, conditioning on global survival there is a positive probability of non-local survival. This implies $q(\cdot, y) \ne \bar{q}$ for every $y \in \mathbb{N}$. Note that, according to Theorem 3.2, $\sup_{x\in\mathbb{N}} q(x, x) = 1$.

The key in the previous example is that the total number of particles alive at time n is bounded. This is not an essential assumption. The following example shows that, given any law ρ of a surviving branching process (that is, $\bar{\rho} = \sum_{n\in\mathbb{N}} n\rho(n) > 1$), it is possible to construct an irreducible BRW which is locally isomorphic to a branching process with no strong local survival (which means that if there is local survival then it is non-strong local survival).

Example 3.9. *Let $X = \mathbb{N}$ and $\rho_x := \rho$ for all $x \in \mathbb{N}$; ρ being the law of a surviving branching process. We know that $\bar{q}(x) \equiv \bar{q}$ for all $x \in \mathbb{N}$ where $\bar{q} < 1$ is the smallest fixed point of $z \mapsto \sum_{n \, in\mathbb{N}} \rho(n) z^n$. Pick a sequence of natural numbers $\{N_i\}_{i\in\mathbb{N}}$ satisfying*

$$\prod_{i\in\mathbb{N}} \rho([0, N_{i+1}])^{\prod_{j=0}^{i} N_j} > \bar{q}, \tag{3.6}$$

where $N_0 := 1$. Note that the probability of the event $\mathcal{A}$="every particle alive at time i has at most N_{i+1} children for all $i \in \mathbb{N}$" is bounded from below by the LHS of equation (3.6). Thus, from equation (3.6), with a probability larger than $\prod_{i\in\mathbb{N}} \rho([0, N_{i+1}])^{\prod_{j=0}^{i} N_j} - \bar{q} > 0$ the colony survives globally and the total size of the population at time n is not larger than $\prod_{j=0}^{n} N_j$ (i.e. the intersection between $\mathcal{A}$ and global survival has positive probability).

We define a BRW with independent diffusion where P is as follows

$$p(i,j) := \begin{cases} p_i & j=i+1,\, i\geq 0 \\ 1-p_i & j=i-1,\, i\geq 1 \\ 1-p_0 & i=j=0. \end{cases}$$

Let p_0 such that $(1-p_0)\bar\rho > 1$; this implies local survival. We choose the sequence $\{p_i\}_{i\in\mathbb{N}}$, where $p_i \in (0,1)$ in such a way that

$$\prod_{i\in\mathbb{N}} p_i^{\prod_{j=0}^i N_j} > 0 \tag{3.7}$$

(or, equivalently, $\sum_{i\in\mathbb{N}}(1-p_i)\prod_{j=0}^i N_j < \infty$). Using equation (3.7), if we condition on $\mathcal{A}$, the probability that, every particle places its children (if any) to its right is bounded from below by $\prod_{i\in\mathbb{N}} p_i^{\prod_{j=0}^i N_j}$. This implies that there is a positive probability of global, non-local survival.

The choice of the sequences $\{N_i\}_{i\in\mathbb{N}}$ and $\{p_i\}_{i\in\mathbb{N}}$ satisfying equations (3.6) and (3.7) respectively can be done as follows. Choose a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ such that $\alpha_i\in(0,1)$ for all $i\in\mathbb{N}$ and $\prod_{i\in\mathbb{N}}\alpha_i > 1-\bar q$. Then, iteratively, if we fixed $N_0,\ldots,N_k$, since $\lim_{x\to\infty}\rho([0,x])=1$ there exists $N_{k+1}\in\mathbb{N}$ such that $\rho([0,N_{k+1}]) > \alpha_{k+1}^{1/\prod_{j=0}^k N_j}$. Let us take, for instance, $p_i > 1/(i\cdot\prod_{j=0}^i N_j)$.

We note that the class constructed in this example includes discrete-time counterparts of continuous-time BRWs where ρ can be chosen as in equation (2.2) where $k(x)\equiv k$ does not depend on x, $k_{xy} := k\cdot p(x,y)$ (where P is defined as before) and $\lambda > \lambda_s$ is fixed. Finally we observe that this example extends naturally to an example of a site-breeding BRW on a radial tree where the number of branches of a vertex at distance k from the root is at least $1/p(k,k+1)$.

4. PROOFS

Here we sketch the proofs.

Proof of Proposition 2.4. Without loss of generality we can suppose that $\bar q(x) < 1$ for all $x\in X$. Indeed, given x_0 such that $\bar q(x_0)=1$ then for all $x\in\mathcal{N}_{x_0}$ we have $\bar q(x)=1$. Since we defined $\widehat z(x) := 1$ whenever $\bar q(x)=1$ we can remove these vertices obtaining a new set $X'\subseteq X$. Consider the restricted BRW on X' (obtained by killing all the particle going outside X'). It is clear that $q^X(x,A)\leq q^{X'}(x,A)$ for all $x\in X'$, $A\subseteq X'$. The generating function G' of the new BRW satisfies $G'((z|_{X'})|x)\geq G(z|x)$ for all $x\in X'$, hence $G(z)\geq z$ implies $G'(z|_{X'})\geq z|_{X'}$. Moreover $\widehat z$ satisfies the conclusions of the proposition if and only if $\widehat{z|_{X'}}\equiv \widehat z|_{X'}$ does. Thus, it is enough to prove the result for the BRW restricted to X'.

Note that $\widehat z := T_{\bar q}^{-1}(z)$, thus $G(z)\geq z$ is equivalent to $\widehat G(\widehat z)\geq\widehat z$. Hence it is enough to prove the proposition when $\mu_x(\mathbf{0})=0$ for all $x\in X$ which implies $\bar q=\mathbf{0}$ and $\widehat z = z$. Suppose that $\mathcal{N}_x$ is nonempty, $z(y)\leq z(x)$ for all $y\in\mathcal{N}_x$ and $z(y_0)<z(x)$ for some $y_0\in\mathcal{N}_x$. Then, using the fact that $z\leq\mathbf{1}$ and that $\prod_{y\in X} z(y)^{f(y)}\leq z(x)$ if $\mathcal{H}(f)\geq 1$, we have that $z(x)\leq G(z|x)\leq \sum_{f\in S_X: f(y_0)=0}\mu_x(f)z(x)+\sum_{f\in S_X:f(y_0)>0}\mu_x(f)z(y_0) < z(x)$ which is a contradiction. As for the second part, since $z(y)\leq 1 = z(x)$ for all $y\in X$ then we have $z(y)=1$ for all $y\in X$. Finally, by induction we obtain the result for the set $\{y\in X: x\to y\}$. $\square$

Proof of Theorem 3.1. Indeed, since $\{q_n(\cdot,A)\}_{n\in\mathbb{N}}$ is non decreasing, $q_n(\cdot,A)=G(q_{n-1}(\cdot,A))$ and $\bar q$ is the smallest fixed point of G, we have immediately that

$$q(\cdot,A)=\bar q(\cdot)\iff q_0(\cdot,A)\leq\bar q(\cdot), \tag{4.8}$$

that is, (1)$\iff$(2). Moreover the event "local survival in A starting from x" implies both "global survival starting from x" and "visiting A at least once starting from x", hence $q(x,A)=\bar q(x)$ if and only if the probability of visiting A infinitely many times starting from x conditioned on global survival is 1 and (1)$\iff$(5)$\Longrightarrow$(4). Trivially (2)$\iff$(3) and (4)$\Longrightarrow$(3). This proves the equivalence. $\square$

Before proving Corollary 3.3 and Theorem 3.2 we need two technical lemmas.

Lemma 4.1. *Let (X, μ) be a BRW and fix $z, v \in [0,1]^X$ such that $z + \varepsilon v \in [0,1]^X$ for some $\varepsilon > 0$. Then the function $t \mapsto G(z + vt|x)$ is strictly convex if and only if*

$$\exists f : \mu_x(f) > 0, \quad \sum_{y \in \mathrm{supp}(v)} f(y) \geq 2, \ \mathrm{supp}(z) \cup \mathrm{supp}(v) \supseteq \mathrm{supp}(f). \tag{4.9}$$

Proof of Lemma 4.1. Let us evaluate the function G on the line $t \mapsto z + tv$ where $t \in [0, T)$ and $T := \sup\{s > 0 : z + sv \in [0,1]^X\}$.

$$\begin{aligned}
G(z+tv|x) &= \sum_{f \in S_X} \mu_x(f) \prod_{y \in X} \sum_{i=0}^{f(y)} \binom{f(y)}{i} z(y)^{f(y)-i} v(y)^i t^i \\
&= \sum_{f \in S_X} \mu_x(f) \sum_{g \in S_X : g \leq f} \prod_{y \in X} \binom{f(y)}{g(y)} z(y)^{f(y)-g(y)} v(y)^{g(y)} t^{g(y)} \\
&= \sum_{f \in S_X} \mu_x(f) \sum_{g \in S_X : g \leq f} t^{\mathcal{H}(g)} \prod_{y \in X} \binom{f(y)}{g(y)} z(y)^{f(y)-g(y)} v(y)^{g(y)} \\
&= \sum_{f \in S_X} \mu_x(f) \sum_{i=0}^{\infty} \sum_{g \in S_X : \mathcal{H}(g) = i, g \leq f} t^i \prod_{y \in X} \binom{f(y)}{g(y)} z(y)^{f(y)-g(y)} v(y)^{g(y)} \\
&= \sum_{i=0}^{\infty} t^i \left(\sum_{f, g \in S_X : \mathcal{H}(g) = i, g \leq f} \mu_x(f) \prod_{y \in X} \binom{f(y)}{g(y)} z(y)^{f(y)-g(y)} v(y)^{g(y)} \right)
\end{aligned}$$

The strict convexity of a power series in t with nonnegative coefficients is equivalent to the strict positivity of at least one coefficient corresponding to t^i with $i \geq 2$. Hence it is easy to show that each of the following assertions is equivalent to the next one and that they are all equivalent to the strict convexity of $t \mapsto G(z + vt|x)$

(1) $\exists f, g : \mathcal{H}(g) \geq 2,\ f \geq g,\ \mu_x(f) > 0 : \mathrm{supp}(v) \supseteq \mathrm{supp}(g),\ \mathrm{supp}(z) \supseteq \mathrm{supp}(f - g)$;
(2) $\exists f, g : \mathcal{H}(g) \geq 2,\ f \geq g,\ \mu_x(f) > 0 : g = f \mathbb{1}_{\mathrm{supp}(v)},\ \mathrm{supp}(z) \supseteq \mathrm{supp}(f) \setminus \mathrm{supp}(v)$;
(3) $\exists f : \mu_x(f) > 0 : \sum_{y \in \mathrm{supp}(v)} f(y) \geq 2,\ \mathrm{supp}(z) \supseteq \mathrm{supp}(f) \setminus \mathrm{supp}(v)$;
(4) $\exists f : \mu_x(f) > 0 : \sum_{y \in \mathrm{supp}(v)} f(y) \geq 2,\ \mathrm{supp}(z) \cup \mathrm{supp}(v) \supseteq \mathrm{supp}(f)$;

□

Lemma 4.2. *Let (X, μ) be a BRW and fix $x_0 \in X$. Suppose that for some $\bar{x}$ in the same irreducible class of x_0 and $f \in S_X$ we have that $\mu_{\bar{x}}(f) > 0$, $\sum_{w : w \rightleftharpoons x_0} f(w) \geq 2$. We can fix $\bar{n} \in \mathbb{N}$ such that if the process starts with one particle at $x_0 \in X$ then we have at least 2 particles at x_0 in the generation $\bar{n}$ wpp.*

Proof of Lemma 4.2. Consider a path $x_0, x_1, \ldots, x_m = \bar{x}$ and let $f \in S_X$ be such that $\mu_{\bar{x}}(f) > 0$ and $\sum_{w : w \rightleftharpoons x_0} f(w) \geq 2$. We can have two cases.

(a). There exists $x_{m+1} \in X$ such that $x_{m+1} \rightleftharpoons x_0$ and $f(x_{m+1}) \geq 2$; in this case consider the closed path $x_0, x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n = x_0$ and take $\bar{n} := n$. Since any particle at x_i has at least one child at x_{i+1} wpp and a particle at $\bar{x}$ has at least 2 children at x_{m+1} wpp, then any particle at x_0 has at least 2 descendants at x_0 in the $\bar{n}$th generation. Indeed, denote by $f_i \in S_X$ such that $\mu_{x_i}(f_i) > 0$, $f_i(x_{i+1}) \geq 1$ for all $i = 0, \ldots \bar{n} - 1$ (f_m being f), then the probability that a particle at x_0 has at least 2 particle at x_0 in the $\bar{n}$th generation is bounded from below by $\prod_{i=0}^{m} \mu_i(f_i) \prod_{j=m+1}^{\bar{n}-1} \mu_j(f_i)^2$.

(b). There exists a couple of different vertices x_{m+1}, y_{m+1} such that $x_{m+1}, y_{m+1} \rightleftharpoons x_0$ and $f(x_{m+1}), f(y_{m+1}) \geq 1$; in this case consider the paths $x_0, x_1, \ldots x_m, x_{m+1}, \ldots, x_{n_1} = x_0$ and $x_0, x_1, \ldots x_m, y_{m+1}, \ldots, y_{n_2} = x_0$ and take $\bar{n} := GCD(n_1, n_2)$ (the conclusion is similar as before). □

Proof of Theorem 3.2. For every z fixed point of G, we know that $z \geq \bar{q}$ and $z \leq \mathbf{1}_X$; this implies that if $\sup_{x\in X} z(x) < 1$ for some fixed point then necessarily $\sup_{x\in X} \bar{q}(x) < 1$. Hence, if $\bar{q} = \mathbf{1}$ there is nothing to prove. Otherwise, we show that if $G(z) = z$ and $z \neq \bar{q}$ then $\sup_{w\in X} z(w) = 1$. Suppose that the BRW is locally isomorphic to (Y, ν) through the map g and define $h(y) := \sup_{w\in g^{-1}(y)} z(w)$. Clearly $h \in [0,1]^Y$ and $h \circ g \geq z$ which implies that $G_Y(h) \geq h$. Indeed

$$\begin{aligned} G_Y(h|y) &= \sup_{x\in g^{-1}(y)} G_Y(h|g(x)) = \sup_{x\in g^{-1}(y)} G(h\circ g|x) \\ &\geq \sup_{x\in g^{-1}(y)} G(z|x) = \sup_{x\in g^{-1}(y)} z(x) = h(y). \end{aligned}$$

If Y finite then we can choose $\widetilde{y} \in Y$ which minimizes

$$t(y) := \frac{1 - \bar{q}^Y(y)}{h(y) - \bar{q}^Y(y)}$$

(where $t(y) := +\infty$ if $h(y) = \bar{q}^Y(y)$); note that $t(y) \geq 1$ for all $y \in Y$ and $t(\widetilde{y}) < +\infty$. By applying the maximum principle (Proposition 2.4) to the function $1/t(y)$ (where y is ranging in the set $\{w : \bar{q}^Y(w) < 1\}$) we have that it is constant on $\{y : \widetilde{y} \to y\}$. Since $\bar{q}^Y(\widetilde{y}) < 1$ and Y is finite, then there exists y_0 such that $\widetilde{y} \to y_0$ and there is local survival at y_0 starting from y_0. Since (Y, ν) satisfies Assumption 2.2 then there exists $\bar{y} \rightleftharpoons y_0$ such that a particle living at $\bar{y}$ wpp has at least 2 children in the irreducible class of y_0. Then by taking y_0 instead of x_0 in Lemma 4.2 we have that we can find $\bar{n} \in \mathbb{N}$ such that the function

$$\phi(t) := G_Y^{(\bar{n})}(\bar{q}^Y + t(h - \bar{q}^Y)|y_0) - \bar{q}^Y(y_0) - t(h(y_0) - \bar{q}^Y(y_0))$$

is strictly convex by Lemma 4.1. Indeed $G_Y^{(\bar{n})}$ is the generating function of the BRW constructed by considering the n-th generations of the original BRW where $\bar{n}|n$ and, under our hypotheses, it satisfies equation (4.9).

Note that ϕ is well defined in $[0, t(y_0)]$ since

$$r_t(y) := \bar{q}^Y(y) + t(h(y) - \bar{q}^Y(y)) \leq \bar{q}^Y(y) + t(y_0)(h(y) - \bar{q}^Y(y)) \leq 1$$

hence $r_t \in [0,1]^Y$ for all $t \in [0, t(y_0)]$.

Clearly every fixed point of G_Y is a fixed point of $G_Y^{(\bar{n})}$; in particular, $G^{(\bar{n})}(z) = z$ and $G_Y^{(\bar{n})}(\bar{q}^Y) = \bar{q}^Y$, whence $\phi(0) = 0$ and $\phi(1) = G_Y^{(\bar{n})}(h|y_0) - h(y_0)$. Now, using equation (2.4), $G_Y^{(\bar{n})}(h) \geq h$ and this, in turn, implies $\phi(1) \geq 0$. Since ϕ is strictly convex we have that $\phi(t) > 0$ for all $t \in (1, t(y_0)]$. If $t(y_0) > 1$ then $0 < \phi(t(y_0)) = G_Y^{(\bar{n})}(r_{t(y_0)}|y_0) - 1$ but this is a contradiction since $r_{t(y_0)} \in [0,1]^Y$ and $G_Y^{(\bar{n})}(r_{t(y_0)}) \in [0,1]^Y$. In the end $t(y_0) = 1$, thus $1 = h(y_0) = \sup_{w\in X} z(w)$. ☐

Note that, from the previous proof, if the BRW on Y is irreducible then by the maximum principle we have that $(h - \bar{q}^Y)/(\mathbf{1} - \bar{q}^Y)$ is a constant function, thus $h(y) = \sup_{w\in g^{-1}(y)} z(w) = 1$ for all $y \in Y$.

Proof of Corollary 3.3. Since (X, E_μ) is irreducible we have that $q(x,y) = q(x,x)$ for all $x, y \in X$ and if $\bar{q} < \mathbf{1}$ (resp. $q(\cdot, y) < \mathbf{1}$) then $\bar{q}(x) < 1$ (resp. $q(x,y) < 1$) for all $x \in X$. Moreover, quasi transitivity implies that if $q(\cdot, y) < \mathbf{1}$ then $\sup_{x\in X} q(x,y) < 1$. Thus, according to Theorem 3.2, $q(\cdot, y) \neq \mathbf{1}$ implies $q(\cdot, y) = \bar{q}$. ☐

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