

# ON THE PLANE PROBLEM OF THE FLOW AROUND A SUBMERGED BEAM

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## Abstract

We consider the problem of the steady flow of an ideal heavy fluid around a submerged beam. The problem is obtained from the free-boundary problem of the flow past a submerged obstacle in the limit of bodies of vanishing thickness. We introduce a special Sobolev space formulation of the problem in term of a perturbed stream function and prove its unique solvability for every value of the unperturbed flow velocity, with the possible exception of a discrete set depending on the geometry of the domain. The asymptotic properties of the solutions are discussed.

## 1. INTRODUCTION.

A classical problem in hydrodynamics is the study of the plane stationary flow of a heavy fluid over submerged obstacles. Assuming the usual hypotheses, i.e., irrotational and divergence-free flow, non viscous fluid and negligible surface tension, the velocity field can be described by a complex function, holomorphic in the domain occupied by the fluid; such a domain has rigid boundaries, where the "no-flow condition" is imposed, and a free boundary where, in addition, the non linear dynamical condition (Bernoulli condition) holds. When no obstacle is contained in the fluid, the only rigid boundary is a horizontal bottom and we have the well-known steady water-wave problem. Although substantial progress has been achieved in the rigorous treatment of the latter problem (even in the three-dimensional case) [1], little is known about the interaction of water waves with submerged or semi-submerged rigid bodies. There are some results concerning steady flows under localized pressure perturbations on the free surface, which may simulate the action of a surface-piercing obstacle [2], [3]; more recently, exact solvability has been proved for the complementary physical problem of the *ship waves* generated by the uniform horizontal motion of a *thin semi-submerged body* [4]. In the case of a completely submerged cylinder, exact solutions have been found in [5] only for supercritical flow velocities (see below).

Nevertheless, most of the mathematical treatment of the problem deals with linearized versions, whose solutions are called *linear water waves* [6]. Even

in dimension two, one of the still open questions in the linear theory is to determine whether the problem for a given obstacle in a current is uniquely solvable for all values of the flux velocity. A positive answer is known for special geometries [7], [8], but there are examples of non trivial, finite energy solutions of the homogeneous problem (trapped modes) in the presence of multiple obstacles and in the case of a submerged hollow [9]. In general, the connection between unique solvability and the geometry of the obstacles is not completely understood.

The standard approach to solvability of the linear problem relies on integral equation techniques, which apply when the boundaries of the obstacles are sufficiently regular, including piecewise smooth contours with corner points (but no cusps) that are quite usual in the applications. Recently, an alternative variational approach has been proposed, which is suitable for a large set of obstacles (particularly when the perturbed flow has finite energy [10]) including the limit case of a *surface beam* [11].

The aim of this paper is to apply the same technique to the problem for a *submerged horizontal beam*, which cannot be treated by standard integral equation methods as in the case of a body with smooth boundary [12]. The interest in this problem is that it is obtained from the exact, free-boundary problem with an obstacle of thickness of order  $\epsilon$ , in the limit  $\epsilon \rightarrow 0$ ; thus, the study of this limit case is a crucial step in treating the non linear problem of the flow past a *thin submerged body* by local methods, following the same pattern which led us to prove the solvability of the problem for a surface-piercing body [4].

Our main conclusion is that the linearized problem for a flow with (unperturbed) velocity  $c$  and with an *assigned value of the circulation around the obstacle* is uniquely solvable, with the possible exception of a discrete set of values of  $c$  depending on the depth of the fluid and on the length and position of the beam. We also describe the asymptotic form of the solution at infinity downstream.

## 2. THE PROBLEM.

We consider a horizontal beam contained in a strip

$$\Sigma \equiv \{x \in \mathbb{R}; -H < y < 0\},$$

whose upper and lower boundary consists of a plane surface  $F = \mathbb{R} \times \{0\}$  (which will be improperly called free boundary) and of a flat bottom  $B = \mathbb{R} \times \{-H\}$ ; the beam is represented by a segment  $I = (-x_0, x_0) \times \{-b\}$ , where  $x_0 > 0$  and  $0 < b < H$ ; finally, we define the *strip with a cut*

$$S = \Sigma \setminus I.$$

As mentioned above, we can consider the beam  $I$  as the limit, for  $\epsilon \rightarrow 0$ , of a thin, rigid body represented by the domain

$$\{(x, y) \in \Sigma : -x_0 < x < x_0, -b + \epsilon f_-(x) < y < -b + \epsilon f_+(x)\},$$

where  $f_{\pm} \in \mathcal{C}^1([-x_0, x_0])$  (see fig.1 below).

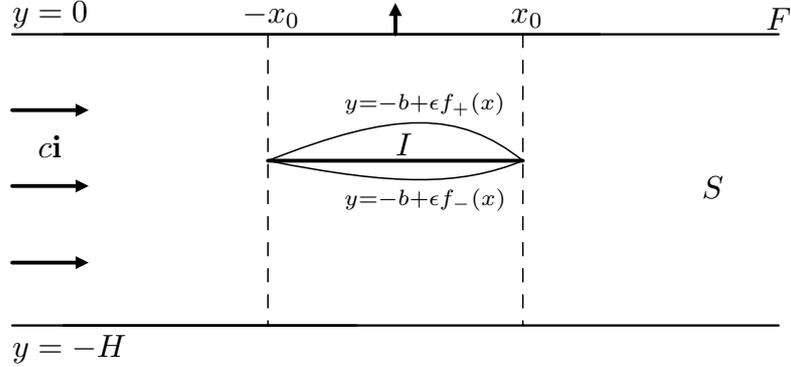


Fig. 1.

We assume that the body is surrounded by a steady flow with (unperturbed) field velocity  $\mathbf{ci}$  at upstream infinity and we denote by  $\epsilon(u\mathbf{i} + v\mathbf{j})$  the perturbed velocity field; moreover, for a zero circulation flow one can define a potential  $\phi$  such that  $\nabla\phi = u\mathbf{i} + v\mathbf{j}$ . Then, we obtain in the limit  $\epsilon \rightarrow 0$ , the following linear system (see [5]):

$$\Delta\phi = 0 \quad \text{in } S \quad (2.1)$$

$$\phi_{xx} + \nu\phi_y = 0 \quad \text{on } F \quad (2.2)$$

$$\partial_y\phi_{\pm} = K_{\pm} \quad \text{on } I \quad (2.3)$$

$$\phi_y = 0 \quad \text{on } B, \quad (2.4)$$

where  $\nu = g/c^2$  ( $g$  is the acceleration of gravity)  $\partial_y\phi_{\pm}$  are the limit values of  $\partial_y\phi = v$  respectively on the upper and lower side of the beam and

$$K_{\pm} = cf'_{\pm} \quad (2.5)$$

by the no-flow condition. The problem is completed by the asymptotic conditions

$$\lim_{x \rightarrow -\infty} \nabla\phi(x, y) = 0 \quad (\text{uniformly in } y), \quad (2.6)$$

$$\sup_{(x, y) \in S \setminus A} |\nabla\phi(x, y)| < \infty, \quad (2.7)$$

where  $A$  is any neighborhood of  $I$ .

The problem (2.1)-(2.4) (for any  $\nu > 0$  and regular enough  $K_{\pm}$ ) with the asymptotic conditions (2.6), (2.7), is called the *Neumann-Kelvin problem* (for a beam); this problem has been widely studied in the literature, since its solution also represents the potential of the *linear ship waves* generated by a body moving in a fluid with uniform motion [6]. However, in two dimensions one can provide an alternative description of the problem by considering the *stream function*  $\psi$  (harmonic conjugate of the potential  $\phi$ ) satisfying  $\psi_x = v$ ,  $-\psi_y = u$ . We recall that the function  $\psi$  is determined (except for an arbitrary constant) by the velocity field of an incompressible fluid (such that  $u_x + v_y = 0$ ) whenever the *flux*  $\oint_{\mathcal{C}} \{v dx - u dy\}$  through a closed curve  $\mathcal{C}$  surrounding the obstacle is vanishing. In the case of a beam, by shrinking  $\mathcal{C}$  to a segment we get

$$\int_I (v_+ - v_-) dx = 0, \quad (2.8)$$

where  $v_{\pm}$  are the limit values of  $v$  respectively on the upper and lower side of the beam; as expected, (2.8) holds for a solution satisfying the "no-flow" condition (i.e. with boundary data of the form (2.5), see below). It follows that the stream function can be defined also in the *more general* case of flows with non zero circulation around the obstacle. We also remark that the stream function was recently employed to provide new variational formulations for nonlinear steady water waves with vorticity [13]. Moreover, as we now show, the linear problem for the function  $\psi$  has the advantage that the second order boundary condition (for the potential) on the free surface is replaced by a first order condition (Steklov condition). For, by the definition of  $\psi$  and observing that both  $F$  and  $B$  are *connected sets*, it follows by (2.2) that  $\psi_y - \nu\psi$  is constant on  $F$  and by (2.4) that  $\psi$  is constant on  $B$ ; by suitably choosing the arbitrary constant in the definition of  $\psi$ , we may assume  $\psi = 0$  on  $B$ . Then, since  $\lim_{x \rightarrow -\infty} \psi_y = 0$  (by (2.6)) we also get  $\psi \rightarrow 0$  for  $x \rightarrow -\infty$  (uniformly in  $y$ ) so that  $\psi_y - \nu\psi = 0$  on  $F$ . Finally, on the two sides of the beam we may assign to  $\psi$  a pair of *Dirichlet boundary data* which correspond to condition (2.3) (see below). Summing up the discussion, we have the problem:

**Problem  $P_{\nu}$  :** For every  $\nu > 0$ , find  $\psi$  satisfying

$$\Delta\psi = 0 \quad \text{in } S \quad (2.9)$$

$$\psi_{\pm} = \mathcal{K}_{\pm} \quad \text{on } I \quad (2.10)$$

$$\psi_y - \nu\psi = 0 \quad \text{on } F \quad (2.11)$$

$$\psi = 0 \quad \text{on } B, \quad (2.12)$$

$$\lim_{x \rightarrow -\infty} \nabla \psi(x, y) = 0 \quad (\text{uniformly in } y), \quad (2.13)$$

$$\sup_{(x, y) \in S \setminus A} |\nabla \psi(x, y)| < \infty, \quad (2.14)$$

where  $A$  is any neighborhood of  $I$ . By the definition of  $\psi$  and by (2.3) the boundary data  $\mathcal{K}_\pm$  in condition (2.10) have to satisfy

$$\mathcal{K}'_\pm = v_\pm = K_\pm; \quad (2.15)$$

hence, by (2.8) we obtain (for absolutely continuous  $\mathcal{K}_\pm$ )

$$(\mathcal{K}_+ - \mathcal{K}_-)(x_0) = (\mathcal{K}_+ - \mathcal{K}_-)(-x_0) \quad (2.16)$$

Note that by assuming  $K_\pm$  of the form (2.5), one can satisfy both (2.15) and (2.16) by taking  $\mathcal{K}_\pm = cf_\pm$ , as the obstacle's profile described by  $f_\pm$  must be a closed curve.

**Remark.** From the previous discussion, it is clear that the Dirichlet data in condition (2.10) are *not* uniquely determined by condition (2.3). In fact, we can add to  $\mathcal{K}_\pm$  two constant  $\gamma_\pm$  without changing (2.15) and (2.16); however, if we look for regular enough ( $H^1$ ) solutions in  $S$ , the two constants are no longer independent (see equation (3.4) below). As we will discuss later, the value of the undetermined constant can be related to the circulation of the velocity field generated by  $\psi$ .

### §3. VARIATIONAL FORMULATION FOR FINITE ENERGY SOLUTIONS.

We are concerned with unique solvability of problem  $P_\nu$  for any value of the positive parameter  $\nu$  (equivalently, for any velocity of the unperturbed flux at upstream infinity). As it is known, a critical value of this parameter is  $\nu = 1/H$ ; in fact, when  $\nu < 1/H$  (supercritical flow) it can be proved that the problem is uniquely solvable and that every solution is exponentially vanishing at infinity (this corresponds to the case of a solitary wave in the water wave problem). This result relies on a variational formulation of the problem, which can also be applied to other types of obstacles in a supercritical stream [10]; on the contrary, there are no results on the resolvability of the problem for  $\nu > 1/H$  (subcritical flow). In fact, the standard methods mentioned in the introduction, which (partially) solve the problem in the case of smooth obstacles, are not suitable for a beam; thus, it seems reasonable to consider the extension of the variational approach to the subcritical flow. In this case, however, the perturbed flow will not vanish (in general) at infinity downstream, since steady periodic water waves are known to exist in the subcritical regime; therefore, the solution will not belong to a functional space with finite Dirichlet norm. Nevertheless, as it was shown in the case of

semi-submerged obstacles [8], one can search *weak solutions of finite energy* belonging to some subspace of the Sobolev space  $H^1(S)$ ; subsequently, by a suitable regularization procedure, one uniquely associates to a given variational solution a solution of problem  $P_\nu$ . In this section, we introduce the variational formulation of the problem and discuss its solvability on suitable subsets of  $H^1(S)$ . As we will see, the value  $\nu = 1/b$  (where  $b$  is the depth of the beam) is also critical in the proof of solvability.

To begin with, we take into account condition (2.12) by defining the subspace of the functions with vanishing trace on  $B$ :

$$H_B^1(S) := \{\psi \in H^1(S), \psi|_B = 0\}. \quad (3.1)$$

We now show that the functions in  $H_B^1(S)$  satisfy the Poincarè inequality; hence, the Sobolev norm on  $H_B^1$  is equivalent to the Dirichlet norm :

$$\|\psi\|^2 = \int_S |\nabla\psi|^2 dx dy. \quad (3.2)$$

**Proposition 3.1** *For any  $\psi \in H_B^1(S)$  there holds*

$$\int_S |\psi|^2 dx dy \leq C \int_S |\nabla\psi|^2 dx dy, \quad (3.3)$$

for some positive constant  $C$ .

**Proof:** Fix  $R > x_0$  and consider the restrictions of  $\psi$  to

$$S_R = S \cap \{(x, y), |x| < R\}$$

and to  $S \setminus S_R$ . In the latter (unbounded) region the bound

$$\int_{S \setminus S_R} |\psi|^2 dx dy \leq C \int_{S \setminus S_R} |\nabla\psi|^2 dx dy$$

is directly verified by elementary estimates on smooth functions vanishing at the edge of a strip and by standard density arguments; moreover, we can divide  $S_R$  into two *bounded and Lipschitz* domains (e.g., the rectangle  $R_0 = (-x_0, x_0) \times (-H, -b)$  and its complement in  $S_R$ ) such that  $\psi$  vanishes on some part of their boundaries with positive one-dimensional measure. Since the Poincaré inequality holds for such functions and domains [14, ch.4 §7], the bound (3.3) follows. •

In order to properly define a weak formulation of the problem in  $H_B^1(S)$ , we must take into account the trace properties of  $H^1$  functions in a domain

with a cut. In fact, if we look for solutions in  $H^1(S)$  satisfying the boundary condition (2.10), it is necessary to assume  $\mathcal{K}_\pm \in H^{1/2}(I)$ , together with certain *compatibility conditions* for the data in the neighborhood of the end points of the beam  $(\pm x_0, -b)$  [15]; in the case of continuous data  $\mathcal{K}_\pm$ , these conditions simply become

$$\mathcal{K}_+(\pm x_0) = \mathcal{K}_-(\pm x_0), \quad (3.4)$$

which obviously imply the previous zero flux condition (2.16). Note further that assuming the no-flow data (2.5) in (2.15) we can choose  $\mathcal{K}_\pm = cf_\pm + \gamma$  in (2.10), where  $\gamma$  is an arbitrary constant. For, in that case the compatibility conditions become  $f_+(x_0) = f_-(x_0)$ ,  $f_+(-x_0) = f_-(-x_0)$ ; as previously remarked, this simply means that boundary of the obstacle (before shrinking it to a beam, see the previous section) is a closed curve.

We can now state the *variational form of problem  $P_\nu$*  :

Find  $\psi \in H_B^1(S)$  satisfying  $\psi_\pm = \mathcal{K}_\pm$  on the two sides of  $I$  and such that

$$\int_S \nabla \psi \nabla v \, dx dy - \nu \int_F \psi v \, dx = 0, \quad (3.5)$$

for every  $v \in H_B^1(S)$ , with  $v_\pm = 0$ .

By our previous assumptions on the boundary data, there exists  $\psi_0 \in H_B^1$  satisfying the boundary conditions (2.10); then, we can write  $\psi = \psi_1 + \psi_0$ , where  $\psi_1$  vanishes on  $I$  and satisfies

$$\int_S \nabla \psi_1 \nabla v \, dx dy - \nu \int_F \psi_1 v \, dx = - \int_S \nabla \psi_0 \nabla v \, dx dy + \nu \int_F \psi_0 v \, dx, \quad (3.6)$$

for every  $v \in H_B^1(S)$ , with  $v_\pm = 0$ . It is readily verified that the bilinear form and the linear functional in equation (3.6) are continuous on  $H_B^1(S)$  for every value of the parameter  $\nu$ ; on the contrary, *coercivity depends critically on  $\nu$* . To see this in detail, let us define as before

$$S_{x_0} = S \cap \{(x, y), |x| < x_0\},$$

and let  $Q_0 = (-x_0, x_0) \times (-b, 0)$ ; note that  $S_{x_0} = R_0 \cup Q_0$ , where  $R_0$  was defined in the proof of proposition 3.1.

Then, by Hölder inequality, it readily verified that every  $v \in H_B^1$  satisfies the following bound:

$$\nu \int_{F \setminus [-x_0, x_0]} |v|^2 dx \leq \nu H \int_{|x| > x_0} dx \int_{-H}^0 dy |v_y(x, y)|^2$$

$$\leq \nu H \int_{S \setminus S_{x_0}} |\nabla v|^2 dx dy,$$

that is

$$\int_{S \setminus S_{x_0}} |\nabla v|^2 dx dy - \nu \int_{F \setminus [-x_0, x_0]} |v|^2 dx \geq (1 - \nu H) \int_{S \setminus S_{x_0}} |\nabla v|^2 dx dy. \quad (3.7)$$

Similarly, if  $v$  vanishes on the upper side of  $I$ :

$$\int_{Q_0} |\nabla v|^2 dx dy - \nu \int_{-x_0}^{x_0} |v|^2 dx \geq (1 - \nu b) \int_{Q_0} |\nabla v|^2 dx dy. \quad (3.8)$$

By the previous estimates, we are led to consider three disjoint intervals of values for the parameter  $\nu$ :  $0 < \nu < 1/H$ ,  $1/H < \nu < 1/b$  and  $\nu > 1/b$ .

Actually, if  $\nu$  belongs to the first interval, we get  $1 - \nu b > 1 - \nu H > 0$  in (3.7), (3.8), so that we can use both the estimates to achieve coercivity of the bilinear form (3.6). In the second interval we have  $1 - \nu H < 0$ , so that we loose the coercivity estimate in  $S \setminus S_{x_0}$ ; finally, in the last interval both the coefficients at the right hand sides of (3.7), (3.8) become negative.

As previously remarked, when  $\nu < 1/H$  one easily gets (via the Lax-Milgram theorem) a unique variational solution  $\psi \in H_B^1(S)$ ; then, by standard methods, one shows that  $\psi$  is harmonic in  $S$ , smooth outside any neighborhood of the beam and exponentially vanishing at infinity [11]. We will now focus on the remaining two cases.

#### §4. WEAK SOLUTIONS FOR $1/H < \nu < 1/b$ .

We first remark that for every  $\nu > 1/H$  there are two independent solutions of the "free problem", i.e.,  $\Delta \psi = 0$  in the strip  $\Sigma$  defined in section 2,  $\psi = 0$  on  $B$  and  $\psi_y = \nu \psi$  on  $F$ ; they are given by

$$\begin{aligned} S_0(x, y) &= \sin(\nu_0 x) \sinh(\nu_0(y + H)); \\ C_0(x, y) &= \cos(\nu_0 x) \sinh(\nu_0(y + H)), \end{aligned} \quad (4.1)$$

where  $\nu_0$  is the positive solution of

$$\tanh(\nu_0 H) = \frac{\nu_0}{\nu}. \quad (4.2)$$

It is convenient to define, for a fixed  $\xi \in \mathbb{R}$ , two linear combinations of these functions, namely

$$\begin{aligned} \Phi_0(x, y; \xi) &= \cos(\nu_0 \xi) S_0(x, y) + \sin(\nu_0 \xi) C_0(x, y); \\ \Psi_0(x, y; \xi) &= \cos(\nu_0 \xi) S_0(x, y) - \sin(\nu_0 \xi) C_0(x, y). \end{aligned} \quad (4.3)$$

Let us fix a point  $(\xi, 0) \in F$ , with  $|\xi| \geq x_0$  and let  $\psi$  be a function harmonic in  $S \setminus \overline{S}_{x_0}$ , satisfying the conditions (2.11), (2.12) and vanishing at infinity; by applying Green's theorem to  $\psi$  and to the second of (4.3) we get

$$\int_{\partial \mathcal{Q}_\xi} \left( \psi \partial_n \Psi_0 - \Psi_0 \partial_n \psi \right) = 0,$$

where  $\mathcal{Q}_\xi$  denotes the half strip  $(\xi, +\infty) \times (-H, 0)$  if  $\xi > x_0$  and the half strip  $(-\infty, \xi) \times (-H, 0)$  if  $\xi < -x_0$ .

Developing the calculations (see also [8] §2.1) we get the formula

$$\int_{-H}^0 \sinh[\nu_0(y + H)] \psi(\xi, y) dy = 0, \quad \forall |\xi| \geq x_0. \quad (4.4)$$

By integration by parts we have

$$\cosh(\nu_0 H) \psi(x, 0) = \int_{-H}^0 \cosh \nu_0(y + H) \psi_y(\xi, y) dy. \quad (4.5)$$

By squaring both members of (4.5), integrating on  $F$  and using Cauchy-Schwarz inequality taking account of (4.2), we get

$$\begin{aligned} & \nu \int_{F \setminus [-x_0, x_0]} |\psi|^2 \\ & \leq \frac{\nu}{\cosh^2(\nu_0 H)} \int_{\mathbb{R} \setminus [-x_0, x_0]} \left\{ \int_{-H}^0 \cosh^2(\nu_0(y + H)) dy \int_{-H}^0 |\psi_y(x, y)|^2 dy \right\} \\ & \leq \frac{1}{2} \left( 1 + \frac{2\nu_0 H}{\sinh(2\nu_0 H)} \right) \|\nabla \psi\|_{L^2(S \setminus S_{x_0})}^2. \end{aligned} \quad (4.6)$$

Notice that the constant appearing at the right hand side is *strictly less than 1* for every  $\nu_0 > 0$ .

Formula (4.6) and the form (3.6) of the variational problem suggest the definition of the following linear closed subspace of  $H_B^1(S)$ :

$$U_* = \left\{ \psi \in H_B^1(S) : \psi|_I = 0, \int_{-H}^0 \sinh[\nu_0(y + H)] \psi(x, y) dy = 0 \quad \forall |x| \geq x_0. \right\} \quad (4.7)$$

Now, the bound (4.6) can be exploited to obtain a *new coercivity estimate* on  $U_*$  which replaces (3.7). Then, one can prove the following

**Theorem 4.1 :** *Let  $1/H < \nu < 1/b$  and let  $\mathcal{K}_\pm \in H^{1/2}(I)$  be continuous functions satisfying (3.4). Then, there is  $\psi \in H_B^1(S)$  which satisfies the boundary conditions (2.10), (2.12) (in the sense of the traces of  $H^1$  functions) and such that (3.5) holds  $\forall v \in U_*$ . Moreover,  $\psi$  is uniquely determined if one requires the additional condition (4.4).*

**Proof :** We write  $\psi = \psi_1 + \psi_0$  as before equation (3.6), where  $\psi_0 \in H_B^1$  satisfies the boundary conditions (2.10) and  $\psi_1|_I = 0$ . We first show that  $\psi_0$  can be chosen to satisfy (4.4); actually, for every  $\psi_0$  we may define

$$\tilde{\psi}_0(x, y) = \psi_0(x, y) - \int_{-H}^0 \psi_0(x, s) \sinh[\nu_0(s + H)] \chi_0(y) dy,$$

where  $\chi_0$  is a smooth function such that  $\text{supp } \chi_0 \subset (-H, -b) \cup (-b, 0)$  and

$$\int \sinh[\nu_0(y + H)] \chi_0(y) dy = 1.$$

Then,  $\tilde{\psi}_0$  belongs to  $H_B^1(S)$ , satisfies (4.4) (for every  $\xi \in \mathbb{R}$ ) and the same boundary conditions as  $\psi_0$  (in the sequel, we still denote by  $\psi_0$  such a function).

The problem is now reduced to find  $\psi_1 \in U_*$  satisfying (3.6) for every  $v \in U_*$ . We claim that the bilinear form at the left hand side of (3.6) is coercive on  $U_*$ ; in fact, every  $v \in U_*$  satisfies (4.6), so that, by (3.8) and by the definition (3.2) of the norm, we get

$$\begin{aligned} & \int_S |\nabla v|^2 dx dy - \nu \int_F |v|^2 dx = \\ &= \int_{S \setminus S_{x_0}} |\nabla v|^2 dx dy - \nu \int_{F \setminus [-x_0, x_0]} |v|^2 dx + \int_{S_{x_0}} |\nabla v|^2 dx dy - \nu \int_{-x_0}^{x_0} |v|^2 dx \\ &\geq \frac{1}{2} \left( 1 - \frac{2\nu_0 H}{\sinh(2\nu_0 H)} \right) \|\nabla v\|_{L^2(S \setminus S_{x_0})}^2 + (1 - \nu b) \|\nabla v\|_{L^2(S_{x_0})}^2 \geq \alpha \|v\|^2, \quad (4.8) \end{aligned}$$

where

$$\alpha = \min \left\{ 1 - \nu b, \frac{1}{2} \left( 1 - \frac{2\nu_0 H}{\sinh(2\nu_0 H)} \right) \right\} > 0.$$

Hence, our claim follows. Then, (3.6) has the unique solution  $\psi_1 \in U_*$ , so that  $\psi = \psi_1 + \psi_0$  satisfies (3.5) (for every  $v \in U_*$ ), (4.4) and the boundary conditions. It remains to prove uniqueness; if  $\hat{\psi}$  is another solution satisfying (4.4) and the same boundary conditions as  $\psi$ , it follows that  $\psi - \hat{\psi}$  is a solution in  $U_*$  of the variational equation (3.5), so that  $\psi - \hat{\psi} = 0$  by coercivity. •

§5. WEAK SOLUTIONS FOR  $\nu > 1/b$ .

In this case, we have to introduce further restrictions on the test functions in the variational equation in order to achieve an additional coercivity estimate for  $|x| \leq x_0$  similar to (4.8). To this aim, we proceed as in the previous section and consider an *a priori relation* valid for any harmonic function  $\psi$  in the rectangle  $Q_0 = (-x_0, x_0) \times (-b, 0)$ , which satisfies the first of (2.10) and (2.11).

Let us now define the functions

$$\begin{aligned} S_1(x, y) &= \sin(\nu_1 x) \sinh(\nu_1(y + b)); \\ C_1(x, y) &= \cos(\nu_1 x) \sinh(\nu_1(y + b)), \end{aligned} \quad (5.1)$$

where  $\nu_1$  is the positive solution of

$$\tanh(\nu_1 b) = \frac{\nu_1}{\nu}. \quad (5.2)$$

As before, we define the two linear combinations

$$\begin{aligned} \Phi_1(x, y; \xi) &= \cos(\nu_1 \xi) S_1(x, y) + \sin(\nu_1 \xi) C_1(x, y); \\ \Psi_1(x, y; \xi) &= \cos(\nu_1 \xi) S_1(x, y) - \sin(\nu_1 \xi) C_1(x, y). \end{aligned} \quad (5.3)$$

Then, for any fixed point  $(\xi, 0) \in F$ , with  $|\xi| \leq x_0$  we apply Green's theorem to  $\psi$  and  $\Psi_1$  in the rectangle  $(\xi, x_0) \times (-b, 0)$ ; by elementary calculations we get (see also [9], Lemma 2.3)

$$\int_{-b}^0 \psi(\xi, y) \sinh[\nu_1(y + b)] dy = \mathcal{H}(\xi), \quad (5.4)$$

where

$$\mathcal{H}(\xi) = \int_{\xi}^{x_0} \mathcal{K}_+(x) \sin[\nu_1(\xi - x)] dx + \alpha \cos(\nu_1 \xi) + \beta \sin(\nu_1 \xi) \quad (5.5)$$

and  $\alpha, \beta$  are suitable constants.

Now, for any given  $\mathcal{H} \in H^1(-x_0, x_0)$  we define the subset

$$\mathcal{W}_{\mathcal{H}} \subset H_B^1$$

of the functions  $\psi$  satisfying (5.4) for every  $\xi \in [-x_0, x_0]$  and (4.4) for  $|\xi| \geq x_0$ . Note that, being the intersection of closed hyperplanes in  $H_B^1$ ,  $\mathcal{W}_{\mathcal{H}}$  is a

closed and convex subset. If  $\mathcal{H} = 0$ , the homogeneous conditions (5.4) and (4.4) define a subspace  $\mathcal{W}_0$ ; we further define the subspace

$$V_* = \{v \in \mathcal{W}_0, v|_I = 0\}. \quad (5.6)$$

Clearly, we have  $V_* \subset U_*$  (see (4.7)). We can now state

**Theorem 5.1 :** *Let  $\nu > 1/b$  and let  $\mathcal{K}_\pm \in H^{1/2}(I)$  be continuous functions satisfying (3.4). Then, for every given  $\mathcal{H} \in H^1(-x_0, x_0)$ , there is a unique  $\psi \in \mathcal{W}_\mathcal{H}$  satisfying the boundary conditions (2.10), (2.12) (in the sense of the traces of  $H^1$  functions) and such that (3.5) holds  $\forall v \in V_*$ .*

**Remark :** We stress that, in order to uniquely determine the weak solution  $\psi$  of the previous theorem, we need a supplementary condition specified by the function  $\mathcal{H}$  in (5.4); clearly, if we require that  $\psi$  is harmonic in  $Q_0$ , it is necessary to choose  $\mathcal{H}$  as in (5.5). However, even in this case we are left with two undetermined constants  $\alpha, \beta$ . In particular, we have a two-dimensional subspace of weak solutions of the problem with *homogenous boundary data* (see below).

**Proof of theorem 5.1 :** We write again  $\psi = \psi_1 + \psi_0$ , where  $\psi_0 \in H_B^1$  satisfies the boundary conditions (2.10) and  $\psi_1|_I = 0$ . We now show that we can choose  $\psi_0 \in \mathcal{W}_\mathcal{H}$ . Let  $\chi_0, \chi_1$ , be smooth functions with  $\text{supp}\chi_0 \in (-H, -b)$ ,  $\text{supp}\chi_1 \in (-b, 0)$  and such that

$$\int \sinh[\nu_0(y + H)]\chi_0(y)dy = \int \sinh[\nu_1(y + b)]\chi_1(y)dy = 1,$$

$$\int \sinh[\nu_0(y + H)]\chi_1(y)dy = 0,$$

where  $\nu_0, \nu_1$  satisfy respectively (4.2) and (5.2); moreover, let  $\tilde{\mathcal{H}} \in H^1(\mathbb{R})$  be any extension of  $\mathcal{H}$ . Then, for every  $\psi_0 \in H_B^1$  we define

$$\tilde{\psi}_0(x, y) = \psi_0(x, y) - \left[ \int_{-H}^0 \psi_0(x, s) \sinh[\nu_0(s + H)]ds \right] \chi_0(y)$$

$$+ \left[ \tilde{\mathcal{H}}(x) - \int_{-b}^0 \psi_0(x, s) \sinh[\nu_1(s + b)]ds \right] \chi_1(y).$$

It is readily verified that  $\tilde{\psi}_0$  belongs to  $\mathcal{W}_\mathcal{H}$  and satisfies the same boundary conditions as  $\psi_0$  (again, we redefine  $\tilde{\psi}_0 \rightarrow \psi_0$ ).

We now have to find  $\psi_1 \in V_*$  satisfying (3.6) for every  $v \in V_*$ . By recalling the discussion of the previous section, it is straightforward to check the coercivity

of the left hand side of (3.6) in  $V_*$ ; actually, by the same calculations that led from (4.4) to (4.5), we also get

$$\cosh(\nu_1 b)v(x, 0) = \int_{-b}^0 \cosh[\nu_1(y + b)]v_y(x, y)dy,$$

for  $|x| \leq x_0$ . By squaring both members, integrating on  $(-x_0, x_0)$  and using Cauchy-Schwarz inequality we obtain

$$\nu \int_{-x_0}^{x_0} |v|^2 dx \leq \frac{1}{2} \left(1 + \frac{2\nu_1 b}{\sinh(2\nu_1 b)}\right) \|\nabla v\|_{L^2(Q_0)}^2,$$

for every  $v \in V_*$ . By this bound, by (4.6) and by splitting the integrals as in the second line of (4.8) we now find

$$\begin{aligned} & \int_S |\nabla v|^2 dx dy - \nu \int_F |v|^2 dx \geq \\ & \geq \frac{1}{2} \left(1 - \frac{2\nu_0 H}{\sinh(2\nu_0 H)}\right) \|\nabla v\|_{L^2(S \setminus S_{x_0})}^2 + \frac{1}{2} \left(1 - \frac{2\nu_1 H_1}{\sinh(2\nu_1 b)}\right) \|\nabla v\|_{L^2(S_{x_0})}^2 \end{aligned} \quad (5.7)$$

and we conclude that coercivity holds with the constant

$$\alpha = \frac{1}{2} \left(1 - \frac{2\nu_1 b}{\sinh(2\nu_1 H_1)}\right) > 0.$$

(Note that, by (4.2) and (5.2) we have  $\nu_0 H > \nu_1 b$ ). Then, (3.6) is uniquely solvable in  $V_*$  and  $\psi = \psi_1 + \psi_0 \in \mathcal{W}_{\mathcal{H}}$  satisfies (3.5) for every  $v \in V_*$  and the boundary conditions. Uniqueness follows as in theorem 4.8. •

## §6. PROPERTIES OF THE VARIATIONAL SOLUTIONS.

We now investigate the properties of the solutions given by theorems 4.1 and 5.1 of the previous sections. As we will see, a weak solution is represented by a function which (in general) is not harmonic in the domain  $S$ ; this is not surprising, since we already stressed that for  $\nu > 1/H$  a solution of problem  $P_\nu$  does not vanish at downstream infinity. The subsequent theorems characterize the variational solutions in the two regimes  $1/H < \nu < 1/b$  and  $\nu > 1/b$ .

**Theorem 6.1:** *Assume that  $1/H < \nu < 1/b$  and let  $\psi \in H_B^1(S)$  be defined by theorem 4.1. Then,  $\psi$  satisfies the boundary conditions of problem  $P_\nu$  and there exist real constants  $\lambda_+$ ,  $\lambda_-$  such that*

$$\Delta\psi(x, y) = [\lambda_+\delta(x - x_0) + \lambda_-\delta(x + x_0)] \sinh[\nu_0(y + H)], \quad (6.1)$$

for  $(x, y) \in S$ , where  $\delta$  denotes the Dirac delta function. Moreover, if  $\hat{\psi} \in H_B^1(S)$  solves (6.1) with (possibly different) constants  $\hat{\lambda}_\pm$  and satisfies the same boundary conditions as  $\psi$ , then  $\hat{\psi} = \psi$  (and therefore  $\hat{\lambda}_\pm = \lambda_\pm$ ).

**Proof.** Let  $\varphi_0$  be a smooth function with support in  $S$  and such that

$$\int_{-H}^0 \varphi_0(\pm x_0, y) \sinh[\nu_0(y + H)] dy = 0. \quad (6.2)$$

Furthermore, we set

$$\phi_0(x) = c_0^{-1} \int_{-H}^0 \varphi_0(x, y) \sinh[\nu_0(y + H)] dy, \quad (6.3)$$

where

$$c_0 = \int_{-H}^0 \sinh^2[\nu_0(y + H)] dy = \frac{H}{2} \left( \frac{\sinh(2\nu_0 H)}{2\nu_0 H} - 1 \right).$$

Define now

$$v(x, y) = \begin{cases} \varphi_0(x, y), & \text{if } |x| \leq x_0, \\ \varphi_0(x, y) - \phi_0(x) \sinh[\nu_0(y + H)], & \text{if } |x| \geq x_0. \end{cases} \quad (6.4)$$

Then,  $v \in U_*$  (see (4.7)) and by inserting it in the variational equation (3.5) we find, after suitable integrations by parts (see theorem 6.2 below)

$$\int_S \nabla \psi \nabla \varphi_0 \, dx dy = 0, \quad (6.5)$$

By (6.5) we have that  $\psi$  is (weakly) harmonic for  $x \neq \pm x_0$  in  $S$ ; then, equation (6.1) is obtained by splitting the test functions space as a direct sum of the subspace of the functions satisfying (6.2) with its (two-dimensional) supplementary subspace (see also theorem 6.2 below). The boundary conditions now follow by the same arguments used in [9, theorem 3.1]. Finally, it can be checked that  $\hat{\psi} - \psi \in U_*$  and satisfies the variational equation (3.5) in  $U_*$ ; then, we get  $\hat{\psi} = \psi$  by coercivity.

**Theorem 6.2:** Assume that  $\nu > 1/b$  and let  $\psi \in \mathcal{W}_{\mathcal{H}}$  be defined by theorem 5.1, where  $\mathcal{H}$  is given by (5.5) with arbitrary constants  $\alpha, \beta$ . Then,  $\psi$  satisfies the boundary conditions of problem  $P_\nu$  and there exist real constants  $\lambda_+, \lambda_-, \mu_+, \mu_-$  such that

$$\begin{aligned} \Delta \psi(x, y) &= [\lambda_+ \delta(x - x_0) + \lambda_- \delta(x + x_0)] \sinh[\nu_0(y + H)] \\ &+ [\mu_+ \delta(x - x_0) + \mu_- \delta(x + x_0)] \sinh[\nu_1(y + b)] \chi_{[-b, 0]}(y), \end{aligned} \quad (6.6)$$

for  $(x, y) \in S$ , where  $\chi_{[-b,0]}(y)$  denotes the characteristic function of the interval  $[-b, 0]$ . Moreover, if  $\hat{\psi}$  solves (6.6) with (possibly different) constants  $\hat{\lambda}_{\pm}$ ,  $\hat{\mu}_{\pm}$  and satisfies the same boundary conditions and the same additional condition (5.4) as  $\psi$ , then  $\hat{\psi} = \psi$  (and therefore  $\hat{\lambda}_{\pm} = \lambda_{\pm}$ ,  $\hat{\mu}_{\pm} = \mu_{\pm}$ ).

**Proof.** We provide a detailed proof of equation (6.6); then, the checking of the boundary conditions and of uniqueness will follow as in theorem 6.1.

Let us now denote by  $\varphi_0$  a smooth function with support in  $S$ , satisfying (6.2) and the additional conditions

$$\int_{-b}^0 \varphi_0(\pm x_0, y) \sinh[\nu_1(y+b)] dy = 0. \quad (6.7)$$

We further set

$$\phi_1(x) = c_1^{-1} \int_{-b}^0 \varphi_0(x, y) \sinh[\nu_1(y+b)] dy, \quad (6.8)$$

where

$$c_1 = \int_{-b}^0 \sinh^2[\nu_1(y+b)] dy = \frac{b}{2} \left( \frac{\sinh(2\nu_1 b)}{2\nu_1 b} - 1 \right)$$

and define

$$v(x, y) = \begin{cases} \varphi_0(x, y) - \phi_1(x) \sinh[\nu_1(y+b)] \chi_{[-b,0]}(y), & \text{if } |x| \leq x_0, \\ \varphi_0(x, y) - \phi_0(x) \sinh[\nu_0(y+H)], & \text{if } |x| \geq x_0. \end{cases} \quad (6.9)$$

We readily check that  $v \in V_*$  (see (5.6)); by inserting (6.9) in the variational equation (3.5) we find,

$$\begin{aligned} & \int_S \nabla \psi \nabla \varphi_0 \, dx dy = \\ & \int_{-x_0}^{x_0} \int_{-b}^0 \left\{ \psi_x(x, y) \phi_1'(x) \sinh[\nu_1(y+b)] + \psi_y(x, y) \phi_1(x) \nu_1 \cosh[\nu_1(y+b)] \right\} dx dy \\ & + \int_{|x| \geq x_0} \int_{-H}^0 \left\{ \psi_x(x, y) \phi_0'(x) \sinh[\nu_0(y+H)] \right. \\ & \quad \left. + \psi_y(x, y) \phi_0(x) \nu_0 \cosh[\nu_0(y+H)] \right\} dx dy \\ & - \nu \sinh(\nu_1 b) \int_{-x_0}^{x_0} \psi(x, 0) \phi_1(x) dx - \nu \sinh(\nu_0 H) \int_{|x| \geq x_0} \psi(x, 0) \phi_0(x) dx. \end{aligned}$$

Furthermore, by recalling (4.4), (5.4) and (5.5) and integrating by parts the terms with the derivative  $\psi_y$  taking account of the boundary conditions, we get

$$\int_S \nabla \psi \nabla \varphi_0 \, dx dy = \int_{-x_0}^{x_0} \{ \phi_1'(x) \mathcal{H}'(x) - \phi_1(x) \nu_1 [(\mathcal{K}_+(x) + \nu_1 \mathcal{H}(x))] \} dx,$$

where we also used (4.2) and (5.2) to cancel the integrals on the upper boundary. Finally, by recalling that  $\phi_1(\pm x_0) = 0$  we have

$$\int_S \nabla \psi \nabla \varphi_0 \, dx dy = - \int_{-x_0}^{x_0} \{ \phi_0(x) [\mathcal{H}''(x) + \nu_1^2 \mathcal{H}(x) + \nu_1 \mathcal{K}_+(x)] \} dx$$

By (5.5) we readily verify that  $\mathcal{H}''(x) + \nu_1^2 \mathcal{H}(x) = -\nu_1 \mathcal{K}_+(x)$ ; hence, we still obtain (6.5) for every  $\varphi_0$  satisfying (6.2) and (6.7). In order to prove (6.6) let us fix two pair of test functions  $\eta_i^\pm$ ,  $i = 0, 1$  such that

$$\text{supp } \eta_i^+ \subset \{(x, y) \in S, x > -x_0\}, \quad \text{supp } \eta_i^- \subset \{(x, y) \in S, x < x_0\}$$

and satisfying

$$\int_{-H}^0 \eta_i^\pm(\pm x_0, y) \sinh[\nu_0(y + H)] dy = \delta_{i,0},$$

$$\int_{-b}^0 \eta_i^\pm(\pm x_0, y) \sinh[\nu_1(y + b)] dy = \delta_{i,1}.$$

Furthermore, for any  $\varphi \in \mathcal{D}(S)$  we define

$$\gamma_0^\pm = \int_{-H}^0 \varphi(\pm x_0, y) \sinh[\nu_0(y + H)] dy,$$

$$\gamma_1^\pm = \int_{-b}^0 \varphi(\pm x_0, y) \sinh[\nu_1(y + b)] dy.$$

Then, the function

$$\varphi_0(x, y) = \varphi(x, y) - \sum_{i=0}^1 \{ \gamma_i^+ \eta_i^+(x, y) + \gamma_i^- \eta_i^-(x, y) \}$$

satisfies (6.2) and (6.7) so that, by (6.5),

$$\int_S \nabla \psi \nabla \varphi \, dx dy = \sum_{i=0}^1 \left\{ \gamma_i^+ \int_S \nabla \psi \nabla \eta_i^+(x, y) + \gamma_i^- \int_S \nabla \psi \nabla \eta_i^-(x, y) \right\}. \quad (6.10)$$

Thus, by the definition of  $\gamma_i^\pm$  we obtain (6.6) with

$$\lambda_\pm = \int_S \nabla \psi \nabla \eta_0^\pm, \quad \mu_\pm = \int_S \nabla \psi \nabla \eta_1^\pm.$$

## §7. REGULARIZATION AND SOLVABILITY OF THE PROBLEM.

As we clarified in the previous section, a variational solution is not in general a solution of problem  $P_\nu$ , since it is not harmonic in  $S$ . Roughly speaking, this is because the weak solution does not include the "oscillating parts" of the stream function which appear downstream the beam for  $\nu > 1/H$  (with wave number given by (4.2)) and also between the beam and the free surface when  $\nu > 1/b$  (with wave number given by (5.2)). In this section, we "regularize" the weak solutions described in theorems 6.1 and 6.2 (respectively, in the two regimes  $1/H < \nu < 1/b$  and  $\nu > 1/b$ ) by adding suitable oscillating terms satisfying the *homogeneous boundary conditions* (2.10)-(2.12) in such a way to obtain (unique) solutions of problem  $P_\nu$ .

We first outline the case  $1/H < \nu < 1/b$ , as its regularization is very similar to the existing ones for a surface beam [4], [11] and for a surface-piercing body [8]. The first step is to modify the weak solution  $\psi$  in order to obtain a function harmonic through  $S$ : let us set  $\tilde{\Phi}_0(x, y) = \Phi_0(x, y; x_0)$  and  $\tilde{\Psi}_0(x, y) = \Psi_0(x, y; x_0)$ , where  $\Phi_0, \Psi_0$ , are defined by (4.3). By elementary calculations, the function

$$\tilde{\psi} = \begin{cases} \psi + \frac{\lambda_-}{\nu_0} \tilde{\Phi}_0 & \text{if } x < -x_0, \\ \psi & \text{if } |x| \leq x_0, \\ \psi - \frac{\lambda_+}{\nu_0} \tilde{\Psi}_0 & \text{if } x > x_0, \end{cases} \quad (7.1)$$

is *harmonic in  $S$*  and satisfies the same boundary conditions as  $\psi$ ; we remark, however, that  $\tilde{\psi}$  is *not* a solution to problem  $P_\nu$  (unless  $\lambda_- = 0$ ) since it oscillates *both* at  $+\infty$  and  $-\infty$  and therefore does not satisfy the asymptotic condition (2.13). In order to get rid of the unwanted waves at  $-\infty$ , we exploit the properties of two variational solutions, denoted by  $\psi^s$  and  $\psi^c$ , satisfying special conditions on the beam:  $\psi^s$  and  $\psi^c$  are the two (uniquely determined) functions given by theorem 4.1, where:

$$\begin{aligned} \text{for } \psi^s \text{ we take } \mathcal{K}_\pm &= \sin(\nu_0 x); \\ \text{for } \psi^c \text{ we take } \mathcal{K}_\pm &= \cos(\nu_0 x). \end{aligned} \quad (7.2)$$

By the symmetry properties of this data and by uniqueness, we have  $\psi^s(-x, y) = -\psi^s(x, y)$  and  $\psi^c(-x, y) = \psi^c(x, y)$ ; as a consequence,  $\psi^s$  and

$\psi^c$  satisfy (6.1) with  $\lambda_+ = -\lambda_- \equiv \lambda^s$  and  $\lambda_+ = \lambda_- \equiv \lambda^c$ , respectively. For special values of the parameter  $\nu_0$ , the functions  $\psi^{s,c}$ , and the coefficients  $\lambda^{s,c}$  are explicitly known; for, by defining

$$\tilde{S}_0(x, y) = \frac{\sinh(\nu_0(y + H))}{\sinh[\nu_0(H - b)]} \sin(\nu_0 x), \quad \tilde{C}_0(x, y) = \frac{\sinh(\nu_0(y + H))}{\sinh[\nu_0(H - b)]} \cos(\nu_0 x),$$

we have, for  $\nu_0 x_0 = n\pi$ ,  $n = 1, 2, \dots$  :

$$\psi^s(x, y) = \tilde{S}_0(x, y) \quad \text{for } |x| \leq x_0, \quad \psi^s = 0 \quad \text{for } |x| > x_0; \quad (7.3)$$

with  $\lambda^s = (-1)^{n+1} \nu_0 / \sinh[\nu_0(H - b)]$ . Similarly, if  $\nu_0 x_0 = (n - 1/2)\pi$ , we have

$$\psi^c(x, y) = \tilde{C}_0(x, y) \quad \text{for } |x| \leq x_0, \quad \psi^c = 0 \quad \text{for } |x| > x_0, \quad (7.4)$$

with  $\lambda^c = (-1)^{n+1} \nu_0 / \sinh[\nu_0(H - b)]$ .

Let us now modify  $\psi^s$  and  $\psi^c$  as in (7.1) with the correspondent coefficients  $\lambda^s, \lambda^c$ , in order to obtain functions  $\tilde{\psi}^{s,c}$  harmonic through  $S$ . Finally, we set

$$\zeta_0^s = \tilde{\psi}^s - \tilde{S}_0, \quad \zeta_0^c = \tilde{\psi}^c - \tilde{C}_0. \quad (7.5)$$

The harmonic functions  $\zeta_0^s$  and  $\zeta_0^c$  satisfy conditions (2.9)-(2.12) of problem  $P_\nu$  with vanishing boundary data; furthermore, they have the same symmetry properties as  $\psi^s$  and  $\psi^c$  respectively, and oscillate at both directions at infinity. Hence, by a suitable combination of them, we may suppress the oscillations at  $-\infty$  of  $\tilde{\psi}$  in (7.1). Let us define

$$\omega = \tilde{\psi} + a\zeta_0^s + b\zeta_0^c \quad (7.6)$$

and choose the coefficients  $a$  and  $b$  in such a way that  $\omega$  satisfies (2.13); then,  $\omega$  will be the required solution of problem  $P_\nu$ . By writing the asymptotic expression of  $\omega$  for  $x \rightarrow -\infty$  (which can be derived from (7.1) and (7.5) taking account that the variational parts,  $\psi$  and  $\psi^{s,c}$ , are exponentially vanishing) and imposing that the coefficients of the oscillatory terms vanish, we get the system

$$\begin{cases} [\lambda^s \cos(\nu_0 x_0) + \frac{\nu_0}{\sinh[\nu_0(H-b)]}] a - \lambda^c \cos(\nu_0 x_0) b = \lambda_- \cos(\nu_0 x_0), \\ \lambda^s \sin(\nu_0 x_0) a - [\lambda^c \sin(\nu_0 x_0) - \frac{\nu_0}{\sinh[\nu_0(H-b)]}] b = \lambda_- \sin(\nu_0 x_0). \end{cases} \quad (7.7)$$

This gives

$$a = \frac{\lambda_-}{\Delta} \cos(\nu_0 x_0); \quad b = \frac{\lambda_-}{\Delta} \sin(\nu_0 x_0), \quad (7.8)$$

where

$$\Delta = \frac{\nu_0}{\sinh[\nu_0(H-b)]} + \lambda^s \cos(\nu_0 x_0) - \lambda^c \sin(\nu_0 x_0). \quad (7.9)$$

With this choice, the asymptotic expression of  $\omega$  for  $x \rightarrow +\infty$  writes

$$\omega = \mathcal{O}(e^{-\mu_1|x|}) + AS_0 + BC_0, \quad (7.10)$$

where  $\mu_1$  is the least positive solution of  $\tan(\mu H) = \frac{\mu}{\nu}$  and characterize the decay at infinity of the variational solution [11]. The coefficients of the oscillating terms can be easily calculated from (7.5), (7.6) and (7.8).

We remark that, in the special case  $\nu_0 x_0 = n\pi$  ( $\nu_0 x_0 = (n-1/2)\pi$ ), the function  $\zeta^s$  ( $\zeta^c$ ) identically vanishes. Moreover, we will show below that

$$\Delta = J_0 \frac{\sinh[\nu_0(H-b)]}{c_0} \sin(\nu_0 x_0) \cos(\nu_0 x_0), \quad (7.11)$$

where  $J_0 > 0$  for every  $\nu_0 > 0$ , i.e. for every  $\nu > 1/H$  (see also [10, proposition 4.8]); nevertheless, the solution (7.6) as well as the coefficients  $A$  and  $B$  have well defined limits for  $\nu_0 x_0 \rightarrow m\frac{\pi}{2}$ . The calculations are the same as in the case of the beam [4, proposition A.2] (see also [8], section 2.2). The conclusion of this discussion is :

**Theorem 7.1:** *For  $1/H < \nu < 1/b$ , there is a unique solution  $\omega$  to problem  $P_\nu$  which, for  $x \rightarrow +\infty$ , satisfies (7.10), with suitably defined  $A$  and  $B$ .*

The assertion of uniqueness follows exactly in the same way as it was done for the surface beam ([4, theorem 3.2]).

We now discuss the case  $\nu > 1/b$ , whose regularization requires a more careful analysis. In fact, if  $\psi$  is a weak solution as in theorem 6.2, we now see that the function  $\tilde{\psi}$  defined by (7.1) is no longer harmonic but it satisfies

$$\Delta \tilde{\psi}(x, y) = [\mu_+ \delta(x - x_0) + \mu_- \delta(x + x_0)] \sinh[\nu_1(y+b)] \chi_{[-b,0]}(y). \quad (7.12)$$

On the other hand, we know that a variational solution is not uniquely determined by the boundary data, but it depends on two arbitrary constants (see (5.4), (5.5)); clearly, the same is true for  $\tilde{\psi}$ . Thus, it is natural to ask whether such constants can be determined by requiring  $\mu_\pm = 0$  in the above equation.

To begin with, we still denote by  $\psi^s$ ,  $\psi^c$  the two weak solutions, given by theorem (5.1), which satisfy (7.2) and the additional condition (5.4) respectively in the form :

$$\begin{aligned} \int_{-b}^0 \psi^s(\xi, y) \sinh[\nu_1(y+b)] dy &= \frac{\nu_1}{\nu_0^2 - \nu_1^2} \sin(\nu_0 \xi), \\ \int_{-b}^0 \psi^c(\xi, y) \sinh[\nu_1(y+b)] dy &= \frac{\nu_1}{\nu_0^2 - \nu_1^2} \cos(\nu_0 \xi), \end{aligned} \quad (7.13)$$

for every  $\xi \in [-x_0, x_0]$ . We remark that the right hand sides of (7.13) are obtained from (5.5) by inserting  $\mathcal{K}_\pm$  given by (7.2) and with a suitable choice of the coefficients  $\alpha, \beta$ . It is readily verified that  $\psi^s$  and  $\psi^c$  maintain the previous symmetry properties with respect to  $x$ ; hence, they satisfy (6.6) respectively with  $\lambda_+ = -\lambda_- \equiv \lambda^s, \mu_+ = -\mu_- \equiv \mu^s$  and  $\lambda_+ = \lambda_- \equiv \lambda^c, \mu_+ = \mu_- \equiv \mu^c$ . Moreover, by direct computations it follows that (7.3), (7.4), still hold respectively for  $\nu_0 x_0 = n\pi$  and  $\nu_0 x_0 = (n - 1/2)\pi$ , with the same values of  $\lambda^{s,c}$  and with  $\mu^{s,c} = 0$ .

By recalling the remark following theorem 5.1, we can now consider *another pair of special solutions of the variational problem*, denoted by  $\chi^s, \chi^c$ , with homogeneous boundary data and such that

$$\begin{aligned} \int_{-b}^0 \chi^s(\xi, y) \sinh[\nu_1(y+b)] dy &= \sin(\nu_1 \xi), \\ \int_{-b}^0 \chi^c(\xi, y) \sinh[\nu_1(y+b)] dy &= \cos(\nu_1 \xi), \end{aligned} \quad (7.14)$$

for every  $\xi \in [-x_0, x_0]$ . The following properties are easily verified:

$\chi^s(-x, y) = -\chi^s(x, y), \chi^c(-x, y) = \chi^c(x, y)$ ; then,  $\chi^s$  and  $\chi^c$  satisfy (6.6) with  $\lambda_+ = -\lambda_- \equiv \xi^s, \mu_+ = -\mu_- \equiv v^s$  and  $\lambda_+ = \lambda_- \equiv \xi^c, \mu_+ = \mu_- \equiv v^c$ , respectively. For special values of the parameter  $\nu_1$ , the functions  $\chi^{s,c}$ , and the coefficients  $\xi^{s,c}, v^{s,c}$  are explicitly known; in fact, by setting

$$\tilde{S}_1(x, y) = \frac{\sinh(\nu_1(y+b))}{c_1} \sin(\nu_1 x), \quad \tilde{C}_1(x, y) = \frac{\sinh(\nu_1(y+b))}{c_1} \cos(\nu_1 x),$$

where the constant  $c_1$  was defined after (6.8), we have for  $\nu_1 x_0 = n\pi, n = 1, 2, \dots$ :

$$\chi^s(x, y) = \tilde{S}_1(x, y) \chi_{[-b,0]}(y) \quad \text{for } |x| \leq x_0, \quad \chi^s = 0 \quad \text{for } |x| > x_0; \quad (7.15)$$

with  $\xi^s = 0$  and  $v^s = (-1)^{n+1} \nu_1 / c_1$ . Similarly, if  $\nu_1 x_0 = (n - 1/2)\pi$ , we have

$$\chi^c(x, y) = \tilde{C}_1(x, y) \chi_{[-b,0]}(y) \quad \text{for } |x| \leq x_0, \quad \chi^c = 0 \quad \text{for } |x| > x_0, \quad (7.16)$$

with  $\xi^c = 0$  and  $v^c = (-1)^{n+1} \nu_1 / c_1$ .

Let us now define as before in (7.1) the functions  $\tilde{\psi}^s, \tilde{\psi}^c, \tilde{\chi}^s, \tilde{\chi}^c$ ; furthermore, let  $\zeta_0^s, \zeta_0^c$  be given as in (7.5) and set:

$$\zeta_1^s = \tilde{\chi}^s; \quad \zeta_1^c = \tilde{\chi}^c.$$

Clearly,  $\zeta_0^s, \zeta_0^c$  and  $\zeta_1^s, \zeta_1^c$  satisfy the homogeneous boundary conditions of problem  $P_\nu$  and the equation (7.12) respectively with  $\mu_\pm = \pm \mu^s, \mu_\pm = \mu^c$

and  $\mu_{\pm} = \pm v^s$ ,  $\mu_{\pm} = v^c$ . Then, if  $\psi$  is a weak solution given by theorem 6.2 and  $\tilde{\psi}$  is defined as in (7.1), we set

$$\omega = \tilde{\psi} + a_0 \zeta_0^s + b_0 \zeta_0^c + a_1 \zeta_1^s + b_1 \zeta_1^c. \quad (7.17)$$

We want to choose the four constants  $a_0, a_1, b_0, b_1$ , in such a way to satisfy the two requirements :

- i)  $\omega$  is harmonic in  $S$ ,
- ii)  $\omega$  satisfy the asymptotic condition (2.13).

In that case, the function  $\omega$  will be a solution to problem  $P_{\nu}$ .

Let us now write explicitly the two conditions i) and ii); to this aim, we define the following coefficients:

$$\begin{aligned} \Lambda_c^s &= \lambda^s \cos(\nu_0 x_0) + \frac{\nu_0}{\sinh[\nu_0(H-b)]}; & \Lambda_c^c &= -\lambda^c \cos(\nu_0 x_0); \\ \Lambda_s^s &= \lambda^s \sin(\nu_0 x_0); & \Lambda_s^c &= -\lambda^c \sin(\nu_0 x_0) + \frac{\nu_0}{\sinh[\nu_0(H-b)]}. \\ \Xi_c^s &= \xi^s \cos(\nu_0 x_0); & \Xi_c^c &= -\xi^c \cos(\nu_0 x_0); \\ \Xi_s^s &= \xi^s \sin(\nu_0 x_0); & \Xi_s^c &= -\xi^c \sin(\nu_0 x_0). \end{aligned} \quad (7.18)$$

Then, by explicit calculations and taking account of the asymptotic expression of  $\omega$  as in (7.7), we get:

$$\begin{cases} \mu^s a_0 + \mu^c b_0 + v^s a_1 + v^c b_1 = -\mu_+ \\ -\mu^s a_0 + \mu^c b_0 - v^s a_1 + v^c b_1 = -\mu_- \\ \Lambda_c^s a_0 + \Lambda_c^c b_0 + \Xi_c^s a_1 + \Xi_c^c b_1 = \lambda_- \cos(\nu_0 x_0), \\ \Lambda_s^s a_0 + \Lambda_s^c b_0 + \Xi_s^s a_1 + \Xi_s^c b_1 = \lambda_- \sin(\nu_0 x_0). \end{cases} \quad (7.19)$$

The system (7.19) is uniquely solvable if and only if

$$\begin{vmatrix} \mu^s & \mu^c & v^s & v^c \\ -\mu^s & \mu^c & -v^s & v^c \\ \Lambda_c^s & \Lambda_c^c & \Xi_c^s & \Xi_c^c \\ \Lambda_s^s & \Lambda_s^c & \Xi_s^s & \Xi_s^c \end{vmatrix} \neq 0.$$

By elementary calculations and by the definitions (7.18), we get

$$\mu^c v^s \xi^c \sin(\nu_0 x_0) - \mu^s v^c \xi^s \cos(\nu_0 x_0) + v^s v^c \Delta \neq 0, \quad (7.20)$$

where  $\Delta$  is given by (7.9). In order to write condition (7.20) in a more explicit form, we exploit some features of the coefficients at the left hand side :

**Proposition 7.2 :** *The following relations hold*

$$\begin{aligned}\mu^s &= -I^s \sin(\nu_0 x_0); & \mu^c &= -I^c \cos(\nu_0 x_0); \\ \xi^s &= I^s \frac{\sinh[\nu_0(H-b)]}{c_0} \sin(\nu_1 x_0); & \xi^c &= I^c \frac{\sinh[\nu_0(H-b)]}{c_0} \cos(\nu_1 x_0); \\ v^s &= -\frac{\nu_1}{c_1} \cos(\nu_1 x_0) - J_b^s \sin(\nu_1 x_0); & v^c &= \frac{\nu_1}{c_1} \sin(\nu_1 x_0) - J_b^c \cos(\nu_1 x_0); \\ \Delta &= J_0 \frac{\sinh[\nu_0(H-b)]}{c_0} \sin(\nu_0 x_0) \cos(\nu_0 x_0),\end{aligned}$$

where  $J_0 = J^s - J^c > 0$ ,  $J_b^s > J_b^c > 0$ ,  $(I^s)^2 \leq J^s J_b^s$ ,  $(I^c)^2 \leq J^c J_b^c$ ; the quantities  $J^s$ ,  $J^c$ ,  $J_b^s$ ,  $J_b^c$  are quadratic functionals respectively of  $\psi^s$ ,  $\psi^c$ ,  $\chi^s$ ,  $\chi^c$ , while  $I^s$ ,  $I^c$  are bilinear forms of the arguments  $\psi^s$ ,  $\chi^s$  and  $\psi^c$ ,  $\chi^c$ . All the above terms depend analytically on  $\nu$ .

The proof is given in appendix; we only remark here that the above expressions are compatible with the values previously calculated from the explicit solutions (7.3), (7.4) and (7.15), (7.16).

By proposition 7.2 and by elementary calculations, we can write (7.20) in the form

$$\sin(\nu_0 x_0) \cos(\nu_0 x_0) [\mathcal{P} \cos^2(\nu_1 x_0) + \mathcal{Q} \sin^2(\nu_1 x_0) + \mathcal{R} \sin(\nu_1 x_0) \cos(\nu_1 x_0)] \neq 0, \quad (7.21)$$

where

$$\begin{aligned}\mathcal{P} &= \frac{\nu_1}{c_1} [(I^c)^2 + J_0 J_b^c], & \mathcal{Q} &= \frac{\nu_1}{c_1} [(I^s)^2 - J_0 J_b^s], \\ \mathcal{R} &= (I^c)^2 J_b^s - (I^s)^2 J_b^c + J_0 \left[ J_b^s J_b^c - \left( \frac{\nu_1}{c_1} \right)^2 \right].\end{aligned}$$

Thus, the system (7.19) is uniquely solvable if  $\nu_0 x_0 \neq k\pi/2$  ( $k \in \mathbb{N}$ ) and whenever the factor in the square brackets of (7.21) does not vanish. Unfortunately, it is not clear how to write the latter condition in an explicit form; moreover, one should check whether the solution (7.17) could also be defined by a suitable limit procedure (as described before theorem 7.1) when  $\nu_0$  ( $\nu_1$ ) approaches the roots of the left hand side of (7.21).

We will not pursue here this analysis, but we observe that the left hand side of (7.21) is an analytic function (of the parameter  $\nu$ ) which is not identically vanishing (take  $\nu$  such that  $\nu_1 x_0 = n\pi$ ); thus, by recalling theorem 7.1 we get:

**Theorem 7.3:** *There is a unique solution  $\omega$  to problem  $P_\nu$  for every value of  $\nu$ , with the possible exception of a sequence of singular values  $\nu^{(n)} \rightarrow +\infty$ , with  $\nu^{(n)} > 1/b$  for every  $n$ . For  $x \rightarrow +\infty$  the solution satisfies (7.10).*

§8. FINAL REMARKS AND OPEN PROBLEMS.

We have studied a plane problem describing the linearized steady flow around a "thin" submerged obstacle in a heavy ideal fluid. We proved unique solvability for every value of the unperturbed flow velocity  $c$  with the possible exception of a discrete set in the interval  $0 < c < \sqrt{gb}$ , where  $b$  is the obstacle's depth. This set must be a subset of the sequence  $\{c_n\}_{n=1}^{\infty}$ ,  $c_n \rightarrow 0$ , where  $\nu^{(n)} = g/c_n^2$  are the zero points of the left hand side of (7.21). We stress that the coefficients in (7.21) only depend on the geometric data of the problem, that is the half width of the beam  $x_0$ , its depth  $b$  and the depth  $H$  of the channel's bottom.

It remains an open question to verify if a given  $c_n$  is a true singular value of the problem or if one can recover unique solvability by a suitable limiting procedure as in the case  $\sqrt{gb} < c < \sqrt{gH}$ . Another interesting (and related) issue is the connection between the solutions of problem  $P_\nu$  and those of the problem (2.1)-(2.7) for the velocity potential (see the remark at the end of section 2). In this context, we note that if  $\psi$  solves  $P_\nu$  with boundary data  $\mathcal{K}_\pm = 1$  in condition (2.10) then the corresponding velocity field satisfies the *homogeneous condition*  $\psi_x = v = 0$  on the beam. We conjecture that this solution represents a kind of vortex flow with non trivial circulation around the beam and therefore is not represented by a potential field; actually, the conjecture is verified for a supercritical flow since in that case the circulation  $\oint_{\mathcal{C}} \{u dx + v dy\}$  through a closed curve  $\mathcal{C}$  surrounding the beam is given by

$$\begin{aligned} - \int_I \{u_+ - u_-\} dx &= \int_I \{(\partial_y \psi)_+ - (\partial_y \psi)_-\} dx \\ &= \int_I \psi \{(\partial_y \psi)_+ - (\partial_y \psi)_-\} dx = \int_S |\nabla \psi|^2 dx dy - \nu \int_F |\psi|^2 dx > 0, \end{aligned}$$

where in the last line we used the boundary conditions and the divergence theorem. The strict positivity of the last term follows by  $\nu H < 1$  and by the Hölder inequality applied to the identity  $\psi(x, 0) = \int_{-H}^0 \psi_y(x, s) ds$ , which holds for every  $x \in \mathbb{R}$  since  $\psi$  is constant along the beam.

In the case  $\nu H > 1$  we cannot apply the previous argument, but we expect that the solution  $\psi$  still has non trivial circulation at least for the regular values of the parameter  $\nu$ ; if not, the (well defined) harmonic conjugate  $\phi$  would be a non trivial solution of the homogeneous problem (2.1)-(2.7). From the above discussion it follows in particular that in order to uniquely determine the velocity field of a (linearized) stationary flow around a submerged body, it is necessary to fix the value of the circulation around the obstacle itself.

APPENDIX.

*Proof of proposition 7.2 :*

By the expression of  $\Delta\psi^{s,c}$ ,  $\Delta\chi^{s,c}$ , we have :

$$\begin{aligned}\psi_x^{s,c}(x_0^+, y) - \psi_x^{s,c}(x_0^-, y) &= \lambda^{s,c} \sinh[\nu_0(y+H)] + \mu^{s,c} \sinh[\nu_1(y+b)]\chi_{[-b,0]}(y); \\ \chi_x^{s,c}(x_0^+, y) - \chi_x^{s,c}(x_0^-, y) &= \xi^{s,c} \sinh[\nu_0(y+H)] + v^{s,c} \sinh[\nu_1(y+b)]\chi_{[-b,0]}(y).\end{aligned}\tag{A.1}$$

Now, the explicit forms of  $\psi^{s,c}$  and of  $\chi^{s,c}$ , respectively for  $\nu_0 x_0 = n\pi/2$  and for  $\nu_1 x_0 = n\pi/2$ ,  $n = 1, 2, \dots$  (see section 7) justify the following definitions:

$$\begin{aligned}\psi^s(x, y) &= \begin{cases} \sin(\nu_0 x_0)\hat{\psi}^s(x, y) + \tilde{S}_0(x, y) & \text{for } |x| \leq x_0, \\ \sin(\nu_0 x_0)\psi^s(x, y) & \text{for } |x| \geq x_0. \end{cases} \\ \psi^c(x, y) &= \begin{cases} \cos(\nu_0 x_0)\hat{\psi}^c(x, y) + \tilde{C}_0(x, y) & \text{for } |x| \leq x_0, \\ \cos(\nu_0 x_0)\psi^c(x, y) & \text{for } |x| \geq x_0. \end{cases} \\ \chi^s(x, y) &= \begin{cases} \sin(\nu_1 x_0)\hat{\chi}^s(x, y) + \tilde{S}_1(x, y)\chi_{[-b,0]}(y) & \text{for } |x| \leq x_0, \\ \sin(\nu_1 x_0)\check{\chi}^s(x, y) & \text{for } |x| \geq x_0. \end{cases} \\ \chi^c(x, y) &= \begin{cases} \cos(\nu_1 x_0)\hat{\chi}^c(x, y) + \tilde{C}_1(x, y)\chi_{[-b,0]}(y) & \text{for } |x| \leq x_0, \\ \cos(\nu_1 x_0)\check{\chi}^c(x, y) & \text{for } |x| \geq x_0, \end{cases}\end{aligned}\tag{A.2}$$

where  $\tilde{S}_0$ ,  $\tilde{C}_0$ ,  $\tilde{S}_1$ ,  $\tilde{C}_1$ , were defined in section 7. Clearly, the functions at the right hand sides of (A.2) are harmonic in  $S_{x_0}$  and in  $S \setminus S_{x_0}$  respectively; moreover, they satisfy homogenous boundary conditions on  $F$ ,  $B$  and  $I$ . Finally,  $\hat{\psi}^{s,c}(x, y)$  and  $\hat{\chi}^{s,c}(x, y)$  verify (4.4), while the functions  $\hat{\psi}^{s,c}(x, y)$ ,  $\hat{\chi}^{s,c}(x, y)$  satisfy the *homogeneous condition* (5.4), that is :

$$\begin{aligned}\int_{-b}^0 \hat{\psi}^{s,c}(x, y) \sinh[\nu_1(y+b)]dy &= 0, \\ \int_{-b}^0 \hat{\chi}^{s,c}(x, y) \sinh[\nu_1(y+b)]dy &= 0,\end{aligned}\tag{A.3}$$

for  $|x| \leq x_0$ . In fact, the first line is obtained by (7.13) and by the identity

$$\int_{-b}^0 \sinh[\nu_0(y+H)] \sinh[\nu_1(y+b)]dy = \sinh[\nu_0(H-b)] \frac{\nu_1}{\nu_0^2 - \nu_1^2},$$

which follows by explicit calculations using (4.2) and (5.2). The second equation follows from (7.14) and by the definition of  $c_1$ .

Then, by (4.4), (7.13) and (7.14), we get the following relations

$$\begin{aligned} \int_{-H}^0 [\psi_x^{s,c}(x_0^+, y) - \psi_x^{s,c}(x_0^-, y)] \hat{\psi}^{s,c}(x_0, y) dy &= -\frac{c_0}{\sinh[\nu_0(H-b)]} \lambda^{s,c}; \\ \int_{-H}^0 [\chi_x^{s,c}(x_0^+, y) - \chi_x^{s,c}(x_0^-, y)] \check{\chi}^{s,c}(x_0, y) dy &= v^{s,c}; \\ \int_{-H}^0 [\psi_x^{s,c}(x_0^+, y) - \psi_x^{s,c}(x_0^-, y)] \check{\chi}^{s,c}(x_0, y) dy &= \mu^{s,c}; \\ \int_{-H}^0 [\chi_x^{s,c}(x_0^+, y) - \chi_x^{s,c}(x_0^-, y)] \hat{\psi}^{s,c}(x_0, y) dy &= -\frac{c_0}{\sinh[\nu_0(H-b)]} \xi^{s,c}; \end{aligned}$$

We now transform the left hand sides taking into account (A.2), (A.3) and integrating by parts:

$$\begin{aligned} &\int_{-H}^0 [\psi_x^s(x_0^+, y) - \psi_x^s(x_0^-, y)] \hat{\psi}^s(x_0, y) dy \\ &= \frac{\nu_0 c_0}{\sinh^2[\nu_0(H-b)]} \cos(\nu_0 x_0) - J^s \sin(\nu_0 x_0), \end{aligned}$$

where

$$\begin{aligned} J^s &= -\left[ \int_{-H}^0 \check{\psi}_x^s(x_0^+, y) \check{\psi}^s(x_0, y) dy - \int_{-H}^0 [\check{\psi}_x^s(x_0^-, y)] \hat{\psi}^s(x_0, y) dy \right] \\ &= \frac{1}{2} \left[ \int_{S \setminus S_{x_0}} |\nabla \check{\psi}^s|^2 dx dy - \nu \int_{|x| \geq x_0} |\check{\psi}^s|^2 dx \right] \\ &\quad + \frac{1}{2} \left[ \int_{S_{x_0}} |\nabla \hat{\psi}^s|^2 dx dy - \nu \int_{-x_0}^{x_0} |\hat{\psi}^s|^2 dx \right] \end{aligned}$$

Similarly

$$\begin{aligned} &\int_{-H}^0 [\psi_x^c(x_0^+, y) - \psi_x^c(x_0^-, y)] \hat{\psi}^c(x_0, y) dy \\ &= -\frac{\nu_0 c_0}{\sinh^2[\nu_0(H-b)]} \sin(\nu_0 x_0) - J^c \cos(\nu_0 x_0), \end{aligned}$$

where

$$J^c = \frac{1}{2} \left[ \int_{S \setminus S_{x_0}} |\nabla \check{\psi}^c|^2 dx dy - \nu \int_{|x| \geq x_0} |\check{\psi}^c|^2 dx \right]$$

$$+\frac{1}{2}\left[\int_{Sx_0}|\nabla\hat{\psi}^c|^2dxdy-\nu\int_{-x_0}^{x_0}|\hat{\psi}^c|^2dx\right]$$

By coercivity and by the Dirichlet principle we have (see [8], Lemma 2.4)

$$J^s > J^c > 0.$$

Moreover,

$$\int_{-H}^0[\chi_x^s(x_0^+,y)-\chi_x^s(x_0^-,y)]\check{\chi}^s(x_0,y)dy=-\frac{\nu_1}{c_1}\cos(\nu_1x_0)-J_b^s\sin(\nu_1x_0)$$

$$\int_{-H}^0[\chi_x^c(x_0^+,y)-\chi_x^c(x_0^-,y)]\check{\chi}^c(x_0,y)dy=\frac{\nu_1}{c_1}\sin(\nu_1x_0)-J_b^c\cos(\nu_1x_0),$$

where

$$J_b^{s,c}=\frac{1}{2}\left[\int_{S\setminus Sx_0}|\nabla\check{\chi}^{s,c}|^2dxdy-\nu\int_{|x|\geq x_0}|\check{\chi}^{s,c}|^2dx\right]$$

$$+\frac{1}{2}\left[\int_{Sx_0}|\nabla\hat{\chi}^{s,c}|^2dxdy-\nu\int_{-x_0}^{x_0}|\hat{\chi}^{s,c}|^2dx\right]$$

We still have the inequalities

$$J_b^s > J_b^c > 0.$$

Furthermore:

$$\int_{-H}^0[\psi_x^s(x_0^+,y)-\psi_x^s(x_0^-,y)]\check{\chi}^s(x_0,y)dy=-I^s\sin(\nu_0x_0),$$

$$\int_{-H}^0[\psi_x^c(x_0^+,y)-\psi_x^c(x_0^-,y)]\check{\chi}^c(x_0,y)dy=-I^c\cos(\nu_0x_0),$$

where

$$I^{s,c}=\frac{1}{2}\left[\int_{S\setminus Sx_0}\nabla\check{\psi}^{s,c}\cdot\nabla\check{\chi}^{s,c}dxdy-\nu\int_{|x|\geq x_0}\check{\psi}^{s,c}\check{\chi}^{s,c}dx\right]$$

$$+\frac{1}{2}\left[\int_{Sx_0}\nabla\hat{\psi}^{s,c}\nabla\hat{\chi}^{s,c}dxdy-\nu\int_{-x_0}^{x_0}\hat{\psi}^{s,c}\hat{\chi}^{s,c}dx\right]$$

Similarly,

$$\int_{-H}^0[\chi_x^s(x_0^+,y)-\chi_x^s(x_0^-,y)]\hat{\psi}^s(x_0,y)dy=-I^s\sin(\nu_1x_0),$$

$$\int_{-H}^0[\chi_x^c(x_0^+,y)-\chi_x^c(x_0^-,y)]\hat{\psi}^c(x_0,y)dy=-I^c\cos(\nu_1x_0).$$

Finally, the bounds  $(I^s)^2 \leq J^s J_b^s$ ,  $(I^c)^2 \leq J^c J_b^c$  follows by the positivity of the previous quadratic forms and by elementary calculations.

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