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Quantum measurements in continuous time, non Markovian evolutions and feedback

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Abstract

In this paper we present a non Markovian version of quantum trajectory theory, based on the stochastic Schrödinger equation with stochastic coefficients. In this framework we can describe measurements in continuous time combined with measurement based feedback. Indeed, realistic descriptions of a feedback loop have to include delay and thus need a non Markovian theory. As an application we consider a two-level atom stimulated by a laser. We introduce closed loop control too, via the stimulating laser, with the aim to enhance the "squeezing" of the emitted light, and other typical quantum properties. Let us stress the change of point of view with respect to the usual applications of control theory. Here the "system" is the two-level atom, but we do not want to control its state, to bring the atom to a final target state. Our aim is to control the "Mandel Q-parameter" and the spectrum of the emitted light; in particular the spectrum is not a property at a single time, but involves a long interval of times (a Fourier transform of the autocorrelation function of the observed output is needed).

1 Quantum trajectories and control

Stochastic wave function methods for the description of open quantum systems are now widely used [1–5] and are often referred to as *quantum trajectory theory.* These approaches are very important for numerical simulations and allow the continuous measurement description of detection schemes in quantum optics, namely direct, homodyne and heterodyne photodetection [6–9]. In the Markovian case, the stochastic differential equations of the quantum trajectory theory can be deduced by purely quantum evolution equations, involving quantum fields and quantum stochastic calculus, and can be interpreted in terms of measurements in continuous time because they can be related to *positive operator valued measures* and *instruments*, which are the objects representing observables and state changes in the modern axiomatic formulation of quantum mechanics [8, 10–13].

The whole quantum trajectory theory is well developed in the Markovian case, but to include memory effects is more and more important. Different generalizations have been developed, but often without a measurement interpretation [14, 15]. A way to include non Markovian effects is to start from the *stochastic Schrödinger equation* (SSE) and to generalize it by allowing for stochastic coefficients. This can be done without violating the axiomatic formulation of quantum mechanics and a non Markovian quantum trajectory theory can be developed in a mathematically consistent way [16]. Some applications to systems affected by coloured noises and monitored with continuity have been developed [17, 18].

In quantum optical systems, even when the Markov approximation for the reduced dynamics is well justified, memory can enter into play when imperfections in the stimulating lasers are taken into account [19] and when feedback loops are introduced to control the system [4, 20–25]. The so called *closed loop control* is based on the continuous monitoring of the system and, so, it fits well in the theory of measurements in continuous time. In some approximation, one can consider an instantaneous and very singular feedback and in this case the usual Markov framework is sufficient; however, more realistic descriptions of the feedback loop, including delay, need a non Markovian theory [9, 21, 22, 24, 25].

In this paper we present the non Markovian version of the theory of quantum measurements in continuous time, based on the SSE and the *stochastic master equation* (SME). We show how to get the physical probabilities for the output of the observation and its moments. In quantum optical systems the moments of the stochastic output are connected to the Mandel Q-parameter and to the spectrum of the emitted light (homodyne and heterodyne spectra) and allow for the study of typical quantum properties of the emitted light, such as *squeezing* [9, 26, 27]. The theory allows for the introduction of coloured noises, but our emphasis will be on the possibility of modeling a non perfectly monochromatic and coherent stimulating laser and of modeling a measurement based feedback. To illustrate these concepts we shall use a prototype model, a two-level atom stimulated by a laser, which is known to have a rich spectrum and to emit squeezed light under particular conditions. We shall introduce closed loop control, via the stimulating laser, with the aim to control the squeezing in the observed spectrum, not to control the state of the system. Let us stress the change of point of view with respect to the usual applications of control theory. Here the "system" is the two-level atom, but we do not want to control its state, as to bring the atom to a final target state. Our aim is to control the properties of the emitted light; moreover, we want to control the spectrum, which is not a property at a single time, but involves a long interval of times (a Fourier transform in time is needed).

2 The stochastic Schrödinger equation and the stochastic master equation

The best way to introduce memory in quantum evolutions is to start from a dynamical equation in the Hilbert space; this approach automatically guarantees the complete positivity of the evolution of the state (statistical operator) of the system. Moreover, to consider the linear version of the SSE allows to construct the *instruments* related to the continuous monitoring even in the non Markov case [16–18]. We shall introduce first several mathematical objects, from the linear SSE (1) to the instruments (11), and later, thanks to these latter, we shall give a consistent physical interpretation of the whole construction.

Let $\mathcal{H}$ be a separable complex Hilbert space, the Hilbert space of the quantum system of interest, and let us denote by $\mathcal{L}(\mathcal{H})$ the space of the bounded operators on $\mathcal{H}$, by $\mathcal{T}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ the trace class and by $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ the convex set of the statistical operators.

2.1 The linear SSE and the reference probability

Let us consider a reference probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a filtration of σ-algebras$(\mathcal{F}_t)_t$ satisfying the usual hypotheses, i.e. $A \in \mathcal{F}$ with $\mathbb{Q}(A) = 0$ implies $A \in \mathcal{F}_0$, and $\mathcal{F}_t = \bigcap_{T>t} \mathcal{F}_T$. In this probability space we have d continuous standard Wiener processes $B_1, \ldots, B_d$ and d' Poisson processes $N_1, \ldots, N_{d'}$. Under the reference probability $\mathbb{Q}$, all these processes are independent and are adapted, with increments independent from the past, with respect to the given filtration. Every Poisson process N_k is taken with trajectories continuous from the right and with limits from the left (càdlàg); let $\lambda_k > 0$ be the intensity of N_k.

We assume also to have a set of stochastic processes $L_i(t)$, $R_k(t)$, $H(t)$ with values in $\mathcal{L}(\mathcal{H})$, such that $H(t)^* = H(t)$ and $t \mapsto L_i(t)$, $t \mapsto R_k(t)$, $t \mapsto H(t)$ are adapted processes with trajectories continuous from the left and with limits from the right (càglàd, continuity in the strong operator topology). We assume also

$$\int_0^T \Big(\|H(t)\| + \sum_i \|L_i(t)\|^2 + \sum_k \|R_k(t)\|^2 \Big) \mathrm{d}t \leq M(T) < +\infty, \qquad \forall T > 0,$$

where $M(T)$ is independent of ω.

Then, we introduce the linear SSE (a simplified version of the SSE introduced in Ref. 16)

$$\mathrm{d}\phi(t) = \Big[-\mathrm{i}H(t) - \frac{1}{2}\sum_{i=1}^{d} L_i(t)^* L_i(t) - \frac{1}{2}\sum_{k=1}^{d'} R_k(t)^* R_k(t) + \frac{\lambda}{2}\Big]\phi(t_-)\mathrm{d}t \\ + \sum_{i=1}^{d} L_i(t)\phi(t_-)\,\mathrm{d}B_i(t) + \sum_{k=1}^{d'}\left(\frac{R_k(t)}{\sqrt{\lambda_k}} - \mathbb{1}\right)\phi(t_-)\,\mathrm{d}N_k(t), \quad (1)$$

with initial condition $\phi(0) = \phi_0 \in \mathcal{H}$, $\|\phi_0\|^2 = 1$, and where $\lambda := \sum_{k=0}^{d'} \lambda_k$. Equation (1) is an Itô-type stochastic differential equation admitting a unique strong solution [16, Proposition 2.1].

Here only bounded coefficients are considered, in order not to have mathematical complications, but generalizations to unbounded coefficients are of physical interest. Note that the filtration $(\mathcal{F}_t)$ can be taken bigger than the natural filtration generated by the processes B_i and N_k, so that the processes L_i, R_k, H can depend also on some other external noises.

By normalizing the random vector $\phi(t)$ one gets a norm-one Hilbert space process which turns out to satisfy a non linear stochastic equation (under a new probability); it is this non linear equation which is usually called SSE. It is this equation which gives the starting point for powerful numerical simulations, but we skip it here as it is not essential to introduce the continuous measurements.

2.2 The linear stochastic master equation

The SSE (1) can be translated into a stochastic equation for trace class operators. By stochastic calculus we compute the stochastic differential of $|\phi(t)\rangle\langle\phi(t)|$; in this way we get a closed linear equation, whose initial condition can be generalized to any pure or mixed state. This is the linear stochastic master equation (SME) [16, Propositions 3.2 and 3.4]:

$$\mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t_-)]\mathrm{d}t + \sum_{i=1}^{d}\big(L_i(t)\sigma(t_-) + \sigma(t_-)L_i(t)^*\big)\mathrm{d}B_i(t) \\ + \sum_{k=1}^{d'}\left(\frac{1}{\lambda_k}\,R_k(t)\sigma(t_-)R_k(t)^* - \sigma(t_-)\right)\big(\mathrm{d}N_k(t) - \lambda_k\mathrm{d}t\big), \quad (2)$$

$\sigma(0) = \rho_0 \in \mathcal{S}(\mathcal{H})$; the operator $\mathcal{L}(t)$ is the stochastic Liouvillian

$$\mathcal{L}(t)[\tau] := -\mathrm{i}\,[H(t),\,\tau] - \frac{1}{2}\sum_{i=1}^{d}\{L_i(t)^*L_i(t),\,\tau\} - \frac{1}{2}\sum_{k=1}^{d'}\{R_k(t)^*R_k(t),\tau\} \\ + \sum_{i=1}^{d} L_i(t)\tau L_i(t)^* + \sum_{k=1}^{d'} R_k(t)\tau R_k(t)^*. \quad (3)$$

The SME (2) admits a unique strong solution. Typically, the solution $\sigma(t)$ is not Markovian as $\mathcal{L}(t)$ depends on the past.

The propagator. In the following we shall need the fundamental solution, or *propagator*, $\mathcal{A}(t,s)$ of Eq. (2), i.e. the random linear map on $\mathcal{T}(\mathcal{H})$ defined by $\sigma(s) \mapsto \sigma(t)$. By construction $\mathcal{A}(t,s)$ is completely positive, $\mathcal{A}(t,t) = \mathbb{1}$ and $\mathcal{A}(t,s)\circ\mathcal{A}(s,r) = \mathcal{A}(t,r)$ for $0 \le r \le s \le t$.

2.3 The new probability

Let us fix a non random state $z \in \mathcal{S}(\mathcal{H})$ and define the stochastic processes

$$p(t) := \operatorname{Tr}\{\sigma(t)\}, \qquad \rho(t,\omega) := \begin{cases} p(t,\omega)^{-1}\sigma(t,\omega) & \text{if } p(t,\omega) \neq 0, \\ z & \text{if } p(t,\omega) = 0, \end{cases} \tag{4}$$

$$m_i(t) := 2 \operatorname{Re} \operatorname{Tr}\left\{L_i(t)\rho(t_-)\right\}, \qquad i = 1, \dots, d, \tag{5a}$$

$$i_k(t) := \operatorname{Tr}\left\{R_k(t)^* R_k(t)\rho(t_-)\right\}^2, \qquad k = 1, \dots, d'. \tag{5b}$$

By taking the trace of (2), we have that $p(t)$ satisfies the Doléans equation

$$\mathrm{d}p(t) = p(t_-)\left\{\sum_{i=1}^{d} m_i(t)\mathrm{d}B_i(t) + \sum_{k=1}^{d'}\left(\frac{i_k(t)}{\lambda_k} - 1\right)\left(\mathrm{d}N_k(t) - \lambda_k \mathrm{d}t\right)\right\} \tag{6}$$

with $p(0) = 0$ and where the coefficients $m_i(t)$ and $i_k(t)$ depend on the initial condition ρ_0 in (2). The solution of (6) can be written as

$$\begin{aligned} p(t) = \|\phi(0)\|^2 \exp\Bigg\{&\sum_{i=1}^{d}\int_0^t \left[m_i(s)\mathrm{d}B_i(s) - \frac{1}{2}\,m_i(s)^2\mathrm{d}s\right] \\ &- \sum_{k=0}^{d'}\int_0^t \left(i_k(s) - \lambda_k\right)\mathrm{d}s\Bigg\} \prod_{0<r\leq t}\left(1 + \sum_{k=0}^{d'}\left(\frac{i_k(r)}{\lambda_k} - 1\right)\Delta N_k(r)\right), \end{aligned} \tag{7}$$

where $\Delta N_k(r) = N_k(r) - N_k(r_-)$.

The new probability. The key property of quantum trajectory theory is that Eq. (6) implies that $p(t)$ is a mean-one $\mathbb{Q}$-martingale [16, Theorem 2.4, Section 3.1]. This allows to define the new probabilities

$$\mathbb{P}^T_{\rho_0}(F) := \int_F p(T,\omega)\,\mathbb{Q}(\mathrm{d}\omega) = \mathbb{E}_{\mathbb{Q}}[p(T)1_F], \quad \forall F \in \mathcal{F}_T. \tag{8}$$

Due to the martingale property of $p(t)$, the probabilities $\mathbb{P}^T_{\rho_0}$ are consistent, in the sense that $\mathbb{P}^t_{\rho_0}(F) = \mathbb{P}^s_{\rho_0}(F)$ for $F \in \mathcal{F}_s$, $t \geq s \geq 0$.

It is possible to show that the stochastic state $\rho(t)$ defined by (4) satisfies a non linear SME under the new probability $\mathbb{P}^T_{\rho_0}$ [16, Remark 3.6].

The Girsanov transformation. The new probabilities $\mathbb{P}^T_{\rho_0}$ modify the distribution of the processes B_i and N_k. A very important property is that a Girsanov-type theorem holds [16, Proposition 2.5, Remarks 2.6 and 3.5]. Under $\mathbb{P}^T_{\rho_0}$, in the time interval $[0,T]$, the processes

$$W_j(t) := B_j(t) - \int_0^t m_j(s)\mathrm{d}s, \qquad j = 1, \dots, d, \tag{9}$$

are independent Wiener processes, while $N_1, \dots, N_{d'}$ become simple regular càdlàg counting process of stochastic intensities $i_1, \dots, i_{d'}$.

From this result we have immediately

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}_{\mathbb{P}^T_{\rho_0}}\left[B_i(t)\right] = \mathbb{E}_{\mathbb{P}^T_{\rho_0}}\left[m_i(t)\right] = 2\operatorname{Re}\mathbb{E}_{\mathbb{Q}}\left[\operatorname{Tr}\left\{L_i(t)\sigma(t_-)\right\}\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}_{\mathbb{P}^T_{\rho_0}}\left[N_k(t)\right] = \mathbb{E}_{\mathbb{P}^T_{\rho_0}}\left[i_k(t)\right] = \mathbb{E}_{\mathbb{Q}}\left[\operatorname{Tr}\left\{R_k(t)^* R_k(t)\sigma(t_-)\right\}\right].$$

2.4 The continuous measurement process

Let us introduce now the real processes

$$J_\ell(t) := \sum_{j=1}^{d} \int_0^t a_{\ell j}(t-s)\,\mathrm{d}B_j(s) + e_\ell(t), \qquad \ell = 1,\ldots,m_I, \tag{10a}$$

$$I_h(t) := \sum_{k=0}^{d'} \int_{(0,t]} n_{hk}(t-s)\,\mathrm{d}N_k(s) + w_k(t), \qquad h = 1,\ldots,m_J, \tag{10b}$$

where the integral kernels $a_{\ell j}(t-s)$ and $n_{hk}(t-s)$ are deterministic, and where $e_\ell(t)$ and $w_k(t)$ can be stochastic processes. Even more general expressions than the ones in Eqs. (10) could be considered. What is important is the natural filtration generated by the processes (10) which we denote by $(\mathcal{E}_t)$. We assume $\mathcal{E}_t \subseteq \mathcal{F}_t$, $\forall t \geq 0$.

2.4.1 Instruments and a posteriori states

Now, for $t \geq 0$, let us define the map-valued measure $\mathcal{I}_t$ [16, Remark 4.2]:

$$\mathcal{I}_t(E)[\varrho] = \int_E \mathcal{A}(t,0,\omega)[\varrho]\,\mathrm{d}\mathbb{Q}(\omega), \qquad \forall E \in \mathcal{E}_t, \quad \forall \varrho \in \mathcal{T}(\mathcal{H}). \tag{11}$$

Such a measure has the properties: (i) $\forall E \in \mathcal{E}_t$, $\mathcal{I}_t(E)$ is a completely positive linear map on $\mathcal{T}(\mathcal{H})$, (ii) $\forall \varrho \in \mathcal{T}(\mathcal{H})$, $\forall a \in \mathcal{L}(\mathcal{H})$, $\mathrm{Tr}\left\{a\mathcal{I}_t(\cdot)[\varrho]\right\}$ is σ-additive, (iii) $\forall \varrho \in \mathcal{T}(\mathcal{H})$, $\mathrm{Tr}\left\{\mathcal{I}_t(\Omega)[\varrho]\right\} = \mathrm{Tr}\left\{\varrho\right\}$. Such a map-valued measure $\mathcal{I}_t$ is called an *instrument* with value space $(\Omega, \mathcal{E}_t)$ and it can be consistently interpreted as a quantum mechanical measurement on the system $\mathcal{H}$ of the processes $J_\ell(s)$ and $I_h(s)$ in the time interval $[0,t]$; the instrument gives both the probability distribution of the output and the state changes conditional on the observation.

According to the physical interpretation of the notion of instrument, the probability of the event $E \in \mathcal{E}_t$, when the pre-measurement state is ρ_0, is given by

$$\mathrm{Tr}\left\{\mathcal{I}_t(E)[\rho_0]\right\} = \mathbb{E}_{\mathbb{Q}}\left[1_E \,\mathrm{Tr}\left\{\sigma(t)\right\}\right] = \mathbb{P}^t_{\rho_0}(E) \tag{12}$$

and this shows that the physical probability for the observation of the output up to time t is indeed the one introduced in Eq. (8) restricted to $\mathcal{E}_t$.

Then, Eq. (10) can be interpreted as the effect of the measuring apparatus which processes the ideal outputs $B_i(t)$ and $N_k(t)$ by detector response functions $a_{\ell j}$ and n_{hk}, and which degrades the outputs by adding some more noises $e_\ell(t)$ and $w_k(t)$ due to the physical realization of the apparatus itself. In the final part of this section we shall see how to compute some relevant properties of the outputs under the physical probability.

Moreover, let us take the conditional expectation on $\mathcal{E}_t$ of the random state $\rho(t)$ defined by (4):

$$\hat{\rho}(t) := \mathbb{E}_{\mathbb{P}^t_{\rho_0}}\left[\rho(t)|\mathcal{E}_t\right] \equiv \frac{\mathbb{E}_{\mathbb{Q}}[\sigma(t)|\mathcal{E}_t]}{\mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}[\sigma(t)|\mathcal{E}_t]\right\}}. \tag{13}$$

The interpretation is that $\hat{\rho}(t)$ is the conditional state one attributes to the system at time t having observed the trajectory of the output up to time t. Indeed $\hat{\rho}(t)$ is $\mathcal{E}_t$-measurable, thus depending only on the trajectories of the output in $[0,t]$, and, moreover, one can directly check that [16, Remark 4.4]

$$\int_E \hat{\rho}(t,\omega)\mathbb{P}^t_{\rho_0}(\mathrm{d}\omega) = \mathcal{I}_t(E)[\rho_0], \qquad \forall E \in \mathcal{E}_t. \tag{14}$$

The state $\hat{\rho}(t)$ is the *a posteriori state* at time t.

When t goes from 0 to T, the family of instruments $\mathcal{I}_t$ gives a consistent description of a continuous measurement performed on the system.

In the extreme case $\mathcal{E}_t = \mathcal{F}_t$, which occurs for example when $\mathcal{F}_t$ is generated by the the processes B_i and N_k and just these processes are the observed output, we get $\hat{\rho}(t) = \rho(t)$. Therefore the evolution of the a posteriori state is completely defined by the non linear SME satisfied by $\rho(t)$, or, equivalently, by the SME (2) for $\sigma(t)$, or even, as pure states are mapped to pure states, by the SSE (1) for $\phi(t)$. The randomness of the coefficients $L_i(t)$, $R_k(t)$ and $H(t)$, which typically prevents $\sigma(t)$ from being Markovian, allows us to model non Markovian features of the dynamic due both to some classical noises and to measurement based feedback loops.

When $\mathcal{E}_t \subsetneqq \mathcal{F}_t$, the a posteriori state $\hat{\rho}(t)$ has a non Markovian evolution which typically does not even satisfies a differential equation. In this case the SSE (1) and the SME (2) have to be interpreted as an ideal unravelling of the physical evolution of $\hat{\rho}(t)$ which allows to consistently define it, by (2) and (13), and allows to compute, at least numerically, all the quantities of physical interest (that is to define the instruments $\mathcal{I}_t$).

2.4.2 A priori states

When the output of the continuous measurement is not taken into account, the state of the system at time t is given by the mean state

$$\eta(t) := \mathbb{E}_{\mathbb{P}^t_{\rho_0}}[\hat{\rho}(t)] = \mathbb{E}_{\mathbb{Q}}[\sigma(t)] = \mathbb{E}_{\mathbb{Q}}[\mathcal{A}(t,0)[\rho_0]] = \mathcal{I}_t(\Omega)[\rho_0].$$

The state $\eta(t)$ is the *a priori state* at time t. Note that $\mathbb{E}_{\mathbb{Q}}[\mathcal{A}(t,0)[\cdot]]$ is a completely positive, trace preserving, linear map, i.e. a *quantum channel* in the terminology of quantum information.

From the SME (2) we get $\dot{\eta}(t) = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}(t)[\sigma(t)]\right]$, which is not a closed differential equation when $\mathcal{L}(t)$ is stochastic, contrarily to the Markov case [9, Section 3.5]. By the projection operator technique a closed integro-differential equation for the a priori state $\eta(t)$ could be obtained [18] (an evolution equation with memory), but this equation is too involved to be of practical use. Again (1) and (2) are an unravelling of a non Markovian evolution.

2.4.3 Spectra and moments

In quantum optics, the typical output current of an homodyne or heterodyne detector is of the form (10a) with $a_{\ell j}(t) = \delta_{\ell j} F(t)$ (F is the detector response function). The output current is a stochastic process and its *spectrum* is given by the classical notion [28]. If J_ℓ is at least asymptotically stationary and the limit in (15) exists at least in the sense of distributions in μ, then, the spectrum of J_ℓ is defined by

$$S_\ell(\mu) = \lim_{T\to+\infty} \frac{1}{T}\, \mathbb{E}_{\mathbb{P}^T_{\rho_0}}\left[\left|\int_0^T \mathrm{e}^{-\mathrm{i}\mu t} J_\ell(t)\mathrm{d}t\right|^2\right]. \tag{15}$$

In the pure case of no extra noise, $e_\ell(t) = 0$, and of a detector response function going to a Dirac delta, that is in the pure case $J_\ell = \dot{B}_\ell$, the spectrum becomes

$$S_\ell(\mu) = \lim_{T\to+\infty} \frac{1}{T}\, \mathbb{E}_{\mathbb{P}^T_{\rho_0}}\left[\left|\int_0^T \mathrm{e}^{-\mathrm{i}\mu t} \mathrm{d}B_\ell(t)\right|^2\right]. \tag{16}$$

The spectrum depends on the distribution of the current $J_\ell(t)$, which is the output of a continuous measurement on the system performed by the detection of its emitted light. Thus, S_ℓ gives informations both on the monitored system and on the fluorescence light. For example, $S_\ell < 1$ reveals squeezing of the emitted light.

An expression for the autocorrelation function needed in the computation of the spectrum can be obtained by generalizing the techniques used in the Markovian case [9, Section 4.5]. When the Liouville operator (3) and B_ℓ are independent (which implies that B_ℓ is not used for the feedback), we get

$$\frac{\partial^2}{\partial t \partial s}\, \mathbb{E}_{\mathbb{P}^T_{\rho_0}}[B_\ell(t) B_\ell(s)] = \delta(t-s) + b_\ell(t,s) + b_\ell(s,t), \tag{17a}$$

$$
b_\ell(t,s) = 1_{(0,+\infty)}(t-s)\ \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{Tr}\Big\{\left(L_\ell(t)+L_\ell(t)^*\right) \\
\times \mathcal{A}(t_-,s)\left[L_\ell(s)\sigma(s_-)+\sigma(s_-)L_\ell(s)^*\right]\Big\}\Big]. \qquad (17b)
$$

Let us set $n_\ell(t) = \mathbb{E}_{\mathbb{P}^t_{\rho_0}}[m_\ell(t)]$ and assume that the limit $n_\infty := \lim_{t\to+\infty} n_\ell(t)$ exists. Then, we obtain the decomposition of the spectrum in the elastic part and the inelastic one (the spectrum of the fluctuations)

$$
S_J(\mu) = S_{\mathrm{el}}(\mu) + S_{\mathrm{inel}}(\mu), \qquad S_{\mathrm{el}}(\mu) = 2\pi\, n_\infty^{\,2}\, \delta(\mu), \tag{18a}
$$

$$
S_{\mathrm{inel}}(\mu) = 1 + \lim_{T\to+\infty} \frac{2}{T}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\, \cos\mu(t-s)\, d_\ell(t,s), \tag{18b}
$$

$$
d_\ell(t,s) := b_\ell(t,s) - 1_{(0,+\infty)}(t-s)\, n_\infty n_\ell(t) = 1_{(0,+\infty)}(t-s) \\
\times \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\left(L_\ell(t)+L_\ell(t)^*\right)\mathcal{A}(t_-,s)\left[L_\ell(s)\sigma(s_-)+\sigma(s_-)L_\ell(s)^* - n_\infty\sigma(s_-)\right]\right\}\right].
$$

2.4.4 Mandel Q-parameter

When we consider direct detection in quantum optics, in the ideal case of noiseless counter, the output of the measurement is one of the counting processes, say N_k. In this case a typical quantity is the Mandel Q-parameter, defined by

$$
Q_k(t;t_0) := \frac{\mathrm{Var}_{\mathbb{P}^T_{\rho_0}}[N_k(t_0+t)-N_k(t_0)]}{\mathbb{E}_{\mathbb{P}^T_{\rho_0}}[N_k(t_0+t)-N_k(t_0)]} - 1.
$$

In the case of a Poisson process this parameter is zero; in quantum optics, in the case of positive Q parameter one speaks of super-Poissonian light and of sub-Poissonian light in the other case. Sub-Poissonian light is considered an indication of non-classical effects.

In quantum trajectory theory, one can find expressions for the moments also in the counting case and we get

$$
Q_k(t;t_0) = \frac{V_k(t;t_0)}{M_k(t;t_0)} - M_k(t;t_0), \qquad V_k(t;t_0) := \mathbb{E}_{\mathbb{P}^T_{\rho_0}}\left[\left(N_k(t_0+t)-N_k(t_0)\right)^2\right],
$$

$$
M_k(t;t_0) := \mathbb{E}_{\mathbb{P}^T_{\rho_0}}[N_k(t_0+t)-N_k(t_0)] = \int_{t_0}^{t_0+t} \mathrm{d}s\, \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{R_k(s)^*R_k(s)\sigma(s_-)\right\}\right].
$$

When $R_k(s)^*R_k(s)\mathcal{A}(s_-,r)$ is $\mathbb{Q}$-independent from $N_k(r)$, which happens when N_k is not used for feedback, we get

$$
V_k(t;t_0) = 2\int_{t_0}^{t_0+t} \mathrm{d}s \int_{t_0}^{s} \mathrm{d}r\, \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{R_k(s)^*R_k(s)\mathcal{A}(s_-,r)\left[R_k(r)\sigma(r_-)R_k(r)^*\right]\right\}\right].
$$

3 An example: the two-level atom

As an application of the theory we consider a two-level atom stimulated by a laser; it is an ideal example, but is sufficiently rich and flexible to illustrate the possibilities of the theory. The Hilbert space is $\mathbb{C}^2$ and the Hamiltonian part of the dynamics is given by

$$
H(t) = H_0 + H_f(t), \qquad H_0 = \frac{\omega_0}{2}\,\sigma_z, \quad \omega_0 > 0, \qquad H_f(t) = \overline{f(t)}\,\sigma_- + f(t)\sigma_+,
$$

where σ_z, $\sigma_\pm$ are the usual Pauli matrices. The function f is the laser wave, which can be noisy and can be controlled by the experimenter.

Let us complete the model by choosing the noise-driven terms in the SME (2), which we call channels in the sequel. We consider two diffusive channels realized by heterodyne or homodyne detectors of the emitted light with *local oscillators* represented by the functions h_j:

$$d = 2, \qquad L_j(t) = \overline{h_j(t)}\, \alpha_j \sigma_-, \qquad |h_j(t)| = 1, \quad \alpha_j \in \mathbb{C}.$$

The observation of the light in the diffusive channel 1 will be used to control the stimulating laser light. The light in the diffusive channel 2 is only observed; indeed, we are interested in controlling the properties of this part of the emitted light by controlling the atom via the stimulating laser.

In the case of *homodyne detection* the local oscillator is feeded by the light produced by the stimulating laser and we have $h_j(t) = \mathrm{e}^{-\mathrm{i}\epsilon_j} \frac{f(t)}{|f(t)|}$.

In the case of *heterodyne detection* the local oscillator is feeded by an independent laser. For this light we use the so called phase diffusion model, $h_j(t) = \exp\{-\mathrm{i}\epsilon_j - \mathrm{i}\nu_j t - \mathrm{i}k_{-j} B_{-j}(t)\}$, where B_{-1} and B_{-2} are extra noises. The function h_j represents a nearly monochromatic wave with a Lorentzian spectrum centred on ν_j.

We introduce also four jump channels:

$$d' = 4, \quad R_k(t) = R_k, \quad \beta_k \in \mathbb{C}, \quad \gamma > 0, \quad \bar{n} \geq 0, \quad \sum_{i=1}^{2} |\alpha_i|^2 + \sum_{k=1}^{2} |\beta_k|^2 = \gamma,$$

$$R_k = \beta_k \sigma_-, \quad k = 1, 2, 3 \qquad R_4 = \beta_4 \sigma_+, \qquad |\beta_3|^2 = |\beta_4|^2 = \gamma \bar{n}.$$

The jump channels 1 and 2 are electromagnetic channels: channel 1 represents the emitted light reaching a photo-counter (*direct detection*), while channel 2 represents the lost light, which is not observed. The output of channel 1 could be used again as a possible signal for closed loop control, but we use it here only to see properties of direct detection. The counting channels 3 and 4 are used only to introduce dissipation due to a thermal bath; these channels are not connected to observation.

The stimulating laser light can be noisy and can be controlled by the output of the diffusive channel 1. In mathematical terms, f can be an adapted functional of B_0 (extra-noise) and B_1 (feedback of the diffusive channel 1).

For this model we need a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where $B_{-2}, \ldots, B_2$ are Wiener processes, $N_1, \ldots, N_4$ are Poisson processes and they are all independent. According to the notations of Section 2, the filtration $(\mathcal{F}_t)$ is generated by all these processes, while $(\mathcal{E}_t)$ is generated by B_1, B_2 and N_1.

From Eq. (3) and the assumptions of the model we get the random Liouville operator

$$\mathcal{L}(t)[\tau] = -\mathrm{i}\,[H_0 + H_f(t),\, \tau] + \gamma \sigma_- \tau \sigma_+ - \frac{\gamma}{2}\{P_+, \tau\} + \gamma \bar{n}\,(\sigma_- \tau \sigma_+ + \sigma_+ \tau \sigma_- - \tau)\,. \tag{19}$$

Note that the whole randomness is in the wave $f(t)$ and it is due purely to noise in the laser light and to feedback, but this is enough to have a non Markovian model.

In principle the quantities of interest could be computed by starting from simulations of the non linear SSE of the model. However, in order to have an analytically computable spectrum and to have an idea of the possible behaviours of the model, we take a very simple form for f. We assume the laser to be not perfectly monochromatic and we describe this again by a phase diffusion term; moreover, we take the feedback to act as a very simple phase modulation:

$$f(t) = \frac{\Omega}{2}\,\mathrm{e}^{-\mathrm{i}u(t)}, \qquad u(t) = \vartheta + \omega t + k_0 B_0(t) + k_1 B_1(t), \tag{20}$$

$$\Omega \geq 0, \qquad \omega \geq 0, \quad k_0,\, k_1 \in \mathbb{R}, \quad \vartheta \in (-\pi, \pi].$$

3.1 Control of the homodyne spectrum

According to Eqs. (18), to compute the spectrum of the light in channel 2 we need to compute first the quantities $n_2(t)$ and $d_2(t,s)$. The best way is to make a unitary transformation and to define

$$\Lambda(t,s)[\tau] := \mathrm{e}^{\frac{\mathrm{i}}{2}\,u(t)\sigma_z} \mathcal{A}(t,s) \left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\,u(s)\sigma_z} \tau \mathrm{e}^{\frac{\mathrm{i}}{2}\,u(s)\sigma_z}\right] \mathrm{e}^{-\frac{\mathrm{i}}{2}\,u(t)\sigma_z},$$

$$\xi(t) := \mathrm{e}^{\frac{\mathrm{i}}{2}\,u(t)\sigma_z}\sigma(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\,u(t)\sigma_z} = \Lambda(t,0)\left[\mathrm{e}^{\frac{\mathrm{i}}{2}\,\vartheta\sigma_z}\rho_0\mathrm{e}^{-\frac{\mathrm{i}}{2}\,\vartheta\sigma_z}\right]. \tag{21}$$

By using (2), (20) and computing the stochastic differential of $\xi(t)$ and we obtain

$$\mathrm{d}\xi(t) = \hat{\mathcal{L}}[\xi(t_-)]\mathrm{d}t + \sum_{i=0}^{2}\mathcal{D}_i(t)[\xi(t_-)]\mathrm{d}B_i(t) \\ + \sum_{k=1}^{4}\left(\frac{1}{\lambda_k}\,R_k\xi(t_-)R_k{}^* - \xi(t_-)\right)\left(\mathrm{d}N_k(t) - \lambda_k\mathrm{d}t\right), \tag{22}$$

$$\hat{\mathcal{L}}(t)[\tau] = -\frac{\mathrm{i}}{2}\left[\Delta\omega\sigma_z + \Omega\sigma_x,\,\tau\right] + \gamma\sigma_-\tau\sigma_+ - \frac{\gamma}{2}\left\{P_+,\tau\right\} + \frac{k_0{}^2 + k_1{}^2}{4}\left(\sigma_z\tau\sigma_z - \tau\right) \\ + \frac{\mathrm{i}}{2}\left[\sigma_z,\,\overline{g_1(t)}\,\sigma_-\tau + g_1(t)\tau\sigma_+\right] + \gamma\bar{n}\left(\sigma_-\tau\sigma_+ + \sigma_+\tau\sigma_- - \tau\right), \tag{23}$$

$$\mathcal{D}_0[\tau] = \frac{\mathrm{i}}{2}\,k_0[\sigma_z,\tau], \qquad \mathcal{D}_1(t)[\tau] = \overline{g_1(t)}\,\sigma_-\tau + g_1(t)\tau\sigma_+ + \frac{\mathrm{i}}{2}\,k_1[\sigma_z,\tau],$$

$$\mathcal{D}_2(t)[\tau] = \overline{g_2(t)}\,\sigma_-\tau + g_2(t)\tau\sigma_+, \qquad g_i(t) = \overline{\alpha_i}\,\mathrm{e}^{\mathrm{i}u(t)}\,h_i(t), \qquad \Delta\omega = \omega_0 - \omega.$$

Then, we get

$$n_2(t) = \mathbb{E}_{\mathbb{P}^T_{\rho_0}}\left[m_2(t)\right] = 2\,\mathrm{Re}\,\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{L_2(t)\sigma(t_-)\right\}\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{D}_2(t)\left[\xi(t_-)\right]\right\}\right],$$

$$d_2(t,s) = 1_{(0,+\infty)}(t-s)\,\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathcal{D}_2(t)\left[\Lambda(t_-,s)\left[\mathcal{D}_2(s)\left[\xi(s_-)\right] - n_\infty\xi(s_-)\right]\right]\right\}\right].$$

3.1.1 Homodyning in channels 1 and 2

In the case of homodyning we have $g_j(t) = |\alpha_j|\,\mathrm{e}^{-\mathrm{i}\vartheta_j}$, $j = 1,2$, which gives $\mathcal{D}_1$, $\mathcal{D}_2$, $\hat{\mathcal{L}}$ non random and independent of time, so that $\mathbb{E}_{\mathbb{Q}}[\Lambda(t,s)] = \mathrm{e}^{\hat{\mathcal{L}}(t-s)}$. By Bloch equation techniques we can compute the homodyne spectrum (18) of the light in channel 2 [9, Part II]. The final result is

$$S_{\mathrm{el}}(\mu) = 2\pi\left|\alpha_2\right|^2 v^2\delta(\mu), \qquad v := \cos\vartheta_2\,d_1 + \sin\vartheta_2\,d_2, \tag{24}$$

$$S_{\mathrm{inel}}(\mu) = 1 + \begin{pmatrix}\cos\vartheta_2 & \sin\vartheta_2 & 0\end{pmatrix}\frac{2\left|\alpha_2\right|^2}{A^2+\mu^2}\left(A\begin{pmatrix}\cos\vartheta_2(1+d_3)\\ \sin\vartheta_2(1+d_3)\\ -v\end{pmatrix} + v\vec{u}\right), \tag{25}$$

$$A = \begin{pmatrix}\frac{\Gamma}{2} & \Delta\omega & -k_1\left|\alpha_1\right|\sin\vartheta_1\\ -\Delta\omega & \frac{\Gamma}{2} & \Omega + k_1\left|\alpha_1\right|\cos\vartheta_1\\ 0 & -\Omega & (2\bar{n}+1)\,\gamma\end{pmatrix}, \qquad \vec{u} = \begin{pmatrix}-k_1\left|\alpha_1\right|\sin\vartheta_1\\ k_1\left|\alpha_1\right|\cos\vartheta_1\\ \gamma\end{pmatrix}, \tag{26}$$

$$\vec{d} = -A^{-1}\vec{u}, \qquad \Gamma = (2\bar{n}+1)\,\gamma + k_0{}^2 + k_1{}^2. \tag{27}$$

Now we can study the dependence of this spectrum on our parameters, feedback included, and learn how to control the emitted light. By certain choices of the parameters one can obtain $S_{\mathrm{inel}}(\mu) < 1$, which is interpreted as squeezing of the light in the channel 2. By using quantum fields and quantum stochastic calculus it is possible to prove that the Heisenberg uncertainty principle implies that the product of $S_{\mathrm{inel}}(\mu)$, computed for a certain value ϑ_2, times the same quantity for $\vartheta_2 + \pi/2$ is always not less than 1 [26].

As an example, we take $k_0 = 0$, $\gamma = 1$, $\bar{n} = 0$, $\left|\alpha_1\right|^2 = \left|\alpha_2\right|^2 = 0.45$ and we use the other parameters to enhance the squeezing of the fluorescence light: we fix a value for μ and then we minimize $S_{\mathrm{inel}}(\mu)$ over the other parameters. For $\Omega = 1.6150$, $\Delta\omega = 1.3833$, $k_1 = 0.3213$, $\vartheta_1 = -1.9307$, $\vartheta_2 = -0.1540$ we get a minimum in $\mu = 2$ of $S_{\mathrm{inel}}(2) = 0.8621$. With the same parameters the coefficient of the delta-spike in the elastic part is $2\pi\left|\alpha_2\right|^2 v^2 = 1.4214$. Similarly, for $\Omega = 3.1708$, $\Delta\omega = 2.5576$, $k_1 = 0.3249$, $\vartheta_1 = -1.7863$, $\vartheta_2 = -0.0760$ there is a minimum in $\mu = 4$ of $S_{\mathrm{inel}}(4) = 0.8572$ and $2\pi\left|\alpha_2\right|^2 v^2 = 1.5356$. These values are

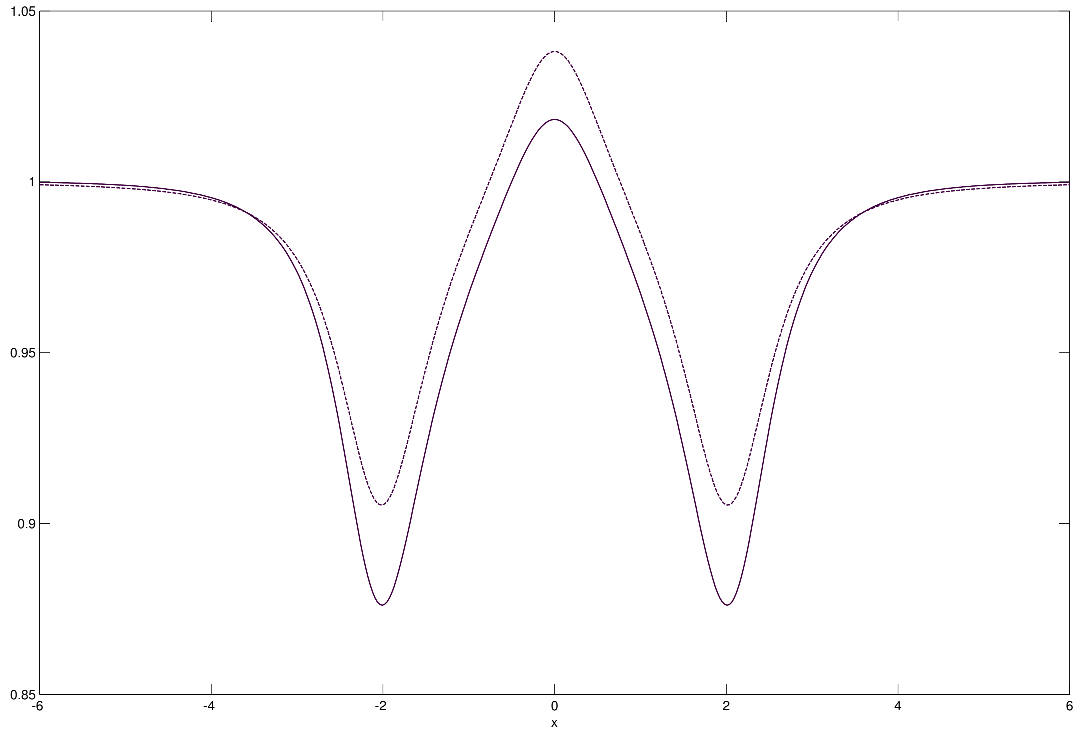

Figure 1: Squeezing control. $S_{\text{inel}}(\mu)$ with and without feedback for $\gamma = 1$, $k_0 = \bar{n} = 0$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$ and: (solid line) $k_1 = 0.3213$, $\vartheta_1 = -1.9307$, $\vartheta_2 = -0.1540$, $\Delta\omega = 1.3833$, $\Omega = 1.6150$; (dotted line) $k_1 = 0$, $\vartheta_2 = -0.1784$, $\Delta\omega = 1.4937$, $\Omega = 1.4360$.

very similar to the ones found in [9, p. 244], where a very different scheme of feedback was considered (the instantaneous and singular feedback à la Wiseman-Milburn [21]). In Figure 1 we plot the inelastic spectrum in function of $x = \mu$ for a case with feedback and a case without feedback. The parameters are optimized to have the largest minimum in $\mu = 2$.

In Figure 2 we plot the previous case with feedback, taken with three different choices of ϑ_2 in order to see the dependence on ϑ_2, which changes the field quadrature under monitoring.

Finally, in Figure 3 we take an high value of Ω and no feedback, $k_1 = 0$. For these parameters, in heterodyne detection, one sees the three-pecked structure of the Mollow spectrum. But here we are in homodyne detection and one sees as the Mollow triplet is built up by the contributions of the various values of ϑ_2.

3.2 Control of direct detection

We maintain the same feedback as before, but now we study the Mandel Q-parameter of the direct detection N_1. We consider only the stationary regime at long times and we take

$$Q_1(t) = \lim_{t_0 \to +\infty} Q_1(t;t_0) = \lim_{t_0 \to +\infty} \left(\frac{V_1(t;t_0)}{M_1(t;t_0)} - M_1(t;t_0) \right).$$

By using again Bloch equation techniques to compute the expressions of Section 2.2.4.4, we get

$$\lim_{t_0 \to +\infty} M_1(t;t_0) = \frac{t}{2} \left|\beta_1\right|^2 (1 + d_3)\,,$$

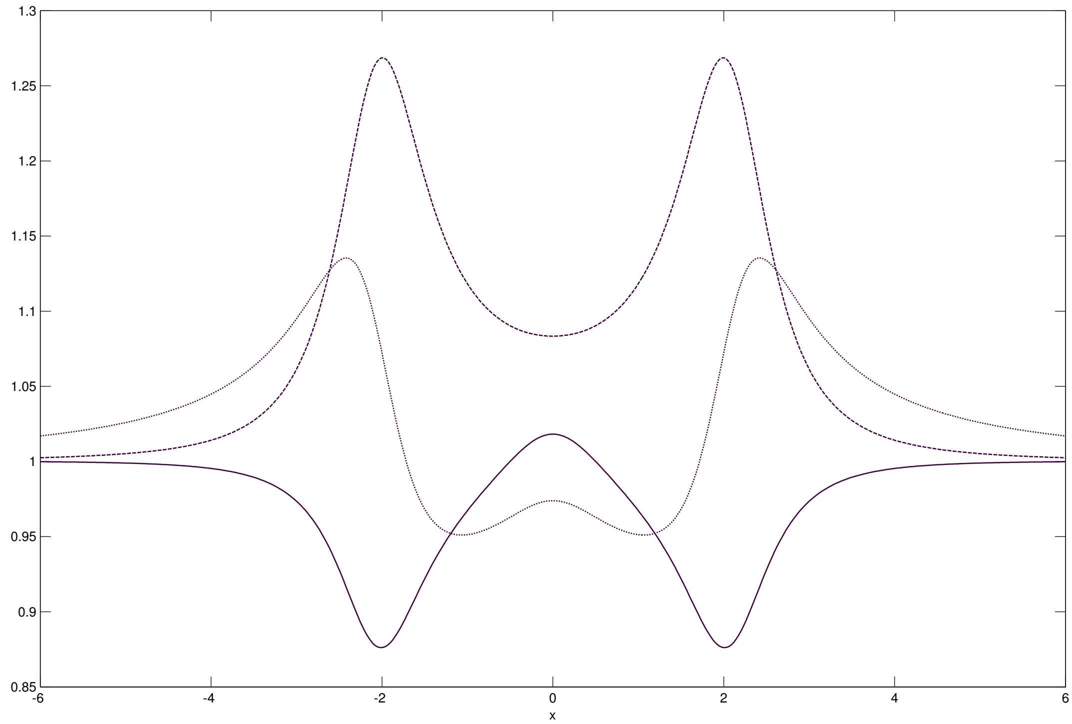

Figure 2: Effect of Heisenberg uncertainty on $S_{\mathrm{inel}}(\mu)$. The parameters are $\gamma = 1$, $k_0 = \bar{n} = 0$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$, $k_1 = 0.3213$, $\vartheta_1 = -1.9307$, $\Delta\omega = 1.3833$, $\Omega = 1.6150$ and: (solid line) $\vartheta_2 = -0.1540$; (dotted line) $\vartheta_2 = \frac{\pi}{4} - 0.1540$; (dashed line) $\vartheta_2 = \frac{\pi}{2} - 0.1540$.

$$Q_1(t) = |\beta_1|^2 \left[\left(\frac{1 - \mathrm{e}^{-At}}{At} - 1 \right) A^{-1} \left(\vec{d} + \vec{e} \right) \right]_3 .$$

where $\vec{d}$, A are defined in Eqs. (26), (27) and $\vec{e}^{\mathrm{T}} = (0,0,1)^{\mathrm{T}}$. By taking a long time interval we get

$$Q_1 := \lim_{t \to +\infty} Q_1(t) = -|\beta_1|^2 \left[A^{-1} \left(\vec{d} + \vec{e} \right) \right]_3 .$$

According to the choice of the parameters we can obtain a sub-Poissonian or a super-Poissonian Q-parameter: $k_0 = 0$, $\gamma = 1$, $\bar{n} = 0$, $|\beta_1|^2 = 0.45$,

- $\Delta\omega = 0$, $k_1 = 1.0126$, $\Omega = 1.0063$, $\vartheta_1 = \pi$: $Q_1 = -0.5094$;
- $\Delta\omega = 0$, $k_1 = 0$, $\Omega = 0.7071$: $Q_1 = -0.3375$;
- $\Delta\omega = 2$, $k_1 = 2.8515$, $\Omega = 2.3516$, $\vartheta_1 = 2.6914$: $Q_1 = -0.4356$;
- $\Delta\omega = 2$, $k_1 = 0$, $\Omega = 2.9155$: $Q_1 = 0.0860$.

Moreover, one can see numerically that in the case of no feedback, $k_1 = 0$, and $\Delta\omega = 2$, we have $Q_1 > 0$ for all $\Omega > 0$. So, in this case feedback control is essential to get sub-Poissonian light. Finally, no relation appears between $Q_1 < 0$ and squeezing; with the parameters which give squeezing we have $Q_1 = 0.0602$ (for $\Omega = 1.6150$, $k_1 = 0.3213$, $\vartheta_1 = -1.9307$, $\Delta\omega = 1.3833$) and $Q_1 = 0.09508$ (for $\Omega = 3.1708$, $k_1 = 0.3249$, $\vartheta_1 = -1.7863$, $\Delta\omega = 2.5576$).

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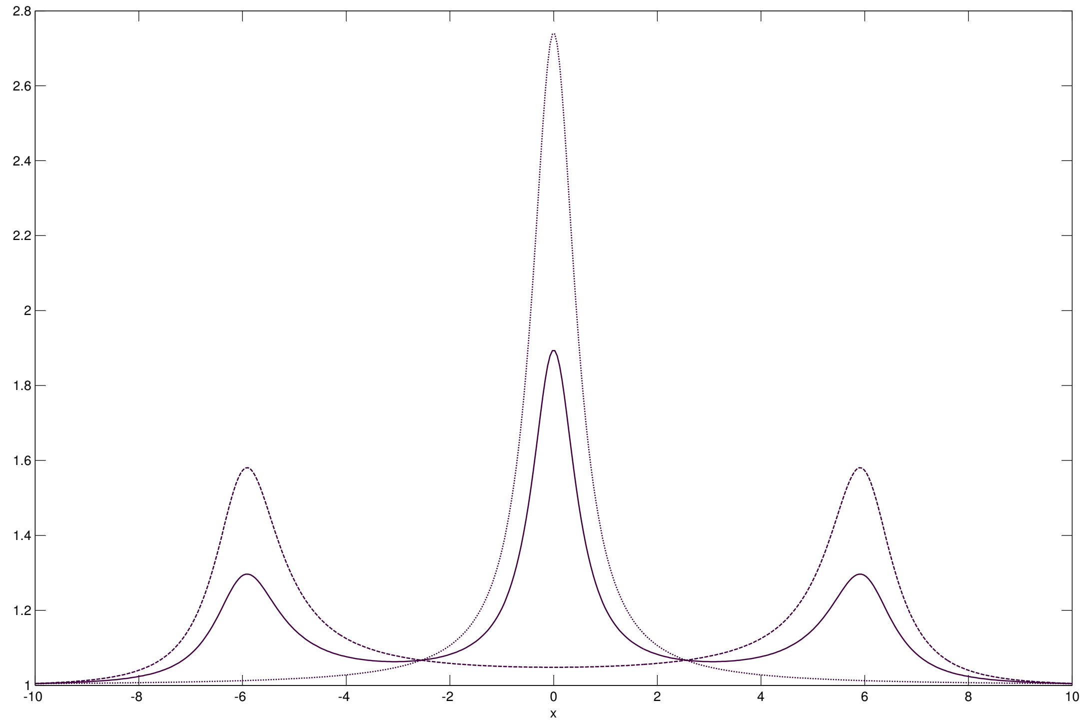

Figure 3: Formation of the Mollow triplet. The parameters are $\gamma = 1$, $k_0 = \bar{n} = 0$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$, $k_1 = 0.0$, $\Delta\omega = 0.0$, $\Omega = 6.0$ and: (dashed line) $\vartheta_2 = 0.0$; (solid line) $\vartheta_2 = \frac{\pi}{4}$; (dotted line) $\vartheta_2 = \frac{\pi}{2}$.

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