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Discrete least squares polynomial approximation with random evaluations - application to parametric and stochastic elliptic PDEs

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Abstract

Motivated by the numerical treatment of parametric and stochastic PDEs, we analyze the least-squares method for polynomial approximation of multivariate functions based on random sampling according to a given probability measure. Recent work has shown that in the univariate case and for the uniform distribution, the least-squares method is optimal in expectation in [1] and in probability in [7], under the condition that the number of samples scales quadratically with respect to the dimension of the polynomial space. Here "optimal" means that the accuracy of the least-squares approximation is comparable with that of the best approximation in the given polynomial space. In this paper, we discuss the optimality of the polynomial least-squares method in arbitrary dimension. Our analysis applies to any arbitrary multivariate polynomial space (including tensor product, total degree or hyperbolic crosses), under the minimal requirement that its associated index set is downward closed. The optimality criterion only involves the relation between the number of samples and the dimension of the polynomial space. We extend our results to the approximation of Hilbert space-valued functions in order to apply them to the approximation of parametric and stochastic elliptic PDEs. As a particular case, we discuss "inclusion type" elliptic PDE models, and derive an exponential convergence estimate for the least-squares method. Numerical results confirm our estimate, yet pointing out a gap between the condition necessary to achieve optimality in the theory, and the condition that in practice yields the optimal convergence rate.

Keywords: approximation theory, polynomial approximation, least squares, parametric and stochastic PDEs, high-dimensional approximation.
AMS classification: 41A10, 41A25, 65N35, 65N12, 65N15, 35J25

1 Introduction

In recent years, various strategies have been proposed for the numerical treatment of parametric and stochastic partial differential equations

$$\mathcal{D}(u, y) = 0, \tag{1}$$

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where $u \mapsto \mathcal{D}(u, y)$ is a partial differential operator depending on a d-dimensional parameter vector

$$y := (y_1, \dots, y_d) \in \Gamma \subset \mathbb{R}^d. \tag{2}$$

Depending on the application, the parameter vector may be deterministic or stochastic. In the latter case y is a random variable distributed over Γ according a probability measure ρ. We denote by (Γ, Σ, ρ) the corresponding probability space, where Σ is the Borel σ-algebra. In certain applications one has to deal with a countable number of parameters $y = (y_j)_{j \geq 1}$ which means that $d = +\infty$.

Assuming well-posedness of the problem in some Banach space X, the solution map

$$y \mapsto u(y), \tag{3}$$

is defined from the parameter domain Γ to the solution space X. In both deterministic and stochastic settings, the main challenge is to approximate the function $y \mapsto u(y)$ with a reasonable cost. In the first setting, one typically searches for approximations that are uniformly accurate over the parameter space Γ, which amounts in measuring the error in $L^\infty(\Gamma, X)$. In the second setting, one is typically interested in approximations that are accurate in a probabilistic sense, such as in the least-squares sense which amounts in measuring the error in $L^2(\Gamma, X, \rho)$.

Polynomial approximation methods of the solution map have been studied for various types of operators $\mathcal{D}$ corresponding to various PDEs. In such methods, the solution map is approximated by polynomial maps of the form

$$u_\Lambda(y) = \sum_{\nu \in \Lambda} u_\nu y^\nu, \tag{4}$$

where $\Lambda \subset \mathcal{F}$ is a finite set of (multi-)indices. The set of multi-indices $\mathcal{F}$ coincides with $\mathbb{N}_0^d$ where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ in the case $d < +\infty$ and denote the countable set of all finitely supported sequences $\nu = (\nu_1, \nu_2, \dots, 0, 0, \dots) \in \mathbb{N}_0^{\mathbb{N}}$ in the case $d = +\infty$. Also, in both cases, the polynomials $y \mapsto y^\nu$ are defined by

$$y^\nu := \prod_{j=1}^{d} y_j^{\nu_j}, \tag{5}$$

with the convention $0^0 = 1$. Note that the coefficients u_ν belong to the Banach space X and therefore the construction of u_Λ requires in principle the computation of $\#(\Lambda)$ such functions. The functions u_Λ are thus selected in $X_\Lambda := X \otimes \mathbb{P}_\Lambda$, where

$$\mathbb{P}_\Lambda := \mathrm{Span}\Big\{ y^\nu \ : \ \nu \in \Lambda \Big\} \tag{6}$$

denotes the polynomial space associated with the index set Λ and with coefficients in $\mathbb{R}$. Throughout this paper, we only work with index sets Λ that have the following natural property.

Definition 1. *The index set Λ is downward closed if*

$$\nu \in \Lambda \quad \text{and} \quad \nu' \leq \nu \Rightarrow \nu' \in \Lambda, \tag{7}$$

where $\nu' \leq \nu$ means that $\nu'_j \leq \nu_j$ for all $j \geq 1$.

Following a more concise and established terminology in the literature, we will also denote by lower set a downward closed set. Note that a lower set always contains the null index

$$0_{\mathcal{F}} := (0, 0, \dots). \tag{8}$$

Considering only polynomial spaces $\mathbb{P}_\Lambda$ associated with such sets is very natural. In particular, the downward closedness property of the set Λ allows us to replace the monomials y^ν in the definition of the spaces $\mathbb{P}_\Lambda$ by any other tensorized basis of the form $P_\nu(y) = \prod_{j\geq 1} P_{\nu_j}(y_j)$ where $(P_k)_{k\geq 0}$ is a sequence of univariate polynomials such that $P_0 = 1$ and P_k has degree exactly equal to k, for example the Legendre polynomials. Polynomial spaces associated with lower index sets have been introduced in [18] in dimension $d = 2$ and in [19] and [15] in higher dimension.

Polynomial approximation is well known to be effective when the solution map has some smoothness. In certain instances, it can even provably break the curse of dimensionality, in the sense that an algebraic convergence rate with respect to $\#(\Lambda)$ can be established even for functions of countably many parameters $d = +\infty$. Such results are proven in [5, 6, 3] for the model parametric elliptic equation

$$-\operatorname{div}(a\nabla u) = f \ \text{ in } \ D \subset \mathbb{R}^q, \ \ u = 0 \ \text{ on } \ \partial D, \tag{9}$$

where $D \subset \mathbb{R}^q$ is a Lipschitz domain, $f \in H^{-1}(D)$, and the diffusion coefficient has the form

$$a(x, y) := \bar{a}(x) + \sum_{j\geq 1} y_j \psi_j(x), \tag{10}$$

with the functions ψ_j and $\bar{a}$ in $L^\infty(D)$, and $y \in \Gamma := [-1, 1]^{\mathbb{N}}$. Assuming the uniform ellipticity assumption

$$0 < r \leq a(x, y) \leq R < +\infty, \quad x \in D, \ y \in \Gamma, \tag{11}$$

the solution map is well defined from Γ to the Hilbert space $X := H^1_0(D)$. Then, it is proved in [3] that if $(\|\psi_j\|_{L^\infty})_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $0 < p < 1$, there exists a sequence of lower sets

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \mathcal{F}, \quad \#(\Lambda_m) = m, \tag{12}$$

such that

$$\inf_{v\in X_{\Lambda_m}} \|u - v\|_{L^\infty(\Gamma,X)} \leq Cm^{-s}, \quad s := \frac{1}{p} - 1 > 0. \tag{13}$$

Similar results with a slightly improved convergence rate are obtained in [5, 6, 2] for the $L^2(\Gamma, X, \rho)$ norm, where ρ denotes the uniform probability measure: under the same assumptions there exists a sequence of lower sets such that

$$\inf_{v\in X_{\Lambda_m}} \|u - v\|_{L^2(\Gamma,X,\rho)} \leq Cm^{-s}, \quad s := \frac{1}{p} - \frac{1}{2} > 0. \tag{14}$$

These general convergence results are extended in [4] to other models than (9).

The construction of sequences of sets $(\Lambda_m)_{m\geq 1}$ which achieve the convergence rates (13) or (14), and therefore of the polynomial spaces $\mathbb{P}_{\Lambda_m}$, is critical in the design of algorithms for high-dimensional approximation. Sequences of quasi-optimal sets giving such rates, with possibly a suboptimal constant $C > 0$ can either be derived from a-priori estimates in [5, 6, 12, 2, 4] or by an adaptive search [16, 3, 4]. The resulting spaces $\mathbb{P}_{\Lambda_m}$ typically differ from the standard multivariate polynomial spaces $\mathbb{P}_k$ of fixed total degree.

Given a finite index set Λ, several strategies can be used to compute $u_\Lambda \in X_\Lambda$:

1. Taylor expansions [3] can be recursively computed in the case of problems with affine parameter dependence such as (9). Adaptive methods based on such expansions have been proved to converge uniformly with the same rate as in (13).

2. Projection methods [11, 12, 5, 16] produce near best approximations in X_Λ for the metric $L^2(\Gamma, X, \rho)$ where ρ is a chosen measure in the parameter space. In addition, in the Galerkin framework, it is possible to use techniques of a-posteriori analysis in order to adaptively build the sequence of index sets $(\Lambda_m)_{m\geq 1}$. This approach was developed in [16] for the problem (9), and proved to converge with the same rate as in (14).

3. Collocation methods [10, 12, 7, 21, 22, 4] produce a polynomial approximation in X_Λ based on the data of particular solution instances $u^i := u(y^i)$ for some chosen values $y^i \in \Gamma$ of the parameter vector with $i = 1, \ldots, n$. One significant advantage of this approach is that it is *non intrusive*: the u^i can be computed by any given numerical solver for the problem (1) and the polynomial approximation is built from these solutions by numerical techniques similar to those employed for scalar-valued maps such as interpolation or least-squares regression.

The convergence analysis of collocation methods is less satisfactory in the sense that convergence rates similar to (13) and (14) do not seem to have been established for such methods. This is in part due to the difficulty to control the stability of interpolation or least-squares projection for general multivariate polynomial spaces. For interpolation methods, several results have been recently established in [4] showing that the convergence rate in (13) can be achieved if the interpolation points are carefully selected. Least-squares methods have been recently analyzed in [1, 7] in the stochastic setting, assuming that the samples y^i are independent realizations of the random variable y, therefore identically distributed according to ρ. This analysis reveals that in the univariate case $\Gamma = [-1, 1]$ and for the uniform distribution, the least-squares method is stable and produces a near best approximation in the $L^2(\Gamma, X, \rho)$ norm, under the condition that the number of samples n scales quadratically (up to a logarithmic factor) with respect to the dimension m of the polynomial space $\mathbb{P}_{m-1}$.

The objective of this paper is to address the problem of the stability and convergence of the polynomial least-squares method in the general context of the spaces X_Λ associated with arbitrary lower sets. We begin in §2 by discussing the least-squares method for a real-valued function in a general framework not limited to polynomials and recalling recent stability and approximation results established in [1]. In §3 we focus on the particular framework of the multivariate polynomial spaces $\mathbb{P}_\Lambda$. Our analysis reveals in particular that, with $\Gamma = [-1, 1]^d$ and the uniform distribution, the same scaling $n \sim (\#(\Lambda))^2$ as in the univariate case, ensures stability and near best approximation of the method *independently of the dimension d*.

Then in §4, we show how a similar analysis applies to X-valued functions, where X is a Hilbert space, and therefore to the solutions of parametric and stochastic PDEs. As a relevant example, the equation (9) with random inclusions in the diffusion coefficient is discussed in §5, and numerical illustration for this example are given in §6.

2 Discrete least-squares approximations

Let (Γ, Σ, ρ) be a probability space. We denote by $L^2(\Gamma, \rho)$ the Hilbert space of real-valued square integrable functions with respect to ρ and denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the associated inner product and norm, i.e.

$$\langle v, w \rangle := \int_\Gamma v(y) w(y) d\rho(y), \quad \|v\| := \sqrt{\langle v, v \rangle}, \quad v, w \in L^2(\Gamma, \rho). \tag{15}$$

We consider V_m a finite dimensional space of $L^2(\Gamma, \rho)$ with $\dim(V_m) = m$. We assume that the functions belonging to V_m are defined everywhere over Γ. We let $\mathcal{B}_L := (L_j)_{1 \le j \le m}$ be any orthonormal basis of V_m with respect to the above inner product. The best approximation of a function $u \in L^2(\Gamma, \rho)$ in the least-squares sense is given by

$$P_m u = \sum_{j=1}^m c_j L_j, \quad c_j = \langle u, L_j \rangle, \tag{16}$$

and its best approximation error by

$$e_m(u) := \inf_{v \in V_m} \|u - v\| = \|u - P_m u\|. \tag{17}$$

If u is unknown and if $(z^i)_{i=1,\cdots,n}$ are noiseless or noisy observations of u at the points $(y^i)_{i=1,\cdots,n}$ where the y^i are i.i.d. random variables distributed according to ρ, we introduce the discrete least-squares approximation

$$w := \underset{v \in V_m}{\operatorname{argmin}} \sum_{i=1}^{n} |z^j - v(y^j)|^2. \tag{18}$$

This minimization problem always has a solution, which may not be unique. In particular, it is never unique in the regime $m > n$. In the sequel, we only consider the regime $m \leq n$. In the noise free case, $z^i = u(y^i)$, the solution may be viewed as the orthogonal projection of u onto V_m with respect to the inner product $\langle \cdot, \cdot \rangle_n$ associated with the empirical semi-norm

$$\|v\|_n = \Big(\frac{1}{n} \sum_{i=1}^{n} |v(y^i)|^2 \Big)^{\frac{1}{2}}. \tag{19}$$

In this case, we denote the solution w of the problem (18) by $P_m^n u$. The projection $P_m^n u$ depends on the sample $(y^j)_{1 \leq j \leq n}$, so that $P_m^n u$ is a "random" least-squares projector. In both the noisy and noiseless case, the coordinate vector $\mathbf{w} \in \mathbb{R}^m$ of w in the basis $\mathcal{B}_L$ is the solution to the system

$$\mathbf{G}\mathbf{w} = \mathbf{J}\mathbf{z}, \tag{20}$$

where $\mathbf{G}$ and $\mathbf{J}$ are the $m \times m$ and $m \times n$ matrices given by

$$\mathbf{G}_{ij} := \langle L_i, L_j \rangle_n, \quad \text{and} \quad \mathbf{J}_{ij} := \frac{L_i(y^j)}{n} \tag{21}$$

and $\mathbf{z} \in \mathbb{R}^n$ is the vector of coordinates z^j. When $\mathbf{G}$ is not singular, then the solution w of (18) is given by

$$w = \sum_{j=1}^{n} z^j \pi_j. \tag{22}$$

where $\mathcal{B}_\pi := \{\pi_1, \ldots, \pi_n\}$ are the elements of V_m given by

$$\mathcal{B}_\pi = \left(\mathbf{G}^{-1}\mathbf{J}\right)^t \mathcal{B}_L, \tag{23}$$

with the product matrix-basis to be understood in the obvious sense. In the case where $\mathbf{G}$ is singular, we set by convention $w := 0$.

If u satisfies a uniform bound $|u(y)| \leq L$ over Γ, where L is known, we introduce the truncated least-squares approximation

$$\tilde{w} = T_L(w), \quad T_L(t) := \operatorname{sign}(t) \min\{L, |t|\}, \tag{24}$$

which we also denote by $\tilde{P}_m^n u$ in the noiseless case.

The analysis in [1, 7] investigates the minimal amount of sampling $n(m) \geq m$ that allows an accurate approximation of the unknown function u by the random approximations w or $\tilde{w}$. The accuracy here is to be understood in the sense of a comparison between the error $\|u - w\|$ and the best approximation error $e_m(u)$. This analysis is based on probabilistic estimates comparing the norm $\|\cdot\|$ and its empirical counterpart $\|\cdot\|_n$ uniformly over the space V_m. This comparison amounts in estimating the deviation of the random matrix $\mathbf{G}$ from its expectation $\mathbb{E}(\mathbf{G}) = \mathbf{I}$, where $\mathbf{I}$ is the $m \times m$ identity matrix, since for $v \in V_m$ and $\mathbf{v}$ the vector representing v in the basis $\mathcal{B}_L$, one has

$$\|v\|_n^2 = \mathbf{v}^T \mathbf{G} \mathbf{v} \quad \text{and} \quad \|v\|^2 = \mathbf{v}^T \mathbf{I} \mathbf{v}, \tag{25}$$

so that, for any $0 < \delta < 1$,

$$|||\mathbf{G} - \mathbf{I}||| \leq \delta \Leftrightarrow |||v||_n^2 - \|v\|^2| \leq \delta \|v\|^2, \quad v \in V_m, \tag{26}$$

where $|||\cdot|||$ denotes the spectral norm of a matrix. For this purpose, one introduces the quantity

$$K(V_m) := \sup_{y \in \Gamma} \sum_{j=1}^{m} |L_j(y)|^2. \tag{27}$$

One can easily check, using Cauchy-Schwartz inequality, that

$$K(V_m) = \sup_{v \in V_m, \|v\|=1} \|v\|_{L^\infty(\Gamma)}^2, \tag{28}$$

from which we deduce that $K(V_m)$ does not depend on the choice of the orthonormal basis $\mathcal{B}_L$ and only depends on V_m and ρ.

The main results in [1] imply that for any $r > 0$ and the number of samples n large enough such that

$$\frac{n}{\ln n} \geq \frac{K(V_m)}{\kappa}, \quad \kappa := \frac{1 - \ln 2}{2 + 2r}, \tag{29}$$

the following hold:

- The deviation between $\mathbf{G}$ and $\mathbf{I}$ satisfies

$$\Pr\left\{|||\mathbf{G} - \mathbf{I}||| > \frac{1}{2}\right\} \leq 2n^{-r}. \tag{30}$$

- In the noiseless case, if u satisfies a uniform bound L over Γ, then

$$\mathbb{E}(\|u - \tilde{P}_m^n u\|^2) \leq (1 + \epsilon(n)) e_m(u)^2 + 8L^2 n^{-r}, \tag{31}$$

  where $\epsilon(n) := \frac{4\kappa}{\ln(n)}$.

- In the noisy case, if u satisfies a uniform bound L over Γ, then

$$\mathbb{E}(\|u - \tilde{w}\|^2) \leq (1 + 2\epsilon(n)) e_m(u)^2 + 8\Big(L^2 n^{-r} + \sigma^2 \frac{m}{n}\Big), \tag{32}$$

  where $\sigma^2 := \max_{y \in \Gamma} \mathbb{E}(|z - u(y)|^2 | y)$ is the noise level.

It is also desirable to estimate the error between u and its estimator in probability rather than in expectation. In the following we give such an estimate, in the noisless case and for the non-truncated estimator $w = P_m^n u$, however using the best approximation error in the uniform norm

$$e_m(u)_\infty := \inf_{v \in V_m} \|u - v\|_{L^\infty(\Gamma)}, \tag{33}$$

which is obviously larger than $e_m(u)$. The following result was stated in the particular case of polynomial least squares in [7].

Theorem 1. *Under condition* (29)*, one has*

$$\Pr\Big(\|u - P_m^n u\| \geq (1 + \sqrt{2}) e_m(u)_\infty\Big) \leq 2n^{-r}. \tag{34}$$

Proof: Introducing the event set $\Omega_+^n := \{|||\mathbf{G} - \mathbf{I}||| \leq \frac{1}{2}\}$, we know from (30) that $\Pr(\Omega_+^n) \geq 1 - 2n^{-r}$. Given any draw in Ω_+^n, we have for any $v \in V_m$

$$\|u - P_m^n u\| \leq \|u - v\| + \|v - P_m^n u\| \leq \|u - v\| + \sqrt{2}\|v - P_m^n u\|_n, \tag{35}$$

where we have used (26). By the orthogonality identity $\|u - v\|_n^2 = \|u - P_m^n u\|_n^2 + \|P_m^n u - v\|_n^2$, we deduce

$$\|u - P_m^n u\| \leq \|u - v\| + \sqrt{2}\|u - v\|_n \leq (1 + \sqrt{2})\|u - v\|_\infty,$$

which completes the proof. ∎

All these results above lead to the problem of understanding which minimal amount n of sample ensures the validity of condition (29). In the one-dimensional case $d = 1$, with $V_m = \mathbb{P}_{m-1}$ and ρ being the uniform density over $\Gamma = [-1, 1]$, elementary computations using the Legendre polynomials show that $K(V_m) = m^2$ and therefore (29) holds for $\frac{n}{\ln n} \sim m^2$, meaning that n scales like m^2 up to a logarithmic factor. This relation between n and m was also used in [7] to establish (30) and (34) by arguments which are more tied to the use of univariate polynomials and the uniform measure. The next section discusses the implications of condition (29) for the multivariate polynomial spaces $\mathbb{P}_\Lambda$.

3 Least-squares approximation with multivariate polynomials

In this section, we investigate the implications of the condition (29) in the setting of multivariate polynomial spaces $\mathbb{P}_\Lambda$. We consider the domain $\Gamma := [-1, 1]^d$ with $d \in \mathbb{N}$ and the uniform measure ρ over Γ, i.e.

$$d\rho := \otimes_{j=1}^d \frac{dy_j}{2}. \tag{36}$$

We may also consider the case $\Gamma := [-1, 1]^{\mathbb{N}}$ for which $d = +\infty$ and ρ is the uniform measure defined over Γ in the usual manner.

We use the notations $L^2(\Gamma, \rho)$, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ of the previous section and denote $\mathcal{F}$ the set of multi-indices in the cases $d < +\infty$ and $d = +\infty$ as explained in the introduction. Given Λ a finite subset of $\mathcal{F}$, u the unknown function and $(z^i)_{i=1,\cdots,n}$ noiseless or noisy observations of u at the points $(y^i)_{i=1,\cdots,n}$ where the y^i are i.i.d. random variables distributed according to ρ, we introduce the polynomial discrete least-squares approximation

$$w_\Lambda := \operatorname*{argmin}_{v \in \mathbb{P}_\Lambda} \sum_{i=1}^n |z^j - v(y^j)|^2, \tag{37}$$

where the polynomial space $\mathbb{P}_\Lambda$ is defined as in (6). In order to study the optimality of the least-squares approximation, we need to investigate the growth of the quantity of interest $K(\mathbb{P}_\Lambda)$ introduced in (27) in the present setting. We shall show that, under the minimal requirement that the index set Λ is downward closed, we have as in the one-dimensional case that $K(\mathbb{P}_\Lambda) \leq (\#\Lambda)^2$.

We introduce $(L_k)_{k \geq 0}$ the univariate Legendre polynomials normalized according to

$$\int_{-1}^1 |L_k(t)|^2 \frac{dt}{2} = 1, \tag{38}$$

and introduce $(L_\nu)_{\nu \in \mathcal{F}}$ the multivariate Legendre polynomials defined by

$$L_\nu(y) := \prod_{j=1}^d L_{\nu_j}(y_j). \tag{39}$$

The family $(L_\nu)_{\nu\in\mathcal{F}}$ is an orthonormal basis of the space $L^2(\Gamma,\rho)$. Using the remarks on lower sets given in the introduction, one has that $(L_\nu)_{\nu\in\Lambda}$ is an orthonormal basis of $\mathbb{P}_\Lambda$ if the index set Λ is downward closed. Therefore

$$K(\mathbb{P}_\Lambda) := \sum_{\nu\in\Lambda} \|L_\nu\|^2_{L^\infty(\Gamma)}. \tag{40}$$

To lighten the notation, we have shortened $K(\mathbb{P}_\Lambda)$ to $K(\Lambda)$. Since the univariate Legendre polynomials satisfy $\|L_k\|_{L^\infty([-1,1])} = \sqrt{2k+1}$, then

$$K(\Lambda) = \sum_{\nu\in\Lambda} \prod_{j:\nu_j\neq 0} (2\nu_j+1) \tag{41}$$

Theses quantities have already been studied in [4] and proved to have moderate growth for finite lower sets. To keep our document self contained, we recall the result of [4] with its proof in the case $d = +\infty$. The case $d < +\infty$ is a straightforward consequence.

Lemma 1. *For any finite lower set $\Lambda \subset \mathcal{F}$, the quantity $K(\Lambda)$ satisfies*

$$\#(\Lambda) \leq K(\Lambda) \leq (\#(\Lambda))^2. \tag{42}$$

Proof: The first inequality is obvious. To prove the second inequality, we use induction on $n_\Lambda := \#(\Lambda) \geq 1$. When $n_\Lambda = 1$, then $\Lambda = \{0_\mathcal{F}\}$ and an equality holds. Let $n \geq 1$ and Λ denote a lower set with $n_\Lambda = n+1$. Without loss of generality, we suppose that $\nu_1 \neq 0$ for some $\nu \in \Lambda$. We introduce the index sets

$$\Lambda_k := \left\{ \hat{\nu} \in \mathcal{F} : (k,\hat{\nu}) \in \Lambda \right\}, \quad k \geq 0. \tag{43}$$

Here $(k,\hat{\nu})$ denote the multi-index $(k,\hat{\nu}_1,\hat{\nu}_2,\cdots)$. Since Λ is downward closed and finite, then it is easy to check that the sets Λ_k are finite, downward closed (when not empty) and satisfy

$$\cdots \subset \Lambda_k \subset \cdots \subset \Lambda_1 \subset \Lambda_0. \tag{44}$$

Let us also remark that there exists $J \geq 0$ such that $\Lambda_k = \emptyset$ for any $k > J$ and that $\#(\Lambda_0) \leq n_\Lambda - 1 = n$ since $\nu_1 \neq 0$ for some $\nu \in \Lambda$. Therefore the induction hypothesis applied to the sets Λ_k, implies

$$K(\Lambda) = \sum_{k=0}^{J} (2k+1) K(\Lambda_k) \leq \sum_{k=0}^{J} (2k+1)(\#(\Lambda_k))^2 \,. \tag{45}$$

Now, by the nestedness of the sets Λ_k, we have

$$k(\#(\Lambda_k))^2 \leq \#(\Lambda_k)\#(\Lambda_0) + ... + \#(\Lambda_k)\#(\Lambda_{k-1}), \quad 1 \leq k \leq J. \tag{46}$$

Therefore

$$K(\Lambda) \leq \sum_{k=0}^{J} (\#(\Lambda_k))^2 + 2\sum_{k=1}^{J}\sum_{k'=0}^{k-1} \#(\Lambda_k)\#(\Lambda_{k'}) = \Big(\sum_{k=0}^{J} \#(\Lambda_k)\Big)^2. \tag{47}$$

Since $\#(\Lambda) = \sum_{k=0}^{J} \#(\Lambda_k)$, we conclude the proof. ∎

The previous bound is valid for any lower set independently of its shape. In addition, the inequality is sharp, in the sense that the equality holds for certain types of lower sets. Indeed, given $\nu \in \mathcal{F}$ supported in $\{1,\cdots,J\}$ and considering the *rectangle* index set

$$\mathcal{R}_\nu := \{\mu \in \mathcal{F} : \mu \leq \nu\}, \tag{48}$$

one has

$$K(\mathcal{R}_\nu) = \sum_{\mu \le \nu} \prod_{1 \le j \le J} (2\mu_j + 1) = \prod_{1 \le j \le J} \sum_{\mu_j \le \nu_j} (2\mu_j + 1) = \prod_{1 \le j \le J} (\nu_j + 1)^2 = (\#(\mathcal{R}_\nu))^2. \tag{49}$$

However, we expect this bound to be pessimistic for lower sets that have shapes very different from rectangles. For instance, let $k \ge 1$ and consider the lower set

$$S_{k,d} := \{\nu \in \mathbb{N}_0^d : |\nu| \le k\}, \tag{50}$$

where $|\nu| := \sum_{j=1}^d \nu_j$, associated with the polynomial space $\mathbb{P}_{S_{k,d}}$ of *total degree* k in dimension d.

By the inequality between the arithmetic and geometric means, one has for any $\nu \in S_{k,d}$

$$\prod_{1 \le j \le d} (2\nu_j + 1) \le \Big(\frac{1}{d} \sum_{1 \le j \le d} (2\nu_j + 1)\Big)^d = \Big(\frac{2|\nu|}{d} + 1\Big)^d \le \Big(\frac{2k}{d} + 1\Big)^d. \tag{51}$$

Therefore (see also [23, Chapter 2 and Chapter 3])

$$K(S_{k,d}) \le \Big(\frac{2k}{d} + 1\Big)^d \#(S_{k,d}), \tag{52}$$

and $\Big(\frac{2k}{d} + 1\Big)^d$ is very small compared to $\#(S_{k,d}) = \binom{d+k}{k}$ for large values of d. On Figure 1, we provide a comparison between $\#(S_{k,d})$, $K(S_{k,d})$ and $(\#(S_{k,d}))^2$ for various dimensions.

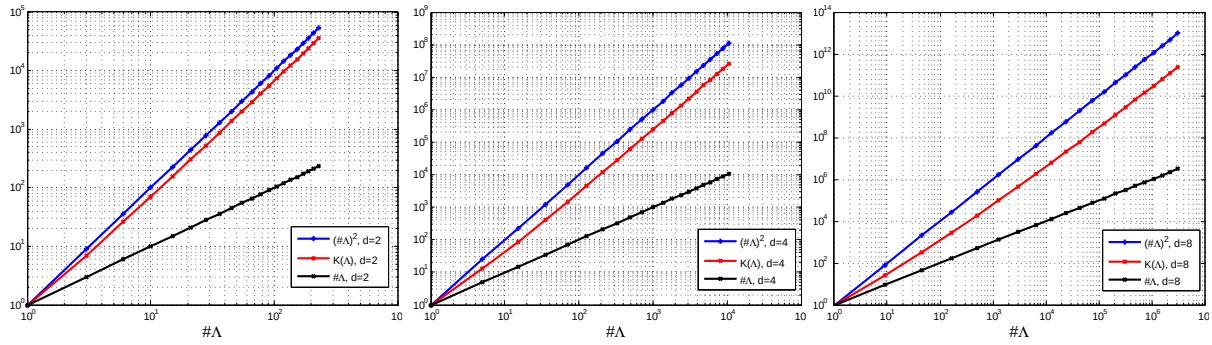

Figure 1: Comparison between $\#(\Lambda)$, $K(\Lambda)$ and $(\#(\Lambda))^2$ in the case where $\Lambda = S_{k,d}$ (see (50)). Left: $d = 2$. Center: $d = 4$. Right: $d = 8$.

In light of Lemma 1, given a finite lower set Λ, if the number of samples n scales like $(\#(\Lambda))^2$ up to a logarithmic factor, according to

$$\frac{n}{\ln n} \ge \frac{(\#(\Lambda))^2}{\kappa}, \quad \kappa := \frac{1 - \ln 2}{2 + 2r}, \tag{53}$$

then the results (30), (31), (32) and (34) hold in the the present setting of multivariate polynomial least-squares approximation.

It is interesting to see if the estimates on the quantity $K(\mathbb{P}_\Lambda)$ can be improved when using other standard probability measures over Γ. In what follows, we study this quantity when the measure ρ is the tensorized Chebyshev measure, i.e.

$$d\rho := \otimes_{j=1}^d \varrho(y_j) dy_j, \quad \text{with} \quad \varrho(t) := \frac{1}{\pi} \frac{1}{\sqrt{1 - t^2}}. \tag{54}$$

Using in this case the notation $K_T(\Lambda) = K(\mathbb{P}_\Lambda)$, we have

$$K_T(\Lambda) := \Big\| \sum_{\nu\in\Lambda} |T_\nu|^2 \Big\|_{L^\infty(\Gamma)}, \tag{55}$$

where $T_\nu(y) = \prod_{j\geq 1} T_{\nu_j}(y_j)$ is the tensorization of the Chebyshev polynomials $(T_k)_{k\geq 0}$ normalized according to

$$\int_{-1}^{1} |T_k(t)|^2 \varrho(t)dt = 1. \tag{56}$$

It is easily checked that these polynomials are related to the classical Chebyshev polynomials of the first kind by $T_k(\cos\theta) = \sqrt{2}\cos(k\theta)$ for any $k \geq 1$ and $T_0 = 1$. It follows that

$$K_T(\Lambda) := \sum_{\nu\in\Lambda} 2^{\#(\mathrm{supp}(\nu))} \tag{57}$$

where $\mathrm{supp}(\nu) := \{1 \leq j \leq d : \nu_j \neq 0\}$ is the support of $\nu \in \mathcal{F}$. Given ν in Λ, with Λ being a lower set, the multi-index μ that has the same support as ν and has entries 1 satisfies $\mu \leq \nu$, so that $\mu \in \Lambda$ and $\mathcal{R}_\mu \subset \Lambda$. This implies that $2^{\#(\mathrm{supp}(\nu))} = \#(\mathcal{R}_\mu) \leq \#(\Lambda)$. Therefore we obtain

$$K_T(\Lambda) \leq (\#(\Lambda))^2, \tag{58}$$

which is the same bound as for the uniform measure.

Sharper bounds can be established by a finer analysis. We first prove an elementary lemma.

Proposition 1. *For any real positive numbers $a_0 \geq a_1 \geq ... \geq a_k$ and any $\alpha \geq \frac{\ln 3}{\ln 2}$, one has*

$$a_0^\alpha + 2(a_1^\alpha + \ldots + a_k^\alpha) \leq (a_0 + \ldots + a_k)^\alpha. \tag{59}$$

Proof: We use induction on k. For $k = 0$, an equality holds in (59). For $k = 1$, since the function $x \mapsto (x + a_1)^\alpha - x^\alpha$ is increasing in $[a_1, +\infty[$ then its value at a_0 is greater than its value at a_1, that is

$$2a_1^\alpha \leq (2^\alpha - 1)a_1^\alpha \leq (a_0 + a_1)^\alpha - a_0^\alpha \tag{60}$$

where we have used $2^\alpha > 3$. Now let $k \geq 1$ and $a_0 \geq a_1 \geq ... \geq a_{k+1}$ be real positive numbers. By the induction hypothesis at steps 1 and k, we infer

$$\begin{aligned} (a_0 + ... + a_{k+1})^\alpha &= \Big((a_0 + ... + a_k) + a_{k+1}\Big)^\alpha \\ &\geq (a_0 + ... + a_k)^\alpha + 2a_{k+1}^\alpha \\ &\geq a_0^\alpha + 2(a_1^\alpha ... + a_k^\alpha) + 2a_{k+1}^\alpha \\ &= a_0^\alpha + 2(a_1^\alpha ... + a_{k+1}^\alpha). \end{aligned} \tag{61}$$

The proof is then complete. ∎

Lemma 2. *For any lower set $\Lambda \subset \mathcal{F}$, the quantity $K_T(\Lambda)$ satisfies*

$$K_T(\Lambda) \leq (\#(\Lambda))^\beta, \quad \text{with} \quad \beta = \frac{\ln 3}{\ln 2}. \tag{62}$$

Proof: We use induction on $n_\Lambda := \#(\Lambda)$. When $n_\Lambda = 1$, then $\Lambda = \{0_\mathcal{F}\}$ and an equality holds. Let $n \geq 1$ and Λ denote a lower set with $n_\Lambda = n + 1$. Without loss of generality, we suppose that $\nu_1 \neq 0$ for some

$\nu \in \Lambda$. Defining $J \geq 0$ and the sets Λ_k as in the proof of Lemma 1 and using the induction hypothesis with these sets, we obtain

$$K_T(\Lambda) = \sum_{k=0}^{J} \gamma(k) K_T(\Lambda_k) \leq \sum_{k=0}^{J} \gamma(k) (\#(\Lambda_k))^{\frac{\ln 3}{\ln 2}}, \tag{63}$$

where γ is defined by $\gamma(0) = 1$ and $\gamma(k) = 2$ for $k \geq 1$. Using (59), we infer

$$K_T(\Lambda) \leq (\#(\Lambda_0))^{\frac{\ln 3}{\ln 2}} + 2\sum_{k=1}^{J} (\#(\Lambda_k))^{\frac{\ln 3}{\ln 2}} \leq \Big(\#(\Lambda_0) + \#(\Lambda_1) + \cdots + \#(\Lambda_J)\Big)^{\frac{\ln 3}{\ln 2}} = (\#(\Lambda))^{\frac{\ln 3}{\ln 2}}. \tag{64}$$

The proof is then complete. ∎

The bound (62) is sharp for certain type of lower sets. For instance if ν is the multi-index such that $\nu_1 = \cdots = \nu_J = 1$ and $\nu_j = 0$ for $j > J$, then

$$K_T(\mathcal{R}_\nu) = \sum_{\mu \leq \nu} 2^{\#(\mathrm{supp}(\mu))} = \sum_{\mu \leq \nu} 2^{\mu_1 + \cdots + \mu_J} = \prod_{j=1}^{J} (1+2) = 3^J = (2^J)^\beta = (\#(\mathcal{R}_\nu))^\beta. \tag{65}$$

In the case of finite dimension $d < +\infty$, the following bound can be easily obtained from the result of Lemma 2:

$$K_T(\Lambda) \leq \min\left\{(\#(\Lambda))^{\frac{\ln 3}{\ln 2}}, 2^d \#(\Lambda)\right\}.$$

Let us note that algebraic bounds can also be obtained for the quantity $K(\mathbb{P}_\Lambda)$ when the measure ϱ is any measure of the type

$$\varrho(t) = \frac{(1-t)^{\alpha_1}(1+t)^{\alpha_2}}{W_{\alpha_1,\alpha_2}}, \quad W_{\alpha_1,\alpha_2} := \int_{-1}^{1} (1-t)^{\alpha_1}(1+t)^{\alpha_2} dt, \quad \alpha_1, \alpha_2 > -1. \tag{66}$$

Indeed, for such measures, the Jacobi polynomials $(P_k^{\alpha_1,\alpha_2})_{k\geq 0}$ that are orthonormal with respect to ϱ satisfy $P_0^{\alpha_1,\alpha_2} = 1$ and

$$\|P_k^{\alpha_1,\alpha_2}\|_{L^\infty[-1,1]} \leq C(k+1)^\theta, \quad k \geq 1, \tag{67}$$

for some constant $C \geq 1$, depending on α_1 and α_2, and some constant $\theta \geq 0$. Using similars arguments to the proof of Lemmas 1 and 2, one can derive in this case the algebraic bound

$$K(\mathbb{P}_\Lambda) \leq (\#(\Lambda))^\beta, \quad \beta = 2\theta + \frac{\ln(C^2+1)}{\ln 2}. \tag{68}$$

4 Discrete least-squares approximation of Hilbert-valued functions

In sections 2 and 3, the functions that we propose to approximate using the least-squares method are real valued. Motivated by the application to parametric PDEs, we investigate the applicability of the least-squares method in the approximation of X-valued functions, with X being any Hilbert space. Similar to §2, we work in the abstract setting of a probability space (Γ, Σ, ρ). We study the least-squares approximation of functions u belonging to the Bochner space

$$L^2(\Gamma, X, \rho) := \left\{u : \Gamma \to X, \|u\| := \int_\Gamma \|u(y)\|_X^2 d\rho(y) < +\infty\right\}. \tag{69}$$

Therefore $L^2(\Gamma, X, \rho) = X \otimes L^2(\Gamma, \rho)$ and we are interested in the least-squares approximation in spaces of type $X \otimes V_m$ where V_m is an m-dimensional subspace of $L^2(\Gamma, \rho)$. Given $u \in L^2(\Gamma, X, \rho)$ an unknown function and $(z^i)_{i=1,\cdots,n}$ noiseless or noisy observations of u at the points $(y^i)_{i=1,\cdots,n}$ where the y^i are i. i. d. random variables distributed according to ρ, we consider the discrete least-squares approximation

$$w := \operatorname*{argmin}_{v \in X \otimes V_m} \sum_{i=1}^{n} \|z^j - v(y^j)\|_X^2. \tag{70}$$

The purpose of this section is to briefly discuss the extension of the results from §2 to this framework.

Let $\mathcal{B}_L$ be an orthonormal basis of the space V_m with respect to the measure ρ and consider the matrices $\mathbf{G}$ and $\mathbf{J}$ and the family $\mathcal{B}_\pi \subseteq V_m$ obtained from the basis $\mathcal{B}_L$ and the points $(y^i)_{i=1,\cdots,n}$ as in §2. When the matrix $\mathbf{G}$ is not singular, we claim that the solution to (70) has the same form

$$\sum_{k=1}^{n} z^k \pi_k, \tag{71}$$

with $z^k \in X$ for all $k = 1, \ldots, n$, as in the real-valued case. Indeed, for any $g \in X$, the real-valued function $w_g := \sum_{k=1}^n \langle z^k, g\rangle \pi_k \in V_m$ is the solution to the least-squares problem

$$w_g = \operatorname*{argmin}_{h \in V_m} \sum_{i=1}^{n} |\langle z^i, g\rangle - h(y^i)|^2, \tag{72}$$

which implies the orthogonality relations

$$\sum_{i=1}^{n} \langle \sum_{k=1}^{n} z^k \pi_k(y^i), g L_j(y^i)\rangle = \sum_{i=1}^{n} \langle z^i, g L_j(y^i)\rangle, \quad g \in X,\ j \in \{1, \cdots, m\}, \tag{73}$$

showing that $\sum_{k=1}^n z^k \pi_k$ is the solution to (70). When the matrix $\mathbf{G}$ is singular, the solution (70) is non-unique and we set by convention $w := 0$.

The explicit formula of the least-squares approximation (70) being established, we are interested in the stability and accuracy of the approximation. Similarly to the analysis in §2, we investigate the comparability over $X \otimes V_m$ of the norm $\|\cdot\|$ and its empirical counterpart $\|\cdot\|_n$ defined by

$$\|v\|_n = \Big(\frac{1}{n} \sum_{j=1}^{n} \|v(y^j)\|_X^2\Big)^{\frac{1}{2}}, \quad v \in L^2(\Gamma, X, \rho). \tag{74}$$

It is easily checked that given $v := \sum_{j=1}^{m} v_j L_j \in X \otimes V_m$, one has

$$\|v\|_n^2 - \|v\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} (\mathbf{G} - \mathbf{I})_{ij} \langle v_i, v_j\rangle_X = \langle \mathbf{v}, (\mathbf{G} - \mathbf{I})\mathbf{v}\rangle_{X^m}, \tag{75}$$

where $\mathbf{v} := (v_1, \cdots, v_m)^t \in X^m$ and the matrix-vector product is defined as in the real case. Here the inner product $\langle \cdot, \cdot \rangle_{X^m}$ is the standard inner product over X^m constructed from $\langle \cdot, \cdot \rangle_X$. Note that we have $\|v\| = \|\mathbf{v}\|_{X^m}$. We next observe that if $\mathbf{M}$ is an $m \times m$ real symmetric matrix, one has

$$\sup_{\|\mathbf{v}\|_{X^m} = 1} |\langle \mathbf{v}, \mathbf{M}\mathbf{v}\rangle_{X^m}| = |||\mathbf{M}|||, \tag{76}$$

where $|||\mathbf{M}|||$ is the spectral norm of M (this is immediately checked by diagonalizing M in an orthonormal basis). Therefore it holds that

$$\|v\|_n^2 - \|v\|^2 \leq |||\mathbf{G} - \mathbf{I}|||\, \|v\|^2, \tag{77}$$

and, similarly to the results discussed in §2, we find that under condition (29) the norm $\|\cdot\|$ and its counterpart $\|\cdot\|_n$ are equivalent over $X \otimes V_m$ with probability greater than $1 - 2n^{-r}$, with

$$\Big|\|v\|_n^2 - \|v\|^2\Big| \leq \frac{1}{2}\|v\|^2. \tag{78}$$

We now compare the accuracy of the least-squares approximation (70) with the error of best approximation

$$e_m(u) := \inf_{v \in X \otimes V_m} \|u - v\| = \|u - P_m u\|, \tag{79}$$

where P_m is the orthogonal projector onto $X \otimes V_m$. We again use the notation $P_m^n u$ for the least-squares solution in the noiseless case. If u satisfies a unifom bound $\|u(y)\|_X \leq L$ over Γ where L is known, we define the truncated least-squares approximation

$$\tilde{w} = T_L(w), \tag{80}$$

also denoted by $\tilde{P}_m^n u$ in the noiseless case, where T_L is the trunction operator, now defined as follows

$$T_L(v) = \begin{cases} v \quad \text{if} \quad \|v\| \leq L, \\ \\ \frac{v}{\|v\|} L \quad \text{if} \quad \|v\| > L. \end{cases} \tag{81}$$

Note that T_L is the projection map onto the closed disc $\{\|v\| \leq L\}$ and is therefore Lipschitz continuous with constant equal to 1. The following counterparts to the results of §2 are proven in the same way and therefore we only state them:

- Under condition (29), and if u satisfies a unifom bound $\|u(y)\|_X \leq L$ over Γ, one has

$$\mathbb{E}(\|u - \tilde{P}_m^n u)\|^2) \leq (1 + \epsilon(n)) e_m(u)^2 + 8L^2 n^{-r}. \tag{82}$$

- In the noisy case, and under the same conditions as above, one has

$$\mathbb{E}(\|u - \tilde{w}\|^2) \leq (1 + 2\epsilon(n)) e_m(u)^2 + 8\Big(L^2 n^{-r} + \sigma^2 \frac{m}{n}\Big), \tag{83}$$

  where $\sigma^2 := \max_{y \in \Gamma} \mathbb{E}(\|z - u(y)\|_X^2 | y)$ is the noise level.

- Under condition (29), one has

$$\Pr\Big(\|u - P_m^n u\| \geq (1 + \sqrt{2}) e_m(u)_\infty\Big) \leq 2n^{-r}, \tag{84}$$

  where $e_m(u)_\infty = \inf_{v \in X \otimes V_m} \|u - v\|_{L^\infty(\Gamma, X)}$.

As a general example of application, consider the model stochastic elliptic boundary value problem (9) with a diffusion coefficient given by (10) and satisfying (11). As recalled in the introduction, if $(\|\psi_j\|_{L^\infty(D)})_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p < 1$, then there exists a nested sequence of lower sets

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \mathcal{F}, \quad \#(\Lambda_m) = m, \tag{85}$$

such that with $X := H_0^1(D)$ and $V_m := \mathbb{P}_{\Lambda_m}$ one has

$$e_m(u) \leq Cm^{-s}, \quad s := \frac{1}{p} - \frac{1}{2} > 0. \tag{86}$$

Since the solution satisfies the uniform bound $\|u(y)\|_X \leq L := \frac{\|f\|_{V^*}}{r}$, we can compute its trunctated least-squares approximation $\tilde{P}_m^n u$ based on n observations $u^i = u(y^i)$ where the y^i are i.i.d. with respect to the uniform measure over $\Gamma := [-1,1]^{\mathbb{N}}$. Combining (82) and (42), it follows that

$$\mathbb{E}(\|u - \tilde{P}_m^n u)\|^2) \leq (1 + \epsilon(n))C^2 m^{-2s} + 8L^2 n^{-r}, \tag{87}$$

provided that $\frac{n}{\ln n} \geq \frac{m^2}{\kappa}$ with $\kappa := \frac{1-\ln 2}{2+2r}$. In particular, taking $r = s$, we obtain the estimate

$$\mathbb{E}(\|u - \tilde{P}_m^n u)\|^2) \lesssim m^{-2s}. \tag{88}$$

Taking the minimal amount of sample n such that $\frac{n}{\ln n} \geq \frac{m^2}{\kappa}$, this gives the convergence estimate

$$\mathbb{E}(\|u - \tilde{P}_m^n u)\|^2) \lesssim \Big(\frac{n}{\ln n}\Big)^{-s}. \tag{89}$$

Remark 1. *An analysis of the Chebyshev coefficients of u reveals that the same approximation rate as* (86) *holds for the L^2 norm with respect to the tensorized Chebyshev measure. However, in view of* (62)*, the condition between m and n is now $\frac{n}{\ln n} \geq \frac{m^\beta}{\kappa}$ with $\beta := \frac{\ln 3}{\ln 2}$. It follows that the rate in* (89) *can be improved into*

$$\mathbb{E}(\|u - \tilde{P}_m^n u)\|^2) \lesssim \Big(\frac{n}{\ln n}\Big)^{-\frac{2\ln 3}{\ln 2}s}, \tag{90}$$

if we use samples y^i that are i.i.d. with respect to the tensorized Chebyshev measure and if we use the L^2 error with respect to this measure.

5 Application to elliptic PDEs with random inclusions

5.1 The case of non-overlapping inclusions: approximation in total degree polynomial spaces

In this section, we focus on the subclass of stochastic PDEs (9)–(10) characterized by the fact that the functions ψ_j have nonoverlapping support. This situation allows to model, for instance, the diffusion process in a medium with nonoverlapping inclusions of random conductivity (see *e.g.* Fig. 2). In this case, a priori estimates on the Legendre coefficients have been obtained *e.g.* in [9] and have been shown numerically to be quite sharp. They read:

$$\|u_\nu\|_X \leq C \prod_{j=1}^{d} \exp\{-\nu_j\, g_j\}, \quad \forall\, \nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d,$$

with $X = H_0^1(D)$. Explicit expressions for the constant C can be found in [9, Corollary 8]. The coefficients $(g_j)_{1 \leq j \leq d}$ can be estimated through an *a posteriori* procedure, that requires to solve only "one-dimensional" problems, *i.e.* analyzing the convergence when considering one random variable at a time and freezing all other variables to their expected value. As a consequence, quasi-optimal index sets associated with the problems in the aforementioned class are of the form

$$\Lambda_{\mathrm{w}} = \Big\{\nu \in \mathbb{N}_0^d \,:\, \sum_{j=1}^{d} g_j \nu_j \leq \mathrm{w}\Big\}, \qquad \mathrm{w} = 1, 2, \ldots$$

and correspond to anisotropic total degree spaces, *i.e.* the anisotropic variants of (50). Analogous estimates, showing the optimality of the total degree space, have been presented in [3].

In the next discussion, we consider the simple isotropic case where $g_j = g$ for all $j = 1, \ldots, d$. For convenience we introduce the quantities τ, ϕ defined as:

$$\tau := \frac{gd}{e}, \tag{91}$$

$$\phi := \frac{\widehat{C}_{\text{Leg}}^2}{(1-e^{-g})^d} \exp\left\{\frac{2\,e^2\,(1-e^{-1})\,\tau}{5}\right\}. \tag{92}$$

The expression of $\widehat{C}_{\text{Leg}}$ can be recovered from [9, Corollary 8] and depends on d and g.

Lemma 3. *In the isotropic case,* i.e. $g_j = g$ *for all* $j = 1, \ldots, d$*, the following estimate on the error of the* L^2 *projection* P_m *on the quasi-optimal TD set* (91) *with* $\#(\Lambda) = m$*, it holds*

$$\|u - P_m u\|^2 \leq \phi \exp\left\{-\tau m^{\frac{1}{d}}\right\} \tag{93}$$

for $m > (2e/5)^d$*.*

Proof. The following estimate has been obtained in [9, Theorem 22]:

$$\|u - P_\Lambda u\|^2 \leq \frac{\widehat{C}_{\text{Leg}}^2}{(1-e^{-g})^d} \exp\left\{-\tau \ln\left((1-\eta(m))^{-1}\right) m^{\frac{1}{d}}\right\}, \tag{94}$$

with

$$\eta(m) = (1-e^{-1})\left(1 - \frac{2e}{5m^{\frac{1}{d}}}\right). \tag{95}$$

When $(2e/5)^d < m$ then $(1-\eta(m)) < 1$, and moreover $\lim_{m\to+\infty} (1-\eta(m))^{\left(\tau m^{\frac{1}{d}}\right)} = 0$. Introducing the change of variable z as

$$z = m^{\frac{1}{d}}, \tag{96}$$

using the definition of τ in (91) and replacing η by (95), then the exponential term on the right side in (94) can be manipulated as

$$\left(e^{-1} + \frac{2\,e\,(1-e^{-1})}{5\,z}\right)^{\tau\,z} = \left(1 + \frac{2\,e^2\,(1-e^{-1})}{5\,z}\right)^{\tau\,z} \cdot e^{-\tau\,z} < e^{\frac{2\,e^2\,(1-e^{-1})}{5}\,\tau} \cdot e^{-\tau\,z}. \tag{97}$$

Notice from (96) that the limit $m \to +\infty$ is equivalent to $z \to +\infty$. Thanks to (97) we can bound the exponential term on the right hand side of (94), and using the definition (92) of ϕ we obtain (93). ∎

5.2 Convergence of the discrete least-squares approximation

In this subsection we derive an estimate for the expected L^2 error $\mathbb{E}(\|u - \tilde{P}_m^n u\|^2)$ of the discrete least-squares approximation in terms of the number of sampling points n. To do this we rely on the estimates regarding the exact L^2 projection on total degree polynomial spaces that have been recalled in Section 5.1. To begin with, we will use the isotropic estimate (93). The extension to anisotropic problems can be obtained following the estimates presented in [9].

To lighten the notation we introduce the constant

$$\zeta := \frac{1-\ln 2}{2} \approx 0.15 \tag{98}$$

and the factor

$$\varphi = \varphi(n) := \big(1 + \epsilon(n)\big)\, \phi. \tag{99}$$

Notice that $\epsilon = \epsilon(n)$ is a decreasing function converging to zero as n increases, and in practice its value is such that $\varphi \approx \phi$.

Theorem 2. *In the aforementioned PDE model class, the convergence rate of the discrete least-squares approximation with an optimal choice of the polynomial space satisfies*

$$\mathbb{E}\left(\|u - \tilde{P}_m^n u\|^2\right) \leq (\varphi + 8L^2)\, \exp\left\{-\left(\frac{\tau^{2d}\, \zeta\, n}{2}\right)^{\frac{1}{2d+1}}\right\}, \tag{100}$$

with τ as in (91) *and ζ as in* (98).

Proof. The first step to characterize the optimal convergence rate with respect to n is to impose a relation between n, ζ, r and m to have a stable least-squares approximation. In the case of polynomial approximation, the relation (29) holds choosing the multi-index set Λ such that

$$m = \left\lfloor \frac{\zeta}{r+1} \frac{n}{\ln n} \right\rfloor^{\frac{1}{2}}. \tag{101}$$

Therefore, the constraint (101) prescribes how to enlarge the dimension of the polynomial space as n increases, to ensure stability and optimality of the discrete least-squares projection thanks to (30). The value of r can be chosen optimally, and we will pursue this strategy in the following. Replacing m with (101) in the right hand side of (93) we have

$$\|u - P_m u\|^2 \leq \phi \exp\left\{-\tau \left(\frac{\zeta n}{(r+1)\ln n}\right)^{\frac{1}{2d}}\right\} \leq \phi \exp\left\{-\tau \left(\frac{\zeta n}{2r \ln n}\right)^{\frac{1}{2d}}\right\}, \quad \text{for } r > 1. \tag{102}$$

Since we embedded the stability condition (29) as a constraint, then we can apply (82) and use (102) to bound the error on the right hand side, obtaining

$$\mathbb{E}\left(\|u - \tilde{P}_m^n u\|^2\right) \leq \varphi e^{-\tau\left(\frac{\zeta n}{2r \ln n}\right)^{\frac{1}{2d}}} + 8L^2\, e^{-r \ln n}. \tag{103}$$

Notice the factor $r \ln n$ in both the exponents of (103). Now we can choose $r = r(n, d)$ so that the exponents of the two exponential terms in (103) are equal, *i.e.*,

$$r = \frac{1}{\ln n} \left(\frac{\tau^{2d}\, \zeta\, n}{2}\right)^{\frac{1}{2d+1}}. \tag{104}$$

Finally, substituting the expression (104) of r in (103), we obtain the convergence rate (100) optimized with respect to r. ∎

Looking at (100) we observe that:

- the error converges to zero sub-exponentially as $\exp\{-\alpha n^{\frac{1}{2d+1}}\}$ with $\alpha := (dg/e)^{\frac{2d}{2d+1}} (\zeta/2)^{\frac{1}{2d+1}}$,
- the dimension d appears in α in favor of the convergence, and in the exponent of $n^{\frac{1}{2d+1}}$ which slows down the convergence,
- The error of the best m-term approximation converges to zero with the rate $\exp\big\{-\tau m^{\frac{1}{d}}\big\}$ (see (93)), whereas the error of the random discrete projection converges to zero with the rate $\exp\Big\{-\alpha n^{\frac{1}{2d+1}}\Big\}$, with $n \sim m^2$.

6 Numerical experiments

In this section we present some numerical examples that confirm the theoretical findings presented in Sections 2–5. In particular, we check that the convergence rate (100) is sharp when the number of sampling points n is chosen as

$$n = \frac{2\tau}{\zeta}\Big(m\Big)^{\left(2+\frac{1}{d}\right)}, \tag{105}$$

that comes from the optimal choice of r prescribed by (104). To investigate the sensitivity with respect to ζ, we denote again $\zeta = \zeta(\delta)$, for different choices of δ such that $\zeta = 0.15$ or $\zeta = 0.25$.

We consider the elliptic model (9) on the bounded domain $D \subset \mathbb{R}^2$ shown in Fig. 2, with the random diffusion coefficient a defined in (106) by means of the geometry displayed in Fig. 2. The eight inclusions $D_1, \ldots, D_8$ are circles with radius equal to 0.13, and are centered in the points $\mathbf{x} = (0.5, 0.5 \pm 0.3)$, $\mathbf{x} = (0.5 \pm 0.3, 0.5)$ and $\mathbf{x} = (0.5 \pm 0.3, 0.5 \pm 0.3)$. The 0.2-by-0.2 inner square D_0 lies in the center of D. The forcing term f is equal to 100 in D_0 and zero in $D \setminus D_0$. The random diffusion coefficient depends on a multivariate uniform random variable $\mathbf{Y} \sim \mathcal{U}([-1,1]^d)$, and is defined as

$$a(\mathbf{x}, \mathbf{Y}) = \begin{cases} 0.395\Big(Y_i + 1\Big) + 0.01, & \mathbf{x} \in D_i, \quad i = 1, \ldots, 8, \\ 1, & \mathbf{x} \in D_0, \end{cases} \tag{106}$$

such that each random variable is associated with an inclusion. The range of variation of the coefficient in each inclusion is therefore $[0.01, 0.8]$. This test case has been used in [12], and allows a direct comparison of our results with those obtained when employing the classical Stochastic Galerkin method. The monodimensional convergence rate $g = 1.9$ of this example has been estimated in [9, Fig.7-left]. Notice that the coefficient a in (106) satisfies Assumption 11.

We consider the following Quantity of Interest, related to the solution of the elliptic model (9):

$$\mathrm{QOI}_1(u(\mathbf{Y})) = \frac{1}{|D|}\int_D u(\mathbf{x}, \mathbf{Y})\, d\mathbf{x},$$

and present the results obtained when approximating this function on TD polynomial spaces. Similar results hold also with other Quantities of Interest, *e.g.*

$$\mathrm{QOI}_2(u(\mathbf{Y})) = \frac{1}{|\Omega|}\int_\Omega \Big|\nabla u(\mathbf{x}, \mathbf{Y})\Big|^2 d\mathbf{x}, \quad \mathrm{QOI}_3(u(\mathbf{Y})) = \frac{1}{|\Omega_0|}\int_{\Omega_0} u(\mathbf{x}, \mathbf{Y})\, d\mathbf{x},$$

which will not be shown here. We consider three cases with $d = 2$, $d = 4$, $d = 8$ independent random variables. In the case $d = 2$, the first random variable describes the diffusion coefficient in the four inclusions at the top, bottom, left, right of the center square D_0. The second random variable describes the diffusion coefficient in the other four inclusions. In the case $d = 4$, each one of the four random variables is associated with two opposite inclusions with respect to the center of the domain. When $d = 8$ each one of the random variables is associated with a different inclusion.

The Figs. 3, 4, 5 show the convergence plots obtained by the discrete least-squares approximation using a number of samples as in (105) with two different choices of ζ (*i.e.* $\zeta = 0.15$ and $\zeta = 0.25$). The theoretical bound (100) is also shown as well as the reference slope $n^{-1/2}$ of a standard Monte Carlo method. In the same figures we also show the convergence plots obtained when using a simple linear rule $n = 3m$ or $n = 10m$.

The approximation error of the discrete least-squares projection

is approximated as

$$\mathbb{E}\left(\|\mathrm{QOI}_1(u) - \tilde{P}_m^n \mathrm{QOI}_1(u)\|\right) \approx \mathbb{E}\left(\|\mathrm{QOI}_1(u) - \tilde{P}_m^n \mathrm{QOI}_1(u)\|_{cv}\right),$$

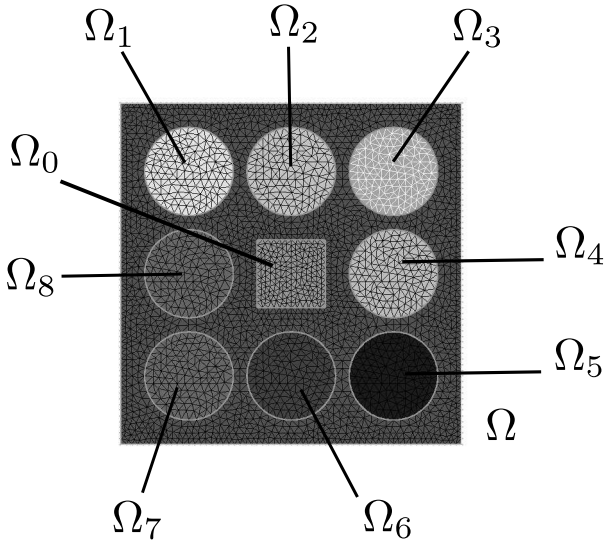

Figure 2: Mesh discretization and geometries of the inclusions. The domain D is the unitary square. The inner square is named D_0, the eight circular inclusions are $D_1, \ldots, D_8$.

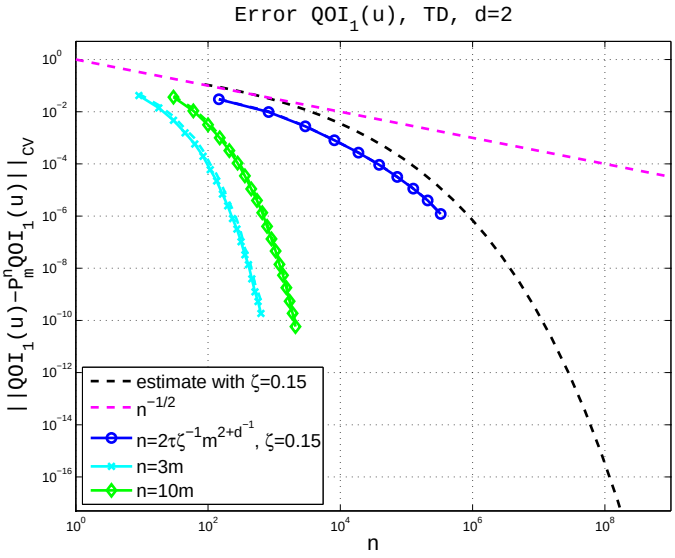

Figure 3: Error $\mathbb{E}(\|u - \tilde{P}_m^n u\|^2)$ in the case $d = 2$. Different relations between the number of samples n and the dimension of the polynomial space m are tested. The black dash line is the bound (100). The magenta dash line is the Monte Carlo convergence rate $n^{-1/2}$.

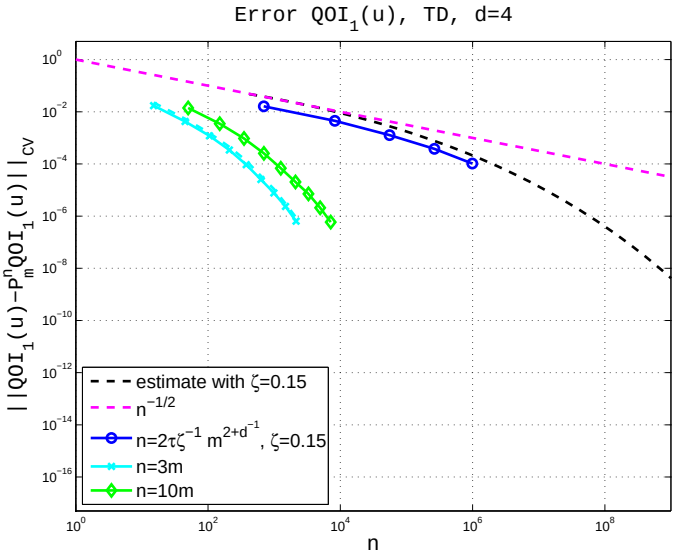

Figure 4: Error $\mathbb{E}(\|u - \tilde{P}_m^n u\|^2)$ in the case $d = 4$. Different relations between the number of samples n and the dimension of the polynomial space m are tested. The black dash line is the bound (100). The magenta dash line is the Monte Carlo convergence rate $n^{-1/2}$.

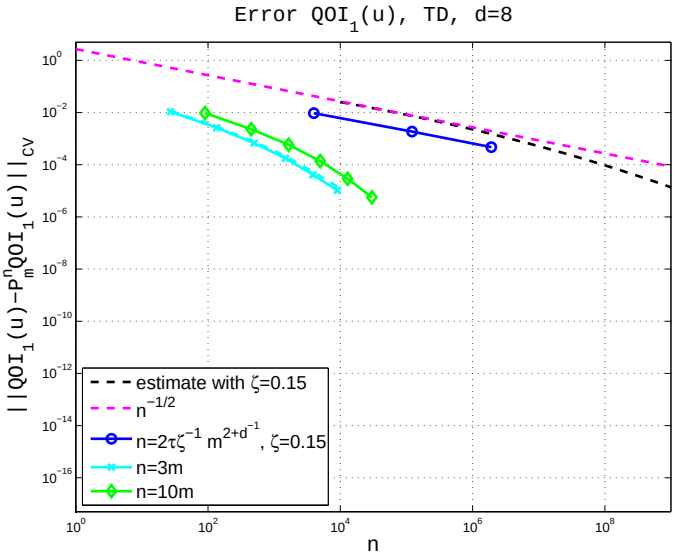

Figure 5: Error $\mathbb{E}(\|u - \tilde{P}_m^n u\|^2)$ in the case $d = 8$. Different relations between the number of samples n and the dimension of the polynomial space m are tested. The black dash line is the bound (100). The magenta dash line is the Monte Carlo convergence rate $n^{-1/2}$.

employing the cross-validation procedure described in [7, Section 4], where the expectation in the previous formula has been replaced by a sample average of the discrete least-squares approximation using 1000 independent samples of size n.

The results presented in Figs. 3, 4, 5 show that the bound (100) proposed predicts very sharply the error $\mathbb{E}(\|u - \tilde{P}_m^n u\|^2)$, when the number of sampling points n is chosen according to (105). The bound accurately describes the effect of the dimension d as well, in the cases of moderately high dimensions.

On the other hand, a faster convergence of the error $\mathbb{E}(\|u - \tilde{P}_m^n u\|^2)$ with respect to n is observed, with the linear scaling $n \sim m$ that yields a lower number of sampling points than (105), for a given set Λ. The efficiency of the linear scaling has been pointed out in [8], and its importance is motivated by the impossibility to employ the number of sampling points (105) when the dimension d is large. Fig. 5 shows that already when $d = 8$, the exponential gain of the bound (100) with respect to a Monte Carlo rate becomes perceivable only with an astronomical number of samples, making the choice (105) less attractive for the applications, whereas a linear scaling, even with $n = 3m$ leads to very good results.

7 Conclusion

In this work the approximation technique based on least squares with random evaluations has been analyzed. The condition between the number of sampling points and the dimension of the polynomial space, which is necessary to achieve stability and optimality, has been extended to any lower set of multi-indices identifying the polynomial space, in any dimension of the parameter set, and with the uniform and Chebyshev densities. When the density is uniform, this condition requires the number of sampling points to scale as the square of the dimension of the polynomial space.

Afterwards, this technique has been applied to a class of "inclusion-type" elliptic PDE models with stochastic coefficients, and an exponential convergence rate in expectation has been derived. This estimate clarifies the dependence of the convergence rate on the number of sampling points and on the dimension of the parameter set. Moreover, this estimate establishes a relation between the convergence rate of the least-squares approximation with random evaluations and the convergence rate of the best m-term "exact" L^2 projection.

The numerical tests presented show that the proposed estimate is sharp, when the number of sampling points is chosen according to the condition that ensures stability and optimality. In addition, these results show that, in the aforementioned model class, a linear scaling of the number of sampling points with respect to the dimension seems to be sufficient to ensure the stability of the discrete projection, thus leading to faster convergence rates, although we have no rigourous explaination of this fact.

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