MOX-Report No. 34/2017

Analysis of a Discontinuous Galerkin Finite Element discretization of a degenerate Cahn-Hilliard equation with a single-well potential

Agosti, A.

MOX, Dipartimento di Matematica
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

mox-dmat@polimi.it http://mox.polimi.it

Analysis of a Discontinuous Galerkin Finite Element discretization of a degenerate Cahn-Hilliard equation with a single-well potential

Abramo Agosti[‡]

[‡] MOX– Modellistica e Calcolo Scientifico
Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano
via Bonardi 9, 20133 Milano, Italy
abramo.agosti@polimi.it

Keywords: Degenerate Cahn Hilliard equation; cellular potential; tumor growth models; Discontinuous Galerkin finite element approximation; Variational inequality; Complementarity systems; Convergence analysis.

AMS Subject Classification: 35Q92, 35K35, 35K65, 65M60, 65K15, 65G99.

Abstract

This work concerns the construction and the convergence analysis of a Discontinuous Galerkin Finite Element approximation of a Cahn-Hilliard type equation with degenerate mobility and single-well singular potential of Lennard-Jones type. This equation has been introduced in literature as a diffuse interface model for the evolution of solid tumors. Differently from the Cahn-Hilliard equation analyzed in the literature, in this model the singularity of the potential does not compensate the degeneracy of the mobility at zero by constraining the solution to be strictly positive. In previous works a finite element approximation with continuous elements of the problem has been developed by the author and co-authors. In the latter case, the positivity of the solution is enforced through a discrete variational inequality, which is solved only on *active nodes* of the triangulation where the degenerate operator can be inverted. Moreover, a lumping approximation of the L^2 scalar product is introduced in the formulation in order to select the solutions with a moving support with finite speed of velocity from the unphysical solutions with fixed support. As a consequence of this approximation, the order of convergence of the method is lowered down with respect to the case of the classical Cahn-Hilliard equation with constant mobility. In the present discretization with discontinuous elements, the concept of *active nodes* is delocalized to the concept of *active elements* of the triangulation and no lumping approximation of the mass products is needed to select the physical solutions. The well posedness of the discrete formulation is shown, together with the convergence to the weak solution. Different algorithms to solve the discrete variational inequality, based on iterative solvers of the associated complementarity system, are derived and implemented. Simulation results in two space dimensions are reported in order to test the validity of the proposed algorithms, in which the dynamics of the spinodal decomposition and the evolution behaviour in the coarsening regime are studied. Similar results to the ones obtained in standard phase ordering dynamics are found, which highlight nucleation and pattern formation phenomena and the evolution of single domains to steady state with constant curvature. Since the present formulation does not depend on the particular form of the potential, but it's based on the fact that the singularity set of the potential and the degeneracy set of the mobility do not coincide, it can be applied also to the degenerate CH equation with smooth potential.

1 Introduction

In this paper a Discontinuous Galerkin finite element approximation of the following initial and boundary value problem for a Cahn-Hilliard type equation with degenerate mobility and single-well potential of Lennard-Jones type is considered:

Problem **P**: Find $c(\mathbf{x},t)$ such that

$$\frac{\partial c}{\partial t} = \nabla \cdot (b(c)\nabla(-\gamma\Delta c + \psi'(c))) \quad \text{in } \Omega_T := \Omega \times (0,T), \tag{1}$$

$$c(\mathbf{x},0) = c_0(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega, \tag{2}$$

$$\nabla c \cdot \boldsymbol{\nu} = b(c)\nabla(-\gamma\Delta c + \psi'(c)) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{3}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$ is a given bounded domain with a Lipschitz boundary $\partial\Omega$, $\boldsymbol{\nu}$ is the unit normal vector pointing outward to $\partial\Omega$, c is the volume fraction of cancerous cells, c_0 is a given initial concentration and

$$\psi(c) = \psi_1(c) + \psi_2(c), \tag{4}$$

where

$$\psi_1(c) = -(1-c^*)\log(1-c), \tag{5}$$
$$\psi_2(c) = -\frac{c^3}{3} - (1-c^*)\frac{c^2}{2} - (1-c^*)c + k.$$

Here c^* is the volume fraction at which the cells would naturally be at mutual equilibrium and $k > 0$. A spinodal decomposition can be triggered if $c < \bar{c}$, where $\psi''(\bar{c}) = 0$. The derivative of the potential is

$$\psi'(c) = \frac{c^2(c-c^*)}{1-c}. \tag{6}$$

Correspondingly, the mobility is given by

$$b(c) = c(1-c)^2. \tag{7}$$

In [8, 9] Problem **P** is derived as a result of the application of mixture model to solid tumors. Note that ψ_1 is a convex function defined on $[0,1)$ while ψ_2 is concave. Also, the product $b\psi''$ is a continuous function in $[0,1]$.

In [4] the existence of different classes of weak solutions of Problem **P** and their positivity properties, for the cases of spatial dimension $d = 1$ and $d = 2,3$ separately, were studied, and a continuous finite element approximation of the problem was formulated, studying its well posedness in $d = 1,2,3$ spatial dimensions and its convergencence, in $d = 1$ spatial dimension, to the weak solutions. In [3] the error analysis of the discretization introduced in [4] was studied, showing that in that case the order of convergence of the approximation method is lowered down with respect to the case of the classical Cahn-Hilliard equation with constant mobility. As a consequence of the fact that (1) degenerates on the set $\{c = 0; c = 1\}$, and the singularity is concentrated on the set $\{c = 1\}$ only, one cannot exploit the relationship between b and ψ at 0 in order to ensure that $c > 0$ at a discrete level. Moreover, the Entropy inequalities obtained in [4], which guarantee the positivity property of the continuous solutions, are not straightforwardly available at the discrete level. Therefore, following [6], this condition was imposed in [4] as a constraint and a discrete variational inequality was formulated, solved only on the *active nodes* of the triangulation where the degenerate operator can be inverted. Moreover, a lumping approximation of the L^2 scalar product was introduced in the formulation in order for the discrete solution to be able to track compactly supported solutions of (1) with a free boundary which moves with a finite speed of velocity (whose existence is discussed in, e.g., [6]).

In this work a finite element approximation of Problem **P** with discontinuous elements, which is an extension of the finite element approximation with continuous elements introduced in [4], is formulated.

Discontinuous Galerkin Finite element discretizations of the Cahn-Hilliard equation with non degenerate mobility and a smooth or a double-well logarithmic type potential have been studied previously in [13, 18, 19]. In the present paper, the discretization with discontinuous elements is studied for the degenerate case. Note that, since the present formulation does not depend on the particular form of the potential, but it's based on the fact that the singularity set of the potential and the degeneracy set of the mobility do not coincide, it can be applied also to the degenerate CH equation with smooth potential. The well-posedness of the formulation in $d = 1, 2, 3$ spatial dimensions and its convergence in one spatial dimension is proved. In particular, in order to handle the singular cellular potential, a regularized problem is introduced as an intermediate step.

The discretization with discontinuous elements, delocalizing the concept of *passive* and *active nodes*, introduced in the case of the discretization with continuous elements, to *passive* and *active elements* of the partition of the domain, eliminates the necessity of using a lumped mass scalar product in order to obtain an expanding support of the discrete solution with finite velocity, thus avoiding to introduce the lumping approximation in the variational formulation of the problem. Moreover, the discretization with discontinuous elements is useful when advective terms are present, ensuring local mass conservation and regularizing hyperbolic instabilities; a fully coupled model of the Cahn-Hilliard equation for cellular mixtures coupled with transport equations will be studied in a furthcoming work.

The paper is organized as follows. In section 2 we introduce the Discontinuous Galerkin finite element approximation of (1)$-$(7), showing its well posedness in d spatial dimensions and its convergence in one space dimension. In particular, due to the singularity in the cellular potential, a regularized problem is studied in subsection 2.1 as an intermediate step. Section 3 is devoted to the convergence analysis in one space dimension. In Section 4 we present different numerical algorithms which can be used to solve the discrete variational inequality, based on proper iterative solvers of the corresponding complementarity system. In Section 5 we present some numerical simulations in two dimensions, in order to discuss the dynamics of the spinodal decomposition and the evolution behaviour in the coarsening regime of the obtained discrete solution and to show the validity of the proposed numerical algorithms. The final Section 6 contains a discussion of the main results.

1.1 Notation and functional setting

For a given domain $\omega \subset \Omega$, $d = 1, 2, 3$, let's indicate with $L^p(\omega)$, $W^{m,p}(\omega)$, $H^m(\omega) = W^{m,2}(\omega)$ and $L^p((0,T);V)$ the usual Lebesgue, Sobolev and Bochner spaces, (see, e.g., [2]), for a $p \in [1, \infty]$ and $m \in \mathbb{N}$, endowed with the corresponding canonical norms and seminorms $||\cdot||_{m,p,\omega}$, $||\cdot||_{m,\omega}$, $|\cdot|_{m,p,\omega}$ and $|\cdot|_{m,\omega}$, respectively. Throughout, $(\cdot,\cdot)_\omega$ denotes the standard L^2 inner product over ω, and $< \cdot,\cdot >_\omega$ denotes the duality pairing between $(H^1(\omega))'$ and $(H^1(\omega))$. We omit the index ω when $\omega = \Omega$. Let's moreover denote by $< \cdot,\cdot >_\sigma$ the $(d-1)$ dimensional $L^2(\sigma)$ inner product on $\sigma \subset \mathbb{R}^{d-1}$. With $C(\bar{\omega})$, $C^n(I_1, I_2)$, $n \geq 0$, and $C^{s_1,s_2}_{x,t}(\bar{\omega}_T)$, $0 < s_1, s_2 < 1$, let's indicate the space of continuous functions from $\bar{\omega}$ to $\mathbb{R}$, the space of C^n continuous functions from interval $I_1 \subset \mathbb{R}$ to interval $I_2 \subset \mathbb{R}$, and the space of Hölder continuous functions from $\bar{\omega}_T$ to $\mathbb{R}$ with Hölder exponents s_1 and s_2 in the arguments x and t, respectively.

Furthermore, C denotes throughout a generic positive constant independent of the unknown variables, the discretization and the regularization parameters, the value of which might change from line to line; $C_1, C_2, \ldots$ indicate generic positive constants whose particular value must be tracked through the calculations; $C(a)$ denotes a constant depending on the non-negative parameter a, such that, for $C_1 > 0$, if $a \leq C_1$, there exists a $C_2 > 0$ such that $C(a_1) \leq C_2$.

We will use the following Sobolev interpolation result, (see, e.g., [2]): let $p \in [1, \infty]$, $m \geq 1$,

$$\begin{cases} r \in [p, \infty] & \text{if} \quad m - \frac{d}{p} > 0, \\ [p, \infty) & \text{if} \quad m - \frac{d}{p} = 0, \\ [p, -\frac{d}{m-(d/p)}] & \text{if} \quad m - \frac{d}{p} < 0, \end{cases}$$

and $\mu = \frac{d}{m}\left(\frac{1}{p} - \frac{1}{r}\right)$. Then there is a constant C such that for all $v \in W^{m,p}(\omega)$

$$||v||_{0,r,\omega} \le C||v||_{0,p,\omega}^{1-\mu}||v||_{m,p,\omega}^{\mu}. \tag{8}$$

Let $\mathcal{T}_h$ be a quasi-uniform conforming decomposition of Ω into disjoint open simplices K, and let's introduce the following discontinuous finite element spaces:

$$S(\Omega,\mathcal{T}_h) := \{\chi \in L^2(\Omega) : \chi|_K \in \mathbb{P}^1(K)\ \forall K \in \mathcal{T}_h\},$$
$$K(\Omega,\mathcal{T}_h) := \{\chi \in S(\Omega,\mathcal{T}_h) : \chi \ge 0 \text{ in } \Omega\}$$

where $\mathbb{P}_1(K)$ indicates the space of polynomials of total order 1 on K. Let's also define the broken Sobolev spaces

$$H^{\mathbf{s}}(\Omega,\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_K \in H^{s_K}(K)\ \forall K \in \mathcal{T}_h\},$$

where $\mathbf{s} = \{s_K\}_{K\in\mathcal{T}_h}$ is a set of positive integers, endowed with the broken norms

$$||v||_{\mathbf{s},\mathcal{T}_h} := \left(\sum_{K\in\mathcal{T}_h} ||v||_{s_K,K}^2\right)^{1/2}.$$

The finite element space $S(\Omega,\mathcal{T}_h)$ is a subset of $H^{\mathbf{s}}(\Omega,\mathcal{T}_h)$ for any set $\{s_K\}_{K\in\mathcal{T}_h}$.
The traces of functions in $H^1(\Omega,\mathcal{T}_h)$ belong to the space $T(\Gamma) := \prod_{K\in\mathcal{T}_h} L^2(\partial K)$, where $\Gamma := \cup_{K\in\mathcal{T}_h}\partial K$.
For $v \in H^1(\Omega,\mathcal{T}_h)$, let's define the piecewise gradient $\nabla_h v$ of v by $(\nabla_h v)|_K := \nabla(v|_K)$, for $K \in \mathcal{T}_h$. For $q \in T(\Gamma)$, let's define the average $\{q\}$ and the jump $[[q]]$ of q on $\Gamma^0 := \Gamma \setminus \partial\Omega$ as follows. Let e be an interior edge shared by elements K^+ and K^-. Assuming that the unit normal vector $\mathbf{n}_e$ is oriented from K^+ to K^-, with $q^\pm := q|_{\partial K^\pm}$, let's set

$$[[q]]_e := q^+|_e - q^-|_e; \qquad \{q\}_e := \frac{1}{2}(q^+|_e + q^-|_e).$$

For ease of writing we shall suppress the subscript e in the notations.
In the particular case of $d = 1$, denoting by $0 = x_0 < x_1 < \cdots < x_N = L$ a partition $\mathcal{T}_h$ of the interval $[0, L]$, with $I_n = (x_n, x_{n+1})$, and, given a function $v|_{I_n} \in \mathbb{P}_1(I_n)$, denoting with $v(x_n^+) := \lim_{\epsilon\to 0,\epsilon>0} v(x_n + \epsilon)$ and with $v(x_n^-) := \lim_{\epsilon\to 0,\epsilon>0} v(x_n - \epsilon)$, let's define the jump and average of v at the endpoints of I_n as

$$[[v(x_n)]] := v(x_n^-) - v(x_n^+); \qquad \{v(x_n)\} := \frac{1}{2}(v(x_n^-) + v(x_n^+)).$$

Let's now define the DG SIPG isotropic and anisotropic bilinear forms, (see, e.g., [16]), $B_{\mathcal{T}_h}(v,w) : S(\Omega,\mathcal{T}_h)\times S(\Omega,\mathcal{T}_h) \to \mathbb{R}$ and $B_{\mathcal{T}_h}(u; v, w) : S(\Omega,\mathcal{T}_h) \times S(\Omega,\mathcal{T}_h) \times S(\Omega,\mathcal{T}_h) \to \mathbb{R}$, which penalizes the continuity of the discrete solutions at the interelement boundaries,

$$B_{\mathcal{T}_h}(v,w) := \sum_{K\in\mathcal{T}_h} (\nabla v, \nabla w)_K - \sum_{e\in\Gamma_0} [([[v]], \{\nabla w \cdot \mathbf{n}_e\})_e + ([[w]], \{\nabla v \cdot \mathbf{n}_e\})_e - \frac{\sigma}{|e|}([[v]], [[w]])_e], \tag{9}$$

$$B_{\mathcal{T}_h}(u; v, w) := \sum_{K\in\mathcal{T}_h} (b(u)\nabla v, \nabla w)_K - \sum_{e\in\Gamma_0} [([[v]], \{b(u)\nabla w \cdot \mathbf{n}_e\})_e + ([[w]], \{b(u)\nabla v \cdot \mathbf{n}_e\})_e - \frac{\sigma}{|e|}([[v]], [[w]])_e], \tag{10}$$

where σ is a sufficiently large positive constant. In the particular case of $d = 1$, denoting by $h_n = x_{n+1} - x_n$, $h_{n-1,n} = \max(h_{n-1}, h_n)$, $h = \max_{0\le n\le N-1} h_n$, let's define the following DG SIPG isotropic and anisotropic bilinear forms:

$$
\begin{aligned}
B_{\mathcal{T}_h}(v,w) &:= \sum_{n=0}^{N-1} (v'(x), w'(x))_{I_n} \\
&- \sum_{n=1}^{N-1} \Big([[v(x_n)]]\{w'(x_n)\} + [[w(x_n)]]\{v'(x_n)\} - \frac{\sigma}{h_{n-1,n}} [[v(x_n)]][[w(x_n)]] \Big),
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
B_{\mathcal{T}_h}(u;v,w) &:= \sum_{n=0}^{N-1} (b(u(x))v'(x), w'(x))_{I_n} \\
- \sum_{n=1}^{N-1} \Big([[v(x_n)]]&\{b(u(x_n))w'(x_n)\} + [[w(x_n)]]\{b(u(x_n))v'(x_n)\} - \frac{\sigma}{h_{n-1,n}} [[v(x_n)]][[w(x_n)]] \Big).
\end{aligned} \tag{12}
$$

Let's define the energy seminorm on $H^2(\Omega, \mathcal{T}_h)$

$$
|||v|||_c := \Big(|v|^2_{1,\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} h_K^2 |v|^2_{2,K} + J_p(v,v) \Big)^{1/2}, \tag{13}
$$

where

$$
\begin{aligned}
J_p(v,v) &:= \sum_{e \in \Gamma_0} \frac{\sigma}{|e|} ([[v]], [[v]])_e \quad \text{for } d > 1, \\
J_p(v,v) &:= \sum_{n=1}^{N-1} \frac{\sigma}{h_{n-1,n}} [[v(x_n)]][[v(x_n)]] \quad \text{for } d = 1.
\end{aligned} \tag{14}
$$

On $S(\Omega, \mathcal{T}_h)$, by application of the local inverse inequality (19) with $m = 2$, the norm (13) is equivalent to the weaker norm

$$
|||v|||_d := \Big(|v|^2_{1,\mathcal{T}_h} + J_p(v,v) \Big)^{1/2}, \tag{15}
$$

Remark 1.1 *The bilinear form $B_{\mathcal{T}_h}(\cdot,\cdot)$ and the seminorms $|||\cdot|||_c$ and $|||\cdot|||_d$ have the following properties (see, e.g., [16]):*

1. *Continuity: There exists a positive constant C, independent of the discretization parameter h, such that*
$$|B_{\mathcal{T}_h}(v,w)| \le C |||v|||_c \, |||w|||_c \quad \forall v, w \in H^2(\Omega, \mathcal{T}_h).$$

2. *Coercivity: There exists a positive constant σ_0, and for each $\sigma \ge \sigma_0$ there exists a positive constant $C_0 = C_0(\sigma)$, independent of the discretization parameter h, such that*
$$C_0 |||v|||_d^2 \le B_{\mathcal{T}_h}(v,v) \quad \forall v \in S(\Omega, \mathcal{T}_h).$$

Henceforth, we shall assume that $\sigma = \sigma_0$ in the definition of the penalty parameter in (9) and (11). In the following we will indicate both the seminorms (13) and (15) with the notation $|||\cdot|||$, meaning $|||v||| \equiv |||v|||_c$ if $v \in H^2(\Omega, \mathcal{T}_h)$, and $|||v||| \equiv |||v|||_d$ if $v \in S(\Omega, \mathcal{T}_h)$.

Analogous results will be derived and used later for the anisotropic bilinear form $B_{\mathcal{T}_h}(\cdot,\cdot,\cdot)$. Whenever $B_{\mathcal{T}_h}(v,v) \ge 0$ let's also define the seminorm on $H^2(\Omega, \mathcal{T}_h)$

$$
|||v|||_B := B_{\mathcal{T}_h}(v,v). \tag{16}
$$

As a consequence of remark 1.1, $|||\cdot|||_B$ is equivalent to $|||\cdot|||$ on $S(\Omega,\mathcal{T}_h)$ as a seminorm.

Let J_K be the set of local nodes of $K \subset \mathcal{T}_h$, with $|J_K| = d+1$, and let $\{\phi_j^K\}_{K\in\mathcal{T}_h}$, $j = 1,\ldots,d+1$ be the standard basis functions for $S(\Omega,\mathcal{T}_h)$, with $\overline{\text{supp}}[\phi_j^K] \equiv K$. Let's introduce the local Lagrangian interpolation operator $\pi_K^h : C(\bar{K}) \to S(\Omega,\mathcal{T}_h)$

$$\pi_K^h(v) := \sum_{j=1}^{d+1} v(x_j^K)\phi_j^K,$$

where x_j^K are the coordinates of the nodes in the set J_K. Let's introduce also the L^2 projection operator $p^h : L^2(\Omega) \to S(\Omega,\mathcal{T}_h)$ defined by

$$(p^h(\eta),\chi) = (\eta,\chi) \quad \forall\chi \in S(\Omega,\mathcal{T}_h). \tag{17}$$

We recall some well known local inverse, interpolation and trace inequalities on $S(\Omega,\mathcal{T}_h)$, (see, e.g., [15, 16]).

Lemma 1.1

$$|\chi|_{m,p_2,K} \le Ch_K^{-d(\frac{1}{p_1}-\frac{1}{p_2})}|\chi|_{m,p_1,K} \quad \forall\chi \in S(\Omega,\mathcal{T}_h),\ 1 \le p_1 \le p_2 \le \infty,\ m = 0\ or 1; \tag{18}$$

$$|\chi|_{m,p,K} \le Ch_K^{-1}|\chi|_{m-1,p,K} \quad \forall\chi \in S(\Omega,\mathcal{T}_h),\ 1 \le p \le \infty\ m \ge 1; \tag{19}$$

$$\lim_{h\to 0}||(I-\pi_K^h)\eta||_{0,\infty,K} = 0 \quad \forall\eta \in C(\bar{K}); \tag{20}$$

$$||(I-p^h)\eta||_{0,K} + h||(I-p^h)\eta||_{1,K} \le Ch_K^m||\eta||_{m,K} \quad \forall\eta \in H^m(K),\ m = 1\ or 2; \tag{21}$$

$$||(I-\pi_K^h)\eta||_{m,r,K} \le Ch^{1-m}||\eta||_{1,r,K} \quad \forall\eta \in W^{1,r}(K),\ m = 0\ or 1,\ r \in [1,\infty]\ if d = 1; \tag{22}$$

$$\lim_{h\to 0}||(I-\pi_K^h)\eta||_{1,K} = 0 \quad \forall\eta \in H^1(K)\ if d = 1. \tag{23}$$

$$||v||_e \le C|e|^{1/2}|K|^{-1/2}(||v||_K + h_K||\nabla v||_K) \quad \forall v \in H^s(K),\ s \ge 1,\ \forall e \subset \partial K; \tag{24}$$

$$||\nabla v\cdot\mathbf{n}||_e \le C|e|^{1/2}|K|^{-1/2}(||\nabla v||_K + h_K||\nabla^2 v||_K) \quad \forall v \in H^s(K),\ s \ge 2,\ \forall e \subset \partial K; \tag{25}$$

where $h_K := \sup_{x,y\in K}||x-y||$, and in (24), (25) we have indicated, with an abuse of notation, $v|_e \equiv \gamma_0 v$, $(\nabla v\cdot\mathbf{n})|_e \equiv \gamma_1 v$, with γ_0 and γ_1 the usual trace operators onto Sobolev spaces on the boundary of a domain.

Using (19) in (24) and (25) we get

$$||v||_e \le C|e|^{1/2}|K|^{-1/2}||v||_K \quad \forall v \in \mathbb{P}_k(K),\ \forall e \subset \partial K; \tag{26}$$

$$||\nabla v\cdot\mathbf{n}||_e \le C|e|^{1/2}|K|^{-1/2}||\nabla v||_K \quad \forall v \in \mathbb{P}_k(K),\ \forall e \subset \partial K; \tag{27}$$

In the particular case $d = 1$ we have

$$|v(x_n)| \le Ch_{n-1,n}^{-1/2}||v||_{I_n} \quad \forall v \in \mathbb{P}_k(I_n), \tag{28}$$

$$|v'(x_n)| \le Ch_{n-1,n}^{-1/2}||v'||_{I_n} \quad \forall v \in \mathbb{P}_k(I_n), \tag{29}$$

Let's introduce also the following broken Friedrichs' inequality, (see, e.g., [16]),

$$||v|| \le C\bigg(|v|_{1,\mathcal{T}_h}^2 + J_p(v,v)\bigg)^{1/2} \quad \forall v \in H^1(\Omega,\mathcal{T}_h) \text{ with } (v,1) = 0. \tag{30}$$

Using (19), (26), (27), the fact that $|e| \le h_K^{d-1}$, the regularity and quasi-uniformity of the partition $\mathcal{T}_h$, the Cauchy-Schwarz and Young inequalities we obtain

$$|||v||| \le Ch^{-1}||v||, \quad |||v|||_B \le Ch^{-1}||v|| \quad \forall v \in S(\Omega,\mathcal{T}_h). \tag{31}$$

Let's define the operators $\mathcal{G}_{\mathcal{T}_h} : \mathcal{F} \cap L^2(\Omega) \to H^2_*(\Omega, \mathcal{T}_h)$ and $\mathcal{G}^h_{\mathcal{T}_h} : \mathcal{F} \cap L^2(\Omega) \to V(\Omega, \mathcal{T}_h)$, where $\mathcal{F} := \{v \in (H^1(\Omega))' :< v, 1 >= 0\}$, $H^2_*(\Omega, \mathcal{T}_h) := \{v \in H^2(\Omega, \mathcal{T}_h) : (v, 1) = 0\}$ and $V(\Omega, \mathcal{T}_h) := \{v^h \in S(\Omega, \mathcal{T}_h) : (v^h, 1) = 0\}$, such that

$$B_{\mathcal{T}_h}(\mathcal{G}_{\mathcal{T}_h} v, \chi) = (v, \chi) \quad \forall \chi \in H^2(\Omega, \mathcal{T}_h), \tag{32}$$

$$B_{\mathcal{T}_h}(\mathcal{G}^h_{\mathcal{T}_h} v, \chi) = (v, \chi) \quad \forall \chi \in S(\Omega, \mathcal{T}_h). \tag{33}$$

Existence and uniqueness of $\mathcal{G}_{\mathcal{T}_h} v$ and $\mathcal{G}^h_{\mathcal{T}_h} v$, for any $v \in \mathcal{F} \cap L^2(\Omega)$, follows from the Lax-Milgram theorem on $H^2_*(\Omega, \mathcal{T}_h)$ and $V(\Omega, \mathcal{T}_h)$, on noting that $B_{\mathcal{T}_h}(\cdot, \cdot)$ is continuous and coercive on $V(\Omega, \mathcal{T}_h) \times V(\Omega, \mathcal{T}_h)$ with respect to the energy norm $|||\cdot|||$, and is continuous and coercive on $H^2_*(\Omega, \mathcal{T}_h) \times H^2_*(\Omega, \mathcal{T}_h)$ with respect to the stronger norm $|||\cdot||| := (|v|^2_{1,\mathcal{T}_h} + h^2_K |v|^2_{2,\mathcal{T}_h} + J_p(v, v))^{1/2}$, (cf. the trace inequality (25). See, e.g., [16]). Note that, since $v \in \mathcal{F} \cap L^2(\Omega)$, from elliptic regularity and the fact that $B_{\mathcal{T}_h}(\mathcal{G}_{\mathcal{T}_h} v, 1) = 0$, we have effectively that $\mathcal{G}_{\mathcal{T}_h} v \in H^2_*(\Omega, \mathcal{T}_h)$.
We can define a norm on $\mathcal{F} \cap L^2(\Omega)$ by setting

$$||v||_{\mathcal{F} \cap L^2(\Omega)} := \left(B_{\mathcal{T}_h}(\mathcal{G}_{\mathcal{T}_h} v, \mathcal{G}_{\mathcal{T}_h} v) \right)^{1/2} = (v, \mathcal{G}_{\mathcal{T}_h} v)^{1/2} \quad \forall v \in \mathcal{F} \cap L^2(\Omega). \tag{34}$$

2 Discontinuous Galerkin Finite Element approximation

In this section we introduce the finite element and time discretization of (1)-(3). While at the continuous level Entropy estimates guarantee the positivity of the solution (see [4], in particular Estimate (2.24) and Theorems 2.2 and 2.3 therein), at a discrete level such estimates are not straightforwardly available. In [12], a suitable approximation of the mobility has been introduced in order to guarantee the validity of an Entropy estimate and consequently the positivity of the solution also at a discrete level, which consists of an harmonic average of the mobility on a structured mesh.

Following [6] and [4], let's impose this property as a constraint through a variational inequality. In the sequel we will show that the solution of the discrete formulation, for the discretization parameters tending to zero, satisfies a mixed weak formulation of (1)-(3) and is thus consistent.

Let's set $\Delta t = T/N$ for a $N \in \mathbb{N}$, and $t_n = n\Delta t$, $n = 1, ..., N$. For $d = 1, 2, 3$, starting from a datum $c_0 \in H^1(\Omega)$ and $c^0_h|_K = \pi^h_K c_0$ (if $d = 1$) or $c^0_h|_K = p^h(c_0)|_K$, with $0 \leq c^0_h < 1$, consider the following fully discretized problem:

Problem $\mathbf{P}^h$. For $n = 1, \ldots, N$, given $c^{n-1}_h \in K(\Omega, \mathcal{T}_h)$, find $(c^n_h, w^n_h) \in K(\Omega, \mathcal{T}_h) \times S(\Omega, \mathcal{T}_h)$ such that, for all $(\chi, \phi) \in S(\Omega, \mathcal{T}_h) \times K(\Omega, \mathcal{T}_h)$,

$$\begin{cases} \left(\dfrac{c^n_h - c^{n-1}_h}{\Delta t}, \chi \right) + B_{\mathcal{T}_h}(c^{n-1}_h; w^n_h, \chi) = 0, \\ \gamma B_{\mathcal{T}_h}(c^n_h, \phi - c^n_h) + (\psi'_1(c^n_h), \phi - c^n_h) \geq (w^n_h - \psi'_2(c^{n-1}_h), \phi - c^n_h) \end{cases} \tag{35}$$

Defining the discrete Energy functional $F_1 : S(\Omega, \mathcal{T}_h) \to \mathbb{R}^+$ by

$$F_1[c^n_h] = \frac{\gamma}{2} B_{\mathcal{T}_h}(c^n_h, c^n_h) + \int_\Omega \{\psi_1(c^n_h) + \chi_{\mathbb{R}^+}(c^n_h)\} dx, \tag{36}$$

where $\chi_{\mathbb{R}^+}(\cdot)$ is the indicator function of the closed and convex set $\mathbb{R}^+$, we can rewrite, using the simmetry of the bilinear form $B_{\mathcal{T}_h}(\cdot, \cdot)$, the second equation of system (35) as

$$(w^n_h - \psi'_2(c^{n-1}_h), \phi - c^n_h) + F_1[c^n_h] \leq F_1[\phi], \quad \forall \phi \in S(\Omega, \mathcal{T}_h), \tag{37}$$

which is equivalent to

$$w^n_h - \psi'_2(c^{n-1}_h) \in \partial F_1[c^n_h], \tag{38}$$

where ∂ is the subdifferential of the convex and lower-semicontinuous function F_1. The convexity and lower-semicontinuity properties of F_1 are a consequence of remark 1.1 and of the properties of ψ_1 and $\chi_{\mathbb{R}^+}$. Note

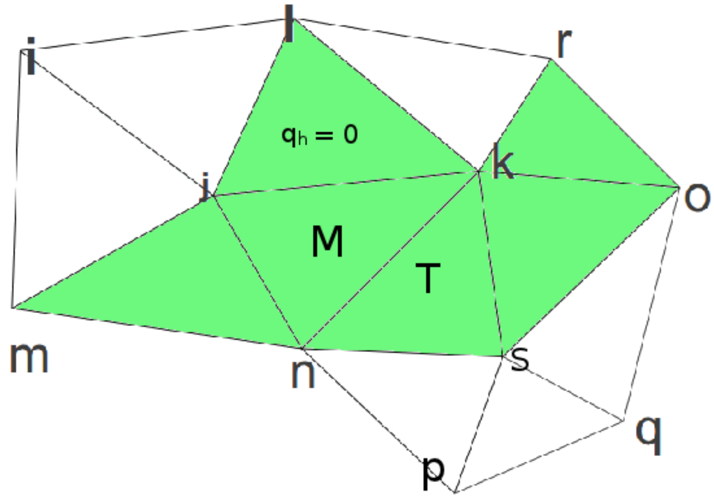


Figure 1: A partition of the domain Ω into regions where $q_h \equiv 0$ (the colored region) and $q_h \neq 0$.

that the formulation (38) represents the generalized discrete analogous of the subdifferential approach to the standard Cahn-Hilliard equation with constraints introduced in [14]. Here that approach is generalized to the present case. Inequality (37) will be used in (97) and (98) in order to study the convergence of a suitable regularized problem to the original one.

Remark 2.1 *Given the assumption $0 \leq c_h^0 < 1$, the term $(\psi_1'(c_h^n), \chi - c_h^n)$ in the second equation of (35) is well defined, since we will show that $|c_h^0|_{0,\infty} < 1$ implies that $|c_h^n|_{0,\infty} < 1$ for all $n \geq 1$ (see Lemma 2.4). From now on, let's assume that $0 \leq c_h^0 < 1$. Notice that this is a physically-consistent assumption, since subregions in the domains where the cellular phase concentrates against infinite cell-cell repulsion are unphysical.*

We introduce now the broken discrete Green operator of the degenerate elliptic form in the first equation of system (35), which will be used to express the chemical potential w_h^n in terms of $\frac{c_h^n - c_h^{n-1}}{\Delta t}$ and to show the well posedness of Problem $\mathbf{P}^h$. We modify the approach with continuous elements introduced in [4] to invert the degenerate elliptic form on a proper closed and convex subset of $S(\Omega, \mathcal{T}_h)$.

In order to introduce the subset of $S(\Omega, \mathcal{T}_h)$ on which the degenerate elliptic form $B_{\mathcal{T}_h}(c_h^{n-1}; w_h^n, \chi)$ can be inverted, we must subdivide properly the partition $\mathcal{T}^h$ of Ω into elements on which $c_h^{n-1} \equiv 0$ and elements on which $c_h^{n-1} \neq 0$. Given $q^h \in K(\Omega, \mathcal{T}_h)$, let's define a set of *passive elements* $K_0(q^h) \subset \mathcal{T}_h$ by

$$K \in K_0(q^h) \Leftrightarrow \pi_K^h q^h \equiv 0 \wedge \pi_{\mathcal{N}(\partial K, K)}^h q^h \equiv 0 \Leftrightarrow (q^h, \phi_j^K) = (q^h, \phi_j^{\mathcal{N}(\partial K, K)}) = 0, \tag{39}$$

for each $j = 1, \ldots, d+1$, where $\mathcal{N}(\partial K, K)$ denotes the elements in $\mathcal{T}_h$ that are neighbors of K through an edge $e \subset \partial K \setminus \partial\Omega$. The elements in the set $K_+(q^h) = \mathcal{T}_h \setminus K_0(q^h)$ are called *active elements*; these elements can be partitioned into mutually disjoint and maximally connected subsets $K_+(q^h) \equiv \bigcup_{m=1}^M K_m(q^h)$, for which any couple of elements K_1 and K_2 in $K_m(q^h)$ belongs to a maximally connected set of elements of $\mathcal{T}^h$ on which $(q^h \neq 0)$ or $(q^h = 0 \wedge [[q^h]]_e \neq 0)$ for some $e \subset \partial K_i$, $i = 1, 2$. In Figure 1 we show a possible partition of the domain in regions where $q_h \equiv 0$ (the colored region) and $q_h \neq 0$. Note that the element $M \in K_0(q^h)$, and all other elements are in $K_+(q^h)$. In particular, $(q^h = 0 \wedge [[q^h]]_{\bar{ns}} \neq 0)$ on the element T.

Defining

$$\Sigma_m(q^h) := \sum_{K \in K_m(q^h)} \sum_{j \in J_{K_m(q^h)}} \phi_j^K,$$

note that

$$\Sigma_m(q^h) \equiv 1 \quad \text{on } K_m(q^h). \tag{40}$$

Finally we introduce the set

$$V^h(q^h) := \{v^h \in S(\Omega, \mathcal{T}^h) : \pi_K^h v^h \equiv 0 \wedge [[v^h]]_e = 0 \; \forall e \subset \partial K \setminus \partial\Omega, \, \forall K \in K_0(q^h), \quad \text{and } (v^h, \Sigma_m(q^h)) = 0, \, m = 1, \dots, M\} \tag{41}$$

Observe that any $v^h \in S(\Omega, \mathcal{T}^h)$ can be written as

$$\begin{aligned} v^h &\equiv \sum_{K \in \mathcal{T}^h} \sum_{j \in J_K} v^h(x_j^K)\phi_j^K \\ &\equiv \bar{v}^h + \sum_{K \in K_0(q^h)} \sum_{j \in J_K} v^h(x_j^K)\phi_j^K + \sum_{m=1}^{M} \left[\fint_{K_m(q^h)} v^h \right] \Sigma_m(q^h) \\ &+ \sum_{m=1}^{M} \sum_{K \in K_m(q^h)} \sum_{j \in J_K} \sum_{K' \in K_0(q^h)} \sum_{e \subset \partial K'} \left[v^h(x_j^K) - \fint_{K_m(q^h)} v^h \right] \left(\phi_j^K \Big|_e \right), \end{aligned} \tag{42}$$

where

$$\fint_{K_m(q^h)} v^h := \frac{(v^h, \Sigma_m(q^h))}{(1, \Sigma_m(q^h))}. \tag{43}$$

and $\bar{v}^h$ is the p^h projection of v^h onto $V^h(q^h)$, i.e.

$$\begin{aligned} \bar{v}^h &:= \sum_{m=1}^{M} \sum_{K \in K_m(q^h)} \sum_{j \in J_K} \left[v^h(x_j^K) - \fint_{K_m(q^h)} v^h \right] \phi_j^K \\ &- \sum_{m=1}^{M} \sum_{K \in K_m(q^h)} \sum_{j \in J_K} \sum_{K' \in K_0(q^h)} \sum_{e \subset \partial K'} \left[v^h(x_j^K) - \fint_{K_m(q^h)} v^h \right] \left(\phi_j^K \Big|_e \right). \end{aligned} \tag{44}$$

Note that $[[\bar{v}^h]]_e = 0 \; \forall e \subset \partial K \setminus \partial\Omega, \, \forall K \in K_0(q^h)$, and in particular

$$\bar{v}^h(x_j^K) = 0 \quad \forall x_j^K \in e \subset \partial K' \setminus \partial\Omega, \, \forall K' \in K_0(q^h), \forall K \in K_m(q^h). \tag{45}$$

We can now define, for all $q^h \in K(\Omega, \mathcal{T}_h)$ with $q^h < 1$, the broken discrete anisotropic Green's operator $G^h_{(q^h, \mathcal{T}_h)} : V^h(q^h) \to V^h(q^h)$ such that

$$B_{\mathcal{T}_h}(q^h; G^h_{(q^h, \mathcal{T}_h)} v^h, \chi) = (v^h, \chi) \quad \forall \chi \in S(\Omega, \mathcal{T}_h). \tag{46}$$

To show the well posedness of $G^h_{(q^h, \mathcal{T}_h)}$ we need the following lemma.

Lemma 2.1 *There exists a positive constant σ_1, and for each $\sigma \geq \sigma_1$ there exists a positive constant $C_1 = C_1(\sigma)$, independent of the discretization parameter h, such that*

$$C_1 |||v|||_{q^h}^2 \leq B_{\mathcal{T}_h}(q^h; v, v) \quad \forall v \in V^h(q^h), \tag{47}$$

where the anisotropic energy seminorm $|||v|||_{q^h}$ is defined as

$$|||v|||_{q^h} := \left(||[b(q^h)]^{1/2} \nabla v||_{0,\mathcal{T}_h}^2 + J_p(v, v) \right)^{1/2}, \tag{48}$$

Henceforth, we shall assume that $\sigma = \sigma_1$ in the definition of the penalty parameter in (10) *and* (12).

Proof. Let's show the proof for the case $d = 2, 3$. The proof can be adapted straightforwardly to the case $d = 1$. Let's rewrite the energy seminorm (48) and the anisotropic bilinear form (10) as follows:

$$|||v|||_{q^h} = \Bigg(\sum_{K\in K_0(q^h)} ||[b(q^h)]^{1/2}\nabla v||^2_K + \sum_{K\in K_0(q^h)} \sum_{e\in\partial K} \frac{\sigma}{|e|}([[v]],[[v]])_e \tag{49}$$

$$+ \sum_{m=1}^{M} \sum_{K'\in K_m(q^h)} ||[b(q^h)]^{1/2}\nabla v||^2_{K'} + \sum_{m=1}^{M} \sum_{K'\in K_m(q^h)} \sum_{K\in K_0(q^h)} \sum_{e\in\partial K'\setminus\partial K} \frac{\sigma}{|e|}([[v]],[[v]])_e \Bigg)^{1/2},$$

$$B_{\mathcal{T}_h}(q^h; v, v) = \sum_{K\in K_0(q^h)} (b(q^h)\nabla v, \nabla v)_K \tag{50}$$

$$- \sum_{K\in K_0(q^h)} \sum_{e\in\partial K} \left[2([[v]], \{b(q^h)\nabla v\cdot \mathbf{n}_e\})_e - \frac{\sigma}{|e|}([[v]],[[v]])_e \right] + \sum_{m=1}^{M} \sum_{K'\in K_m(q^h)} (b(q^h)\nabla v, \nabla v)_{K'}$$

$$- \sum_{m=1}^{M} \sum_{K'\in K_m(q^h)} \sum_{K\in K_0(q^h)} \sum_{e\in\partial K'\setminus\partial K} \left[2([[v]], \{b(q^h)\nabla v\cdot \mathbf{n}_e\})_e - \frac{\sigma}{|e|}([[v]],[[v]])_e \right].$$

Since $v \in V^h(q^h)$, from (41) it can be seen that the first two terms on the right hand sides of equations (49) and (50) are identically equal to zero. For this same reason we have omitted to write the contributions from the trace terms on $e \in \partial K' \cap \partial K$ in the last terms on the right hand side of (49) and (50). Using now the trace inequality (27) and the Cauchy-Schwarz inequality for an $e \in \partial K' \setminus \partial K$ shared by elements K^+ and K^- in $K_m(q^h)$ on which $q^h \neq 0$, we get

$$([[v]], \{b(q^h)\nabla v\cdot \mathbf{n}_e\})_e \leq C||b(q^h)||_{0,\infty}(|v|^2_{1,K^+} + |v|^2_{1,K^-})^{1/2}\left(\frac{1}{|e|}\right)^{1/2} ||[[v]]||_e. \tag{51}$$

Let's define the set $K^*_m(q^h) := \{K \in K_m(q^h) \,:\, q^h|_K \neq 0\}$. Analogously, for an $e \in \partial K' \setminus \partial K$ shared by elements $K^+ \in K^*_m(q^h)$ and $K^- \in K_m(q^h) \setminus K^*_m(q^h)$, for which $q^h \equiv 0 \wedge [[q^h]]_e \neq 0$ on K^- (note that $K^- \cap K \neq \emptyset$ for some $K \in K_0(q^h)$), we get

$$([[v]], \{b(q^h)\nabla v\cdot \mathbf{n}_e\})_e \leq C||b(q^h)||_{0,\infty}(|v|_{1,K^+})\left(\frac{1}{|e|}\right)^{1/2} ||[[v]]||_e. \tag{52}$$

Using (51) and (52) in (50), using Young inequality and denoting

$$b^*_{\min}(q^h) := \min_{K\in K^*_m(q^h)} \frac{1}{|K|}\int_K b(q^h)dx,$$

We get

$$B_{\mathcal{T}_h}(q^h; v, v) \geq \frac{1}{2}\sum_{m=1}^{M} \sum_{K'\in K_m(q^h)} ||[b(q^h)]^{1/2}\nabla v||^2_{K'} \tag{53}$$

$$+ \sum_{m=1}^{M} \sum_{K'\in K_m(q^h)} \sum_{K\in K_0(q^h)} \sum_{e\in\partial K'\setminus\partial K} \left[\frac{\sigma - 2\frac{C^2||b(q^h)||^2_{0,\infty}}{b^*_{\min}(q^h)}}{|e|} \right]([[v]],[[v]])_e.$$

Choosing

$$\sigma_1 = 2\frac{C^2||b(q^h)||^2_{0,\infty}}{b^*_{\min}(q^h)},$$

we get (47). □

Let's show now the well posedness of the operator $G^h_{(q^h,\mathcal{T}_h)}$. Choosing in (46) $\chi \equiv \phi^K_j$, for $K \in K_0(q^h)$, $j \in J_K$, leads to both sides vanishing, on noting (39) and (41). Choosing $\chi \equiv \Sigma_m(q^h)$, $m = 1, \ldots, M$ leads to both sides of (46) vanishing. Indeed, observing that the sum of the trace terms on the boundaries of the elements in the set $K_m(q^h)$ which are not shared with elements in $K_0(q^h)$ cancels out, (which is a consequence of the local conservativity of the method), we get

$$-\sum_{K\in K_0(q^h)}\sum_{e\in\partial K}\left[\{b(q^h)\nabla G^h_{(q^h,\mathcal{T}_h)}v^h\cdot\mathbf{n}_e\}_e-\frac{\sigma}{|e|}[[G^h_{(q^h,\mathcal{T}_h)}v^h]]_e\right]=(v^h,\Sigma_m(q^h)), \tag{54}$$

and all the terms in (54) are identically equal to zero, due to (39) and (41).
Therefore, to prove the well posedness of the operator $G^h_{(q^h,\mathcal{T}_h)}: V^h(q^h)\to V^h(q^h)$ it remains to prove uniqueness, as $V^h(q^h)$ has finite dimension. If there exist two solutions $Z^h_i\in V^h(q^h)$, $i=1,2$, with $B_{\mathcal{T}_h}(q^h;Z^h_i,\chi)=(v^h,\chi)\quad\forall\chi\in S(\Omega,\mathcal{T}_h)$, then $Z^h:=Z^h_1-Z^h_2\in V^h(q^h)$ satisfies, choosing $\chi\equiv Z^h$ and using lemma 2.1,

$$C_1|||Z^h|||^2_{q^h}\le B_{\mathcal{T}_h}(q^h;Z^h,Z^h)=0. \tag{55}$$

Using (49), (39) and (41), we can rewrite (55) as

$$\sum_{m=1}^{M}\sum_{K'\in K_m(q^h)\setminus K^*_m(q^h)}\sum_{K\in K_0(q^h)}\sum_{e\in\partial K'\setminus\partial K}\frac{\sigma}{|e|}|||[[Z^h]]|||^2_e+\sum_{m=1}^{M}\sum_{K'\in K^*_m(q^h)}||[b(q^h)]^{1/2}\nabla Z^h||^2_{K'} \tag{56}$$
$$+\sum_{m=1}^{M}\sum_{K'\in K^*_m(q^h)}\sum_{e\in\partial K'}\frac{\sigma}{|e|}|||[[Z^h]]|||^2_e=0.$$

As a first consequence of (56) we get that $[[Z^h]]_e=0$ in $L^2(e)$ on $e\in\partial K'\setminus\partial K$, where $K'\in K_m(q^h)\setminus K^*_m(q^h)$ and $K\in K_0(q^h)$ (i.e. $[[Z^h]]_e$ on the boundaries of elements on which $q^h\equiv 0\wedge[[q^h]]_e\neq 0$). Since $Z^h\in\mathbb{P}_1$, we effectively have $[[Z^h]]_e=0$. Similarly, as a second consequence of (56), since $b(q^h)>0$ on $K^*_m(q^h)$, we obtain that Z^h is equal to a constant on each $K^*_m(q^h)$, $m=1,\ldots,M$. Given an $e\in\partial K'\setminus\partial K$, where $K'\in K_m(q^h)\setminus K^*_m(q^h)$ and $K\in K_0(q^h)$, shared by two elements $K^-\in K_m(q^h)\setminus K^*_m(q^h)$ and $K^+\in K^*_m(q^h)$, indicating $Z^h_{K^\pm}:=Z^h|_{K^\pm}$, the following facts hold:

1. $Z^h_{K^+}|_e$ is equal to a constant;
2. $[[Z^h]]_e=0$;
3. since $K^-\cap K\neq\emptyset$ for some $K\in K_0(q^h)$, we have from (45) that $Z^h_{K^-}(x^{K^-}_j)=0$ for $x^{K^-}_j\in\partial K^-\cap\partial K$.

These facts together imply that $Z^h=0$ on $K_m(q^h)$, and hence $Z^h\equiv 0$ on Ω. Thus $G^h_{(q^h,\mathcal{T}_h)}$ is well posed. Note that the previous reasoning is valid only for $d>1$, since in this case the set Γ is connected. In the case $d=1$ we have to use (55) and the property that $(Z^h,1)=0$ on $K_m(q^h)$ in order to simply show that $Z^h\equiv 0$ on $[0,L]$.

Remark 2.2 *By comparing* (46) *with* (35)*, choosing* $q^h\equiv c^{n-1}_h$ *and* $v^h\equiv\frac{c^n_h-c^{n-1}_h}{\Delta t}$ *in* (46)*, note that the definition of the space* $V^h(c^{n-1}_h)$ *introduces the property of a moving support of the discrete solution of* (35) *with a finite speed of velocity, since the support can expand at most of a length* h_K *locally at each time step.*

Let's proceed now by studying a regularized version of problem (35), in order to deal with the singularity in the cellular potential and to show the well posedness of problem (35) when the regularization parameter tends to zero.

2.1 Regularized problem

Let's introduce the following regularization of the cellular potential near $c^n_h=1$: for $\epsilon>0$, set

$$\psi''_{1,\epsilon}(c^n_h):=\begin{cases}\psi''_1(1-\epsilon) & \text{for}\quad c^n_h\ge 1-\epsilon,\\ \psi''_1(c^n_h) & \text{for}\quad c^n_h<1-\epsilon.\end{cases} \tag{57}$$

By expanding $\psi_1(c_h^n)$ in (5) in a neighborhood of $(1-\epsilon)$ when $c_h^n \geq 1-\epsilon$, $\psi_{1,\epsilon}$ is obtained, i.e.

$$\psi_{1,\epsilon}(c_h^n) := \begin{cases} -(1-c*)\log\epsilon + \frac{3}{2}(1-c*) - \frac{2}{\epsilon}(1-c*)(1-c_h^n) + \frac{1-c*}{2\epsilon^2}(1-c_h^n)^2, \\ \psi_1(c_h^n), \end{cases} \tag{58}$$

and $\psi'_{1,\epsilon}$, i.e.

$$\psi'_{1,\epsilon}(c_h^n) := \begin{cases} \frac{2}{\epsilon}(1-c*) - \frac{1-c*}{\epsilon^2}(1-c_h^n), \\ \psi'_1(c_h^n), \end{cases} \tag{59}$$

for $c_h^n \geq 1-\epsilon$ and $c_h^n < 1-\epsilon$ respectively. Furthermore, expanding $\psi_{1,\epsilon}(c_h^n)$ in the Taylor series around $(1-\epsilon)$, with an argument $s > 1$ and with $\epsilon < 1$, using (57), (58) and (59) we obtain

$$\begin{aligned} \psi_{1,\epsilon}(s) &= \psi_{1,\epsilon}(1-\epsilon) + \psi'_{1,\epsilon}(1-\epsilon)(s-(1-\epsilon)) + \frac{1}{2}\psi''_{1,\epsilon}(1-\epsilon)(s-(1-\epsilon))^2 \\ &= -(1-c*)\log\epsilon + \frac{1-c*}{\epsilon}(s-(1-\epsilon)) + \frac{1-c*}{2\epsilon^2}(s-(1-\epsilon))^2 \geq \frac{1-c*}{2\epsilon^2}(s-1)^2. \end{aligned}$$

Hence we have that

$$\psi_{1,\epsilon}(s) \geq \frac{1-c*}{2\epsilon^2}([s-1]_+)^2 \quad \forall s \in \mathbb{R}, \tag{60}$$

where $[\cdot]_+ = \max\{0,\cdot\}$. Introducing the concave preserving extension $\bar{\psi}_2 \in C^1(\mathbb{R}^+)$ of $\psi_2 \in C^1([0,1])$,

$$\bar{\psi}_2(c_h^n) := \begin{cases} \psi_2(1) + (c_h^n - 1)\psi'_2(1) & \text{for} \quad c_h^n \geq 1, \\ \psi_2(c_h^n) & \text{for} \quad 0 \leq c_h^n < 1, \end{cases} \tag{61}$$

let's set $\psi_\epsilon(c_h^n) := \psi_{1,\epsilon}(c_h^n) + \bar{\psi}_2(c_h^n)$. Note that

$$\psi_{1,\epsilon}(r) + \bar{\psi}_2(r) = -(1-c^*)\log\epsilon - \frac{1}{3} + \left[\frac{2}{\epsilon}(1-c^*) + (2c^*-3)\right](r-1) + \left(\frac{1-c^*}{2\epsilon^2}\right)(r-1)^2 \tag{62}$$

for $r \geq 1$. Since there exists a sufficiently small positive value ϵ_0 such that the expression in the square brackets in (62) is positive $\forall \epsilon \leq \epsilon_0$, we obtain that

$$\psi_\epsilon(s) + \frac{1}{3} \geq \frac{1-c*}{2\epsilon^2}([s-1]_+)^2 \quad \forall s \in \mathbb{R},\ \epsilon \leq \epsilon_0. \tag{63}$$

In order to show the well posedness of Problem $\mathbf{P}^h$, let's introduce the following regularized version of (35):

Problem $\mathbf{P}^h_\epsilon$**.** For $n = 1, \ldots, N$, given $c_h^{n-1} \in K(\Omega, \mathcal{T}_h)$, with $c_h^{n-1} < 1$ and $|||c_h^{n-1}||| \leq C$, find $(c_{h,\epsilon}^n, w_{h,\epsilon}^n) \in K(\Omega, \mathcal{T}_h) \times S(\Omega, \mathcal{T}_h)$ such that for all $(\chi, \phi) \in S(\Omega, \mathcal{T}_h) \times K(\Omega, \mathcal{T}_h)$,

$$\begin{cases} \left(\dfrac{c_{h,\epsilon}^n - c_h^{n-1}}{\Delta t}, \chi\right) + B_{\mathcal{T}_h}(c_h^{n-1}; w_{h,\epsilon}^n, \chi) = 0, \\ \gamma B_{\mathcal{T}_h}(c_{h,\epsilon}^n, \phi - c_{h,\epsilon}^n) + (\psi'_{1,\epsilon}(c_{h,\epsilon}^n), \phi - c_{h,\epsilon}^n) \geq (w_{h,\epsilon}^n - \hat{\psi}'_2(c_h^{n-1}), \phi - c_{h,\epsilon}^n) \end{cases} \tag{64}$$

The following result shows that Problem $\mathbf{P}^h_\epsilon$ is well posed.

Lemma 2.2 *There exists a solution $(c_{h,\epsilon}^n, w_{h,\epsilon}^n)$ to Problem $\mathbf{P}^h_\epsilon$. Moreover, the solution $\{c_{h,\epsilon}^n\}_{n=1}^N$ is unique, and $w_{h,\epsilon}^n$ is unique on $K_m(c_h^{n-1})$, for $m = 1, \ldots, M$, $n = 1, \ldots, N$.*

Proof. From the first equation in system (64) and from (46) it follows that, given $c_h^{n-1} \in K(\Omega, \mathcal{T}_h)$, $c_h^{n-1} < 1$, a $c_{h,\epsilon}^n \in K^h(c_h^{n-1})$ is searched, where

$$K^h(c_h^{n-1}) := \{\chi \in K(\Omega, \mathcal{T}_h) \,:\, \chi - c_h^{n-1} \in V^h(c_h^{n-1})\}. \tag{65}$$

Moreover, a solution $w^n_{h,\epsilon} \in S(\Omega, \mathcal{T}_h)$ can be expressed in terms of $c^n_{h,\epsilon} - c^{n-1}_h$ through the discrete anisotropic Green operator (46), recalling (42), as

$$w^n_{h,\epsilon} \equiv -\mathcal{G}^h_{(c^{n-1}_h, \mathcal{T}_h)}\left[\frac{c^n_{h,\epsilon} - c^{n-1}_h}{\Delta t}\right] + \sum_{K \in K_0(c^{n-1}_h)} \sum_{j \in J_K} \mu^{K,n}_{j,\epsilon} \phi^K_j + \sum_{m=1}^{M} \lambda^n_{m,\epsilon} \Sigma_m(c^{n-1}_h) \tag{66}$$
$$+ \sum_{m=1}^{M} \sum_{K \in K_m(c^{n-1}_h)} \sum_{j \in J_K} \sum_{K' \in K_0(c^{n-1}_h)} \sum_{e \subset \partial K'} \delta^{K,n}_{j,\epsilon} \left(\phi^K_j \bigg|_e \right),$$

where $\{\mu^{K,n}_{j,\epsilon}\}_{j \in J_K, K \in K_0(c^{n-1}_h)}$, $\{\lambda^n_{m,\epsilon}\}^M_{m=1}$ and $\{\delta^{K,n}_{j,\epsilon}\}_{j \in J_K, K \in K_m(c^{n-1}_h), m=1,\ldots,M}$ are constants which express the values of $w^n_{h,\epsilon}$ on the nodes of *passive elements*, its average value on $K_m(c^{n-1}_h)$ and its values on the nodes on boundaries between *active* and *passive elements* respectively. Hence, Problem $\mathbf{P}^h_\epsilon$ can be restated as follows:
given $c^{n-1}_h \in K(\Omega, \mathcal{T}_h)$, with $c^{n-1}_h < 1$, find $c^n_{h,\epsilon} \in K^h(c^{n-1}_h)$ and constant Lagrange multipliers $\{\mu^{K,n}_{j,\epsilon}\}_{j \in J_K, K \in K_0(c^{n-1}_h)}$, $\{\lambda^n_{m,\epsilon}\}^M_{m=1}$ and $\{\delta^{K,n}_{j,\epsilon}\}_{j \in J_K, K \in K_m(c^{n-1}_h), m=1,\ldots,M}$ such that, for all $\phi \in K(\Omega, \mathcal{T}_h)$,

$$\gamma B_{\mathcal{T}_h}(c^n_{h,\epsilon}, \phi - c^n_{h,\epsilon}) + \left(\mathcal{G}^h_{(c^{n-1}_h, \mathcal{T}_h)}\left[\frac{c^n_{h,\epsilon} - c^{n-1}_h}{\Delta t}\right] + \psi'_{1,\epsilon}(c^n_{h,\epsilon}), \phi - c^n_{h,\epsilon} \right) \tag{67}$$
$$\geq \left(\sum_{K \in K_0(c^{n-1}_h)} \sum_{j \in J_K} \mu^{K,n}_{j,\epsilon} \phi^K_j + \sum_{m=1}^{M} \lambda^n_{m,\epsilon} \Sigma_m(c^{n-1}_h) - \hat{\psi}'_2(c^{n-1}_h), \phi - c^n_{h,\epsilon} \right)$$
$$+ \sum_{m=1}^{M} \sum_{K \in K_m(c^{n-1}_h)} \sum_{j \in J_K} \sum_{K' \in K_0(c^{n-1}_h)} \sum_{e \subset \partial K'} \delta^{K,n}_{j,\epsilon} (\phi^K_j, \phi - c^n_{h,\epsilon})_e.$$

Note that (67) represents, together with $c^n_{h,\epsilon} \in K^h(c^{n-1}_h)$, the Karush-Kuhn-Tucker optimality conditions, (see, e.g. [10]), of the minimization problem

$$\inf_{v_{h,\epsilon} \in S(\Omega, \mathcal{T}_h)} \sup_{\mu^K_{j,\epsilon}, \lambda_{m,\epsilon}, \delta^K_{j,\epsilon}, \nu_\epsilon \geq 0} \Bigg\{ \gamma |||v_{h,\epsilon}|||^2_B + 2(\psi_{1,\epsilon}(v_{h,\epsilon}) + \hat{\psi}'_2(c^{n-1}_h) v_{h,\epsilon}, 1) \tag{68}$$
$$+ \frac{1}{\Delta t} B_{\mathcal{T}_h}\left(c^{n-1}_h; \mathcal{G}^h_{(c^{n-1}_h, \mathcal{T}_h)}(v_{h,\epsilon} - c^{n-1}_h), \mathcal{G}^h_{(c^{n-1}_h, \mathcal{T}_h)}(v_{h,\epsilon} - c^{n-1}_h) \right)$$
$$- (\nu_\epsilon, v_{h,\epsilon}) - \sum_{K \in K_0(c^{n-1}_h)} \sum_{j \in J_K} \mu^K_{j,\epsilon}(\phi^K_j, v_{h,\epsilon}) - \sum_{m=1}^{M} \lambda_{m,\epsilon}(\Sigma_m(c^{n-1}_h), v_{h,\epsilon})$$
$$- \sum_{m=1}^{M} \sum_{K \in K_m(c^{n-1}_h)} \sum_{j \in J_K} \sum_{K' \in K_0(c^{n-1}_h)} \sum_{e \subset \partial K'} \delta^K_{j,\epsilon}([[\phi^K_j]], [[v_{h,\epsilon}]])_e \Bigg\},$$

with $\nu_\epsilon \in K(\Omega, \mathcal{T}_h)$ the Lagrange multiplier of the inequality constraint. Noting the convexity of $\psi_{1,\epsilon}(\cdot)$, remark 1.1, lemma 2.1 and the fact that $c^{n-1}_h \in K(\Omega, \mathcal{T}_h)$, the primal form associated to the Lagrangian (68) is a convex, proper, lower semi continuous and coercive function from the closed convex set $K^h(c^{n-1}_h)$ to $\mathbb{R}$, and the primal problem is stable. Hence, from the Kuhn-Tucker theorem, (see, e.g., [10]), there exist $c^n_{h,\epsilon} \in K^h(c^{n-1}_h)$, solution of the primal problem, and Lagrange multipliers $\{\mu^{K,n}_{j,\epsilon}\}_{j \in J_K, K \in K_0(c^{n-1}_h)}$, $\{\lambda^n_{m,\epsilon}\}^M_{m=1}$, $\{\delta^{K,n}_{j,\epsilon}\}_{j \in J_K, K \in K_m(c^{n-1}_h), m=1,\ldots,M}$ and $\nu_\epsilon \in -\partial \chi_{\mathbb{R}^+}(c^n_{h,\epsilon})$, for each n. Therefore, from (66) follows the existence of a solution $(c^n_{h,\epsilon}, w^n_{h,\epsilon})^N_{n=1}$ to Problem $\mathbf{P}^h_\epsilon$.

For what concerns uniqueness, if, for fixed $n \geq 1$, (67) has two solutions

$$(c^{n,i}_{h,\epsilon}, \{\mu^{K,n,i}_{j,\epsilon}\}_{j \in J_K, K \in K_0(c^{n-1}_h)}, \{\lambda^{n,i}_{m,\epsilon}\}^M_{m=1}, \{\delta^{K,n,i}_{j,\epsilon}\}_{j \in J_K, K \in K_m(c^{n-1}_h), m=1,\ldots,M}), \quad i = 1, 2,$$

by taking $\phi = c^{n,2}_{h,\epsilon}$ in the inequality for $c^{n,1}_{h,\epsilon}$ and $c^{n,1}_{h,\epsilon}$ in the inequality for $c^{n,2}_{h,\epsilon}$ and taking the difference between the two inequalities, on noting that $c^{n,1}_{h,\epsilon} - c^{n,2}_{h,\epsilon} := \bar{c}^n_{h,\epsilon} \in V^h(c^{n-1}_h)$ and (41), it follows that

$$
\begin{aligned}
&\gamma|||\bar{c}^n_{h,\epsilon}|||^2_B + (\psi'_{1,\epsilon}(c^{n,1}_{h,\epsilon}) - \psi'_{1,\epsilon}(c^{n,2}_{h,\epsilon}), \bar{c}^n_{h,\epsilon}) \\
&+ \frac{1}{\Delta t} B_{\mathcal{T}_h}\left(c^{n-1}_h; \mathcal{G}^h_{(c^{n-1}_h,\mathcal{T}_h)}\bar{c}^n_{h,\epsilon}, \mathcal{G}^h_{(c^{n-1}_h,\mathcal{T}_h)}\bar{c}^n_{h,\epsilon}\right) \leq 0 \\
&\rightarrow \gamma|||\bar{c}^n_{h,\epsilon}|||^2_B + \frac{1}{\Delta t} B_{\mathcal{T}_h}\left(c^{n-1}_h; \mathcal{G}^h_{(c^{n-1}_h,\mathcal{T}_h)}\bar{c}^n_{h,\epsilon}, \mathcal{G}^h_{(c^{n-1}_h,\mathcal{T}_h)}\bar{c}^n_{h,\epsilon}\right) \leq 0,
\end{aligned}
$$

where we have used the monotonicity of $\psi'_{1,\epsilon}(\cdot)$ in the second step. Therefore the uniqueness of $c^n_{h,\epsilon}$ follows from remark 1.1, lemma 2.1, the fact that, since $\bar{c}^n_{h,\epsilon} \in V^h(c^{n-1}_h)$, $(\bar{c}^n_{h,\epsilon}, 1) = 0$, and (30). For any $\delta \in (0,1)$, choosing $\phi = c^n_{h,\epsilon} \pm \delta[c^n_{h,\epsilon}\Sigma_m(c^{n-1}_h)]$ in (67), for $m = 1,\ldots,M$, yields uniqueness of the Lagrange multipliers $\{\lambda^n_{m,\epsilon}\}^M_{m=1}$, $\{\delta^{K,n}_{j,\epsilon}\}_{j\in J_K, K\in K_m(c^{n-1}_h), m=1,\ldots,M}$. Hence the uniqueness result for $w^n_{h,\epsilon}$ follows from (66). □

In order to pass to the limit for $\epsilon \to 0$ in system (64), we need to deduce suitable $\epsilon-$independent bounds for the solution $(c^n_{h,\epsilon}, w^n_{h,\epsilon})$. The following result holds.

Lemma 2.3 *For every sequence $\epsilon \to 0$, there exist a subsequence $\epsilon' \to 0$ and a $c^n_h \in K(\Omega,\mathcal{T}_h)$ such that*

$$c^n_{h,\epsilon'} \to c^n_h \quad \text{and} \quad \nabla c^n_{h,\epsilon'}|_K \to \nabla c^n_h|_K \quad \text{for } \epsilon' \to 0, K \in \mathcal{T}_h. \tag{69}$$

For the case $d > 1$, there exists a subsequence $\{c^n_{h,\epsilon'}\}$ such that, for each $e \in \Gamma_0$,

$$[[c^n_{h,\epsilon'}]]_e \to [[c^n_h]]_e \quad \text{and} \quad \{\nabla c^n_{h,\epsilon'} \cdot \mathbf{n}_e\}_e \to \{\nabla c^n_h \cdot \mathbf{n}_e\}_e, \tag{70}$$

for $\epsilon' \to 0$. For the case $d = 1$, there exists a subsequence $\{c^n_{h,\epsilon'}\}$ such that

$$[[c^n_{h,\epsilon'}(x_{\bar{n}})]] \to [[c^n_h(x_{\bar{n}})]] \quad \text{and} \quad \{c'^n_{h,\epsilon'}(x_{\bar{n}})\} \to \{c'^n_h(x_{\bar{n}})\} \ \text{for } \epsilon' \to 0, \bar{n} = 1,\ldots,\bar{N}-1, \tag{71}$$

where $0 = x_0 < x_1 < \cdots < x_{\bar{N}} = L$ is a partition of $\mathcal{T}_h$.
For every sequence $\epsilon \to 0$, there exist a subsequence $\epsilon' \to 0$ and a $w^n_h \in S(\Omega,\mathcal{T}_h)$ such that

$$w^n_{h,\epsilon'} \to w^n_h \quad \text{and} \quad \nabla w^n_{h,\epsilon'}|_K \to \nabla w^n_h|_K \ \text{for } K \in K^*_m(c^{n-1}_h), \quad \text{for } \epsilon' \to 0. \tag{72}$$

For the case $d > 1$, there exists a subsequence $\{w^n_{h,\epsilon'}\}$ such that, for each $e \in \partial K$ and for each $e' \in \partial K'$, $K' \in K^*_m(c^{n-1}_h)$, $m = 1,\ldots,M$,

$$
\begin{aligned}
&[[w^n_{h,\epsilon'}]]_e \to [[w^n_h]]_e \quad \text{and} \\
&\{b(c^{n-1}_h)\nabla w^n_{h,\epsilon'} \cdot \mathbf{n}_{e'}\}_{e'} \to \{b(c^{n-1}_h)\nabla w^n_h \cdot \mathbf{n}_{e'}\}_{e'},
\end{aligned} \tag{73}
$$

for $\epsilon' \to 0$. For the case $d = 1$, there exists a subsequence $\{w^n_{h,\epsilon'}\}$ such that

$$[[w^n_{h,\epsilon'}(x_{\bar{n}})]] \to [[w^n_h(x_{\bar{n}})]] \text{ and } \{b(c^{n-1}_h)(x_{\bar{n}*})w'^n_{h,\epsilon'}(x_{\bar{n}*})\} \to \{b(c^{n-1}_h)(x_{\bar{n}*})w'^n_h(x_{\bar{n}*})\}, \tag{74}$$

for $\epsilon' \to 0$, $\bar{n} = 1,\ldots,\bar{N}-1$, $\bar{n}* = 1,\ldots,\bar{N}-1 \wedge c^{n-1}_h(I_{\bar{n}*}) \neq 0$.

Proof. Let's start by proving stability bounds for the regularized problem (64). Choosing $\chi = w^n_{h,\epsilon}$ in the first equation of (64) and $\phi = c^{n-1}_h$ in the second equation of (64), it follows that

$$\gamma B_{\mathcal{T}_h}(c^n_{h,\epsilon}, c^n_{h,\epsilon} - c^{n-1}_h) + (\psi'_{1,\epsilon}(c^n_{h,\epsilon}) + \hat{\psi}'_2(c^{n-1}_h), c^n_{h,\epsilon} - c^{n-1}_h) + \Delta t B_{\mathcal{T}_h}(c^{n-1}_h; w^n_{h,\epsilon}, w^n_{h,\epsilon}) \leq 0.$$

Using now the identity $2s(s-r) = s^2 - r^2 + (s-r)^2$, $\forall r,s \in \mathbb{R}$, and the convexity and the concavity properties of $\psi_{1,\epsilon}(\cdot)$ and $\hat{\psi}_2(\cdot)$, it follows that

$$
\begin{aligned}
&\frac{\gamma}{2}|||c^n_{h,\epsilon}|||^2_B + \frac{\gamma}{2}|||c^n_{h,\epsilon} - c^{n-1}_h|||^2_B + (\psi_\epsilon(c^n_{h,\epsilon}), 1) + \Delta t\, B_{\mathcal{T}_h}(c^{n-1}_h; w^n_{h,\epsilon}, w^n_{h,\epsilon}) \\
&\leq (\psi_\epsilon(c^{n-1}_h), 1) + \frac{\gamma}{2}|||c^{n-1}_h|||^2_B \leq C.
\end{aligned} \tag{75}
$$

From (75) and (63) it follows that

$$||[c_{h,\epsilon}^n - 1]_+||^2 \leq C\epsilon^2. \tag{76}$$

Let's introduce the function

$$f_{1,\epsilon}(r) := -(1-c^*)\log\epsilon + \frac{3}{2}(1-c^*) - \frac{2}{\epsilon}(1-c^*)(1-r).$$

Note from (58) that

$$\psi_{1,\epsilon}(c_{h,\epsilon}^n) = f_{1,\epsilon}(c_{h,\epsilon}^n) + \frac{1-c^*}{2\epsilon^2}(1-c_{h,\epsilon}^n)^2, \tag{77}$$

for $c_{h,\epsilon}^n \geq 1-\epsilon$, and that there exists a value ϵ_0 such that $f_{1,\epsilon}(c_{h,\epsilon}^n) \geq 0$ for $c_{h,\epsilon}^n \geq 1-\epsilon$ and $\epsilon \leq \epsilon_0$. Calling Ω_ϵ the support of the base functions corresponding to nodes on which $c_{h,\epsilon}^n \geq 1-\epsilon$, we get from (75), (77) and (76) that

$$(f_{1,\epsilon}(c_{h,\epsilon}^n), 1)_{\Omega_\epsilon} \leq C, \tag{78}$$

independently on ϵ. Since $f_{1,\epsilon}(c_{h,\epsilon}^n) \in S(\Omega, \mathcal{T}_h)$ and $f_{1,\epsilon}(c_{h,\epsilon}^n) \geq 0$ for $c_{h,\epsilon}^n \geq 1-\epsilon$ and $\epsilon \leq \epsilon_0$, using (18) it follows that

$$||f_{1,\epsilon}(c_{h,\epsilon}^n)||_{0,\infty,\Omega_\epsilon} \leq Ch^{-d}, \tag{79}$$

independently on ϵ. Due to the logarithmic term in $f_{1,\epsilon}(\cdot)$, (79) implies that, for each $\epsilon \leq \epsilon_0$,

$$c_{h,\epsilon}^n < 1, \tag{80}$$

uniformly in ϵ. It follows from (75), the fact that $(c_{h,\epsilon}^n, 1) = (c_h^{n-1}, 1)$, remark 1.1, (30) and the Bolzano-Weierstrass theorem that there exist a subsequence $\{c_{h,\epsilon'}^n\}$ and a $c_h^n \in K(\Omega, \mathcal{T}_h)$ such that (69) holds.

For what concerns the convergence properties of $w_{h,\epsilon}^n$, let's start by using (30) on the sets $K_m^*(c_h^{n-1})$, $m = 1, \ldots, M$, obtaining, using lemma 2.1 and noting (75), that

$$\begin{aligned} &||[(I - ⨍_{K_m^*(c_h^{n-1})})w_{h,\epsilon}^n]\Sigma^*(c_h^{n-1})||^2 \leq C(h^{-1})\Big(|w_{h,\epsilon}^n|_{1,K_m^*(c_h^{n-1})} + J_p(w_{h,\epsilon}^n, w_{h,\epsilon}^n)\Big) \\ &\leq C(h^{-1})[b_{\min}^*(c_h^{n-1})]^{-1}|||w_{h,\epsilon}^n|||_{c_h^{n-1}}^2 dx \leq C(h^{-1}, (\Delta t)^{-1})[b_{\min}^*(c_h^{n-1})]^{-1}, \end{aligned} \tag{81}$$

where $\Sigma^*(c_h^{n-1}) := \sum_{K \in K_m^*(q^h)} \sum_{j \in J_{K_m^*(q^h)}} \phi_j^K$. Let's now bound $⨍_{K_m^*(c_h^{n-1})} w_{h,\epsilon}^n$.
Let's take

$$K(\Omega, \mathcal{T}_h) \ni \phi = c_{h,\epsilon}^n + \Sigma_m^*(c_h^{n-1})$$

in the second equation of system (64). We get

$$(w_{h,\epsilon}^n, \Sigma_m^*(c_h^{n-1})) \leq \gamma B_{\mathcal{T}_h}(c_{h,\epsilon}^n, \Sigma_m^*(c_h^{n-1})) + (\psi_{1,\epsilon}'(c_{h,\epsilon}^n), \Sigma_m^*(c_h^{n-1})) + (\hat{\psi}_2'(c_h^{n-1}), \Sigma_m^*(c_h^{n-1})). \tag{82}$$

On noting that $\Sigma_m^*(c_h^{n-1}) \equiv 1$ on $K_m^*(c_h^{n-1})$, $m = 1, \ldots, M$, and is zero elsewhere, using (9), (26), (27) and (75), the Young inequality, the facts that $\hat{\psi}_2'(c_h^{n-1})$ and $\psi_{1,\epsilon}'(c_{h,\epsilon}^n)$ are bounded, due to (80), it follows that

$$\begin{aligned} &|(w_{h,\epsilon}^n, \Sigma_m^*(c_h^{n-1}))| \leq Ch^{-1}||c_{h,\epsilon}^n||_{1,\mathcal{T}_h}^2 + C + |(\psi_{1,\epsilon}'(c_{h,\epsilon}^n), \Sigma_m^*(c_h^{n-1}))| + \\ &C||\Sigma_m^*(c_h^{n-1})||_{0,\infty} \leq C + C(h^{-1}). \end{aligned} \tag{83}$$

Now, combining (81) with (83), on noting the definition (43) , we get

$$||(w_{h,\epsilon}^n \Sigma_m^*(c_h^{n-1}))|| \leq C + C(h^{-1}) + C(h^{-1}, (\Delta t)^{-1})[b_{\min}^*(c_h^{n-1})]^{-1}. \tag{84}$$

From (75) and (47) it follows that, in the case $d > 1$,

$$||[[w_{h,\epsilon}^n]]_e||_e \leq Ch_e^{1/2}(\Delta t)^{-1/2}. \tag{85}$$

From (84) and the fact that $w_{h,\epsilon}^n \in S(\Omega, \mathcal{T}_h)$ has finite dimension we deduce that $w_{h,\epsilon}^n$ is bounded on $K_m^*(c_h^{n-1})$, uniformly in ϵ. Moreover, from (85) it can be deduced that $[[w_{h,\epsilon}^n]]_e$ is bounded for each $e \in \Gamma_0$, uniformly in ϵ. Hence, it follows from (84), (85) and (75) that there exist a subsequence $\{w_{h,\epsilon'}^n\}$ and a $w_h^n \in S(\Omega, \mathcal{T}_h)$ such that (72) holds. This is valid also for the case $d = 1$, where (85) takes the form $[[w_{h,\epsilon}^n(x_n)]] \leq Ch_{n-1,n}^{1/2}(\Delta t)^{-1/2}$, for each $n = 1, \ldots, N-1$.

Remark 2.3 *Note that the presence of the jump terms in* (10), (12) *and* (75) *makes it possible to have a convergence* $w^n_{h,\epsilon'} \to w^n_h$ *on the whole domain* Ω*, whereas in the case of discretization with continuous elements this convergence property is valid only on subdomanis where* $c_h^{n-1} \neq 0$ *(see [4]).*

From (75) and (27) we have that (in the case $d > 1$)

$$||[[c^n_{h,\epsilon}]]_e||_e \leq Ch_e^{1/2}, \quad ||\{\nabla c^n_{h,\epsilon} \cdot \mathbf{n}_e\}_e||_e \leq Ch_e^{-1/2} \tag{86}$$

Hence from the Bolzano-Weierstrass theorem there exist two elements $\xi_1, \xi_2 \in \mathbb{P}_1(e)$ and a subsequence $\{c^n_{h,\epsilon'}\}$ such that, for each $e \in \Gamma_0$,

$$[[c^n_{h,\epsilon'}]]_e \to \xi_1 \quad \text{and} \quad \{\nabla c^n_{h,\epsilon'} \cdot \mathbf{n}_e\}_e \to \xi_2, \tag{87}$$

for $\epsilon' \to 0$. Moreover, from (26), the second bound in (86) and an interpolation inequality (see [2]) it follows that $[[c^n_{h,\epsilon}]]_e \in H^{1/2}(e) \hookrightarrow L^2(e)$ and

$$||\gamma_0 c^n_{h,\epsilon'}||_{1/2,e} \leq C||\gamma_0 c^n_{h,\epsilon}||_e^{1/2} |\gamma_0 c^n_{h,\epsilon}|_{1,e}^{1/2} \leq Ch_e^{-1/2}. \tag{88}$$

Hence, from (69), the linearity of the trace operator $\gamma_0 : H^1(K) \to H^{1/2}(e)$ and the bound (88), we get $\xi_1 \equiv [[c^n_h]]_e$. Let's now introduce the lifting operator (see [5]) $l : L^2(\Gamma_0) \to S(\Omega, \mathcal{T}_h)$, defined by

$$\int_\Omega l(q)\tau dx = -\int_{\Gamma_0} q[[\tau]]ds \quad \forall \tau \in S(\Omega, \mathcal{T}_h). \tag{89}$$

From an inverse inequality (see [5]) and the second bound in (86) it is obtained that

$$||l(\{\nabla c^n_{h,\epsilon} \cdot \mathbf{n}_e\}_e)||_K \leq Ch_e^{-1/2}||\{\nabla c^n_{h,\epsilon} \cdot \mathbf{n}_e\}_e||_e \leq Ch_e^{-1} \tag{90}$$

From (69), the linearity of the lifting operator and bound (90) we have that there exists a subsequence $\{c^n_{h,\epsilon'}\}$ such that

$$l(\{\nabla c^n_{h,\epsilon'} \cdot \mathbf{n}_e\}_e) \to (\{\nabla c^n_h \cdot \mathbf{n}_e\}_e), \tag{91}$$

for $\epsilon' \to 0$. Applying the lifting operator defined in (89) to both sides of the second limit in (87) and using (91) we get

$$l(\{\nabla c^n_{h,\epsilon'} \cdot \mathbf{n}_e\}_e) \to l(\xi_2) \equiv l(\{\nabla c^n_h \cdot \mathbf{n}_e\}_e),$$

for $\epsilon' \to 0$. Hence, it follows that $\xi_2 \equiv \{\nabla c^n_h \cdot \mathbf{n}_e\}_e$. For the case $d = 1$, from (75) and (29) we have that

$$[[c^n_{h,\epsilon}(x_{\bar{n}})]] \leq Ch^{1/2}_{\bar{n}-1,\bar{n}}, \quad \{c'^n_{h,\epsilon}(x_{\bar{n}})\} \leq Ch^{-1/2}_{\bar{n}-1,\bar{n}} \tag{92}$$

Hence from the Bolzano-Weierstrass theorem there exist two elements $\eta_1, \eta_2 \in \mathbb{R}$ and a subsequence $\{c^n_{h,\epsilon'}\}$ such that

$$[[c^n_{h,\epsilon'}(x_{\bar{n}})]] \to \eta_1 \text{ and } \{c'^n_{h,\epsilon'}(x_{\bar{n}})\} \to \eta_2 \text{ for } \epsilon' \to 0, \bar{n} = 1, \dots, \bar{N} - 1. \tag{93}$$

From (75), the Sobolev embedding $H^1((I_n)) \subset\subset C^0([I_n])$, and using a lifting operator technique analogous to that used for the case $d > 1$, it follows that $\eta_1 \equiv [[c^n_h(x_{\bar{n}})]]$, $\eta_2 \equiv \{c'^n_h(x_{\bar{n}})\}$. Note that the limit point $c^n_h \in K^h(c_h^{n-1})$.

The convergence properties (73), (74) can be obtained analogously to (70), (71), on noting that $0 < b(c_h^{n-1}) < 1$ on $K^*_m(c_h^{n-1})$ and using (27) and (29) on $K^*_m(c_h^{n-1})$, using moreover (75) and (72). □

Lemma 2.4 *The limit point* c^n_h *introduced in Lemma 2.3 satisfies the property that* $||c^n_h||_{0,\infty} < 1$.

Proof. Since $\psi_{1,\epsilon}(c^n_{h,\epsilon}) \geq 0$, from the Fatou's Lemma and (75) it follows that

$$\int_\Omega \liminf_{\epsilon \to 0} \psi_{1,\epsilon}(c^n_{h,\epsilon}) \leq \liminf_{\epsilon \to 0} \int_\Omega \psi_{1,\epsilon}(c^n_{h,\epsilon}) \leq C. \tag{94}$$

From the convergence property (69) and from (58) it follows that

$$\liminf_{\epsilon \to 0} \psi_{1,\epsilon}(c^n_{h,\epsilon}) = \begin{cases} \psi_1(c^n_h) & \text{if} \quad c^n_h < 1, \\ \infty & \text{elsewhere.} \end{cases} \tag{95}$$

Hence, from (95) and (94) it follows that the set $\{x \,|\, c^n_h(x,t) = 1\}$ has zero measure. □

2.2 Well posedness of Problem P^h

We are now in a position to show the well posedness of problem (35).

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^d$, and let $c_h^0 \in K(\Omega, \mathcal{T}_h)$, with $c_h^0 < 1$ and $|||c_h^0||| \leq C$. Then, for all $\Delta t > 0$, there exists a solution (c_h^n, w_h^n) to problem (35).*

Moreover, the solution $\{c_h^n\}_{n=1}^N$ is unique, while w_h^n is unique on $K_m(c_h^{n-1})$, for $m = 1, \ldots, M$ and $n = 1, \ldots, N$.

Proof. Let's take the limit for $\epsilon \to 0$ in (64). Recalling the definition (36), on noting the convexity of $\psi_{1,\epsilon}(\cdot)$, it can be introduced a regularized convex energy functional

$$F_{1,\epsilon}[c_{h,\epsilon}^n] = \frac{\gamma}{2} B_{\mathcal{T}_h}(c_{h,\epsilon}^n, c_{h,\epsilon}^n) + \int_\Omega \{\psi_{1,\epsilon}(c_{h,\epsilon}^n) + \chi_{\mathbb{R}^+}(c_{h,\epsilon}^n)\} dx, \tag{96}$$

and rewrite system (64) as

$$\begin{cases} \left(\dfrac{c_{h,\epsilon}^n - c_h^{n-1}}{\Delta t}, \chi\right) + B_{\mathcal{T}_h}(c_h^{n-1}; w_{h,\epsilon}^n, \chi) = 0, \\ (w_{h,\epsilon}^n - \hat{\psi}_2'(c_h^{n-1}), \phi - c_{h,\epsilon}^n) + F_{1,\epsilon}[c_{h,\epsilon}^n] \leq F_{1,\epsilon}[\phi], \end{cases} \tag{97}$$

for each $\chi, \phi \in S(\Omega; \mathcal{T}_h)$. We may now pass to the limit in (97), considering the convergence properties introduced in lemma 2.3 and the uniform boundedness of $c_{h,\epsilon}^n$. For any $(\chi, \phi) \in S(\Omega, \mathcal{T}_h) \times S(\Omega, \mathcal{T}_h) \subset H^2(\Omega, \mathcal{T}_h) \times H^2(\Omega, \mathcal{T}_h)$, we have

$$\begin{aligned} &\lim_{\epsilon \to 0} \left(\frac{c_{h,\epsilon}^n - c_h^{n-1}}{\Delta t}, \chi\right) = \left(\frac{c_h^n - c_h^{n-1}}{\Delta t}, \chi\right); \\ &\lim_{\epsilon \to 0} B_{\mathcal{T}_h}(c_h^{n-1}; w_{h,\epsilon}^n, \chi) = B_{\mathcal{T}_h}(c_h^{n-1}; w_h^n, \chi); \\ &\lim_{\epsilon \to 0} (w_{h,\epsilon}^n - \hat{\psi}_2'(c_h^{n-1}), \phi - c_{h,\epsilon}^n) = (w_h^n - \psi_2'(c_h^{n-1}), \phi - c_h^n). \end{aligned}$$

Since $\psi_{1,\epsilon}$ is uniformly bounded and $c_{h,\epsilon}^n \geq 0$, since moreover $\psi_{1,\epsilon}(\cdot) \to \psi_1(\cdot)$ uniformly for $\epsilon \to 0$, from the convergence properties introduced in lemma 2.3, the dominated convergence theorem and the semi continuity property of the indicator function $\chi_{\mathbb{R}^+}(\cdot)$, it can be deduced that

$$\begin{aligned} &\lim_{\epsilon \to 0} F_{1,\epsilon}[c_{h,\epsilon}^n] \geq F_1[c_h^n]; \\ &\lim_{\epsilon \to 0} F_{1,\epsilon}[\phi] = F_1[\phi]. \end{aligned}$$

Hence, the limit point (c_h^n, w_h^n) satisfies

$$\begin{cases} \left(\dfrac{c_h^n - c_h^{n-1}}{\Delta t}, \chi\right) + B_{\mathcal{T}_h}(c_h^{n-1}; w_h^n, \chi) = 0, \\ (w_h^n - \psi_2'(c_h^{n-1}), \phi - c_h^n) + F_1[c_h^n] \leq F_1[\phi] \end{cases} \tag{98}$$

Finally, since $||c_h^n||_{0,\infty} < 1$ (see Lemma 2.4) and $\psi_1(c_h^n)$ is convex and Lipschitzian for $c_h^n < 1$, system (98) is equivalent to system (35) (see (37)), hence the limit point (c_h^n, w_h^n) is the unique solution of Problem $\mathbf{P}^h$. □

We now proceed to obtain the energy estimates.

Lemma 2.5 (Energy estimates) *Let (c_h^n, w_h^n), $n = 1, \ldots, N$ be the solution of system (35). Then, the following stability bounds hold:*

$$\begin{aligned} &\max_{n=1 \to N} |||c_h^n|||^2 + (\Delta t)^2 \sum_{n=1}^N |||\frac{c_h^n - c_h^{n-1}}{\Delta t}|||^2 + \Delta t \sum_{n=1}^N B_{\mathcal{T}_h}(c_h^{n-1}; w_h^n, w_h^n) \\ &+ \Delta t \sum_{n=1}^N [b_{\max}^{n-1}]^{-1} |||\mathcal{G}_{\mathcal{T}_h}^h [\frac{c_h^n - c_h^{n-1}}{\Delta t}]|||^2 \leq C(|||c_h^0|||^2), \end{aligned} \tag{99}$$

where $b_{max} \geq \max_{n=1 \to N} ||b(c_h^{n-1})||_{0,\infty}$.

Proof. Taking the limit for $\epsilon \to 0$ in (75) we get

$$\frac{\gamma}{2}|||c_h^n|||_B^2 + \frac{\gamma}{2}|||c_h^n - c_h^{n-1}|||_B^2 + (\psi(c_h^n), 1) + \Delta t\, B_{\mathcal{T}_h}(c_h^{n-1}; w_h^n, w_h^n) \leq (\psi(c_h^{n-1}), 1) + \frac{\gamma}{2}|||c_h^{n-1}|||_B^2. \tag{100}$$

Summing from $n = 1 \to m$, for $m = 1 \to N$, noting that $0 \leq c_h^0 < 1$ and $\psi(c_h^0) \leq C$, that $|||c_h^0|||_B \leq C$, that $||c_h^n||_{0,\infty} < 1$, using a Poincaré inequality (on noting that $\fint c_h^n = \fint c_h^0$) and remark 1.1, we get the first three bounds in (99).

Choosing now $\chi = \mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]$ in the first equation of system (35), using (33), remark 1.1, lemma 2.1, (27), the facts that $b(c_h^{n-1} > 0$ and $||b(c_h^{n-1})||_{0,\infty} < 1$, Cauchy-Schwarz and Young inequalities, we get

$$\begin{aligned}
&C_0\left|\left|\left|\mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]\right|\right|\right|^2 \leq \left|\left|\left|\mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]\right|\right|\right|_B^2 = \left(\frac{c_h^n - c_h^{n-1}}{\Delta t}, \mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]\right) = \\
&- B_{\mathcal{T}_h}\left(c_h^{n-1}; w_h^n, \mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]\right) \leq \sum_{K \in \mathcal{T}_h} ||b(c_h^{n-1})\nabla w_h^n||_K \left|\left|\nabla \mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]\right|\right|_K \\
&+ C \sum_{K \in \mathcal{T}_h} \sum_{e \in \Gamma_0} ||b(c_h^{n-1})\nabla w_h^n||_K \left(\frac{1}{h_e}\left|\left|\left[\left[\mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]\right]\right]\right|\right|_e^2\right)^{1/2} \\
&+ D \sum_{K \in \mathcal{T}_h} \sum_{e \in \Gamma_0} \left|\left|\nabla \mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]\right|\right|_K \left(\frac{1}{h_e}|||[[w_h^n]]|||_e^2\right)^{1/2} + \\
&\sum_{e \in \Gamma_0} \frac{\sigma}{h_e}|||[[w_h^n]]|||_e \left|\left|\left[\left[\mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]\right]\right]\right|\right|_e \leq C b_{\max}^{n-1} B_{\mathcal{T}_h}(c_h^{n-1}; w_h^n, w_h^n) + \\
&\frac{C_0}{2}\left|\left|\left|\mathcal{G}_{\mathcal{T}_h}^h[\frac{c_h^n - c_h^{n-1}}{\Delta t}]\right|\right|\right|^2.
\end{aligned}$$

Summing from $n = 1 \to N$ and using the third bound in (99) we get the last bound in (99). □

Remark 2.4 *Note from* (100) *that the function* $\frac{\gamma}{2}|||c_h^n|||_B^2 + (\psi(c_h^n), 1)$ *is a decreasing (Lyapunov) function for the discrete solutions. Hence the finite element and time discretization* (35) *has the gradient stability property in the sense of Eyre (see [11]).*

3 Convergence analysis

In this section we present the convergence analysis for the discrete scheme (35). The analysis will be restricted to the $d = 1$ case (see Remark 3.1).

To the sequence of discrete solutions c_h^n of Problem $\mathbf{P}^h$ it can be associated the following time continuous approximation:

$$C_h(t) := \frac{t - t_{n-1}}{\Delta t} c_h^n + \frac{t_n - t}{\Delta t} c_h^{n-1}, \tag{101}$$

for $t \in [t_{n-1}, t_n]$, $n = 1, \cdots, N$, which is a family of linear time interpolants that depend on the parameters h and Δt. Let's also define the piecewise constant-in-time functions

$$\begin{aligned} C_h^+(t) &:= c_h^n, \quad C_h^-(t) := c_h^{n-1}, \\ W_h^+(t) &:= w_h^n, \quad W_h^-(t) := w_h^{n-1}, \end{aligned} \tag{102}$$

for $t \in (t_{n-1}, t_n]$, $n = 1, \cdots, N$.

By multiplying system (35) by a $C_0^\infty([0, T])$ function, and integrating in time from 0 to T, it is obtained that (C_h, W_h) satisfies the following weak formulation:

Find $\{C_h, W_h\} \in L^2(0,T;K(\Omega,\mathcal{T}_h)) \times L^2(0,T;S(\Omega,\mathcal{T}_h))$ such that, for all $(\chi,\phi) \in L^2(0,T;S(\Omega,\mathcal{T}_h)) \times L^2(0,T;K(\Omega,\mathcal{T}_h))$,

$$\begin{cases} \int_0^T \left[\left(\frac{\partial C_h}{\partial t}, \chi \right) + B_{\mathcal{T}_h}(C_h^-; W_h^+, \chi) \right] dt = 0, \\ \int_0^T [\gamma B_{\mathcal{T}_h}(C_h^+, \phi - C_h^+) + (\psi_1'(C_h^+), \phi - C_h^+)] dt \geq \int_0^T (W_h^+ - \psi_2'(C_h^-), \phi - C_h^+), \end{cases} \tag{103}$$

with $C_h(0) = c_h^0$.

In order to pass to the limit in (103), for $h, \Delta t \to 0$, and identify the system satisfied by the limit points, we need the following results.

Lemma 3.1 *Let $d = 1$ and $c_h^0|_{I_n} = \pi_{I_n}^h(c_0)$, with $0 \leq c_0 < 1$ and $|||c_0||| \leq C$. Then there exist a subsequence of continuous and piecewise constant in time interpolants, with $C_h(0) \to c_0$ strongly in $H^1([0,L])$ as $h \to 0$, and functions $c \in L^\infty(0,T;H^1([0,L])) \cap H^1(0,T;(H^1([0,L]))') \cap C_{x,t}^{\frac{1}{2},\frac{1}{8}}([0,L]_T)$ and $w \in L^2_{\text{loc}}(0 < c < 1)$ with $\frac{\partial w}{\partial x} \in L^2_{\text{loc}}(0 < c < 1)$, such that, for $(h, \Delta t) \to 0$,*

$$C_h, C_h^\pm \rightharpoonup c \text{ weakly in } L^2(0,T;H^1([0,L],\mathcal{T}_h)), \tag{104}$$

$$C_h, C_h^\pm \to c \text{ uniformly on } \prod_{I_n \in \mathcal{T}_h} \bar{I}_n \times [0,T], \tag{105}$$

$$W_h^+ \rightharpoonup w, \quad \frac{\partial W_h^+}{\partial x} \rightharpoonup \frac{\partial w}{\partial x} \quad \text{weakly in } L^2_{\text{loc}}(0 < c < 1)_{\mathcal{T}_h}, \tag{106}$$

where $\{0 < q < 1\}_{\mathcal{T}_h} := \{(x,t) \in \prod_{I_n \in \mathcal{T}_h} \bar{I}_n \times [0,T] : 0 < q(x,t) < 1\}$.

Proof. From the definition (101) we have

$$|||C_h|||^2 = |||c_h^{n-1} + [c_h^n - c_h^{n-1}] \frac{t - t_{n-1}}{\Delta t}|||^2 \leq 2|||c_h^{n-1}|||^2 + 2\frac{(t - t_{n-1})^2}{(\Delta t)^2}|||c_h^n - c_h^{n-1}|||^2.$$

Hence, using the first bound in (99) and the parallelogram identity, we get

$$|||C_h|||^2 \leq C, \quad |||C_h^\pm|||^2 \leq C. \tag{107}$$

From (23) it follows that $C_h(0) \to c_0$ strongly in $H^1([0,L])$ as $h \to 0$. This implies that $⨍ C_h(0) = ⨍ C_h = ⨍ C_h^\pm \in (0,1)$, and hence, by (107) and (30), that

$$||C_h||^2_{L^\infty(0,T;H^1([0,L],\mathcal{T}_h))} \leq C, \tag{108}$$

and

$$[[C_h(x_{\bar{n}})]]^2 \leq Ch^2, \quad \text{for } \bar{n} = 1, \ldots, \bar{N}. \tag{109}$$

Furthermore, using (99) and the definition (101) it follows that

$$\int_0^T |||\partial_t C_h|||^2 dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} |||\frac{c_h^n - c_h^{n-1}}{\Delta t}|||^2 dt \leq \sum_{n=1}^N \Delta t |||\frac{c_h^n - c_h^{n-1}}{\Delta t}|||^2 \leq C(\Delta t)^{-1},$$

$$\int_0^T B_{\mathcal{T}_h}(c_h^{n-1}; w_h^n, w_h^n) dt =$$
$$\sum_{n=1}^N \int_{t_{n-1}}^{t_n} B_{\mathcal{T}_h}(c_h^{n-1}; w_h^n, w_h^n) dt \leq \sum_{n=1}^N \Delta t B_{\mathcal{T}_h}(c_h^{n-1}; w_h^n, w_h^n) dt \leq C,$$

$$\int_0^T \left|\left|\left| \mathcal{G}_{\mathcal{T}_h}^h \left[\frac{c_h^n - c_h^{n-1}}{\Delta t} \right] \right|\right|\right|^2 dt =$$
$$\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left|\left|\left| \mathcal{G}_{\mathcal{T}_h}^h \left[\frac{c_h^n - c_h^{n-1}}{\Delta t} \right] \right|\right|\right|^2 dt \leq \sum_{n=1}^N \Delta t \left|\left|\left| \mathcal{G}_{\mathcal{T}_h}^h \left[\frac{c_h^n - c_h^{n-1}}{\Delta t} \right] \right|\right|\right|^2 \leq Cb_{\max} \leq C. \tag{110}$$

Hence, we have, using (108), (110) and lemma 2.1

$$||C_h||^2_{L^\infty(0,T;H^1([0,L],\mathcal{T}_h))} + \Delta t||C_h||^2_{H^1(0,T;H^1([0,L],\mathcal{T}_h))} + \left|\left|[b(C_h^-)]^{1/2}\frac{\partial W_h^+}{\partial x}\right|\right|^2_{L^2(0,T;L^2((0,L),\mathcal{T}_h))} \leq C, \tag{111}$$

and

$$||[[W_h^+(x_{\bar{n}})]]||^2_{L^2((0,T))} \leq Ch^2, \quad \text{for } \bar{n} = 1, \dots, \bar{N}. \tag{112}$$

In the next step we show that the continuous interpolants C_h are uniformly Hölder continuous on $\prod_{I_{\bar{n}}\in\mathcal{T}_h} \bar{I}_{\bar{n}} \times [0,T]$. The first bound in (111) gives

$$\begin{aligned}
|C_h(x_2,t) - C_h(x_1,t)| &= \left|\int_{x_1}^{x_2} \frac{\partial C_h}{\partial x}(s,t)ds\right| \leq |x_2 - x_1|^{1/2}\left(\int_{x_1}^{x_2}\left|\frac{\partial C_h}{\partial x}(s,t)\right|^2 dx\right)^{1/2} \\
&\leq |x_2 - x_1|^{1/2}\left|\left|\frac{\partial C_h}{\partial x}\right|\right|_{L^\infty(0,T;L^2(I_{\bar{n}}))} \leq Ch|x_2 - x_1|^{1/2} \quad \forall x_1, x_2 \in \bar{I}_{\bar{n}},\ \forall t \geq 0,
\end{aligned}$$

for each $\bar{n} = 1, \dots, \bar{N}$. In addition it follows from (8), the definition (33), remark 1.1, the Cauchy-Schwarz inequality, (107) and the third bound in (110) that

$$\begin{aligned}
&||C_h(\cdot,t_2) - C_h(\cdot,t_1)||_{0,\infty,I_{\bar{n}}} \leq Ch||C_h(\cdot,t_2) - C_h(\cdot,t_1)||^{1/2}||C_h(\cdot,t_2) - C_h(\cdot,t_1)||^{1/2}_{1,\mathcal{T}_h} \\
&\leq Ch\, B_{\mathcal{T}_h}(\mathcal{G}^h_{\mathcal{T}_h}(C_h(\cdot,t_2) - C_h(\cdot,t_1)), (C_h(\cdot,t_2) - C_h(\cdot,t_1)))^{1/4}|||C_h(\cdot,t_2) - C_h(\cdot,t_1)|||^{1/2} \\
&\leq Ch|||\mathcal{G}^h_{\mathcal{T}_h}(C_h(\cdot,t_2) - C_h(\cdot,t_1))|||^{1/4}|||C_h(\cdot,t_2) - C_h(\cdot,t_1)|||^{3/4} \\
&\leq Ch\left|\left|\left|\int_{t_1}^{t_2}\mathcal{G}^h_{\mathcal{T}_h}\frac{\partial C_h}{\partial t}(\cdot,t)dt\right|\right|\right|^{1/4} \leq Ch(t_2 - t_1)^{1/8}\left(\int_{t_1}^{t_2}\left|\left|\left|\mathcal{G}^h_{\mathcal{T}_h}\frac{\partial C_h}{\partial t}(\cdot,t)\right|\right|\right|^2 dt\right)^{1/8} \\
&\leq Ch(t_2 - t_1)^{1/8} \quad \forall t_2 \geq t_1 \geq 0,
\end{aligned} \tag{113}$$

for each $\bar{n} = 1, \dots, \bar{N}$. From the first bound in (111) and the Sobolev embedding theorem (8) with $r = \infty, m = 1$, we get that the norm of C_h is uniformly bounded on $\prod_{I_n\in\mathcal{T}_h} \bar{I}_n \times [0,T]$ independently on $h, \Delta t$ and T; moreover, from the previous bounds we have that its $C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\prod_{I_n\in\mathcal{T}_h} \bar{I}_n \times [0,T])$ norm is uniformly bounded independently on $h, \Delta t$ and T. Hence, every sequence C_h is uniformly bounded and equicontinuous on $\prod_{I_n\in\mathcal{T}_h} \bar{I}_n \times [0,T]$, and by the Ascoli-Arzelá theorem and (109) there exists a subsequence of C_h such that (105) holds, with $c \geq 0$, $||c||_{0,\infty} < 1$ and $c \in C^{\frac{1}{2},\frac{1}{8}}_{x,t}([0,L]_T)$. Moreover, the first bound in (111) and (109) implies, by means of the Banach-Alaoglu theorem and the fact that $H^1([0,L])$ is the subset of $H^1([0,L],\mathcal{T}_h)$ characterized by interelement continuity , that this same subsequence satisfies (104).
From the fact that

$$C_h - C_h^\pm = (t - t_n^\pm)\frac{\partial C_h}{\partial t}, \quad t \in (t_{n-1}, t_n),\ n \geq 1,$$

we deduce, using the second bound in (111) and taking $t_1 = t_n^\pm$ in (113), that

$$\begin{aligned}
&||C_h - C_h^\pm||^2_{L^\infty(0,T;H^1([0,L],\mathcal{T}_h))} \leq (\Delta t)^2||\frac{\partial C_h}{\partial t}||^2_{L^\infty(0,T;H^1([0,L],\mathcal{T}_h))} \leq C\Delta t; \\
&||C_h - C_h^\pm||_{L^\infty(\bar{I}_n\times[0,T])} \leq C(\Delta t)^{1/8}.
\end{aligned}$$

Hence, the same convergence results (105) and (104) hold for the piecewise constant interpolants $C_h^\pm$.

We now show the compactness of $\{W_h^+\}_h$ on compact subsets of $\{0 < c < 1\}$. For any $\delta > 0$, let's set

$$\begin{aligned}
D^+_{\delta,\mathcal{T}_h} &:= \{(x,t) \in \prod_{I_n\in\mathcal{T}_h} \bar{I}_n \times [0,T] : \delta < c(x,t) < 1\}, \\
D^+_{\delta,\mathcal{T}_h}(t) &:= \{x \in \prod_{I_n\in\mathcal{T}_h} \bar{I}_n : \delta < c(x,t) < 1\}.
\end{aligned}$$

From the uniform convergence (105) it follows that, for a fixed $\delta > 0$, there exists a $h(\delta) \in \mathbb{R}^+$ such that, for all $h \le h(\delta)$,

$$0 \le C_h^{\pm}(x,t) < \min\{2\delta, 1\} \quad \forall (x,t) \notin D^+_{\delta,\mathcal{T}_h}, \tag{114}$$
$$\frac{1}{8}\delta \le C_h^{\pm}(x,t) < 1 \quad \forall (x,t) \in D^+_{\frac{\delta}{4},\mathcal{T}_h}.$$

From the third bound in (111) and from (114) we have

$$b_{\min}(\frac{\delta}{8})\Big(\int_{D^+_{\frac{\delta}{4},\mathcal{T}_h}} \Big|\frac{\partial W_h^+}{\partial x}\Big|^2 dxdt\Big) \le \int_{D^+_{\frac{\delta}{4},\mathcal{T}_h}} b(C_h^-)\Big|\frac{\partial W_h^+}{\partial x}\Big|^2 dxdt \le C, \tag{115}$$

where $b_{\min}(\delta) := \min_{\delta \le z < 1} b(z)$.
From (114) we have that for all $h \le h(\delta)$ and for a.e. $t \in (0,T)$

$$\phi(\cdot,t) \equiv C_h^+(\cdot,t) \pm \frac{1}{8}\frac{\eta^h(\cdot,t)}{||\eta^h(\cdot,t)||_\infty} \in K(\Omega,\mathcal{T}_h), \quad \forall \eta^h \in L^2(0,T;S(\Omega,\mathcal{T}_h)),$$

with $\mathrm{supp}(\eta^h) \subset\subset D^+_{\frac{\delta}{4},\mathcal{T}_h}$.

Choosing such ϕ in the second equation of system (103) yields, $\forall h < h(\delta)$, that

$$\int_0^T \Big[\gamma\, B_{\mathcal{T}_h}(C_h^+,\eta^h) + (\psi_1'(C_h^+) + \psi_2'(C_h^-),\eta^h)\Big]dt = \int_0^T (W_h^+,\eta^h)dt. \tag{116}$$

We introduce now a cut-off function $\theta_\delta \in C_0^\infty(D^+_{\frac{\delta}{2},\mathcal{T}_h})$ such that

$$\theta_\delta(\cdot,t) \equiv 1 \quad \text{on } D^+_{\delta,\mathcal{T}_h}(t), \quad 0 \le \theta_\delta(\cdot,t) \le 1. \tag{117}$$

Noting that, since $c \in C^{\frac{1}{2},\frac{1}{8}}_{x,t}([0,L]_T)$, it follows that $C\delta \le |x_2 - x_1|^{1/2}$ for $x_1, x_2 \in D^+_{\frac{\delta}{2},\mathcal{T}_h} \setminus D^+_{\delta,\mathcal{T}_h}$, we can choose a $\theta_\delta(\cdot,t)$ such that

$$|\nabla\theta_\delta(\cdot,t)| \le C\delta^{-2}. \tag{118}$$

Since $\theta_\delta^2 W_h^+ \in L^2(\Omega)$, and there exists an $h_1(\delta) \le h(\delta)$ such that $\mathrm{supp}(p^h(\theta_\delta^2 W_h^+)) \subset \mathrm{supp}(\theta_\delta^2 \mathrm{W}_\mathrm{h}^+) \subset\subset \mathrm{D}^+_{\frac{\delta}{4},\mathcal{T}_\mathrm{h}}$, for all $h \le h_1(\delta)$, we can choose $\eta^h = p^h(\theta_\delta^2 W_h^+)$ in (116). Using the definition (17), the fact that $||C_h^{\pm}||_{0,\infty} < 1$ and that $\psi_i(\cdot) \in C^1([0,1))$, $i = 1,2$, the regularity of θ_δ, remark 1.1, the estimates (107), (112) and the following inequality, obtained from (21) and (24),

$$|||(I - p^h)\eta||| \le C|\eta|_{1,\mathcal{T}_h} \quad \forall \eta \in H^1([0,L],\mathcal{T}_h), \tag{119}$$

we get

$$\begin{aligned}
&\int_0^T (W_h^+, p^h(\theta_\delta^2 W_h^+))dt = \int_{[0,L]_T} \theta_\delta^2 (W_h^+)^2 dxdt\\
&\int_0^T \Big[\gamma\, B_{\mathcal{T}_h}\big(C_h^+, p^h(\theta_\delta^2 W_h^+)\big) + (\psi_1'(C_h^+) + \psi_2'(C_h^-), p^h(\theta_\delta^2 W_h^+))\Big]dt\\
&\le C\int_0^T |||C_h^+|||\, |\theta_\delta^2 W_h^+|_{1,\mathcal{T}_h} + C\int_0^T |||C_h^+||| + E||\theta_\delta W_h^+||_{L^2([0,L]_T)}\\
&\le C(1+\delta^{-2})||\theta_\delta W_h^+||_{L^2([0,L]_T)} + C\int_{D^+_{\frac{\delta}{4},\mathcal{T}_h}} \Big|\frac{\partial W_h^+}{\partial x}\Big|^2 dxdt.
\end{aligned}$$

Now, using a Young inequality and bound (115), it follows that

$$\int_{[0,L]_T} \theta_\delta^2 (W_h^+)^2 dxdt \le C(\delta)^{-1}. \tag{120}$$

Therefore, combining (120) and (115) and recalling the definition of $\theta_\delta(\cdot,t)$, it follows that, for all $\delta > 0$,

$$||W_h^+||_{L^2(0,T;H^1(D^+_{\delta,\mathcal{T}_h}(t)))} \le C(\delta^{-1}) \quad \forall h \le h_1(\delta). \tag{121}$$

Applying the Banach Alaoglu theorem on compact subsets of the set $\{0 < c < 1\}_{\mathcal{T}_h}$ and using (112) we get finally (106). □

Remark 3.1 *In the case $d = 1$ the uniform convergence* (105)*, together with the convergence result* (106)*, makes it possible to calculate the $h, \Delta t \to 0$ limit of the degenerate elliptic term in the first equation of system* (103) *on the set $\{0 < c < 1\}_{\mathcal{T}_h}$. To the best of our knowledge, in the case $d > 1$ there does not exist in literature a convergence result which shows the convergence of the discrete solution of* (103) *to the continuous solution of a weak formulation of* (1)*.*

We can now obtain the limit equations of system (103) as $(h, \Delta t) \to (0, 0)$. Indeed, setting $\int_{0<c<1}(\ ,\ )dt := \int_0^T (\ ,\ )_{D_0^+(t)} dt$, we have

Theorem 3.1 *The limit point (c, w) of lemma 3.1 satisfies the weak formulation*

$$\begin{cases} \displaystyle\int_0^T \left\langle \frac{\partial c}{\partial t}, \eta \right\rangle dt + \int_{0<c<1} \left(b(c) \frac{\partial w}{\partial x}, \frac{\partial \eta}{\partial x} \right) dt = 0, \quad \forall \eta \in L^2(0,T; H^1([0,L])), \\ \displaystyle\int_{0<c<1} \gamma \left(\frac{\partial c}{\partial x}, \frac{\partial \theta}{\partial x} \right) dt + \int_{0<c<1} (\psi'(c), \theta) dt - \int_{0<c<1} (w, \theta) dt = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \theta \in L^2(0,T; H^1([0,L])), \end{cases} \tag{122}$$

with $c(\cdot, 0) = c_0(\cdot)$, and with $\mathrm{supp}(\theta) \subset \{0 < \mathrm{c} < 1\}$.

Proof. Let's choose $\eta \in H^1(0,T; H^1([0,L]))$ and $\theta \in L^2(0,T; H^1([0,L]))$, with $\mathrm{supp}(\theta) \subset \mathrm{D}^+_{\delta, \mathcal{T}_\mathrm{h}}$. Choosing $\chi|_{I_{\bar{n}}} = \pi^h_{I_{\bar{n}}} \eta$, $\phi|_{I_{\bar{n}}} = \pi^h_{I_{\bar{n}}} \theta$ in (103), considering (116), (11), (12) and using the fact that $[[\eta(x_{\bar{n}})]] = [[\theta(x_{\bar{n}})]] = [[\pi^h(\eta(x_{\bar{n}}))]] = [[\pi^h(\theta(x_{\bar{n}}))]] = 0$, for $\bar{n} = 1, \ldots, \bar{N} - 1$, where π^h is the global continuous interpolant, we rewrite (103) as

$$\begin{cases} \displaystyle\int_0^T \left(\frac{\partial C_h}{\partial t}, \eta \right) dt + \sum_{\bar{n}=0}^{\bar{N}-1} \int_0^T \left(b(C_h^-) \frac{\partial W_h^+}{\partial x}, \frac{\partial \eta}{\partial x} \right)_{I_{\bar{n}}} dt \\ \displaystyle - \sum_{\bar{n}=1}^{\bar{N}-1} \int_0^T \left([[W_h^+(x_{\bar{n}})]] \left\{ b(C_h^-(x_{\bar{n}})) \frac{\partial \eta}{\partial x}(x_{\bar{n}}) \right\} \right) dt = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left(\frac{\partial C_h}{\partial t}, (\eta - \pi^h \eta) \right) dt \\ \displaystyle + \sum_{n=1}^{N} \sum_{\bar{n}=0}^{\bar{N}-1} \int_{t_{n-1}}^{t_n} \left(b(C_h^-) \frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x}(\eta - \pi^h_{I_{\bar{n}}} \eta) \right)_{I_{\bar{n}}} dt \\ \displaystyle - \sum_{n=1}^{N} \sum_{\bar{n}=1}^{\bar{N}-1} \int_{t_{n-1}}^{t_n} \left([[W_h^+(x_{\bar{n}})]] \left\{ b(C_h^-(x_{\bar{n}})) \frac{\partial}{\partial x}(\eta - \pi^h_{I_{\bar{n}}} \eta)(x_{\bar{n}}) \right\} \right) dt \\ \\ \displaystyle \sum_{\bar{n}=0}^{\bar{N}-1} \int_0^T \gamma \left(\frac{\partial C_h^+}{\partial x}, \frac{\partial \theta}{\partial x} \right)_{I_{\bar{n}}} dt - \gamma \sum_{\bar{n}=1}^{\bar{N}-1} \int_0^T \left([[C_h^+(x_{\bar{n}})]] \left\{ \frac{\partial \theta}{\partial x}(x_{\bar{n}}) \right\} \right) dt \\ \displaystyle + \int_0^T ([\psi_1'(C_h^+) + \psi_2'(C_h^-)], \theta) dt - \int_0^T (W_h^+, \theta) dt = \\ \displaystyle \sum_{n=1}^{N} \sum_{\bar{n}=0}^{\bar{N}-1} \int_{t_{n-1}}^{t_n} \gamma \left(\frac{\partial C_h^+}{\partial x}, \frac{\partial}{\partial x}(\theta - \pi^h \theta) \right)_{I_{\bar{n}}} dt \\ \displaystyle - \gamma \sum_{n=1}^{N} \sum_{\bar{n}=1}^{\bar{N}-1} \int_{t_{n-1}}^{t_n} \left([[C_h^+(x_{\bar{n}})]] \left\{ \frac{\partial}{\partial x}(\theta - \pi^h_{I_{\bar{n}}} \theta)(x_{\bar{n}}) \right\} \right) dt \\ \displaystyle + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\psi_1'(C_h^+) + \psi_2'(C_h^-), \theta - \pi^h \theta) dt - \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (W_h^+, \theta - \pi^h \theta) dt \end{cases} \tag{123}$$

Let's start from considering the first equation of system (123). The left hand side converges to the limit, for $h, \Delta t \to 0$,

$$(c(\cdot, T), \eta(\cdot, T)) - (c(\cdot, 0), \eta(\cdot, 0)) - \int_0^T (c, \frac{\partial \eta}{\partial t}) dt + \int_{0<c<1} \left(b(c) \frac{\partial w}{\partial x}, \frac{\partial \eta}{\partial x} \right) dt \tag{124}$$

For the first term we have

$$\int_0^T \left(\frac{\partial C_h}{\partial t}, \eta \right) dt = - \int_0^T \left(C_h, \frac{\partial \eta}{\partial t} \right) dt + (C_h(\cdot, T), \eta(\cdot, T)) - (C_h(\cdot, 0), \eta(\cdot, 0)). \tag{125}$$

From the uniform convergence (105) and the regularity of η we deduce the first limit term in (124). For the second term, we write the domain of integration as $\prod_{I_{\bar{n}}\in\mathcal{T}_h}\bar{I}_{\bar{n}}\times[0,T]=\left(\prod_{I_{\bar{n}}\in\mathcal{T}_h}\bar{I}_{\bar{n}}\times[0,T]\setminus D^+_{\delta,\mathcal{T}_h}\right)\cup D^+_{\delta,\mathcal{T}_h}$. On noting the third bound in (111) and (114), we get

$$\Big|\sum_{I_{\bar{n}}\in\mathcal{T}_h}\int_{\prod_{I_{\bar{n}}\in\mathcal{T}_h}\bar{I}_{\bar{n}}\times[0,T]\setminus D^+_{\delta,\mathcal{T}_h}}\left(b(C_h^-)\frac{\partial W_h^+}{\partial x}\frac{\partial\eta}{\partial x}\right)dxdt\Big|\leq \tag{126}$$

$$||(b(C_h^-))^{1/2}||_{L^\infty(\prod_{I_n\in\mathcal{T}_h}\bar{I}_{\bar{n}}\times[0,T]\setminus D^+_{\delta,\mathcal{T}_h})}\Big|\Big|(b(C_h^-))^{1/2}\frac{\partial W_h^+}{\partial x}\Big|\Big|_{L^2(0,T;L^2((0,L),\mathcal{T}_h))}$$

$$||\eta||_{L^2(0,T;H^1([0,L],\mathcal{T}_h))}\leq C(b_{\max}(2\delta))^{1/2}||\eta||_{L^2(0,T;H^1([0,L],\mathcal{T}_h))},$$

where $b_{\max}(2\delta)=\max_{0\leq z\leq 2\delta}b(z)$, for all $h\leq h(\delta)$. Next, we write

$$\sum_{I_{\bar{n}}\in\mathcal{T}_h}\int_{D^+_{\delta,\mathcal{T}_h}}\left(b(C_h^-)\frac{\partial W_h^+}{\partial x}\frac{\partial\eta}{\partial x}\right)dxdt= \tag{127}$$

$$\sum_{I_{\bar{n}}\in\mathcal{T}_h}\int_{D^+_{\delta,\mathcal{T}_h}}\left(b(c)\frac{\partial W_h^+}{\partial x}\frac{\partial\eta}{\partial x}\right)dxdt+\sum_{I_{\bar{n}}\in\mathcal{T}_h}\int_{D^+_{\delta,\mathcal{T}_h}}\left([b(C_h^-)-b(c)]\frac{\partial W_h^+}{\partial x}\frac{\partial\eta}{\partial x}\right)dxdt$$

Due to the uniform convergence (105), the Lipshitz continuity of $b(\cdot)$ and the bound (115) it follows, concerning the second term on the right hand side of equation (127), that

$$\Big|\sum_{I_{\bar{n}}\in\mathcal{T}_h}\int_{D^+_{\delta,\mathcal{T}_h}}\left([b(C_h^-)-b(c)]\frac{\partial W_h^+}{\partial x}\frac{\partial\eta}{\partial x}\right)dxdt\Big|\leq$$

$$||b(C_h^-)-b(c)||_{L^\infty([0,L]_T)}\sum_{I_{\bar{n}}\in\mathcal{T}_h}\left(\int_{D^+_{\delta,\mathcal{T}_h}}\left|\frac{\partial W_h^+}{\partial x}\right|^2dxdt\right)||\eta||_{L^2(0,T;H^1([0,L],\mathcal{T}_h))} \tag{128}$$

$$\leq C[b_{\min}(\frac{\delta}{8})]^{-1}||b(C_h^-)-b(c)||_{L^\infty([0,L]_T)}||\eta||_{L^2(0,T;H^1([0,L],\mathcal{T}_h))}\to 0\quad\text{for } h,\Delta t\to 0. \tag{129}$$

Hence, from (106), (127) and (128), we get

$$\sum_{I_{\bar{n}}\in\mathcal{T}_h}\int_{D^+_{\delta,\mathcal{T}_h}}\left(b(C_h^-)\frac{\partial W_h^+}{\partial x}\frac{\partial\eta}{\partial x}\right)dxdt\to\int_{0<c<1}\left(b(c)\frac{\partial w}{\partial x}\frac{\partial\eta}{\partial x}\right)dxdt\quad\text{for } h,\Delta t\to 0. \tag{130}$$

For the third term, let's consider for a moment $\eta\in L^2(0,T;H^2([0,L]))\cap H^1(0,T;L^2((0,L)))\hookrightarrow C([0,T],H^1([0,L]))$. Using (112), (25) and the Cauchy-Schwarz inequality we get

$$\Big|\sum_{\bar{n}=1}^{\bar{N}-1}\int_0^T\left([[W_h^+(x_{\bar{n}})]]\left\{b(C_h^-(x_{\bar{n}}))\frac{\partial\eta}{\partial x}(x_{\bar{n}})\right\}\right)dt\Big|\leq \tag{131}$$

$$Ch\sum_{\bar{n}=0}^{\bar{N}-1}b_{\max,\pm}(C_h^-(x_{\bar{n}}))h_{\bar{n}-1,\bar{n}}^{-1/2}\int_0^T\left(|\eta|^2_{1,I_{\bar{n}}}+h^2_{\bar{n}-1,\bar{n}}|\eta|^2_{2,I_{\bar{n}}}\right)^{1/2}\leq Ch^{1/2}\to 0\quad\text{for } h\to 0,$$

where $b_{\max,\pm}(C_h^-(x_{\bar{n}})):=\max[b(C_h^-(x_{\bar{n}}^-)),b(C_h^-(x_{\bar{n}}^+))]$.

Note. It should be sufficient to consider $\eta\in L^2(0,T;H^2([0,L],\mathcal{T}_h))\cap H^1(0,T;L^2((0,L)))$ or $\eta\in L^2(0,T;H^{3/2}([0,L],\mathcal{T}_h))\cap H^1(0,T;L^2((0,L)))$. Moreover, from (131), it should be sufficient to consider $|\eta|_{2,I_{\bar{n}}}\sim h^{-1}$.

Considering (126), (130) and (131) for all $\delta>0$, on noting that $b_{\max}(2\delta)\to 0$ as $\delta\to 0$, we get (124) for $\eta\in L^2(0,T;H^2([0,L]))\cap H^1(0,T;L^2((0,L)))\hookrightarrow C([0,T],H^1([0,L]))$. Note that (124) for $\eta\in H^1(0,T;H^1([0,L]))$ can be recovered by density arguments.

We now show that the terms in the right hand side of the first equation of system (123) converge to zero for $(h,\Delta t)\to 0$. Let's denote these terms by the notation $\mathcal{I}_1,\cdots,\mathcal{I}_3$.

Taking an integration by parts in time, considering (105), (22), the regularity of η and the Cauchy-Schwarz inequality we get

$$|\mathcal{I}_1|\leq Ch\left(\int_0^T||C_h||^2dt\right)^{1/2}\left(\sum_{n=1}^N\int_{t_{n-1}}^{t_n}||\frac{\partial\eta}{\partial t}||_1^2dt\right)^{1/2}$$

$$+Ch||C_h(\cdot,T)||\,||\eta(\cdot,T)||_1+Ch||C_h(\cdot,0)||\,||\eta(\cdot,0)||_1\leq Ch||\eta||_{H^1(0,T;H^1(\Omega))}\to 0 \tag{132}$$

Using the third bound in (111), (23) and the Cauchy-Schwarz inequality we can write

$$|\mathcal{I}_2| \le \tag{133}$$
$$||(b(C_h^-)^{1/2})||_{L^\infty([0,L]_T)}||(b(C_h^-))^{1/2}\frac{\partial W_h^+}{\partial x}||_{L^2(0,T;L^2((0,L),\mathcal{T}_h)}\left(\sum_{n=1}^{N}\sum_{\bar n=0}^{\bar N-1}\int_{t_{n-1}}^{t_n}||\frac{\partial}{\partial x}(\eta-\pi_{I_{\bar n}}^h\eta)||_{I_{\bar n}}^2 dt\right)^{1/2}$$
$$\le C||\eta-\pi^h\eta||_{L^2(0,T;H^1([0,L]))}\to 0.$$

Restricting to $\eta\in L^2(0,T;H^2([0,L]))\cap H^1(0,T;L^2((0,L)))$, using (112), (25), (22) and the Cauchy-Schwarz inequality we get

$$|\mathcal{I}_3|\le Ch\sum_{\bar n=0}^{\bar N-1} b_{\max,\pm}(C_h^-(x_{\bar n}))h_{\bar n-1,\bar n}^{-1/2}\int_0^T\left(h_{\bar n-1,\bar n}^2|\eta|_{2,I_{\bar n}}^2\right)^{1/2}\to 0. \tag{134}$$

Hence, the first equation of system (123) converges to the limit, for $h,\Delta t\to 0$,

$$(c(\cdot,T),\eta(\cdot,T))-(c(\cdot,0),\eta(\cdot,0))-\int_0^T(c,\frac{\partial\eta}{\partial t})dt+\int_{0<c<1}\left(b(c)\frac{\partial w}{\partial x},\frac{\partial\eta}{\partial x}\right)dt=0. \tag{135}$$

Since, from lemma 3.1, $b(c)\frac{\partial w}{\partial x}\in L^2(0<c<1)_{\mathcal{T}_h}$, from (131) we deduce that $c\in H^1(0,T;(H^1([0,L]))')$, and the first equation in (122) is valid. Moreover, due to the uniform convergence (105), $c(\cdot,0)=c_0(\cdot)$.

We consider now the second equation in (123). The left hand side converges to the limit, for $h,\Delta t\to 0$,

$$\int_{0<c<1}\gamma\left(\frac{\partial c}{\partial x},\frac{\partial\theta}{\partial x}\right)dt+\int_{0<c<1}(\psi'(c),\theta)dt-\int_{0<c<1}(w,\theta)dt,$$
$$\forall\theta\in L^2(0,T;H^1(\Omega))\,\text{with}\,\text{supp}(\theta)\subset \mathrm{D}_0^+. \tag{136}$$

The first and the fourth terms of the second equation in (123) converges to the first and the third terms of (136) as a direct consequence of the convergence results (104) and (106). Restricting to $\theta\in L^2(0,T;H^2([0,L]))$, using (109), (25) and the Cauchy-Schwarz inequality we get

$$\left|\sum_{\bar n=1}^{\bar N-1}\int_0^T\left([[C_h^+(x_{\bar n})]]\left\{\frac{\partial\theta}{\partial x}(x_{\bar n})\right\}\right)dt\right|\le \tag{137}$$
$$Ch\sum_{\bar n=0}^{\bar N-1}h_{\bar n-1,\bar n}^{-1/2}\int_0^T\left(|\theta|_{1,I_{\bar n}}^2+h_{\bar n-1,\bar n}^2|\theta|_{2,I_{\bar n}}^2\right)^{1/2}\le Ch^{1/2}\to 0\quad\text{for } h\to 0.$$

From the facts that $\psi_1(\cdot)\in C^1([0,1))$, $\psi_2(\cdot)\in C^1([0,1])$, that $C_h^\pm\ge 0$, $||C_h^\pm||_{0,\infty}<1$ and from the uniform convergence (105) we have that

$$\left|\int_0^T([\psi_1'(C_h^+)+\psi_2'(C_h^-)-\psi_1'(c)-\psi_2'(c)],\theta)dt\right|\le$$
$$C||\psi_1'(C_h^+)-\psi_1'(c)||_{L^\infty([0,L]_T)}||\theta||_{L^2([0,L]_T)}+C||\psi_2'(C_h^-)-\psi_2'(c)||_{L^\infty([0,L]_T)}||\theta||_{L^2([0,L]_T)}\to 0.$$

Hence the third term on the left hand side of (123) converges to the second term in (136).
We now show that the terms in the right hand side converge to zero for $(h,\tau)\to 0$. Let's denote these terms by the notation $\mathcal{I}_1,\cdots,\mathcal{I}_4$. Using the first bound in (111), (23) and the Cauchy-Schwarz inequality it can be deduced, similarly to (133), that $|\mathcal{I}_1|\to 0$. Restricting to $\theta\in L^2(0,T;H^2([0,L]))$, using (109), (25), (22) and the Cauchy-Schwarz inequality we deduce, similarly to (134), that $|\mathcal{I}_2|\to 0$. Using the facts that $\psi_1(\cdot)\in C^1([0,1))$, $\psi_2(\cdot)\in C^1([0,1])$, that $C_h^\pm\ge 0$, $||C_h^\pm||_{0,\infty}<1$, the first bound in (111), the bound (121), (22) and the Cauchy-Schwarz inequality, on noting that $\text{supp}(\theta)\subset\mathrm{D}_\delta^+$ we deduce that $|\mathcal{I}_3|\to 0$ and $|\mathcal{I}_4|\to 0$.

Collecting (135) and (136) we obtain (122). □

4 Algorithms for solving the variational inequality

In this section we propose different algorithms for solving the variational inequality at each time level in Problem $\mathbf{P}^h$.

4.1 Algorithm 1

The first algorithm proposed is based on solving directly the KKT conditions of the functional (68) without regularization, (i.e. with $\epsilon = 0$). A null space method is used, i.e. the KKT system is reduced onto the null space of the operator associated to the equality constraints imposed on the nodes of *passive elements* and on the nodes on boundaries between *active* and *passive elements*.
Let's introduce the following reduced matrices:

$$\bar{M}_{ij} := \sum_K (\phi_i^K, \phi_j^K); \quad \bar{A}_{ij} := B_{\mathcal{T}_h}(\phi_i^K, \phi_j^K); \quad \bar{A}_{c_h^{n-1},ij} := B_{\mathcal{T}_h}(c_h^{n-1}; \phi_i^K, \phi_j^K); \tag{138}$$

for $j = 1, \ldots, d+1, K \in K_+(c_h^{n-1})$, with $x_j^K \notin e \subset \partial K', K' \in K_0(c_h^{n-1})$. These are the matrices of the algebraic formulation of system (35) reduced on the domain $\Omega \setminus K_0(c_h^{n-1})$, with homogeneous Dirichlet boundary conditions on the internal boundaries $\partial K, K \in K_0(c_h^{n-1})$. Indicating with $\bar{v}_h$ the vector of components $v_h(x_j^K)$, for a generic $v_h \in S(\Omega, \mathcal{T}_h)$, we can express $w_h^n = \sum_{j,K} w_h^n(x_j^K)\phi_j^K$, $c_h^n = \sum_{j,K} c_h^n(x_j^K)\phi_j^K$, $c_h^{n-1} = \sum_{j,K} c_h^{n-1}(x_j^K)\phi_j^K$, for $j = 1, \ldots, d+1, K \in K_+(c_h^{n-1})$, with $x_j^K \notin e \subset \partial K', K' \in K_0(c_h^{n-1})$, in the first equation of (35), and obtain, on noting that $\bar{A}_{c_h^{n-1}}$ is invertible, that

$$\bar{w}_h^n = -\frac{1}{\Delta t} \bar{A}^{-1}_{c_h^{n-1}} \bar{M}(\bar{c}_h^n - \bar{c}_h^{n-1}).$$

Substituting this relation in the second inequality of (35), we get

$$B_{\mathcal{T}_h}(c_h^n, \phi - c_h^n) + \Bigg(\psi_1'(c_h^n) + \frac{1}{\Delta t}\sum_{j,K}\left[\bar{A}^{-1}_{c_h^{n-1}}\bar{M}\bar{c}_h^n\phi_j^K\right] + \psi_2'(c_h^{n-1}) - \frac{1}{\Delta t}\sum_{j,K}\left[\bar{A}^{-1}_{c_h^{n-1}}\bar{M}\bar{c}_h^{n-1}\phi_j^K\right], \phi - c_h^n\Bigg) \geq 0. \tag{139}$$

Let's now introduce the symmetric and positive definite matrix

$$Q := \bar{A} + \frac{1}{\Delta t}\bar{M}\bar{A}^{-1}_{c_h^{n-1}}\bar{M}.$$

Inequality (139) is equivalent to the following complementarity problem:

$$\begin{cases} Q\bar{c}_h^n + \bar{M}\psi_1'(\bar{c}_h^n) + \bar{M}\left(\psi_2'(\bar{c}_h^{n-1}) - \frac{1}{\Delta t}\bar{A}^{-1}_{c_h^{n-1}}\bar{M}\bar{c}_h^{n-1}\right) - \bar{\lambda} = 0, \\ 0 \leq \bar{\lambda} \perp \bar{c}_h^n \geq 0, \end{cases} \tag{140}$$

where $\bar{\lambda}$ is the vector of Lagrange multipliers of the inequality contstraint $\bar{c}_h^n \geq 0$. System (140) can be solved by a preconditioned accelerated gradient method.

We finally derive the following algorithm:

Require: $\alpha_0 > 0, c_h^{n-1}, w_h^{n-1}, J' := \{j \in J_K,\, K \in K_0(c_h^{n-1}) \wedge \big(j \in J_K,\, K \in K_m(c_h^{n-1}),\, m = 1, \ldots, M,\, x_j \in e \subset \partial K',\, K' \in K_0(c_h^{n-1})\big)\}$;
Step 1
for $k \geq 0$ **do**
Initialization

$$c_h^{n,0} = c_h^{n-1}, w_h^{n,0} = w_h^{n-1};$$

Find $c_h^{n,k+1} \in K(\Omega, \mathcal{T}_h)$ such that:
if $j \in J'$ **then**
$c_h^{n,k+1}(x_j) \leftarrow c_h^{n-1}(x_j)$,

else

$$\bar{c}_{h,j}^{n,k+1} = \max\Bigg(0, \bar{c}_{h,j}^{n,k} - \alpha_k (\mathrm{diag}(Q))^{-1} \times \Bigg[Q\bar{c}_h^{n,k} + \bar{M}\psi_1'(\bar{c}_h^{n,k}) + \bar{M}\Bigg(\psi_2'(\bar{c}_h^{n-1}) - \frac{1}{\Delta t}\bar{A}^{-1}_{c_h^{n-1}}\bar{M}\bar{c}_h^{n-1}\Bigg)\Bigg]_j\Bigg), \tag{141}$$

end if

if $||c_h^{n,K+1} - c_h^{n,K}||_{0,\infty} < 10^{-6}$ **then**
$c_h^n \leftarrow c_h^{n,K+1}$; **break**.
end if
end for
Step 2 Find $w_h^n \in S_h$, such that:

$$\bar{w_h}^n = -\frac{1}{\Delta t}\bar{A}^{-1}_{c_h^{n-1}}\bar{M}(\bar{c}_h^n - \bar{c}_h^{n-1})$$

.

The acceleration parameter α_k is dinamically chosen by a projected search in such a way that the functional associated to (140) is decreased at each iterative step k (see e.g. [1], chapter 12, for details). Note that, since the operator acting on $\bar{c}_h^{n,k}$ in the square bracket in (141) is continuous and strictly monotone, the projection map defined in (141) has a unique fixed point (see e.g. [17], chapter 2, for details). The main drawback of Algorithm 1 is the necessity of assembling and calculating the inverse of the matrix $\bar{A}_{c_h^{n-1}}$ at each time step, which renders the algorithm very time demanding. Another drawback is the presence of the non linearity in (140), which makes the complementarity problem non linear and the convergence of the map (141) quite slow. In order to deal with the latter problem, we formulate in the following subsection an alternative algorithm.

4.2 Algorithm 2

The second algorithm proposed is based on solving the complementarity problem (140) by means of a Newton like method. Let's introduce the symmetric and positive definite matrix

$$Q_{\text{lin}} := Q + \psi_1''(c_h^n)\bar{M},$$

and the vector

$$\nabla f(\bar{c}_h^n) := Q\bar{c}_h^n + \bar{M}\psi_1'(\bar{c}_h^n) + \bar{M}\Bigg(\psi_2'(\bar{c}_h^{n-1}) - \frac{1}{\Delta t}\bar{A}^{-1}_{c_h^{n-1}}\bar{M}\bar{c}_h^{n-1}\Bigg).$$

The following algorithm is derived:

Require: $\alpha_0 > 0, c_h^{n-1}, w_h^{n-1}$, $J' := \{j \in J_K,\, K \in K_0(c_h^{n-1}) \wedge \big(j \in J_K,\, K \in K_m(c_h^{n-1}),\, m = 1,\ldots,M,\, x_j \in e \subset \partial K',\, K' \in K_0(c_h^{n-1})\big)\}$;
Step 1
for $k \geq 0$ **do**
Initialization

$$c_h^{n,0} = c_h^{n-1}, w_h^{n,0} = w_h^{n-1}; \bar{\lambda}_i^0 = \begin{cases} 0 & \text{if } \ \bar{c}_{h,i}^{n,0} > 0, \\ \nabla f(\bar{c}_h^{n,0})_i & \text{if } \ \bar{c}_{h,i}^{n,0} = 0. \end{cases}$$

Find $c_h^{n,k+1} \in K(\Omega, \mathcal{T}_h)$ such that:
if $j \in J'$ **then**
$c_h^{n,k+1}(x_j) \leftarrow c_h^{n-1}(x_j)$,

else

for $l \geq 0$ **do**

Initialization

$$\Delta c^{k,0} = 0$$

$$\Delta c_j^{k,l+1} = \max\left(0, \bar{c}_{h,j}^{n,k} + \Delta c_j^{k,l} - \alpha_l (\mathrm{diag}(Q_{\mathrm{lin}}))^{-1}\left[Q_{\mathrm{lin}}\Delta c^{k,l} + \nabla f(\bar{c}_h^{n,k}) - \bar{\lambda}^k\right]_j\right) - \bar{c}_{h,j}^{n,k}, \tag{142}$$

if $||\Delta c^{k,L+1} - \Delta c^{k,L}||_{0,\infty} < 10^{-6}$ **then**

$\Delta c^k \leftarrow \Delta c^{k,L+1}$; **break**.

$$\Delta\bar{\lambda}_j^k = \begin{cases} 0 \quad \text{if} \quad \Delta c_j^k > 0, \\ Q_{\mathrm{lin}}\Delta c_j^k + \nabla f(\bar{c}_h^{n,k})_j - \bar{\lambda}_j^k \quad \text{if} \quad \Delta c_j^k = 0. \end{cases}$$

end if

end for

$$\begin{cases} c_h^{n,k+1} = c_h^{n,k} + \Delta c^k, \\ \bar{\lambda}^{k+1} = \bar{\lambda}^k + \Delta\bar{\lambda}^k. \end{cases}$$

end if

if $||c_h^{n,K+1} - c_h^{n,K}||_{0,\infty} < 10^{-6}$ **then**

$c_h^n \leftarrow c_h^{n,K+1}$; **break**.

end if

end for

Step 2 Find $w_h^n \in S_h$, such that:

$$\bar{w_h}^n = -\frac{1}{\Delta t}\bar{A}^{-1}_{c_h^{n-1}}\bar{M}(\bar{c}_h^n - \bar{c}_h^{n-1})$$

.

The acceleration parameter α_l is dinamically chosen such that (142) defines a steepest-descent gradient method (see e.g. [15], chapter 2, for details). Note that, thanks to the definition of *passive elements* (39), $\bar{\lambda}_i^0 > 0$ if $\bar{c}_{h,i}^{n,0} = 0$ and $\Delta\bar{\lambda}_j^k > 0$ if $\Delta c_j^k = 0$. The main drawback of Algorithm 2 is the necessity of assembling and calculating the inverse of the matrix $\bar{A}_{c_h^{n-1}}$ at each time step, which again renders the algorithm very time demanding. In order to deal with this problem, we formulate in the following subsection an alternative algorithm.

4.3 Algorithm 3

The third algorithm proposed is the generalization to discontinuous elements of the splitting algorithm proposed in [6].

The following algorithm is formulated:

Require: $\mu > 0$ (a relaxation parameter), $\alpha_0 > 0$, c_h^{n-1}, w_h^{n-1}, $J' := \{j \in J_K,\, K \in K_0(c_h^{n-1}) \wedge \big(j \in J_K,\, K \in K_m(c_h^{n-1}),\, m = 1, \ldots, M,\, x_j \in e \subset \partial K',\, K' \in K_0(c_h^{n-1})\big)\}$;

for $k \geq 0$ **do**

Initialization

$$c_h^{n,0} = c_h^{n-1}, w_h^{n,0} = w_h^{n-1};$$

Step 1 Find $Z^{n,k} \in S(\Omega, \mathcal{T}_h)$ such that $\forall q \in S(\Omega, \mathcal{T}_h)$:

$$(Z^{n,k}, q) = (c_h^{n,k}, q) - \mu[\lambda B_{\mathcal{T}_h}(c_h^{n,k}, q) + (\psi_2'(c_h^{n-1}) - w_h^{n,k}, q)];$$

Step 2 Find $c_h^{n,k+1/2} \in K(\Omega, \mathcal{T}_h)$ such that:

if $j \in J'$ **then**

$c_h^{n,k+1/2}(x_j) \leftarrow c_h^{n-1}(x_j)$

else

for $l \geq 0$ **do**

Initialization

$$\bar{c}_{h,j}^{n,k+1/2,0} = \bar{c}_{h,j}^{n,k}$$

$$\bar{c}_{h,j}^{n,k+1/2,l+1} = \max\Bigg(0, \bar{c}_{h,j}^{n,k+1/2,l} - \alpha_l(\mathrm{diag}(\bar{M}))^{-1} \times \left[\bar{M}\bar{c}_h^{n,k+1/2,l} + \mu\bar{M}\psi_1'(\bar{c}_h^{n,k+1/2,l}) - \bar{M}\bar{Z}^{n,k}\right]_j\Bigg) \tag{143}$$

if $||c_h^{n,k+1/2,L+1} - c_h^{n,k+1/2,L}||_{0,\infty} < 10^{-6}$ **then**

$c_h^{n,k+1/2} \leftarrow c_h^{n,k+1/2,L+1}$; **break**.

end if

end for

end if

Step 3 Find $(c_h^{n,k+1}, w_h^{n,k+1}) \in S(\Omega, \mathcal{T}_h) \times S(\Omega, \mathcal{T}_h)$, $\forall q \in S(\Omega, \mathcal{T}_h)$, such that:

$$\begin{cases} \left(\dfrac{c_h^{n,k+1} - c_h^{n-1}}{\Delta t}, q\right) + B_{\mathcal{T}_h}(w_h^{n,k+1}, q) = B_{\mathcal{T}_h}(w_h^{n,k}, q) - B_{\mathcal{T}_h}(c_h^{n-1}; w_h^{n,k}, q), \\ (c_h^{n,k+1}, q) + \mu[\lambda\, B_{\mathcal{T}_h}(c_h^{n,k+1}, q) + (\psi_2'(c_h^{n-1}) - w_h^{n,k+1}, q) = (2c_h^{n,k+1/2} - Z^{n,k}, q). \end{cases}$$

if $||c_h^{n,K+1} - c_h^{n,K}||_{0,\infty} < 10^{-6}$ **then**

$(c_h^n, w_h^n) \leftarrow (c_h^{n,K+1}, w_h^{n,K+1})$; **break**.

end if

end for

The acceleration parameter α_l is dinamically chosen by a projected search in such a way that the functional

$$\frac{1}{2}(c_h^{n,k+1/2,l}, c_h^{n,k+1/2,l}) + \mu(\psi_1(c_h^{n,k+1/2,l}), 1) - (Z^{n,k}, c_h^{n,k+1/2,l})$$

is decreased at each iterative step l (see e.g. [1], chapter 12, for details). Note that, since the operator acting on $\bar{c}_h^{n,k+1/2,l}$ in the square bracket in (143) is continuous and strictly monotone, the projection map defined in (143) has a unique fixed point (see e.g. [17], chapter 2, for details). The present algorith does not require the necessity of assembling and calculating the inverse of the matrix $\bar{A}_{c_h^{n-1}}$ at each time step, and, since in (143) no elliptic term is present, the projection step (143) can be solved element by element independently. These features make Algorithm 3 much faster than Algorithm 1 and 2, even if it converges more slowly, since it requires the convergence of the fixed point iteration associated to the splitting step.

5 Numerical results

In this section we investigate the evolution dynamics of the solution of Problem P^h with an initial concentration characterized by a small uncorrelated white noise over a constant value c_0, for different average values $c_0 < \bar{c}$ and homogeneous Neumann boundary conditions for the $2-d$ case. In these cases the system undergoes a spinodal decomposition and evolves, after a transitory regime, towards an equilibrium state consisting of regions which are rich ($c \sim c^*$) or empty ($c = 0$) of cells.

Moreover, we also study the evolution for long time scales of the solution of a test case with an initial condition of cross-like shape.

We report results of numerical tests in which Algorithm 3 has been implemented, even if we have tested also Algorithms 1 and 2 on the three proposed test cases for the spinodal decomposition dynamics. Even if Algorithms 1 and 2 solves directly, by a preconditioned projected gradient method, the original variational inequality, whereas Algorithm 3 solves it indirectly through a splitting method, Algorithm 3 has the advantage that it doesn't need to assemble and invert the degenerate elliptic operator on its proper domain at each time step and it can be solved element by element independently, needing much less computational resources than Algorithms 1 and 2. In the perspective of a parallel implementation of Algorithms 1 and 2, their faster convergence should make them more performing than Algorithm 3.

Note that an approximative analogue of the sets $K_0(c_h^{n-1})$ and $K_+(c_h^{n-1})$ has been introduced where $c_h^{n-1} > 10^{-6}$ is meant for $c_h^{n-1} > 0$. We remark that this approximation introduces a small error in the mass conservation of the algorithms.

Note moreover from Remark 2.2 that the discrete solution is able to track compactly supported solutions of (1) with a free boundary which moves with a finite speed of velocity if

$$\Delta t = C h_{\min}, \quad C < \frac{1}{\max_{K\in\mathcal{T}_h} \max_K v_{\mathrm{supp,K}}}, \tag{144}$$

where $h_{\min} := \min_{k\in\mathcal{T}_h} h_K$ and $v_{\mathrm{supp},K} := -(1-(c_h^{n-1})^2)(\nabla_h w_h^n)|_K$ (see, e.g., [8] for the definition of the expanding velocity of the cancerous cells). In the implementation of Algorithms 1, 2 and 3 the condition (144) is checked at each time step.

5.1 Spinodal decomposition

Let's consider three test cases in which the initial value c_0 is chosen to be a small uniformly distributed random perturbation around the values $c_0 = 0.05$, $c_0 = c^*/2 = 0.3$ and $c_0 = 0.36$. We consider homogeneous Neumann boundary conditions. We set $\gamma = 0.000196$ and $\Delta t = 10\gamma$. The relaxation parameter is chosen to be $\mu = 1/2$, and $\alpha_0 = 1/2$. The domain is $\Omega = (-3,3)\times(-3,3)$, and a uniform partition of 32-by-32 triangular elements is introduced. The results are collected in Figures 2, 3, 4, showing that the system exhibits two kinds of subregions after a transitory regime, one empty in cells, i.e. $c = 0$, and the other rich in cells, with $c \sim c^*$. The initial separation of the two phases is fast compared to the overall growth timescale of the segregated pattern.

In Figures 2, 3, 4 it can be observed that if $c_0 < c^*/2$ (resp. $c > c^*/2$) then the segregated solution is made of isolated clusters of cells (resp. voids), while if $c_0 = c^*/2$ the system forms maze-like patterns and the domain is equally spaced in subregions rich in each phase. These behaviours reply the main features of the phase order dynamics as predicted by the classical theory of coarsening in systems with a locally conserved order parameter, described, e.g., in [7, 8].

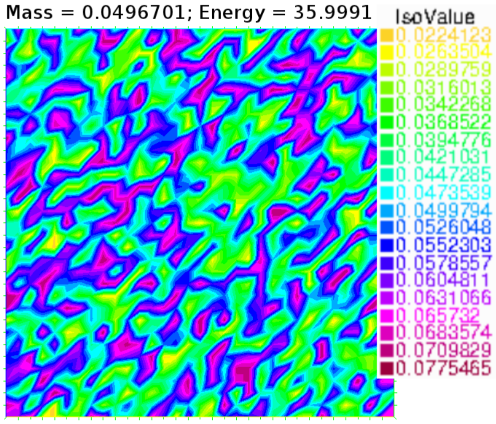
Check also that the mass, i.e. the value of $(c_h^n, 1)$, is conserved up to a small error, and that the value of the Energy $\frac{\gamma}{2} B_{\mathcal{T}_h}(c_h^n, c_h^n) + (\psi(c_h^n), 1)$ decreases.

5.2 Evolution of a cross-like shape

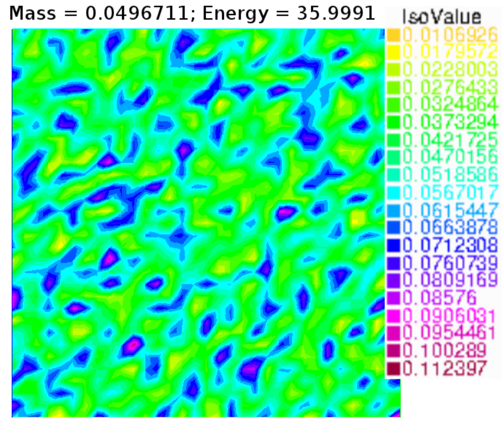
In this test case an initial datum c_0 is chosen given by a piecewise constant function whose jump set has the shape of a cross, with values $c_0 = 0.55$ inside the cross and $c_0 = 0$ outside it. Homogeneous Neumann boundary conditions are considered. We set $\gamma = 0.000196$ and $\Delta t = 10\gamma$ in the initial stages of the evolution, in which the concentration relaxes fastly to the equilibrium value $c*$, and $\Delta t = 50\gamma$ in the late coarsening stages. The relaxation parameter is chosen to be $\mu = 1/2$, and $\alpha_0 = 1/2$. The domain is $\Omega = (-3,3)\times(-3,3)$, and a uniform partition of 64-by-64 triangular elements is introduced. The results are collected in Figure 5, showing as expected that the system evolves to a steady state exhibiting a circular interface (see, e.g., [7] for a description of the coarsening dynamics associated to the degenerate CH equation).

Check also that the mass, i.e. the value of $(c_h^n, 1)$, is conserved up to a small error, and that the value of the Energy $\frac{\gamma}{2} B_{\mathcal{T}_h}(c_h^n, c_h^n) + (\psi(c_h^n), 1)$ decreases.

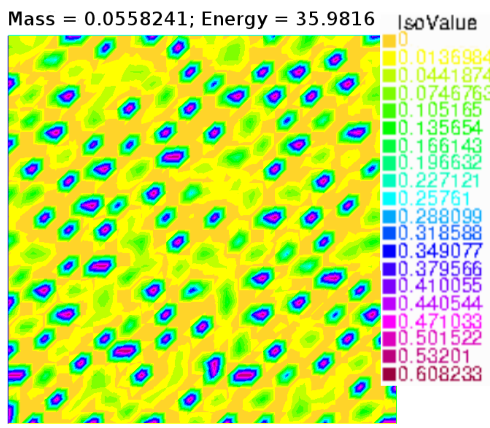
$c_0 = 0.05$



(a) $t = 0$.



(b) $t = 1.0586$.



(c) $t = 6.28004$.

Figure 2: Values of $c(x, y)$ for $c_0 = 0.05$ at different instants of time. The values of mass and energy are reported. The values of the parameters are $\gamma = 0.000196$, $c^* = 0.6$ and $\Delta t = 10\gamma$.

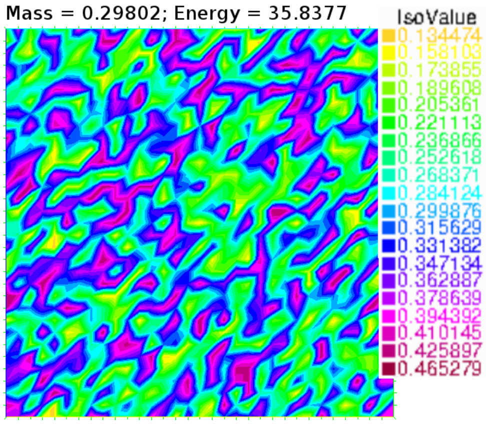
6 Conclusions

This work investigated a discontinuous Galerkin finite element approximation of a degenerate CahnHilliard equation with a single-well potential, which models the evolution and growth of biological cells such as solid tumors. In contrast to the models studied in the literature, where the degeneracy and the singularity sets coincide, here the degeneracy set is $\{c = 0, c = 1\}$ and the singularity set is $\{c = 1\}$. This constitutive choice introduces further complications, since the singularity in $c = 1$ does not guarantee that $c > 0$ and at the discrete level no Entropy estimates are straightforwardly available to guarantee the positivity of the solution.

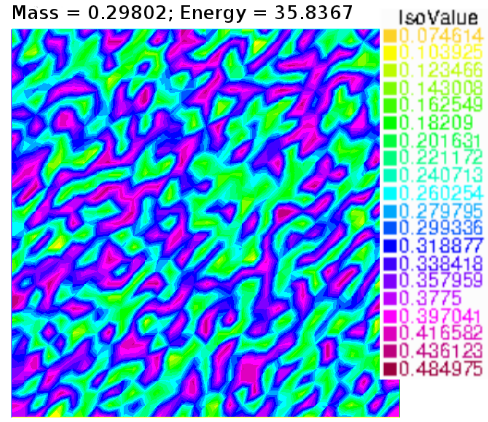
Unlike the standard discontinuous finite element methods proposed in the literature for the non degenerate CH equation, the discontinuous Galerkin method proposed here for the degenerate case is non standard and consists of a discrete variational inequality, in which the positivity of the solution is imposed as a constraint, solved on the *active elements* of the triangulation on which the degenerate elliptic operator can be inverted. The proposed discretization method does not require the additional approximation of the mass scalar products by a lumping procedure, which was needed in the approximation with continuous elements in order to select the physical solutions with compact support and moving boundary from the ones with fixed support.

A suitable formulation of the discrete variational inequality with discontinuous elements has been pro-

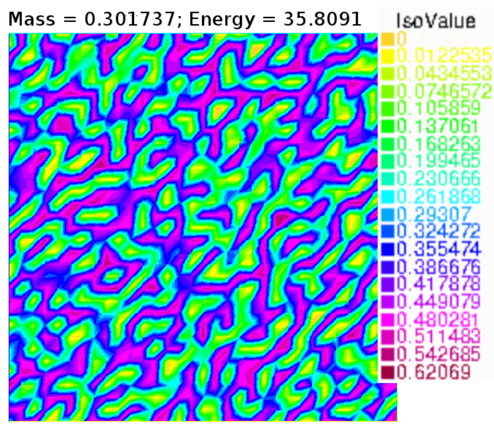
$c_0 = 0.3$



(a) $t = 0$.



(b) $t = 0.0196$.



(c) $t = 0.317716$.

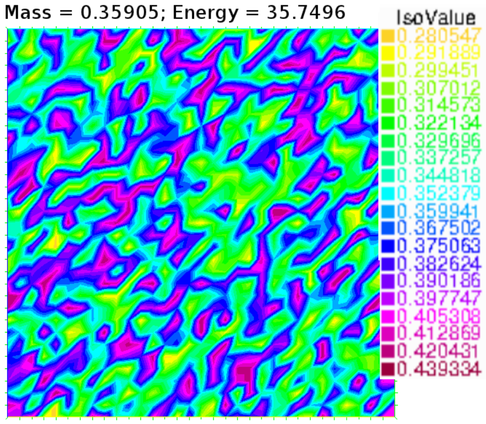
Figure 3: Values of $c(x, y)$ for $c_0 = 0.3$ at different instants of time. The values of mass and energy are reported. The values of the parameters are $\gamma = 0.000196$, $c^* = 0.6$ and $\Delta t = 10\gamma$.

posed, proving the existence and uniqueness of the discrete solution, using a regularization approach. Moreover, the convergence in one space dimension of the discrete solution to the weak solution of the continuous problem has been established.

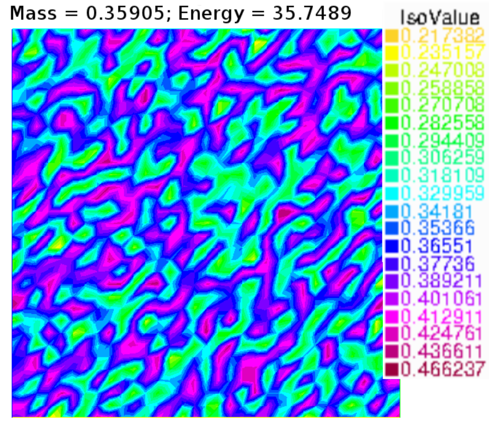
Three numerical algorithms have been proposed to solve the discrete variational inequality, based on different iterative solvers of the corresponding complementarity system. In particular, the Algorithms 1 and 2 solves directly, by a preconditioned projected gradient method, the original variational inequality, whereas Algorithm 3 solves it indirectly through a splitting method, which is a generalization to discontinuous elements of the one proposed in [6]. Algorithm 3 has the advantage that it can be solved element by element independently, but as a disadvantage it needs the convergence of a further fixed point iteration associated to the splitting step. Algorithms 1 and 2 converge faster than Algorithm 3, but they require heavy computational resources in order to assemble and invert the degenerate elliptic operator on its proper domain at each time step.

Finally, some numerical results for different test cases in two space dimensions have been reported in order to discuss the validity of the proposed numerical algoritms. It is found that the dynamics of the spinodal decomposition for the discrete solution is analogous to the one obtained in standard phase ordering dynamics. In fact the geometry of the segregated domains is driven by the initial value of the concentration,

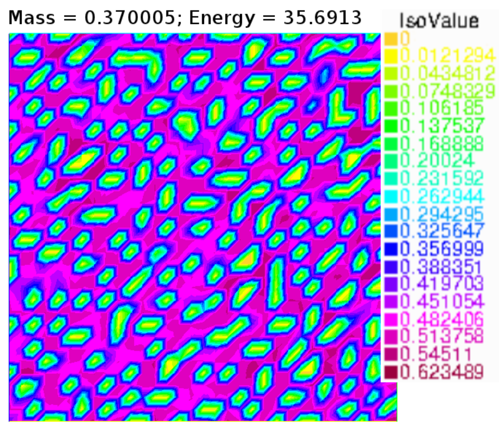
$c_0 = 0.36$



(a) $t = 0$.



(b) $t = 0.0784$.



(c) $t = 1.1076$.

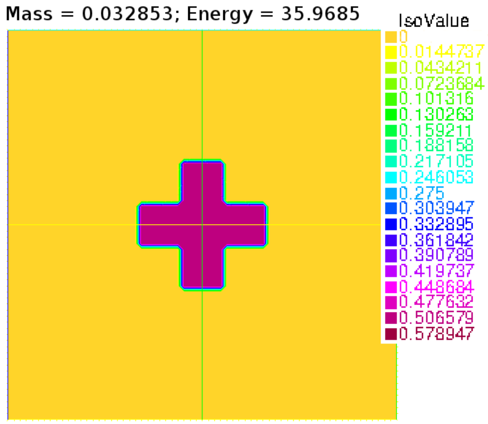
Figure 4: Values of $c(x, y)$ for $c_0 = 0.36$ at different instants of time. The values of mass and energy are reported. The values of the parameters are $\gamma = 0.000196$, $c^* = 0.6$ and $\Delta t = 10\gamma$.

with the appearance of isolated clusters of cells for $c_0 < c^*/2$, maze-like patterns for $c_0 = c^*/2$, and isolated clasters void in cells for $c^*/2 < c_0 < \bar{c}$. A different feature of this model concerns the evolution and growth of single domains in the coarsening regime of the dynamics. As expected, the evolution of a single domain with a cross-like shape to a steady state with constant curvature has been highlighted.

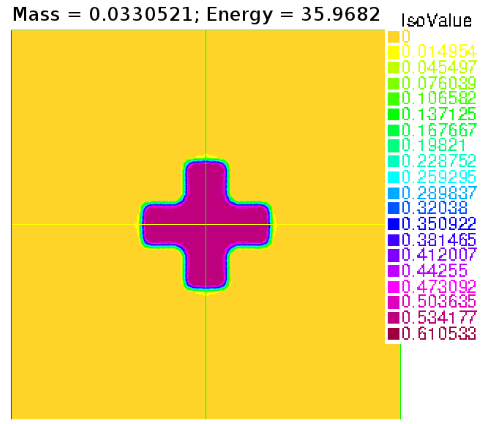
A further development of this work will concern the error analysis of the discrete solution, which will be presented in a forthcoming paper.

Acknowledgements

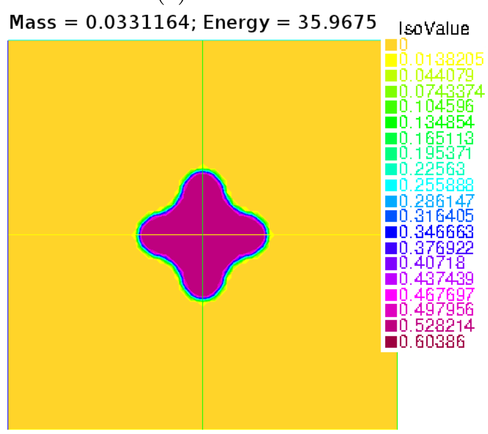
The author acknowledges P. Ciarletta, M. Grasselli, M. Verani and P. F. Antonietti for some useful discussions. The author is a member of GNFM of the Istituto Nazionale di Alta Matematica (INdAM).



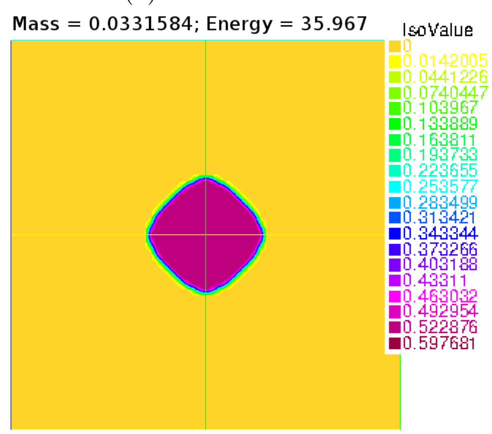
(a) $t = 0$.



(b) $t = 2.39453$.



(c) $t = 23.9643$.



(d) $t = 99.0323$.

Figure 5: Values of $c(x, y)$ for different instant of times with an condition given by a piecewise constant function with cross like shape, with values $c_0 = 0.55$ inside the cross and $c_0 = 0$ outside it. The values of the parameters are $\gamma = 0.000196$, $c^* = 0.6$, $\Delta t = 10\gamma$ at first stages of simulation and $\Delta t = 50\gamma$ for later stages.

References

[1] V. Acary and B. Brogliato. *Numerical methods for nonsmooth dynamical systems: Applications in Mechanics and Electronics. Vol. 35.* Springer-Verlag Berlin Heidelberg, 2008.

[2] R.A. Adams. *Sobolev spaces.* Academic press, New York, 1975.

[3] A. Agosti. Error analysis of a finite element approximation of a degenerate Cahn-Hilliard equation. *Submitted to Esaim-math. model. num., also available at*, 2017.

[4] A. Agosti, P. F. Antonietti, P. Ciarletta, M. Grasselli, and M. Verani. A Cahn-Hilliard type equation with application to tumor growth dynamics. *Submitted to Math. Methods Appl. Sci.*, 2017.

[5] D.N. Arnold, F. Brezzi, B. Cockburn, and D. Marini. Unified analysis of discontinuous galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2008.

[6] J.W. Barrett, J.F. Blowey, and H. Garcke. Finite element approximation of the Cahn-Hilliard equation with degenerate mobility. *SIAM J. Numer. Anal.*, 37(1):286–318, 1999.

[7] A. J. Bray. Theory of phase-ordering kinetics. *Adv. Phys.*, 43(3):357–459, 1994.

[8] C. Chatelain, T. Balois, P. Ciarletta, and M. Ben Amar. Emergence of microstructural patterns in skin cancer: a phase separation analysis in a binary mixture. *New J. Phys.*, 115013(13):21pp, 2011.

[9] M.C. Colombo, C. Giverso, E. Faggiano, C. Boffano, F. Acerbi, and P. Ciarletta. Towards the personalized treatment of glioblastoma: Integrating patient-specific clinical data in a continuous mechanical model. *PLoS One*, 10, 2015.

[10] I. Ekeland and R. Temam. *Convex analysis and variational problems.* siam, 1999.

[11] D.J. Eyre. An unconditionally stable one-step scheme for gradient systems. `http://www.math.utah.edu/~eyre/research/methods/stable.ps`, 1997.

[12] G. Grün and M. Rumpf. Nonnegativity preserving convergent schemes for the thin film equation. *Numer. Math.*, 87:113–152, 2000.

[13] D. Kay, V. Styles, and E. Süli. Discontinuous galerkin finite element approximation of the Cahn-Hilliard equation with convection. *Siam J. Numer. Anal.*, 47(4):2660–2685, 2009.

[14] N. Kenmochi, M. Niezgódka, and I. Pawlow. Subdifferential operator approach to the Cahn-Hilliard equation with constraint. *J. Differential Equations*, 117:320–354, 1995.

[15] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations.* Springer, 2008.

[16] B. Riviere. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations.* Siam, 2008.

[17] R. Temam. *Navier-Stokes equations and nonlinear functional analysis.* siam, 1983.

[18] G. N. Wells, E. Kuhl, and K. A. Garikipati. A discontinuous Galerkin method for the Cahn-Hilliard equation. *J. Comput. Phys.*, 218(2):860–877, 2006.

[19] Y. Xia, Y. Xu, and C.W. Shu. Local discontinuous Galerkin methods for the Cahn-Hilliard type equations. *J. Comput. Phys.*, 227(1):472–491, 2007.

MOX Technical Reports, last issues

Dipartimento di Matematica
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

33/2017 Fumagalli, I.
A free-boundary problem with moving contact points

32/2017 Riccobelli, D.; Ciarletta, P.
Shape transitions in a soft incompressible sphere with residual stresses

31/2017 Schiltz, F.; Masci, C.; Agasisti, T.; Horn, D.
Using Machine Learning to Model Interaction Effects in Education: a Graphical Approach.

29/2017 Antonietti, P.F.; Mascotto, L.; Verani, M.
A multigrid algorithm for the p--version of the Virtual Element Method

30/2017 Bacchelli, V.; Micheletti, S.; Perotto, S.; Pierotti, D.
Parameter identification for the linear wave equation with Robin boundary condition

27/2017 Bonaventura, L.; Ferretti, R.; Rocchi L.;
A fully semi-Lagrangian discretization for the 2D Navier--Stokes equations in the vorticity--streamfunction formulation

28/2017 Pini, A.; Spreafico, L.; Vantini, S.; Vietti, A.
Multi-aspect local inference for functional data: analysis of ultrasound tongue profiles

26/2017 Masci, C.; Johnes, G.; Agasisti, T.
Student and School Performance in the OECD: a Machine Learning Approach.

25/2017 Paulon, G.; De Iorio, M.; Guglielmi, A.; Ieva, F.
Joint modelling of recurrent events and survival: a Bayesian nonparametric approach

24/2017 Domanin, M.; Bissacco, D.; Le Van, D.; Vergara, C.
Computational fluid-dynamic comparison between patch-based and direct suture closure techniques after carotid endarterectomy