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Abstract

We study the mixed formulation of the stochastic Hodge-Laplace problem defined on a n-dimensional domain D ($n \geq 1$), with random forcing term. In particular, we focus on the magnetostatic problem and on the Darcy problem in the three dimensional case. We derive and analyze the moment equations, that is the deterministic equations solved by the m-th moment ($m \geq 1$) of the unique stochastic solution of the stochastic problem. We find stable tensor product finite element discretizations, both full and sparse, and provide optimal order of convergence estimates. In particular, we prove the inf-sup condition for sparse tensor product finite element spaces.

1

1 Introduction

Many engineering applications are affected by uncertainty. This uncertainty may be due to the incomplete knowledge on the input data or some intrinsic variability of them. For example, if we model the two-phase flow in a porous medium, randomness arises in the permeability tensor, due to impossibility of a full characterization of conductivity properties of subsurface media, but also in the source term, typically pressure gradients or impervious boundaries. See for example [45, 42, 20, 21, 44, 38, 18, 7]. Similar situations appear in many other applications, such as combustion flows, earthquake engineering, biomedical engineering and finance. Probability theory provides an effective tool to include uncertainty in the model. We refer to [30, 9, 1] for probability measures on Banach spaces, and to [28, 27, 36, 16] and the references therein for stochastic partial differential equations. We notice that the SPDEs that we consider in this work differ from those in [28, 27, 36, 16] since we are taking L^m-intregrable processes.

In this work we focus on the linear Hodge-Laplace problem in mixed formulation, with stochastic forcing term and homogeneous boundary conditions. This problem includes the magnetostatic and electrostatic equations as well as the Darcy problem for monophase flows in saturated media.

The mathematical framework involving the Hodge-Laplace is the exterior calculus, a theoretical approach that, using tools from differential geometry, allows to simultaneously treat many different problems. In particular, the Hodge Laplacian $d\delta + \delta d$, where δ is the formal adjoint of the exterior derivative d, maps differential k-forms to differential k-forms, and unifies some important second-order differential operators, such as the Laplacian and curl $-$ curl problems arising in electromagnetics. For more details, see [3, 4, 14].

The solution of the mixed formulation of the stochastic Hodge-Laplace problem is a couple (u, p) of random fields taking values in a suitable space of differential forms. The description of these random fields requires the knowledge of their moments. A possible approach is to compute the moments by the Monte-Carlo method in which, after sampling the probability space, the deterministic PDE is solved for each sample and the results are combined to obtain statistical information about the random field. This is a widely used technique, but it features a very slow convergence rate. Improvements can be achieved by several techniques. We mention for instance the Multilevel Monte-Carlo method appeared in recent years in literature, and applied to both stochastic ODEs and PDEs: see [24, 19, 8, 26, 13] and the references therein.

An alternative strategy is to directly calculate the moments of interest of the stochastic solution without doing any sampling. Indeed, the aim of the present work is to derive the moment equations, that is the deterministic equations solved by the m-points correlation functions of the stochastic solution, show their well-posedness and propose a stable sparse finite element approximation.

The stochastic problem has the form

$$T \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \text{ a.e. in } D,$$

where T is a second order linear differential operator, D is a domain in $\mathbb{R}^n$, and the forcing terms $f_1(\omega, x)$, $f_2(\omega, x)$ are random fields, with $x \in D$, $\omega \in \Omega$ and Ω indicating the set of possible outcomes. The m-th moment equation involves the tensor product operator $T^{\otimes m} := \underbrace{T \otimes \cdots \otimes T}_{m \text{ times}}$ and the forcing term is given by the m-points correlation function of the couple $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

We start proving the well-posedness of the m-th moment equation. Although this comes easily from a tensorial argument, we also present a direct proof of the int-sup condition for the tensor operator $T^{\otimes m}$. This proof generalizes to some extent to the case of non tensor product spaces and will be the key tool to show the stability of a sparse finite element approximation.

Concerning the numerical approximation of the m-th moment equation, a tensorized FE approach for the numerical approximation of the moment equations is viable only for small m, as the number of degrees of freedom increases exponentially in m. For large m one should consider instead sparse approximations (see e.g. [41, 12, 34, 35, 40] and the references therein). We consider both full tensor product and sparse tensor product finite element approximations, and prove their stability using the tools from the finite element exterior calculus. See [2, 3, 4, 15]. In particular, the stability of a full tensor product approximation is a simple consequence of a tensor product argument. On the contrary, a tensor product argument does not apply if sparse tensor product approximations are considered and a direct proof of the inf-sup condition is needed, and will be proved in Section 6. We also provide optimal order of convergence estimates both for the full and the sparse approximations.

The analysis on well-posedness and stable discretization for the m-points correlation problem developed in this work will be necessary to analyze more complex situations with randomness appearing in the operator itself instead of simply in the right hand side. This case can be treated for small randomness by a perturbation approach (Taylor or Neumann expansions, see e.g. [6, 42, 20] and the references therein) and is currently under investigation.

The outline of the paper is the following: in Section 2 after recalling the definitions of the classical Sobolev spaces, we generalize them to the Sobolev spaces of differential forms. We then recall the main results on the mixed formulation of the Hodge-Laplace problem in the deterministic setting, stating the well-posedness of the problem and translating it to the language of partial differential equations using the proxy fields. In Section 3 we consider the stochastic counterpart of the mixed Hodge Laplacian problem, and we prove the well-posedness of its weak formulation. Section 4 is dedicated to the analysis of the moment equations where we provide in particular the constructive proof of the

inf-sup condition for the tensor product operator $T^{\otimes m}$. In Section 5 we focus on two problems of particular interest from the point of view of applications: the stochastic magnetostatic equations and the stochastic Darcy problem. Finally, in Section 6, we provide both full and sparse finite element discretizations for the deterministic m-th moment problem, we prove their stability and optimal order of convergence estimates.

2 Sobolev spaces of differential forms and the deterministic Hodge-Laplace problem

In this section we first recall the main concepts and definitions concerning the finite element exterior calculus and the Sobolev spaces of differential forms, which generalize the classical Sobolev spaces. We prove the inf-sup condition for the mixed formulation of the Hodge-Laplace problem providing a choice of test functions different from the classical one proposed in [3]. This will be needed later on to prove the equivalent inf-sup condition for the m-points correlation problem. Finally, we use the proxy fields correspondences to translate the Hodge-Laplace problem in the three dimensional case to the language of partial differential equations with the aim of showing that this general setting includes some important problems of practical interest. For more details we refer to [3, 4, 14].

2.1 Classical Sobolev spaces

Let $D \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. We denote with $L^m(D)$ the Lebesgue space of index m with $1 \le m < \infty$. $L^m(D)$ is a Banach space endowed with the standard norm

$$\|f\|_{L^m(D)} := \left(\int_D |f(x)|^m dx \right)^{1/m}. \tag{1}$$

When $p = 2$ we obtain the only Hilbert space of this class, with inner product given by

$$(f, g)_{L^2(D)} := \int_D f(x)g(x)dx, \quad f, g \in L^2(D).$$

We denote with $H^s(D)$ the Sobolev space defined as:

$$H^s(D) := \left\{ f \in L^2(D) | \, D^\alpha f \in L^2(D) \text{ for all } |\alpha| \le s \right\}. \tag{2}$$

$H^s(D)$ is a Hilbert space with the natural inner product

$$(f, g)_{H^s(D)} := \sum_{|\alpha| \le s} \langle D^\alpha f, D^\alpha g \rangle_{L^2(D)}, \quad \text{for } f, g \in H^s(D).$$

For more on the Lebesgue spaces $L^m(D)$ and the Sobolev spaces $H^s(D)$ see for example [29]. As it will be useful later on, we also recall the following Sobolev

spaces constrained by boundary conditions on $\Gamma_D \subset \partial D$:

$$H^1_{\Gamma_D}(D) = \left\{ v \in L^2(D) \mid \nabla v \in L^2(D), \ v|_{\Gamma_D} = 0 \right\},$$

$$H_{\Gamma_D}(\mathrm{curl}\,, D) = \left\{ v \in (L^2(D))^n \mid \mathrm{curl}\, v \in (L^2(D))^n, \ v \times \nu|_{\Gamma_D} = 0 \right\},$$

$$H_{\Gamma_D}(\mathrm{div}\,, D) = \left\{ v \in (L^2(D))^n \mid \mathrm{div}\, v \in L^2(D), \ v \cdot \nu|_{\Gamma_D} = 0 \right\},$$

where ν is the outer-pointing normal versor. These spaces are Hilbert spaces with respect to the graph norm.

Considering now a probability space $(\Omega, d\mathbb{P})$, the definition of L^m generalizes immediately. In this case we will use the notation $\left(L^m(\Omega, d\mathbb{P}), \| \cdot \|_{L^m(\Omega, d\mathbb{P})} \right)$ to denote the Banach space of real random variables on Ω with finite m-th moment. If $m = 2$, $\left(L^2(\Omega, d\mathbb{P}), \| \cdot \|_{L^2(\Omega, d\mathbb{P})} \right)$ is the Hilbert space of all real random variables on Ω with finite second moment, equipped with the usual inner product

$$(f(\omega), g(\omega))_{L^2(\Omega, d\mathbb{P})} := \int_\Omega f(\omega) g(\omega) d\mathbb{P}(\omega), \text{ for } f, g \in L^2(\Omega, d\mathbb{P}).$$

2.2   Sobolev spaces of differential forms

In order to generalize the definitions of the Sobolev spaces $H^s(D)$ to differential forms, we need to briefly recall the basic objects and results of exterior algebra and exterior calculus, inspired by [3]. The natural setting is a sufficiently smooth finite dimensional manifold D with or without boundary. For our purposes, we can restrict ourselves to the particular case of a n-dimensional bounded domain $D \subset \mathbb{R}^n$ with boundary denoted by $\partial D \subset \mathbb{R}^{n-1}$. In this way, at each point $x \in D$ the tangent space is naturally identified with $\mathbb{R}^n$ and we make this assumption throughout the paper. We denote by $Alt^k \mathbb{R}^n$, $1 \le k \le n$, the space of alternating k-linear maps on $\mathbb{R}^n$. Clearly, $Alt^0 \mathbb{R}^n = \mathbb{R}$ and $Alt^n \mathbb{R}^n = \mathbb{R}$, and the unique element in $Alt^n \mathbb{R}^n$ is a volume form vol_n. We recall the wedge product $\wedge : Alt^k \mathbb{R}^n \times Alt^l \mathbb{R}^n \to Alt^{k+l} \mathbb{R}^n$ and the inner product $(\cdot, \cdot)_{Alt^k \mathbb{R}^n} : Alt^k \mathbb{R}^n \times Alt^k \mathbb{R}^n \to \mathbb{R}$ for $k+l \le n$. Starting from this inner product, the Hodge star operator $\star : Alt^k \mathbb{R}^n \to Alt^{n-k} \mathbb{R}^n$ is defined: $u \wedge \star w = (u, w)_{Alt^k \mathbb{R}^n} vol_n$ (see e.g. [3]).

A differential k-form on D is a map u which associates to each $x \in D$ an element $u_x \in Alt^k \mathbb{R}^n$. We denote by $\Lambda^k(D)$ the space of all smooth differential k-forms on D. The wedge product of alternating k-forms may be applied pointwise to define the wedge product of differential forms: $(u \wedge w)_x = u_x \wedge w_x$. The exterior derivative d^k maps $\Lambda^k(D)$ into $\Lambda^{k+1}(D)$ for each $k \ge 0$ and it is defined as

$$\mathrm{d}^k u_x(v_1, \ldots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \partial_{v_j} u_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1}), \quad u \in \Lambda^k(D),$$

$v_1, \ldots, v_{k+1} \in \mathbb{R}^n$, where the hat is used to indicate a suppressed argument. The exterior derivative satisfies the key property $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$, $\forall\, k$. The coderivative

operator $\delta^k : \Lambda^k(D) \to \Lambda^{k-1}(D)$ is the formal adjoint of the exterior derivative and it is defined by

$$\star\delta^k u = (-1)^k \mathrm{d}^{n-k} \star u, \quad u \in \Lambda^k(D). \tag{3}$$

To lighten the notation, in the following we omit the k when no ambiguity arises. The trace operator $\mathrm{Tr} : \Lambda^k(D) \to \Lambda^k(\partial D)$ is defined as the pullback of the inclusion $\partial D \hookrightarrow D$. We denote with vol the unique volume form in $\Lambda^n(D)$ such that at each $x \in D$, vol_n is the unique form associated with $Alt^n\mathbb{R}^n$. Given two differential k-forms on D it is possible to define their L^2-inner product as the integral of their pointwise inner product in $Alt^k\mathbb{R}^n$:

$$(u,w) := \int_D (u_x, w_x)_{Alt^k\mathbb{R}^n} vol = \int_D u \wedge \star w, \ u,w \in \Lambda^k(D). \tag{4}$$

In the following we will denote with $\| \cdot \|$ the norm induced by the L^2-inner product $(\cdot, \cdot)$. The following integration by parts formula holds:

$$(\mathrm{d}u, v) = (u, \delta v) + \int_{\partial D} \mathrm{Tr}(u) \wedge \mathrm{Tr}(\star v), \ u \in \Lambda^k(D), \ v \in \Lambda^{k+1}(D). \tag{5}$$

The completion of $\Lambda^k(D)$ in the norm induced by the scalar product (4) defines the Hilbert space $L^2\Lambda^k(D)$. The Sobolev space of square integrable k-forms whose exterior derivative is also square integrable is given by

$$H\Lambda^k(D) = \left\{ u \in L^2\Lambda^k(D) | \ \mathrm{d}u \in L^2\Lambda^{k+1}(D) \right\}. \tag{6}$$

It is a Hilbert space equipped with the inner product

$$(u,w)_{H\Lambda^k(D)} := (u,w) + (\mathrm{d}u, \mathrm{d}w).$$

In analogy with $H\Lambda^k(D)$, it is possible to define the Hilbert space

$$H^*\Lambda^k(D) := \left\{ u \in L^2\Lambda^k(D) | \ \delta u \in L^2\Lambda^{k-1}(D) \right\}. \tag{7}$$

Let $\partial D = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. As it is standard ([3]), the spaces (6) and (7) can be endowed with boundary conditions:

$$H_{\Gamma_D}\Lambda^k(D) := \left\{ u \in H\Lambda^k(D) | \ \mathrm{Tr}(u)|_{\Gamma_D} = 0 \right\}. \tag{8}$$

$$H^*_{\Gamma_N}\Lambda^k(D) := \left\{ u \in H^*\Lambda^k(D) | \ \mathrm{Tr}(\star u)|_{\Gamma_N} = 0 \right\}.$$

With the spaces defined in (8) and the exterior derivative operator, we can construct the L^2 de Rham complex:

$$0 \to H_{\Gamma_D}\Lambda^0(D) \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} H_{\Gamma_D}\Lambda^n(D) \to 0. \tag{9}$$

Since $\mathrm{d} \circ \mathrm{d} = 0$, we have

$$\mathfrak{B}_k \subseteq \mathfrak{Z}_k, \tag{10}$$

where $\mathfrak{B}_k$ is the image of d in $H_{\Gamma_D}\Lambda^k(D)$ while $\mathfrak{Z}_k$ is the kernel of d in $H_{\Gamma_D}\Lambda^k(D)$.

The following orthogonal decomposition of $L^2\Lambda^k(D)$, known as Hodge decomposition, holds:

$$L^2\Lambda^k(D) = \mathfrak{B}_k \oplus \mathfrak{B}_k^\perp \tag{11}$$

where $\mathfrak{B}_k^\perp$ is the L^2-complement of $\mathfrak{B}_k$.
We define two projection operators $\pi^\perp$ and π° as follows:

$$\pi^\perp : \mathfrak{B}_k \oplus \mathfrak{B}_k^\perp \to \mathfrak{B}_k^\perp \tag{12}$$
$$v = \mathrm{d}v^\circ + v^\perp \mapsto v^\perp$$

$$\pi^\circ : \mathfrak{B}_k \oplus \mathfrak{B}_k^\perp \to \mathfrak{B}_{k-1}^\perp \tag{13}$$
$$v = \mathrm{d}v^\circ + v^\perp \mapsto v^\circ.$$

We recall a classical result in the theory of Sobolev spaces:

Lemma 2.1 (Poincaré inequality) *There exists a positive constant C_P that depends only on the domain D such that*

$$\|v\| \le C_P \|\mathrm{d}v\| \quad \forall v \in \mathfrak{Z}_k^\perp \tag{14}$$

where $\mathfrak{Z}_k^\perp$ is the orthogonal complement of $\mathfrak{Z}_k$ in $H_{\Gamma_D}\Lambda^k(D)$.

For the sake of simplicity, we consider only the case of geometries which are trivial from the topological point of view. More precisely, from now on, we make the following

Assumption 2.1 *The domain $D \subset \mathbb{R}^n$ is bounded, Liptschitz and contractible. Its boundary ∂D is given by the disjoint union of two open sets Γ_D and Γ_N, with $\Gamma_D, \Gamma_N \ne \emptyset$, Γ_D contractible as well and with boundary sufficiently regular (at least piecewise C^1).*

Under assumption 2.1, $\mathfrak{B}_k^\perp = \mathfrak{B}_k^*$, where $\mathfrak{B}_k^*$ is the image of δ in $H_{\Gamma_N}^*\Lambda^k(D)$. This relation is proved in the three dimensional case in [17], and generalizes to the n dimensional case (see e.g. [32]).

Remark 2.1 *The case of non-trivial topology can likely be treated following [4], but it would make the exposition of our results much more difficult.*

Remark 2.2 *We assume $\Gamma_D, \Gamma_N \ne \emptyset$, but the two limit cases treated in [3] can be considered with suitable modifications of our argument.*

We end the section by introducing the following notations for two Hilbert spaces we will use later on:

$$W_k := \begin{bmatrix} L^2\Lambda^k(D) \\ L^2\Lambda^{k-1}(D) \end{bmatrix}, \quad V_k := \begin{bmatrix} H_{\Gamma_D}\Lambda^k(D) \\ H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix}, \tag{15}$$

with the inner products $(\cdot,\cdot)_{W_k}$, $(\cdot,\cdot)_{V_k}$, and the norms $\|\cdot\|_{W_k}$, $\|\cdot\|_{V_k}$.

2.3 Mixed formulation of the Hodge-Laplace problem

The Hodge Laplacian is the differential operator $\delta d + d\delta$ mapping k-forms into k-forms, and the Hodge-Laplace problem is the boundary value problem for the Hodge Laplacian. Suppose we have a domain $D \subset \mathbb{R}^n$ satisfying Assumption 2.1. We consider a particular case of the mixed formulation of the Hodge-Laplace problem with variable coefficients described in [3, 4, 14], which allows to include the Darcy problem (see Section 2.3.1). Given a non negative coefficient $\alpha \in \mathbb{R}_+$ and source terms $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in W_k$, find $\begin{bmatrix} u \\ p \end{bmatrix}$ such that

$$\begin{cases} \delta du + dp = f_1 \text{ in } D \\ \delta u - \alpha p = f_2 \text{ in } D \\ \begin{cases} \text{Tr}(u) = 0 \text{ on } \Gamma_D \\ \text{Tr}(p) = 0 \text{ on } \Gamma_D \end{cases} \begin{cases} \text{Tr}(\star u) = 0 \text{ on } \Gamma_N \\ \text{Tr}(\star du) = 0 \text{ on } \Gamma_N \end{cases} \end{cases} \tag{16}$$

We introduce $T : V_k \to V_k'$, the linear operator of order two represented by the matrix:

$$T := \begin{bmatrix} \delta d & d \\ \delta & -\alpha \text{Id} \end{bmatrix} = \begin{bmatrix} A & B^* \\ B & -\alpha \text{Id} \end{bmatrix}, \tag{17}$$

where $V_k' = \begin{bmatrix} (H_{\Gamma_D}\Lambda^k(D))' \\ (H_{\Gamma_D}\Lambda^{k-1}(D))' \end{bmatrix}$ is the dual space of V_k defined in (15), the operators A and B are defined as:

$$A : H_{\Gamma_D}\Lambda^k(D) \to (H_{\Gamma_D}\Lambda^k(D))' \tag{18}$$
$$\langle Av, w \rangle := (dv, dw)$$

$$B : H_{\Gamma_D}\Lambda^k(D) \to (H_{\Gamma_D}\Lambda^{k-1}(D))' \tag{19}$$
$$\langle Bv, q \rangle := (v, dq)$$

and B^* is the adjoint of B. Moreover we introduce the linear operators $F_1 \in (H_{\Gamma_D}\Lambda^k)'$ and $F_2 \in (H_{\Gamma_D}\Lambda^{k-1})'$ defined as:

$$F_1 : H_{\Gamma_D}\Lambda^k(D) \to \mathbb{R} \tag{20}$$
$$F_1(v) := (f_1, v)$$

$$F_2 : H_{\Gamma_D}\Lambda^{k-1}(D) \to \mathbb{R} \qquad (21)$$
$$F_2(q) := (f_2, q)$$

The mixed formulation of the deterministic Hodge Laplacian with homogeneous essential boundary conditions on Γ_D and homogeneous natural boundary conditions on Γ_N is

Deterministic Problem:

$$
\boxed{
\begin{array}{c}
Given \; \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in V_k', \; find \; \begin{bmatrix} u \\ p \end{bmatrix} \in V_k \; s.t. \\[2em]
T \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \; in \; V_k',
\end{array}
} \qquad (22)
$$

Theorem 2.1 *For every $\alpha > 0$, problem (22) is well-posed, so that there exists a unique solution that depends continuously on the data. In particular, there exist positive constants C_1, C_1' that depend only on the Poincaré constant C_P and on the parameter α, such that for any $\begin{bmatrix} u \\ p \end{bmatrix} \in V_k$ there exists $\begin{bmatrix} v \\ q \end{bmatrix} \in V_k$ with*

$$\left\langle T \begin{bmatrix} u \\ p \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix} \right\rangle_{V_k', V_k} \geq C_1 \left\| \begin{bmatrix} u \\ p \end{bmatrix} \right\|_{V_k}^2 = C_1 \left(\|u\|_{H\Lambda^k}^2 + \|p\|_{H\Lambda^{k-1}}^2 \right), \qquad (23)$$

$$\left\| \begin{bmatrix} v \\ q \end{bmatrix} \right\|_{V_k} \leq C_1' \left\| \begin{bmatrix} u \\ p \end{bmatrix} \right\|_{V_k}. \qquad (24)$$

The same result holds with $\alpha = 0$ provided that F_2 corresponds to $f_2 \in \delta H_{\Gamma_D}\Lambda^k(D)$.

The well-posedness of problem (22) is proved in [3] by showing the equivalent inf-sup condition for the bounded bilinear and symmetric form $\langle T\cdot, \cdot \rangle : V_k \times V_k \to \mathbb{R}$ (23), (24) (see [5, 11]). However, we report it entirely (with a slightly different choice of test functions) as a preparatory step for the proofs we will propose later on.

Proof. We need to show (23) and (24). Let us start considering $\alpha > 0$. For a given $\begin{bmatrix} u \\ p \end{bmatrix}$ we use the Hodge decomposition (11):

$$\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} \mathrm{d}u^\circ + u^\perp \\ \mathrm{d}p^\circ + p^\perp \end{bmatrix}, \qquad (25)$$

with $\mathrm{d}u^\circ \in \mathfrak{B}_k$, $\mathrm{d}p^\circ \in \mathfrak{B}_{k-1}$, $u^\perp \in \mathfrak{B}_k^\perp$ and $p^\perp \in \mathfrak{B}_{k-1}^\perp$. We choose as test functions

$$\begin{bmatrix} v \\ q \end{bmatrix} = \begin{bmatrix} u^\perp + \mathrm{d}p^\perp \\ \gamma u^\circ - \mathrm{d}p^\circ \end{bmatrix}, \qquad (26)$$

where γ is a positive parameter to be set later. Relation (26) can also be written in a compact form as

$$\begin{bmatrix} v \\ q \end{bmatrix} = P \begin{bmatrix} u \\ p \end{bmatrix}, \qquad (27)$$

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where

$$P = \begin{bmatrix} \pi^\perp & \mathrm{d}\pi^\perp \\ \gamma\pi^\circ & -\mathrm{d}\pi^\circ \end{bmatrix} \qquad (28)$$

and the operators $\pi^\perp$, π° are defined in (12) and (13) respectively. Substituting (26) into (23), using the property $\mathrm{d} \circ \mathrm{d} = 0$, the Hodge decomposition (11) and the Poincaré inequality (14) we find

$$\left\langle T \begin{bmatrix} u \\ p \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix} \right\rangle_{V_k', V_k} = (\mathrm{d}u, \mathrm{d}v) + (v, \mathrm{d}p) + (u, \mathrm{d}q) - \alpha(p, q)$$

$$= \|\mathrm{d}u^\perp\|^2 + \|\mathrm{d}p^\perp\|^2 + \gamma\|\mathrm{d}u^\circ\|^2 + \alpha\|\mathrm{d}p^\circ\|^2 - \alpha\gamma\,(p^\perp, u^\circ)$$

$$\geq \|\mathrm{d}u^\perp\|^2 + \|\mathrm{d}p^\perp\|^2 + \gamma\|\mathrm{d}u^\circ\|^2 + \alpha\|\mathrm{d}p^\circ\|^2$$

$$- \frac{\alpha\gamma^{1/2}}{2}\left(C_P^2\|\mathrm{d}p^\perp\|^2 + \gamma C_P^2\|\mathrm{d}u^\circ\|^2\right)$$

$$\geq \|\mathrm{d}u^\perp\|^2 + \left(1 - \frac{\alpha}{2}\gamma^{1/2}C_P^2\right)\|\mathrm{d}p^\perp\|^2 +$$

$$\gamma\left(1 - \frac{\alpha\gamma^{1/2}C_P^2}{2}\right)\|\mathrm{d}u^\circ\|^2 + \alpha\|\mathrm{d}p^\circ\|^2.$$

It is possible to choose γ in order to make (23) true with $C_1 = C_1(C_P, \alpha)$. The inequality (24) with $C_1 = C_1'(C_P, \alpha)$ follows from the Hodge decomposition (11) and Poincaré inequality (14).

The proof in the case $\alpha = 0$ is very similar. Suppose $f_2 \in \delta H_{\Gamma_D}\Lambda^k(D)$. In order to have a unique solution, we need to look for $p \in \mathfrak{B}_{k-1}^\perp$. Fixed $u = \mathrm{d}u^\circ + u^\perp \in H_{\Gamma_D}\Lambda^k(D)$ we again choose the test functions as in (27): $v = \mathrm{d}p + u^\perp \in H_{\Gamma_D}\Lambda^k(D)$ and $q = u^\circ \in \mathfrak{B}_{k-1}^\perp$. Using the Poincaré inequality (14) and the orthogonal decomposition (11) we are able to prove the relations (23) and (24). $\qquad\square$

A simple consequence of Theorem 2.1 (see [11]) is that there exists a positive constant $K = K(C_P, \alpha)$ such that

$$\left\| \begin{bmatrix} u \\ p \end{bmatrix} \right\|_{V_k} \leq K \left\| \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \right\|_{V_k'}. \qquad (29)$$

Another way to express the result given in Theorem 2.1 is given by the following

Proposition 2.1 *Given T as in (17) and P as in (28), $\forall \begin{bmatrix} u \\ p \end{bmatrix} \in V_k$ it holds*

$$\left\langle T \begin{bmatrix} u \\ p \end{bmatrix}, P \begin{bmatrix} u \\ p \end{bmatrix} \right\rangle_{V_k', V_k} \geq C_1 \left\| \begin{bmatrix} u \\ p \end{bmatrix} \right\|_{V_k}^2 \qquad (30)$$

$$\|P\|_{\mathcal{L}(V_k, V_k)} \leq C_1'. \qquad (31)$$

2.3.1 Translation to the language of partial differential equations

Let us consider the case $D \subset \mathbb{R}^3$, naturally identifying the space $T_x D$ with $\mathbb{R}^3$. Thanks to the identification of $Alt^0\mathbb{R}^3$ and $Alt^3\mathbb{R}^3$ with $\mathbb{R}$, and of $Alt^1\mathbb{R}^3$ and

	$H_{\Gamma_D}\Lambda^k(D)$	d	$Tr\vert_{\Gamma_D}u$
$k=0$	$H^1_{\Gamma_D}(D)$	∇	$u\vert_{\Gamma_D}$
$k=1$	$H_{\Gamma_D}(\mathrm{curl}\,,D)$	curl	$u\times n\vert_{\Gamma_D}$
$k=2$	$H_{\Gamma_D}(\mathrm{div}\,,D)$	div	$u\cdot n\vert_{\Gamma_D}$
$k=3$	$L^2(D)$	0	0

Table 1: Correspondences in terms of proxy fields between the space of differential forms $H\Lambda^k(D)$ and the classical spaces of functions and vector fields, in the case $n=3$.

$Alt^2\mathbb{R}^3$ with $\mathbb{R}^3$, we can establish correspondences between the spaces of differential forms and scalar or vector fields. These fields are called proxy fields. In particular, we can identify each 0-form and 3-form with a scalar-valued function, and each 1-form and 2-form with a vector-valued function. Table 1 summarizes the correspondences in terms of proxy fields for the spaces of differential forms $H_{\Gamma_D}\Lambda^k(D)$, the exterior derivative operators and the trace operators. Based on the identifications in Table 1 we can reinterpret the de Rham complex (9) as follows:

$$0 \to H^1_{\Gamma_D}(D) \xrightarrow{\nabla} H_{\Gamma_D}(\mathrm{curl}\,,D) \xrightarrow{\mathrm{curl}} H_{\Gamma_D}(\mathrm{div},D) \xrightarrow{\mathrm{div}} L^2(D) \to 0 \qquad (32)$$

In this section we will use the symbol $(\cdot,\cdot)$ to denote the inner product in $L^2(D)$, that corresponds by proxy identifications to the inner product in $L^2\Lambda^k(D)$.

- Let us start with $k=0$. In this case $H_{\Gamma_D}\Lambda^{-1}(D)=0$, so $p=0$. Then $u \in H^1_{\Gamma_D}(D)$ satisfies

$$(\nabla u, \nabla v) = (f_1, v) \quad \forall v \in H^1_{\Gamma_D}(D). \qquad (33)$$

  We obtain the usual weak formulation of the Poisson equation equipped with homogeneous Dirichlet boundary conditions on Γ_D and homogeneous Neumann boundary conditions on Γ_N.

- For $k=1$ and $\alpha=0$, the linear operator T of order two defined in (17) is represented by the matrix

$$T = \begin{bmatrix} \mathrm{curl}^{\,2} & \nabla \\ -\mathrm{div} & 0 \end{bmatrix}. \qquad (34)$$

  Problem (22) is the weak formulation of the magnetostatic/electrostatic equations (see for example [33, 10, 25]). Indeed, $V_1 = \begin{bmatrix} H_{\Gamma_D}(\mathrm{curl}\,,D) \\ H^1_{\Gamma_D}(D) \end{bmatrix}$ and $\begin{bmatrix} u \\ p \end{bmatrix} \in V_1$ satisfies

$$\begin{cases} (\mathrm{curl}\,u, \mathrm{curl}\,v) + (\nabla p, v) = (f_1, v) \\ (u, \nabla q) = (f_2, q). \end{cases} \quad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_1. \qquad (35)$$

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- When $k = 2$,
$$T = \begin{bmatrix} -\nabla \text{div} & \text{curl} \\ \text{curl} & -\alpha \text{Id} \end{bmatrix}.$$

Problem (22) is the mixed formulation of the vectorial Poisson equation: find $\begin{bmatrix} u \\ p \end{bmatrix} \in V_2 = \begin{bmatrix} H_{\Gamma_D}(\text{div}, D) \\ H_{\Gamma_D}(\text{curl}, D) \end{bmatrix}$ s.t.

$$\begin{cases} (\text{div } u, \text{div } v) + (\text{curl } p, v) = (f_1, v) \\ (u, \text{curl } q) - \alpha (p, q) = (f_2, q) \end{cases} \quad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_2. \quad (36)$$

- Finally, for $k = 3$, problem (22) models the flow in porous media. We can reinterpret the linear tensor operator of order two T as

$$T = \begin{bmatrix} 0 & \text{div} \\ -\nabla & -\alpha \text{Id} \end{bmatrix}, \quad (37)$$

where $\alpha > 0$ is linked to the inverse of the permeability. Hence, problem (22) is the Darcy equations: find $\begin{bmatrix} u \\ p \end{bmatrix} \in V_3 = \begin{bmatrix} L^2(D) \\ H_{\Gamma_D}(\text{div}, D) \end{bmatrix}$ s.t.

$$\begin{cases} (\text{div } p, v) = (f_1, v) \\ (u, \text{div } q) - \alpha (p, q) = 0 \end{cases} \quad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_3. \quad (38)$$

3 Stochastic Sobolev spaces of differential forms and stochastic Hodge Laplacian

3.1 Stochastic Sobolev spaces of differential forms

Let $v_1 \in V_1$ and $v_2 \in V_2$, where V_1, V_2 are Hilbert spaces. Let $v_1 \otimes v_2 : V_1 \times V_2 \to \mathbb{R}$ denote the symmetric bilinear form which acts on each couple $(w_1, w_2) \in V_1 \times V_2$ by

$$v_1 \otimes v_2(w_1, w_2) = (v_1, w_1)_{V_1} (v_2, w_2)_{V_2},$$

where $(\cdot, \cdot)_{V_1}$ denotes the inner product in V_1 and $(\cdot, \cdot)_{V_2}$ the inner product in V_2. Let us define an inner product $(\cdot, \cdot)_{V_1 \otimes V_2}$ on the set of such symmetric bilinear forms as

$$\left(v_1 \otimes v_2, v_1' \otimes v_2' \right)_{V_1 \otimes V_2} = \left(v_1, v_1' \right)_{V_1} \left(v_2, v_2' \right)_{V_2}, \quad (39)$$

and extend it by linearity to the set

$$\text{span} \left\{ v_1 \otimes v_2 \ : \ v_1 \in V_1, v_2 \in V_2 \right\} \quad (40)$$

composed of finite linear combinations of such symmetric bilinear forms.

Definition 3.1 *Given V_1 and V_2 Hilbert spaces, the tensor product $V_1 \otimes V_2$ is the Hilbert space defined as the completition of the set* (40) *under the inner product* $(\cdot, \cdot)_{V_1 \otimes V_2}$ *in* (39).

In the following we will denote with $\| \cdot \|_{V_1 \otimes V_2}$ the norm induced by the inner product $(\cdot, \cdot)_{V_1 \otimes V_2}$. Definition 3.1 naturally generalizes to the tensor product of m Hilbert spaces, with $m \geq 2$ integer. For more details on tensor product spaces and on norms on tensor product spaces see for example [37, 31] and the references therein.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space and V a separable Hilbert space. The stochastic counterpart of V is the Hilbert space given by the tensor product $V \otimes L^2(\Omega, d\mathbb{P})$, where $L^2(\Omega, d\mathbb{P})$ is the Hilbert space defined in Section 2.1. Let $L^2(\Omega; V)$ be the Bochner space composed of functions u such that $\omega \mapsto \|u(\omega)\|_V^2$ is measurable and integrable, so that $\|u\|_{L^2(\Omega;V)} := \left(\int_\Omega \|u(\omega)\|_V^2 d\mathbb{P}(\omega) \right)^{1/2}$ is finite. We observe that there is a unique isomorphism from $V \otimes L^2(\Omega, d\mathbb{P})$ to $L^2(\Omega; V)$ which maps $\psi \otimes \mu \in V \otimes L^2(\Omega, d\mathbb{P})$ onto the function $\omega \mapsto \mu(\omega)\psi \in V$.

The definition of the Hilbert space $L^2(\Omega; V)$ easily generalizes to the Banach space $L^m(\Omega; V)$ with $m \geq 1$ integer. We say that a random field $u : \Omega \to V$ is in the Bochner space $L^m(\Omega; V)$ if $\omega \mapsto \|u(\omega)\|_V^m$ is measurable and integrable, so that $\|u\|_{L^m(\Omega;V)} := \left(\int_\Omega \|u(\omega)\|_V^m d\mathbb{P}(\omega) \right)^{1/m}$ is finite.

In the following we focus on two stochastic Sobolev spaces of differential forms, namely $L^m(\Omega; W_k)$ and $L^m(\Omega; V_k)$ with $m \geq 1$ integer, where W_k and V_k are Sobolev spaces of differential forms defined in (15).

3.2 Stochastic mixed Hodge-Laplace problem

Let D be a domain in $\mathbb{R}^n$ satisfying assumption 2.1. Let be given $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k')$, with $m \geq 1$, defined as the stochastic version of (20) and (21):

$$F_1(\omega) : H_{\Gamma_D} \Lambda^k(D) \to \mathbb{R}$$
$$F_1(\omega)(v) := (f_1(\omega), v)$$

$$F_2(\omega) : H_{\Gamma_D} \Lambda^{k-1}(D) \to \mathbb{R}$$
$$F_2(\omega)(q) := (f_2(\omega), q)$$

where $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^m(\Omega; V_k)$ is given. The stochastic counterpart of problem (22) is:

Stochastic Problem:

$$\boxed{
\begin{array}{c}
\textit{Given } m \geq 1 \textit{ and } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k'), \textit{ find } \begin{bmatrix} u \\ p \end{bmatrix} \in L^m(\Omega; V_k) \textit{ s.t.} \\[3mm]
T \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} = \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix} \textit{ in } V_k', \textit{ a.e. in } \Omega.
\end{array}
} \quad (41)$$

Theorem 3.1 (Well-posedness of the stochastic Hodge Laplacian) *For every $\alpha > 0$ problem (41) is well-posed, so that there exists a unique solution that depends continuously on the data. The same result holds with $\alpha = 0$ provided that F_2 corresponds to $f_2 \in L^m\left(\Omega; \delta H_{\Gamma_D}\Lambda^k(D)\right)$.*

Proof. Thanks to Theorem 2.1, for almost all $\omega \in \Omega$, problem (41) admits a unique solution $\begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} \in V_k$, the mapping $\omega \mapsto \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix}$ is measurable and we have:

$$\left\| \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} \right\|_{V_k} \leq K \left\| \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix} \right\|_{V_k'} \qquad \text{a.e. in } \Omega \tag{42}$$

with $K = K(C_P, \alpha)$ independent of ω (see (29)). For any $m \geq 1$,

$$\left(\mathbb{E}\left[\left\| \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} \right\|_{V_k}^m \right] \right)^{1/m} \leq K \left(\mathbb{E}\left[\left\| \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix} \right\|_{V_k'}^m \right] \right)^{1/m}.$$

By hypothesis $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m\left(\Omega; V_k'\right)$, hence we conclude that $\begin{bmatrix} u \\ p \end{bmatrix} \in L^m\left(\Omega; V_k\right)$. $\qquad \square$

4 Deterministic problems for the statistics of u and p

We are interested in the statistical moments of the unique stochastic solution $\begin{bmatrix} u \\ p \end{bmatrix}$ of the stochastic problem (41). We exploit the linearity of the system $T \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} = \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix}$ to derive the moment equations, that is the deterministic equations solved by the statistical moments of the unique stochastic solution $\begin{bmatrix} u \\ p \end{bmatrix}$. At the beginning we focus on the first moment equation. Then, after recalling the definition of the m-th statistical moment ($m \geq 2$ integer) and the main concepts about the tensor product of operators defined on Hilbert spaces, we establish the well-posedness of the m-th moment problem. The main achievement is the constructive proof of the inf-sup condition for the tensor product operator $T^{\otimes m}$ stated in Theorem 4.2. Indeed, this proof extends to the case of sparse tensor product approximations (see Section 6.4).

4.1 Equations for the mean

Following [43, 41], we provide a way to compute the first statistical moment of the unique stochastic solution of the stochastic Hodge Laplace problem (41).

Given a random field $v \in L^1\left(\Omega; V\right)$, where V in a Hilbert space, its first statistical moment $\mathbb{E}[v] \in V$ is well defined, and is given by:

$$\mathbb{E}[v](x) := \int_\Omega v(\omega, x) d\mathbb{P}, \quad x \in D. \tag{43}$$

Definition (43) easily applies to the vector case ($V = V_k, V = W_k$).

Suppose that $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^1\left(\Omega; V_k'\right)$, so that the unique solution of the stochastic problem is such that $\begin{bmatrix} u \\ p \end{bmatrix} \in L^1\left(\Omega; V_k\right)$. To derive the first moment equation we simply apply the mean operator to the stochastic problem (41). We exploit the commutativity between the operators T defined in (17) and $\mathbb{E}$ defined in (43), so that $\mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix}$ is a solution of:

Mean Problem

$$
\boxed{
\begin{aligned}
Given \ & \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^1\left(\Omega; V_k'\right), \ find \ E_s \in V_k \ s.t. \\
& T(E_s) = \mathbb{E}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \ in \ V_k',
\end{aligned}
}
\tag{44}
$$

where $\mathbb{E}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in V_k'$ is defined as:

$$
\mathbb{E}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\left(\begin{bmatrix} v \\ q \end{bmatrix}\right) := \left(\mathbb{E}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix}\right)_{W_k} \quad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_k.
$$

Theorem 2.1 states the well-posedness of problem (44), hence $\mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix}$ is the unique solution. We notice that problem (44) has exactly the same structure of problem (41) with loading terms given by the mean of the loading terms in (41).

4.2 Statistical moments of a random function

Let $u \in L^m\left(\Omega; V\right)$, where V is a Hilbert space and $L^m\left(\Omega; V\right)$ is defined as in Section 3.1. Then $u^{\otimes m} := \underbrace{u \otimes \cdots \otimes u}_{m \text{ times}} \in L^1(\Omega, V^{\otimes m})$, where from now on $V^{\otimes m}$ denotes the tensor product space $\underbrace{V \otimes \cdots \otimes V}_{m \text{ times}}$. Hence we can give the following definition:

Definition 4.1 *Given $u \in L^m\left(\Omega; V\right)$, $m \geq 2$ integer, then the m-th moment of $u(\omega)$ is defined by*

$$
\mathcal{M}^m\left[u\right] := \mathbb{E}\left[u \otimes \cdots \otimes u\right] = \int_\Omega u(\omega) \otimes \cdots \otimes u(\omega)d\mathbb{P}(\omega) \in V^{\otimes m}. \tag{45}
$$

It clearly holds $\|\mathcal{M}^m\left[u\right]\|_{V^{\otimes m}} \leq \|u\|_{L^m(\Omega;V)}^m$. Definition 4.1 with $m = 1$ is (43). Moreover, Definition 4.1 easily generalizes to the vector case.

4.3 Tensor product of operators on Hilbert spaces

We will see that the deterministic equation for the m-th moment involves the tensor product of the operator T. Hence, we need to describe some aspects of the theory of tensor product operators on Hilbert spaces. For more details see for example [37] and the references therein.

Suppose that $T_1 : V_1 \to V_1'$, $T_2 : V_2 \to V_2'$ are continuous operators on the Hilbert spaces V_1 and V_2 respectively. $T_1 \otimes T_2$ is defined on functions of the type $\phi \otimes \psi$, with $\phi \in V_1$, $\psi \in V_2$ as:

$$(T_1 \otimes T_2)(\phi \otimes \psi) = T_1\phi \otimes T_2\psi \in V_1' \otimes V_2'.$$

This definition extends to $V_1 \otimes V_2$ by linearity and density. The tensor product of two bounded operators on Hilbert space is still a bounded operator, as stated by the following

Proposition 4.1 *Let $T_1 : V_1 \to V_1'$, $T_2 : V_2 \to V_2'$ be bounded operators on Hilbert spaces V_1 and V_2 respectively. Then*

$$\|T_1 \otimes T_2\|_{\mathcal{L}(V_1 \otimes V_2, V_1' \otimes V_2')} = \|T_1\|_{\mathcal{L}(V_1, V_1')}\|T_2\|_{\mathcal{L}(V_2, V_2')}.$$

Proof. See [37]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The definition of the tensor product of two operators on Hilbert spaces and Proposition 4.1 generalize to tensor product of any finite number of operators defined on Hilbert spaces.

We detail now the vector case, since it will be useful in the next section. Let $V_1 = V_2 = V_k$, where V_k is defined in (15), and $T_1 = T_2 = T$, where $T = (T)_{i,j=1,2} : V_k \to V_k'$ is the linear operator of order two defined in (17). The tensor product operator $T^{\otimes m} := \underbrace{T \otimes \cdots \otimes T}_{m}$, $(m \geq 1$ integer$)$, is the operator of order $2m$ that maps tensors in $V_k^{\otimes m}$ to tensors is $(V_k')^{\otimes m}$ defined as

$$(T^{\otimes m})_{i_1 \ldots i_{2m}} = T_{i_1 i_2} \otimes \cdots \otimes T_{i_{2m-1} i_{2m}}. \tag{46}$$

Given $X \in V_k^{\otimes m}$, $T^{\otimes m}X$ is a tensor of order m in $(V_k')^{\otimes m}$ given by

$$(T^{\otimes m}X)_{i_1 \ldots i_m} = \sum_{j_1, \ldots, j_m = 1}^{2} (T_{i_1 j_1} \otimes \cdots \otimes T_{i_m j_m})X_{j_1 \ldots j_m}, \quad i_1, \ldots, i_m = 1, 2. \tag{47}$$

Definition 4.2 *Let T and V_k be as before and let $X \in V_k^{\otimes m}$ and $Y \in V_k^{\otimes m}$. We define*

$$\langle T^{\otimes m}X, Y \rangle = \sum_{i_1, \ldots, i_m = 1}^{2} \sum_{j_1, \ldots, j_m = 1}^{2} \langle T_{i_1, j_1} \cdots T_{i_m, j_m} X_{j_1, \ldots, j_m}, Y_{i_1, \ldots, i_m} \rangle. \tag{48}$$

4.4 Equations for the m-th moment

Following [43], we analyze the m-th moment equation for $m \geq 2$. Suppose $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k')$ so that $\begin{bmatrix} u \\ p \end{bmatrix} \in L^m(\Omega; V_k)$. To derive the deterministic m-th moment problem we tensorize the stochastic problem (41) with itself m times:

$$\underbrace{T \otimes \ldots \otimes T}_{m \text{ times}} \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix}^{\otimes m} = \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix}^{\otimes m} \text{ in } (V_k')^{\otimes m}, \text{ for a.e. } \omega \in \Omega.$$

We take the expectation on both sides and we exploit the commutativity between the operators T and $\mathbb{E}$. By definition, $\mathbb{E} \begin{bmatrix} u \\ p \end{bmatrix}^{\otimes m} = \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix}$. Thus, $\mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix}$ is a solution of

m-Points Correlation Problem:

> *Given $m \geq 2$ integer and $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k')$, find $M_s^{\otimes m} \in V_k^{\otimes m}$ s.t.*
>
> $$T^{\otimes m} M_s^{\otimes m} = \mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \text{ in } (V_k')^{\otimes m},$$

(49)

where $\mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in (V_k')^{\otimes m}$ is defined as:

$$\mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \left(\begin{bmatrix} v \\ q \end{bmatrix} \right) := \left(\mathcal{M}^m \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix} \right)_{W_k^{\otimes m}} \quad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_k^{\otimes m}.$$

We notice that in the right-hand side of (49) we have the m-points correlation of the loading terms of problem (41).

Remark 4.1 *Note that problem (44) is a saddle-point problem, and (49) is composed of m "nested" saddle-point problems. Indeed, if for example $m = 2$, $T \otimes T$ can be represented by the matrix*

$$T \otimes T = \left[\begin{array}{cc|cc} \delta\mathrm{d} \otimes \delta\mathrm{d} & \delta\mathrm{d} \otimes \mathrm{d} & \mathrm{d} \otimes \delta\mathrm{d} & \mathrm{d} \otimes \mathrm{d} \\ \delta\mathrm{d} \otimes \delta & \delta\mathrm{d} \otimes -\alpha\mathrm{Id} & \mathrm{d} \otimes \delta & \mathrm{d} \otimes -\alpha\mathrm{Id} \\ \hline \delta \otimes \delta\mathrm{d} & \delta \otimes \mathrm{d} & -\alpha\mathrm{Id} \otimes \delta\mathrm{d} & -\alpha\mathrm{Id} \otimes \mathrm{d} \\ \delta \otimes \delta & \delta \otimes -\alpha\mathrm{Id} & -\alpha\mathrm{Id} \otimes \delta & -\alpha\mathrm{Id} \otimes -\alpha\mathrm{Id} \end{array} \right]. \quad (50)$$

Theorem 4.1 (Well-posedness of the m-th problem) *For every $\alpha > 0$, problem (49) is well-posed, so that there exists a unique solution that depends continuously on the data. The same result holds with $\alpha = 0$ provided that F_2 corresponds to $f_2 \in L^m\left(\Omega; \delta H_{\Gamma_D} \Lambda^k(D)\right)$.*

Proof. Theorem 4.1 can be proved by a simple tensor product argument, as follows. Since problem (22) is well-posed, the inverse operator T^{-1} exists and is linear and bounded. Now we take into account the tensor operator $(T^{-1})^{\otimes m} = \underbrace{T^{-1} \otimes \ldots \otimes T^{-1}}_{m \text{ times}}$.

It is the inverse operator of $T^{\otimes m}$. Moreover, it is linear and bounded (Proposition 4.1). Hence we can immediately conclude the well-posedness of problem (49). $\square$

Remark 4.2 *The approach presented in the proof is not completely satisfactory in view of a finite dimensional approximation. Indeed, when considering a finite dimensional version of the operator, $T_h := T|_{V_{k,h}} : V_{k,h} \to V'_{k,h}$, where $V_{k,h}$ is a finite dimensional subspace of V_k, and aiming at proving the well-posedness of the tensor operator $(T_h)^{\otimes m} = \underbrace{T_h \otimes \ldots \otimes T_h}_{m \text{ times}}$, this tensor product argument applies only if the finite dimensional subspace is a tensor product space $V_{k,h}^{\otimes m}$. It will not apply straightforwardly if sparse tensor product spaces are considered instead.*

4.4.1 Constructive proof of inf-sup condition for the tensorized problem

Here we propose an alternative proof of Theorem 4.1 that consists in showing the inf-sup condition for $T^{\otimes m}$. This proof will be used later on to prove the stability of a sparse tensor product finite element discretization, which is of practical interest for moderately large m as it reduces considerably the curse of dimensionality with respect to a full tensor product approximation.

A result equivalent to Theorem 4.1 is the following

Theorem 4.2 (Tensorial inf-sup condition) *For every $M_s^{\otimes m} \in V_k^{\otimes m}$, there exist a test function $M_t^{\otimes m} \in V_k^{\otimes m}$ and positive constants $C_m = C_m(\alpha, C_{P,1}, \|T\|, \|P\|)$, $C'_m = C'_m(\alpha, C_{P,1}, \|T\|, \|P\|)$ s.t.*

$$\left\langle T^{\otimes m} M_s^{\otimes m}, M_t^{\otimes m} \right\rangle_{(V'_k)^{\otimes m}, V_k^{\otimes m}} \geq C_m \|M_s^{\otimes m}\|^2_{V_k^{\otimes m}}, \tag{51}$$

$$\|M_t^{\otimes m}\|_{V_k^{\otimes m}} \leq C'_m \|M_s^{\otimes m}\|_{V_k^{\otimes m}}, \tag{52}$$

where $C_{P,1}$ is introduced in (60) and P is defined in (28).

Before presenting the proof we state the tensorized versions of the Hodge decomposition and the Poincaré inequality, which are two keys ingredients in the proof of the inf-sup condition for the deterministic problem (22).

Let us write the space $V_k^{\otimes m}$ as

$$V_k^{\otimes m} = V_k \otimes V_k^{\otimes(m-1)} = \begin{bmatrix} H_{\Gamma_D}\Lambda^k(D) \\ H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix} \otimes V_k^{\otimes(m-1)} = \begin{bmatrix} U_k^m \\ U_{k-1}^m \end{bmatrix} \tag{53}$$

where we defined

$$U_k^m := H_{\Gamma_D}\Lambda^k(D) \otimes V_k^{\otimes(m-1)}, \qquad (54)$$

$$U_{k-1}^m := H_{\Gamma_D}\Lambda^{k-1}(D) \otimes V_k^{\otimes(m-1)} \qquad (55)$$

We obtain the tensorial Hodge decomposition following the idea of the one dimensional Hodge decomposition (11). Indeed, for every integer $m \geq 2$, we split U_k^m (U_{k-1}^m is analogous) as:

$$\textbf{Tensorial Hodge Decomposition:} \qquad (56)$$

$$U_k^m = \mathfrak{B}_k^m \oplus \mathfrak{B}_k^{m,\perp} \qquad (57)$$

where

$$\mathfrak{B}_k^m := \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} U_{k-1}^m = \mathfrak{B}_k \otimes V_k^{\otimes(m-1)}$$
$$\mathfrak{B}_k^{m,\perp} := \mathfrak{B}_k^\perp \otimes V_k^{\otimes(m-1)}$$

and $\mathfrak{B}_k$, $\mathfrak{B}_k^\perp$ are defined in Section 2.

The tensor operators $\pi^\perp \otimes \mathrm{Id}^{\otimes(m-1)}$ and $\pi^\circ \otimes \mathrm{Id}^{\otimes(m-1)}$ respectively defined in (12) and (13) act on U_k^m (U_{k-1}^m is analogous) as:

$$\pi^\perp \otimes \mathrm{Id}^{\otimes(m-1)} : U_k^m = \mathfrak{B}_k^m \oplus \mathfrak{B}_k^{m,\perp} \to \mathfrak{B}_k^{m,\perp} \qquad (58)$$
$$v = \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)}v^\circ + v^\perp \mapsto v^\perp$$

$$\pi^\circ \otimes \mathrm{Id}^{\otimes(m-1)} : U_k^m = \mathfrak{B}_k^m \oplus \mathfrak{B}_k^{m,\perp} \to \mathfrak{B}_{k-1}^{m,\perp} \qquad (59)$$
$$v = \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)}v^\circ + v^\perp \mapsto v^\circ.$$

The tensorial Poincaré inequality is proved in the following lemma.

Lemma 4.1 (Tensorial Poincaré inequality) *For every integer $m \geq 2$, there exists a positive constant $C_{P,1}$ such that*

$$\|v\|_{(L^2\Lambda^k(D))^{\otimes m}} \leq C_{P,1}\|\mathrm{Id}\otimes\ldots\otimes\underbrace{\mathrm{d}}_{i}\otimes\ldots\otimes\mathrm{Id}\, v\|_{L^2\Lambda^k\otimes\ldots\otimes\underbrace{L^2\Lambda^{k+1}}_{i}\otimes\ldots\otimes L^2\Lambda^k}, \quad (60)$$

$\forall v \in L^2\Lambda^k \otimes \ldots \otimes \underbrace{(\mathfrak{Z}_k^\perp)}_{i} \otimes \ldots \otimes L^2\Lambda^k(D)$, *where $\mathfrak{Z}_k^\perp$ is defined in Section 2.2.*

Proof. We know that $H\Lambda^k(D)$ is a Hilbert space with the inner product $(u,v)_{H\Lambda^k}$ and $(u,u)_{H\Lambda^k} = \|u\|_{H\Lambda^k}^2$. Besides, we know that $\mathfrak{Z}_k^\perp$ is a Hilbert space with the equivalent inner product $(\mathrm{d}u, \mathrm{d}v)$ and norm $\|\mathrm{d}u\| = (\mathrm{d}u, \mathrm{d}u)$. A consequence of the Open Mapping Theorem states that given m Hilbert spaces $H_1, \ldots, H_m$, the topology of $H_1 \otimes \ldots \otimes H_m$ depends only on the topology and not on the choice of the inner products of $H_1, \ldots, H_m$.

If we apply this statement with $H_i = \mathfrak{Z}_k^\perp$ and $H_j = H\Lambda^k(D)$, $i \neq j$, we can conclude the inequality (60). $\qquad\square$

A simple consequence of the previous lemma is:

$$\|v\|_{(L^2\Lambda^k(D))^{\otimes m}} \leq C_{P,m}\|\mathrm{d}^{\otimes m}v\|_{(L^2\Lambda^{k+1}(D))^{\otimes m}} \quad \forall v \in \left(\mathfrak{Z}_k^\perp\right)^{\otimes m}, \qquad (61)$$

where $C_{P,m} > 0$ depends only on the domain D and on m.

Proof. [Proof of Theorem 4.2]

As shown before, $\mathcal{M}^m\begin{bmatrix} u \\ p \end{bmatrix}$ is a solution of (49). Uniqueness of the solution of problem (49) is related to the global inf-sup condition (51), (52) (see [5, 11]). Suppose $\alpha > 0$ (the case $\alpha = 0$ is analogous). To lighten the notations, in the proof we use the brackets $\langle \cdot, \cdot \rangle$ without specifying the spaces we consider, when no ambiguity arises. We use the tensorial Hodge decomposition (57) and the tensorial Poicaré inequality (Lemma 4.1). We prove (51) by induction. In Theorem 2.1 we already proved the inf-sup condition with $m = 1$. Now suppose $m = 2$. We fix $M_s^{\otimes 2} = \begin{bmatrix} (M_s^{\otimes 2})_{1:} \\ (M_s^{\otimes 2})_{2:} \end{bmatrix}$ where $(M_s^{\otimes 2})_{1:}$ (respectively $(M_s^{\otimes 2})_{2:}$) means that in the tensor of order two $M_s^{\otimes 2} = (M_s^{\otimes 2})_{ij=1,2}$ we fix $i = 1$ (respectively $i = 2$) and let j vary. Using (53) and (57) with $m = 2$ we decompose

$$M_s^{\otimes 2} = \begin{bmatrix} \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{1:} + (M_s^\perp)_{1:} \\ \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{2:} + (M_s^\perp)_{2:} \end{bmatrix} \in \begin{bmatrix} U_k^2 \\ U_{k-1}^2 \end{bmatrix},$$

where

$$(M_s^\perp)_{1:} = \pi^\perp \otimes \mathrm{Id}(M_s^{\otimes 2})_{1:} \in \mathfrak{B}_k^{2,\perp}$$
$$(M_s^\perp)_{2:} = \pi^\perp \otimes \mathrm{Id}(M_s^{\otimes 2})_{2:} \in \mathfrak{B}_{k-1}^{2,\perp}$$
$$(M_s^\circ)_{1:} = \pi^\circ \otimes \mathrm{Id}(M_s^{\otimes 2})_{1:} \in \mathfrak{B}_{k-1}^{2,\perp}$$
$$(M_s^\circ)_{2:} = \pi^\circ \otimes \mathrm{Id}(M_s^{\otimes 2})_{2:} \in \mathfrak{B}_{k-2}^{2,\perp}.$$

We choose $M_t^{\otimes 2} = P \otimes PM_s^{\otimes 2}$, where P is defined in (28), so that:

$$\left\langle T \otimes TM_s^{\otimes 2}, M_t^{\otimes 2} \right\rangle = \left\langle T \otimes TM_s^{\otimes 2}, P \otimes PM_s^{\otimes 2} \right\rangle$$

$$= \sum_{i,j=1}^{2} \left\langle T_{ij} \otimes T(M_s^{\otimes 2})_{j:}, (P \otimes PM_s^{\otimes 2})_{i:} \right\rangle. \qquad (62)$$

Let $\left\langle T_{ij} \otimes T(M_s^{\otimes 2})_{j:}, (P \otimes PM_s^{\otimes 2})_{i:} \right\rangle = \mathcal{I}_{ij}$. We will bound each term $\mathcal{I}_{ij}$ for $i, j = 1, 2$.

Using (47) we explicit the term $(P \otimes PM_s^{\otimes 2})_{i:}$:

$$(P \otimes PM_s^{\otimes 2})_{i:} = P_{i1} \otimes P(M_s^{\otimes 2})_{1:} + P_{i2} \otimes P(M_s^{\otimes 2})_{2:}. \qquad (63)$$

Let us start from the case $i = j = 1$.

$$\mathcal{I}_{11} = \left\langle A \otimes T(M_s^{\otimes 2})_{1:}, (\pi^\perp \otimes P(M_s^{\otimes 2})_{1:} + \mathrm{d}\pi^\perp \otimes P(M_s^{\otimes 2})_{2:}) \right\rangle. \qquad (64)$$

Since $\mathrm{d} \circ \mathrm{d} = 0$, $\left\langle A \otimes T(M_s^{\otimes 2})_{1:}, \mathrm{d}\pi^\perp \otimes P(M_s^{\otimes 2})_{2:} \right\rangle = 0$ and $A \otimes T(\mathrm{d} \otimes \mathrm{Id}M_s^\circ)_{1:} \equiv 0$. Hence,

$$\mathcal{I}_{11} = \left\langle A \otimes T(M_s^\perp)_{1:}, \mathrm{Id} \otimes P(M_s^\perp)_{1:} \right\rangle$$
$$= \left\langle \mathrm{d} \otimes T(M_s^\perp)_{1:}, \mathrm{d} \otimes P(M_s^\perp)_{1:} \right\rangle$$
$$\geq C_1\|\mathrm{d} \otimes \mathrm{Id}(M_s^\perp)_{1:}\|_{L^2\Lambda^{k+1}\otimes V_k}^2.$$

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The last step follows from Proposition 2.1. If $i = 1$ and $j = 2$ we find

$$\mathcal{I}_{12} = \left\langle B^* \otimes T(M_s^{\otimes 2})_{2:}, \pi^\perp \otimes P(M_s^{\otimes 2})_{1:} + \mathrm{d}\pi^\perp \otimes P(M_s^{\otimes 2})_{2:} \right\rangle. \tag{65}$$

Since $\pi^\perp \otimes P(M_s^{\otimes 2})_{1:} \in \mathfrak{B}_k^{2,\perp}$, $\left\langle B^* \otimes T(M_s^{\otimes 2})_{2:}, \pi^\perp \otimes P(M_s^{\otimes 2})_{1:} \right\rangle = 0$. Hence,

$$\begin{aligned}
\mathcal{I}_{12} &= \left\langle B^* \otimes T(M_s^\perp)_{2:}, \mathrm{d} \otimes P(M_s^\perp)_{2:} \right\rangle \\
&= \left\langle \mathrm{d} \otimes T(M_s^\perp)_{2:}, \mathrm{d} \otimes P(M_s^\perp)_{2:} \right\rangle \\
&\geq C_1 \| \mathrm{d} \otimes \mathrm{Id}(M_s^\perp)_{2:} \|^2_{L^2 \Lambda^k \otimes V_k}.
\end{aligned}$$

If $i = 2$ and $j = 1$ we find

$$\mathcal{I}_{21} = \left\langle B \otimes T(M_s^{\otimes 2})_{1:}, \gamma \pi^\circ \otimes P(M_s^{\otimes 2})_{1:} - \mathrm{d}\pi^\circ \otimes P(M_s^{\otimes 2})_{2:} \right\rangle. \tag{66}$$

Since $\left\langle B \otimes T(M_s^{\otimes 2})_{1:}, \mathrm{d}\pi^\circ \otimes P(M_s^{\otimes 2})_{2:} \right\rangle = 0$, and $\left\langle B \otimes T(M_s^\perp)_{1:}, \mathrm{Id} \otimes P(M_s^\circ)_{1:} \right\rangle = 0$, we have:

$$\begin{aligned}
\mathcal{I}_{21} &= \gamma \left\langle B \otimes T(\mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{1:}), \mathrm{Id} \otimes P(M_s^\circ)_{1:} \right\rangle \\
&= \gamma \left\langle \mathrm{d} \otimes T(M_s^\circ)_{1:}, \mathrm{d} \otimes P(M_s^\circ)_{1:} \right\rangle \\
&\geq \gamma C_1 \| \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{1:} \|^2_{L^2 \Lambda^k \otimes V_k}.
\end{aligned}$$

If $i = j = 2$

$$\begin{aligned}
\mathcal{I}_{22} &= -\alpha \left\langle \mathrm{Id} \otimes T(M_s^{\otimes 2})_{2:}, \gamma \pi^\circ \otimes P(M_s^{\otimes 2})_{1:} - \mathrm{d}\pi^\circ \otimes P(M_s^{\otimes 2})_{2:} \right\rangle \\
&= \alpha \left\langle \mathrm{Id} \otimes T(M_s^{\otimes 2})_{2:}, \mathrm{d}\pi^\circ \otimes P(M_s^{\otimes 2})_{2:} \right\rangle \tag{67} \\
&\quad - \alpha \left\langle \mathrm{Id} \otimes T(M_s^{\otimes 2})_{2:}, \gamma \pi^\circ \otimes P(M_s^{\otimes 2})_{1:} \right\rangle. \tag{68}
\end{aligned}$$

Since $\left\langle \mathrm{Id} \otimes T(M_s^\perp)_{2:}, \mathrm{d}\pi^\circ \otimes P(M_s^{\otimes 2})_{2:} \right\rangle = 0$, we find

$$\begin{aligned}
(67) &= \alpha \left\langle \mathrm{d} \otimes T(M_s^\circ)_{2:}, \mathrm{d} \otimes P(M_s^\circ)_{2:} \right\rangle \\
&\geq \alpha C_1 \| \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{2:} \|^2_{L^2 \Lambda^k \otimes V_k}.
\end{aligned}$$

Moreover, since $\left\langle \mathrm{Id} \otimes T(\mathrm{d}\pi^\circ \otimes \mathrm{Id}(M_s^{\otimes 2})_{2:}), \pi^\circ \otimes P(M_s^{\otimes 2})_{1:} \right\rangle = 0$, we find

$$\begin{aligned}
(68) &= -\alpha\gamma \left\langle \mathrm{Id} \otimes T(M_s^\perp)_{2:}, \mathrm{Id} \otimes P(M_s^\circ)_{1:} \right\rangle \\
&\geq -\frac{\alpha}{2}\gamma^{1/2} \left(\| \mathrm{Id} \otimes T(M_s^\perp) \|^2_{L^2 \Lambda^{k-1} \otimes V_k'} + \gamma \| \mathrm{Id} \otimes P(M_s^\circ)_{1:} \|^2_{L^2 \Lambda^k \otimes V_k} \right) \\
&\geq -\frac{\alpha}{2}\gamma^{1/2} \left(C_{P,1}^2 \|T\|^2_{\mathcal{L}(V_k, V_k')} \| \mathrm{d} \otimes \mathrm{Id}(M_s^\perp)_{2:} \|^2_{L^2 \Lambda^k \otimes V_k} \right. \\
&\quad \left. + \gamma C_{P,1}^2 \|P\|^2_{\mathcal{L}(V_k, V_k)} \| \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{1:} \|^2_{L^2 \Lambda^{k+1} \otimes V_k} \right),
\end{aligned}$$

where we used Proposition 4.1 and Lemma 4.1. Using the lower bounds on $\mathcal{I}_{11}$, $\mathcal{I}_{12}$, $\mathcal{I}_{21}$ and $\mathcal{I}_{22}$, we can now conclude that:

$$\begin{aligned}
(62) &\geq C_1 \| \mathrm{d} \otimes \mathrm{Id}(M_s^\perp)_{1:} \|^2_{L^2 \Lambda^{k+1} \otimes V_k} \\
&\quad + \left(C_1 - \frac{\alpha}{2}\gamma^{1/2} C_{P,1}^2 \|T\|^2_{\mathcal{L}(V_k, V_k')} \right) \| \mathrm{d} \otimes \mathrm{Id}(M_s^\perp)_{2:} \|^2_{L^2 \Lambda^k \otimes V_k} \\
&\quad + \gamma \left(C_1 - \frac{\alpha}{2}\gamma^{1/2} C_{P,1}^2 \|P\|^2_{\mathcal{L}(V_k, V_k)} \right) \| \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{1:} \|^2_{L^2 \Lambda^k \otimes V_k} \\
&\quad + \alpha C_1 \| \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{2:} \|^2_{L^2 \Lambda^k \otimes V_k}.
\end{aligned}$$

Hence, if we choose γ sufficiently small, condition (51) is satisfied for $m = 2$. Now suppose that the problem for the $(m-1)$-th moment is well-posed, and in particular that the inf-sup condition is verified with the test function $M_t^{\otimes(m-1)} = P^{\otimes(m-1)} M_s^{\otimes(m-1)}$:

$$\left\langle T^{\otimes(m-1)} M_s^{\otimes(m-1)}, P^{\otimes(m-1)} M_s^{\otimes(m-1)} \right\rangle \geq C_{m-1} \| M_s^{\otimes(m-1)} \|_{V_k^{\otimes(m-1)}}^2, \qquad (69)$$

where $C_{m-1} = C_{m-1}(C_{P,1}, \alpha, \|T\|, \|P\|) > 0$. We want to prove (51). As before, we fix $M_s^{\otimes m} = \begin{bmatrix} (M_s^{\otimes m})_{1:} \\ (M_s^{\otimes m})_{2:} \end{bmatrix}$ where $(M_s^{\otimes m})_{1:}$ (respectively $(M_s^{\otimes m})_{2:}$) means that in the tensor of order m, $M_s^{\otimes m} = (M_s^{\otimes m})_{i_1 \dots i_m = 1,2}$, we fix $i_1 = 1$ (respectively $i_1 = 2$) and let $i_2, \dots, i_m$ vary. Using (53) and (57) we decompose

$$M_s^{\otimes m} = \begin{bmatrix} (M_s^{\perp})_{1:} + \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\circ})_{1:} \\ (M_s^{\perp})_{2:} + \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\circ})_{2:} \end{bmatrix} \in \begin{bmatrix} U_k^m \\ U_{k-1}^m \end{bmatrix},$$

where now

$$(M_s^{\perp})_{1:} = \pi^{\perp} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\otimes m})_{1:} \in \mathfrak{B}_k^{m,\perp}$$
$$(M_s^{\perp})_{2:} = \pi^{\perp} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\otimes m})_{1:} \in \mathfrak{B}_{k-1}^{m,\perp}$$
$$(M_s^{\circ})_{1:} = \pi^{\perp} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\otimes m})_{1:} \in \mathfrak{B}_{k-1}^{m,\perp}$$
$$(M_s^{\circ})_{2:} = \pi^{\perp} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\otimes m})_{1:} \in \mathfrak{B}_{k-2}^{m,\perp}.$$

We choose $M_t^{\otimes m} = P^{\otimes m} M_s^{\otimes m}$, so that:

$$\left\langle T^{\otimes m} M_s^{\otimes m}, M_t^{\otimes m} \right\rangle = \left\langle T^{\otimes m} M_s^{\otimes m}, P^{\otimes m} M_s^{\otimes m} \right\rangle$$
$$= \sum_{i,j=1}^{2} \left\langle T_{i,j} \otimes T^{m-1} (M_s^{\otimes m})_{j:}, (P^{\otimes m} M_s^{\otimes m})_{i:} \right\rangle. \qquad (70)$$

Let $\mathcal{J}_{ij} = \left\langle T_{i,j} \otimes T^{m-1} (M_s^{\otimes m})_{j:}, (P^{\otimes m} M_s^{\otimes m})_{i:} \right\rangle$. We follow a completely similar reasoning as before, and we apply (69). If $i = j = 1$,

$$\mathcal{J}_{11} = \left\langle A \otimes T^{\otimes(m-1)} (M_s^{\otimes m})_{1:}, (P \otimes P^{\otimes(m-1)} M_s^{\otimes m})_{1:} \right\rangle$$
$$\geq C_{m-1} \| \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\perp})_{1:} \|_{L^2 \Lambda^{k+1} \otimes V_k^{\otimes(m-1)}}^2.$$

If $i = 1$ and $j = 2$,

$$\mathcal{J}_{12} = \left\langle B^* \otimes T^{\otimes(m-1)} (M_s^{\otimes m})_{2:}, (P \otimes P^{\otimes(m-1)} M_s^{\otimes m})_{1:} \right\rangle$$
$$\geq C_{m-1} \| \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\perp})_{2:} \|_{L^2 \Lambda^k \otimes V_k^{\otimes(m-1)}}^2.$$

If $i = 2$ and $j = 1$,

$$\mathcal{J}_{21} = \left\langle B \otimes T^{\otimes(m-1)} (M_s^{\otimes m})_{1:}, (P \otimes P^{\otimes(m-1)} M_s^{\otimes m})_{2:} \right\rangle$$
$$\geq \gamma C_{m-1} \| \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\circ})_{1:} \|_{L^2 \Lambda^k \otimes V_k^{\otimes(m-1)}}^2.$$

If $i = j = 2$,

$$
\begin{aligned}
\mathcal{J}_{22} = {} & -\alpha \left\langle \mathrm{Id} \otimes T^{\otimes(m-1)} (M_s^{\otimes m})_{2:}, (P \otimes P^{\otimes(m-1)} M_s^{\otimes m})_{2:} \right\rangle \\
\geq {} & \alpha C_{m-1} \|\mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\circ})_{2:}\|^2_{L^2\Lambda^k \otimes V_k^{\otimes(m-1)}} + \\
& -\frac{\alpha}{2} \gamma^{1/2} \left(C_{P,1}^2 \|T\|^{2(m-1)}_{\mathcal{L}(V_k, V_k')} \|\mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\perp})_{2:}\|^2_{L^2\Lambda^k \otimes V_k^{\otimes(m-1)}} + \right. \\
& \left. +\gamma C_{P,1}^2 \|P\|^{2(m-1)}_{\mathcal{L}(V_k, V_k)} \|\mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\circ})_{1:}\|^2_{L^2\Lambda^k \otimes V_k^{\otimes(m-1)}} \right).
\end{aligned}
$$

Hence, if we choose γ sufficiently small, condition (51) is satisfied. Relation (52) follows from the orthogonal decomposition (57) and the tensorial Poincaré inequality in Lemma 4.1. $\qquad \square$

Another way to express the result given in Theorem 4.2 is the following

Proposition 4.2 *Given T as in (17) and P as in (28), $\forall M_s^m$ it holds*

$$
\left\langle T^{\otimes m} M_s^{\otimes m}, P^{\otimes m} M_s^{\otimes m} \right\rangle_{(V_k')^{\otimes m}, V_k^{\otimes m}} \geq C_m \left\| M_s^{\otimes m} \right\|^2_{V_k^{\otimes m}}.
$$

As a simple consequence of Proposition 4.1 we have also the bound on $P^{\otimes m}$.

Remark 4.3 *We underline that the operator P is not the classical one presented in [3] to prove the well-posedness of the deterministic Hodge-Laplace problem. Indeed it is the minimal one such that the inf-sup condition for $\langle T^{\otimes m}\cdot, \cdot \rangle : V_k^{\otimes m} \times V_k^{\otimes m} \to \mathbb{R}$ (for every finite $m \geq 1$) is satisfied. With the classical operator, the inf-sup condition for $m \geq 2$ is not automatically satisfied.*

5 Some three-dimensional problems important in applications

In Section 2.3.1 we have reinterpreted the deterministic Hodge-Laplace problem in $n = 3$ dimensions in terms of PDEs. Here we translate in terms of partial differential equations the stochastic Hodge-Laplace problem. In particular, we focus on the two problems obtained for $k = 1$ and $k = 3$: the stochastic magnetostatic/electrostatic equations and the stochastic Darcy equations, and we explicitly write the systems solved by the mean and the two-points correlation of the unique stochastic solution of the stochastic problem.

5.1 The stochastic magnetostatic/electrostatic equations

Take $k = 1$ and $\alpha = 0$. Let $f_1 \in L^m \left(\Omega; L^2\Lambda^1(D)\right)$, $f_2 \in L^m \left(\Omega; L^2\Lambda^0(D)\right)$ be stochastic functions with, $m \geq 1$ integer, representing an uncertain current and an uncertain charge respectively. The stochastic magnetostatic/electrostatic problem is the stochastic counterpart of problem (35). Thanks to Theorem 3.1, the stochastic magnetostatic/electrostatic problem admits a unique stochastic

solution that depends continuously on the data. If $m \geq 1$, the first statistical moment $\mathcal{M}^1 \begin{bmatrix} u \\ p \end{bmatrix} = \mathbb{E} \begin{bmatrix} u \\ p \end{bmatrix}$ is well-defined, and is the unique solution of (see (44)): find $E_s = \begin{bmatrix} E_{s,1} \\ E_{s,2} \end{bmatrix} \in V_1$ such that

$$\begin{cases} (\text{curl } E_{s,1}, \text{curl } v) + (\nabla E_{s,2}, v) = (\mathbb{E}\,[f_1]\,, v) \\ (E_{s,1}, \nabla q) = (\mathbb{E}\,[f_2]\,, q)\,. \end{cases} \qquad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_1, \qquad (71)$$

where the parenthesis in (71) mean the L^2-inner product. In the case $m \geq 2$, the second statistical moment $\mathcal{M}^2 \begin{bmatrix} u \\ p \end{bmatrix}$ is well-defined, and is the unique solution of (see (49) with $m = 2$): find

$$M_s^{\otimes 2} \in V_1 \otimes V_1 = \begin{bmatrix} H_{\Gamma_D}(\text{curl}\,, D) \otimes H_{\Gamma_D}(\text{curl}\,, D) & H_{\Gamma_D}(\text{curl}\,, D) \otimes H^1_{\Gamma_D}(D) \\ H^1_{\Gamma_D}(D) \otimes H_{\Gamma_D}(\text{curl}\,, D) & H^1_{\Gamma_D}(D) \otimes H^1_{\Gamma_D}(D) \end{bmatrix}$$

such that

$$\begin{cases}
\begin{aligned}
& \big(\text{curl } \otimes \text{curl } (M_s^{\otimes 2})_{11}, \text{curl } \otimes \text{curl } (M_t^{\otimes 2})_{11}\big) + \\
& \quad \big(\text{curl } \otimes \nabla(M_s^{\otimes 2})_{12}, \text{curl } \otimes \text{Id}(M_t^{\otimes 2})_{11}\big) + \\
& \quad \big(\nabla \otimes \text{curl } (M_s^{\otimes 2})_{21}, \text{Id} \otimes \text{curl } (M_t^{\otimes 2})_{11}\big) + \big(\nabla \otimes \nabla(M_s^{\otimes 2})_{22}, (M_t^{\otimes 2})_{11}\big) = \\
& \quad \big(\mathcal{M}^2\,[f_1]\,, (M_t^{\otimes 2})_{11}\big) \\[6pt]
& - \big(\text{curl } \otimes \text{Id}(M_s^{\otimes 2})_{11}, \text{curl } \otimes \nabla(M_t^{\otimes 2})_{12}\big) - \\
& \quad \big(\nabla \otimes \text{Id}(M_s^{\otimes 2})_{12}, \text{Id} \otimes \nabla(M_t^{\otimes 2})_{12}\big) = \big(\mathbb{E}\,[f_1 f_2]\,, (M_t^{\otimes 2})_{12}\big) \\[6pt]
& - \big(\text{Id} \otimes \text{curl } (M_s^{\otimes 2})_{12}, \nabla \otimes \text{curl } (M_t^{\otimes 2})_{21}\big) - \\
& \quad \big(\text{Id} \otimes \nabla M_s^{\otimes 2})_{21}, \nabla \otimes \text{Id}(M_t^{\otimes 2})_{21}\big) = \big(\mathbb{E}\,[f_2 f_1]\,, (M_t^{\otimes 2})_{21}\big) \\[6pt]
& \big((M_s^{\otimes 2})_{11}, \nabla \otimes \nabla(M_t^{\otimes 2})_{22}\big) = \big(\mathcal{M}^2\,[f_2]\,, (M_t^{\otimes 2})_{22}\big)
\end{aligned}
\end{cases}$$
$$(72)$$

$\forall M_t^{\otimes 2} \in V_1 \otimes V_1$, where the parenthesis in (72) have to be intended as inner product in $(L^2(D))^3 \otimes (L^2(D))^3$.

5.2 The stochastic Darcy problem

Let $k = 3$, $f_2 \equiv 0$ and $f_1 \in L^m\left(\Omega; L^2 \Lambda^3(D)\right)$, $m \geq 1$ integer, representing an uncertain source in porous media flow. The stochastic Darcy problem is the stochastic counterpart of problem (38). Thanks to Theorem 3.1, the stochastic Darcy problem admits a unique stochastic solution that depends continuously on the data. If $m \geq 1$, the first statistical moment $\mathcal{M}^1 \begin{bmatrix} u \\ p \end{bmatrix} = \mathbb{E} \begin{bmatrix} u \\ p \end{bmatrix}$ is well-defined, and is the unique solution of (see (44)): find $E_s = \begin{bmatrix} E_{s,1} \\ E_{s,2} \end{bmatrix} \in V_3$

such that

$$\begin{cases} (\text{div } E_{s,2}, v) = (\mathbb{E}\,[f_1]\,, v) \\ (E_{s,1}, \text{div } q) - \alpha\,(E_{s,2}, q) = 0 \end{cases} \qquad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_3. \qquad (73)$$

where the parenthesis in (73) mean the L^2-inner product. In the case $m \geq 2$, the second statistical moment $\mathcal{M}^2 \begin{bmatrix} u \\ p \end{bmatrix}$ is well-defined, and is the unique solution of (see (49) with $m = 2$): find

$$M_s^{\otimes 2} \in V_3 \otimes V_3 = \begin{bmatrix} L^2(D) \otimes L^2(D) & L^2(D) \otimes H_{\Gamma_D}(\text{div }, D) \\ H_{\Gamma_D}(\text{div }, D) \otimes L^2(D) & H_{\Gamma_D}(\text{div }; D) \otimes H_{\Gamma_D}(\text{div }; D) \end{bmatrix}$$

such that

$$\begin{cases} \big(\text{div } \otimes \text{div } (M_s^{\otimes 2})_{22}, (M_t)_{11}\big) = \big(\mathcal{M}^2\,[f_1]\,, (M_t)_{11}\big) \\[2mm] \big(\text{div } \otimes \text{Id}(M_s^{\otimes 2})_{21}, \text{Id} \otimes \text{div } (M_t^{\otimes 2})_{12}\big) - \alpha\,\big(\text{div } \otimes \text{Id}(M_s^{\otimes 2})_{22}, (M_t^{\otimes 2})_{12}\big) = 0 \\[2mm] \big(\text{Id} \otimes \text{div } (M_s^{\otimes 2})_{12}, \text{div } \otimes \text{Id}(M_t^{\otimes 2})_{21}\big) - \alpha\,\big(\text{Id} \otimes \text{div } (M_s^{\otimes 2})_{22}, (M_t^{\otimes 2})_{21}\big) = 0 \\[2mm] \big((M_s^{\otimes 2})_{11}, \text{div } \otimes \text{div } (M_t^{\otimes 2})_{22}\big) - \alpha\,\big((M_s^{\otimes 2})_{12}, \text{div } \otimes \text{Id}(M_t^{\otimes 2})_{22}\big) \\ \quad - \alpha\,\big((M_s^{\otimes 2})_{21}, \text{Id} \otimes \text{div } (M_t^{\otimes 2})_{22}\big) + \alpha^2\,\big((M_s^{\otimes 2})_{22}, (M_t^{\otimes 2})_{22}\big) = 0 \end{cases}$$
$$(74)$$

$\forall M_t^{\otimes 2} \in V_3 \otimes V_3$, where the parenthesis in (72) have to be intended as inner product in $L^2(D) \otimes L^2(D)$.

6 Finite element discretization of the moment equations

In this section we aim to derive a stable discretization for the moment equations, i.e. the deterministic problems solved by the statistics of the unique stochastic solution $\begin{bmatrix} u \\ p \end{bmatrix}$. First we recall the main concepts concerning the finite element differential forms and the existence of a stable finite element discretization for the mean problem (44). Then we construct both a full and a sparse tensor product finite element discretization for the m-th problem, with $m \geq 2$ integer, we prove their stability and provide optimal order of convergence estimates.

6.1 Finite element differential forms

Following [3], throughout this section we assume that the domain $D \subset \mathbb{R}^n$ satisfying Assumption 2.1 is a polyhedral domain in $\mathbb{R}^n$ which is partitioned into a finite set of n-simplices. These simplices are such that their union is the closure of D and the intersection of any two of them, if non-empty, is a common

$k = 0$	$\mathcal{P}_r^-\Lambda^0(\mathcal{T}_h)$	Lagrangian elements of degree $\leq r$
$k = 1$	$\mathcal{P}_r^-\Lambda^1(\mathcal{T}_h)$	Nédélec 1-nd kind $H(\mathrm{curl}\ )$ elements of order $r - 1$
$k = 2$	$\mathcal{P}_r^-\Lambda^2(\mathcal{T}_h)$	Nédélec 1-nd kind $H(\mathrm{div}\ )$ elements of order $r - 1$
$k = 3$	$\mathcal{P}_r^-\Lambda^3(\mathcal{T}_h)$	Discontinuous elements of degree $\leq r - 1$

Table 2: Proxy fields correspondences between finite element differential forms $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ and the classical finite element spaces for $n = 3$.

sub simplex. We denote the partition with $\mathcal{T}_h$ and the discretization parameter with h. To discretize the moment equations we use the finite element differential forms

$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) = \left\{ v \in H\Lambda^k(D)|\ v|_T \in \mathcal{P}_r^-\Lambda^k(T)\ \forall\ T \in \mathcal{T}_h \right\}, \qquad (75)$$

where the space $\mathcal{P}_r^-\Lambda^k(T)$ and the de Rham subcomplex

$$0 \to \mathcal{P}_r^-\Lambda^0(\mathcal{T}_h) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{P}_r^-\Lambda^n(\mathcal{T}_h) \to 0$$

are treated in [3, 25]. Since we are particularly interested in the $n = 3$ case, we resume in Table 2 the correspondences between the finite element differential forms (75) and the classical finite element spaces of scalar and vector functions. The spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ are not the only choice. Indeed, in [3, 4, 25, 14] the authors present other finite element differential forms to discretize the deterministic Hodge Laplacian.

In [4] the authors propose the construction of a projector $\Pi_{k,h} : H\Lambda^k(D) \to \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ which is a cochain map, that is it commutes with the exterior derivative, and such that the following approximation property holds:

$$\|v - \Pi_{k,h}v\|_{H\Lambda^k(D)} \leq Ch^s\|v\|_{H^{s+1}\Lambda^k(D)}, \quad \forall\ v \in H^{s+1}\Lambda^k(D),\ 0 \leq s \leq r, \quad (76)$$

where $H^s\Lambda^k(D)$ is the space of differential k-forms with square integrable partial derivatives of order at most s, and C is independent of h. Moreover, $\Pi_{k,h}$ is bounded by a constant C_π independent of h:

$$\|\Pi_{k,h}v\|_{H\Lambda^k} \leq C_\pi\|v\|_{H\Lambda^k} \quad \forall\ v \in H_{\Gamma_D}\Lambda^k(D). \qquad (77)$$

Since we are dealing with Dirichlet boundary conditions on Γ_D, we make the following

Assumption 6.1 *There exists a bounded (see (77)) cochain projector, that by abuse of notation we denote still by $\Pi_{k,h}$,*

$$\Pi_{k,h} : H_{\Gamma_D}\Lambda^k(D) \to \mathcal{P}_{r,\Gamma_D}^-\Lambda^k(\mathcal{T}_h) := \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) \cap H_{\Gamma_D}\Lambda^k(D), \qquad (78)$$

such that (76) is satisfied for every $v \in H^{s+1}\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D),\ 0 \leq s \leq r$.

Assumption 6.1 is satisfied in the two and three dimensional case: see [39]. The n dimensional case is still a topic of current research, whereas if natural boundary conditions are imposed on ∂D, the existence of such an operator is proved in [3], and if essential boundary conditions are imposed on ∂D, the existence of such an operator is proved in [15].

6.2   Discrete mean problem

The problem solved by the mean of the unique stochastic solution of the stochastic Hodge Laplacian turns out to be the deterministic Hodge Laplacian. In [3] the authors study the finite element formulation of the deterministic Hodge Laplacian with natural boundary conditions on ∂D ($\Gamma_D = \emptyset$). In [4] all the results obtained in [3] for $\Gamma_D = \emptyset$ are extended to include the case of essential boundary conditions on ∂D ($\Gamma_N = \emptyset$). Under Assumption 6.1, all the results in [3, 4] apply to the general case $\Gamma_D,\ \Gamma_N \neq \emptyset$.

Let $\left(\mathcal{P}_{r,\Gamma_D}^{-}\Lambda^k(\mathcal{T}_h), \mathrm{d}\right)$ be the finite element de Rham subcomplex, h the discretization parameter, and $V_{k,h} = \left[\begin{array}{c} \mathcal{P}_{r,\Gamma_D}^{-}\Lambda^k(\mathcal{T}_h) \\ \mathcal{P}_{r,\Gamma_D}^{-}\Lambda^{k-1}(\mathcal{T}_h) \end{array} \right]$. The finite element formulation of problem (44) is:

Mean Problem - FE Formulation

$$
\boxed{
\begin{array}{l}
Given \ \left[\begin{array}{c} F_1 \\ F_2 \end{array} \right] \in L^1\left(\Omega; V_k'\right),\ find\ E_{s,h} \in V_{k,h}\ s.t. \\[4mm]
T(E_{s,h}) = \mathbb{E}\left[\begin{array}{c} F_1 \\ F_2 \end{array} \right] \ in\ V_{k,h}'.
\end{array}
}
\tag{79}
$$

In [3] the authors show the stability of (79) by proving the inf-sup condition for the bounded bilinear and symmetric form $\langle T\cdot, \cdot \rangle$ restricted to the finite element spaces. Moreover, using a quasi-optimal error estimate and the interpolation property (76), the authors deduce the following order of convergence estimate:

$$
\left\| \mathbb{E}\left[\begin{array}{c} u \\ p \end{array} \right] - E_{s,h} \right\|_{V_k} = \mathcal{O}(h^r),\ for\ \mathbb{E}\left[\begin{array}{c} u \\ p \end{array} \right] \in \left[\begin{array}{c} H^{r+1}\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \\ H^{r+1}\Lambda^{k-1}(D) \cap H_{\Gamma_D}\Lambda^{k-1}(D) \end{array} \right],
\tag{80}
$$

where $\mathbb{E}\left[\begin{array}{c} u \\ p \end{array} \right]$ and $E_{s,h}$ are the unique solutions of problems (44) and (79) respectively.

6.3   Discrete m-th moment problem: full tensor product approximation

The full tensor product finite element formulation (FTP-FE) of problem (49) is:

m-Points Correlation Problem (FTP-FE):

$$\boxed{\begin{array}{c} \text{Given } m \geq 2 \text{ integer and } \left[\begin{array}{c} F_1 \\ F_2 \end{array} \right] \in L^m\left(\Omega; V_k'\right), \text{ find } M_{s,h}^{\otimes m} \in V_{k,h}^{\otimes m} \text{ s.t.} \\[2em] T^{\otimes m} M_{s,h}^{\otimes m} = \mathcal{M}^m \left[\begin{array}{c} F_1 \\ F_2 \end{array} \right] \text{ in } (V_{k,h}')^{\otimes m} \end{array}}$$

$$(81)$$

Theorem 4.1 applies to problem (81), as a consequence of a tensor product structure (see Remark 4.2). We conclude therefore the stability of the full tensor product finite element discretization $V_{k,h}^{\otimes m}$.

Let $\mathcal{M}^m \left[\begin{array}{c} u \\ p \end{array} \right]$ be the unique solution of problem (49) and $M_{s,h}^{\otimes m}$ be the unique solution of problem (81). Exploiting the Galerkin orthogonality and the stability of the discretization, we can obtain the following quasi-optimal convergence estimate:

$$\left\| \mathcal{M}^m \left[\begin{array}{c} u \\ p \end{array} \right] - M_{s,h}^{\otimes m} \right\|_{V_k^{\otimes m}} \leq C \inf_{M_h^{\otimes m} \in V_{k,h}^{\otimes m}} \left\| \mathcal{M}^m \left[\begin{array}{c} u \\ p \end{array} \right] - M_h^{\otimes m} \right\|_{V_k^{\otimes m}}. \quad (82)$$

To study the approximation properties of the space $V_{k,h}^{\otimes m}$ we construct the tensorial projection operator $\Pi_{k,h}^{\otimes m}$ as follows.

Definition 6.1 *Let* $\Pi_{k,h} : H_{\Gamma_D}\Lambda^k(D) \rightarrow \mathcal{P}_{r,\Gamma_D}^-\Lambda^k(\mathcal{T}_h)$ *be a bounded cochain projector satisfying Assumption 6.1. Given* $m \geq 2$ *integer, we define*

$$\Pi_{k,h}^{\otimes m} := \underbrace{\Pi_{k,h} \otimes \ldots \otimes \Pi_{k,h}}_{m \text{ times}} : \left(H_{\Gamma_D}\Lambda^k(D) \right)^{\otimes m} \rightarrow \left(\mathcal{P}_{r,\Gamma_D}^-\Lambda^k(\mathcal{T}_h) \right)^{\otimes m}. \quad (83)$$

Since $\Pi_{k,h}$ is bounded in $H\Lambda^k$-norm by C_π, $\Pi_{k,h}^{\otimes m}$ is bounded in $(H\Lambda^k)^{\otimes m}$-norm by $(C_\pi)^m$ (Proposition 4.1). Moreover, since it is the tensor product of cochain projectors, it is itself a cochain projector. We state the approximation properties of $\Pi_{k,h}^{\otimes m}$ in the following

Proposition 6.1 *The projector* $\Pi_{k,h}^{\otimes m}$ *introduced in Definition 6.1 is such that*

$$\|v - \Pi_{k,h}^{\otimes m} v\|_{(H\Lambda^k)^{\otimes m}} \leq Ch^s \|v\|_{(H^{s+1}\Lambda^k)^{\otimes m}} \quad (84)$$

for all $v \in (H^{s+1}\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D))^{\otimes m}$, $0 \leq s \leq r$, *where* C *is independent of* h.

Proof. We already know the result for $m = 1$ (see (76)). Let $m = 2$. By triangle inequality,

$$\|v - \Pi_{k,h}^{\otimes 2} v\|_{H\Lambda^k \otimes H\Lambda^k}$$
$$\leq \|v - \Pi_{k,h} \otimes \mathrm{Id}\ v\|_{H\Lambda^k \otimes H\Lambda^k} + \|\Pi_{k,h} \otimes (\mathrm{Id} - \Pi_{k,h})\ v\|_{H\Lambda^k \otimes H\Lambda^k}$$
$$\leq Ch^s \|v\|_{H^{s+1}\Lambda^k \otimes H\Lambda^k} + C_\pi \|v - \mathrm{Id} \otimes \Pi_{k,h} v\|_{H\Lambda^k \otimes H\Lambda^k}$$
$$\leq Ch^s \|v\|_{H^{s+1}\Lambda^k \otimes H\Lambda^k} + CC_\pi h^s \|v\|_{H\Lambda^k \otimes H^{s+1}\Lambda^k}$$
$$\leq Ch^s (1 + C_\pi) \|v\|_{H^{s+1}\Lambda^k \otimes H^{s+1}\Lambda^k},$$

where we used (76). By induction on m, we conclude (84). $\qquad\square$

From the approximation properties of the projector $\Pi_{k,h}^{\otimes m}$ (84), it follows

Theorem 6.1 (Order of convergence of the FTP-FE discretization)

$$\left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{s,h}^{\otimes m} \right\|_{V_k^{\otimes m}} = \mathcal{O}(h^r), \tag{85}$$

provided that $\begin{bmatrix} u \\ p \end{bmatrix} \in L^m \left(\Omega; \begin{bmatrix} H^{r+1}\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \\ H^{r+1}\Lambda^{k-1}(D) \cap H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix} \right).$

6.4 Discrete m-th moment problem: sparse tensor product approximation

In Section 6.3 we proved the stability of the full tensor product finite element discretization $V_{k,h}^{\otimes m} = \underbrace{V_{k,h} \otimes \ldots \otimes V_{k,h}}_{m \text{ times}}$. The main problem of this approach is that it is strongly affected by the curse of dimensionality. Indeed, if $\dim(V_{k,h}) = N_h$, the space $V_{k,h}^{\otimes m}$ has dimension $(N_h)^m$ which is impractical for m moderately large. A reduction in the dimensionality of the problem is possible if we consider a sparse tensor product finite element (STP-FE) approximation instead (see e.g. [43, 41, 12, 40, 23] and the references therein).

Let $\mathcal{T}_0$ be a regular mesh of the physical domain $D \subset \mathbb{R}^n$, and $\{\mathcal{T}_l\}_{l=0}^\infty$ be a sequence of partitions obtained by uniform mesh refinement, that is $h_l = h_{l-1}/2$, where h_l is the discretization parameter of $\mathcal{T}_l$. We have a sequence $\left\{ \mathcal{P}_r^- \Lambda^k(\mathcal{T}_l) \right\}_{l=0}^\infty$ of finite dimensional subspaces of the space V_k, which are nested and dense in V_k. Let us define the orthogonal complement of $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_{l-1})$ in $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_l)$: $S_{k,l} = \mathcal{P}_r^- \Lambda^k(\mathcal{T}_l) \setminus \mathcal{P}_r^- \Lambda^k(\mathcal{T}_{l-1})$, and set $Z_{k,l} = \begin{bmatrix} S_{k,l} \\ S_{k-1,l} \end{bmatrix}$. For every integer $m \geq 2$, we define the sparse tensor product finite element space of level $L > 0$, $V_{k,L}^{(m)}$, as:

$$V_{k,L}^{(m)} := \bigoplus_{|\underline{l}| \leq L} \left(Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_m} \right), \tag{86}$$

where $\underline{l}$ is a multi index in $\mathbb{N}_0^m$ and $|\underline{l}|$ is its length $l_1 + \ldots + l_m$. At the numerical level it may not be needed to explicitly build a basis for $Z_{k,l}$. In [22] the authors

propose to use a redundant basis for the space (86) and an algorithm to solve the m-th moment problem in the sparse tensor product framework.

The sparse tensor product finite element (STP-FE) approximation of problem (49) is:

m-Points Correlation Problem (STP-FE):

$$
\boxed{
\begin{array}{c}
\text{Given } m \geq 2 \text{ integer and } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m\left(\Omega; V_k\right), \text{ find } M_{s,L}^{(m)} \in V_{k,L}^{(m)} \text{ s.t.} \\[2ex]
T^{\otimes m} M_{s,L}^{(m)} = \mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad in \ \left(V_{k,L}^{(m)}\right)'
\end{array}
}
$$
(87)

Let us fix $j \in \{1, \ldots, m\}$. We observe that:

$$
V_{k,L}^{(m)} := \bigoplus_{|\underline{l}| \leq L} \left(Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_m}\right)
$$

$$
= \bigoplus_{\sum_{i \neq j} l_i \leq L} Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_{j-1}} \otimes \left(\bigoplus_{l_j=0}^{\left(L - \sum_{i \neq j} l_i\right)} Z_{k,l_j}\right) \otimes Z_{k,l_{j+1}} \otimes \ldots \otimes Z_{k,l_m}
$$

$$
= \bigoplus_{\sum_{i \neq j} l_i \leq L} \left(Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_{j-1}} \otimes V_{k,L-\sum_{i \neq j} l_i} \otimes Z_{k,l_{j+1}} \otimes \ldots \otimes Z_{k,l_m}\right)
$$

$$
= \bigoplus_{|\underline{l}|=L} \left(Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_{j-1}} \otimes V_{k,l_j} \otimes Z_{k,l_{j+1}} \otimes \ldots \otimes Z_{k,l_m}\right).
$$

Hence, we obtain an equivalent representation of the space $V_{k,L}^{(m)}$:

$$
V_{k,L}^{(m)} = \bigoplus_{|\underline{l}|=L} \left(Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_{j-1}} \otimes V_{k,l_j} \otimes Z_{k,l_{j+1}} \otimes \ldots \otimes Z_{k,l_m}\right) = \begin{bmatrix} U_{k,L,j}^{(m)} \\ U_{k-1,L,j}^{(m)} \end{bmatrix},
$$
(88)

where

$$
U_{k,L,j}^{(m)} := \bigoplus_{|\underline{l}|=L} \left(Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_{j-1}} \otimes \mathcal{P}_r^- \Lambda^k(\mathcal{T}_{l_j}) \otimes Z_{k,l_{j+1}} \otimes \ldots \otimes Z_{k,l_m}\right), \quad (89)
$$

and $U_{k-1,L,j}^{(m)}$ is defined analogously. In [3], the authors prove the discrete Hodge decomposition

$$
\mathcal{P}_r^- \Lambda^k(\mathcal{T}_l) = \mathfrak{B}_{k,l} \oplus \mathfrak{B}_{k,l}^\perp,
$$
(90)

where $\mathfrak{B}_{k,l}$ is the image of d in $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_l)$ and $\mathfrak{B}_{k,l}^\perp$ is its orthogonal complement. Using (90) we can carry the tensorial Hodge decomposition (57) to the discrete level.

Lemma 6.1 (Discrete Tensorial Hodge Decomposition) *For every integer $m \geq 2$, the space $U_{k,L,j}^{(m)}$ admits the following orthogonal decomposition:*

$$U_{k,L,j}^{(m)} = \mathfrak{B}_{k,L,j}^{(m)} \oplus \mathfrak{B}_{k,L,j}^{(m),\perp} \tag{91}$$

where

$$\mathfrak{B}_{k,L,j}^{(m)} := \bigoplus_{|\underline{l}|=L} \left(Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_{j-1}} \otimes \mathfrak{B}_{k,l_j} \otimes Z_{k,l_j} \otimes \ldots \otimes Z_{k,l_m} \right) \tag{92}$$

$$\mathfrak{B}_{k,L,j}^{(m),\perp} := \bigoplus_{|\underline{l}|=L} \left(Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_{j-1}} \otimes \mathfrak{B}_{k,l_j}^{\perp} \otimes Z_{k,l_j} \otimes \ldots \otimes Z_{k,l_m} \right) \tag{93}$$

$$\tag{94}$$

Proof. Let us suppose $j = 1$ w.l.o.g. We need to show that the spaces $\mathfrak{B}_{k,L,1}^{(m)}$ and $\mathfrak{B}_{k,L,1}^{(m),\perp}$ are orthogonal. Let us take $v = \sum_{|\underline{l}|=L} b_{k,l_1} \otimes z_{k,l_2} \otimes \ldots \otimes z_{k,l_m} \in \mathfrak{B}_{k,L,1}^{(m)}$ and $\tilde{v} = \sum_{|\underline{p}|=L} \tilde{b}_{k,p_1} \otimes \tilde{z}_{k,p_2} \otimes \ldots \otimes \tilde{z}_{k,p_m} \in \mathfrak{B}_{k,L,1}^{(m),\perp}$, then

$$(v,\tilde{v}) = \sum_{|\underline{l}|=L} \sum_{|\underline{p}|=L} \left(b_{k,l_1}, \tilde{b}_{k,p_1} \right) \prod_{i=2}^{m} (z_{k,l_i}, \tilde{z}_{k,p_i}) = \sum_{|\underline{l}|=L} \sum_{|\underline{p}|=L} \left(b_{k,l_1}, \tilde{b}_{k,p_1} \right) \prod_{i=2}^{m} \delta_{l_i,p_i}.$$

In the second equality we used that $(z_{k,l_i}, \tilde{z}_{k,p_i}) = 0$ if $l_i \neq p_i$. But $l_2 = p_2, \ldots, l_m = p_m$ imply $l_1 = p_1$. Since b_{l_1} and $\tilde{b}_{l_1}$ belong to orthogonal spaces, then $(v,\tilde{v}) = 0$. By linearity, the previous relation extents to all elements of the spaces $\mathfrak{B}_{k,L,1}^{(m)}$ and $\mathfrak{B}_{k,L,1}^{(m),\perp}$. $\qquad\square$

The tensor product operators $\mathrm{Id}^{\otimes(j-1)} \otimes \pi^{\perp} \otimes \mathrm{Id}^{\otimes(m-j)}$ and $\mathrm{Id}^{\otimes(j-1)} \otimes \pi^{\circ} \otimes \mathrm{Id}^{\otimes(m-j)}$ map each element of the space $U_{k,L,j}^{(m)}$ onto the spaces $U_{k,L,j}^{(m)}$ and $U_{k-1,L,j}^{(m)}$ respectively. Indeed, let us take $j = 1$ w.l.o.g and $v = \sum_{|\underline{l}|=L} p_{k,l_1} \otimes z_{k,l_2} \otimes \ldots \otimes z_{k,l_m} \in U_{k,L,1}^{(m)}$. Using (90), we split v as

$$v = \sum_{|\underline{l}|=L} \mathrm{d}p_{k-1,l_1}^{\circ} \otimes z_{k,l_2} \otimes \ldots \otimes z_{k,l_m} + \sum_{|\underline{l}|=L} p_{k,l_1}^{\perp} \otimes z_{k,l_2} \otimes \ldots \otimes z_{k,l_m},$$

where $\mathrm{d}p_{k-1,l_1}^{\circ} \in \mathfrak{B}_{k,l_1} = \mathrm{d}\mathfrak{B}_{k-1,l_1}^{\perp}$, $p_{k,l_1}^{\perp} \in \mathfrak{B}_{k,l_1}^{\perp}$, $\sum_{|\underline{l}|=L} \mathrm{d}p_{k-1,l_1}^{\circ} \otimes z_{k,l_2} \otimes \ldots \otimes z_{k,l_m} \in \mathfrak{B}_{k,L,1}^{(m)}$ and $\sum_{|\underline{l}|=L} p_{k,l_1}^{\perp} \otimes z_{k,l_2} \otimes \ldots \otimes z_{k,l_m} \in \mathfrak{B}_{k,L,1}^{(m),\perp}$. Then,

$$\left(\pi^{\perp} \otimes \mathrm{Id}^{\otimes(m-1)} \right) v := \sum_{|\underline{l}|=L} \left(p_{k,l_1}^{\perp} \otimes z_{k,l_2} \otimes \ldots \otimes z_{k,l_m} \right) \in \mathfrak{B}_{k,L,1}^{(m),\perp} \subset U_{k,L,1}^{(m)}$$

$$\left(\pi^{\circ} \otimes \mathrm{Id}^{\otimes(m-1)} \right) v := \sum_{|\underline{l}|=L} \left(p_{k,l_1}^{\circ} \otimes z_{k,l_2} \otimes \ldots \otimes z_{k,l_m} \right) \in \mathfrak{B}_{k-1,L,1}^{(m),\perp} \subset U_{k-1,L,1}^{(m)}.$$

By linearity, the previous definitions extent to all elements of the spaces $U_{k,L,1}^{(m)}$. Observe that the operators $\mathrm{Id}^{\otimes(j-1)} \otimes \mathrm{d}\pi^{\perp} \otimes \mathrm{Id}^{\otimes(m-j)}$ and $\mathrm{Id}^{\otimes(j-1)} \otimes \mathrm{d}\pi^{\circ} \otimes$

$\mathrm{Id}^{\otimes(m-j)}$ map each element of the space $U_{k,L,j}^{(m)}$ onto the spaces $U_{k+1,L,j}^{(m)}$ and $U_{k,L,j}^{(m)}$ respectively.

As a consequence, the tensor product operator

$$P^{\otimes m} = \left(P \otimes \mathrm{Id}^{\otimes(m-1)} \right) \left(\mathrm{Id} \otimes P \otimes \mathrm{Id}^{\otimes(m-2)} \right) \ldots \left(\mathrm{Id}^{\otimes(m-1)} \otimes P \right)$$

is well defined on the sparse tensor product finite element space $V_{k,L}^{(m)}$, and it maps $V_{k,L}^{(m)}$ onto $V_{k,L}^{(m)}$.

In [3] the authors prove the discrete Poincaré inequality. It directly follows the discrete tensorial Poincaré inequality.

Lemma 6.2 (Discrete Tensorial Poincaré inequality) *Let $\mathcal{T}_h$ be a regular mesh on the physical domain $D \subset \mathbb{R}^n$. For every integer $m \geq 2$, there exists a positive constant $C_{P,disc,1}$ that depends only on C_π, defined in (77) and $C_{P,1}$, defined in (60), and is otherwise independent of h, such that:*

$$\|u_h\|_{(L^2\Lambda^k(D))^{\otimes m}} \leq C_{P,disc,1} \|\mathrm{Id} \otimes \ldots \otimes \underbrace{\mathrm{d}}_{i} \otimes \ldots \otimes \mathrm{Id}\, u_h\|_{L^2\Lambda^k \otimes \ldots \otimes \underbrace{L^2\Lambda^{k+1}}_{i} \otimes \ldots \otimes L^2\Lambda^k},$$

$$\tag{95}$$

$\forall\, u_h \in L^2\Lambda^k \otimes \ldots \otimes \underbrace{(\mathfrak{Z}_{k,h}^\perp)}_{i} \otimes \ldots \otimes L^2\Lambda^k(D)$, *where $\mathfrak{Z}_{k,h}$ is the kernel of d in*

$\mathcal{P}_{r,\Gamma_D}^-\Lambda^k(\mathcal{T}_h)$ *and $\mathfrak{Z}_{k,h}^\perp$ is its L^2-orthogonal complement.*

As simple consequence of the previous lemma and of the properties of the operator $\Pi_{k,h}^{\otimes m}$, we have:

$$\|u_h\|_{(L^2\Lambda^k(D))^{\otimes m}} \leq C_{P,disc,m} \|\mathrm{d}^{\otimes m} u_h\|_{(L^2\Lambda^{k+1}(D))^{\otimes m}} \quad \forall u_h \in \left(\mathfrak{Z}_{k,h}^\perp \right)^{\otimes m},$$

where $C_{P,disc,m}$ depends only on $C_{P,m}$ (defined in (61)) and C_π.

We are now ready to state the main result of this section.

Theorem 6.2 (Stability of the STP-FE discretization) *For every $\alpha \geq 0$, problem (87) is a stable discretization for the m-th moment problem (49). In particular, for every $M_{s,L}^{(m)} \in V_{k,L}^{(m)}$, there exist a test function $M_{t,L}^{(m)} \in V_{k,L}^{(m)}$ and positive constants $C_{m,disc}$, $C'_{m,disc}$ independent of h_L, s.t.*

$$\left\langle T^{\otimes m} M_{s,L}^{(m)}, M_{t,L}^{(m)} \right\rangle_{\left(V_{k,L}^{(m)}\right)', V_{k,L}^{(m)}} \geq C_{m,disc} \|M_{s,L}^{(m)}\|_{V_k^{\otimes m}}^2, \tag{96}$$

$$\|M_{t,L}^{(m)}\|_{V_k^{\otimes m}} \leq C'_{m,disc} \|M_{s,L}^{(m)}\|_{V_k^{\otimes m}}. \tag{97}$$

To prove the stability of problem (87), we can not use a tensor product argument as we did to prove the stability of the FTP-FE discretization. We need to explicitly prove the inf-sup condition for the tensor product operator $T^{\otimes m}$

restricted to the STP-FE space $V_{k,L}^{(m)}$. As we mentioned in section 4.4.1, the constructive proof of the inf-sup condition for $T^{\otimes m}$ defined on $V_k^{\otimes m}$ extends to $V_{k,L}^{(m)}$.

Proof. Suppose $\alpha > 0$ (the case $\alpha = 0$ is analogous). We prove (96) for $m = 2$. The general result follows by induction on m. We fix $M_{s,L}^{(2)} = \begin{bmatrix} (M_{s,L}^{(2)})_{1:} \\ (M_{s,L}^{(2)})_{2:} \end{bmatrix} \in \begin{bmatrix} U_{k,L,1}^{(2)} \\ U_{k-1,L,1}^{(2)} \end{bmatrix}$, where $U_{k,L,1}^{(2)}$ is defined by (89) with $m = 2$ and $j = 1$, and we decompose it using (91):

$$M_{s,L}^{(2)} = \begin{bmatrix} \mathrm{d} \otimes \mathrm{Id}(M_{s,L}^{\circ})_{1:} + (M_{s,L}^{\perp})_{1:} \\ \mathrm{d} \otimes \mathrm{Id}(M_{s,L}^{\circ})_{2:} + (M_{s,L}^{\perp})_{2:} \end{bmatrix},$$

where

$$(M_{s,L}^{\perp})_{1:} = \pi^{\perp} \otimes \mathrm{Id}(M_{s,L}^{(2)})_{1:} \in \mathfrak{B}_{k,L,1}^{(2),\perp}$$
$$(M_{s,L}^{\perp})_{2:} = \pi^{\perp} \otimes \mathrm{Id}(M_{s,L}^{(2)})_{2:} \in \mathfrak{B}_{k-1,L,1}^{(2),\perp}$$
$$(M_{s,L}^{\circ})_{1:} = \pi^{\circ} \otimes \mathrm{Id}(M_{s,L}^{(2)})_{1:} \in \mathfrak{B}_{k-1,L,1}^{(2),\perp}$$
$$(M_{s,L}^{\circ})_{2:} = \pi^{\circ} \otimes \mathrm{Id}(M_{s,L}^{(2)})_{2:} \in \mathfrak{B}_{k-2,L,1}^{(2),\perp}.$$

We choose $M_{t,L}^{(2)} = P \otimes P M_{s,L}^{(2)}$. By performing the same steps of the constructive proof of the inf-sup condition (Section 4.4.1), we conclude (96). Relation (97) follows from the orthogonal decomposition (91) and the tensorial discrete Poincaré inequality in Lemma 6.2. $\square$

Another way to express the result given in Theorem 6.2 is the following

Proposition 6.2 *Given T as in (17) and P as in (28), $\forall M_{s,L}^{(m)}$ it holds*

$$\left\langle T^{\otimes m} M_{s,L}^{(m)}, P^{\otimes m} M_{s,L}^{(m)} \right\rangle_{\left(V_{k,L}^{(m)}\right)', V_{k,L}^{(m)}} \geq C_{m,disc} \left\| M_{s,L}^{(m)} \right\|_{V_k^{\otimes m}}^2.$$

Let $\mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix}$ be the unique solution of problem (49) and $M_{s,L}^{(m)}$ be the unique solution of problem (87). Exploiting the Galerkin orthogonality and the stability of the discretization, we can obtain the following quasi-optimal convergence estimate:

$$\left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{s,L}^{(m)} \right\|_{V_k^{\otimes m}} \leq C \inf_{M_{t,L}^{(m)} \in V_{k,L}^{(m)}} \left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{t,L}^{(m)} \right\|_{V_k^{\otimes m}}. \tag{98}$$

To study the approximation properties of the space $V_{k,L}^{(m)}$ we construct the sparse tensorial projection operator $\Pi_{k,L}^{(m)}$ as follows.

Definition 6.2 *Let $\Pi_{k,h} : H_{\Gamma_D}\Lambda^k(D) \to \mathcal{P}_{r,\Gamma_D}^{-}\Lambda^k(\mathcal{T}_h)$ be a bounded cochain projector satisfying Assumption 6.1. Given $m \geq 2$ integer, we define*

$$\Pi_{k,L}^{(m)} := \sum_{|\underline{l}| \leq L} \otimes \Delta_{k,l_j}, \tag{99}$$

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where $\Delta_{k,l} := \Pi_{k,h_l} - \Pi_{k,h_{l-1}}$.

$\Pi_{k,L}^{(m)}$ is a bounded cochain projector. To state its approximation properties, we need the following technical lemma.

Lemma 6.3 *It holds:*

$$\sum_{|\underline{l}|>L} 2^{-\gamma|\underline{l}|} = \sum_{i=0}^{m-1} \left(\frac{1}{2^\gamma - 1}\right)^{m-i} \left(\begin{array}{c} L+m \\ i \end{array}\right) 2^{-\gamma L} \leq \left(\frac{1}{1-2^{-\lambda\gamma}}\right)^m 2^{-L\gamma(1-\lambda)}$$

(100)

for every real $\gamma > 0$ and integer $L > 0$, with $0 < \lambda < 1$.

Proof. To prove the equality, we observe that

$$\sum_{|\underline{l}|>L} 2^{-\gamma|\underline{l}|} = \sum_{j=L+1}^{\infty} \sum_{|l|=j} 2^{-\gamma j} = \sum_{j=L+1}^{\infty} \left(\begin{array}{c} j+m-1 \\ m-1 \end{array}\right) 2^{-\gamma j}.$$

It is sufficient to show that

$$\sum_{j=L+1}^{\infty} \left(\begin{array}{c} j+K-1 \\ m-1 \end{array}\right) 2^{-\gamma j} = \sum_{i=0}^{m-1} \left(\frac{1}{2^\gamma-1}\right)^{m-i} \left(\begin{array}{c} L+K \\ i \end{array}\right) 2^{-\gamma L}. \qquad (101)$$

for every integer K. We prove (101) by induction on m. If $m = 1$,

$$\sum_{j=L+1}^{\infty} \left(\begin{array}{c} j+K-1 \\ 0 \end{array}\right) 2^{-\gamma j} = \frac{2^{-\gamma(L+1)}}{1-2^{-\gamma}} = \frac{1}{2^\gamma-1} 2^{-\gamma L}.$$

Let us assume the result true for $m-1$. Then

$$\sum_{j=L+1}^{\infty} \left(\begin{array}{c} j+K-1 \\ m-1 \end{array}\right) 2^{-\gamma j}$$

$$= \sum_{j=L+1}^{\infty} \left(\begin{array}{c} j+K-1 \\ m-1 \end{array}\right) \frac{2^\gamma-1}{2^\gamma-1} 2^{-\gamma j}$$

$$= \frac{1}{2^\gamma-1} \left(\sum_{j=L+1}^{\infty} \left(\begin{array}{c} j+K-1 \\ m-1 \end{array}\right) \left(2^{-\gamma(j-1)} - 2^{-\gamma j}\right)\right)$$

$$= \frac{1}{2^\gamma-1} \left(\sum_{i=L}^{\infty} \left(\begin{array}{c} i+K \\ m-1 \end{array}\right) 2^{-\gamma i} - \sum_{j=L+1}^{\infty} \left(\begin{array}{c} j+K-1 \\ m-1 \end{array}\right) 2^{-\gamma j}\right)$$

$$= \frac{1}{2^\gamma-1} \left(\left(\begin{array}{c} L+K \\ m-1 \end{array}\right) 2^{-\gamma L} + \sum_{j=L+1}^{\infty} \left(\left(\begin{array}{c} j+K \\ m-1 \end{array}\right) - \left(\begin{array}{c} j+K-1 \\ m-1 \end{array}\right)\right) 2^{-\gamma j}\right)$$

$$= \frac{1}{2^\gamma-1} \left(\left(\begin{array}{c} L+K \\ m-1 \end{array}\right) 2^{-\gamma L} + \sum_{j=L+1}^{\infty} \left(\begin{array}{c} j+K-1 \\ m-2 \end{array}\right) 2^{-\gamma j}\right).$$

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Applying recursively the previous equality we get:

$$\sum_{j=L+1}^{\infty} \binom{j+K-1}{m-1} 2^{-\gamma j} = \sum_{i=0}^{m-1} \left(\frac{1}{2^\gamma-1}\right)^{m-i} \binom{L+K}{i} 2^{-\gamma L}.$$

Let us now show the inequality. Let $0 < \lambda < 1$.

$$\sum_{i=0}^{m-1} \left(\frac{1}{2^\gamma-1}\right)^{m-i} \binom{L+m}{i} 2^{-\gamma L} \leq \sum_{i=0}^{m-1} \left(\frac{1}{2^{\lambda\gamma}-1}\right)^{m-i} \binom{L+m}{i} 2^{-\gamma L}$$

$$= \frac{2^{-\gamma L}}{(2^{\lambda\gamma}-1)^m} \sum_{i=0}^{m-1} (2^{\lambda\gamma}-1)^i \binom{L+m}{i}$$

$$\leq \frac{2^{-\gamma L}}{(2^{\lambda\gamma}-1)^m} \left(2^{\lambda\gamma}\right)^{L+m}$$

$$= \left(\frac{1}{1-2^{-\lambda\gamma}}\right)^m 2^{-L\gamma(1-\lambda)}.$$

$\square$

Proposition 6.3 *The projector $\Pi_{k,L}^{(m)}$ introduced in Definition 6.2 is such that*

$$\|v - \Pi_{k,L}^{(m)} v\|_{(H\Lambda^k)^{\otimes m}} \leq C h_L^{s(1-\lambda)} \|v\|_{(H^{s+1}\Lambda^k)^{\otimes m}}, \tag{102}$$

$0 < \lambda < 1$, *for all* $v \in (H_{\Gamma_D}^{s+1}\Lambda^k(D))^{\otimes m}$, $0 \leq s \leq r$, *where* $C = C(m,\lambda,s)$ *is independent of* h_L.

Proof.   Following [12], we proceed in three steps. We start considering the approximation properties of $\Delta_{k,l}$. Using (76) we have:

$$\left\|\Delta_{k,l} \otimes \mathrm{Id}^{\otimes(m-1)} v\right\|_{(H\Lambda^k)^{\otimes m}}$$

$$= \left\|\Pi_{k,h_l} \otimes \mathrm{Id}^{\otimes(m-1)} v - \Pi_{k,h_{l-1}} \otimes \mathrm{Id}^{\otimes(m-1)} v\right\|_{(H\Lambda^k)^{\otimes m}}$$

$$\leq \left\|v - \Pi_{k,h_l} \otimes \mathrm{Id}^{\otimes(m-1)} v\right\|_{(H\Lambda^k)^{\otimes m}} + \left\|v - \Pi_{k,h_{l-1}} \otimes \mathrm{Id}^{\otimes(m-1)} v\right\|_{(H\Lambda^k)^{\otimes m}}$$

$$\leq C h_l^s \|v\|_{H^{s+1}\Lambda^k \otimes (H\Lambda^k)^{\otimes(m-1)}} + C h_{l-1}^s \|v\|_{H^{s+1}\Lambda^k \otimes (H\Lambda^k)^{\otimes(m-1)}}$$

$$\leq C h_{l-1}^s \|v\|_{H^{s+1}\Lambda^k \otimes (H\Lambda^k)^{\otimes(m-1)}},$$

for every $0 < s < r$.

Now, we consider the tensor product $\otimes_{j=1}^m \Delta_{k,l_j}$. By recursion,

$$\left\|\otimes_{j=1}^m \Delta_{k,l_j} v\right\|_{(H\Lambda^k)^{\otimes m}} \leq C h_{\underline{l}-1}^s \|v\|_{(H^{s+1}\Lambda^k)^{\otimes m}},$$

where $h_{\underline{l}-1}^s = h_{l_1-1}^s \ldots h_{l_m-1}^s$. Finally, using (100):

$$\left\| v - \Pi_{k,L}^{(m)} v \right\|_{(H\Lambda^k)^{\otimes m}}$$

$$= \left\| \sum_{|\underline{l}|>L} \otimes_{j=1}^m \Delta_{k,l_j} v \right\|_{(H\Lambda^k)^{\otimes m}}$$

$$\leq \sum_{|\underline{l}|>L} \left\| \otimes_{j=1}^m \Delta_{k,l_j} v \right\|_{(H\Lambda^k)^{\otimes m}} \leq \sum_{|\underline{l}|>L} C h_{\underline{l}-1}^s \|v\|_{(H^{s+1}\Lambda^k)^{\otimes m}}$$

$$= C \|v\|_{(H^{s+1}\Lambda^k)^{\otimes m}} h_0^{sm} \sum_{|\underline{l}|>L} 2^{-s|\underline{l}-1|} = C \|v\|_{(H^{s+1}\Lambda^k)^{\otimes m}} h_0^{sm} 2^{sm} \sum_{|\underline{l}|>L} 2^{-s|\underline{l}|}$$

$$\leq C \|v\|_{(H^{s+1}\Lambda^k)^{\otimes m}} h_0^{sm} 2^{sm} 2^{-Ls(1-\lambda)} \left(\frac{1}{1-2^{-s\lambda}} \right)^m$$

$$= C \|v\|_{(H^{s+1}\Lambda^k)^{\otimes m}} \left(\frac{2^s h_0^s}{1-2^{-s\lambda}} \right)^m 2^{-Ls(1-\lambda)}$$

for every $0 \leq s \leq r$. $\qquad\square$

It follows

Theorem 6.3 (Order of convergence of the STP-FE discretization)

$$\left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{s,L}^{(m)} \right\|_{V_k^{\otimes m}} = \mathcal{O}(h_L^{r(1-\lambda)}), \tag{103}$$

$0 < \lambda < 1$, *provided that* $\begin{bmatrix} u \\ p \end{bmatrix} \in L^m \left(\Omega; \begin{bmatrix} H^{r+1}\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \\ H^{r+1}\Lambda^{k-1}(D) \cap H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix} \right).$

The previous theorem states that the STP - FE approximation has almost the same rate of convergence as the FTP - FE. On the other hand, the great advantage of the sparse approximation with respect to the full one is represented by a drastic reduction of the dimensionality of the sparse finite element space.

7 Conclusions

The present work addresses the mixed formulation of the Hodge Laplacian defined on a n-dimensional domain $D \subseteq \mathbb{R}^n$, ($n \geq 1$), with stochastic forcing terms. The well-posedness of this problem is equivalent to the inf-sup condition of a suitable bounded bilinear and symmetric form $\langle T\cdot, \cdot \rangle$ coming from the weak formulation of the mixed Hodge Laplacian.

We have studied the moment equations, i.e. the deterministic equations solved by the statistical moments of the unique stochastic solution. In particular, if T is the (deterministic) operator that defines the starting problem, we show that the m-th moment equation involves the tensor product operator $T^{\otimes m} := \underbrace{T \otimes \cdots \otimes T}_{m \text{ times}}$. The main achievement of the paper has been to characterize an operator P and its tensorial version $P^{\otimes m}$ that allows us to construct

suitable test functions to prove the inf-sup condition for the tensor problem $\langle T^{\otimes m} \cdot, \cdot \rangle$ both at the continuous level and at the discrete level with full or sparse FE discretizations. By this tool we have been able to show that known stable FE approximations for the deterministic problem are also stable and optimally convergent for the tensorial problem both in the full and sparse versions.

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ERRATUM: Moment equations for the mixed formulation of the Hodge Laplacian with stochastic data

Francesca Bonizzoni, Annalisa Buffa, Fabio Nobile

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In Section 6.4 of the report "Moment equations for the mixed formulation of the Hodge Laplacian with stochastic data" we constructed the sparse tensor product space $V_{k,L}^{(m)}$ to discretize the tensor product space $V_k^{\otimes m}$. The proof of the stability of the sparse discretization (Theorem 6.2) is based on the discrete tensorial Hodge decomposition stated in Lemma 6.1. Since, in general, the operators $\pi^\perp$ and π° do not map each element of the discrete one-dimensional space $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_l)$ onto a discrete k-form, the result in Lemma 6.1 is not true. As a consequence, also the proof of Theorem 6.2 is wrong.

Here we propose the correct proof of the stability of the sparse discretization (Theorem 6.2, now Theorem E.1), which requires Lemma E.1. On the other hand, the final estimates on the order of convergence still hold (Proposition 6.3 and Theorem 6.3). However, for completeness, we restate also these results (now Proposition E.1 and Theorem E.2) in a slightly improved version.

E.1 Discrete m-th moment problem: sparse tensor product approximation

We briefly recall the construction of the sparse tensor product space $V_{k,L}^{(m)}$ of level $L > 0$ with $m \geq 2$ integer (see Section 6.4, formula (86)):

$$V_{k,L}^{(m)} := \bigoplus_{|\underline{l}| \leq L} (Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_m}), \tag{E.1}$$

where $Z_{k,l} = \begin{bmatrix} S_{k,l} \\ S_{k-1,l} \end{bmatrix}$, $S_{k,l} = \mathcal{P}_r^- \Lambda^k(\mathcal{T}_l) \setminus \mathcal{P}_r^- \Lambda^k(\mathcal{T}_{l-1})$ and $\left\{\mathcal{P}_r^- \Lambda^k(\mathcal{T}_l)\right\}_{l=0}^{\infty}$ is a sequence of nested and dense finite dimensional subspaces of the space V_k ($h_l = h_{l-1}/2$). We also recall the sparse tensor product finite element (STP-FE) approximation of problem (49):

m-Points Correlation Problem (STP-FE):

$$\boxed{\begin{array}{c} \textit{Given } m \geq 2 \textit{ integer and } \left[\begin{array}{c} F_1 \\ F_2 \end{array} \right] \in L^m\left(\Omega; V_k\right), \textit{ find } M_{s,L}^{(m)} \in V_{k,L}^{(m)} \textit{ s.t.} \\[4mm] T^{\otimes m} M_{s,L}^{(m)} = \mathcal{M}^m \left[\begin{array}{c} F_1 \\ F_2 \end{array} \right] \quad \textit{in } \left(V_{k,L}^{(m)}\right)' \end{array}}$$

(E.2)

From now on we make the following regularity assumption on the domain D, which will be needed to prove the stability of the numerical schemes we propose in this paper.

Assumption E.1 *For every $0 \leq k \leq n$, there exists $0 < s \leq 1$ such that*

$$H_{\Gamma_D}\Lambda^k(D) \cap H^*_{\Gamma_N}\Lambda^k(D) \subseteq H^s\Lambda^k(D). \tag{E.3}$$

Inclusion (E.3) is verified for an s-regular domain s.t. $\Gamma_D = \partial D$ and $\Gamma_N = \emptyset$. In particular, if ∂D is smooth, then D is 1-regular, and if D is Lipschitz, then D is 1/2-regular. See [3] and the references therein. We assume the second inclusion to be verified in our more general setting where $\Gamma_N \neq \emptyset$ and $\Gamma_D \subsetneq \partial D$.

To prove the stability of (E.2) we can not use a tensor product argument as we did to prove the stability of the FTP-FE discretization. We need to explicitly prove the inf-sup condition for the tensor product operator $T^{\otimes m}$ restricted to the STP-FE space $V_{k,L}^{(m)}$. The proof of this sparse inf-sup condition rests on two key ingredients. On one hand, we make use of the continuous inf-sup operator $P^{\otimes m}$ introduced in the proof of Theorem 4.2. On the other hand, we use a reasoning similar to the one proposed in [12] which defines and uses the so-called *GAP property* (see [12] and the references therein): we seek for its analogue in the case of STP-FE space, which will be called in what follows *STP-GAP property*. The main ingredient of the STP-GAP property is the sparse tensorial projection operator $\Pi_{k,L}^{(m)}$ (see Definition 6.2). It is defined starting from the projector $\Pi_{k,h}$, for which we recall a result sharper than (76):

$$\|v - \Pi_{k,h}v\|_{L^2\Lambda^k} \leq Ch^s \|v\|_{H^s\Lambda^k}, \quad \forall\, v \in H^s\Lambda^k,\ 0 \leq s \leq r. \tag{E.4}$$

It is easy to verify that $\Pi_{k,L}^{(m)}$ is a bounded cochain projector. Moreover,

$$\Pi_{k,L}^{(m)} \left(H_{\Gamma_D}\Lambda^k(D)\right)^{\otimes m} = \bigoplus_{|\underline{l}| \leq L} \left(S_{k,l_1} \otimes \ldots \otimes S_{k,l_m}\right).$$

We state the STP-GAP property for $m = 2$, but its generalization to $m \geq 2$ is straightforward.

Lemma E.1 (STP-GAP property) *For every*

$$v_h \in \Pi_{k,L}^{(2)} \left(H_{\Gamma_D} \Lambda^k(D) \otimes H_{\Gamma_D} \Lambda^k(D) \right)$$

there exist $0 < s \le 1$ and positive constants $C^{(1)}$, $C^{(2)}$, $C^{(3)}$, $C^{(4)}$ independent of h_0 such that

$$\left\| \mathrm{d}\pi^\circ \otimes \mathrm{d}\pi^\circ v_h - \Pi_{k,L}^{(2)} \left(\mathrm{d}\pi^\circ \otimes \mathrm{d}\pi^\circ v_h \right) \right\|_{H\Lambda^k \otimes H\Lambda^k} \le C^{(1)} h_0^s \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} , \quad \text{(E.5)}$$

$$\left\| \mathrm{d}\pi^\circ \otimes \pi^\perp v_h - \Pi_{k,L}^{(2)} \left(\mathrm{d}\pi^\circ \otimes \pi^\perp v_h \right) \right\|_{H\Lambda^k \otimes H\Lambda^k} \le C^{(2)} h_0^s \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} , \quad \text{(E.6)}$$

$$\left\| \pi^\perp \otimes \mathrm{d}\pi^\circ v_h - \Pi_{k,L}^{(2)} \left(\pi^\perp \otimes \mathrm{d}\pi^\circ v_h \right) \right\|_{H\Lambda^k \otimes H\Lambda^k} \le C^{(3)} h_0^s \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} , \quad \text{(E.7)}$$

$$\left\| \pi^\perp \otimes \pi^\perp v_h - \Pi_{k,L}^{(2)} \left(\pi^\perp \otimes \pi^\perp v_h \right) \right\|_{H\Lambda^k \otimes H\Lambda^k} \le C^{(4)} h_0^s \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} , \quad \text{(E.8)}$$

where $\pi^\perp$, π° are defined in (12) and (13), respectively. Note that v_h is uniquely expressed as $v_h = \mathrm{d}\pi^\circ \otimes \mathrm{d}\pi^\circ v_h + \mathrm{d}\pi^\circ \otimes \pi^\perp v_h + \pi^\perp \otimes \mathrm{d}\pi^\circ v_h + \pi^\perp \otimes \pi^\perp v_h$ thanks to the continuous Hodge decomposition (57).

Proof. Let $v_h \in \Pi_{k,L}^{(2)} \left(H_{\Gamma_D} \Lambda^k(D) \otimes H_{\Gamma_D} \Lambda^k(D) \right)$, so that $\Pi_{k,L}^{(2)} v_h = v_h$. Since $\Pi_{k,L}^{(2)}$ is a cochain map, it holds:

$$\mathrm{d} \otimes \mathrm{d}\, v_h = \mathrm{d} \otimes \mathrm{d}\Pi_{k,L}^{(2)}\, v_h = \Pi_{k,L}^{(2)} \mathrm{d} \otimes \mathrm{d}\, v_h, \qquad \text{(E.9)}$$

$$\mathrm{d} \otimes \mathrm{Id}\, v_h = \mathrm{d} \otimes \mathrm{Id}\Pi_{k,L}^{(2)}\, v_h = \Pi_{k,L}^{(2)} \mathrm{d} \otimes \mathrm{Id}\, v_h, \qquad \text{(E.10)}$$

$$\mathrm{Id} \otimes \mathrm{d}\, v_h = \mathrm{Id} \otimes \mathrm{d}\Pi_{k,L}^{(2)}\, v_h = \Pi_{k,L}^{(2)} \mathrm{Id} \otimes \mathrm{d}\, v_h. \qquad \text{(E.11)}$$

By definition of $\mathfrak{B}_k^\perp$ and Assumption 2.1, $\mathfrak{B}_k^\perp \subset H_{\Gamma_D}\Lambda^k \cap H_{\Gamma_N}^*\Lambda^k$, so that, thanks to Assumption E.1,

$$\|\Delta_{k,l} w\|_{L^2\Lambda^k} \le C\, h_{l-1}^s \|w\|_{H^s\Lambda^k} \le \tilde{C}\, h_{l-1}^s \|w\|_{H\Lambda^k} \quad \forall w \in \mathfrak{B}_k^\perp. \qquad \text{(E.12)}$$

- Let us start proving inequality (E.8). To this end, we need to bound four quantities:

$$\left\| \pi^\perp \otimes \pi^\perp v_h - \Pi_{k,L}^{(2)} \left(\pi^\perp \otimes \pi^\perp v_h \right) \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} , \qquad \text{(E.13)}$$

$$\left\| \mathrm{d}\pi^\perp \otimes \pi^\perp v_h - \Pi_{k,L}^{(2)} \left(\mathrm{d}\pi^\perp \otimes \pi^\perp v_h \right) \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k} , \qquad \text{(E.14)}$$

$$\left\| \pi^\perp \otimes \mathrm{d}\pi^\perp v_h - \Pi_{k,L}^{(2)} \left(\pi^\perp \otimes \mathrm{d}\pi^\perp v_h \right) \right\|_{L^2\Lambda^k \otimes L^2\Lambda^{k+1}} , \qquad \text{(E.15)}$$

$$\left\| \mathrm{d}\pi^\perp \otimes \mathrm{d}\pi^\perp v_h - \Pi_{k,L}^{(2)} \left(\mathrm{d}\pi^\perp \otimes \mathrm{d}\pi^\perp v_h \right) \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^{k+1}} . \qquad \text{(E.16)}$$

Using that $v_h = \sum_{L=0}^{+\infty} \sum_{|\underline{l}|=L} \Delta_{k,l_1} \otimes \Delta_{k,l_2} v_h$, the triangular inequality and

(E.12),

$$(\text{E.}13) \leq \sum_{|\underline{l}|>L} \left\| (\Delta_{k,l_1} \otimes \Delta_{k,l_2}) \left(\pi^\perp \otimes \pi^\perp\right) v_h \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k}$$

$$= \sum_{|\underline{l}|>L} \left\| \left(\Delta_{k,l_1}\pi^\perp \otimes \text{Id}\right) \left(\text{Id} \otimes \Delta_{k,l_2}\pi^\perp\right) v_h \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k}$$

$$\leq \sum_{|\underline{l}|>L} C\, h_{l_1-1}^s \left\| \left(\text{Id} \otimes \Delta_{k,l_2}\pi^\perp\right) v_h \right\|_{H\Lambda^k \otimes L^2\Lambda^k}$$

$$\leq \sum_{|\underline{l}|>L} C\, h_{l_1-1}^s \ h_{l_2-1}^s \left\| v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \tag{E.17}$$

where $C>0$ is independent on $h_l \ \forall l$. Observing that

$$(\text{d} \otimes \text{Id})\left(\pi^\perp \otimes \pi^\perp v_h\right) = \text{d} \otimes \pi^\perp v_h \in \Pi_{k,L}\left(H_{\Gamma_D}\Lambda^k(D)\right) \otimes \mathfrak{B}_k^\perp$$

so that $(\Delta_{k+1,l_1} \otimes \text{Id})\left(\text{d} \otimes \pi^\perp v_h\right) = 0$ if $l_1 > L$, we can bound (E.14):

$$(\text{E.}14) = \left\| \text{d} \otimes \pi^\perp v_h - \Pi_{k,L}^{(2)}\left(\text{d} \otimes \pi^\perp v_h\right) \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k}$$

$$\leq \sum_{l_1=0}^{L} \sum_{l_2=L-l_1+1}^{+\infty} \left\| (\Delta_{k+1,l_1} \otimes \Delta_{k,l_2})\left(\text{d} \otimes \pi^\perp\right) v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k}$$

$$\leq \sum_{l_1=0}^{L} \sum_{l_2=L-l_1+1}^{+\infty} \left\| \Delta_{k+1,l_1} \right\|_{\mathcal{L}(L^2\Lambda^{k+1}, L^2\Lambda^{k+1})} \left\| (\text{Id} \otimes \Delta_{k,l_2})\left(\text{d} \otimes \pi^\perp\right) v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k}$$

$$\leq C \sum_{l_1=0}^{L} \sum_{l_2=L-l_1+1}^{+\infty} h_{l_2-1} \left\| \text{d} \otimes \text{Id}\ v_h \right\|_{L^2\Lambda^{k+1} \otimes H\Lambda^k}$$

$$\leq C(L+1) \sum_{l_2=1}^{+\infty} h_{l_2-1}^s \left\| v_h \right\|_{H\Lambda^k \otimes H\Lambda^k}$$

$$\leq C h_0^s \left\| v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \tag{E.18}$$

where we have used that $\left\| \Delta_{k+1,l_1} \right\|_{\mathcal{L}(L^2\Lambda^{k+1}, L^2\Lambda^{k+1})}$ is bounded by a constant independent of h_{l_1}. By symmetry, we can obtain that

$$(\text{E.}15) \leq C h_0^s \left\| v_h \right\|_{H\Lambda^k \otimes H\Lambda^k}. \tag{E.19}$$

Finally, using (E.9), we have

$$(\text{d} \otimes \text{d})\left(\pi^\perp \otimes \pi^\perp\right) v_h = \text{d} \otimes \text{d}\ v_h = \text{d} \otimes \text{d}\ \Pi_{k,L}^{(2)}\ v_h = \Pi_{k,L}^{(2)}\ (\text{d} \otimes \text{d})\left(\pi^\perp \otimes \pi^\perp\right) v_h,$$

so that the quantity in (E.16) vanishes. Thus, putting together (E.17), (E.18), (E.19), we conclude (E.8).

- Let us prove inequality (E.7). We need to bound two quantities:

$$\left\| \pi^\perp \otimes \text{d}\pi^\circ\ v_h - \Pi_{k,L}^{(2)}\left(\pi^\perp \otimes \text{d}\pi^\circ v_h\right) \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k}, \tag{E.20}$$

$$\left\| \text{d}\pi^\perp \otimes \text{d}\pi^\circ\ v_h - \Pi_{k,L}^{(2)}\left(\text{d}\pi^\perp \otimes \text{d}\pi^\circ v_h\right) \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k}. \tag{E.21}$$

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Since $\pi^\perp \otimes \mathrm{d}\pi^\circ \; v_h = \pi^\perp \otimes \mathrm{Id} \; v_h - \pi^\perp \otimes \pi^\perp \; v_h$ and $\pi^\perp \otimes \mathrm{Id} \; v_h \in \mathfrak{B}_k^\perp \otimes \Pi_{k,L}(H_{\Gamma_D}\Lambda^k(D))$, and using (E.8),

$$
\begin{aligned}
\text{(E.20)} &\leq \left\| \pi^\perp \otimes \mathrm{Id} \; v_h - \Pi_{k,L}^{(2)} \; \pi^\perp \otimes \mathrm{Id} \; v_h \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&\quad + \left\| \pi^\perp \otimes \pi^\perp \; v_h - \Pi_{k,L}^{(2)} \; \pi^\perp \otimes \pi^\perp \; v_h \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&\leq \sum_{l_2=0}^{L} \sum_{l_1=L+1-l_2}^{+\infty} \left\| (\Delta_{k,l_1} \otimes \Delta_{k,l_2})\left(\pi^\perp \otimes \mathrm{Id}\right) \; v_h \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} + C \; h_0^s \; \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\leq \sum_{l_2=0}^{L} \sum_{l_1=L+1-l_2}^{+\infty} \|\Delta_{k,l_2}\|_{\mathcal{L}(L^2\Lambda^k, L^2\Lambda^k)} \; h_{l_1-1}^s \; \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} + C \; h_0^s \; \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\leq C \; h_0^s \; \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} \, . \tag{E.22}
\end{aligned}
$$

Moreover, using (E.8)

$$
\begin{aligned}
\text{(E.21)} &\leq \left\| \mathrm{d}\pi^\perp \otimes \mathrm{Id} \; v_h - \Pi_{k,L}^{(2)} \; \mathrm{d}\pi^\perp \otimes \mathrm{Id} \; v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k} \\
&\quad + \left\| \mathrm{d}\pi^\perp \otimes \pi^\perp \; v_h - \Pi_{k,L}^{(2)} \; \mathrm{d}\pi^\perp \otimes \pi^\perp \; v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k} \\
&\leq C \; h_0^s \; \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} \, . \tag{E.23}
\end{aligned}
$$

In the last inequality we exploited (E.10), which implies that $\mathrm{d}\pi^\perp \otimes \mathrm{Id} \; v_h = \mathrm{d} \otimes \mathrm{Id} \; v_h = \mathrm{d} \otimes \mathrm{Id} \; \Pi_{k,L}^{(2)} \; v_h = \Pi_{k,L}^{(2)} \; \mathrm{d}\pi^\perp \otimes \mathrm{Id} \; v_h$, so that

$$
\left\| \mathrm{d}\pi^\perp \otimes \mathrm{Id} \; v_h - \Pi_{k,L}^{(2)} \; \mathrm{d}\pi^\perp \otimes \mathrm{Id} \; v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k} = 0.
$$

Using (E.22) and (E.23) we conclude (E.7).

- To show (E.6), we write v_h as $v_h = \mathrm{Id} \otimes \mathrm{d}\pi^\circ \; v_h + \mathrm{Id} \otimes \pi^\perp \; v_h$ and proceed as in the proof of (E.7).

- To show (E.5) we observe that

$$
\begin{aligned}
&\left\| \mathrm{d}\pi^\circ \otimes \mathrm{d}\pi^\circ v_h - \Pi_{k,L}^{(2)} \left(\mathrm{d}\pi^\circ \otimes \mathrm{d}\pi^\circ v_h\right) \right\|_{H\Lambda^k \otimes H\Lambda^k} \\
&= \left\| \left(\mathrm{Id} \otimes \mathrm{Id} - \Pi_{k,L}^{(2)}\right) \left(\mathrm{Id} \otimes \mathrm{Id} - \mathrm{d}\pi^\circ \otimes \pi^\perp - \pi^\perp \otimes \mathrm{d}\pi^\circ - \pi^\perp \otimes \pi^\perp\right) v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\leq \left\| v_h - \Pi_{k,L}^{(2)} \; v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} + \left\| \mathrm{d}\pi^\circ \otimes \pi^\perp \; v_h - \Pi_{k,L}^{(2)} \mathrm{d}\pi^\circ \otimes \pi^\perp \; v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\quad + \left\| \pi^\perp \otimes \mathrm{d}\pi^\circ \; v_h - \Pi_{k,L}^{(2)} \pi^\perp \otimes \mathrm{d}\pi^\circ \; v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\quad + \left\| \pi^\perp \otimes \pi^\perp \; v_h - \Pi_{k,L}^{(2)} \pi^\perp \otimes \pi^\perp \; v_h \right\|_{H\Lambda^k \otimes H\Lambda^k}
\end{aligned}
$$

and we conclude (E.5) using that $v_h = \Pi_{k,L}^{(2)} \; v_h$, and (E.6), (E.7), (E.8).

$\square$

We are now ready to prove the main result of this section

Theorem E.1 (Stability of the STP-FE discretization) *For every $\alpha \geq 0$ there exists $\bar{h}_0 > 0$ such that for all $h_0 \leq \bar{h}_0$ problem (E.2) is a stable discretization for the m-th moment problem (49). In particular, for every $M_{s,L}^{(m)} \in V_{k,L}^{(m)}$, there exists a test function $M_{t,L}^{(m)} \in V_{k,L}^{(m)}$ and positive constants $C_{m,disc} = C_{m,disc}(C_m)$ (C_m is introduced in (51)), $C'_{m,disc} = C'_{m,disc}\left(\alpha, \|P\|, \left\|\Pi_{k,L}^{(2)}\right\|\right)$ s.t.*

$$\left\langle T^{\otimes m} M_{s,L}^{(m)}, M_{t,L}^{(m)}\right\rangle_{\left(V_{k,L}^{(m)}\right)', V_{k,L}^{(m)}} \geq C_{m,disc}\|M_{s,L}^{(m)}\|^2_{V_k^{\otimes m}}, \qquad \text{(E.24)}$$

$$\|M_{t,L}^{(m)}\|_{V_k^{\otimes m}} \leq C'_{m,disc}\|M_{s,L}^{(m)}\|_{V_k^{\otimes m}}. \qquad \text{(E.25)}$$

Proof. Suppose $\alpha > 0$ (the case $\alpha = 0$ is analogous). We fix $M_{s,L}^{(m)} \in V_{k,L}^{(m)}$ and look for a sparse test function $M_{t,L}^{(m)} \in V_{k,L}^{(m)}$ such that (E.24) and (E.25) are satisfied. We choose $M_{t,L}^{(m)} = \Pi_{k,L}^{(m)} P^{\otimes m} M_{s,L}^{(m)}$. Thanks to Proposition 4.1 and the boundness of the operators P and $\Pi_{k,L}^{(m)}$, we immediately conclude (E.25). In the proof of (E.24), we use brackets $\langle \cdot, \cdot \rangle$ without specifying the spaces taken into account, when no ambiguity arises.

$$\begin{aligned}\left\langle T^{\otimes m} M_{s,L}^{(m)}, M_{t,L}^{(m)}\right\rangle &= \left\langle T^{\otimes m} M_{s,L}^{(m)}, \Pi_{k,L}^{(m)} P^{\otimes m} M_{s,L}^{(m)}\right\rangle \\ &= \left\langle T^{\otimes m} M_{s,L}^{(m)}, P^{\otimes m} M_{s,L}^{(m)}\right\rangle \\ &\quad - \left\langle T^{\otimes m} M_{s,L}^{(m)}, \left(\mathrm{Id}^{\otimes m} - \Pi_{k,L}^{(m)}\right) P^{\otimes m} M_{s,L}^{(m)}\right\rangle.\end{aligned}$$

We observe that, thanks to the continuous inf-sup condition (51),

$$\left\langle T^{\otimes m} M_{s,L}^{(m)}, P^{\otimes m} M_{s,L}^{(m)}\right\rangle \geq C_m \left\|M_{s,L}^{(m)}\right\|^2_{V_k^{\otimes m}}, \qquad \text{(E.26)}$$

and, from Lemma E.1,

$$\begin{aligned}&\left\langle T^{\otimes m} M_{s,L}^{(m)}, \left(\mathrm{Id}^{\otimes m} - \Pi_{k,L}^{(m)}\right) P^{\otimes m} M_{s,L}^{(m)}\right\rangle \\ &\leq \|T\|^m_{\mathcal{L}(V_k, V_k')} \left\|M_{s,L}^{(m)}\right\|_{V_k^{\otimes m}} \left\|\left(\mathrm{Id}^{\otimes m} - \Pi_{k,L}^{(m)}\right) P^{\otimes m} M_{s,L}^{(m)}\right\|_{V_k^{\otimes m}} \\ &\leq C h_0^s \|T\|^m_{\mathcal{L}(V_k, V_k')} \left\|M_{s,L}^{(m)}\right\|^2_{V_k^{\otimes m}}.\end{aligned}$$

Therefore, for h_0 sufficiently small, (E.24) follows. $\qquad \square$

Another way to express the result given in Theorem E.1 is the following: $\forall M_{s,L}^{(m)}$ it holds

$$\left\langle T^{\otimes m} M_{s,L}^{(m)}, P^{\otimes m} M_{s,L}^{(m)}\right\rangle_{\left(V_{k,L}^{(m)}\right)', V_{k,L}^{(m)}} \geq C_{m,disc} \left\|M_{s,L}^{(m)}\right\|^2_{V_k^{\otimes m}}.$$

Let $\mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix}$ be the unique solution of problem (49) and $M_{s,L}^{(m)}$ be the unique solution of problem (E.2). Exploiting the Galerkin orthogonality and

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the stability of the discretization, we can obtain the following quasi-optimal convergence estimate:

$$\left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{s,L}^{(m)} \right\|_{V_k^{\otimes m}} \leq C \inf_{M_{t,L}^{(m)} \in V_{k,L}^{(m)}} \left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{t,L}^{(m)} \right\|_{V_k^{\otimes m}}. \quad \text{(E.27)}$$

To state the approximation properties of the sparse projector $\Pi_{k,L}^{(m)}$ and, as a consequence, of the sparse space $V_{k,L}^{(m)}$, we use Lemma 6.3 and the following

Proposition E.1 *The projector $\Pi_{k,L}^{(m)}$ introduced in Definition 6.2 is such that*

$$\|v - \Pi_{k,L}^{(m)} v\|_{(L^2 \Lambda^k)^{\otimes m}} \leq C h_L^{s(1-\lambda)} \|v\|_{(H^s \Lambda^k)^{\otimes m}}, \quad \text{(E.28)}$$

$0 < \lambda < 1$, *for all* $v \in (H_{\Gamma_D}^s \Lambda^k(D))^{\otimes m}$, $0 < s \leq r$, *where* $C = C(m, \lambda, s)$ *is independent of* h_L.

Proof. Following [13], we proceed in three steps. We start considering the approximation properties of $\Delta_{k,l}$. Using the triangular inequality and (E.4) we have:

$$\left\| \Delta_{k,l} \otimes \text{Id}^{\otimes(m-1)} v \right\|_{(L^2 \Lambda^k)^{\otimes m}} \leq C h_{l-1}^s \|v\|_{H^s \Lambda^k \otimes (L^2 \Lambda^k)^{\otimes(m-1)}},$$

for every $0 < s \leq r$. Now, we consider the tensor product $\otimes_{j=1}^m \Delta_{k,l_j}$. By recursion,

$$\left\| \otimes_{j=1}^m \Delta_{k,l_j} v \right\|_{(L^2 \Lambda^k)^{\otimes m}} \leq C h_{\underline{l}-1}^s \|v\|_{(H^s \Lambda^k)^{\otimes m}},$$

where $h_{\underline{l}-1}^s = h_{l_1-1}^s \ldots h_{l_m-1}^s$. Finally, using Lemma 6.3:

$$\begin{aligned}
\left\| v - \Pi_{k,L}^{(m)} v \right\|_{(L^2 \Lambda^k)^{\otimes m}} &= \left\| \sum_{|\underline{l}| > L} \otimes_{j=1}^m \Delta_{k,l_j} v \right\|_{(L^2 \Lambda^k)^{\otimes m}} \leq \sum_{|\underline{l}| > L} \left\| \otimes_{j=1}^m \Delta_{k,l_j} v \right\|_{(L^2 \Lambda^k)^{\otimes m}} \\
&\leq \sum_{|\underline{l}| > L} C h_{\underline{l}-1}^s \|v\|_{(H^s \Lambda^k)^{\otimes m}} = C \|v\|_{(H^s \Lambda^k)^{\otimes m}} h_0^{sm} \sum_{|\underline{l}| > L} 2^{-s|\underline{l}-1|} \\
&= C \|v\|_{(H^s \Lambda^k)^{\otimes m}} h_0^{sm} 2^{sm} \sum_{|\underline{l}| > L} 2^{-s|\underline{l}|} \\
&\leq C \|v\|_{(H^s \Lambda^k)^{\otimes m}} h_0^{sm} 2^{sm} 2^{-Ls(1-\lambda)} \left(\frac{1}{1 - 2^{-s\lambda}} \right)^m \\
&= C \|v\|_{(H^s \Lambda^k)^{\otimes m}} \left(\frac{2^s h_0^s}{1 - 2^{-s\lambda}} \right)^m 2^{-Ls(1-\lambda)}
\end{aligned}$$

for every $0 < s \leq r$. $\qquad \square$

It follows

Theorem E.2 (Order of convergence of the STP-FE discretization)

$$\left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{s,L}^{(m)} \right\|_{V_k^{\otimes m}} = \mathcal{O}(h_L^{r(1-\lambda)}), \quad \text{(E.29)}$$

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$0 < \lambda < 1$, *provided that*

$$\begin{bmatrix} u \\ p \end{bmatrix} \in L^m \left(\Omega; \begin{bmatrix} H^r \Lambda^k(D) \cap H_{\Gamma_D} \Lambda^k(D) \\ H^r \Lambda^{k-1}(D) \cap H_{\Gamma_D} \Lambda^{k-1}(D) \end{bmatrix} \right)$$

$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}p \end{bmatrix} \in L^m \left(\Omega; \begin{bmatrix} H^r \Lambda^{k+1}(D) \cap H_{\Gamma_D} \Lambda^{k+1}(D) \\ H^r \Lambda^k(D) \cap H_{\Gamma_D} \Lambda^k(D) \end{bmatrix} \right).$$

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