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Spectral Element Discretization of Optimal Control Problems

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Abstract

In this work we consider the numerical solution of a distributed optimal control problem associated with an elliptic partial differential equation. We approximate the optimality system by the spectral element method and derive *a posteriori* error estimates with respect to the cost functional. Then we use an hN adaptive refinement technique to reduce this error: the error indicator is used to mark what elements must be refined. The choice between an h or N refinement is based on the use of a predicted error reduction algorithm. Numerical results show the way this algorithm works.

Keywords: optimal control, spectral element method, a *posteriori* error estimates, mesh refinement

Introduction

We present an hN adaptive algorithm for a linear optimal control problem discretized by spectral element method. The use of adaptive algorithms to reduce the error on the cost functional is generally accepted in the context of finite element methods, see eg. [1, 2]. Very few results exist on the use of spectral

elements discretization of optimal control problems. In [4, 7] error estimates are obtained for the control, state and adjoint variables in the natural norms of the corresponding spaces. However, these results do not guarantee an error bound on the cost functional, a quantity of interest in many applications. The purpose of this paper is to derive *a posteriori* error estimates for the error on the objective functional besides those on energy norm error estimates in the context of spectral approximation, then to use them to guide an hN adaptive design mesh. Starting by an initial conforming spectral element mesh we solve the optimal control problem and estimate the error on the cost functional. When necessary we adapt the mesh to improve the discretization error and we solve again the optimal control problem on the new mesh until convergence within error tolerance.
The paper is organized as follows. In Section 1 we introduce the model linear optimal control problem. In Section 2 we introduce the spectral element approximation space and the discrete problem formulation. In Section 3, the error on the cost functional is estimated by the sum of two contributions: the iteration and discretization errors. In Section 4, some numerical results are presented to show how the algorithm works.

1 Linear Optimal Control Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with a Lipschitz boundary $\partial\Omega$ and V, U be the Hilbert spaces of state and control functions respectively. On the product space $V \times U$ we introduce a functional J that represents the quantity of physical interest, the objective of the control problem. The state problem describes, for each given control variable $u \in U$, the way the system evolves. The model problem considered features a distributed observation and a distributed control problem, in which:

- the functional J is quadratic:

$$J(y,u) = \frac{1}{2}\left\|Cy - z_d\right\|^2_{L^2(\Omega)} + \frac{1}{2}\alpha\left\|u - u_d\right\|^2_{L^2(\Omega)}$$

  where for a given Hilbert space of observations Z, $z_d \in Z$ is an assigned desired function, $C : V \to Z$ a bounded operator, $\alpha > 0$ is a penalization factor, $u_d \in U$ a given desired control (possibly zero);

- the state problem is an elliptic partial differential equation:

$$A(y(u), u; f) = 0$$

  where A is the linear differential operator defined on the domain Ω and f is a given source term. If we introduce the bilinear form $a : V \times V \to \mathbb{R}$, $\quad a(u,v) =< Au, v >_{V',V}$, with $< \cdot, \cdot >_{V',V}$ the duality pairing between V and V', then the variational formulation of the problem is

$$\text{find } y \in V : \quad a(y,v) =< f, v >_{V',V} + < Bu, v >_{V',V} \quad \forall v \in V,$$

where $B : U \to V'$ is a bounded linear functional. We assume a to be a bilinear continuous coercive form to ensure the well posedness of the state problem for each control.

Our optimal control problem reads as follows: look for $(y, u) \in V \times U$ such that

$$\begin{aligned} &\min_{(y,u)} J(y(u), u) \\ &\text{sbj to } A(y(u), u; f) = 0. \end{aligned}$$

Under the assumptions on the bilinear form a and on the functional, it is well-know that this problem is well-posed, see eg. [8].
Our approach to solve the problem is to introduce a Lagrangian functional $\mathcal{L}$ and to transform the optimal control problem as the search for the saddle-point of $\mathcal{L}$. We define $\mathcal{L} : V \times V \times U \to \mathbb{R}$

$$\mathcal{L}(y, p, u) := J(y, u) + < p, A(y, u) >_{V' \times V},$$

where p is the Lagrange multiplier, also called the adjoint variable.
If $x = (y, p, u)$ is the optimal solution then $\nabla\mathcal{L}(x)[\phi, \mu, \psi] = 0$ where the derivative is of Fréchet type. Upon taking the derivatives with respect to each variable, this yields the KKT (Karush-Kuhn-Tucker) optimality system:

$$\begin{cases} \nabla_p \mathcal{L}(x^*)[\phi] = 0 & \forall \phi \in V & \longmapsto & \text{state problem} \\ \nabla_y \mathcal{L}(x^*)[\mu] = 0 & \forall \mu \in V & \longmapsto & \text{adjoint problem} \\ \nabla_u \mathcal{L}(x^*)[\psi] = 0 & \forall \psi \in U & \longmapsto & \text{optimality conditions.} \end{cases}$$

For the specific model problem at hand the KKT system is: find $(y, p, u) \in V \times V \times U$ s.t.

$$\begin{cases} a(y, v) & = & < f + Bu, v >_{V',V} & \forall v \in V \\ a^*(p, v) & = & < C'\Lambda_Z(Cy - z_d), v >_{V',V} & \forall v \in V \\ < B'p + \alpha\Lambda_U u, \tilde{v} >_{U',U} & = & 0 & \forall \tilde{v} \in U \end{cases}$$

where $a^*(\cdot, \cdot)$ is the adjoint bilinear form of a, whereas $\Lambda_Z : Z \to Z'$ and $\Lambda_U : U \to U'$ are the Riesz inclusion operators, see [11]. To solve this problem we use an iterative method: given u^0, we solve the state and the adjoint problem according to the optimality conditions, then we update the derivative functional $\nabla_u J$. If $\|\nabla_u J\| \leq tol$ (for an assigned tolerance) we stop else we update the control variable u by a steepest-descent method $u^{j+1} = u^j - \tau\nabla_u J(u^j)$, whit τ being a relaxation parameter.

2 SEM discretization

At each step of the iterative method used for the solution of the KKT system we solve the state and the dual problem by a spectral element method.

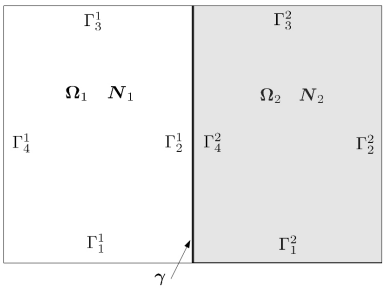

Figure 1: Decomposition of Ω, $K = 2$.

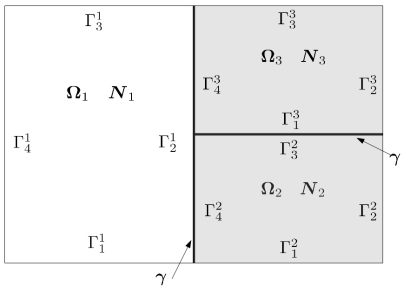

Figure 2: Decomposition of Ω, $K = 3$.

Let us decompose Ω into K spectral elements: $\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}_k$, such that $\forall \Omega_k$ there exists a bijective transformation $\varphi_k : \widehat{\Omega} \longrightarrow \Omega_k$, $\widehat{\Omega} = (-1, 1)^2$. We denote with $\overrightarrow{\delta} = \{\delta_k\}_{k=1}^{K}$ the vector of discretization parameters, $\delta_k = (h_k, N_k)$, h_k being the diameter of Ω_k and N_k the degree of the polynomial in Ω_k. For each couple of neighboring elements, say Ω_k, Ω_m, three different situations may occur:

1. either geometric and polynomial conformity, that is $\gamma = \overline{\Omega_k} \cap \overline{\Omega_m}$ is a common (full) side of Ω_k and Ω_m, and $N_k = N_m$. In this case we enforce $\mathcal{C}^0$ continuity across γ;

2. geometrical conformity but polynomial non-conformity, that is $N_k \neq N_m$. Then, we enforce continuity only at $\overline{N} + 1$ LGL nodes on γ, where $\overline{N} = \min(N_k, N_m)$;

3. full non-conformity, both geometrical and polynomial. In this case one chooses the longest edge and we call it γ. Then on γ we choose the smallest value $\overline{N}$ of the polynomial degree among those of all the spectral elements sharing the edge γ. Finally, we enforce continuity at $\overline{N} + 1$ LGL nodes on γ, see [5].

For the sake of illustration, two examples are shown in Figure 1 and 2, where we denote with Γ_q^k the $q-th$ side (according to the local side numbering) of the element Ω_k. With reference to Figure 1 we have $K = 2$, $N_1 \neq N_2$, $\gamma = \Gamma_2^1 = \Gamma_4^2$, $\overline{N} = \min(N_1, N_2)$. With reference to Figure 2 we have $K = 3$, $N_1 \neq N_2 \neq N_3$. in Figure 2 we have two different interfaces on which we enforce pointwise continuity. The former is $\gamma = \Gamma_3^2 = \Gamma_1^3$ for which we set $\overline{N} = \min(N_2, N_3)$, the latter is $\gamma = \Gamma_2^1$ on which we set $\overline{N} = \min(N_1, N_2, N_3)$.
The general situation can be regarded as the union of the two previous cases. On a general interface γ shared by (at least) two spectral elements, Ω_k and Ω_m, after defining $\overline{N}$ as before, the pointwise matching condition read

$$v_\delta^{(k)}|_\gamma(\xi_q) = v_\delta^{(m)}|_\gamma(\xi_q) \text{ with } \xi_q \text{ being a LGL node on } \gamma,\ q = 1, \ldots, \overline{N} + 1 \quad (1)$$

We introduce the spectral element space X_δ:

$$X_\delta := \{v_\delta : \ v_\delta \circ \varphi_k \in \mathbb{Q}_{N_k}(\widehat{\Omega}) \text{ and } v_{\delta_k} = 0 \quad \text{on} \ \ \Gamma_D \cap \partial\Omega_k, \forall k = 1, \ldots, K\}$$

where $\mathbb{Q}_N(\Omega)$ is the set of polynomials of two variables with degree $\leq N_k$ with respect to each variable and Γ_D is the Dirichlet boundary. Then the SEM space is:

$$V_\delta := \{v_\delta \in X_\delta : \ \text{for every selected interface } \gamma \ (1) \ \text{ is satisfied}\}.$$

With this formulation we could have nonconformity for both mesh and functional space, this is a natural situation that may arise after every step of an adaptive algorithm, when only some elements are refined. To ensure comparable mesh diameters between neighboring elements only one hanging node for side is allowed. So in addition to the elements marked by the *a posteriori* indicator, some further refinements could be made.
Now given V_δ, $U_{\tilde{\delta}}$ two suitable finite discretization of the state and control space V, U respectively, we search $(y_\delta, p_\delta, u_{\tilde{\delta}}) \in V_\delta \times V_\delta \times U_{\tilde{\delta}}$:

$$\begin{cases} a(y_\delta, v_\delta) & =< f + Bu_\delta, v_\delta >_{V',V} & \forall v_\delta \in V_\delta \\ a^*(p_\delta, v_\delta) & =< C'\Lambda_Z(Cy_\delta - z_d), v_\delta >_{V',V} & \forall v_\delta \in V_\delta \\ < B'p_\delta + \alpha\Lambda_U u_\delta, \tilde{v}_\delta >_{U',U} & = 0 & \forall \tilde{v}_\delta \in U_{\tilde{\delta}} \end{cases} \quad (2)$$

3 Iteration and discretization error estimates

After the discrete KKT system (2), we analyze the accuracy on the functional that we have achieved. By proceeding as done in [6], we split the functional error into two parts:

$$\left|J(y,u) - J(y_\delta^j, u_\delta^j)\right| \leq \underbrace{\left|J(y,u) - J(y^j,u^j)\right|}_{\epsilon_{\text{iter}}^{(j)}} + \underbrace{\left|J(y^j,u^j) - J(y_\delta^j, u_\delta^j)\right|}_{\epsilon_{\text{dis}}^{(j)}}$$

where (y,u) is the exact optimal control solution, (y^j,u^j) are the hypothetical continuous solutions at the iterative step j and (y_δ^j, u_δ^j) is the discrete optimal control solution. The first part represents the iteration error and the second the discretization error. We will estimate each term as follows.

Theorem 3.1 *For linear control problems, the iteration error at the j-th iteration has the following expression:*

$$\epsilon_{iter}^{(j)} = |J(y^*,u^*) - J(y^j,u^j)| = \frac{1}{2}(\nabla_u J(p^j,u^j), u^* - u^j).$$

Corollary 3.2 *If a steepest-descent iterative method with constant relaxation parameter τ is used, $\epsilon_{iter}^{(j)}$ can be estimated as:* $\left|\epsilon_{iter}^{(j)}\right| \simeq \frac{1}{2}\tau \left\|\nabla_u J(p^j,u^j)\right\|^2$.

See [6] for the proofs. Then the first part of the error is minimized during the iterative solution of the KKT system, accordingly with the stopping criterium, $\|\nabla_u J\| \leq tol_{iter}$. For the $\epsilon_{\text{dis}}^{(j)}$ we use a dual weighted estimation.

Theorem 3.3 *Assume the mesh to be γ shape regular, that is $\exists \gamma > 0 :\ \gamma^{-1} h_k \leq h_{k'} \leq \gamma h_k$ if k and k' are such that $\overline{\Omega}_k \cap \overline{\Omega}_{k'} \neq \emptyset$, with polynomial degrees of neighboring elements comparable $\gamma^{-1}(N_k + 1) \leq N_{k'} + 1 \leq \gamma(N_k + 1)$.*
Then for the spectral element discretization we have:

$$\epsilon_{dis}^{(j)} = \left| J(y^j, u^j) - J(y_\delta^j, u_\delta^j) \right| \leq \eta_{dis}^j := C \sum_{k=1}^{K} \eta_{dis}^k$$

$$\eta_{dis}^k := \rho_k^y \tfrac{h_k}{N_k} \|\nabla p_\delta^j\|_{L^2(\omega_k^1)} + \rho_k^p \tfrac{h_k}{N_k} \|\nabla y_\delta^j\|_{L^2(\omega_k^1)} + \rho_k^u \tfrac{h_k}{N_k} \|\nabla u_{\tilde{\delta}}^j\|_{L^2(\omega_k^1)}$$

where

$$\rho_k^y := \|R(y_\delta^j, u_\delta^j)\|_{\Omega_k} + (\tfrac{h_k}{N_k})^{-\frac{1}{2}} \|r(y_\delta^j)\|_{\partial\Omega_k}$$
$$\omega_k^y := \|y^j - I_\delta y^j\|_{\Omega_k} + (\tfrac{h_k}{N_k})^{\frac{1}{2}} \|y^j - I_\delta y^j\|_{\partial\Omega_k}$$
$$\rho_k^p := \|R(p_\delta^j, y_\delta^j)\|_{\Omega_k} + (\tfrac{h_k}{N_k})^{-\frac{1}{2}} \|r(p_\delta^j)\|_{\partial\Omega_k}$$
$$\omega_k^p := \|p^j - I_\delta p^j\|_{\Omega_k} + (\tfrac{h_k}{N_k})^{\frac{1}{2}} \|p^j - I_\delta p^j\|_{\partial\Omega_k}$$
$$\rho_k^u := (\tfrac{h_k}{N_k})^{-\frac{1}{2}} \|r(u_\delta^j)\|_{\partial\Omega_k}\,,\ \omega_k^u := (\tfrac{h_k}{N_k})^{\frac{1}{2}} \|u^j - I_\delta u^j\|_{\partial\Omega_k}$$

and $R(\cdot,\cdot)$ are the residuals of the state or dual equations, $r(\cdot,\cdot)$ are the jumps for state, dual and control variables, $I_\delta y^j$,$I_\delta p^j$,$I_\delta u^j$ are some suitable approximations of y^j, p^j, u^j, respectively. Here ω_k^1 is the union of the patches associated at each vertex of the element Ω_k.

Proof. According to [2] for the Galerkin element discretization we have $\epsilon_{\text{dis}}^{(j)} := \left| J(y^j, u^j) - J(y_\delta^j, u_\delta^j) \right| \leq \eta_{dis}^{(j)} := \sum_{k=1}^{K} \eta_{dis}^k$, where $\eta_{dis}^k := \{\rho_k^y \omega_k^p + \rho_k^p \omega_k^y + \rho_k^u \omega_k^u\}$. Now using hp-Clement interpolant estimates, see eg. [9], each term in the weighs ω_k^y, ω_k^p, ω_k^u can be estimated by the norms of the gradients in the ω_k^1 domain associated to Ω_k. □ The choice between h or N refinement strategy is made according to a predictable error estimates. For both the state and the adjoint equations we construct *a posteriori* residual estimates and a predictable estimates. Then we define the total residual and total predictable estimates as the sum of the two contributions by the state and adjoint problems. Comparing this total estimates following the algorithm proposed in [10] we choose between a spatial h or functional N refinement.

4 Numerical results

We present some numerical results to show how the algorithm works. Let $\Omega = (0,1)^2$ and consider an initial conform mesh. More particularly Ω is subdivided in $K = 4$ spectral elements and on each element we use a uniform degree $N = 2$. The state equation is:

$$\begin{cases} -\Delta y = u & \text{in} \quad \Omega \\ y = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

it_{iter}	it_{dis}	ϵ_{iter}	ϵ_{dis}	#{ref h}	#{ref N}
1	0	$1.4725e-008$	$1.2677e-005$	0	0
1	1	$1.3425e-005$	$1.9999e-006$	1	0
1	2	$5.5197e-006$	$3.5798e-007$	1	0
2	0	$1.8497e-008$	$8.0828e-007$	0	0
2	1	$1.0809e-007$	$7.6918e-007$	1	0
2	2	$6.3807e-007$	$3.0858e-007$	2	6
3	0	$3.072e-008$	$3.4665e-007$	0	0
3	1	$4.3163e-007$	$1.3016e-006$	3	1
3	2	$4.1335e-007$	$5.3046e-008$	5	0
4	0	$1.7597e-008$	$7.0768e-008$	0	0

Table 1: The error estimates at each optimization and adaptive step and the number of elements refined in h, N at each adaptive step

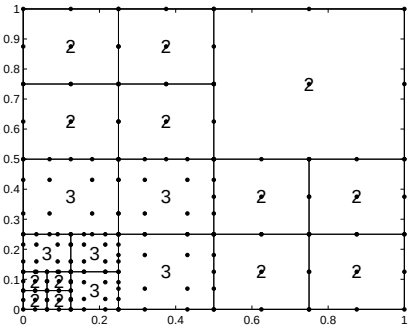

Figure 3: After 3 steps of the algorithm

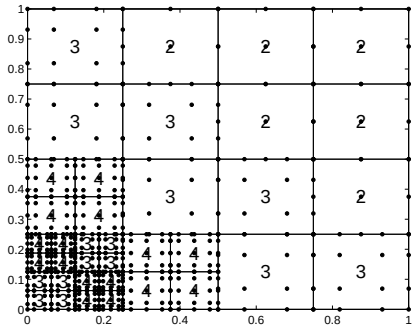

Figure 4: Final mesh and degrees for every element

For the quadratic functional J we fix $\alpha = 0.1$ and $z_d = \exp(-(x^2+y^2)/0.04)$. We solve both the optimization and the adaptive process in an iterative way, the two tolerances are $tol_{iter} = tol_{dis} = 1e-7$, an we start with an initial control $u_0 = 1$. In the adaptive process we admit at maximum $it_{dis}^{max} = 2$ iterations because changing the approximation of the functional J the optimal control calculated on the old mesh could be very different from the one on the new mesh. In Table 1 we report the results obtained during the process.
In the figures below we report an intermediate mesh in Figure 3 and the final mesh in Figure 4. For each element we plot the degrees of freedom and the local polynomial degree. In Figure 5 the final control function and in Figure 6 the associated final state function.

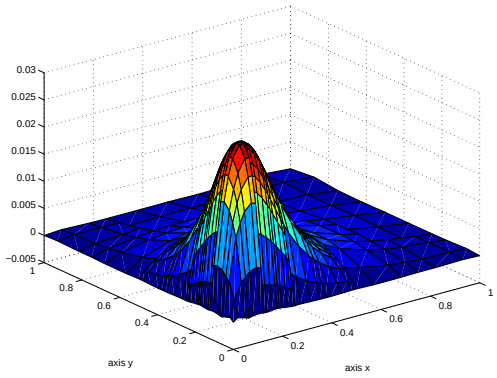

Figure 5: Final control function

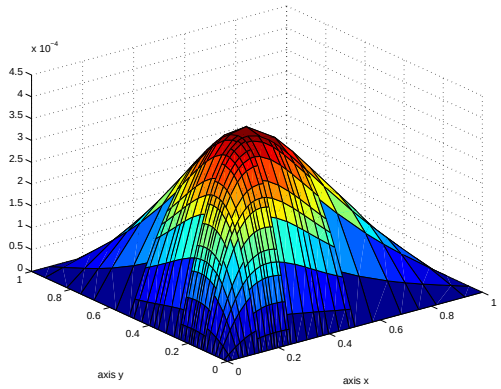

Figure 6: Final state function

Conclusions

In this note we have presented a spectral element method for the discretization of an elliptic optimal control problem and the use of hN adaptivity to reduce the error on the cost functional. The proposed estimate for the discretization part has driven to an automatic design of either the mesh and the polynomial degrees in a configuration strictly dependent on the problem considered. More information are used near the corner where the desired functions and the control variables change more rapidly.

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