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Abstract

We investigate a finite element approximation of an initial boundary value problem associated with parabolic Partial Differential Equations endowed with mixed time varying boundary conditions, switching from essential to natural and viceversa. The switching occurs both in time and in different portions of the boundary. For this problem, we apply and extend the Nitsche's method presented in [Juntunen and Stenberg, Mathematics of Computation, 2009] to the case of mixed time varying boundary conditions. After proving existence and numerical stability of the full discrete numerical solution obtained by using the θ-method for time discretization, we present and discuss a numerical test that compares our method to a standard approach based on remeshing and projection procedures.

Key words. Nitsche's method; parabolic problems; mixed time varying boundary conditions; stability analysis; Finite Element method.

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1   Introduction and motivation

The study of initial boundary value problems associated with parabolic Partial Differential Equations (PDEs) and complemented with mixed time varying boundary conditions (BCs) represents a challenging task both for the theoretical analysis and the numerical approximation. This kind of model may be suitable to describe physical problems as, e.g. the distribution of the temperature in a body with temperature and heat flux prescribed on a portion of the boundary changing in time, or flows in cavities (e.g. the heart ventricles and atria) with opening and closing valves. Despite its applicative interest, to the best of our knowledge, neither numerical examples nor numerical studies are available in literature. On the other hand, in the last decades, several works have been devoted to the analysis of this class of problems, even if only from a theoretical point of view [1, 5, 9, 10, 18, 21, 29]. In particular, the focus has been on the existence and regularity properties of the solution; in this respect, since the 1970s, different techniques have been developed. Starting from the standard theory of abstract evolution equations of Kato, Lions, and Magens [18, 19], works as [9, 10] adapted these arguments to the case of time varying mixed BCs, leading to solutions with properties connected to the geometrical structure of the boundaries. Then, by developing a more general result on abstract evolution equations in variable domains, Baiocchi [5] investigated the solution of these problems in spaces independent of the geometry under relatively weak assumptions on the data and on the regularity of the boundary. This latter approach has been further extended by Savaré in [29].

In [29], it is noted that the proposed abstract approach for the proof of the existence of the solution of a parabolic PDE defined by a second order elliptic spatial operator and endowed with mixed time varying BCs can be extended to a Finite Element (FE) formulation [25]. Starting from the framework of [29], one can consider a numerical method based on the backward Euler (BE) method [25] for the discretization in time and on the FE method for the spatial approximation. Specifically, one can use the standard (strong) imposition of Dirichlet BCs in the discretization of the parabolic PDE with mixed time varying BCs, thus building a family of closed time varying FE function subspaces. This method, which we identify with the name *time varying FE method*, exhibits some drawbacks which may considerably affect the computational performances of the simulations. Indeed, at each time step, the number of degrees of freedom associated to the FE space changes and, as well as, the size and the structure of the associated discrete problem. Moreover, when implementing the time varying FE method by using commercial codes, it may be necessary to re-mesh the domain and use extension and projection operators of the intermediate solutions for all the time steps.

In order to overcome the drawbacks of the time varying FE method, in this work we propose and analyse a new numerical approach for mixed time varying BCs which yields a FE semi-discretized spatial approximation with time independent function spaces and is based on the Nitsche's method [22]. Moreover, even if in this paper we focus on the θ-method [25, 26] for the time discretization, we highlight that any other scheme could similarly be used. We remark that the Nitsche's method for parabolic problems with mixed time varying BCs is formulated in the framework of [17], but considering the more general case of space-time weighting functions in place of constant coefficients, characterizing both the type of data and the degree of penalization; in addition we consider a general advection-diffusion-reaction PDE. We remark that the proposed approach introduces penalty and consistency terms in the variational formulation of the problem. Specifically, the Dirichlet BCs are imposed weakly in the variational formulation rather than strongly in the space of test functions by means of penalty terms; for the Neumann BCs,

additional terms are introduced with respect to those of the classical formulation.

In our theoretical analysis we prove the well posedness of the semi-discrete problem; then, we study the stability of the full discrete problem with stability conditions depending on both the data and the penalty functions. We show results in accordance with the ones presented in literature involving the weak imposition of essential BCs, which is commonly used in the context of the Discontinuous Galerkin (DG) method [25, 27]. As a matter of fact, the weak imposition of essential BCs (see e.g. [6]) represents a particular case of the more general scheme proposed in this paper.

This paper is organized as follows. In Section 2, we describe the continuous problem and we recall the results on the existence and regularity properties of the solution already established in [29]. Then, we present the standard numerical method based on the BE scheme for the time discretization and the spatial approximation by means of the FE method with time varying spaces in the framework of [29]. In Section 3, we introduce the proposed approach for mixed time varying BCs based on the Nitsche's method. We analyse the proposed semi-discrete formulation, then we introduce the approximation in time by using the θ-method, and we finally analyse the fully discretized problem; specifically, we provide results on the numerical stability of the method. Finally, in Section 4, we report and discuss some numerical results to highlight the efficiency of the proposed method. Conclusions follow.

2 The continuous problem

In this section we introduce, the parabolic problem with mixed time varying BCs. In Sec. 2.1 we recall some notions and notations on Banach spaces by referring in particular to [2], we describe the problem defined by a PDE with a second order elliptic spatial operator and we recall a result on the existence and uniqueness of the solution of Savaré in [29]. Finally, in Sec. 2.2, we present the FE method with time varying function spaces.

2.1 The continuous problem: preliminaries and well-posedness analysis

We assume that $\Omega \subseteq \mathbb{R}^n$, with $n \in \mathbb{N}$, is an open connected bounded set with regular boundary $\Gamma \equiv \partial\Omega$, satisfying a strong local Lipschitz condition. The independent spatial variable $\mathbf{x}$ will be defined in Ω, while the independent time variable t will take values in an interval $I := (0, T) \subseteq \mathbb{R}$ with $0 < T < \infty$. For any positive integer $m \geq 0$ and any real number $1 \leq p \leq \infty$, we denote the Sobolev space of order m as $W^{m,p}(\Omega) := \left\{ v \in L^p(\Omega) \ : \ D^{(\iota)}v \in L^p(\Omega) \text{ for all } \boldsymbol{\iota} = (\iota_1, \ldots, \iota_d) \in \mathbb{N}^n \text{ with } 0 \leq |\boldsymbol{\iota}| = \iota_1 + \ldots + \iota_n \leq m \right\}$, with $D^{(\iota)}$ denoting the distributional partial differential operator. The corresponding norm and semi-norm are indicated as $\|\cdot\|_{W^{m,p}(\Omega)}$ and $|\cdot|_{W^{m,p}(\Omega)}$, respectively. Moreover, for $p = 2$, we denote by $H^m(\Omega)$ the Hilbert space $W^{m,2}(\Omega)$ and by $H^m_{\Gamma_D}(\Omega)$ the space of functions $v \in H^m(\Omega)$ vanishing on $\Gamma_D \subseteq \Gamma \equiv \partial\Omega$ in the sense of the traces, i.e. $H^m_{\Gamma_D} := \{v \in H^m : v|_{\Gamma_D} \equiv 0\}$.

Let $\mathcal{H}$ be a Banach space endowed with norm $\|\cdot\|_{\mathcal{H}}$ defined over Ω. Then, the space of measurable functions with respect to the Lebesgue measure, defined on the interval I and having values in $\mathcal{H}$ with finite Bochner integral on I, is denoted by $L^1(I; \mathcal{H}) := \{v : I \to \mathcal{H} : \int_I \|v(s)\|_{\mathcal{H}} \, ds < \infty\}$. More generally, for any p such that $1 \leq p < \infty$, we have $L^p(I; \mathcal{H}) := \{v : I \to \mathcal{H} : \int_I \|v(s)\|_{\mathcal{H}}^p \, ds < \infty\}$, and $L^\infty(I; \mathcal{H}) := \{v : I \to \mathcal{H} : \operatorname*{ess\,sup}_{t \in I} \|v(t)\|_{\mathcal{H}} < \infty\}$. Analogously, for any positive integer $m \geq 0$, we consider the Hilbert spaces of measurable functions defined on I with values in $\mathcal{H}$,

say $H^m(I; \mathcal{H}) = \left\{ v \in L^2(I; \mathcal{H}) \; : \; D_t^{(\iota)} v \in L^2(I; \mathcal{H}) \text{ for all } \iota = (\iota_1, \ldots, \iota_d) \in \mathbb{N}^d \text{ with } 0 \le |\iota| = \iota_1 + \ldots + \iota_d \le m \right\}$, where $D_t^{(\iota)}$ is the distributional partial derivative with respect to the independent variable $t \in I$. Finally, for any fractional numbers, r and $s \in \mathbb{Q}$, we denote by $H^{r,s}(\Omega)$ the space defined as $H^{r,s}(\Omega) = L^2(I; H^r(\Omega)) \cap H^s(I; L^2(\Omega))$.

Now, let $Q_T = \Omega \times (0, T)$ be the space-time domain. We consider the linear parabolic equation in the unknwn $u : Q_T \to \mathbb{R}$, which reads:

$$\frac{\partial u}{\partial t}(\mathbf{x}, t) + \mathcal{L}u(\mathbf{x}, t) = f(\mathbf{x}, t) \quad \text{in } Q_T, \tag{2.1}$$

endowed with suitable initial and boundary conditions; $f : Q_T \to \mathbb{R}$ is the source term and $\mathcal{L}$ is a linear elliptic second order partial differential operator in the form:

$$\mathcal{L}u = -\nabla \cdot (\boldsymbol{\sigma} \nabla u) + \boldsymbol{\beta} \cdot \nabla u + \kappa u. \tag{2.2}$$

Specifically, the diffusivity tensor $\boldsymbol{\sigma}$, with $\sigma_{ij} \in W^{1,\infty}(Q_T)$, is assumed to be symmetric and elliptic, i.e. $\sigma_{ij} = \sigma_{ji}$ and:

$$\exists \alpha > 0 : \sum_{i,j=1}^{n} \sigma_{ij}(\mathbf{x}, t)\xi_i \xi_j \ge \alpha \, |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \; \forall (\mathbf{x}, t) \in Q_T; \tag{2.3}$$

moreover, we require a global Lipschitz condition on the coefficients σ_{ij} for any $i, j = 1, \ldots, n$. The vector $\boldsymbol{\beta}$ is supposed to be solenoidal ($\nabla \cdot \boldsymbol{\beta} = 0$) with components $\beta_i \in L^\infty(Q_T)$; finally, $\kappa \in L^\infty(Q_T)$ represents the reaction term. We highlight that, under the hypothesis on $\boldsymbol{\beta}$, $\mathcal{L}u$ can be rewritten as $\mathcal{L}u = -\nabla \cdot (\boldsymbol{\sigma} \nabla u - u\boldsymbol{\beta}) + \kappa u$.

In order to define the mixed time varying boundary value Cauchy problem associated with Eq. (2.1), we introduce, analogously to [29], a family of $C^{1,1}$ submanifolds with boundary $\Gamma_D^{(t)}$ for $t \in (0, T)$ on the lateral boundary $\Sigma = \Gamma \times (0, T)$. We denote by Σ_D and Σ_N the subsets of Σ covered by these submanifolds and their complement defined as:

$$\Sigma_D := \bigcup_{t \in (0,T)} \Gamma_D^{(t)} \times \{t\} \quad \text{and} \quad \Sigma_N := \Sigma \backslash \overline{\Sigma_D}, \tag{2.4}$$

respectively; Σ_D and Σ_N represent the lateral time varying parts of the boundary on which we prescribe Dirichlet and Neumann BCs as:

$$u(\mathbf{x}, t) = g(\mathbf{x}, t) \qquad\qquad\qquad \text{on } \Sigma_D, \tag{2.5a}$$

$$-\boldsymbol{\Phi}_{diff}(u(\mathbf{x}, t)) \cdot \mathbf{n} = G(\mathbf{x}, t) \qquad\qquad \text{on } \Sigma_N, \tag{2.5b}$$

respectively, where $\mathbf{n}$ indicates the outward directed unit vector normal to Σ_N, while $\boldsymbol{\Phi}_{diff}$ denotes the diffusive flux tensor associated to $\mathcal{L}$, which reads $\boldsymbol{\Phi}_{diff}(u) := -\boldsymbol{\sigma} \nabla u$. We suppose that the functions $g : \Sigma_D \to \mathbb{R}$ and $G : \Sigma_N \to \mathbb{R}$ are defined in suitable trace spaces. Finally, the problem (2.1) endowed with mixed time varying BCs and initial condition $u_0 : \Omega \to \mathbb{R}$ reads:

$$\text{find } u : Q_T \to \mathbb{R} : \begin{cases} \dfrac{\partial u}{\partial t}(\mathbf{x}, t) + \mathcal{L}u(\mathbf{x}, t) = f(\mathbf{x}, t) & \text{in } \Omega \times (0, T), \quad (2.6a) \\[2mm] u(\mathbf{x}, t) = g(\mathbf{x}, t) & \text{on } \Sigma_D, \quad (2.6b) \\[2mm] -\boldsymbol{\Phi}_{diff}(u(\mathbf{x}, t)) \cdot \mathbf{n} = G(\mathbf{x}, t) & \text{on } \Sigma_N. \quad (2.6c) \\[2mm] u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \text{in } \Omega. \quad (2.6d) \end{cases}$$

We recall a result on the existence and uniqueness of the solution of Eq. (2.6) provided by Savaré in [29].[a] Specifically, by introducing the distance $d(\cdot, \cdot)$ defining a metric space over Σ_D, we focus on the case in which a two-sided condition on the dilatation of the Dirichlet boundary $\Gamma_D^{(t)}$ is defined by the Hausdorff distance as:

$$d_H(\Gamma_D^{(t)}, \Gamma_D^{(s)}) = \max\{ \sup_{\mathbf{y} \in \Gamma_D^{(s)}} \inf_{\mathbf{x} \in \Gamma_D^{(t)}} d(\mathbf{x}, \mathbf{y}), \sup_{\mathbf{x} \in \Gamma_D^{(t)}} \inf_{\mathbf{y} \in \Gamma_D^{(s)}} d(\mathbf{x}, \mathbf{y})\}. \qquad (2.7)$$

Theorem 2.1. *Let the source term $f \in L^2(Q_T)$, the initial condition $u_0 \in H^1(\Omega)$, and the boundary data $g \in H^{3/2, 3/4}(\Sigma_D)$ and $G \in H^{1/2, 1/4}(\Sigma_N)$, with the initial compatibility condition $u_0(\mathbf{x}) = g(\mathbf{x}, 0)$ on $\Gamma_D^{(0)}$ satisfied. If the excess of dilatation $d_H(\Gamma_D^{(t)}, \Gamma_D^{(s)})$ of Eq. (2.7) is controlled by a weighted measure of the interval of time, i.e.:*

$$\exists \rho \in L^4(0, T) \quad : \quad d_H(\Gamma_D^{(t)}, \Gamma_D^{(s)}) \leq \int_s^t \rho(\tau) d\tau, \qquad 0 \leq s < t \leq T, \qquad (2.8)$$

then, Eq. (2.6) possesses a unique solution $u : Q_T \to \mathbb{R}$ such that:

$$\frac{\partial u}{\partial t}, \ \mathcal{L}u \in L^2(Q_T), \ \text{and} \ u \in C^0(0, T; H^1(\Omega)). \qquad (2.9)$$

2.2 FE method with time varying function spaces

In order to introduce a spatial FE approximation of the PDE (2.6), firstly, we recast this problem in weak formulation. For any $t \in (0, T)$, we define the trial affine space $V_g^{(t)} := \left\{ v \in H^1(\Omega) \ : \ v(t, \mathbf{x})|_{\Gamma_D^{(t)}} = g(t, \mathbf{x}) \right\}$ and we introduce a lifting function $\bar{g}(t, \mathbf{x}) \in H^1(\Omega)$ satisfying the essential BC (2.6b) in the sense of the traces, i.e. such that $\gamma_{\Gamma_D^{(t)}} \bar{g}(t, \mathbf{x}) = g(t, \mathbf{x})$. Moreover, $\bar{g}$ is such that for all the functions $u \in V_g^{(t)}$ there exists a unique $w \in V_0^{(t)} := \left\{ v \in H^1(\Omega) \ : \ v(t, \mathbf{x})|_{\Gamma_D^{(t)}} = 0 \right\}$ for which $u = \bar{g} + w$. We remark that, the Dirichlet BC (2.6b) is imposed strongly in the space of test functions, while the Neumann BC (2.6c) is weakly introduced in the variational formulation. The weak formulation of Eq. (2.6) reads for a.e. $t \in (0, T)$:

$$\text{find } w(t) \in V_0^{(t)} : \quad \left(\frac{\partial w}{\partial t}(t), \varphi \right) + a_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa}(t; w(t), \varphi) = \widetilde{\mathcal{F}}(t; \varphi) \quad \forall \varphi \in V_0^{(t)}, \qquad (2.10)$$

with

$$a_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa}(t; w, \varphi) := \int_\Omega \nabla \varphi \cdot (\boldsymbol{\sigma} \nabla w) \ \mathrm{d}\Omega + \int_\Omega \varphi \boldsymbol{\beta} \cdot \nabla w \ \mathrm{d}\Omega + \int_\Omega \varphi \kappa w \ \mathrm{d}\Omega \qquad (2.11)$$

and

$$\widetilde{\mathcal{F}}(t; \varphi) := \int_\Omega \varphi f(t) \ \mathrm{d}\Omega + \int_{\Gamma_N^{(t)}} \varphi G(t) \ \mathrm{d}\Gamma - a_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa}(t; \bar{g}(t), \varphi). \qquad (2.12)$$

At this point, we introduce a time discretization of Eq. (2.10), by using the BE scheme[b], yielding the semi-discrete problem. Let $N_T \in \mathbb{N}$ be a given integer and consider an uniform partition of

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[a] A more general result is provided in [29] considering a weaker condition on the dilatation of the Dirichlet boundary and solutions in Besov spaces [2, 8, 20].

[b] We use the BE scheme in the framework of the proof of Theorem 2.1 in [29], for which the author shows that, under suitable hypotheses, the continuous and piecewise linear (with respect to time) solution of the semi-discrete problem converges to the solution $u(t)$ of Eq. (2.6) when the time step tends to zero.

the time interval $(0, T)$ into subintervals $\{[t_n, t_{n+1})\}_{n=0}^{N_T-1}$ of size $\Delta t := T/N_T$, with $t_n := n\Delta t$, for $n = 0, \ldots, N_T$. By evaluating the data f, g, and G at the time t_n as f^n, g^n, and G^n, for $n = 1, \ldots, N_T$, and by setting $u^0 = u_0$ in Ω, we solve recursively the following family of elliptic problems in the unknowns $u^n \in V^{(t^n)}$, for $n = 1, \ldots, N_T$, reading:

$$\text{find } w^n \in V_0^{(t^n)}: \quad \frac{1}{\Delta t}\left((w^n - w^{n-1}), \varphi\right)_{L^2(\Omega)} + a_{\boldsymbol{\sigma},\boldsymbol{\beta},\kappa}(t^n; w^n, \varphi) = \widetilde{\mathcal{F}}(t^n; \varphi) \quad \forall \varphi \in V_0^{(t^n)}; \quad (2.13)$$

with $u^n = w^n + \bar{g}^n$.

Finally, in order to obtain the full discrete FE formulation, we consider a regular family of quasi-uniform triangulations $\{\mathcal{T}_h^n\}_h$ of Ω, denoting with $h_T := \operatorname{diam} T$ the diameter of any mesh element $T \in \mathcal{T}_h^n$ and by $h^n := \max_{T \in \mathcal{T}_h^n} h_T$ the mesh size at time t_n ([26, 33]). Moreover, we assume that for any mesh size h, the mesh $\mathcal{T}_h^n$ is conforming with the boundary $\Gamma \equiv \partial\Omega$ and we denote by $\mathcal{E}_h^n$ the corresponding triangulation on the interface, whose elements (edges or faces) are denoted by Γ_b for $b = 1, \ldots, N_b^h$ with associated diameter $h_b^n := \operatorname{diam} \Gamma_b^n$, respectively. We remark that the mesh $\mathcal{T}_h^n$ varies or changes with the time t^n, as well as $\mathcal{E}_h^n$. Therefore, for any $n = 1, \ldots, N_T$, the full discrete problem (2.6) is defined by looking for a FE approximated solution u_h^n of Eq. (2.13) in the FE space of degree $k \geq 1$ defined as:

$$\mathcal{V}_{0,h}^n := \mathcal{X}_h^n \cap V_0^{(t^n)}, \quad (2.14)$$

where $\mathcal{X}_h^n := \left\{\varphi_h \in \mathbb{C}^0(\overline{\Omega}) : \varphi_h|_T \in \mathbb{P}_k \quad \forall T \in \mathcal{T}_h^n\right\}$ and $\mathbb{P}_k$ denotes the space of polynomials of degree less than or equal to k. We remark that the FE space $\mathcal{X}_h^n$ depends on the mesh $\mathcal{T}_h^n$, which may change at each time step in order to match $\Gamma_D^{(t^n)}$. Then, the full discrete problem reads, for $n = 1, \ldots, N_T$:

$$\text{find } u_h^n \in V^{(t^n)}: \quad (w_h^n, \varphi_h) + \Delta t\, a_{\boldsymbol{\sigma},\boldsymbol{\beta},\kappa}(t^n; w_h^n, \varphi_h) = \Delta t\, \widetilde{\mathcal{F}}(t^n; \varphi_h) + \left(w_h^{n-1}, \varphi_h\right) \quad \forall \varphi_h \in \mathcal{V}_{0,h}^n;$$
$$(2.15)$$

with $u_h^n = w_h^n + \bar{g}_h^n$, given $u_h^0 \in \mathcal{V}_h$, where $\widetilde{\mathcal{F}}$ involves the approximate lifting function $\bar{g}^n \in \mathcal{V}_{g,h}^n := \mathcal{X}_h \cap V_g^{(t^n)}$, e.g. obtained by a L^2 projection technique.

Remark 2.1. *The FE spaces of Eq. (2.14) are time varying function spaces. Specifically, depending on the measure of $\Gamma_D^{(t_n)}$, the number of degrees of freedom associated to the FE space $\mathcal{V}_{0,h}^{(n)}$ changes accordingly, as well as the size and the structure of the associated discrete problem.*

Remark 2.2. *When solving problem (2.15) using available commercial software, the FE space can only be defined after the Dirichlet boundary $\Gamma_D^{(t^n)}$ is prescribed at time t^n. Therefore, one needs to re-mesh and re-define a new FE space for any discrete time t^n, for all $n = 1, \ldots, N_T$. Moreover, for any t^n, suitable injection and projection operators need to be introduced to account for the variation of the mesh $\mathcal{T}_h^n$. In practice, to account for this variation, we need to introduce a mapping T^n acting on the data u^{n-1} as $T^n : \mathcal{V}_{g,h}^{n-1} \xrightarrow{I_h^n} H^1(\Omega) \xrightarrow{P_h^n} \mathcal{V}_{g,h}^n$, where I_h^n denotes the injection operator of the solution $u^{n-1} \in \mathcal{V}_{g,h}^{n-1}$ into $H^1(\Omega)$, while P_h^n is the projection operator from $H^1(\Omega)$ onto the FE space $\mathcal{V}_{g,h}^n$. Accounting for such operators in commercial software or FE codes is neither straightforward nor computationally efficient.*

3   Nitsche's method for mixed time varying BCs

In this section, we propose the Nitsche's method for the treatment of the mixed time varying BCs of Eq. (2.6). Following what done in [17] for elliptic PDEs, we consider Eq. (2.6a) endowed with a generalized Robin BCs in the form:

$$- \mathbf{\Phi}_{diff}(u(\mathbf{x},t)) \cdot \mathbf{n} + \gamma(\mathbf{x},t)u(\mathbf{x},t) = G(\mathbf{x},t) + \gamma(\mathbf{x},t)g(\mathbf{x},t) \quad \text{on } \Sigma, \tag{3.1}$$

where $\gamma : \Sigma \to (0, +\infty) \subseteq \mathbb{R}$. Indeed, given $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in H^1(\Omega)$, $g \in L^2(0,T;H^{1/2}(\Gamma_D^{(t)}))$, and $G \in L^2(0,T;L^2(\Gamma_N^{(t)}))$, we consider the problem:
    find $u : \Omega \times (0,T) \to \mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(\mathbf{x},t) + \mathcal{L}u(\mathbf{x},t) = f(\mathbf{x},t) & \text{in } \Omega \times (0,T), & (3.2\text{a}) \\[2mm] -\mathbf{\Phi}_{diff}(u(\mathbf{x},t)) \cdot \mathbf{n} + \gamma(\mathbf{x},t)u(\mathbf{x},t) = G(\mathbf{x},t) + \gamma(\mathbf{x},t)g(\mathbf{x},t) & \text{on } \Sigma, & (3.2\text{b}) \\[2mm] u(0) = u_0 & \text{in } \Omega, & (3.2\text{c}) \end{cases}$$

with $u_0(\mathbf{x}) = g(\mathbf{x},0)$ on $\Gamma_D^{(0)}$. We notice that in the limit $\gamma \to 0$, Eq. (3.2b) tends to the *pure* Neumann BC of Eq. (2.6c), while, in the limit $\gamma \to \infty$, we recover the *pure* Dirichlet BC of Eq. (2.6b). Problem (3.2) can therefore be regarded as more general than (2.6).

3.1   The Nitsche's method: spatial discretization

For the spatial discretization of Eq. (3.2), we introduce a FE approximation based on the Nitsche's method [17, 22] to treat the time varying BCs by looking for a solution of the weak counterpart of Eq. (3.2) in $H^1(\Omega)$ with function spaces independent of time. We introduce the FE space of degree $k \geq 1$ defined as:

$$\mathcal{V}_h := \mathcal{X}_h \cap H^1(\Omega), \tag{3.3}$$

with $\mathcal{X}_h^n$ being the FE space of Lagrangian basis functions defined over the mesh $\mathcal{T}_h$, which is now fixed in Ω and in time; $h := \max_{T \in \mathcal{T}_h} h_T$ represents the mesh size, $h_T := \operatorname{diam} T$ the diameter of any element $T \in \mathcal{T}_h$, $\mathcal{E}_h$ the triangulation on the boundary, whose elements (edges or faces) are denoted by Γ_b for $b = 1, \ldots, N_b^h$ with associated diameter $h_b := \operatorname{diam} \Gamma_b$. Moreover, let T_b, for $b = 1, \ldots, N_b^h$ be the boundary triangle of $\mathcal{T}_h$ associated to Γ_b, i.e. $\Gamma_b := T_b \cap \Gamma$.

3.1.1   The semi-discrete problem

We consider the following problem, for all $t \in (0,T)$:

$$\text{find } u_h(t) \in \mathcal{V}_h : \quad \left(\frac{\partial u_h}{\partial t}(t), \varphi_h \right) + a_h(t; u_h(t), \varphi_h) = \mathcal{F}_h(t; \varphi_h) \quad \forall \varphi_h \in \mathcal{V}_h, \tag{3.4}$$

where:

$$\begin{aligned} a_h(t; u_h(t), \varphi_h) := \; & a_{\boldsymbol{\sigma},\boldsymbol{\beta},\kappa}(t; u_h(t), \varphi_h) + \sum_{b=1}^{N_b^h} \left[\int_{\Gamma_b} \varphi_h \left(-\frac{\gamma h_b}{\xi + \gamma h_b} \right) \boldsymbol{\sigma} \nabla u_h \cdot \mathbf{n} \, \mathrm{d}\Gamma \right. \\ & + \int_{\Gamma_b} (\mathbf{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) \left(-\frac{\gamma h_b}{\xi + \gamma h_b} \right) u_h \, \mathrm{d}\Gamma + \int_{\Gamma_b} \varphi_h \left(\frac{|\boldsymbol{\sigma}| \xi \gamma}{\xi + \gamma h_b} \right) u_h \, \mathrm{d}\Gamma \\ & \left. + \int_{\Gamma_b} (\mathbf{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) \left(-\frac{h_b}{|\boldsymbol{\sigma}| (\xi + \gamma h_b)} \right) \boldsymbol{\sigma} \nabla u_h \cdot \mathbf{n} \, \mathrm{d}\Gamma \right] \end{aligned} \tag{3.5}$$

and:

$$
\begin{aligned}
\mathcal{F}_h(t; \varphi_h) := \; & \mathcal{F}(t; \varphi_h) + \sum_{b=1}^{N_b^h} \left[\int_{\Gamma_b} (\boldsymbol{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) \left(-\frac{\gamma h_b}{\xi + \gamma h_b} \right) g \, d\Gamma + \int_{\Gamma_b} \varphi_h \left(\frac{|\boldsymbol{\sigma}| \xi \gamma}{\xi + \gamma h_b} \right) g \, d\Gamma \right. \\
& \left. + \int_{\Gamma_b} (\boldsymbol{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) \left(-\frac{h_b}{|\boldsymbol{\sigma}| (\xi + \gamma h_b)} \right) G \, d\Gamma + \int_{\Gamma_b} \varphi_h \left(\frac{\xi}{\xi + \gamma h_b} \right) G \, d\Gamma \right],
\end{aligned}
\tag{3.6}
$$

with $a_{\boldsymbol{\sigma},\boldsymbol{\beta},\kappa}(t; u_h, \varphi_h)$ defined in Eq. (2.11), $\mathcal{F}(t; \varphi_h) := \int_\Omega \varphi_h f \, d\Omega$, χ_{in} and χ_{out} the characteristic functions of the subsets of the boundary corresponding to the inflow and outflow parts, respectively:

$$
\chi_{in} : \Sigma \to \{0, 1\}, \quad \chi_{in}(\mathbf{x}, t) := \begin{cases} 1 & \text{if } \boldsymbol{\beta}(\mathbf{x}, t) \cdot \mathbf{n} < 0, \\ 0 & \text{otherwise,} \end{cases}, \qquad \text{and} \qquad \chi_{out} := 1 - \chi_{in}, \tag{3.7}
$$

and $\boldsymbol{\Phi}_{in}^*$ is the adjoint of the inflow flux tensor:

$$
\boldsymbol{\Phi}_{in}^*(u(\mathbf{x}, t)) := \boldsymbol{\sigma}(\mathbf{x}, t) \nabla u(\mathbf{x}, t) + u(\mathbf{x}, t) \boldsymbol{\beta}(\mathbf{x}, t) \chi_{in}(\mathbf{x}, t); \tag{3.8}
$$

moreover, $\xi : \bigcup_{b=1}^{N_b^h} \Gamma_b \times (0, T) \to (0, +\infty)$ is the penalty function.

Remark 3.1. *The first boundary term of Eq. (3.5) is a consistency term, while the remaining terms of Eqs. (3.5) and (3.6) ensure a weak enforcement of the BCs. The case of pure Neumann BC, i.e. when $\Sigma_N \equiv \Sigma$, is recovered for $\gamma(\mathbf{x}, t) \to 0^+$ a.e. $(\mathbf{x}, t) \in \Sigma$. Indeed, in that case, Eq. (3.4) becomes:*

$$
\text{find } u_h(t) \in \mathcal{V}_h : \; \left(\frac{\partial u_h}{\partial t}(t), \varphi_h \right) + a_{\boldsymbol{\sigma},\boldsymbol{\beta},\kappa}(t; u_h, \varphi_h) + \sum_{b=1}^{N_b^h} \int_{\Gamma_b} (\boldsymbol{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) \left(-\frac{h_b}{|\boldsymbol{\sigma}| \xi} \right) \boldsymbol{\sigma} \nabla u_h \cdot \mathbf{n} \, d\Gamma
$$

$$
= \mathcal{F}(t; \varphi_h) + \int_\Gamma \varphi_h G \, d\Gamma + \sum_{b=1}^{N_b^h} \int_{\Gamma_b} (\boldsymbol{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) \left(-\frac{h_b}{|\boldsymbol{\sigma}| \xi} \right) G \, d\Gamma \quad \forall \varphi_h \in \mathcal{V}_h.
\tag{3.9}
$$

We notice that the last terms on either side of Eq. (3.9) represent additional terms with respect to the standard formulation, which however do not affect the consistency of the method. The case of a pure Dirichlet BC, i.e. when $\Sigma_D \equiv \Sigma$, is recovered for $\gamma(\mathbf{x}, t) \to +\infty$ a.e. $(\mathbf{x}, t) \in \Sigma$. In this case, Eq. (3.4) reads:

$$
\begin{aligned}
\text{find } u_h(t) \in \mathcal{V}_h : \quad & \left(\frac{\partial u_h}{\partial t}(t), \varphi_h \right) + a_{\boldsymbol{\sigma},\boldsymbol{\beta},\kappa}(t; u_h, \varphi_h) + \sum_{b=1}^{N_b^h} \int_{\Gamma_b} -\varphi_h \left(\boldsymbol{\sigma} \nabla u_h \cdot \mathbf{n} \right) d\Gamma \\
& + \sum_{b=1}^{N_b^h} \int_{\Gamma_b} - (\boldsymbol{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) u_h \, d\Gamma + \sum_{b=1}^{N_b^h} \int_{\Gamma_b} \varphi_h \left(\frac{|\boldsymbol{\sigma}| \xi}{h_b} \right) u_h \, d\Gamma \\
& = \mathcal{F}(t; \varphi_h) + \sum_{b=1}^{N_b^h} \int_{\Gamma_b} - (\boldsymbol{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) g \, d\Gamma + \sum_{b=1}^{N_b^h} \int_{\Gamma_b} \varphi_h \left(\frac{|\boldsymbol{\sigma}| \xi}{h_b} \right) g \, d\Gamma.
\end{aligned}
\tag{3.10}
$$

Eq. (3.10) corresponds to the weak treatment of the Dirichlet BCs as considered e.g. in [6]. As a matter of fact, the weak imposition of Dirichlet BCs of [6] represents a particular case of the more general scheme proposed in this paper. Weak imposition of Dirichlet BCs is easily accommodated in the context of the Discontinuous Galerkin (DG) method [25, 27], where the inter-element continuity of the solution is weakly enforced. In particular, in view of the analysis of the penalty terms added to the variational formulation, we refer the interested reader to [3] and [27].

Remark 3.2. *Concerning the penalty function $\xi = \xi(\mathbf{x}, t)$, we attribute a different role to ξ on the boundary of the space-time cylinder Σ where we impose either Dirichlet or Neumann BCs. Indeed, from definitions (3.5) and (3.6), we infer that, for the weak imposition of the Dirichlet BC on Γ, the bigger the value of ξ, the more significant is the penalization on the Dirichlet data g. Conversely, for the Neumann BC on Γ, the larger the value of ξ, the smaller is the contribution of the additional consistency terms, see Eq. (3.9).*

We remark that Eq. (3.4) considers a different treatment of the inflow boundary part with respect to the outflow one, with the aim of controlling both the regimes of dominating advection or diffusion. The scaling with respect to $|\boldsymbol{\sigma}|$ of the boundary terms of Eqs. (3.5) and (3.6) allows to consider the correct dimensions in the formulation. The weak treatment of the Dirichlet BCs at the inflow and outflow boundaries, which is obtained for $\gamma(\mathbf{x}, t) \to +\infty$, can be justified analogously to e.g. [6]. For the analysis of the Nitsche's method, we impose the following restrictions on $\xi : \bigcup_{b=1}^{N_b^h} \Gamma_b \times (0, T) \to (0, +\infty)$; ξ is a measurable function for which there exists two positive constants $\overline{\xi}_0$ and $\overline{\xi}_\infty$ such that:

$$0 < \overline{\xi}_0 \leq \xi(\mathbf{x}, t) \leq \overline{\xi}_\infty < +\infty, \quad \forall (\mathbf{x}, t) \in \bigcup_{b=1}^{N_b^h} \Gamma_b \times (0, T). \tag{3.11}$$

Then, let us define the following weighting functions for the Nitsche's method:

$$\xi_k^b : \Gamma \times (0, T) \to \mathbb{R} : (\mathbf{x}, t) \mapsto \frac{\delta_k^b}{\xi(\mathbf{x}, t) + \gamma(\mathbf{x}, t) h_b \chi_{\Gamma_b}(\mathbf{x})}, \tag{3.12}$$

for $k = 1, \ldots, 4$, with:

$$\delta_1^b := \gamma(\mathbf{x}, t) h_b \chi_{\Gamma_b}(\mathbf{x}), \ \delta_2^b := \xi(\mathbf{x}, t) \gamma(\mathbf{x}, t), \ \delta_3^b := h_b \chi_{\Gamma_b}(\mathbf{x}), \quad \text{and} \quad \delta_4^b := \xi(\mathbf{x}, t), \tag{3.13}$$

for all $b = 1, \ldots, N_b^h$ and with χ_{Γ_b} the characteristic functions of the subset Γ_b, i.e. $\chi_{\Gamma_b} : \Gamma \to \{0, 1\}$: $\mathbf{x} \mapsto \begin{cases} 1 & \text{if } \mathbf{x} \in \Gamma_b, \\ 0 & \text{otherwise} \end{cases}$. For the weighting functions (3.12), we have the following upper bounds:

$$\begin{aligned} \left| \xi_1^b(\mathbf{x}, t) \right| &\leq 2, & \left| \xi_2^b(\mathbf{x}, t) \right| &\leq 2 \frac{\overline{\xi}_\infty}{\min_b h_b} \lesssim \frac{\overline{\xi}_\infty}{h}, \\ \left| \xi_3^b(\mathbf{x}, t) \right| &\leq \frac{\max_b h_b}{\overline{\xi}_0} \lesssim \frac{h}{\overline{\xi}_0}, & \left| \xi_4^b(\mathbf{x}, t) \right| &\leq 1, & \forall (\mathbf{x}, t) \in \Gamma \times (0, T), \end{aligned} \tag{3.14}$$

where Eq. (3.14) for ξ_2^b and ξ_3^b follow from the quasi-uniformity of the family of triangulations $\mathcal{T}_h$.

3.1.2 Some preliminary lemmas

We recall the following inverse inequalities for traces of finite element functions from [3, 33]. Let $\widetilde{\Gamma} \subseteq \partial\Omega$, then there exist positive constants $\widetilde{C}_1$ and $\widetilde{C}_2 > 0$ independent of h, but possibly dependent on the FE degree k such that:

$$\|\phi\|^2_{L^2(\widetilde{\Gamma})} \leq \widetilde{C}_1 h^{-1} \|\phi\|^2_{L^2(\Omega)} \quad \text{and} \quad \|\nabla\phi \cdot \mathbf{n}\|^2_{L^2(\widetilde{\Gamma})} \leq \widetilde{C}_2 h^{-1} |\phi|^2_{H^1(\Omega)} \qquad \forall\phi \in \mathcal{V}_h. \tag{3.15}$$

Moreover, for $b = 1, \ldots, N_b^h$, there exist constants $\overline{C}_1$ and $\overline{C}_2 > 0$ dependent on k, but independent of h_b, such that:

$$\|\phi\|^2_{L^2(\Gamma_b)} \leq \overline{C}_1 h_b^{-1} \|\phi\|^2_{L^2(T_b)} \qquad \forall\phi \in \mathcal{V}_h, \tag{3.16a}$$

$$\|\nabla\phi \cdot \mathbf{n}\|^2_{L^2(\Gamma_b)} \leq \overline{C}_2 h_b^{-1} |\phi|^2_{H^1(T_b)} \qquad \forall\phi \in \mathcal{V}_h. \tag{3.16b}$$

Finally, thanks to the quasi-uniformity of $\mathcal{T}_h$, the following inverse inequality holds [33] for a positive constant $\bar{C}$ independent of h:

$$|\phi|^2_{H^1(\Omega)} \leq \bar{C} h^{-2} \|\phi\|^2_{L^2(\Omega)} \qquad \forall\phi \in \mathcal{V}_h. \tag{3.17}$$

For our analysis, we introduce the following h-dependent norm:

$$\|v\|_h := \left(|v|^2_{H^1(\Omega)} + \sum_{b=1}^{N_b^h} \|\zeta v\|^2_{L^2(\Gamma_b)} \right)^{1/2}, \tag{3.18}$$

where, for the sake of simplicity, we have introduced from Eq. (3.12), with $k = 2$, the function:

$$\zeta : \Sigma \to \mathbb{R} : (\mathbf{x}, t) \mapsto \sqrt{\xi_2^b(\mathbf{x}, t)}. \tag{3.19}$$

We recall the following result, consequence of the Peetre-Tartar Lemma [15], from which we deduce the successive lemmas.

Lemma 3.1. *Let $\mathcal{O}$ be a general open bounded connected set with Lipschitz boundary and let F be a linear functional on $H^1(\mathcal{O})$ whose restriction on constant functions is not null; then, there exists a positive constant $C_\mathcal{O} > 0$, dependent on the domain $\mathcal{O}$, such that:*

$$C_\mathcal{O} \|\varphi\|_{H^1(\mathcal{O})} \leq |\varphi|_{H^1(\mathcal{O})} + |F(\varphi)| \quad \forall\varphi \in H^1(\mathcal{O}). \tag{3.20}$$

Lemma 3.2. *Under the hypotheses of Lemma 3.1, for $\mathcal{O} = T_b$, there exists a constant $C_{\Omega_B} > 0$, related to the boundary elements of the mesh T_b for $b = 1, \ldots, N_b^h$, such that for the norm $\|\cdot\|_h$ of Eq. (3.18):*

$$\sum_{b=1}^{N_b^h} \|\varphi_h\|^2_{H^1(T_b)} \leq C_{\Omega_B} \|\varphi_h\|^2_h \qquad \forall\varphi_h \in \mathcal{V}_h. \tag{3.21}$$

Proof. By applying Lemma 3.1 for $\mathcal{O} \equiv T_b$ and $F(\varphi) = \dfrac{1}{h_b} \displaystyle\int_{\Gamma_b} \zeta\varphi \, d\Gamma$, we have:

$$|F(\varphi)| = \frac{1}{h_b} \left| \int_{\Gamma_b} \zeta\varphi \, d\Gamma \right| \leq \frac{1}{h_b} \|1\|_{L^2(\Gamma_b)} \|\zeta\varphi\|_{L^2(\Gamma_b)} = \|\zeta\varphi\|_{L^2(\Gamma_b)}, \tag{3.22}$$

and, for any $b = 1, \ldots, N_b^h$, there exists $C_{T_b} > 0$, such that:

$$C_{T_b} \|\varphi_h\|_{H^1(T_b)} \leq |\varphi_h|_{H^1(T_b)} + |F(\varphi_h)| \leq |\varphi_h|_{H^1(T_b)} + \|\zeta\varphi_h\|_{L^2(\Gamma_b)}; \tag{3.23}$$

therefore, $\dfrac{C_{T_b}^2}{2} \|\varphi_h\|_{H^1(T_b)}^2 \leq |\varphi_h|_{H^1(T_b)}^2 + \|\zeta\varphi_h\|_{L^2(\Gamma_b)}^2$. By summing the latter over $b = 1, \ldots, N_b^h$, we have:

$$\sum_{b=1}^{N_b^h} \frac{C_{T_b}^2}{2} \|\varphi_h\|_{H^1(T_b)}^2 \leq |\varphi_h|_{H^1(\Omega)}^2 + \sum_{b=1}^{N_b^h} \|\zeta\varphi_h\|_{L^2(\Gamma_b)}^2 = \|\varphi_h\|_h^2; \tag{3.24}$$

by setting $C_{\Omega_B} := \dfrac{2}{\min_b C_{T_b}^2}$, Eq. (3.21) follows. $\qquad \square$

Lemma 3.3. *Under the hypotheses of Lemma 3.1, for $\mathcal{O} = \Omega$, there exists a constant $C_\Omega > 0$ such that, for the norm $\|\cdot\|_h$ of Eq. (3.18):*

$$\|\varphi_h\|_{H^1(\Omega)}^2 \leq \frac{2}{C_\Omega^2} \|\varphi_h\|_h^2 \qquad \forall \varphi_h \in \mathcal{V}_h, \tag{3.25a}$$

$$\|\varphi_h\|_{L^2(\Gamma)}^2 \leq \frac{2C_{tr}^2}{C_\Omega^2} \|\varphi_h\|_h^2 \qquad \forall \varphi_h \in \mathcal{V}_h, \tag{3.25b}$$

with C_{tr} trace constant, such that:

$$\|\varphi_h\|_{L^2(\Gamma)} \leq C_{tr} \|\varphi_h\|_{H^1(\Omega)}. \tag{3.26}$$

Proof. By applying Lemma 3.1 for $\mathcal{O} \equiv \Omega$ and $F(\varphi) := \dfrac{1}{\sqrt{2N_b^h \max_b h_b}} \displaystyle\sum_{b=1}^{N_b^h} \int_{\Gamma_b} \zeta\varphi \, d\Gamma$, we obtain:

$$C_\Omega \|\varphi_h\|_{H^1(\Omega)} \leq |\varphi_h|_{H^1(\Omega)} + |F(\varphi_h)| \leq |\varphi_h|_{H^1(\Omega)} + \frac{1}{\sqrt{2N_b^h}} \sum_{b=1}^{N_b^h} \|\zeta\varphi_h\|_{L^2(\Gamma_b)}, \tag{3.27}$$

where the upper bound for $|F(\varphi_h)|$ follows by the triangle inequality and Schwarz inequality $|F(\varphi)| \leq \dfrac{1}{\sqrt{2N_b^h}} \displaystyle\sum_{b=1}^{N_b^h} \|\zeta\varphi\|_{L^2(\Gamma_b)}$. Furthermore, since $\left(\displaystyle\sum_{b=1}^{N_b^h} \|\zeta\varphi\|_{L^2(\Gamma_b)}\right)^2 \leq 2N_b^h \displaystyle\sum_{b=1}^{N_b^h} \|\zeta\varphi\|_{L^2(\Gamma_b)}^2$, we obtain:

$$C_\Omega^2 \|\varphi_h\|_{H^1(\Omega)}^2 \leq 2|\varphi_h|_{H^1(\Omega)}^2 + 2\sum_{b=1}^{N_b^h} \|\zeta\varphi_h\|_{L^2(\Gamma_b)}^2 = 2\|\varphi_h\|_h^2, \tag{3.28}$$

yielding (3.25a); (3.25b) follows from (3.25a) and (3.26). $\qquad \square$

Lemma 3.4. *There exist positive constants $C_1^{(1)}$ and $C_{1,h}^{(2)} > 0$ such that:*

$$C_1^{(1)} \|\varphi_h\|_{H^1(\Omega)} \leq \|\varphi_h\|_h \leq C_{1,h}^{(2)} \|\varphi_h\|_{H^1(\Omega)} \qquad \forall \varphi_h \in \mathcal{V}_h, \tag{3.29}$$

where $C_1^{(1)}$ is independent of h, while $C_{1,h}^{(2)} := \sqrt{1 + \dfrac{2C_{tr}^2 \overline{\xi}_\infty}{\min_b h_b}}$ scales as $h^{-1/2}$ for $h \to 0$.

Proof. By using Lemma 3.3, we obtain the left inequality of Eq. (3.29), with $C_1^{(1)} := \dfrac{C_\Omega}{\sqrt{2}}$. Using the bound (3.14) for the function ξ_2^b and the trace inequality (3.26), we obtain:

$$\sum_{b=1}^{N_b^h} \|\zeta\varphi\|_{L^2(\Gamma_b)}^2 \leq \frac{2\overline{\xi}_\infty}{\min_b h_b} \sum_{b=1}^{N_b^h} \|\varphi\|_{L^2(\Gamma_b)}^2 \leq \frac{2\overline{\xi}_\infty}{\min_b h_b} \|\varphi\|_{L^2(\Gamma)}^2 \leq \frac{2C_{tr}^2 \overline{\xi}_\infty}{\min_b h_b} \|\varphi\|_{H^1(\Omega)}^2. \tag{3.30}$$

$\square$

Lemma 3.5. *For the norm* $\|\cdot\|_h$*, there exists a constant* $\widetilde{C}_0 > 0$ *independent of* h *such that:*

$$\|\varphi_h\|_h \leq \widetilde{C}_0 h^{-1} \|\varphi_h\|_{L^2(\Omega)} \qquad \forall \varphi_h \in \mathcal{V}_h. \tag{3.31}$$

Proof. By considering the inverse inequalities (3.16a) and (3.17) together with the second upper bound of the inequality (3.30), we have:

$$\|\varphi_h\|_h^2 \leq |\varphi_h|_{H^1(\Omega)}^2 + \frac{2\overline{\xi}_\infty}{\min_b h_b} \|\varphi\|_{L^2(\Gamma)}^2 \leq \bar{C} h^{-2} \|\varphi_h\|_{L^2(\Omega)}^2 + 2\overline{\xi}_\infty \widetilde{C}_1 h^{-2} \|\varphi_h\|_{L^2(\Omega)}^2. \tag{3.32}$$

The latter inequality follows from the hypothesis of the quasi-uniformity of $\{\mathcal{T}_h\}_h$; indeed, there exists a constant $\tau > 0$ such that $\min_{T \in \mathcal{T}_h} h_T \geq \tau h$ for $h := \max_{T \in \mathcal{T}_h} h_T$ for which we embed the constant τ in $\widetilde{C}_1$. The thesis follows with $\widetilde{C}_0^2 := \max\{\bar{C}, 2\overline{\xi}_\infty \widetilde{C}_1\}$. $\square$

3.1.3 Analysis of the Nitsche's method: consistency and well posedness

We start by verifying the consistency of the formulation (3.4) with respect to problem (2.6).

Lemma 3.6. *Let* $u_h(t) \in \mathcal{V}_h$ *be solution of* (3.4) *and* $u(t) : \Omega \times (0, T) \to \mathbb{R}$ *the solution of* (3.2), *then, for all* $t \in (0, T)$*, it holds:*

$$\left(\frac{\partial}{\partial t}\left(u_h(t) - u(t)\right), \varphi_h\right) + a_h(t; \left(u_h(t) - u(t)\right), \varphi_h) = 0 \qquad \forall \varphi_h \in \mathcal{V}_h, \tag{3.33}$$

where $a_h(t; \cdot, \cdot)$ *is defined in Eq.* (3.5).

Proof. By multiplying Eq. (3.2a) by $\varphi_h \in \mathcal{V}_h$, integrating over Ω, and using the Green's integration formula, we have:

$$\int_\Omega \varphi_h \frac{\partial u}{\partial t} \, d\Omega + \int_\Omega \nabla\varphi_h \cdot \boldsymbol{\sigma}\nabla u \, d\Omega + \int_\Omega \varphi_h \, \boldsymbol{\beta} \cdot \nabla u \, d\Omega + \int_\Omega \varphi_h \, \kappa u \, d\Omega$$
$$- \int_\Gamma \varphi_h(\boldsymbol{\sigma}\nabla u \cdot \mathbf{n}) \, d\Gamma = \int_\Omega \varphi_h \, f \, d\Omega. \tag{3.34}$$

Next, we consider the BC of Eq. (3.2b) multiplied by the function $\xi_4^b \varphi_h$, where ξ_4^b is defined in Eq. (3.12), with $k = 4$. By integrating this equation over Γ_b, for all $b = 1, \ldots, N_b^h$, we obtain:

$$\int_{\Gamma_b} \varphi_h \left(\frac{\xi}{\xi + \gamma h_b}\right) (\boldsymbol{\sigma}\nabla u \cdot \mathbf{n}) \, d\Gamma + \int_{\Gamma_b} \varphi_h \left(\frac{\xi\gamma |\boldsymbol{\sigma}|}{\xi + \gamma h_b}\right) u \, d\Gamma = \int_{\Gamma_b} \varphi_h \left(\frac{\xi}{\xi + \gamma h_b}\right) (G + \gamma |\boldsymbol{\sigma}| g) \, d\Gamma. \tag{3.35}$$

Similarly, by multiplying the BC of Eq. (3.2b) by the function $\left(-\dfrac{1}{|\boldsymbol{\sigma}|}\xi_3^b\,\boldsymbol{\Phi}_{in}^*(\varphi_h)\cdot\mathbf{n}\right)$, where ξ_3^b is defined in Eq. (3.12), with $k=3$, and $\boldsymbol{\Phi}_{in}^*(\cdot)$ in Eq. (3.8), and by integrating the resulting equation over Γ_b, for all $b=1,\ldots,N_b^h$, we have:

$$
\int_{\Gamma_b}\boldsymbol{\Phi}_{in}^*(\varphi_h)\cdot\mathbf{n}\left(-\frac{1}{|\boldsymbol{\sigma}|}\frac{h_b}{\xi+\gamma h_b}\right)(\boldsymbol{\sigma}\nabla u\cdot\mathbf{n})\,\mathrm{d}\Gamma+\int_{\Gamma_b}\boldsymbol{\Phi}_{in}^*(\varphi_h)\cdot\mathbf{n}\left(-\frac{\gamma h_b}{\xi+\gamma h_b}\right)u\,\mathrm{d}\Gamma
$$
$$
=\int_{\Gamma_b}\boldsymbol{\Phi}_{in}^*(\varphi_h)\cdot\mathbf{n}\left(-\frac{1}{|\boldsymbol{\sigma}|}\frac{h_b}{\xi+\gamma h_b}\right)(G+\gamma\,|\boldsymbol{\sigma}|\,g)\,\mathrm{d}\Gamma.
\tag{3.36}
$$

Therefore, by summing side by side Eqs. (3.34), (3.35), and (3.36), by noticing that $\displaystyle\sum_{b=1}^{N_b^h}\int_{\Gamma_b}\cdot\,\mathrm{d}\Gamma\equiv$ $\displaystyle\int_{\Gamma}\cdot\,\mathrm{d}\Gamma$, and by using the definitions of Eqs. (3.5) and (3.6), the thesis follows since $\displaystyle\int_{\Omega}\varphi_h\,\frac{\partial u_h}{\partial t}\,\mathrm{d}\Omega+$ $a_h(t;u_h,\varphi_h)=\mathcal{F}_h(t;\varphi_h)$ for all $\varphi_h\in\mathcal{V}_h$. $\qquad\square$

In the following two lemmas we show the continuity and coercivity of the bilinear form $a_h(\cdot;\cdot,\cdot)$ in Eq. (3.5) and the continuity the functional $\mathcal{F}_h(\cdot;\cdot)$ in Eq. (3.6) both in terms of the norm $\|\cdot\|_h$.

Lemma 3.7. *Let* $\boldsymbol{\sigma}_0:=\inf\limits_{(\mathbf{x},t)\in Q_T}\{\boldsymbol{\sigma}(\mathbf{x},t)\}$, $\kappa_0:=\inf\limits_{(\mathbf{x},t)\in Q_T}\{\kappa(\mathbf{x},t)\}$ *and*

$$
\bar{\alpha}:=\frac{\boldsymbol{\sigma}_0+\frac{1}{2}\inf_{Q_T}\{\boldsymbol{\beta}\cdot\mathbf{n}\}C_{tr}^2-\frac{3}{2}\|\boldsymbol{\beta}\cdot\mathbf{n}\|_{L^\infty(Q_T)}C_{tr}^2}{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}}.
\tag{3.37}
$$

Moreover, let the data of problem (3.4) satisfy the following conditions:

$$
\bar{\alpha}>0\quad\text{and}\quad\kappa_0+\frac{1}{2}\inf_{Q_T}\{\boldsymbol{\beta}\cdot\mathbf{n}\}C_{tr}^2-\frac{3}{2}\|\boldsymbol{\beta}\cdot\mathbf{n}\|_{L^\infty(Q_T)}C_{tr}^2>0.
\tag{3.38}
$$

In addition, let the penalty function ξ satisfy the condition:

$$
\xi(\mathbf{x},t)>h_b\,\chi_{\Gamma_b}(\mathbf{x})\quad\forall(\mathbf{x},t)\in\Sigma,\ \forall b=1,\ldots,N_b^h,
\tag{3.39}
$$

and its lower bound $\bar{\xi}_0$ satisfy the following one:

$$
\bar{\xi}_0>\frac{\overline{C}_2}{\bar{\alpha}}\left(\frac{\|\boldsymbol{\beta}\cdot\mathbf{n}\|_{L^\infty(Q_T)}}{2\boldsymbol{\sigma}_0}+\frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}}{2\boldsymbol{\sigma}_0}\right),
\tag{3.40}
$$

with $\overline{C}_2$ the constant associated to the trace inequality (3.16b). Then, there exist positive constants $M^a>0$ and $\alpha^a>0$, both independent of h, such that:

    i. $|a_h(t;u_h,\varphi_h)|\leq M^a\,\|u_h\|_h\,\|\varphi_h\|_h\qquad\forall u_h,\varphi_h\in\mathcal{V}_h$ *and a.e. in $(0,T)$;*

    ii. $a_h(t;\varphi_h,\varphi_h)\geq\alpha^a\,\|\varphi_h\|_h^2\qquad\forall\varphi_h\in\mathcal{V}_h$ *and a.e. in $(0,T)$.*

Proof. i.) For $u_h, \varphi_h \in \mathcal{V}_h$ and by applying the triangle inequality to $|a_h(t; u_h, \varphi_h)|$, we have:

$$
|a_h(t; u_h, \varphi_h)| \leq |\, a_{\boldsymbol{\sigma},\boldsymbol{\beta},\kappa}(t; u_h, \varphi_h)| + \sum_{b=1}^{N_b^h} \left[\left| \int_{\Gamma_b} \varphi_h \left(-\frac{\gamma h_b}{\xi + \gamma h_b} \right) (\boldsymbol{\sigma} \nabla u_h \cdot \mathbf{n}) \, \mathrm{d}\Gamma \right| \right.
$$
$$
+ \left| \int_{\Gamma_b} (\boldsymbol{\sigma} \nabla \varphi_h \cdot \mathbf{n} + \varphi_h (\boldsymbol{\beta} \cdot \mathbf{n}) \chi_{in}) \left(-\frac{\gamma h_b}{\xi + \gamma h_b} \right) u_h \, \mathrm{d}\Gamma \right| + \left| \int_{\Gamma_b} \varphi_h \left(\frac{|\boldsymbol{\sigma}| \xi \gamma}{\xi + \gamma h_b} \right) u_h \, \mathrm{d}\Gamma \right|
$$
$$
+ \left. \left| \int_{\Gamma_b} (\boldsymbol{\sigma} \nabla \varphi_h \cdot \mathbf{n} + \varphi_h (\boldsymbol{\beta} \cdot \mathbf{n}) \chi_{in}) \left(-\frac{h_b}{|\boldsymbol{\sigma}| (\xi + \gamma h_b)} \right) (\boldsymbol{\sigma} \nabla u_h \cdot \mathbf{n}) \, \mathrm{d}\Gamma \right| \right].
$$

Moreover, by using the hypothesis (3.39), we have $\xi_1^b(\mathbf{x}, t) \leq \xi_2^b(\mathbf{x}, t)$. Then, in combination with the bounds (3.14), we have:

$$
|a_h(t; u_h, \varphi_h)| \leq \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} |\varphi_h|_{H^1(\Omega)} |u_h|_{H^1(\Omega)} + \|\boldsymbol{\beta}\|_{L^\infty(Q_T)} \|\varphi_h\|_{L^2(\Omega)} |u_h|_{H^1(\Omega)}
$$
$$
+ \|\kappa\|_{L^\infty(Q_T)} \|\varphi_h\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)}
$$
$$
+ \sum_{b=1}^{N_b^h} \left[\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \|\zeta \varphi_h\|_{L^2(\Gamma_b)} \left\| \sqrt{\frac{h_b}{\xi}} \sqrt{\frac{\gamma h_b}{\xi + \gamma h_b}} \, \nabla u_h \cdot \mathbf{n} \right\|_{L^2(\Gamma_b)} \right.
$$
$$
+ \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \left\| \sqrt{\frac{h_b}{\xi}} \sqrt{\frac{\gamma h_b}{\xi + \gamma h_b}} \, \nabla \varphi_h \cdot \mathbf{n} \right\|_{L^2(\Gamma_b)} \|\zeta u_h\|_{L^2(\Gamma_b)}
$$
$$
+ \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} \|\zeta \varphi_h\|_{L^2(\Gamma_b)} \|\zeta u_h\|_{L^2(\Gamma_b)} + |\boldsymbol{\sigma}| \|\zeta \varphi_h\|_{L^2(\Gamma_b)} \|\zeta u_h\|_{L^2(\Gamma_b)}
$$
$$
+ \frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}^2}{|\boldsymbol{\sigma}|} \frac{h}{\bar{\xi}_0} \|\nabla \varphi_h \cdot \mathbf{n}\|_{L^2(\Gamma_b)} \|\nabla u_h \cdot \mathbf{n}\|_{L^2(\Gamma_b)}
$$
$$
+ \left. \frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}}{|\boldsymbol{\sigma}|} \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} \frac{h}{\bar{\xi}_0} \|\varphi_h\|_{L^2(\Gamma_b)} \|\nabla u_h \cdot \mathbf{n}\|_{L^2(\Gamma_b)} \right].
$$

By using the Hölder inequality, (3.16), (3.25b), and (3.21), we have:

$$
|a_h(t; u_h, \varphi_h)| \leq \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} |\varphi_h|_{H^1(\Omega)} |u_h|_{H^1(\Omega)} + \|\boldsymbol{\beta}\|_{L^\infty(Q_T)} \frac{\sqrt{2}}{C_\Omega} \|\varphi_h\|_h |u_h|_{H^1(\Omega)}
$$
$$
+ \|\kappa\|_{L^\infty(Q_T)} \frac{2}{C_\Omega^2} \|\varphi_h\|_h \|u_h\|_h
$$
$$
+ \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \|\varphi_h\|_h \sqrt{\frac{2\overline{C}_2 C_{\Omega_B}}{\bar{\xi}_0}} \|u_h\|_h + \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \sqrt{\frac{2\overline{C}_2 C_{\Omega_B}}{\bar{\xi}_0}} \|\varphi_h\|_h \|u_h\|_h
$$
$$
+ \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} \|\varphi_h\|_h \|u_h\|_h + |\boldsymbol{\sigma}| \|\varphi_h\|_h \|u_h\|_h + \frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}^2}{|\boldsymbol{\sigma}|} \frac{\overline{C}_2}{\bar{\xi}_0} C_{\Omega_B} \|\varphi_h\|_h \|u_h\|_h
$$
$$
+ \frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}}{|\boldsymbol{\sigma}|} \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} \frac{C_{\Omega_B} \sqrt{\overline{C}_1 \overline{C}_2}}{\bar{\xi}_0} \|\varphi_h\|_h \|u_h\|_h.
$$

Since $|\cdot|_{H^1(\Omega)} \leq \|\cdot\|_h$, the continuity of $a_h(\cdot; \cdot, \cdot)$ follows by setting:

$$M^a := \max\left\{ \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \max\left\{1, \sqrt{\frac{2\overline{C_2}C_{\Omega_B}}{\overline{\xi}_0}}, \frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}}{|\boldsymbol{\sigma}|}\frac{\overline{C_2}}{\overline{\xi}_0}C_{\Omega_B}, \frac{\|\boldsymbol{\beta}\cdot\mathbf{n}\|_{L^\infty(Q_T)}}{|\boldsymbol{\sigma}|}\frac{C_{\Omega_B}\sqrt{\overline{C_1}\overline{C_2}}}{\overline{\xi}_0}\right\}, \right.$$
$$\left. \|\boldsymbol{\beta}\|_{L^\infty(Q_T)}\frac{\sqrt{2}}{C_\Omega}, \|\kappa\|_{L^\infty(Q_T)}\frac{2}{C_\Omega^2}, \|\boldsymbol{\beta}\cdot\mathbf{n}\|_{L^\infty(Q_T)}, |\boldsymbol{\sigma}|\right\}.$$
$$(3.41)$$

$ii.)$ To show the coercivity of the bilinear form, we firstly focus on its boundary terms:

$$a_\gamma(t; \varphi_h, \varphi_h) := \sum_{b=1}^{N_b^h} \int_{\Gamma_b} \varphi_h\left(-\frac{\gamma h_b}{\xi + \gamma h_b}\right)(\boldsymbol{\sigma}\nabla\varphi_h \cdot \mathbf{n})\,\mathrm{d}\Gamma + \int_{\Gamma_b}(\boldsymbol{\Phi}_{in}^*(\varphi_h)\cdot\mathbf{n})\left(-\frac{\gamma h_b}{\xi + \gamma h_b}\right)\varphi_h\,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_b} \varphi_h\left(\frac{|\boldsymbol{\sigma}|\xi\gamma}{\xi + \gamma h_b}\right)\varphi_h\,\mathrm{d}\Gamma + \int_{\Gamma_b}(\boldsymbol{\Phi}_{in}^*(\varphi_h)\cdot\mathbf{n})\left(-\frac{h_b}{|\boldsymbol{\sigma}|(\xi + \gamma h_b)}\right)(\boldsymbol{\sigma}\nabla\varphi_h\cdot\mathbf{n})\,\mathrm{d}\Gamma.$$
$$(3.42)$$

We have:

$$a_\gamma(t; \varphi_h, \varphi_h) \geq \sum_{b=1}^{N_b^h} \underbrace{(-2)\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}\left\|\xi_1^b\right\|_{L^\infty(\Gamma_b)}\|\varphi_h\|_{L^2(\Gamma_b)}\|\nabla\varphi_h\cdot\mathbf{n}\|_{L^2(\Gamma_b)}}_{a}$$
$$\underbrace{-\frac{\|\boldsymbol{\beta}\cdot\mathbf{n}\|_{L^\infty(Q_T)}}{\boldsymbol{\sigma}_0}\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}\left\|\xi_3^b\right\|_{L^\infty(\Gamma_b)}\|\varphi_h\|_{L^2(\Gamma_b)}\|\nabla\varphi_h\cdot\mathbf{n}\|_{L^2(\Gamma_b)}}_{(b)}$$
$$+ \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}\|\zeta\varphi_h\|_{L^2(\Gamma_b)}^2 \underbrace{-\int_{\Gamma_b}(\boldsymbol{\beta}\cdot\mathbf{n})\chi_{in}\,\xi_1^b\,\varphi_h^2\,\mathrm{d}\Gamma}_{(c)}$$
$$\underbrace{-\frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}^2}{\boldsymbol{\sigma}_0}\left\|\xi_3^b\right\|_{L^\infty(\Gamma_b)}\|\nabla\varphi_h\cdot\mathbf{n}\|_{L^2(\Gamma_b)}^2}_{(d)}.$$
$$(3.43)$$

The term (a) is bounded using the Young's inequalty $\iota\upsilon \leq \delta\iota^2 + \frac{1}{4\delta}\upsilon^2$ for $\iota, \upsilon \in \mathbb{R}$, for all $\delta > 0$, where we set $\iota = \frac{h}{\overline{\xi}_0}\|\zeta\|_{L^\infty(\Gamma_b)}\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}^{1/2}\|\nabla\varphi_h\cdot\mathbf{n}\|_{L^2(\Gamma_b)}$ and $\upsilon = 2\|\zeta\|_{L^\infty(\Gamma_b)}\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}^{1/2}\|\varphi_h\|_{L^2(\Gamma_b)}$. Then, by considering (3.16b), we have:

$$(a) \geq -\sum_{b=1}^{N_b^h}\left\{\frac{1}{\delta}\frac{h^2}{\overline{\xi}_0^2}\|\zeta\|_{L^\infty(\Gamma_b)}^2\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}\|\nabla\varphi_h\cdot\mathbf{n}\|_{L^2(\Gamma_b)}^2 + \delta\|\zeta\|_{L^\infty(\Gamma_b)}^2\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}\|\varphi_h\|_{L^2(\Gamma_b)}^2\right\}$$
$$\geq -\sum_{b=1}^{N_b^h}\left\{\frac{1}{\delta}\frac{h}{\overline{\xi}_0^2}\|\zeta\|_{L^\infty(\Gamma_b)}^2\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}\overline{C_2}|\varphi_h|_{H^1(T_b)}^2 + \delta\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}\|\zeta\varphi_h\|_{L^2(\Gamma_b)}^2\right\};$$
$$(3.44)$$

The term (b) is bounded by applying the previous Young's inequalty with $\iota = \|\varphi_h\|_{L^2(\Gamma_b)}$, $\upsilon = \|\nabla\varphi_h \cdot \mathbf{n}\|_{L^2(\Gamma_b)}$, and $\delta = \dfrac{1}{2}$, together with (3.16b), as:

$$
\begin{aligned}
(b) &\geq -\sum_{b=1}^{N_b^h} \frac{\|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)}}{\boldsymbol{\sigma}_0} \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \left\|\xi_3^b\right\|_{L^\infty(\Gamma_b)} \left(\frac{\|\varphi_h\|_{L^2(\Gamma_b)}^2}{2} + \frac{\|\nabla\varphi_h \cdot \mathbf{n}\|_{L^2(\Gamma_b)}^2}{2} \right) \\
&\geq -\sum_{b=1}^{N_b^h} \frac{\|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)}}{\boldsymbol{\sigma}_0} \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \left(\frac{\overline{\xi}_\infty}{2\overline{\xi}_0} \|\zeta\varphi_h\|_{L^2(\Gamma_b)}^2 - \left\|\frac{1}{\xi + \gamma h_b}\right\|_{L^\infty(\Gamma)} \overline{C}_2 \, |\varphi_h|_{H^1(T_b)}^2 \right).
\end{aligned}
\tag{3.45}
$$

As for term (c), we simply observe that:

$$
(c) \geq -2\sum_{b=1}^{N_b^h} \int_{\Gamma_b} (\boldsymbol{\beta} \cdot \mathbf{n})\chi_{in}\varphi_h^2 \, \mathrm{d}\Gamma \geq -2 \int_{\Gamma} (\boldsymbol{\beta} \cdot \mathbf{n})\chi_{in}\varphi_h^2 \, \mathrm{d}\Gamma.
\tag{3.46}
$$

Finally, the term (d) can be bounded similarly to term (b), for which we have:

$$
(d) \geq -\sum_{b=1}^{N_b^h} \left\|\frac{1}{\xi + \gamma h_b}\right\|_{L^\infty(\Gamma)} \frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}^2}{\boldsymbol{\sigma}_0} \overline{C}_2 \, |\varphi_h|_{H^1(T_b)}^2.
\tag{3.47}
$$

On the other hand, the hypothesis $\nabla \cdot \boldsymbol{\beta} = 0$ a.e. in Ω yields the identity $\displaystyle\int_{\Omega} \varphi_h \boldsymbol{\beta} \cdot \nabla\varphi_h \, \mathrm{d}\Omega = \dfrac{1}{2} \int_{\Gamma} \varphi_h^2 \, \boldsymbol{\beta} \cdot \mathbf{n} \, \mathrm{d}\Gamma$. Then, using the trace inequality (3.26), we can bound the sum of $a_{\boldsymbol{\sigma},\boldsymbol{\beta},\kappa}(t; \varphi_h, \varphi_h)$ and term (c) as:

$$
\begin{aligned}
a_{\boldsymbol{\sigma},\boldsymbol{\beta},\kappa}(t; \varphi_h, \varphi_h) + (c) &\geq \boldsymbol{\sigma}_0 \, |\varphi_h|_{H^1(\Omega)}^2 + \kappa_0 \, \|\varphi_h\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Gamma} (\boldsymbol{\beta} \cdot \mathbf{n})[\chi_{in} + \chi_{out}]\varphi_h^2 \, \mathrm{d}\Gamma \\
&\quad - 2 \int_{\Gamma} (\boldsymbol{\beta} \cdot \mathbf{n})\chi_{in}\varphi_h^2 \, \mathrm{d}\Gamma \\
&\geq \boldsymbol{\sigma}_0 \, |\varphi_h|_{H^1(\Omega)}^2 + \kappa_0 \, \|\varphi_h\|_{L^2(\Omega)}^2 + \left(\frac{1}{2} \inf_{Q_T}\{\boldsymbol{\beta} \cdot \mathbf{n}\} - \frac{3}{2} \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} \right) \|\varphi_h\|_{L^2(\Gamma)}^2 \\
&\geq \left(\boldsymbol{\sigma}_0 + \frac{1}{2} \inf_{Q_T}\{\boldsymbol{\beta} \cdot \mathbf{n}\} - \frac{3}{2} \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} \right) C_{tr}^2 \, |\varphi_h|_{H^1(\Omega)}^2 \\
&\quad + \left(\kappa_0 + \frac{1}{2} \inf_{Q_T}\{\boldsymbol{\beta} \cdot \mathbf{n}\} - \frac{3}{2} \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} \right) C_{tr}^2 \, \|\varphi_h\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{3.48}
$$

By combining (3.48), (3.44), (3.45), and (3.46), under the hypothesis (3.38), we have:

$$
\begin{aligned}
a_h(t; \varphi_h, \varphi_h) = \ & a_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa}(t; \varphi_h, \varphi_h) + a_\gamma(t; \varphi_h, \varphi_h) \\
\geq \ & \left(\boldsymbol{\sigma}_0 + \frac{1}{2} \inf_{Q_T} \{ \boldsymbol{\beta} \cdot \mathbf{n} \} C_{tr}^2 - \frac{3}{2} \| \boldsymbol{\beta} \cdot \mathbf{n} \|_{L^\infty(Q_T)} C_{tr}^2 \right) | \varphi_h |_{H^1(\Omega)}^2 \\
& - \frac{1}{\delta} \frac{h}{\overline{\xi}_0^2} \| \zeta \|_{L^\infty(\Gamma_b)}^2 \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \overline{C}_2 | \varphi_h |_{H^1(\Omega)}^2 - \delta \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \sum_{b=1}^{N_b^h} \| \zeta \varphi_h \|_{L^2(\Gamma_b)}^2 \\
& - \frac{\| \boldsymbol{\beta} \cdot \mathbf{n} \|_{L^\infty(Q_T)}}{2 \boldsymbol{\sigma}_0} \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \frac{\overline{\xi}_\infty}{\overline{\xi}_0} \sum_{b=1}^{N_b^h} \| \zeta \varphi_h \|_{L^2(\Gamma_b)}^2 \\
& - \frac{\| \boldsymbol{\beta} \cdot \mathbf{n} \|_{L^\infty(Q_T)}}{2 \boldsymbol{\sigma}_0} \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \left\| \frac{1}{\xi + \gamma h_b} \right\|_{L^\infty(\Gamma)} \overline{C}_2 | \varphi_h |_{H^1(\Omega)}^2 \cdot \\
& + \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \sum_{b=1}^{N_b^h} \| \zeta \varphi_h \|_{L^2(\Gamma_b)}^2 - \left\| \frac{1}{\xi + \gamma h_b} \right\|_{L^\infty(\Gamma)} \frac{\| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)}^2}{\boldsymbol{\sigma}_0} \overline{C}_2 | \varphi_h |_{H^1(\Omega)}^2 ,
\end{aligned}
\tag{3.49}
$$

Introducing the constant $\bar{\alpha} > 0$ of (3.37) into (3.49); we obtain:

$$
\begin{aligned}
a_h(t; \varphi_h, \varphi_h) \geq \ & \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \bar{\alpha} | \varphi_h |_{H^1(\Omega)}^2 \\
& - \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \overline{C}_2 \left[\frac{1}{\delta} \frac{h}{\overline{\xi}_0^2} \| \zeta \|_{L^\infty(\Gamma)}^2 + \left\| \frac{1}{\xi + \gamma h_b} \right\|_{L^\infty(\Gamma)} \left(\frac{\| \boldsymbol{\beta} \cdot \mathbf{n} \|_{L^\infty(Q_T)}}{2 \boldsymbol{\sigma}_0} + \frac{\| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)}}{2 \boldsymbol{\sigma}_0} \right) \right] | \varphi_h |_{H^1(\Omega)}^2 \\
& + \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \left(1 - \delta - \frac{\| \boldsymbol{\beta} \cdot \mathbf{n} \|_{L^\infty(Q_T)}}{2 \boldsymbol{\sigma}_0} \frac{\overline{\xi}_\infty}{\overline{\xi}_0} \right) \sum_{b=1}^{N_b^h} \| \zeta \varphi_h \|_{L^2(\Gamma_b)}^2 \\
\geq \ & \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \bar{\alpha} \| \zeta \|_{L^\infty(\Gamma_b)}^2 \left[\left(\underset{\mathbf{x} \in \Gamma}{\operatorname{ess\,inf}} \frac{1}{\gamma} \right) \left(1 - \frac{\overline{C}_2}{\bar{\alpha}} \frac{1}{\overline{\xi}_0} \left(\frac{\| \boldsymbol{\beta} \cdot \mathbf{n} \|_{L^\infty(Q_T)}}{2 \boldsymbol{\sigma}_0} + \frac{\| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)}}{2 \boldsymbol{\sigma}_0} \right) \right) \right. \\
& \left. + \frac{h}{\overline{\xi}_\infty} \left(1 - \frac{\overline{C}_2}{\bar{\alpha}} \frac{1}{\delta} \frac{\overline{\xi}_\infty}{\overline{\xi}_0^2} \right) \right] | \varphi_h |_{H^1(\Omega)}^2 + \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \left(1 - \delta - \frac{\| \boldsymbol{\beta} \cdot \mathbf{n} \|_{L^\infty(Q_T)}}{2 \boldsymbol{\sigma}_0} \frac{\overline{\xi}_\infty}{\overline{\xi}_0} \right) \sum_{b=1}^{N_b^h} \| \zeta \varphi_h \|_{L^2(\Gamma_b)}^2 \cdot
\end{aligned}
\tag{3.50}
$$

By the hypothesis (3.40) on $\overline{\xi}_0$ and selecting δ such that $\dfrac{\overline{C}_2}{\bar{\alpha}} \dfrac{\overline{\xi}_\infty}{\overline{\xi}_0^2} < \delta < 1 - \dfrac{\| \boldsymbol{\beta} \cdot \mathbf{n} \|_{L^\infty(Q_T)}}{2 \boldsymbol{\sigma}_0} \dfrac{\overline{\xi}_\infty}{\overline{\xi}_0}$, the coercivity follows with constant:

$$
\alpha^a := \| \boldsymbol{\sigma} \|_{W^{1,\infty}(Q_T)} \min \left\{ \bar{\alpha} \left(1 - \frac{\overline{C}_2}{\bar{\alpha}} \frac{1}{\delta} \frac{\overline{\xi}_\infty}{\overline{\xi}_0^2} \right), \left(1 - \delta - \frac{\| \boldsymbol{\beta} \cdot \mathbf{n} \|_{L^\infty(Q_T)}}{2 \boldsymbol{\sigma}_0} \frac{\overline{\xi}_\infty}{\overline{\xi}_0} \right) \right\} > 0.
\tag{3.51}
$$

$\square$

Lemma 3.8. *Under the hypotheses of Lemma 3.7, there exists a positive constant $M_h^{\mathcal{F}}$, dependent on h and on the data of the problem, such that:*

$$
| \mathcal{F}_h(t; \varphi_h) | \leq M_h^{\mathcal{F}} \| \varphi_h \|_h \qquad \forall \varphi_h \in \mathcal{V}_h \text{ and a.e. in } (0, T).
$$

Proof. The continuity of the functional $\mathcal{F}_h$ follows by the triangle and Schwarz inequalities and by applying (3.14):

$$
\begin{aligned}
|\mathcal{F}_h(t; \varphi_h)| \leq & \left| \int_{\Omega} \varphi_h \, f \, \mathrm{d}\Omega \right| + \left| \sum_{b=1}^{N_b^h} \left[\int_{\Gamma_b} (\mathbf{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) \left(-\frac{\gamma h_b}{\xi + \gamma h_b} \right) g \, \mathrm{d}\Gamma + \int_{\Gamma_b} \varphi_h \left(\frac{|\boldsymbol{\sigma}| \xi \gamma}{\xi + \gamma h_b} \right) g \, \mathrm{d}\Gamma \right. \right. \\
& \left. \left. + \int_{\Gamma_b} (\mathbf{\Phi}_{in}^*(\varphi_h) \cdot \mathbf{n}) \left(-\frac{h_b}{|\boldsymbol{\sigma}| (\xi + \gamma h_b)} \right) G \, \mathrm{d}\Gamma + \int_{\Gamma_b} \varphi_h \left(\frac{\xi}{\xi + \gamma h_b} \right) G \, \mathrm{d}\Gamma \right] \right| \\
\leq & \, \|\varphi_h\|_{L^2(\Omega)} \|f\|_{L^\infty(0,T;L^2(\Omega))} + \left| \int_{\Gamma} \varphi_h \, G \, \mathrm{d}\Gamma \right| \\
& + \sum_{b=1}^{N_b^h} \left[2 \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \|g\|_{L^2(\Gamma_b)} \|\nabla \varphi_h \cdot \mathbf{n}\|_{L^2(\Gamma_b)}^2 \right. \\
& + \sqrt{\frac{2 h_b}{\overline{\xi}_0}} \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} \|g\|_{L^2(\Gamma_b)} \|\zeta \varphi_h\|_{L^2(\Gamma_b)} \\
& + |\boldsymbol{\sigma}| \sqrt{\frac{\overline{\xi}_\infty}{h_b}} \|g\|_{L^2(\Gamma_b)} \|\zeta \varphi_h\|_{L^2(\Gamma_b)} + \frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}}{|\boldsymbol{\sigma}|} \frac{h_b}{\overline{\xi}_0} \|G\|_{L^2(\Gamma_b)} \|\nabla \varphi_h \cdot \mathbf{n}\|_{L^2(\Gamma_b)} \\
& \left. + \frac{\|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)}}{|\boldsymbol{\sigma}|} \frac{h_b}{\overline{\xi}_0} \|G\|_{L^2(\Gamma_b)} \|\varphi_h\|_{L^2(\Gamma_b)} \right].
\end{aligned}
\tag{3.52}
$$

Moreover, by using (3.16), (3.21), (3.25a), and (3.25b), we obtain the thesis by defining the continuity constant $M_h^{\mathcal{F}}$ as:

$$
\begin{aligned}
M_h^{\mathcal{F}} := & \frac{\sqrt{2}}{C_\Omega} \|f\|_{L^\infty(0,T;L^2(\Omega))} \\
& + \left(\frac{\sqrt{2} C_{tr}}{C_\Omega} + \frac{\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)}}{|\boldsymbol{\sigma}|} \sqrt{\frac{C_{\Omega_B} \overline{C}_2}{\overline{\xi}_0}} + \frac{\|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)}}{|\boldsymbol{\sigma}|} \sqrt{\frac{C_{\Omega_B} \overline{C}_1}{\overline{\xi}_0}} \right) \|G\|_{L^\infty(0,T;L^2(\partial\Omega))} \\
& + \left(\sqrt{2} \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} + \frac{1}{\sqrt{h}} \left(2 \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \sqrt{C_{\Omega_B} \overline{C}_2} + |\boldsymbol{\sigma}| \sqrt{\overline{\xi}_\infty} \right) \right) \|g\|_{L^\infty(0,T;L^2(\partial\Omega))} \cdot
\end{aligned}
\tag{3.53}
$$

$\square$

Remark 3.3. *The constant $M_h^{\mathcal{F}}$ of Eq. (3.53) depends on the mesh size h and actually tends to infinity as $h \to 0$. This result is expected, since analogous results are obtained for the weak imposition of the Dirichlet BCs [27] and follows from the fact that we use a mesh dependent norm $\|\cdot\|_h$. A continuity constant $\widetilde{M}^{\mathcal{F}}$ independent of h for the functional $\mathcal{F}_h(t; \cdot)$ can however be ensured for boundary data g and G sufficiently regular, i.e. such that their extensions to the domain Ω are well-defined, as well as their corresponding $\|\cdot\|_h$ norms; see [31].*

The semi-discetized FE solution of problem (3.4) obtained by using the Nitsche's method, can be expressed, for each time $t \in (0,T)$, by using the basis functions $\varphi_{h,i}$ of the FE space $\mathcal{V}_h$ of Eq. (3.3) as $u_h(\mathbf{x}, t) := \sum_{i=1}^{N^{(h)}} d_i(t) \varphi_{h,i}(\mathbf{x})$, where $N^{(h)} := \dim \mathcal{V}_h$. By substituting the expression of

u_h in Eq. (3.4), we obtain a system of linear ordinary differential equations in the unknown vector of functions $\boldsymbol{d} := (d_1, \ldots, d_{N^{(h)}}) : (0, T) \to \mathbb{R}^{N^{(h)}}$, which reads:

$$\begin{cases} \boldsymbol{M}\dot{\boldsymbol{d}}(t) + \boldsymbol{A}(t)\boldsymbol{d}(t) = \boldsymbol{F}(t) & \text{for } t \in (0, T), \\ \boldsymbol{M}\boldsymbol{d}(0) = \widetilde{\boldsymbol{u}}_0, \end{cases} \tag{3.54}$$

where $\boldsymbol{M}$ and $\boldsymbol{A}$ are the mass and stiffness matrices, defined as $\boldsymbol{M}_{i,j} := (\varphi_{h,j}, \varphi_{h,i})$ and $\boldsymbol{A}_{i,j}(t) := a_h(t; \varphi_{h,j}, \varphi_{h,i})$, for all $i, j = 1, \ldots, N^{(h)}$, respectively. Moreover, $\boldsymbol{F}(t)$ is a vector whose components are $\boldsymbol{F}_i(t) := \mathcal{F}_h(t; \varphi_{h,i})$ and the vector $\widetilde{\boldsymbol{u}}_0$ is such that $\widetilde{\boldsymbol{u}}_{0,i} := (\widetilde{u}_0, \varphi_{h,i})$, with $\widetilde{u}_0$ being the L^2 projection of u_0 onto $\mathcal{V}_h$. By definition, the matrix $\boldsymbol{M}$ is symmetric and positive definite; moreover, by the coercivity of the bilinear form $a_h(t; \cdot, \cdot)$ established in Lemma 3.7 (ii.), also the matrix $\boldsymbol{A}$ is positive definite. From the continuity of both $a_h(t; \cdot, \cdot)$ and $\mathcal{F}_h$ shown in Lemmas 3.7 and 3.8, the existence and uniqueness of the solution $\boldsymbol{d}$ of system (3.54), and consequently of the semi-discrete solution u_h, follows from the theory of ordinary differential equations [12, 26, 27].

3.2 The full discrete problem: the Nitsche's-θ method

Let $N_T \in \mathbb{N}$ be a given integer and let us uniformly partition the time interval $(0, T)$ into a set of subintervals $\{[t^n, t^{n+1})\}_{n=0}^{N_T-1}$ of size $\Delta t := T/N_T$, with $t^n := n\Delta t$. By applying the θ-method [25] to the semi-discrete problem (3.4), this yields, for $n = 1, \ldots, N_T$:

$$\text{find } u_h^n \in \mathcal{V}_h : \quad \begin{cases} \dfrac{1}{\Delta t}\left(u_h^n - u_h^{n-1}, \varphi_h\right) + a_h(t_\theta^n; u_{h,\theta}^n, \varphi_h) = \mathcal{F}_h(t_\theta^n; \varphi_h) & \forall \varphi_h \in \mathcal{V}_h, \text{ (3.55a)} \\ \widetilde{u}_h^0 = \widetilde{u}_0; & \text{(3.55b)} \end{cases}$$

where, for $\theta \in [0, 1]$, we have considered $t_\theta^n := \theta t^n + (1-\theta)t^{n-1}$, $u_{h,\theta}^n := \theta u_h^n + (1-\theta)u_h^{n-1}$, being u_h^n the approximation of $u_h(\mathbf{x}, t^n)$. Moreover, for $\widetilde{u}_0 \in \mathcal{V}_h$, the L^2 projection of u_0 onto $\mathcal{V}_h$, we have from [26]:

$$\left\|u_0 - \widetilde{u}_h^0\right\|_{L^2(\Omega)} + h\left\|u_0 - \widetilde{u}_h^{-1}\right\|_{H^1(\Omega)} \leq Ch\left\|u_0\right\|_{H^1(\Omega)}. \tag{3.56}$$

In the following, we will show that Eq. (3.55) is unconditionally stable with respect to the L^2 norm provided that $\theta \in [1/2, 1]$, while is conditionally stable for $\theta \in [0, 1/2)$.

Theorem 3.1. *Let us consider problem* (3.55) *and a regular family of quasi-uniform triangulations* $\{\mathcal{T}_h\}_h$ *of* Ω. *Moreover, for* $\theta \in [0, 1/2)$, *assume that the following restriction holds on the time step:*

$$\Delta t < \frac{2\alpha^a h^2}{(1-2\theta)(M^a)^2 \widetilde{C}_0^2}, \tag{3.57}$$

where α^a *and* M^a, *given in Eqs.* (3.41) *and* (3.51), *are the coercivity and continuity constants of the bilinear form* $a_h(\cdot; \cdot, \cdot)$ *of Eq.* (3.5), *whereas* $\widetilde{C}_0$ *is given in* (3.31). *Then, under the hypotheses of Lemmas 3.7 and 3.8,* u_h^n *satisfies the following estimate for any* $\theta \in [0, 1]$:

$$\begin{aligned} \left\|u_h^n\right\|_{L^2(\Omega)} \leq C_{\alpha^a, \widetilde{C}_0, T} \Bigg[&\left\|u_h^0\right\|_{L^2(\Omega)} + M_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa, \Omega}^{(1)} \left(\left\|G\right\|_{L^\infty(0,T;L^2(\Gamma))} + \left\|f\right\|_{L^\infty(0,T;L^2(\Omega))}\right) \\ &+ \left(M_{\boldsymbol{\beta}} + \frac{1}{\sqrt{h}}M_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa, \Omega}^{(2)}\right)\left\|g\right\|_{L^\infty(0,T;L^2(\Gamma))} \Bigg], \end{aligned} \tag{3.58}$$

where $C_{\alpha^a, \widetilde{C}_0, T}$ is a positive constant depending on α^a, $\widetilde{C}_0$, and T, but independent of N, Δt, and h. Moreover, $M^{(1)}_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa, \Omega}$, $M_{\boldsymbol{\beta}}$, and $M^{(2)}_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa, \Omega}$ are positive constants depending on the domain Ω and the data of the problem, reading:

$$M^{(1)}_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa, \Omega} := \max \left\{ \frac{\sqrt{2}}{C_\Omega}, \frac{\sqrt{2} C_{tr}}{C_\Omega} + \frac{1}{|\boldsymbol{\sigma}|} \sqrt{\frac{C_{\Omega_B}}{\overline{\xi}_0}} \left(\|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \sqrt{\overline{C}_2} + \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)} \sqrt{\overline{C}_1} \right) \right\},$$
(3.59a)

$$M_{\boldsymbol{\beta}} = \sqrt{2} \|\boldsymbol{\beta} \cdot \mathbf{n}\|_{L^\infty(Q_T)},$$
(3.59b)

$$M^{(2)}_{\boldsymbol{\sigma}, \boldsymbol{\beta}, \kappa, \Omega} := 2 \|\boldsymbol{\sigma}\|_{W^{1,\infty}(Q_T)} \sqrt{C_{\Omega_B} \overline{C}_2} + |\boldsymbol{\sigma}| \sqrt{\overline{\xi}_\infty}.$$
(3.59c)

Proof. Let us take $\varphi_h = u^n_{h,\theta}$ in Eq. (3.55), then, for any $n \geq 1$, we have:

$$\frac{1}{2} \left(\|u^n_h\|^2_{L^2(\Omega)} - \|u^{n-1}_h\|^2_{L^2(\Omega)} \right) + \left(\theta - \frac{1}{2} \right) \|u^n_h - u^{n-1}_h\|^2_{L^2(\Omega)} + \Delta t \, a_h(t^n_\theta; u^n_{h,\theta}, u^n_{h,\theta}) = \Delta t \, \mathcal{F}_h(t^n_\theta; u^n_{h,\theta}).$$

By the coercivity of the bilinear form $a_h(\cdot; \cdot, \cdot)$ of Lemma 3.7 and by using the continuity constant $M^{\mathcal{F}}_h$ of Eq. (3.53) for the functional $\mathcal{F}_h(\cdot; \cdot)$, we get, for any $n \geq 1$:

$$\|u^n_h\|^2_{L^2(\Omega)} - \|u^{n-1}_h\|^2_{L^2(\Omega)} + (2\theta - 1) \|u^n_h - u^{n-1}_h\|^2_{L^2(\Omega)} + 2\Delta t \, \alpha^a \|u^n_{h,\theta}\|^2_h \leq 2\Delta t \, M^{\mathcal{F}}_h \|u^n_{h,\theta}\|_h.$$

By using a Young's type inequalty, we have for some $\epsilon > 0$

$$\|u^n_h\|^2_{L^2(\Omega)} - \|u^{n-1}_h\|^2_{L^2(\Omega)} + (2\theta - 1) \|u^n_h - u^{n-1}_h\|^2_{L^2(\Omega)} + \Delta t \, (2 - \epsilon) \alpha^a \|u^n_{h,\theta}\|^2_h \leq \Delta t \, \frac{\left(M^{\mathcal{F}}_h \right)^2}{\epsilon \alpha^a}. \quad (3.60)$$

If $\theta \in [1/2, 1]$, by setting $\epsilon = 2$, the left hand side of (3.60) can be bounded as:

$$\|u^n_h\|^2_{L^2(\Omega)} - \|u^{n-1}_h\|^2_{L^2(\Omega)} \leq \Delta t \, \frac{\left(M^{\mathcal{F}}_h \right)^2}{2\alpha^a}. \quad (3.61)$$

If $\theta \in [0, 1/2)$, we start by considering in Eq. (3.55) $\varphi_h = u^n_h - u^{n-1}_h$; then, we have:

$$\|u^n_h - u^{n-1}_h\|^2_{L^2(\Omega)} = -\Delta t \, a_h(t^n_\theta; u^n_{h,\theta}, u^n_h - u^{n-1}_h) + \Delta t \, \mathcal{F}_h(t^n_\theta; u^n_h - u^{n-1}_h).$$

By using Lemmas 3.5, 3.7, and 3.8, we obtain:

$$\|u^n_h - u^{n-1}_h\|^2_{L^2(\Omega)} \leq \Delta t \, M^a \|u^n_{h,\theta}\|_h \|u^n_h - u^{n-1}_h\|_h + \Delta t \, M^{\mathcal{F}}_h \|u^n_h - u^{n-1}_h\|_h \quad (3.62)$$

$$\leq \Delta t \, M^a \|u^n_{h,\theta}\|_h \frac{\widetilde{C}_0}{h} \|u^n_h - u^{n-1}_h\|_{L^2(\Omega)} + \Delta t \, M^{\mathcal{F}}_h \frac{\widetilde{C}_0}{h} \|u^n_h - u^{n-1}_h\|_{L^2(\Omega)}.$$

Therefore, the left hand side of (3.60) can be bounded from below as:

$$\|u^n_h\|^2_{L^2(\Omega)} - \|u^{n-1}_h\|^2_{L^2(\Omega)} + (2\theta - 1) \Delta t^2 \frac{\widetilde{C}_0^2}{h^2} \left((M^a)^2 \|u^n_{h,\theta}\|^2_h + \left(M^{\mathcal{F}}_h \right)^2 \right.$$

$$\left. + 2 \, M^a M^{\mathcal{F}}_h \|u^n_{h,\theta}\|_h \right) + \Delta t \, (2 - \epsilon) \alpha^a \|u^n_{h,\theta}\|^2_h \leq \Delta t \, \frac{\left(M^{\mathcal{F}}_h \right)^2}{\epsilon \alpha^a},$$

for some $\epsilon > 0$. By rearranging terms, we finally obtain:

$$\|u_h^n\|_{L^2(\Omega)}^2 - \|u_h^{n-1}\|_{L^2(\Omega)}^2 + \Delta t \, \|u_{h,\theta}^n\|_h \left((2-\epsilon)\,\alpha^a - (1-2\theta)\,\Delta t \,(M^a)^2 \frac{\widetilde{C}_0^2}{h^2} \right)$$

$$\leq \left(\frac{\Delta t}{\epsilon\alpha^a} + (1-2\theta)\,\Delta t^2 \, \frac{\widetilde{C}_0^2}{h^2} \right) (M_h^{\mathcal{F}})^2 - 2\Delta t^2 \, M^a M_h^{\mathcal{F}} \, \|u_{h,\theta}^n\|_h \, \frac{\widetilde{C}_0^2}{h^2}$$

$$\leq \Delta t \, \max\left\{ \frac{1}{\epsilon\alpha^a}, (1-2\theta)\,\widetilde{C}_0^2 \right\} \left(1 + \Delta t \, \frac{1}{h^2} \right) (M_h^{\mathcal{F}})^2.$$

Under the hypothesis of Eq. (3.57) and by choosing $\epsilon > \dfrac{1-2\theta}{2\alpha^a} > 0$ we obtain:

$$\|u_h^n\|_{L^2(\Omega)}^2 - \|u_h^{n-1}\|_{L^2(\Omega)}^2 \leq C_{\epsilon,\alpha^a,\widetilde{C}_0} \Delta t \, (M_h^{\mathcal{F}})^2, \tag{3.64a}$$

with $C_{\epsilon,\alpha^a,\widetilde{C}_0} := \Delta t \, \max\left\{ \dfrac{1}{\epsilon\alpha^a}, (1-2\theta)\,\widetilde{C}_0^2 \right\} \left(1 + \Delta t \, \dfrac{1}{h^2} \right).$

Therefore, for any $\theta \in [0,1]$ and any fixed integer m such that $1 \leq m \leq N_T$, we obtain:

$$\|u_h^m\|_{L^2(\Omega)}^2 \leq \|u_h^0\|_{L^2(\Omega)}^2 + C_{\alpha^a,\widetilde{C}_0} \Delta t \sum_{n=0}^{m-1} (M_h^{\mathcal{F}})^2 \leq \left(\|u_h^0\|_{L^2(\Omega)} + \sqrt{C_{\alpha^a,\widetilde{C}_0} m \Delta t} M_h^{\mathcal{F}} \right)^2, \tag{3.65}$$

with $C_{\alpha^a,\widetilde{C}_0}$ a constant depending on $\dfrac{1}{\alpha^a}$ and $\widetilde{C}_0$. The thesis follows by the expression of $M_h^{\mathcal{F}}$ of (3.53) by considering $C_{\alpha^a,\widetilde{C}_0,T} := \sqrt{C_{\alpha^a,\widetilde{C}_0} T}$. $\qquad\square$

We notice that the last term of the bound (3.58) tends to $+\infty$ as the mesh size h tends to zero, due to the dependency of $M_h^{\mathcal{F}}$ on h. Therefore, the result of Theorem 3.1 does not imply the stability of the solution with respect to h. Indeed, the finer the mesh size, the larger is the growth of the constant in the a priori estimate. This result is coherent with that of [27]; as a matter of fact the weak imposition of the Dirichlet BCs represents a particular case of Eq. (3.55), as already observed in Remark 3.1. However, as reported in Sec. 4 and [31], stable (i.e. h independent) numerical results are obtained also in this case.

4   A numerical example

We solve, by the proposed method, the heat equation complemented with mixed time varying BCs. Our example considers an elliptic operator involving only the diffusive term. We compare the solutions obtained to the the standard time varying FE method's solutions and we refer the reader to [31] for a more exhaustive discussion on numerical results.

By referring to Eq. (2.6), we consider a problem defined for $t \in (0,T)$, with $T = 3$, in the domain $\Omega = (0,1)^2$ given in Fig. 1 with boundary defined as in Eq. (2.4), where $\Gamma_D^{(t)} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_{BD}^{(t)} \cup \Gamma_4$; $\Gamma_N^{(t)} := \Gamma_B^{(t)} \setminus \Gamma_{BD}^{(t)}$ represents the Neumann boundary. Specifically, the subset of the boundary
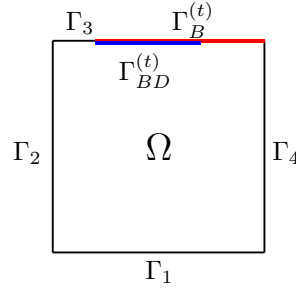
Figure 1: Computational domain: $\Omega = (0,1)^2$; the red line represents the part of the boundary, $\Gamma_B^{(t)}$, on which we set mixed time varying BCs, while $\Gamma_{BD}^{(t)} \subseteq \Gamma_B^{(t)}$, the blue one, its the subset where we impose Dirichlet BCs at time $t \in (0,T)$.

$\Gamma_{BD}^{(t)} \subseteq \Gamma_B^{(t)}$ is defined as:

$$\Gamma_{BD}^{(t)} := \left\{ (x,y) \in \mathbb{R}^2 : x \in I^{(t)}, y = 1 \right\}, \text{ with } I^{(t)} := \begin{cases} \emptyset & \text{for } t < 0.2 \text{ and } 1.5 \leq t < 3, \\ (-2t + 1.4, 1) & \text{for } 0.2 \leq t < 0.6, \\ (0.2, 1) & \text{for } 0.6 \leq t < 1, \\ (1.6t - 1.4, 1) & \text{for } 1 \leq t < 1.5. \end{cases}$$

$$(4.1)$$

As $\Sigma = \Gamma \times (0,T)$ satisfies the hypotheses of Theorem 2.1, the continuous problem is well-posed. On Σ_D and Σ_N, we prescribe homogeneous Dirichlet and Neumann BCs as in Eqs. (2.5) by setting $g = 0$ on Σ_D and $G = 0$ on Σ_N. The initial condition is $u_0 = 0$ in Ω and the source term is defined as $f = 1$ in Q_T. Finally, as for the linear second order elliptic operator $\mathcal{L}$ of Eq. (2.2), we consider the diffusivity tensor to be isotropic, i.e. $\boldsymbol{\sigma} = \sigma \boldsymbol{I}$, with $\boldsymbol{I}$ the identity tensor and $\sigma = 1$; moreover, $\boldsymbol{\beta} = \mathbf{0}$ and $\kappa = 0$.

We use Lagrange $\mathbb{P}_1$ finite elements for the discretization in space, i.e. we consider in Eqs. (2.14) and (3.3), $\mathcal{V}_h^n$ and $\mathcal{V}_h$ defined for degree $k = 1$, respectively, and the BE method ($\theta = 1$ in the general θ-scheme) for the time discretization, with time step $\Delta t = 0.01$. The problem approximated by using the time varying FE method described in Sec. 2.2 has been implemented in `FreeFem++` [16], for which the considerations of Remark 2.2 apply.

In Fig. 2, we report the evolution of the computed solution at different time steps by considering the mesh size $h = 0.00625$. By a qualitative comparison between the Nitsche's method (case (a)) and the time varying FE method (case (b)) the solutions look qualitatively very similar. In Fig. 3, we compare the solutions profile computed at different time steps through the upper bound of the domain $\Gamma_3 \cup \Gamma_B^{(t)}$, by considering the mesh size $h = 0.00625$. We notice that the solutions obtained by using the Nitsche's method (-) result in a smoother curve, especially near the boundary between Σ_D and Σ_N, with respect to the profile of the approximate solution computed with the time varying FE method (-). This is in agreement with the results obtained in [6] for the linear advection-diffusion equation, where the authors observe that the weak imposition of the Dirichlet BCs is able to reduce numerical oscillations, which may be caused by discontinuous boundary data. In Fig. 4, we compare the solutions profile computed at $t = 1.3$ through $\Gamma_3 \cup \Gamma_B^{(t)}$, by considering different mesh sizes $h = 0.025, 0.0125, 0.00625$, and 0.003125. From the comparison between the Nitsche's method (-) and the time varying FE method (-), we can observe that the proposed method shows qualitatively good solutions already with a coarse mesh, thus indirectly showing the effectiveness of the Nitsche's method.
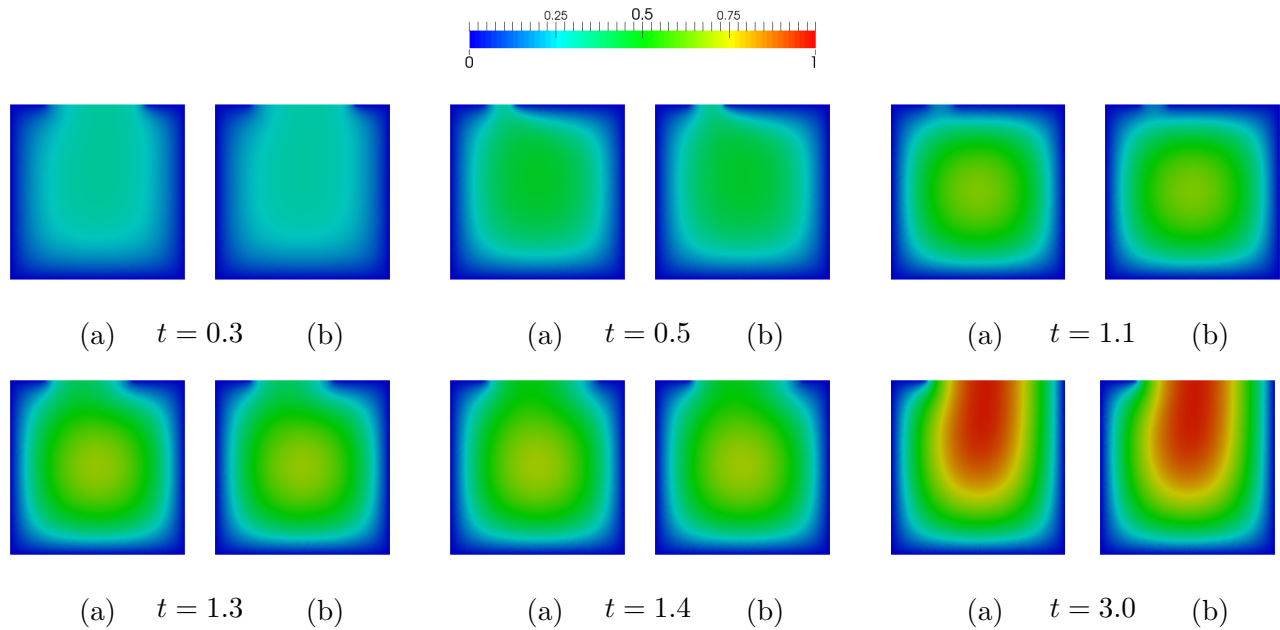
Figure 2: Solutions at different time steps for $h = 0.00625$; comparison between the Nitsche's method (case (a)) and the time varying FE method (case (b)) for some $t \in (0,3)$.
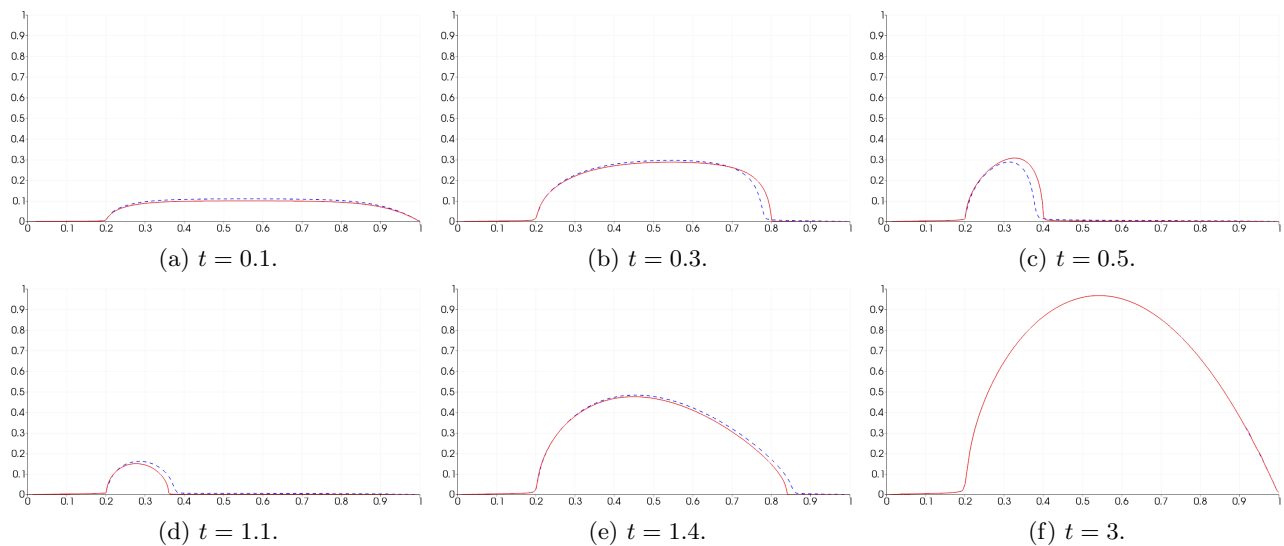

Figure 3: Solutions on $\Gamma_3 \cup \Gamma_B^{(t)}$ at different time steps by considering the mesh size $h = 0.00625$; comparison between the Nitsche's method (- -) and the time varying FE method (-) solutions for some $t \in (0,3)$.
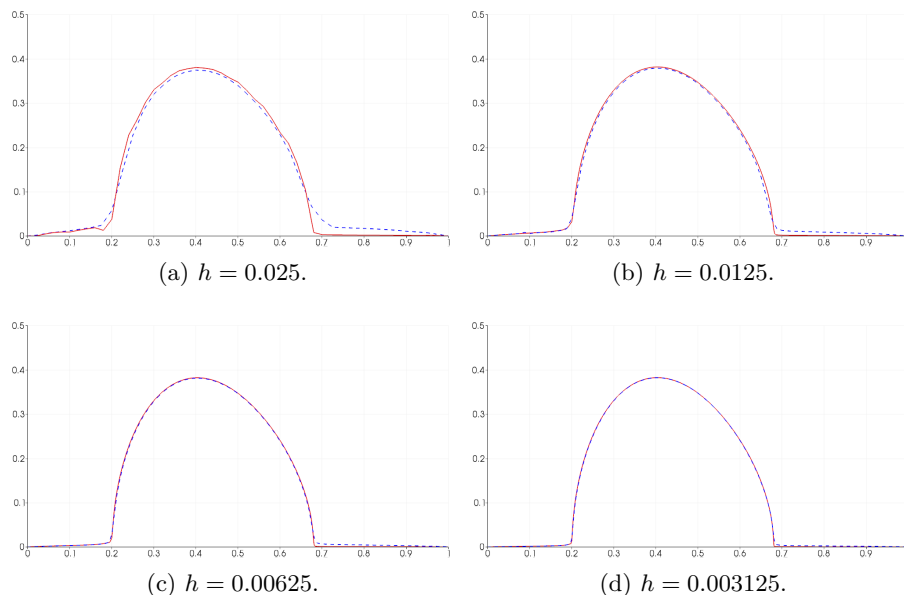
Figure 4: Solutions on $\Gamma_3 \cup \Gamma_B^{(t)}$ at $t = 1.3$ by considering different mesh sizes (h); comparison between the Nitsche's method (--) and the time varying FE method (—) solutions.

Table 1: Comparison of the results (CPU time, number of elements, and DoFs) obtained by using the Nitsche's method and the time varying FE method for different characteristic mesh sizes (h).

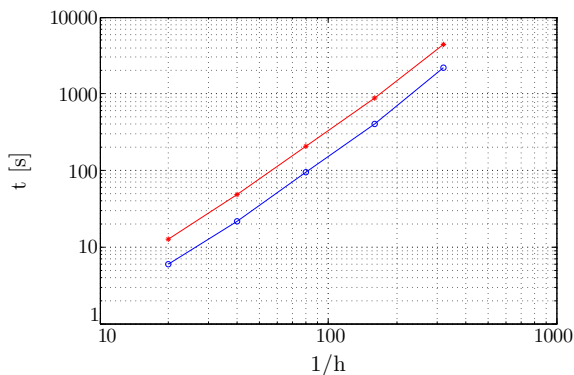| h	Nitsche's method			time varying FE method				
	N_{el}	N_{DoFs}	CPU Time (s)	$\min N_{el}$	$\max N_{el}$	$\min N_{DoFs}$	$\max N_{DoFs}$	CPU Time (s)
0.025	3,808	1,985	22	3,814	10,208	1,988	5,249	48
0.0125	15,176	7,749	95	15,094	40,740	7,708	20,659	208
0.00625	60,594	30,618	409	60,624	164,294	30,633	82,724	887
0.003125	243,178	122,230	2,200	242,642	657,158	121,962	329,732	4,443

Figure 5: Mesh size ($1/h$) vs. CPU time in seconds; comparison between the Nitsche's method (-) and the time varying FE method (-) solutions.

In Table 1, we report, for different values of the mesh size $h = 0.025$, 0.0125, 0.00625, and 0.003125, the number of elements (N_{el}) and the degrees of freedom (N_{DoFs}) associated to the FE spaces $\mathcal{V}_h$ for the Nitsche's method, and the minimum (min) and maximum (max) number of elements and degrees of freedom associated to the time varying FE spaces $\mathcal{V}_h^n$, respectively. Moreover, we record the corresponding CPU times. The smaller the mesh size, the larger are the saving of the computational costs allowed by the Nitsche's method with respect to the time varying FE method, as highlighted in Fig. 5; indeed, for $h = 0.003125$, the number of DoFs is about the half with the proposed Nitsche's method.

5   Conclusions

In this work, we considered a numerical approach developed within the FE method to solve initial boundary value problems associated with parabolic second order PDEs endowed with mixed time varying BCs. Specifically, we proposed a numerical scheme based on the Nitsche's method in the framework of [17], but by considering the more general case of time varying coefficients. We proved the existence and the numerical stability of the solution based on the θ-method for the discretization in time; as expected, we obtained a stability condition of the full discrete scheme dependent on the mesh size, according with the theory developed in the framework of the weak imposition of the Dirichlet BCs [27]. Moreover, we presented some numerical results which highlight the performance and, specifically, the computational efficiency of the proposed full discrete scheme compared to a standard time varying FE method based on remeshing strategies, which represents the typical approach for problems with mixed time varying BCs.

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