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Sobolev--Poincaré inequalities for piecewise $W^{1,p}$ functions over general polytopic meshes

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Sobolev–Poincaré inequalities for piecewise $W^{1,p}$ functions over general polytopic meshes

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Abstract

We establish Sobolev–Poincaré inequalities for piecewise $W^{1,p}$ functions over sequences of fairly general polytopic (thence also shape-regular simplicial and Cartesian) meshes in any dimension; amongst others, they cover the case of standard Poincaré inequalities for piecewise $W^{1,p}$ functions and can be useful in the analysis of nonconforming finite element discretizations of nonlinear problems. Crucial tools in their derivation are novel Sobolev–trace inequalities and Babuška–Aziz inequalities with mixed boundary conditions. We provide estimates that are constant free, i.e., that are fully explicit with respect to the geometric properties of the domain and the underlying sequence of polytopic meshes.

AMS subject classification: 46E35, 65N30.

Keywords: Sobolev–Poincaré inequalities; Sobolev–trace inequalities; Babuška–Aziz inequalities; piecewise $W^{1,p}$ functions; nonconforming finite elements; general polytopic meshes.

Lieber ein Ende mit Schrecken als ein Schrecken ohne Ende

1 Introduction

State-of-the-art: general framework, ... Sobolev–Poincaré inequalities for piecewise $W^{1,p}$ functions are an essential tool in the analysis of several nonconforming methods and have been the objective of extensive studies over the last four decades. In the literature, the adjectives *broken*, *piecewise*, and *discrete* are typically associated with this type of inequalities. On occasions, we shall adopt either of these nomenclatures.

... functions in nonconforming Sobolev spaces, ... Arnold [2] proved a broken Poincaré inequality for piecewise H^1 functions in two dimensions on triangular meshes, using elliptic regularity results in nonsmooth domains. About 20 years later and undertaking a different avenue, Brenner [8] extended Arnold's results to three dimensions and more general meshes; the results in that work are based on compact embedding arguments and the approximation properties of certain DoFs-averaging operators. A similar analysis was carried out for H^2-type inequalities a year later [10]. Shortly after, Lasis and Süli [27] extended with similar arguments the results of the papers mentioned above to the case of Sobolev–Poincaré inequalities for piecewise H^1 functions for partitions of simplices and affine maps of hypercubes. Poincaré inequalities for piecewise $W^{1,p}$ functions in two and three dimensions can be found in [9, Sect. 10.6].

... and piecewise polynomials. Similar results in the context of finite volumes were discussed in [16, Section 4.3]; in particular, the inequality therein was proven for piecewise constant functions. A higher order version of this inequality was given in [29]. Sobolev inequalities involving discrete $W^{1,p}$ norms for finite volumes were detailed in [22, Section 5.1], [24], and [5, Sections 3 and 4]. Buffa and Ortner [11], and Di Pietro and Ern [18, Theorem 6.1] proved similar inequalities for general order piecewise polynomials over polytopic meshes in two and three dimensions satisfying standard regularity assumptions; in both cases, the proof hinged upon the use of a polynomial trace inverse inequality. Finally, discrete Poincaré inequalities in $H(div)$ and $H(curl)$ were recently investigated in [21].

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Main results and goals of the paper. We prove Sobolev–Poincaré inequalities generalizing those in the references above in several respects:

(i) the inequalities are proven for piecewise functions over fairly general polytopic meshes in any dimension in the sense of, e.g., [12, Assumption 2.1];

(ii) the constants appearing in the estimates are fully explicit in terms of the geometry of the mesh and constants appearing in direct estimates;

(iii) the employed broken Sobolev norm possibly contains a minimal number of boundary terms;

(iv) all the results are established in broken $W^{1,p}$ spaces and not for piecewise polynomials or other finite dimensional spaces, which is instead the avenue undertaken in [5,11,18,22,24]; the norms appearing in the estimates are based on the maximal Lebesgue regularity differently from [9, Sect. 10.6], where only p-type norms are used.

Each of these results has important consequences:

(i) these inequalities can be used in the analysis of several polytopic methods, in particular those based on sequences of polytopic meshes with an arbitrary number of facets;

(ii) the well-posedness and convergence analysis of such methods are fully explicit with respect to the geometric properties of the sequence of meshes;

(iii) these inequalities allow for the analysis of problems with mixed boundary conditions;

(iv) they are useful for the analysis of nonconforming methods for nonlinear problems and Galerkin methods that are not polynomial based (such as Trefftz and extended methods).

Important tools in the analysis. The proposed analysis neither relies on enriching operators, as instead done in [8,11,27], nor on continuous embeddings of the space of functions with bounded variation, as done in [5,18,22]. Rather, it requires the generalization of two fundamental results in the Sobolev spaces theory: continuous Sobolev–trace and Babuška–Aziz inequalities.

We extend local continuous trace inequalities as in, e.g., [20, Lemma 12.15] and [17, Lemma 1.31], to Sobolev–trace inequalities that are based on more general meshes, Sobolev norms, and maximal Lebesgue regularity; see Section 2 below.

Babuška–Aziz inequalities express the stability of right-inverses for several differential operators and are strictly related to other results such as Nečas-Lions inequalities, inf-sup conditions, and bounds on the spectrum of the Cosserat operator; see, e.g., [15] and Appendix A. In the literature, the construction of a right-inverse of the divergence is done at least in two different ways: one [3] is based on solving suitable elliptic problems and exploiting the stability of extension operators in Sobolev spaces; the other [6,25] is based on using integral operators. Such right-inverses are typically endowed with homogeneous Dirichlet boundary conditions or without boundary conditions: the corresponding integral operators go under the name of Bogovskiĭ and generalized Poincaré operators. However, we are not aware of Babuška–Aziz inequalities for the case of boundary conditions assigned on part of the boundary of the domain, with constants that are explicit in terms of the shape of the domain. Therefore, in Section 3 below, we shall prove such inequalities with explicit constants under certain regularity properties of the domain and its boundary.

Outline of the remainder of the section and list of the relevant results. In Section 1.1, we introduce some domains of interest, their geometric properties, and establish the notation for Sobolev spaces; Section 1.2 is devoted to introduce sequences of polytopic meshes and corresponding broken Sobolev spaces; the two main results of the paper, whose proofs require two technical tools discussed in Section 1.3, are discussed in Section 1.4. For the reader's convenience, we detail a list of the relevant results of this paper.

- Theorem 1.1 and Corollary 1.2 are novel Sobolev–trace inequalities.
- Theorem 1.4 establishes novel Babuška–Aziz inequalities with mixed boundary conditions.

- Theorems 1.5 and 1.6 are concerned with Sobolev–Poincaré inequalities on broken Sobolev spaces.
- Corollaries 1.7 and 1.8 are variants of the Sobolev–Poincaré inequalities above, which are more suitable for certain nonconforming finite element spaces, including Crouzeix-Raviart spaces.

We are aware that results similar to Theorems 1.5 and 1.6, and Corollaries 1.7 and 1.8 are work in progress also in [13]. The results discussed therein are, however, derived undertaking a different avenue, which is closer to that in [8].

1.1 Domains of interest and Sobolev spaces

Let Ω be a polytopic, open, Lipschitz domain in $\mathbb{R}^d$ with boundary $\partial\Omega$. Furthermore,

$$\Omega \text{ is star-shaped with respect to a ball } B_\rho \text{ of radius } \rho\text{; the diameter of } \Omega \text{ is } h_\Omega. \tag{1}$$

We partition the boundary $\Gamma := \partial\Omega$ of Ω into

$$\Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}, \tag{2}$$

Γ_D having nonzero $(d-1)$ Hausdorff measure for the sake of simplicity, and $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$.

Given $X \subset \overline{\Omega}$ with diameter h_X, we consider Lebesgue spaces of order p consisting of Lebesgue measurable functions with finite norm

$$\|v\|^p_{L^p(X)} := \int_X |v|^p.$$

Analogously, one defines Lebesgue spaces $L^p(\partial X)$ on ∂X. $L^p_0(X)$ is the space of functions in $L^p(X)$ with zero average over X. We denote the gradient and divergence operators by ∇ and $\nabla\cdot$. More in general, given a multi-index $\boldsymbol{\alpha}$ in $\mathbb{N}^d$ with size s, $D^{\boldsymbol{\alpha}}$ denotes the tensor of dimension s containing all mixed derivatives of order s. Given a positive integer k, and a real number p in $[1,\infty)$, $W^{k,p}(X)$ denotes the space of $L^p(X)$ functions with distributional derivatives $D^{\boldsymbol{\alpha}}$ of order k in $L^p(X)$. We introduce the seminorms and norms

$$|v|_{W^{k,p}(X)} := \Big(\sum_{|\boldsymbol{\alpha}|=k} \|D^{\boldsymbol{\alpha}} v\|_{L^p(X)}\Big)^{\frac{1}{p}}, \qquad \|v\|_{W^{k,p}(X)} := \Big(\sum_{\ell=0}^{k} \big(h_X^{\ell-k} |v|_{W^{\ell,p}(X)}\big)^p\Big)^{\frac{1}{p}}.$$

Interpolation theory is used to define Sobolev spaces of positive noninteger order s. We shall use the boldface type to denote vector fields (and the corresponding spaces); for instance, scalar and vector generic Lebesgue and Sobolev spaces are $L^p(X)$ and $\mathbf{L}^p(X)$, and $W^{k,p}(X)$ and $\mathbf{W}^{k,p}(X)$.

There exists a bounded linear map, called trace operator [20, Section 3.2], from $W^{1,p}(X)$ to $L^p(\partial X)$, which acts as the restriction to ∂X for continuous functions. The image of $W^{1,p}(X)$ is $W^{1-\frac{1}{p},p}(\partial X)$; see Corollary 1.2 below for a clearer statement. The subspace of $W^{1,p}(X)$ of functions with zero trace over ∂X is denoted by $W^{1,p}_0(X)$. We also define

$$W^{1,p}_{\widetilde{\Gamma}}(X) := \{v \in W^{1,p}(X) \mid v_{|\widetilde{\Gamma}} = 0\} \qquad \forall \widetilde{\Gamma} \subset \partial X.$$

A Poincaré–Steklov inequality holds true [20, Lemma 3.30]: for all p in $[1,\infty)$, there exists a positive constant $C_{PS}(p,X)$ such that

$$\|v\|_{L^p(X)} \le C_{PS}(p,X) h_X |v|_{W^{1,p}(X)} \qquad \forall v \in W^{1,p}_{\widetilde{\Gamma}}(X), \quad \widetilde{\Gamma} \subseteq \partial X. \tag{3}$$

A similar inequality holds true for functions in $W^{1,p}(X) \cap L^p_0(X)$; with an abuse of notation, we denote the involved constant with the same symbol.

Continuous embeddings (denoted with the symbol $\hookrightarrow$) of Sobolev spaces onto Lebesgue spaces hold true [20, Section 2.3.2]: given ℓ positive and p larger than or equal to 1,

- if $\ell p < d$, $W^{\ell,p}(X) \hookrightarrow L^q(X)$ for all q in $[p, \frac{pd}{d-\ell p}]$;

- if $\ell p = d$, $W^{\ell,p}(X) \hookrightarrow L^q(X)$ for all q in $[p, \infty)$.

For future convenience, we spell out the generic Sobolev embedding estimate:

$$\|v\|_{L^q(X)} \le C_{\mathrm{Sob}}(q, \ell, p, X) h_X^{\frac{d}{q} - \frac{d}{p} + \ell} \|v\|_{W^{\ell,p}(X)} \qquad \forall v \in W^{\ell,p}(X). \tag{4}$$

Given an index p in $[1, \infty)$, we define

$$p' := \frac{p}{p-1}; \qquad p^* := \begin{cases} \frac{pd}{d-p} & \text{if } p < d \\ \infty & \text{otherwise;} \end{cases} \qquad p^\circ := \begin{cases} \frac{p(d-1)}{d-p} & \text{if } p < d \\ \infty & \text{otherwise.} \end{cases} \tag{5}$$

The first index in (5) is the conjugate index of p; the second one relates to Sobolev embeddings in dimension d and $W^{1,p}(X)$; the third one to Sobolev embeddings on the $(d-1)$-dimensional boundary and $W^{1-\frac{1}{p},p}(\partial X)$ (combine the trace and Sobolev embedding theorems).

1.2 Meshes and broken Sobolev spaces

We consider sequences $\{\mathcal{T}_n\}$ where each $\mathcal{T}_n$ is a finite collection of disjoint, closed, polytopic elements such that $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_n} K$. For each K in $\mathcal{T}_n$, ∂K and h_K denote the boundary and the diameter of K, respectively.

We associate each $\mathcal{T}_n$ with a set $\mathcal{F}_n$ covering the mesh skeleton, i.e., $\bigcup_{K \in \mathcal{T}_n} \partial K = \bigcup_{F \in \mathcal{F}_n} F$. A facet F in $\mathcal{F}_n$ is a hyperplanar, closed, and connected subset of $\overline{\Omega}$ with positive $(d-1)$-dimensional Hausdorff measure such that

- either there exist distinct $K_{1,F}$ and $K_{2,F}$ in $\mathcal{T}_n$ such that $F \subseteq \partial K_{1,F} \cap \partial K_{2,F}$ and F is called an *internal facet* (sometimes also an *interface*),
- or there exists K_F in $\mathcal{T}_n$ such that $F \subseteq \partial K_F \cap \partial\Omega$ and F is called a *boundary facet.*

We collect interfaces and boundary facets in $\mathcal{F}_n^I$ and $\mathcal{F}_n^B$. For all K in $\mathcal{T}_n$, $\mathcal{F}^K$ and $\mathbf{n}_K$ denote the set of facets contained in ∂K and the outward unit normal to ∂K. For a given F in $\mathcal{F}_n$, we fix once and for all one of the two unit normal vectors $\mathbf{n}_F$.

We assume that the mesh boundary skeleton is compatible with splitting (2), i.e., $\mathcal{F}_n^B = \mathcal{F}_n^D \cup \mathcal{F}_n^N$ where

$$\mathcal{F}_n^D := \{F \in \mathcal{F}_n^B \mid F \subseteq \Gamma_{\mathrm{D}}\}, \qquad \mathcal{F}_n^N := \{F \in \mathcal{F}_n^B \mid F \subseteq \Gamma_{\mathrm{N}}\}. \tag{6}$$

Following [12, Assumption 2.1], we demand the following regularity assumptions:

- for all K in any $\mathcal{T}_n$, there exists a partition $\mathfrak{T}_K$ of K into non-overlapping d-dimensional simplices;
- there exists a universal, positive constant γ such that, for all K in any $\mathcal{T}_n$ and all T in $\mathfrak{T}_K$ with $\partial T \cap \partial K \neq \emptyset$, given F_K^T the $(d-1)$-dimensional simplex $\partial T \cap \partial K$,
$$\gamma h_K \le d |T| |F_K^T|^{-1}. \tag{7}$$

Remark 1. The regularity assumptions neither impose a restriction on the number of facets per element nor on the facets' size. Moreover, for sequences $\{\mathcal{T}_n\}$ of simplicial meshes, the above assumptions boil down to the standard shape-regularity assumption.

Broken Sobolev spaces associated with $\mathcal{T}_n$ are defined as

$$W^{1,p}(\mathcal{T}_n) := \left\{ u \in L^p(\Omega) \mid u_{|K} \in W^{1,p}(K) \qquad \forall K \in \mathcal{T}_n \right\}.$$

For every $v \in W^{1,p}(\mathcal{T}_n)$ and F in $\mathcal{F}_n$, the jump operator on F is given by

$$[\![v]\!]_F := \begin{cases} v_{|K_{1,F}} \mathbf{n}_{K_{1,F}} \cdot \mathbf{n}_F + v_{|K_{2,F}} \mathbf{n}_{K_{2,F}} \cdot \mathbf{n}_F & \text{if } F \subset \mathcal{F}_n^I,\ F \in \partial K_{1,F} \cap \partial K_{2,F}, \\ v_{|F} \mathbf{n}_{K_F} \cdot \mathbf{n}_F & \text{if } F \in \mathcal{F}_n^B,\ F \subset \partial K_F \cap \partial\Omega. \end{cases}$$

With an abuse of notation, we use the same symbols for vector fields $\mathbf{v}$ in $\mathbf{W}^{1,p}(\mathcal{T}_n)$.

1.3 Main technical tools

We report here the technical tools that are needed to derive the results in Section 1.4 below, and postpone their proofs to Sections 2 and 3 below.

First technical tools. The first technical tools are Sobolev–trace inequalities.

Theorem 1.1 (Sobolev–trace inequalities)**.** *Let $\{\mathcal{T}_n\}$ be a sequence of meshes as in Section 1.2, q be in $[1,\infty)$, s be in $[q,\infty)$, and γ be as in (7). For all K in any $\mathcal{T}_n$ and all v in $W^{1,\frac{s}{s-q+1}}(K)\cap L^s(K)$, we have*

$$\|v\|_{L^q(\partial K)} \le C_{TR}(q,s,d,\gamma)\Big(h_K^{-\frac{1}{q}\left(1-\frac{d}{s}(s-q)\right)}\|v\|_{L^s(K)} + h_K^{\frac{1}{q'}\left(1-\frac{d}{s}(s-q)\right)}\|\nabla v\|_{\mathbf{L}^{\frac{s}{s-q+1}}(K)}\Big), \tag{8}$$

where, given $\Gamma(\cdot)$ the Euler's Gamma function,

$$C_{TR}(q,s,d,\gamma) := \left(\frac{d\pi^{\frac{d(s-q)}{2s}}}{\gamma\Gamma\left(\frac{d}{2}+1\right)^{\frac{s-q}{s}}} + \frac{q}{\gamma}\right)^{\frac{1}{q}}. \tag{9}$$

Two special Sobolev–trace inequalities are an immediate consequence of Theorem 1.1.

Corollary 1.2 (Special Sobolev–trace inequalities)**.** *Let $\{\mathcal{T}_n\}$ be a sequence of meshes as in Section 1.2, q be in $[1,\infty)$, $C_{TR}(\cdot,\cdot,\cdot,\cdot)$ be as in (9), the indices q', q°, and q^* be as in (5), and γ be as in (7). For all K in any $\mathcal{T}_n$ and all v in $W^{1,q}(K)$, we have*

$$\|v\|_{L^q(\partial K)} \le C_{TR}(q,q,d,\gamma)\big(h_K^{-\frac{1}{q}}\|v\|_{L^q(K)} + h_K^{\frac{1}{q'}}\|\nabla v\|_{\mathbf{L}^q(K)}\big), \tag{10}$$

and, if q further belongs to $[1,d)$,

$$\|v\|_{L^{q^\circ}(\partial K)} \le C_{TR}(q^\circ,q^*,d,\gamma)\big(\|v\|_{L^{q^*}(K)} + \|\nabla v\|_{\mathbf{L}^q(K)}\big). \tag{11}$$

Inequality (10) generalizes the standard continuous trace inequality, cf. [20, Lemma 12.15] and [17, Lemma 1.31], to the case of sequences of polytopic meshes as in (7).

Second technical tools. Babuška–Aziz (BA) inequalities are crucial tools in the analysis of PDEs; see Appendix A. The simplest BA inequality states that there exists a stable right-inverse of the divergence operator and typically involves vector fields with either free boundary conditions or imposing homogeneous Dirichlet boundary conditions on the boundary Γ of the domain of interest Ω. We shall be using the following result, whose proof can be found, e.g., in [1, 23].

Lemma 1.3 (Babuška–Aziz inequalities: homogeneous boundary conditions)**.** *There exists a positive constant $C_{BA}(p,\Gamma,\Omega)$ such that for all f in $L^p_0(\Omega)$, p in $(1,\infty)$, it is possible to construct $\mathbf{v}$ in $\mathbf{W}^{1,p}_0(\Omega)$ satisfying*

$$\nabla\cdot\mathbf{v} = f \in L^p_0(\Omega), \qquad\qquad |\mathbf{v}|_{\mathbf{W}^{1,p}(\Omega)} \le C_{BA}(p,\Gamma,\Omega)\|f\|_{L^p(\Omega)}. \tag{12}$$

As discussed in [23, Lemma III.3.1], given h_Ω and ρ as in (1), we have the bound

$$C_{BA}(p,\Gamma,\Omega) \le C_G(d,p)\left(\frac{h_\Omega}{\rho}\right)^d\left(1+\frac{h_\Omega}{\rho}\right), \tag{13}$$

where $C_G(\cdot,\cdot)$ is a positive constant only depending on d and p. As discussed in [19], an estimate that is sharper than that in (13) can be derived in the case p equal to 2.

Recall the splitting (2) of Γ into the union of Γ_D and Γ_N, assume that both sets have nonzero $(d-1)$-dimensional measure in Γ, and recall the compatibility condition (6). We derive BA inequalities for vector fields with homogeneous boundary conditions only on Γ_N. To this aim, we prove a bound with a constant behaving differently depending on the convexity of the domain Ω. In particular, for nonconvex Ω, given $\widetilde{\Gamma}_D$ contained in Γ_D, the regularity of the domain implies the existence of a domain Ω_{ext} such that $(\overline{\Omega} \cap \overline{\Omega_{\text{ext}}})^\circ = \widetilde{\Gamma}_D^\circ$; see Figure 2 below for a graphical representation. Given

$$\widetilde{\Omega} := (\overline{\Omega} \cup \overline{\Omega_{\text{ext}}})^\circ, \tag{14}$$

we observe that the regularity properties of Ω imply that

$$\widetilde{\Omega} \text{ is star-shaped with respect to a ball } B_{\widetilde{\rho}} \text{ of radius } \widetilde{\rho}\text{; the diameter of } \widetilde{\Omega} \text{ is } h_{\widetilde{\Omega}}. \tag{15}$$

Further define

$$\rho_\Gamma := \frac{1}{2} \min \Big\{ h_{\Gamma_j} = \operatorname{diam}(\Gamma_j) \mid \Gamma_j \text{ is a (d-1)-dimensional facet of } \Omega \Big\}. \tag{16}$$

Theorem 1.4 (Babuška–Aziz inequalities with mixed boundary conditions)**.** *There exists a positive constant $C_{BA}(p, \Gamma_N, \Omega)$ such that, for all f in $L^p(\Omega)$, p in $(1,\infty)$, it is possible to construct $\mathbf{v}$ in $\mathbf{W}^{1,p}_{\Gamma_N}(\Omega)$ satisfying*

$$\nabla \cdot \mathbf{v} = f, \qquad\qquad |\mathbf{v}|_{\mathbf{W}^{1,p}(\Omega)} \le C_{BA}(p, \Gamma_N, \Omega) \|f\|_{L^p(\Omega)}. \tag{17}$$

In particular, given $C_G(\cdot,\cdot)$ in (13), h_Ω and ρ in (1), $h_{\widetilde{\Omega}}$ and $\widetilde{\rho}$ in (15), Ω_{ext} in (14), and ρ_Γ in (16), we have the upper bounds

$$C_{BA}(p, \Gamma_N, \Omega) \le \begin{cases} 2^{\frac{1}{p}} C_G(d,p) \Big(\frac{2h_\Omega}{\min(\rho,\rho_\Gamma)} \Big)^d \Big(1 + \frac{2h_\Omega}{\min(\rho,\rho_\Gamma)} \Big) & \text{if } \Omega \text{ is convex} \\ C_G(d,p) \Big(\frac{h_{\widetilde{\Omega}}}{\widetilde{\rho}} \Big)^d \Big(1 + \frac{h_{\widetilde{\Omega}}}{\widetilde{\rho}} \Big) \Big(1 + \frac{|\Omega|^{p-1}}{|\Omega_{ext}|^{p-1}} \Big) & \text{if } \Omega \text{ is not convex.} \end{cases}$$

In the nonconvex case, the bound on $C_{BA}(p, \Gamma_N, \Omega)$ depends heavily on the construction of the extended domain Ω_{ext} as detailed above.

1.4 Main results

We begin by presenting two Sobolev–Poincaré inequalities for functions in broken $W^{1,p}$ spaces and postpone their proofs to Sections 4.1 and 4.2 below. The first result reads as follows.

Theorem 1.5 (1st kind Sobolev–Poincaré inequalities)**.** *Let $C_{BA}(\cdot,\cdot,\cdot)$, $C_{TR}(\cdot,\cdot,\cdot,\cdot)$, $C_{Sob}(\cdot,\cdot,\cdot,\cdot)$, and $C_{PS}(\cdot,\cdot)$ be the constants in (17), (8), (4), and (3), and h_Ω be as in (1). Consider a sequence of polytopic meshes $\{\mathcal{T}_n\}$ as in Section 1.2, p in $[1,d)$, and γ as in (7). Introduce*

$$\begin{aligned} C_1 &:= C_{BA}((p^*)', \Gamma_N, \Omega)\, C_{Sob}(p', 1, (p^*)', \Omega)(1 + h_\Omega^{(p^*)'} C_{PS}((p^*)', \Omega)^{(p^*)'})^{\frac{1}{(p^*)'}}, \\ C_2 &:= C_{TR}((p^\circ)', p', d, \gamma)\big(C_{BA}((p^*)', \Gamma_N, \Omega) + C_1\big). \end{aligned} \tag{18}$$

Then, we have

$$\|v\|_{L^{p^*}(\Omega)} \le C_1 \|\nabla_h v\|_{\mathbf{L}^p(\Omega)} + C_2 \Big(\sum_{F \in \mathcal{F}_n^I \cup \mathcal{F}_n^D} \|[\![v]\!]\|^p_{L^{p^\circ}(F)} \Big)^{\frac{1}{p}} \qquad \forall v \in W^{1,p}(\mathcal{T}_n, \Omega). \tag{19}$$

We further have an inequality involving weaker norms. To this aim, we preliminary introduce

$$\widetilde{h}_F \quad \text{either equal to } h_F \text{ or equal to} \quad \begin{cases} \min(h_{K_1}, h_{K_2}) & \text{if } F \in \mathcal{F}_n^I,\ F = \partial K_1 \cap \partial K_2 \\ h_K & \text{if } F \in \mathcal{F}_n^D,\ F \in \mathcal{F}^K. \end{cases} \tag{20}$$

Theorem 1.6 (2nd kind Sobolev–Poincaré inequalities)**.** *Let $C_{BA}(\cdot,\cdot,\cdot)$, $C_{TR}(\cdot,\cdot,\cdot,\cdot)$, $C_{Sob}(\cdot,\cdot,\cdot,\cdot)$, and $C_{PS}(\cdot,\cdot)$ be the constants in (17), (8), (4), and (3), and h_Ω be as in (1). Consider a sequence of polytopic meshes $\{\mathcal{T}_n\}$ as in Section 1.2, p in $[1,\infty)$, and γ as in (7). Introduce*

$$\begin{aligned} C_3 &:= C_{BA}((p1^*)', \Gamma_N, \Omega)\, h_\Omega^{\frac{d}{p'} - \frac{d}{(p1^*)'} + 1} C_{Sob}(p', 1, (p1^*)', \Omega)(1 + h_\Omega C_{PS}((p1^*)', \Omega)), \\ C_4 &:= C_{BA}((p1^*)', \Gamma_N, \Omega)\, C_{TR}(p', p', d, \gamma)\big(1 + \max_{K \in \mathcal{T}_n} h_K^{p'}\big)^{\frac{1}{p'}}(1 + h_\Omega C_{PS}(p', \Omega)). \end{aligned} \tag{21}$$

Then, given $\widetilde{h}_F$ as in (20), we have

$$\|v\|_{L^{p1^*}(\Omega)} \le C_3 \|\nabla_h v\|_{\mathbf{L}^p(\Omega)} + C_4 \Big(\sum_{F \in \mathcal{F}_n^I \cup \mathcal{F}_n^D} \widetilde{h}_F^{1-p} \|[\![v]\!]\|^p_{L^p(F)} \Big)^{\frac{1}{p}} \qquad \forall v \in W^{1,p}(\mathcal{T}_n, \Omega).$$

Remark 2. Theorem 1.6 involves a weaker Sobolev–Poincaré inequality compared to that in Theorem 1.5. However, if we restrict inequality 19 to functions v_h in a finite dimensional space such that Lebesgue inverse inequalities with explicit constants are available and the facets of the mesh are uniformly shape-regular, then (19) can be improved to

$$\|v_h\|_{L^{p^*}(\Omega)} \lesssim \|\nabla_h v_h\|_{\mathbf{L}^p(\Omega)} + \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} h_F^{1-p} \|[\![v_h]\!]\|^p_{L^p(F)} \Big)^{\frac{1}{p}}.$$

We also present two corollaries to Theorems 1.5 and 1.6 that are more useful, e.g., for Crouzeix-Raviart type discretizations, and are in the spirit of [28] and [8, Remark 1.1]. Their proofs are given in Sections 4.3 and 4.4 below.

Introduce $\Pi^0_{\mathcal{F}_n}$ mapping functions $L^1(\mathcal{F}_n)$ into their piecewise average over the facets in $\mathcal{F}_n$.

Corollary 1.7 (1st kind Sobolev–Poincaré inequalities: averaged version)**.** *Let $C_{TR}(\cdot,\cdot,\cdot,\cdot)$, $C_{Sob}(\cdot,\cdot,\cdot,\cdot)$, $C_{PS}(\cdot,\cdot)$, and C_2 be the constants in (8), (4), (3), and (18), and h_Ω be as in (1). Consider a sequence of polytopic meshes $\{\mathcal{T}_n\}$ as in Section 1.2, p in $[1,\infty)$, and γ as in (7). Introduce*

$$\begin{aligned} C_5 := &\max_{K\in\mathcal{T}_n} \left[C_{Sob}(p^*,1,p,K)(1+h_K C_{PS}(p,K)) \right] \\ &+ 2^{1-\frac{1}{p}} C_2 \Big[\max_{K\in\mathcal{T}_n} (\mathrm{card}(\mathcal{F}^K)) \Big] C_{TR}(p^\circ, p^*, d, \gamma) \cdot \\ &\cdot \Big[\max_{K\in\mathcal{T}_n} (1 + C_{Sob}(p^*,1,p,K))(1+h_K C_{PS}(p^*,K)) \Big], \qquad C_6 := 2^{1-\frac{1}{p}} C_2. \end{aligned} \tag{22}$$

Then, we have

$$\|v\|_{L^{p^*}(\Omega)} \leq C_5 \|\nabla_h v\|_{\mathbf{L}^p(\Omega)} + C_6 \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \big\|\Pi^0_{\mathcal{F}_n} [\![v]\!]\big\|^p_{L^{p^\circ}(F)} \Big)^{\frac{1}{p}}.$$

Corollary 1.8 (2nd kind Sobolev–Poincaré inequalities: averaged version)**.** *Let $C_{TR}(\cdot,\cdot,\cdot,\cdot)$, $C_{Sob}(\cdot,\cdot,\cdot,\cdot)$, $C_{PS}(\cdot,\cdot)$, and C_4 be the constants in (8), (4), (3), and (21), and h_Ω be as in (1). Consider a sequence of polytopic meshes $\{\mathcal{T}_n\}$ as in Section 1.2, p in $[1,\infty)$, and γ as in (7). Given $\widetilde{h}_F$ as in (20), introduce*

$$\begin{aligned} C_7 := &\max_{K\in\mathcal{T}_n} [h_K^{\frac{d}{p1^*}-\frac{d}{p}+1} C_{Sob}(p1^*,1,p,K)(1+h_K C_{PS}(p,K))] \\ &+ 2^{1-\frac{1}{p}} C_4 \Big[\max_{K\in\mathcal{T}_n} \max_{F\in\mathcal{F}^K} \Big(\frac{\widetilde{h}_F}{h_K}\Big)^{-1+\frac{1}{p}} \Big] C_{TR}(p,p,d,\gamma) \big[\max_{K\in\mathcal{T}_n} (1+h_K C_{PS}(p,K)) \big], \qquad C_8 := 2^{1-\frac{1}{p}} C_4. \end{aligned}$$

Then, we have

$$\|v\|_{L^{p1^*}(\Omega)} \leq C_7 \|\nabla_h v\|_{\mathbf{L}^p(\Omega)} + C_8 \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \widetilde{h}_F^{1-p} \big\|\Pi^0_{\mathcal{F}_n} [\![v]\!]\big\|^p_{L^p(F)} \Big)^{\frac{1}{p}}.$$

Remark 3. Compared to Theorem 1.5, Corollary 1.7 displays a bound that is not robust with respect to sequences of meshes of elements with number of facets arbitrarily increasing. This can be seen in the factor $\max_{K\in\mathcal{T}_n}(\mathrm{card}(\mathcal{F}^K))$ appearing in the constant C_5 in (22). On the other hand, Corollary 1.8 contains bounds that are robust with respect to the number of facets of an element and small facets: it suffices to take the second option for $\widetilde{h}_F$ in (20), and assume that the mesh is quasi-uniform in the sense that neighbouring elements have uniformly comparable sizes.

Remark 4. Theorems 1.5 and 1.6, and Corollaries 1.7 and 1.8 can be extended to the $W^{k,p}$ setting by including further jump terms involving higher order derivatives in the estimates using [7, Appendix A].

Remark 5. Theorem 1.6 and Corollary 1.7 generalize the results in [2,8] in the sense that it suffices to pick $p = 2$ and observe that 1^* is larger than 1. Compared to the references above, we admit more general meshes and exhibit bounds with fully explicit constants in terms of the properties of the domain Ω and the underlying sequence of meshes.

Outline of the remainder of the paper. In Sections 2 and 3 we prove the two technical tools discussed in Section 1.3, i.e., Sobolev–trace inequalities and a Babuška–Aziz inequality with mixed boundary condition. These two results are instrumental in proving the main results of the paper, i.e., Theorems 1.5 and 1.6, and Corollaries 1.7 and 1.8, which are the topic of Section 4.

2 Proof of Sobolev–trace inequalities

We prove here Theorem 1.1.
Step 1. We show first an inequality on a simplex T in $\mathfrak{T}_K$ as in Section 1.2 for a given K in any $\mathcal{T}_n$ such that $F_K^T := \partial T \cap \partial K \neq \emptyset$. Let $\mathbf{P}_{F_K^T}$ be the vertex of T opposite to F_K^T. We proceed as in [20, Lemma 12.15] and [17, Lemma 1.31] by considering the lowest order Raviart–Thomas function

$$\boldsymbol{\phi}_{F_K^T}(\mathbf{x}) := \frac{|F_K^T|}{d|T|}(\mathbf{x} - \mathbf{P}_{F_K^T}).$$

The normal component of $\boldsymbol{\phi}_{F_K^T}$ is equal to 1 on F_K^T and zero on the other facets of T; moreover, $\nabla\cdot \boldsymbol{\phi}_{F_K^T} = |F_K^T||T|^{-1}$. The divergence theorem implies

$$\begin{aligned}\|v\|^q_{L^q(F_K^T)} &= \int_{\partial T} |v|^q \boldsymbol{\phi}_{F_K^T} \cdot \mathbf{n}_T = \int_T \nabla\cdot(|v|^q \boldsymbol{\phi}_{F_K^T}) \\ &= \int_T \frac{|F_K^T|}{|T|}|v|^q + \int_T \frac{q|F_K^T|}{d|T|} v|v|^{q-2}\, \nabla v \cdot (\mathbf{x} - \mathbf{P}_{F_K^T}) =: \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

Using Hölder's inequality with indices s/q and $s/(s-q)$, and observing that (7) implies $|F_K^T||T|^{-1} \leq d(\gamma h_K)^{-1}$, we bound the term $\mathcal{I}_1$ as follows:

$$\mathcal{I}_1 \leq \|v\|^q_{L^s(T)} \frac{|F_K^T|}{|T|}|T|^{\frac{s-q}{s}} \leq d\gamma^{-1}h_K^{-1}|T|^{\frac{s-q}{s}}\|v\|^q_{L^s(T)}.$$

As for the term $\mathcal{I}_2$, we remark that $(\mathbf{x} - \mathbf{P}_{F_K^T}) \leq h_K$ for all $\mathbf{x}$ in T, apply Hölder's inequality with indices $s/(q-1)$ and $s/(s-q+1)$, observe that $q'(q^\circ - 1) = q^*$, and use again (7) to infer

$$\mathcal{I}_2 \leq \frac{qh_K|F_K^T|}{d|T|}\int_T |v|^{q-1}|\nabla v| = \frac{qh_K|F_K^T|}{d|T|}\|v\|^{q-1}_{L^s(T)}\|\nabla v\|_{\mathbf{L}^{\frac{s}{s-q+1}}(T)} \leq \frac{q}{\gamma}\|v\|^{q-1}_{L^s(T)}\|\nabla v\|_{\mathbf{L}^{\frac{s}{s-q+1}}(T)}.$$

Gathering the estimate of $\mathcal{I}_1$ and $\mathcal{I}_2$, we obtain a multiplicative trace inequality for simplicial elements K reading

$$\|v\|^q_{L^q(F_K^T)} \leq \gamma^{-1}\|v\|^{q-1}_{L^s(T)}\left(dh_K^{-1}|T|^{\frac{s-q}{s}}\|v\|_{L^s(T)} + q\|\nabla v\|_{\mathbf{L}^{\frac{s}{s-q+1}}(T)}\right). \tag{23}$$

Step 2. Let K be as in Step 1. We define

$$\widetilde{\mathfrak{T}}_K := \{T \in \mathfrak{T}_K \mid F_K^T := \partial T \cap \partial K \neq \emptyset\}.$$

Using (23), we get

$$\|v\|^q_{L^q(\partial K)} = \sum_{T\in\widetilde{\mathfrak{T}}_K}\|v\|^q_{L^q(F_K^T)} \leq \frac{d}{\gamma h_K}\sum_{T\in\widetilde{\mathfrak{T}}_K}|T|^{\frac{s-q}{s}}\|v\|^q_{L^s(T)} + \frac{q}{\gamma}\sum_{T\in\widetilde{\mathfrak{T}}_K}\|v\|^{q-1}_{L^s(T)}\|\nabla v\|_{\mathbf{L}^{\frac{s}{s-q+1}}(T)}.$$

Hölder's inequality for sequences with indices s/q and $s/(s-q)$, and $s/(q-1)$ and $s/(s-q+1)$ implies

$$\begin{aligned}\|v\|^q_{L^q(\partial K)} \leq{}& \frac{d}{\gamma h_K}\Big(\sum_{T\in\widetilde{\mathfrak{T}}_K}|T|\Big)^{\frac{s-q}{s}}\Big(\sum_{T\in\widetilde{\mathfrak{T}}_K}\|v\|^s_{L^s(T)}\Big)^{\frac{q}{s}} \\ &+ \frac{q}{\gamma}\Big(\sum_{T\in\widetilde{\mathfrak{T}}_K}\|\nabla v\|^{\frac{s}{s-q+1}}_{\mathbf{L}^{\frac{s}{s-q+1}}(T)}\Big)^{\frac{s-q+1}{s}}\Big(\sum_{T\in\widetilde{\mathfrak{T}}_K}\|v\|^s_{L^s(T)}\Big)^{\frac{q-1}{s}}.\end{aligned}$$

We further have the trivial inclusion $\bigcup_{T\in\widetilde{\mathfrak{T}}_K} T \subset K$. In light of this, we obtain

$$\|v\|^q_{L^q(\partial K)} \le \frac{d}{\gamma h_K}|K|^{\frac{s-q}{s}}\|v\|^q_{L^s(K)} + \frac{q}{\gamma}\|\nabla v\|_{\mathbf{L}^{\frac{s}{s-q+1}}(K)}\|v\|^{q-1}_{L^s(K)}.$$

Let $\Gamma(\cdot)$ denote the Euler's Gamma function. Given the unit ball $\mathcal{B}_d$ in $\mathbb{R}^d$, we have that

$$|K| \le h_K^d|\mathcal{B}_d| = h_K^d\frac{\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}+1\big)}.$$

Furthermore, the Young inequality $ab \le \frac{a^r}{r} + \frac{b^{r'}}{r'}$ with $r = q/(q-1)$ holds true, which implies the following upper bound for the second term on the right-hand side of (24):

$$\|\nabla v\|_{\mathbf{L}^{\frac{s}{s-q+1}}(K)}\|v\|^{q-1}_{L^s(K)} \le (q-1)h_K^{-1+\frac{d}{s}(s-q)}\frac{\|v\|^q_{L^s(K)}}{q} + h_K^{\frac{q}{q'}\left(1-\frac{d}{s}(s-q)\right)}\frac{\|\nabla v\|^q_{\mathbf{L}^{\frac{s}{s-q+1}}(K)}}{q}.$$

Overall, we get

$$\begin{aligned}\|v\|^q_{L^q(\partial K)} &\le \left(\frac{d\pi^{\frac{d(s-q)}{2s}}}{\gamma\Gamma\big(\frac{d}{2}+1\big)^{\frac{s-q}{s}}} + \frac{q}{\gamma}\right)\left(h_K^{-1+\frac{d}{s}(s-q)}\|v\|^q_{L^s(K)} + h_K^{\frac{q}{q'}\left(1-\frac{d}{s}(s-q)\right)}\|\nabla v\|^q_{\mathbf{L}^{\frac{s}{s-q+1}}(K)}\right)\\ &\le \left(\frac{d\pi^{\frac{d(s-q)}{2s}}}{\gamma\Gamma\big(\frac{d}{2}+1\big)^{\frac{s-q}{s}}} + \frac{q}{\gamma}\right)\left(h_K^{\frac{1}{q}\left(-1+\frac{d}{s}(s-q)\right)}\|v\|_{L^s(K)} + h_K^{\frac{1}{q'}\left(1-\frac{d}{s}(s-q)\right)}\|\nabla v\|_{\mathbf{L}^{\frac{s}{s-q+1}}(K)}\right)^q.\end{aligned} \tag{24}$$

The assertion follows by taking the power q^{-1} on both sides.

3 Babuška–Aziz inequalities with mixed boundary conditions

We prove here Theorem 1.4. Let $\widetilde{\Gamma}_N$ be such that $\Gamma_N \subseteq \widetilde{\Gamma}_N$ and $\widetilde{\Gamma}_D := \Gamma \setminus \widetilde{\Gamma}_N$ is a single $(d-1)$-dimensional facet of Ω. We have

$$C_{BA}(p,\Gamma_N,\Omega) \le C_{BA}(p,\widetilde{\Gamma}_N,\Omega). \tag{25}$$

In fact, using that $W^{1,p}_{\widetilde{\Gamma}_N}(\Omega)$ is contained in $W^{1,p}_{\Gamma_N}(\Omega)$, a right-inverse of the divergence in $W^{1,p}_{\widetilde{\Gamma}_N}(\Omega)$ is also a right-inverse of the divergence in $W^{1,p}_{\Gamma_N}(\Omega)$. In light of (25), it suffices to prove an upper bound on the constant appearing on the right-hand side.

Upper bound on $C_{BA}(p,\widetilde{\Gamma}_N,\Omega)$ on convex domains. Let Ω be convex. Without loss of generality, we assume that $\widetilde{\Gamma}_D$ lies on the hyperplane $x_1 = 0$. In particular, due to the convexity assumption, it is possible to construct

$$\Omega_{\text{ext}} := \{\widetilde{\mathbf{x}} \in \mathbb{R}^d \mid \widetilde{\mathbf{x}} = (-x_1, x_2, \dots, x_d)\ \forall \mathbf{x} \in \Omega\}$$

such that $(\overline{\Omega}\cap\overline{\Omega_{\text{ext}}})^\circ = \widetilde{\Gamma}^\circ_D$. We define $\widetilde{\Omega} := \Omega \cup \Omega_{\text{ext}} \cup \widetilde{\Gamma}^\circ_D$ and set $\widetilde{\Gamma}$ to be its boundary.

The diameter $h_{\widetilde{\Omega}}$ of $\widetilde{\Omega}$ is bounded by $2h_\Omega$. By construction, $\widetilde{\Omega}$ is star-shaped with respect to a ball $B_{\widetilde{\rho}}$ of radius $\widetilde{\rho}$, which is larger than or equal to the minimum between ρ and ρ_Γ in (16). We refer to Figure 1 for a graphical representation of the above construction in two dimensions.

Given f in $L^p(\Omega)$, we define $\widetilde{f}$ in $L^p_0(\widetilde{\Omega})$ as

$$\widetilde{f}(\mathbf{x}) := \begin{cases} f(x_1, x_2, \dots, x_d) & \text{if } \mathbf{x} = (x_1, x_2, \dots, x_d) \in \Omega\\ -f(-x_1, x_2, \dots, x_d) & \text{if } \mathbf{x} = (x_1, x_2, \dots, x_d) \in \Omega_{\text{ext}}.\end{cases} \tag{26}$$

Let $\widetilde{\mathbf{v}}$ in $\mathbf{W}^{1,p}_0(\widetilde{\Omega})$ be the right-inverse of the divergence applied to $\widetilde{f}$ in $L^p_0(\widetilde{\Omega})$. Defining $\mathbf{v} = \widetilde{\mathbf{v}}_{|\Omega}$ in $\mathbf{W}^{1,p}_{\widetilde{\Gamma}_N}(\Omega)$ and using (12), we deduce

$$|\mathbf{v}|_{\mathbf{W}^{1,p}(\Omega)} = |\widetilde{\mathbf{v}}|_{\mathbf{W}^{1,p}(\Omega)} \le |\widetilde{\mathbf{v}}|_{\mathbf{W}^{1,p}(\widetilde{\Omega})} \le C_{BA}(p,\widetilde{\Gamma},\widetilde{\Omega})\left\|\widetilde{f}\right\|_{L^p(\widetilde{\Omega})} \le 2^{\frac{1}{p}}C_{BA}(p,\widetilde{\Gamma},\widetilde{\Omega})\|f\|_{L^p(\widetilde{\Omega})}. \tag{27}$$

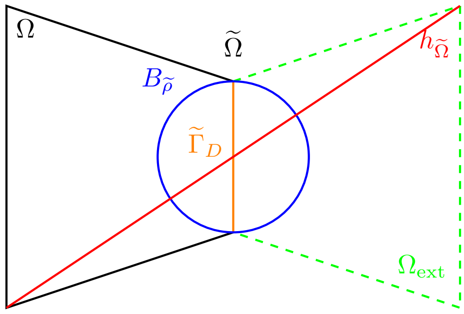

Figure 1: Extended domain $\widetilde{\Omega}$ (with black and dashed green boundary) for a convex domain Ω.

This entails

$$C_{BA}(p,\Gamma_N,\Omega) \overset{(25)}{\leq} C_{BA}(p,\widetilde{\Gamma}_N,\Omega) \overset{(27)}{\leq} 2^{\frac{1}{p}} C_{BA}(p,\widetilde{\Gamma},\widetilde{\Omega}).$$

Further using Lemma 1.3 and recalling that ρ_Γ is defined in (16), we write

$$C_{BA}(p,\widetilde{\Gamma},\widetilde{\Omega}) \leq C_G(d,p)\Big(\frac{h_{\widetilde{\Omega}}}{\widetilde{\rho}}\Big)^d\Big(1+\frac{h_{\widetilde{\Omega}}}{\widetilde{\rho}}\Big) \leq C_G(d,p)\Big(\frac{2h_\Omega}{\min(\rho,\rho_\Gamma)}\Big)^d\Big(1+\frac{2h_\Omega}{\min(\rho,\rho_\Gamma)}\Big).$$

Upper bound on $C_{BA}(\widetilde{\Gamma}_N,p,\Omega)$ on nonconvex domains. Let Ω_{ext} as detailed in Section 1.3; see Figure 2 for a graphical representation of that construction. Given f in $L^p(\Omega)$ and $\overline{f}_\Omega$ its

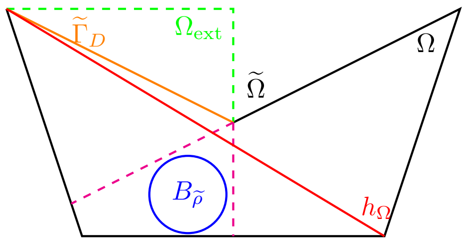

Figure 2: Extended domain $\widetilde{\Omega}$ (with black and dashed green boundary) for a nonconvex Ω.

average over Ω, define $\widetilde{f}$ in $L^p_0(\widetilde{\Omega})$ as

$$\widetilde{f}(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x}\in\Omega \\ -\overline{f}_\Omega\frac{|\Omega|}{|\Omega_{\mathrm{ext}}|} & \text{if } \mathbf{x}\in\Omega_{\mathrm{ext}}. \end{cases} \tag{28}$$

We have the following bounds:

$$\big\|\widetilde{f}\big\|^p_{L^p(\widetilde{\Omega})} \leq \|f\|^p_{L^p(\Omega)} + \big\|\overline{f}_\Omega|\Omega||\Omega_{\mathrm{ext}}|^{-1}\big\|^p_{L^p(\Omega_{\mathrm{ext}})} \leq \|f\|^p_{L^p(\Omega)} + \frac{|\Omega|^{p-1}}{|\Omega_{\mathrm{ext}}|^{p-1}}\|f\|^p_{L^p(\Omega)}.$$

Let $\widetilde{\mathbf{v}}$ in $\mathbf{W}^{1,p}_0(\widetilde{\Omega})$ be the right-inverse of the divergence applied to $\widetilde{f}$ in $L^p_0(\Omega)$. Defining $\mathbf{v}=\widetilde{\mathbf{v}}_{|\Omega}$ in $\mathbf{W}^{1,p}_{\widetilde{\Gamma}_N}(\Omega)$ and using (12), we deduce

$$|\mathbf{v}|_{\mathbf{W}^{1,p}(\Omega)} = |\widetilde{\mathbf{v}}|_{\mathbf{W}^{1,p}(\Omega)} \leq |\widetilde{\mathbf{v}}|_{\mathbf{W}^{1,p}(\widetilde{\Omega})} \leq C_{BA}(p,\widetilde{\Gamma},\widetilde{\Omega})\big\|\widetilde{f}\big\|_{L^p(\widetilde{\Omega})} \leq C_{BA}(p,\widetilde{\Gamma},\widetilde{\Omega})\left(1+\frac{|\Omega|^{p-1}}{|\Omega_{\mathrm{ext}}|^{p-1}}\right)\|f\|_{L^p(\Omega)}. \tag{29}$$

We deduce the upper bound

$$C_{BA}(p,\Gamma_N,\Omega) \overset{(25)}{\leq} C_{BA}(p,\widetilde{\Gamma}_N,\Omega) \overset{(29)}{\leq} C_{BA}(p,\widetilde{\Gamma},\widetilde{\Omega})\left(1+\frac{|\Omega|^{p-1}}{|\Omega_{\mathrm{ext}}|^{p-1}}\right).$$

Further using Lemma 1.3, given $h_{\widetilde{\Omega}}$ and $\widetilde{\rho}$ as in (15), we write

$$C_{BA}(p,\widetilde{\Gamma},\widetilde{\Omega}) \leq C_G(d,p)\Big(\frac{h_{\widetilde{\Omega}}}{\widetilde{\rho}}\Big)^d\Big(1+\frac{h_{\widetilde{\Omega}}}{\widetilde{\rho}}\Big).$$

Remark 6. The construction for nonconvex domains directly applies to the convex case for any extended domain Ω_{ext}. The price to pay is that $\widetilde{f}$ as in (26) is continuous across $\widetilde{\Gamma}_N$, whereas $\widetilde{f}$ as in (28) is not. Therefore, the approach for convex domains yields smoother right-inverses of the divergence with available stability estimates as discussed in [7, Appendix A].

4 Proof of the main results

We prove here the main results of the paper, namely Theorems 1.5 and 1.6, and Corollaries 1.7 and 1.8 in Sections 4.1, 4.2, 4.3, and 4.4.

4.1 Proof of the first main result

We prove here Theorem 1.5. Given p in $(1,\infty)$, let $s=(p^*)'$. By duality, we can write

$$\|v\|_{L^{p^*}(\Omega)} = \sup_{\varphi\in L^s(\Omega)} \frac{{}_{L^{p^*}(\Omega)}(v,\varphi)_{L^s(\Omega)}}{\|\varphi\|_{L^s(\Omega)}}.$$

From Theorem 1.4, we know that there exists $C_{BA}(s,\Gamma_N,\Omega)$ such that for all φ in $L^s(\Omega)$ it is possible to construct $\boldsymbol{\sigma}$ in $\mathbf{W}^{1,s}_{\Gamma_N}(\Omega)$ satisfying

$$\nabla\cdot\boldsymbol{\sigma}=\varphi, \qquad\qquad |\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)} \le C_{BA}(s,\Gamma_N,\Omega)\|\varphi\|_{L^s(\Omega)}.$$

We combine the two displays above and readily deduce

$$\begin{aligned}\|v\|_{L^{p^*}(\Omega)} &\le C_{BA}(s,\Gamma_N,\Omega) \sup_{\boldsymbol{\sigma}\in\mathbf{W}^{1,s}_{\Gamma_N}(\Omega)} \frac{{}_{L^{p^*}(\Omega)}(v,\nabla\cdot\boldsymbol{\sigma})_{L^s(\Omega)}}{|\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)}}\\ &= C_{BA}(s,\Gamma_N,\Omega) \sup_{\boldsymbol{\sigma}\in\mathbf{W}^{1,s}_{\Gamma_N}(\Omega)} \frac{{}_{\mathbf{L}^p(\Omega)}(-\nabla_h v,\boldsymbol{\sigma})_{\mathbf{L}^{p'}(\Omega)} + \sum_{F\in\mathcal{F}_n} {}_{L^{p^\circ}(F)}(\llbracket v\rrbracket,\boldsymbol{\sigma}\cdot\mathbf{n}_F)_{L^{(p^\circ)'}(F)}}{|\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)}}.\end{aligned} \tag{30}$$

We check that the two terms in the numerator on the right-hand side above are well-defined.

- Since v belongs to $W^{1,p}_{\Gamma_N}(\mathcal{T}_n,\Omega)$, we clearly have that $\nabla_h v$ is in $\mathbf{L}^p(\Omega)$.
- Recall that $\boldsymbol{\sigma}$ belongs to $\mathbf{W}^{1,s}(\Omega)$ with $s=(p^*)'$; the Sobolev embedding $\mathbf{W}^{1,s}(\Omega)\hookrightarrow\mathbf{L}^q(\Omega)$ holds true (4) for all q in $[1,s^*]$, where we recall $s^*=(ds)/(d-s)$. To conclude, we have to show that $s^*=p'$. Standard manipulations imply

$$s^* = \frac{sd}{d-s} = \frac{p^*d}{d(p^*-1)-p^*} = \frac{p^*d}{p^*(d-1)-d} = \frac{pd}{pd-p-d+p} = \frac{p}{p-1} = p'.$$

- The trace of v in $W^{1,p}_{\Gamma_N}(\mathcal{T}_n,\Omega)$ belongs to $W^{1-\frac{1}{p},p}(F)$ for all facets F in $\mathcal{F}_n$; see, e.g., [20, Chapter 3]. Using the Sobolev embedding (in dimension $d-1$) $W^{1-\frac{1}{p},p}(F)\hookrightarrow L^q(F)$ for all q such that

$$q \le \frac{p(d-1)}{(d-1)-(p-1)} = \frac{p(d-1)}{d-p} = p^\circ.$$

  In particular, the trace of v belongs to $L^{p^\circ}(F)$ for all facets F.

- The trace of $\boldsymbol{\sigma}$ in $\mathbf{W}^{1,s}(\Omega)$ belongs to $\mathbf{W}^{1-\frac{1}{s},s}(F)$ for all facets F in $\mathcal{F}_n$. Using the Sobolev embedding (4) (in dimension $d-1$) $\mathbf{W}^{1-\frac{1}{s},s}(F)\hookrightarrow\mathbf{L}^q(F)$ for all r such that, proceeding as above, $r\le s^\circ$. We are left with proving $s^\circ=(p^\circ)'$. Standard manipulations imply

$$s^\circ = \frac{s(d-1)}{d-s} = \frac{\left(\frac{pd}{d-p}\right)'(d-1)}{d-\left(\frac{pd}{d-p}\right)'} = \frac{\frac{pd}{pd-d+p}(d-1)}{d-\frac{pd}{pd-d+p}} = \frac{p(d-1)}{pd-d} = \frac{p^\circ}{p^\circ-1} = (p^\circ)'.$$

We now estimate the two terms in the numerator on the right-hand side of (30). The Sobolev embedding (4) $\mathbf{W}^{1,s}(\Omega) \hookrightarrow \mathbf{L}^{p'}(\Omega)$ and the Poincaré–Steklov inequality (3) imply

$$\begin{aligned}\|\boldsymbol{\sigma}\|_{\mathbf{L}^{p'}(\Omega)} &\le h_\Omega^{\frac{d}{p'}-\frac{d}{s}+1} C_{\mathrm{Sob}}(p',1,s,\Omega)\|\boldsymbol{\sigma}\|_{\mathbf{W}^{1,s}(\Omega)}\\ &\le h_\Omega^{\frac{d}{p'}-\frac{d}{s}+1} C_{\mathrm{Sob}}(p',1,s,\Omega)(1+h_\Omega^s C_{PS}(s,\Omega))^s)^{\frac{1}{s}}|\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)}.\end{aligned} \tag{31}$$

Moreover, using two Hölder's inequalities with indices p° and s°, and p and $p' = s^* = s^\circ 1^*$ (for sequences), and Jensen's inequality for sequences, we get

$$\begin{aligned}&\sum_{F\in\mathcal{F}_n} {}_{L^{p^\circ}(F)}(\llbracket v\rrbracket, \boldsymbol{\sigma}\cdot\mathbf{n}_F)_{L^{(p^\circ)'}(F)} = \sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} {}_{L^{p^\circ}(F)}(\llbracket v\rrbracket, \boldsymbol{\sigma}\cdot\mathbf{n}_F)_{L^{(p^\circ)'}(F)}\\ &\le \sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \|\llbracket v\rrbracket\|_{L^{p^\circ}(F)}\|\boldsymbol{\sigma}\cdot\mathbf{n}_F\|_{L^{s^\circ}(F)} \le \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \|\llbracket v\rrbracket\|_{L^{p^\circ}(F)}^p\Big)^{\frac{1}{p}} \Big(\sum_{K\in\mathcal{T}_n}\sum_{F\in\mathcal{F}^K} \|\boldsymbol{\sigma}\cdot\mathbf{n}_F\|_{L^{s^\circ}(F)}^{s^*}\Big)^{\frac{1}{s^*}}\\ &\le \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \|\llbracket v\rrbracket\|_{L^{p^\circ}(F)}^p\Big)^{\frac{1}{p}} \Big(\sum_{K\in\mathcal{T}_n}\big(\sum_{F\in\mathcal{F}^K} \|\boldsymbol{\sigma}\cdot\mathbf{n}_F\|_{L^{s^\circ}(F)}^{s^\circ}\big)^{1^*}\Big)^{\frac{1}{s^*}}\\ &= \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \|\llbracket v\rrbracket\|_{L^{p^\circ}(F)}^p\Big)^{\frac{1}{p}} \Big(\sum_{K\in\mathcal{T}_n} \|\boldsymbol{\sigma}\cdot\mathbf{n}_K\|_{L^{s^\circ}(\partial K)}^{s^*}\Big)^{\frac{1}{s^*}}.\end{aligned}$$

We apply the Sobolev–trace inequality (11) to the last term on the right-hand side and Jensen's inequality for sequences, and get

$$\begin{aligned}&\sum_{F\in\mathcal{F}_n} {}_{L^{p^\circ}(F)}(\llbracket v\rrbracket, \boldsymbol{\sigma}\cdot\mathbf{n}_F)_{L^{(p^\circ)'}(F)}\\ &\le C_{TR}(s^\circ,s^*,d,\gamma)\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \|\llbracket v\rrbracket\|_{L^{p^\circ}(F)}^p\Big)^{\frac{1}{p}} \Big(\sum_{K\in\mathcal{T}_n} (\|\boldsymbol{\sigma}\|_{\mathbf{L}^{s^*}(K)}^{s^*} + |\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(K)}^{s^*})\Big)^{\frac{1}{s^*}}\\ &\le C_{TR}(s^\circ,s^*,d,\gamma)\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \|\llbracket v\rrbracket\|_{L^{p^\circ}(F)}^p\Big)^{\frac{1}{p}} \Big(\|\boldsymbol{\sigma}\|_{\mathbf{L}^{s^*}(\Omega)}^{s^*} + \big(\sum_{K\in\mathcal{T}_n} |\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(K)}^{s}\big)^{\frac{s^*}{s}}\Big)^{\frac{1}{s^*}}\\ &= C_{TR}(s^\circ,s^*,d,\gamma)\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \|\llbracket v\rrbracket\|_{L^{p^\circ}(F)}^p\Big)^{\frac{1}{p}} \Big(\|\boldsymbol{\sigma}\|_{\mathbf{L}^{s^*}(\Omega)} + |\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)}\Big).\end{aligned}$$

We apply a Sobolev embedding as in (4) and the Poincaré–Steklov inequality in (3) to get

$$\begin{aligned}&\sum_{F\in\mathcal{F}_n} {}_{L^{p^\circ}(F)}(\llbracket v\rrbracket, \boldsymbol{\sigma}\cdot\mathbf{n}_F)_{L^{(p^\circ)'}(F)}\\ &\le C_{TR}(s^\circ,s^*,d,\gamma)\big(1+h_\Omega^{\frac{d}{s^*}-\frac{d}{s}+1}C_{\mathrm{Sob}}(s^*,1,s,\Omega)(1+h_\Omega^s C_{PS}(s,\Omega))^s)^{\frac{1}{s}}\big)\cdot\\ &\qquad\cdot\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \|\llbracket v\rrbracket\|_{L^{p^\circ}(F)}^p\Big)^{\frac{1}{p}}|\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)}.\end{aligned} \tag{32}$$

The assertion follows combining (30), (31), and (32), and noting that $\frac{d}{p'} - \frac{d}{(p^*)'} + 1$ equals 0.

4.2 Proof of the second main result

We prove here Theorem 1.6. Throughout, we employ the notation in the proof of Theorem 1.5. Introduce

$$s := (p1^*)' = \left(\frac{dp}{d-1}\right)' = \frac{dp}{dp-d+1}. \tag{33}$$

By duality, we write

$$\|v\|_{L^{p1^*}(\Omega)} = \sup_{\varphi\in L^s(\Omega)} \frac{{}_{L^{p1^*}(\Omega)}(v,\varphi)_{L^s(\Omega)}}{\|\varphi\|_{L^s(\Omega)}}.$$

From Theorem 1.4, we know that there exists $C_{BA}(s,\Gamma_N,\Omega)$ such that for all φ in $L^s(\Omega)$ it is possible to construct $\boldsymbol{\sigma}$ in $\mathbf{W}^{1,s}_{\Gamma_N}(\Omega)$ satisfying

$$\nabla\cdot\boldsymbol{\sigma}=\varphi, \qquad\qquad |\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)}\le C_{BA}(s,\Gamma_N,\Omega)\|\varphi\|_{L^s(\Omega)}.$$

We combine the two displays above and get

$$\begin{aligned}\|v\|_{L^{p1^*}(\Omega)} &\le C_{BA}(s,\Gamma_N,\Omega)\sup_{\boldsymbol{\sigma}\in\mathbf{W}^{1,s}_{\Gamma_N}(\Omega)}\frac{{}_{L^{p1^*}(\Omega)}(v,\nabla\cdot\boldsymbol{\sigma})_{L^s(\Omega)}}{|\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)}}\\ &= C_{BA}(s,\Gamma_N,\Omega)\sup_{\boldsymbol{\sigma}\in\mathbf{W}^{1,s}_{\Gamma_N}(\Omega)}\frac{{}_{\mathbf{L}^p(\Omega)}(-\nabla_h v,\boldsymbol{\sigma})_{\mathbf{L}^{p'}(\Omega)}+\sum_{F\in\mathcal{F}_n}{}_{L^p(F)}(\llbracket v\rrbracket,\boldsymbol{\sigma}\cdot\mathbf{n}_F)_{L^{p'}(F)}}{|\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)}}.\end{aligned}\tag{34}$$

We check that the inner products in the numerator on the right-hand side above are well-defined.

- Since v belongs to $W^{1,p}(\mathcal{T}_n,\Omega)$, we clearly have that $\nabla_h v$ is in $\mathbf{L}^p(\Omega)$.
- Recall that $\boldsymbol{\sigma}$ belongs to $\mathbf{W}^{1,s}_{\Gamma_N}(\Omega)$ with s as in (33); the Sobolev embedding $\mathbf{W}^{1,s}(\Omega)\hookrightarrow\mathbf{L}^q(\Omega)$ holds true (4) for all q in $[1,s^*]$, where we recall $s^*=(ds)/(d-s)$. To conclude, we show that $s^*=p'1^*$, which is larger than p'. Standard manipulations imply
$$s^*\overset{(33)}{=}\frac{(p1^*)'d}{d-(p1^*)'}=\frac{d^2p}{d^2p-d^2+d-dp}=\frac{dp}{(d-1)(p-1)}=d'p'=1^*p'.$$
- The trace of functions in $W^{1,p}(\Omega)$ belongs to $L^p(F)$ for all facets F in $\mathcal{F}_n$.
- The trace of $\mathbf{W}^{1,s}(\Omega)$ belongs to $\mathbf{W}^{1-\frac{1}{s},s}(F)$ for all facets F in $\mathcal{F}_n$. Using the Sobolev embedding (4) (in dimension $d-1$) $\mathbf{W}^{1-\frac{1}{s},s}(F)\hookrightarrow\mathbf{L}^q(F)$ for all q such that
$$q\le\frac{s(d-1)}{(d-1)-(s-1)}\overset{(33)}{=}\frac{(d-1)dp}{d^2p-d^2+d-dp}=\frac{(d-1)p}{(d-1)(p-1)}=\frac{p}{p-1}=p'.$$

As for the first term on the right-hand side of (34), we exploit the Sobolev embedding (4) and the Poincaré–Steklov inequality (3), and arrive at

$$\begin{aligned}&{}_{\mathbf{L}^p(\Omega)}(-\nabla_h v,\boldsymbol{\sigma})_{\mathbf{L}^{p'}(\Omega)}\le\|\nabla_h v\|_{\mathbf{L}^p(\Omega)}\|\boldsymbol{\sigma}\|_{\mathbf{L}^{p'}(\Omega)}\\ &\le h_\Omega^{\frac{d}{p'}-\frac{d}{(p1^*)'}+1}C_{\mathrm{Sob}}(p',1,(p1^*)',\Omega)\big(1+h_\Omega C_{PS}((p1^*)',\Omega)\big)\big)\|\nabla_h v\|_{\mathbf{L}^p(\Omega)}|\boldsymbol{\sigma}|_{\mathbf{W}^{1,s}(\Omega)}.\end{aligned}\tag{35}$$

Recall that $\widetilde{h}_F$ is defined in (20). As for the second term on the right-hand side of (34), using two Hölder's inequalities with indices p and p', and p and p' (for sequences), we get

$$\begin{aligned}&\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}{}_{L^p(F)}(\llbracket v\rrbracket,\boldsymbol{\sigma}\cdot\mathbf{n}_F)_{L^{p'}(F)}\le\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\widetilde{h}_F^{\frac{1-p}{p}}\|\llbracket v\rrbracket\|_{L^p(F)}\widetilde{h}_F^{-\frac{1-p}{p}}\|\boldsymbol{\sigma}\cdot\mathbf{n}_F\|_{L^{p'}(F)}\\ &\le\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\widetilde{h}_F^{1-p}\|\llbracket v\rrbracket\|^p_{L^p(F)}\Big)^{\frac{1}{p}}\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\widetilde{h}_F\|\boldsymbol{\sigma}\cdot\mathbf{n}_F\|^{p'}_{L^{p'}(F)}\Big)^{\frac{1}{p'}}.\end{aligned}\tag{36}$$

We cope with the second term on the right-hand side of (36) separately. Using that $\widetilde{h}_F$ is smaller than or equal to h_K for K such that F belongs to $\mathcal{F}^K$, the trace inequality (10), Jensen's inequality for sequences, the Poincaré–Steklov inequality (3), and standard manipulations yields

$$\begin{aligned}&\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\widetilde{h}_F\|\boldsymbol{\sigma}\cdot\mathbf{n}_F\|^{p'}_{L^{p'}(F)}\Big)^{\frac{1}{p'}}\le\Big(\sum_{K\in\mathcal{T}_n}h_K\sum_{F\in\mathcal{F}^K}\|\boldsymbol{\sigma}\cdot\mathbf{n}_F\|^{p'}_{L^{p'}(F)}\Big)^{\frac{1}{p'}}\\ &=\Big(\sum_{K\in\mathcal{T}_n}h_K\|\boldsymbol{\sigma}\cdot\mathbf{n}_F\|^{p'}_{L^{p'}(\partial K)}\Big)^{\frac{1}{p'}}\le C_{TR}(p',p',d,\gamma)\Big(\sum_{K\in\mathcal{T}_n}h_K\Big(h_K^{-1}\|\boldsymbol{\sigma}\|^{p'}_{L^{p'}(K)}+h_K^{\frac{p'}{p}}|\boldsymbol{\sigma}|^{p'}_{\mathbf{W}^{1,p'}(K)}\Big)\Big)^{\frac{1}{p'}}\\ &=C_{TR}(p',p',d,\gamma)\Big(\sum_{K\in\mathcal{T}_n}\Big(\|\boldsymbol{\sigma}\|^{p'}_{L^{p'}(K)}+h_K^{p'}|\boldsymbol{\sigma}|^{p'}_{\mathbf{W}^{1,p'}(K)}\Big)\Big)^{\frac{1}{p'}}\\ &\le C_{TR}(p',p',d,\gamma)\big(1+\max_{K\in\mathcal{T}_n}h_K^{p'}\big)^{\frac{1}{p'}}\Big(\|\boldsymbol{\sigma}\|^{p'}_{L^{p'}(\Omega)}+|\boldsymbol{\sigma}|^{p'}_{\mathbf{W}^{1,p'}(\Omega)}\Big)^{\frac{1}{p'}}\\ &\le C_{TR}(p',p',d,\gamma)\big(1+\max_{K\in\mathcal{T}_n}h_K^{p'}\big)^{\frac{1}{p'}}(1+h_\Omega C_{PS}(p',\Omega))|\boldsymbol{\sigma}|_{\mathbf{W}^{1,p'}(\Omega)}.\end{aligned}$$

A combination of this display with (36) gives

$$\begin{aligned}&\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} {}_{L^p(F)}(\llbracket v\rrbracket, \boldsymbol{\sigma}\cdot\mathbf{n}_F)_{L^{p'}(F)}\\ &\le C_{TR}(p',p',d,\gamma)\big(1+\max_{K\in\mathcal{T}_n} h_K^{p'}\big)^{\frac{1}{p'}}(1+h_\Omega C_{PS}(p',\Omega))|\boldsymbol{\sigma}|_{\mathbf{W}^{1,p'}(\Omega)}\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\widetilde{h}_F^{1-p}\|\llbracket v\rrbracket\|^p_{L^p(F)}\Big)^{\frac{1}{p}}.\end{aligned} \tag{37}$$

The assertion follows combining (34), (35), and (37).

4.3 Proof of the third main result

We prove here Corollary 1.7. Here and in the following section, let $\Pi^0_{\mathcal{T}_n}$ be the piecewise average operator over $\mathcal{T}_n$. The triangle inequality gives

$$\|v\|_{L^{p^*}(\Omega)} \le \big\|v-\Pi^0_{\mathcal{T}_n}v\big\|_{L^{p^*}(\Omega)} + \big\|\Pi^0_{\mathcal{T}_n}v\big\|_{L^{p^*}(\Omega)}. \tag{38}$$

We focus on the first term on the right-hand side: using the Sobolev embedding (4) and the Poincaré-Steklov inequality (3) yields

$$\big\|v-\Pi^0_{\mathcal{T}_n}v\big\|_{L^{p^*}(\Omega)} \le \max_{K\in\mathcal{T}_n}\big(h_K^{\frac{d}{p^*}-\frac{d}{p}+1}C_{\text{Sob}}(p^*,1,p,K)(1+h_K C_{PS}(p,K))\big)\|\nabla_h v\|_{\mathbf{L}^p(\Omega)}.$$

Observing that $\frac{d}{p^*}-\frac{d}{p}+1$ equals 0 and combining the two displays above entail

$$\|v\|_{L^{p^*}(\Omega)} \le \max_{K\in\mathcal{T}_n}\big(C_{\text{Sob}}(p^*,1,p,K)(1+h_K C_{PS}(p,K))\big)\|\nabla_h v\|_{\mathbf{L}^p(\Omega)} + \big\|\Pi^0_{\mathcal{T}_n}v\big\|_{L^{p^*}(\Omega)}. \tag{39}$$

We are left with estimating the second term on the right-hand side. We apply Theorem 1.5 and get, for C_2 as in (18),

$$\big\|\Pi^0_{\mathcal{T}_n}v\big\|_{L^{p^*}(\Omega)} \le C_2\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\|\,[[\Pi^0_{\mathcal{T}_n}v]]\,\|^p_{L^{p^\circ}(F)}\Big)^{\frac{1}{p}}.$$

Recall that $\Pi^0_{\mathcal{F}_n}$ is the piecewise average operator over $\mathcal{F}_n$. We estimate each jump term separately:

$$\|\,[[\Pi^0_{\mathcal{T}_n}v]]\,\|_{L^{p^\circ}(F)} \le \big\|\,[[\Pi^0_{\mathcal{T}_n}v]] - \Pi^0_{\mathcal{F}_n}\llbracket v\rrbracket\big\|_{L^{p^\circ}(F)} + \big\|\Pi^0_{\mathcal{F}_n}\llbracket v\rrbracket\big\|_{L^{p^\circ}(F)}.$$

Using the stability of the L^2 projector in any L^{p° norm, we arrive at

$$\big\|\,[[\Pi^0_{\mathcal{T}_n}v]] - \Pi^0_{\mathcal{F}_n}\llbracket v\rrbracket\big\|_{L^{p^\circ}(F)} = \big\|\Pi^0_{\mathcal{F}_n}\,[[v-\Pi^0_{\mathcal{T}_n}v]]\,\big\|_{L^{p^\circ}(F)} \le \big\|\,[[v-\Pi^0_{\mathcal{T}_n}v]]\,\big\|_{L^{p^\circ}(F)}.$$

We combine the three displays above and get

$$\begin{aligned}\big\|\Pi^0_{\mathcal{T}_n}v\big\|_{L^{p^*}(\Omega)} &\le 2^{1-\frac{1}{p}}C_2\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\big\|\Pi^0_{\mathcal{F}_n}\llbracket v\rrbracket\big\|^p_{L^{p^\circ}(F)}\Big)^{\frac{1}{p}}\\ &\quad + 2^{1-\frac{1}{p}}C_2\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\|\,[[v-\Pi^0_{\mathcal{T}_n}v]]\,\|^p_{L^{p^\circ}(F)}\Big)^{\frac{1}{p}}.\end{aligned} \tag{40}$$

We are left to handle the second term on the right-hand side, as the first one is fine as it is. Hölder's inequality for sequences with indices p°/p and $p^\circ/(p^\circ-p)$ reveals that

$$\begin{aligned}&\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\|\,[[v-\Pi^0_{\mathcal{T}_n}v]]\,\|^p_{L^{p^\circ}(F)}\Big)^{\frac{1}{p}} \le \Big(\sum_{K\in\mathcal{T}_n}\sum_{F\in\mathcal{F}^K}\|v-\Pi^0_{\mathcal{T}_n}v\|^p_{L^{p^\circ}(F)}\Big)^{\frac{1}{p}}\\ &\le \Big[\max_{K\in\mathcal{T}_n}(\text{card}(\mathcal{F}^K))^{\frac{p^\circ-p}{p}}\Big]\Big(\sum_{K\in\mathcal{T}_n}\|v-\Pi^0_{\mathcal{T}_n}v\|^p_{L^{p^\circ}(\partial K)}\Big)^{\frac{1}{p}}.\end{aligned}$$

Further using the trace inequality (11), the Poincaré-Steklov inequality (3), and the Sobolev embedding (4), and recalling that $\frac{d}{p^*} - \frac{d}{p} + 1$ equals 0, we deduce

$$\begin{aligned}&\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \| [\![v - \Pi^0_{\mathcal{T}_n} v]\!] \|^p_{L^{p^\circ}(F)} \Big)^{\frac{1}{p}} \\ &\le \Big[\max_{K\in\mathcal{T}_n} (\mathrm{card}(\mathcal{F}^K)) \Big] C_{TR}(p^\circ, p^*, d, \gamma) \Big[\max_{K\in\mathcal{T}_n} (1 + C_{\mathrm{Sob}}(p^*, 1, p, K))(1 + h_K C_{PS}(p^*, K)) \Big] \|\nabla_h v\|_{\mathbf{L}^p(\Omega)}.\end{aligned} \tag{41}$$

We combine (38), (39), (40), and (41), and write

$$\begin{aligned}\|v\|_{L^{p^*}(\Omega)} \le \Big\{ &\max_{K\in\mathcal{T}_n} \big[C_{\mathrm{Sob}}(p^*, 1, p, K)(1 + h_K C_{PS}(p, K)) \big] \\ &+ 2^{1-\frac{1}{p}} C_2 \Big[\max_{K\in\mathcal{T}_n} (\mathrm{card}(\mathcal{F}^K)) C_{TR}(p^\circ, p^*, d, \gamma) \cdot \\ &\quad \cdot \Big[\max_{K\in\mathcal{T}_n} (1 + C_{\mathrm{Sob}}(p^*, 1, p, K))(1 + h_K C_{PS}(p^*, K)) \Big] \Big\} \|\nabla_h v\|_{\mathbf{L}^p(\Omega)} \\ &+ 2^{1-\frac{1}{p}} C_2 \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \|\Pi^0_{\mathcal{F}_n} [\![v]\!] \|^p_{L^{p^\circ}(F)} \Big)^{\frac{1}{p}},\end{aligned}$$

which is the assertion.

4.4 Proof of the fourth main result

We prove here Corollary 1.8. Recall that $\Pi^0_{\mathcal{T}_n}$ denotes the piecewise average operator over $\mathcal{T}_n$. The triangle inequality gives

$$\|v\|_{L^{p1^*}(\Omega)} \le \big\| v - \Pi^0_{\mathcal{T}_n} v \big\|_{L^{p1^*}(\Omega)} + \big\| \Pi^0_{\mathcal{T}_n} v \big\|_{L^{p1^*}(\Omega)}. \tag{42}$$

We focus on the first term on the right-hand side: using the Sobolev embedding (4) and the Poincaré-Steklov inequality (3) yields

$$\big\| v - \Pi^0_{\mathcal{T}_n} v \big\|_{L^{p1^*}(\Omega)} \le \max_{K\in\mathcal{T}_n} \big(h_K^{\frac{d}{p1^*} - \frac{d}{p} + 1} C_{\mathrm{Sob}}(p1^*, 1, p, K)(1 + h_K C_{PS}(p, K)) \big) \|\nabla_h v\|_{\mathbf{L}^p(\Omega)}.$$

Combining the two displays above entails

$$\|v\|_{L^{p1^*}(\Omega)} \le \max_{K\in\mathcal{T}_n} \big(h_K^{\frac{d}{p1^*} - \frac{d}{p} + 1} C_{\mathrm{Sob}}(p1^*, 1, p, K)(1 + h_K C_{PS}(p, K)) \big) \|\nabla_h v\|_{\mathbf{L}^p(\Omega)} + \big\| \Pi^0_{\mathcal{T}_n} v \big\|_{L^{p1^*}(\Omega)}. \tag{43}$$

We are left with estimating the second term on the right-hand side. For C_4 as in (21) and $\widetilde{h}_F$ as in (20), we apply Theorem 1.6 and get

$$\big\| \Pi^0_{\mathcal{T}_n} v \big\|_{L^{p1^*}(\Omega)} \le C_4 \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \widetilde{h}_F^{1-p} \| [\![\Pi^0_{\mathcal{T}_n} v]\!] \|^p_{L^p(F)} \Big)^{\frac{1}{p}}.$$

Recall that $\Pi^0_{\mathcal{F}_n}$ is the piecewise average operator over $\mathcal{F}_n$. We estimate each jump term separately:

$$\| [\![\Pi^0_{\mathcal{T}_n} v]\!] \|_{L^p(F)} \le \big\| [\![\Pi^0_{\mathcal{T}_n} v]\!] - \Pi^0_{\mathcal{F}_n} [\![v]\!] \big\|_{L^p(F)} + \big\| \Pi^0_{\mathcal{F}_n} [\![v]\!] \big\|_{L^p(F)}.$$

Using the stability of the L^2 projector in any L^p norm, we arrive at

$$\big\| [\![\Pi^0_{\mathcal{T}_n} v]\!] - \Pi^0_{\mathcal{F}_n} [\![v]\!] \big\|_{L^p(F)} = \big\| \Pi^0_{\mathcal{F}_n} [\![v - \Pi^0_{\mathcal{T}_n} v]\!] \big\|_{L^p(F)} \le \big\| [\![v - \Pi^0_{\mathcal{T}_n} v]\!] \big\|_{L^p(F)}.$$

We combine the three displays above and get

$$\begin{aligned}\big\| \Pi^0_{\mathcal{T}_n} v \big\|_{L^{p1^*}(\Omega)} \le\ & 2^{1-\frac{1}{p}} C_4 \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \widetilde{h}_F^{1-p} \|\Pi^0_{\mathcal{F}_n} [\![v]\!] \|^p_{L^p(F)} \Big)^{\frac{1}{p}} \\ &+ 2^{1-\frac{1}{p}} C_4 \Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D} \widetilde{h}_F^{1-p} \| [\![v - \Pi^0_{\mathcal{T}_n} v]\!] \|^p_{L^p(F)} \Big)^{\frac{1}{p}}.\end{aligned} \tag{44}$$

We are left to handle the second term on the right-hand side, as the first one is fine as it is:

$$\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\widetilde{h}_F^{1-p}\|[\![v-\Pi^0_{\mathcal{T}_n}v]\!]\|^p_{L^p(F)}\Big)^{\frac1p}\le\Big(\sum_{K\in\mathcal{T}_n}\sum_{F\in\mathcal{F}_n}\widetilde{h}_F^{1-p}\|v-\Pi^0_{\mathcal{T}_n}v\|^p_{L^p(F)}\Big)^{\frac1p}$$
$$\le\Big[\max_{K\in\mathcal{T}_n}\max_{F\in\mathcal{F}^K}\Big(\frac{\widetilde{h}_F}{h_K}\Big)^{-1+\frac1p}\Big]\Big(\sum_{K\in\mathcal{T}_n}h_K^{1-p}\|v-\Pi^0_{\mathcal{T}_n}v\|^p_{L^p(\partial K)}\Big)^{\frac1p}.$$

Further using the trace inequality (10) and the Poincaré-Steklov inequality (3), we deduce

$$\begin{aligned}&\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}\widetilde{h}_F^{1-p}\|[\![v-\Pi^0_{\mathcal{T}_n}v]\!]\|^p_{L^p(F)}\Big)^{\frac1p}\\&\le\Big[\max_{K\in\mathcal{T}_n}\max_{F\in\mathcal{F}^K}\Big(\frac{\widetilde{h}_F}{h_K}\Big)^{-1+\frac1p}\Big]C_{TR}(p,p,d,\gamma)\Big[\max_{K\in\mathcal{T}_n}(1+h_KC_{PS}(p,K))\Big]\|\nabla_hv\|_{\mathbf{L}^p(\Omega)}.\end{aligned}\tag{45}$$

We combine (42), (43), (44), and (45), and write

$$\begin{aligned}\|v\|_{L^{p1^*}(\Omega)}\le\Big\{&\max_{K\in\mathcal{T}_n}[h_K^{\frac{d}{p1^*}-\frac dp+1}C_{\mathrm{Sob}}(p1^*,1,p,K)(1+h_KC_{PS}(p,K))]\\&+2^{1-\frac1p}C_4\Big[\max_{K\in\mathcal{T}_n}\max_{F\in\mathcal{F}^K}\Big(\frac{\widetilde{h}_F}{h_K}\Big)^{-1+\frac1p}\Big]C_{TR}(p,p,d,\gamma)\Big[\max_{K\in\mathcal{T}_n}(1+h_KC_{PS}(p,K))\Big]\Big\}\|\nabla_hv\|_{\mathbf{L}^p(\Omega)}\\&+2^{1-\frac1p}C_4\Big(\sum_{F\in\mathcal{F}_n^I\cup\mathcal{F}_n^D}h_F^{1-p}\|\Pi^0_{\mathcal{F}_n}[\![v]\!]\|^p_{L^p(F)}\Big)^{\frac1p},\end{aligned}$$

which is the assertion.

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A Inequalities involving the right-inverse of the divergence

A.1 Results in the Banach setting

We recall several inequalities: given p in $(1, \infty)$,

- (**lowest order inf-sup condition**) there exists a positive constant $\beta(p, \Gamma, \Omega)$ such that

$$\inf_{q \in L_0^{p'}(\Omega)} \sup_{\mathbf{v} \in \mathbf{W}_0^{1,p}(\Omega)} \frac{{}_{L^p}(\nabla\cdot \mathbf{v}, q)_{L^{p'}}}{|\mathbf{v}|_{\mathbf{W}^{1,p}(\Omega)} \|q\|_{L^{p'}(\Omega)}} \geq \beta(p, \Gamma, \Omega); \tag{46}$$

- (**lowest order Nečas-Lions inequality**) there exists a positive constant $C_{BA}(p, \Gamma, \Omega)$ such that

$$\|q\|_{L^{p'}(\Omega)} \leq C_{BA}(p, \Gamma, \Omega) \|\nabla q\|_{(\mathbf{W}^{1,p}(\Omega))'} \qquad \forall q \in L_0^{p'}(\Omega); \tag{47}$$

- (**lowest order Babuška–Aziz inequality**) there exists a positive constant $C_{BA}(p, \Gamma, \Omega)$ such that for all q in $L_0^p(\Omega)$ it is possible to construct $\mathbf{v}$ in $\mathbf{W}_0^{1,p}(\Omega)$ satisfying

$$\nabla\cdot \mathbf{v} = q, \qquad |\mathbf{v}|_{\mathbf{W}^{1,p}(\Omega)} \leq C_{BA}(p, \Gamma, \Omega) \|q\|_{L^p(\Omega)}. \tag{48}$$

In the remainder of this section, we show

$$\beta(p, \Gamma, \Omega)^{-1} = C_{BA}(p, \Gamma, \Omega) \leq C_{BA}(p, \Gamma, \Omega).$$

Showing $\beta(p,\Gamma,\Omega)^{-1} = C_{BA}(p,\Gamma,\Omega)$**.** For all q in $L_0^{p'}(\Omega)$, an integration by parts and the definition of negative norms yields

$$\sup_{\mathbf{v}\in\mathbf{W}_0^{1,p}(\Omega)} \frac{{}_{L^p}(\nabla\cdot\mathbf{v},q)_{L^{p'}}}{|\mathbf{v}|_{\mathbf{W}^{1,p}(\Omega)}\|q\|_{L^{p'}(\Omega)}} = \sup_{\mathbf{v}\in\mathbf{W}_0^{1,p}(\Omega)} \frac{{}_{(\mathbf{W}^{1,p})'}\langle\nabla q,\mathbf{v}\rangle_{\mathbf{W}^{1,p}}}{|\mathbf{v}|_{\mathbf{W}^{1,p}(\Omega)}\|q\|_{L^{p'}(\Omega)}} = \frac{\|\nabla q\|_{(\mathbf{W}^{1,p}(\Omega))'}}{\|q\|_{L^{p'}(\Omega)}}.$$

Taking the inf over all possible q entails the assertion.

Showing $\beta(p,\Gamma,\Omega)^{-1} \le C_{BA}(p,\Gamma,\Omega)$**.** For all q in $L_0^{p'}(\Omega)$, let $\mathbf{v}(f)$ be a right-inverse of the divergence for an f in $L_0^p(\Omega)$ satisfying (48). Further using the bound in (48) yields

$$\begin{aligned}\sup_{\mathbf{v}\in\mathbf{W}_0^{1,p}(\Omega)} \frac{{}_{L^p}(\nabla\cdot\mathbf{v},q)_{L^{p'}}}{|\mathbf{v}|_{\mathbf{W}^{1,p}(\Omega)}\|q\|_{L^{p'}(\Omega)}} &= \sup_{f\in L_0^p(\Omega)} \frac{{}_{L^p}(f,q)_{L^{p'}}}{|\mathbf{v}(f)|_{\mathbf{W}^{1,p}(\Omega)}\|q\|_{L^{p'}(\Omega)}}\\ \ge C_{BA}(p,\Gamma,\Omega)^{-1} \sup_{f\in L_0^p(\Omega)} \left(\frac{{}_{L^p}(f,q)_{L^{p'}}}{\|f\|_{L^p(\Omega)}}\right)&\frac{1}{\|q\|_{L^{p'}(\Omega)}} = C_{BA}(p,\Gamma,\Omega)^{-1}.\end{aligned}$$

Taking the inf over all possible q entails the assertion.

A.2 Results in the Hilbertian setting

Consider the constants in the inequalities (46), (47), and (48) for $p = 2$. In the remainder of this section, we show

$$\beta(2,\Gamma,\Omega)^{-1} = C_{BA}(2,\Gamma,\Omega) = C_{BA}(2,\Gamma,\Omega); \tag{49}$$

see also the related works [4, 15, 26]. In this section, $H^1(\Omega)$ denotes $W^{1,2}(\Omega)$ with the obvious extensions to the vectorial case and the subspace of functions with zero trace.

Cosserat's operator. Consider the following Stokes' eigenvalue problem: find $\mathbf{u}$ in $\mathbf{H}_0^1(\Omega)$, p in $L_0^2(\Omega)$, and λ in $\mathbb{R}$ such that

$$\begin{cases} -\boldsymbol{\Delta}\mathbf{u} + \nabla p = 0 & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = \lambda p & \text{in } \Omega. \end{cases} \tag{50}$$

Multiplying the first equation by λ and taking the gradient of the second equation give

$$\begin{cases} -\lambda\boldsymbol{\Delta}\mathbf{u} + \lambda\nabla p = 0 & \text{in } \Omega \\ \nabla\,\nabla\cdot\mathbf{u} = \lambda\nabla p & \text{in } \Omega. \end{cases}$$

We deduce

$$\nabla\,\nabla\cdot\mathbf{u} = \lambda\boldsymbol{\Delta}\mathbf{u}.$$

Since $\boldsymbol{\Delta}$ is a bijection between $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}^{-1}(\Omega)$, we deduce

$$\boldsymbol{\Delta}^{-1}\nabla\,\nabla\cdot\mathbf{u} = \lambda\mathbf{u}.$$

Taking the divergence on both sides and replacing $\nabla\cdot\mathbf{u}$ by q in $L_0^2(\Omega)$, we get the equivalent problem $\nabla\cdot\boldsymbol{\Delta}^{-1}\nabla q = \lambda q$. Consider the Cosserat's operators (named after the Cosserat brothers [14])

$$\mathcal{S} := \boldsymbol{\Delta}^{-1}\nabla\,\nabla\cdot : \mathbf{H}_0^1(\Omega) \to \mathbf{H}_0^1(\Omega), \qquad \mathcal{S}^* := \nabla\cdot\boldsymbol{\Delta}^{-1}\nabla : L_0^2(\Omega) \to L_0^2(\Omega). \tag{51}$$

Every eigenvalue λ in (50) is also an eigenvalue of the Cosserat's operators, in the sense that

$$\mathcal{S}\mathbf{u} = \lambda\mathbf{u}, \qquad \mathcal{S}^* q = \lambda q.$$

Introduce

$$\sigma(\mathcal{S},2,\Gamma,\Omega) = \sigma(\mathcal{S}^*,2,\Gamma,\Omega) = \text{minimum eigenvalue of } \mathcal{S}\ (\mathcal{S}^*). \tag{52}$$

Showing $\sigma(\mathcal{S},2,\Gamma,\Omega) = C_{BA}(2,\Gamma,\Omega)^{-2}$**.** We characterize $\sigma(\mathcal{S},2,\Gamma,\Omega)$ in (52) by taking the infimum over $\mathbf{v}$ in the $\mathbf{H}^1$ orthogonal to div-free functions of the Rayleigh quotient

$$\begin{aligned}\sigma(\mathcal{S},2,\Gamma,\Omega) &= \inf_{\mathbf{v}} \frac{(\nabla \mathcal{S}\mathbf{v}, \nabla \mathbf{v})_{0,\Omega}}{|\mathbf{v}|^2_{1,\Omega}} \overset{(51)}{=} \inf_{\mathbf{v}} \frac{(\nabla \mathbf{\Delta}^{-1}\nabla\, \nabla\cdot \mathbf{v}, \nabla \mathbf{v})_{0,\Omega}}{|\mathbf{v}|^2_{1,\Omega}} \overset{\text{(IBP)}}{=} \inf_{\mathbf{v}} \frac{(\mathbf{\Delta}\mathbf{\Delta}^{-1}\nabla\, \nabla\cdot \mathbf{v}, \mathbf{v})_{0,\Omega}}{|\mathbf{v}|^2_{1,\Omega}} \\ &= \inf_{\mathbf{v}} \frac{(\nabla\, \nabla\cdot \mathbf{v}, \mathbf{v})_{0,\Omega}}{|\mathbf{v}|^2_{1,\Omega}} \overset{\text{(IBP)}}{=} \inf_{\mathbf{v}} \frac{(\nabla\cdot \mathbf{v}, \nabla\cdot \mathbf{v})_{0,\Omega}}{|\mathbf{v}|^2_{1,\Omega}} = \inf_{\mathbf{v}} \frac{\|\nabla\cdot \mathbf{v}\|^2_{0,\Omega}}{|\mathbf{v}|^2_{1,\Omega}} = C_{BA}(2,\Gamma,\Omega)^{-2}.\end{aligned}$$

A remark. Let q be in $L^2_0(\Omega)$ and $\boldsymbol{\zeta}$ be the solution to

$$\begin{cases} -\mathbf{\Delta}\boldsymbol{\zeta} = \nabla q & \text{in } \Omega \\ \boldsymbol{\zeta} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

In other terms, $\boldsymbol{\zeta}$ is equal to $-\mathbf{\Delta}^{-1}\nabla q$. Using a standard duality argument, we deduce

$$\begin{aligned}\|\nabla q\|_{-1,\Omega} &= \sup_{\mathbf{\Psi}\in \mathbf{H}^1_0(\Omega)} \frac{{}_{-1}\langle \nabla q, \mathbf{\Psi}\rangle_1}{|\mathbf{\Psi}|_{1,\Omega}} = \sup_{\mathbf{\Psi}\in \mathbf{H}^1_0(\Omega)} \frac{{}_{-1}\langle \mathbf{\Delta}\boldsymbol{\zeta}, \mathbf{\Psi}\rangle_1}{|\mathbf{\Psi}|_{1,\Omega}} \\ &= \sup_{\mathbf{\Psi}\in \mathbf{H}^1_0(\Omega)} \frac{(\nabla\boldsymbol{\zeta}, \nabla\mathbf{\Psi})_{0,\Omega}}{|\mathbf{\Psi}|_{1,\Omega}} = \|\nabla\boldsymbol{\zeta}\|_{0,\Omega} = ({}_{-1}\langle -\mathbf{\Delta}\boldsymbol{\zeta}, \boldsymbol{\zeta}\rangle_1)^{\frac{1}{2}} = ({}_{-1}\langle \nabla q, -\mathbf{\Delta}^{-1}\nabla q\rangle_1)^{\frac{1}{2}}.\end{aligned} \tag{53}$$

Showing $\sigma(\mathcal{S},2,\Gamma,\Omega) = C_{BA}(2,\Gamma,\Omega)^{-2}$**.** Observe

$$(\mathcal{S}^* q, q)_{0,\Omega} = (\nabla\cdot \mathbf{\Delta}^{-1}\nabla q, q)_{0,\Omega} =_{-1} \langle \nabla q, -\mathbf{\Delta}^{-1}\nabla q\rangle_1 \overset{(53)}{=} \|\nabla q\|^2_{-1,\Omega}.$$

We deduce

$$\sigma(\mathcal{S},2,\Gamma,\Omega) = \sigma(\mathcal{S}^*,2,\Gamma,\Omega) = \inf_{q\in L^2_0(\Omega)} \frac{(\mathcal{S}^* q, q)_{0,\Omega}}{\|q\|^2_{0,\Omega}} = \inf_{q\in L^2_0(\Omega)} \frac{\|\nabla q\|^2_{-1,\Omega}}{\|q\|^2_{0,\Omega}} = C_{BA}(2,\Gamma,\Omega)^{-2}.$$

Summarizing. Combining the above identities, we find (49); more precisely

$$\beta(2,\Gamma,\Omega)^{-1} = C_{BA}(2,\Gamma,\Omega) = C_{BA}(2,\Gamma,\Omega) = \sigma(\mathcal{S},2,\Gamma,\Omega)^{-\frac{1}{2}}.$$

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