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A finite element method based on weighted interior penalties for heterogeneous incompressible flows

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A finite element method based on weighted interior penalties for heterogeneous incompressible flows. *

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Abstract

We propose a finite element scheme for the approximation of multidomain heterogeneous problems arising in the general framework of linear incompressible flows (e.g. Stokes' and Darcy's equations). We exploit stabilized mixed finite elements together with Nitsche type matching conditions that automatically adapt to the coupling of different subproblem combinations. Optimal error estimates are derived for the coupled problem. Finally, we propose and analyze an iterative splitting strategy for the approximation of the multidomain solution by means of a sequence of independent and local subproblems. Thanks to the introduction of a suitable relaxation strategy, the iterative method turns out to be convergent for any possible coupling between subproblems.

1 Introduction

We consider an incompressible flow problem, which consists on finding for a.e. x in a regular domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a velocity vector field $\mathbf{u}(x)$ and a scalar

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pressure field $p(x)$ such that,

$$\begin{cases} \eta \mathbf{u} + \nabla \cdot \big(\mathbf{I} p - \nu (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \big) = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{1}$$

being ν and η nonnegative coefficients satisfying $\nu + \eta \geq \mu > 0$ a.e on Ω, $\mathbf{f}$ a vector valued forcing term and $\mathbf{I}$ the identity matrix.

The motivation of the present study arises from the following observation. Several studies have addressed the stability of the linear models for incompressible flows when the viscosity is vanishing. In particular, Payne and Straughan [22] have proved that the solution of the generalized Stokes' equations (u_S), also known as Brinkman's equations, converges to the solution of the Darcy's equation (u_D) in the L^2 norm when the viscosity parameter ν tends to zero, i.e., $\|u_S - u_D\|_{L^2(\Omega)} \leq C\sqrt{\nu}$. This fact highlights the smooth dependence of the physical problem on the viscosity parameter, which so far does not correspond to a general unified scheme at the level of numerical approximation for incompressible flow problems with heterogeneous (possibly vanishing) viscosity coefficients.

A difficulty in the development of a unified discretization framework for the coupling of viscous and inviscid subproblems is due to the treatment of interface conditions. In general, matching conditions can be split into *natural* conditions, which typically represent the equilibrium of forces at the interface and *essential* conditions, which account for the continuity of velocities or fluxes. The treatment of natural conditions is straightforward in the framework of variational formulations and very often these conditions apply to the coupling of both homogeneous and heterogeneous problems. Conversely, for essential conditions there is no smooth transition from the homogeneous to the heterogeneous cases. For example, to match Stokes' flows we require $\mathbf{u}$ to be continuous across the interface, while in the heterogeneous coupling of Darcy's and Stokes' models we only need the continuity of the normal velocity, namely $\mathbf{u}\cdot\mathbf{n}$. At the discrete level, this behavior can be easily reproduced by means of the application of penalty techniques to interface conditions of essential type. In particular, our coupling is based on matching conditions due to Nitsche (see [21]), originally introduced as a means to weakly impose boundary conditions and recently extended to a domain decomposition framework in [1, 26, 11]. By means of this approach we develop a finite element scheme that automatically adapts to the coupling of viscous and inviscid models (such as Stokes', Brinkman's and Darcy's equations) by simply defining on each subregion the physical parameters corresponding to each problem. This scheme is also particularly effective for the treatment of tangential velocities. According to [17], this is an indicator of the robustness of the method. For instance, interface conditions of practical interest, as the ones proposed by Beavers, Joseph and Saffman (see [24]) for the coupling of free flows with porous media can naturally be embedded into the scheme.

In the perspective to address practical applications, we also propose and analyze an iterative splitting method to approximate the global solution of the multidomain problem by means of a sequence of *independent* and *local* subproblems. Thanks to the introduction of a suitable relaxation strategy, the iterative method results to be convergent for any possible coupling between subproblems. The multidomain problem can be thus approximated exploiting a multiprocessor architecture, with straightforward advantages on computational time and memory storage. This technique clearly facilitates the treatment of problems with several domains and different couplings, which typically arise in realistic applications involving the flow through heterogeneous media.

From the point of view of numerical approximation, the design of finite elements that are robust and optimally converging for both viscous and inviscid problems is a nontrivial task. For instance, the discretization of viscous problems requires the satisfaction of the *inf-sup* condition for the velocity and pressure spaces, while for inviscid problems some control of the divergence of velocities is also necessary. This difficulty has been recently addressed in [20], where a new non conforming element with 9 degrees of freedom is proposed. A more practical way for the numerical discretization is to use specific scheme for the viscous case and another one for the inviscid case (see i.e. [19, 23, 3]). This allows the use of standard packages for each subproblem and efficient solving using domain decomposition algorithms. However, from an implementation point of view this may be impractical, because functions from different finite element spaces have to be coupled over the interface and different linear algebra solvers may be needed for the different subproblems. For these reasons, we opt for the construction of a unified family of finite elements for viscous and inviscid problems that is simpler than the one of [20]. To achieve this task, we privilege the satisfaction of the H_{div} stability with respect to the *inf-sup* condition. We consider H^1-conformal finite elements of order $r > 0$ for the velocity approximation and totally discontinuous elements of order $r-1$ for the pressure, such that the pressure space contains the divergence of the discrete velocities. This scheme does not satisfy the *inf-sup* condition and a suitable stabilization term acting on the pressure must be introduced. To this purpose, we apply the edge stabilization technique proposed in [10] with the extension to arbitrary polynomial order studied in [8]. Furthermore, the *inf-sup* stability for a general multidomain decomposition is achieved exploiting the fundamental properties of the so called connectivity matrix, analyzed in [2]. Merging the aforementioned results, we prove that our numerical scheme leads to optimal a priori error estimates for both viscous and inviscid problems in the natural energy norms. This allows us to cover a large range of parameter values using the same numerical method.

2 A multidomain formulation

To set up our multidomain problem, we consider a partition of Ω in N nonoverlapping subregions such that $\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i$ and we denote by $\mathbf{n}_i$ the outer unit normal of Ω_i. Let $\mathcal{N}_i$ be the set of indexes such that $\mathcal{N}_i = \{j = 1, \ldots, N : \; j \neq$

$i,\ \partial\Omega_i \cap \partial\Omega_j \neq \emptyset\}$. Then, we define $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ for all $i = 1, \dots, N$ and $j \in \mathcal{N}_i$. We assign to each interface $\Gamma_{ij} \equiv \Gamma_{ji}$ a unit normal vector $\mathbf{n}_\Gamma$ that may point from Ω_i to Ω_j or vice versa. For instance, we assume $\mathbf{n}_\Gamma = \mathbf{n}_i$ on Γ_{ij} if $i < j$ and $\mathbf{n}_\Gamma = \mathbf{n}_j$ in the opposite case. Nonetheless, the arbitrariness of $\mathbf{n}_\Gamma$ will not influence the set up of the method. On the boundary of Ω, we consider the outward unit normal vector $\mathbf{n}$. To each subregion we associate suitable spaces $\mathbf{V}_i$ and Q_i where we look for a (weak) local solution $(\mathbf{u}_i, p_i)$ on Ω_i. Assuming that $\mathbf{v}_i, q_i$ can be regarded as functions on the whole Ω by means of extension to zero, we introduce the following global spaces on Ω, $\mathbf{V} := \bigoplus_{i=1}^N \mathbf{V}_i$, $Q := \bigoplus_{i=1}^N Q_i$ and $\mathbf{W} := \mathbf{V} \times Q$. The additional constraint $Q \subset L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$ may be necessary depending on the boundary conditions. Let us maintain $\mathbf{V}_i$ and Q_i temporarily unspecified and proceed formally. Their precise definition will be given at the beginning of section 4.

Given $\mathbf{v} \in \mathbf{V}$, we define the neighboring values of $\mathbf{v}$ with respect to Γ_{ij} as follows,

$$\mathbf{v}^{\mp}(\mathbf{x}) = \lim_{\delta \to 0^+} \mathbf{v}(\mathbf{x} \mp \delta \mathbf{n}_\Gamma), \text{ a.e. on } \Gamma_{ij}.$$

The jump, $[\![\cdot]\!]$, and the average, $\{\cdot\}$, of $\mathbf{v}$ across Γ_{ij} are obtained combining these values. More precisely, $[\![\mathbf{v}]\!] := \mathbf{v}^- - \mathbf{v}^+$ and owing to our definition of $\mathbf{n}_\Gamma$ we obtain $[\![\mathbf{v}]\!] := \mathbf{v}_i - \mathbf{v}_j$, a.e. on Γ_{ij} if $i < j$ and $[\![\mathbf{v}]\!] := \mathbf{v}_j - \mathbf{v}_i$ if $j > i$, while for the averages we set,

$$\{\mathbf{v}\}_w := w_i \mathbf{v}_i + w_j \mathbf{v}_j,\ \{\mathbf{v}\}^w := w_j \mathbf{v}_i + w_i \mathbf{v}_j,\ \text{with } w_i + w_j = 1 \text{ a.e. on } \Gamma_{ij}.$$

We say that the averages $\{\cdot\}_w$ and $\{\cdot\}^w$ are conjugate, because they satisfy the following identity, $[\![ab]\!] = \{a\}_w [\![b]\!] + [\![a]\!] \{b\}^w$ for any double valued quantities $a = (a^-, a^+), b = (b^-, b^+)$. We also apply similar definitions for any other quantity depending on $(\mathbf{v}, q)$.

Owing to the identity $\nabla \cdot \big(\nabla \mathbf{u}^T\big) = \nabla\big(\nabla \cdot \mathbf{u}\big) = 0$, we consider for simplicity an alternative formulation of (1), where the first equation is replaced by $\eta\mathbf{u} + \nabla \cdot \sigma(\mathbf{u}, p) = \mathbf{f}$ given $\sigma(\mathbf{u}, p) := \mathbf{I}p - \nu\epsilon(\mathbf{u})$ and $\epsilon(\mathbf{u}) := \nabla\mathbf{u}$. Then, denoting with ν_i, $\eta_i \in L^\infty(\Omega_i)$ the viscosity and the permeability in Ω_i respectively, our multidomain problem reads as follows: given $\mathbf{f}$, $\mathbf{F}$, $\mathbf{U}$ sufficiently regular data, find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that,

$$\eta_i \mathbf{u}_i + \nabla \cdot \sigma_i(\mathbf{u}, p) = \mathbf{f}, \quad \nabla \cdot \mathbf{u}_i = 0, \text{ in } \Omega_i, \text{ for } i = 1, \dots, N,$$

together with the following boundary and interface conditions on $\partial\Omega_i$, which assume different forms in the two cases $\nu_i > 0$ or $\nu_i = 0$, being $\nu_j > 0$. Precisely, the *essential* conditions are,

$$\begin{array}{lll} \text{case } \nu_i > 0, & \text{case } \nu_i = 0 & \\ (\mathbf{u}_i - \mathbf{U}) = 0, & (\mathbf{u}_i - \mathbf{U}) \cdot \mathbf{n} = 0, & \text{on } \partial\Omega, \\ [\![\mathbf{u}]\!] = 0, & [\![\mathbf{u}]\!] \cdot \mathbf{n}_\Gamma = 0, & \text{on } \Gamma_{ij}, \end{array}$$

while possible *natural* conditions read as follows,

$$\begin{array}{llll} \text{case } \nu_i > 0, & \text{case } \nu_i = 0 & & \\ \sigma_i(\mathbf{u},p)\mathbf{n} + \mathbf{F} = 0, & \mathbf{n}^T\sigma_i(\mathbf{u},p)\mathbf{n} + \mathbf{F}\cdot\mathbf{n} = 0, & \mathbf{t}^T\sigma_i(\mathbf{u},p)\mathbf{n} = 0, & \text{on } \partial\Omega, \\ [\![\sigma(\mathbf{u},p)]\!]\mathbf{n}_\Gamma = 0, & \mathbf{n}_\Gamma^T[\![\sigma(\mathbf{u},p)]\!]\mathbf{n}_\Gamma = 0, & \kappa_\Gamma \mathbf{u}_j\cdot\mathbf{t}_\Gamma = \mathbf{t}_\Gamma^T[\![\sigma(\mathbf{u},p)]\!]\mathbf{n}_\Gamma, & \text{on } \Gamma_{ij}, \end{array}$$

where the second column of the case $\nu_i = 0$ should be dropped if also $\nu_j = 0$. We will take into account of that in the following generalized coupling conditions. In the case of heterogeneous problems, $\nu_i = 0$ and $\nu_j > 0$, the natural conditions in the normal direction to the interface Γ_{ij} correspond to the continuity of normal stresses, while for the tangential direction we have adopted the so called Beavers-Joseph-Saffman law, being $\kappa_\Gamma > 0$ a given friction coefficient. In order to develop a unified treatment of such couplings, we introduce a generalized set of conditions that represent either the viscous or the viscous/inviscid couplings, depending on the value of the coefficients ν_i and ν_j solely. To this purpose, we propose the following,

$$\begin{array}{lll} \text{essential} & \chi_{\partial\Omega}(\nu_i)(\mathbf{u}_i - \mathbf{U}) = 0,\ (\mathbf{u}_i - \mathbf{U})\cdot\mathbf{n} = 0, & \text{on } \partial\Omega, \\ & \chi_\Gamma(\nu_i,\nu_j)[\![\mathbf{u}]\!] = 0,\ [\![\mathbf{u}]\!]\cdot\mathbf{n}_\Gamma = 0, & \text{on } \Gamma_{ij}, \end{array}$$

$$\begin{array}{lll} \text{natural} & \sigma_i(\mathbf{u},p)\mathbf{n} + \mathbf{F} = 0, & \text{on } \partial\Omega, \\ & [\![\sigma(\mathbf{u},p)]\!]\mathbf{n}_\Gamma = \kappa_\Gamma\varphi_\Gamma(\nu_i,\nu_j)\mathbf{t}_\Gamma\{\mathbf{u}\cdot\mathbf{t}_\Gamma\}^w, & \text{on } \Gamma_{ij}. \end{array}$$

In case ν_i vanishes in a subdomain, the averaging weights w_i, w_j and the scaling functions $\chi_{\partial\Omega}(\nu_i), \chi_\Gamma(\nu_i,\nu_j), \varphi_\Gamma(\nu_i,\nu_j)$ must be chosen so as to guarantee that the matching conditions recover the physically correct behavior, characterized by the following requirements,

$$\begin{array}{ll} \chi_{\partial\Omega}(\nu_i) > 0 \text{ if } \nu_i > 0, & \chi_{\partial\Omega}(\nu_i) = 0 \text{ if } \nu_i = 0, \\ \chi_\Gamma(\nu_i,\nu_j) > 0 \text{ if } \nu_i\cdot\nu_j > 0, & \chi_\Gamma(\nu_i,\nu_j) = 0 \text{ if } \nu_i\cdot\nu_j = 0, \\ \varphi_\Gamma(\nu_i,\nu_j) = 0 \text{ if } \nu_i = \nu_j, & \varphi_\Gamma(\nu_i,\nu_j) = 1 \text{ if } \nu_i\cdot\nu_j = 0. \end{array}$$

Following [11, 27], where a similar case is studied for advection-diffusion equations, the weights and scaling functions are defined as follows,

$$\begin{array}{ll} \chi_{\partial\Omega}(\nu_i) := \nu_i, & \text{on } \partial\Omega, \\ w_i := \dfrac{\nu_j}{\nu_i + \nu_j}, \quad \chi_\Gamma(\nu_i,\nu_j) := \{\nu\}_w, \quad \varphi_\Gamma(\nu_i,\nu_j) := \dfrac{|[\![\nu]\!]|}{2\{\nu\}}, & \text{on } \Gamma_{ij}, \end{array}$$

with $w_i = 1/2$, $\chi_\Gamma(\nu_i,\nu_j) = 0$ and $\varphi_\Gamma(\nu_i,\nu_j) = 0$ in the homogeneous vanishing viscosity case $\nu_i, \nu_j \to 0$. Finally, we notice that, to our knowledge, a precise characterization of what interface condition is convenient to couple Brinkman's equations for heterogeneous materials is missing. From the mathematical viewpoint, continuity of velocities and normal stresses is admissible. However, this choice doesn't seem to be realistic when $\nu_i/\nu_j \gg 1$ or vice versa. In this case, the condition arising from the application of the Beavers-Joseph-Saffman law seems to be more effective, and we will apply it to our method.

3 A finite element method

Let us now set up the discrete variational formulation of problem (1). Although the computational method can be applied for $d = 1, 2, 3$ space dimensions, without loss of generality we restrict to the case $d = 2$ for the notation and some technical aspects of the analysis. We assume that Ω and Ω_i are convex polygonal domains and we consider a partitioning of each subdomain Ω_i into a conforming triangulation $\mathcal{T}_{h,i}$ of affine simplices K, but the local triangulations do not need to be conforming on Ω. Let $\mathcal{T}_{h,i}$ be shape regular and quasi-uniform, and h_i be the local mesh characteristic parameter, while $h := \max_{i=1,N} h_i$, with the assumption $h \ll 1$. More precisely, being $h_K = \mathrm{diam}(K)$ and $h_E = \mathrm{diam}(E)$ with $E \subset \partial K$ for any simplex K and any edge E of K in $\mathcal{T}_{h,i}$, there exists $0 < \sigma_1 < \infty$ such that $h_E \leq h_K \leq \sigma_1 h_E$ and there exist $\sigma_2 > 0$ such that $h_i \leq h \leq \sigma_2 h_i$. We denote with $\mathcal{G}_{h,i}$ and with $\mathcal{B}_{h,i}$ the trace meshes at the interface and at the boundary of the subdomains. We also denote with $\mathcal{F}_{h,i}$ the set of all interior edges of $\mathcal{T}_{h,i}$ and define the intersection of the trace meshes on Γ_{ij} that is denoted with $\mathcal{G}_{h,ij}$. Precisely, we have,

$$
\begin{aligned}
\mathcal{B}_{h,i} &:= \{E \neq \emptyset : E = \partial K \cap \partial\Omega,\ \forall K \in \mathcal{T}_{h,i}\},\ \mathcal{B}_h := \cup_{i=1}^N \mathcal{B}_{h,i},\\
\mathcal{F}_{h,i} &:= \{E \neq \emptyset : E = \partial K \setminus \partial\Omega_i,\ \forall K \in \mathcal{T}_{h,i}\},\ \mathcal{F}_h := \cup_{i=1}^N \mathcal{F}_{h,i},\\
\mathcal{G}_{h,ij} &:= \{E \neq \emptyset : E = \partial K_i \cap \partial K_j,\ \forall K_i \in \mathcal{T}_{h,i},\ \forall K_j \in \mathcal{T}_{h,j}\}.
\end{aligned}
$$

We assume that $\mathcal{G}_{h,ij}$ is nondegenerate, namely there exists $0 < \sigma_3 < \infty$ such that for all $E \in \mathcal{G}_{h,ij}$ we have $\big(\mathrm{diam}(K_i) + \mathrm{diam}(K_j)\big) \leq \sigma_3 \mathrm{diam}(E)$ being $K_i \in \mathcal{T}_{h,i}$ and $K_j \in \mathcal{T}_{h,j}$ such that $\partial K_i \cap \partial K_j = E$. For any $E \in \mathcal{F}_{h,i}$ with $E = \partial K_r \cap \partial K_s$ with $K_r \neq K_s \in \mathcal{T}_{h,i}$ we define $\mathbf{n}_E$ as the outer unit normal vector of K_r if $r < s$ and of K_s otherwise. Then, the definition of jumps and averages on $E \in \mathcal{F}_{h,i}$ is extended straightforwardly. Finally, to simplify the presentation of the results, we apply the following abridged notation,

$$
\int_\Gamma \mathbf{v} \cdot \mathbf{n}_\Gamma := \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \int_{\Gamma_{ij}} \mathbf{v} \cdot \mathbf{n}_\Gamma,
$$

$$
\int_{\mathcal{B}_{h,i}} \mathbf{v}_h \cdot \mathbf{n} := \sum_{E \in \mathcal{B}_{h,i}} \int_E \mathbf{v}_h \cdot \mathbf{n},\ \int_{\mathcal{F}_{h,i}} \mathbf{v}_h \cdot \mathbf{n}_E := \sum_{E \in \mathcal{F}_{h,i}} \int_E \mathbf{v}_h \cdot \mathbf{n}_E,
$$

$$
\int_{\mathcal{G}_{h,ij}} \mathbf{v}_h \cdot \mathbf{n}_\Gamma := \sum_{E \in \mathcal{G}_{h,ij}} \int_E \mathbf{v}_h \cdot \mathbf{n}_\Gamma,\ \int_{\mathcal{G}_h} \mathbf{v}_h \cdot \mathbf{n}_\Gamma := \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \int_{\mathcal{G}_{h,ij}} \mathbf{v}_h \cdot \mathbf{n}_\Gamma.
$$

Extending the work of Burman and Hansbo, see [10], we define the approximation space for the velocity on Ω_i as follows,

$$
\begin{aligned}
V_{h,i} &:= \{v_h \in H^1(\Omega_i)\ :\ v_h|_K \in \mathbb{P}^{r_i}(K),\ \forall K \in \mathcal{T}_{h,i}\}, \quad \mathbf{V}_{h,i} = [V_{h,i}]^d,\\
Q_{h,i} &:= \{q_h \in L^2(\Omega_i)\ :\ q_h|_K \in \mathbb{P}^{r_i - 1}(K),\ \forall K \in \mathcal{T}_{h,i}\}, \quad \text{with } r_i > 0.
\end{aligned}
$$

On each subdomain we may consider a different polynomial order, r_i for the finite element approximation, and we denote $r := \min_{i=1,N} r_i$. We point out that different finite elements can be combined with different mesh sizes on each subdomain. In practice, it is interesting to increase h_i when the local polynomial order is increased. We will discuss later on how this flexibility may be effectively exploited for the approximation of heterogeneous problems (see Remark 3.1). Since the pressure space consists of discontinuous functions we have $\nabla \cdot \mathbf{V}_{h,i} \subset Q_{h,i}$, a property which guarantees L^2-stability of the divergence of the velocities in the Darcy case. We also introduce $V_h := \bigoplus_{i=1}^N V_{h,i}$ and $Q_h := \bigoplus_{i=1}^N Q_{h,i} \cap Q$. In order to properly account for different polynomial orders in our finite element method, we introduce the quantity r_E for any edge $E \in \mathcal{T}_{h,i}$, $i = 1, \ldots, N$. Precisely, we set $r_E := \max[r_i, r_j]$ if $E \in \mathcal{G}_{h,ij}$ and $r_E := r_i$ if $E \in \mathcal{F}_{h,i}$ or $E \in \mathcal{B}_{h,i}$. We also denote $r_K := r_i$ for any $K \in \mathcal{T}_{h,i}$.

In order to reduce the technical aspects in the analysis of the method, we consider homogeneous Dirichlet data on the entire boundary. Then, for any $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$ and $p_h, q_h \in Q_h$ we define the following bilinear forms,

$$
\begin{aligned}
a(\mathbf{u}_h, \mathbf{v}_h) := &\int_\Omega \Big(\nu\epsilon(\mathbf{u}_h) : \epsilon(\mathbf{v}_h) + \eta \mathbf{u}_h \cdot \mathbf{v}_h\Big) - \int_{\partial\Omega} \Big(\nu\epsilon(\mathbf{u}_h)\mathbf{n} \cdot \mathbf{v}_h + \varsigma\nu\epsilon(\mathbf{v}_h)\mathbf{n} \cdot \mathbf{u}_h\Big) \\
&+ \int_{\mathcal{B}_h} \gamma_u \nu \big(\tfrac{h_E}{r_E^2}\big)^{-1} \mathbf{u}_h \cdot \mathbf{v}_h, && (2)
\end{aligned}
$$

$$
b(p_h, \mathbf{v}_h) := -\int_\Omega p_h \nabla \cdot \mathbf{v}_h + \int_{\partial\Omega} p_h \mathbf{v}_h \cdot \mathbf{n}, \qquad (3)
$$

$$
\begin{aligned}
c(\mathbf{u}_h, \mathbf{v}_h) := &\int_{\mathcal{G}_h} \gamma_u \{\nu\}_w \big(\tfrac{h_E}{r_E^2}\big)^{-1} [\![\mathbf{u}_h]\!] \cdot [\![\mathbf{v}_h]\!] + \int_\Gamma \kappa_\Gamma \tfrac{|[\![\nu]\!]|}{2\{\nu\}} \{\mathbf{u}_h \cdot \mathbf{t}_\Gamma\}^w \{\mathbf{v}_h \cdot \mathbf{t}_\Gamma\}^w && (4) \\
&- \int_\Gamma \big(\{\nu\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_w \cdot [\![\mathbf{v}_h]\!] + \varsigma\{\nu\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\}_w \cdot [\![\mathbf{u}_h]\!]\big),
\end{aligned}
$$

$$
d(p_h, \mathbf{v}_h) := \int_\Gamma \{p_h\}_w [\![\mathbf{v}_h \cdot \mathbf{n}_\Gamma]\!], \qquad (5)
$$

$$
j_u(\mathbf{u}_h, \mathbf{v}_h) := \int_{\mathcal{G}_h} \gamma_u \big(\tfrac{h_E}{r_E^2}\big)^{-1} [\![\mathbf{u}_h \cdot \mathbf{n}_\Gamma]\!][\![\mathbf{v}_h \cdot \mathbf{n}_\Gamma]\!] + \int_{\mathcal{B}_h} \gamma_u \big(\tfrac{h_E}{r_E^2}\big)^{-1} (\mathbf{u}_h \cdot \mathbf{n})(\mathbf{v}_h \cdot \mathbf{n}), \qquad (6)
$$

$$
j_p(p_h, q_h) := \int_{\mathcal{F}_h} \gamma_p \big(\tfrac{h_E}{r_E^2}\big) [\![p_h]\!][\![q_h]\!], \qquad (7)
$$

where γ_u and γ_p are constant parameters that should be suitably set to ensure the stability of the method. We notice that the definition of γ_u and γ_p is independent of ν, h_E and r_E. However, for the ease of notation we introduce $\gamma_{u,i} := \gamma_u r_E^2$ and $\gamma_{p,i} := \gamma_p r_E^{-2}$

Furthermore, the parameter $\varsigma = \pm 1$ is a flag that allows us to switch between the symmetric ($\varsigma = 1$) and the skew-symmetric ($\varsigma = -1$) formulations. From this point forward, we will focus on the symmetric case. Then, we define,

$$\mathcal{A}(\mathbf{u}_h, \mathbf{v}_h) := a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{v}_h), \quad \text{and} \quad \mathcal{B}(p_h, \mathbf{v}_h) := b(p_h, \mathbf{v}_h) + d(p_h, \mathbf{v}_h).$$

and we denote the right hand side by $\mathcal{F}(\mathbf{v}_h) := \int_\Omega \mathbf{f} \cdot \mathbf{v}_h$. Then, the mixed formulation of the discrete problem reads as follows: given a sufficiently regular $\mathcal{F}(\cdot)$, find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that,

$$\begin{cases} \mathcal{A}(\mathbf{u}_h, \mathbf{v}_h) + j_u(\mathbf{u}_h, \mathbf{v}_h) + \mathcal{B}(p_h, \mathbf{v}_h) = \mathcal{F}(\mathbf{v}_h), \ \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \mathcal{B}(q_h, \mathbf{u}_h) - \varsigma \ j_p(p_h, q_h) = 0, \ \forall q_h \in Q_h. \end{cases} \tag{8}$$

Introducing the product space $\mathbf{W}_h := \mathbf{V}_h \times Q_h$, the right hand side $\mathcal{G}(\mathbf{v}_h, q_h) = (\mathcal{F}(\mathbf{v}_h), 0)$ and the bilinear form

$$\mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \mathcal{A}(\mathbf{u}_h, \mathbf{v}_h) + j_u(\mathbf{u}_h, \mathbf{v}_h) + \mathcal{B}(p_h, \mathbf{v}_h) - \mathcal{B}(q_h, \mathbf{u}_h) + j_p(p_h, q_h),$$

problem (8) is equivalent to the following: given a sufficiently regular $\mathcal{G}(\cdot)$, find $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ such that,

$$\mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \mathcal{G}(\mathbf{v}_h, q_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{W}_h. \tag{9}$$

Remark 3.1 The simplest case we may consider is $r = 1$ corresponding to $\mathbb{P}^1 - \mathbb{P}^0$ elements for velocities and pressures respectively, which has already been proposed in [10] for Stokes' and Darcy's equations. Such approximation may be effective for free flows in presence of complex geometries or obstacles, because the low polynomial order allows to reduce the mesh size. However, this stabilized finite element turns out to be inaccurate for the approximation of flow in porous media with high hydraulic resistance. Indeed, the high pressure gradients generated when $\eta \gg 1$ have to be approximated by sufficiently large jumps on the element's edges, but this is in contrast with the presence of the stabilization term, which penalizes them. As confirmed by numerical experiments, $\mathbb{P}^1 - \mathbb{P}^0$ elements are not able to respond to these contradictory requirements. This behavior is easily corrected switching to $r = 2$ for Darcy's equations. In this perspective, the possibility to couple different polynomial orders on different subregions turns out to be particularly effective.

Remark 3.2 Our method can be easily generalized to the case of Oseen's equations, exploiting the continuous interior penalty stabilization, proposed in [9], to cure the instability due to high local Reynolds numbers. This is achieved introducing into the bilinear form $a(\mathbf{u}_h, \mathbf{v}_h)$ a new term that penalizes the jump of velocity gradients over element faces. Although this generalization would not require to modify the present setting of the work, we omit it to limit the technical aspects of the analysis.

4 Analysis of the method

In this section we aim to analyze the stability and the convergence of method (8). First of all, for any $v_h \in Y_h^k(\Omega) = \{v_h \in L^2(\Omega) : v|_K \in \mathbb{P}^k(K),\ k \geq 0,\ \forall K \in \mathcal{T}_{h,i}, i = 1, \dots, N\}$ we introduce the following norms,

$$\|v_h\|^2_{\pm\frac{1}{2},h,\Gamma} := \sum_{E \in \mathcal{G}_h} h_E^{\mp 1} \|v_h\|^2_{0,E}, \quad \|v_h\|^2_{\pm\frac{1}{2},h,\Omega} := \sum_{E \in \mathcal{F}_h} h_E^{\mp 1} \|v_h\|^2_{0,E},$$

where $\|\cdot\|_{0,\Sigma}$ and $\|\cdot\|_{1,\Sigma}$ denote the standard norms in $L^2(\Sigma)$ and $H^1(\Sigma)$. These definitions can be straightforwardly extended to $\mathcal{B}_{h,i}$ and to vector valued functions. Then, we introduce suitable norms in $\mathbf{V}_h$ and $\mathbf{W}_h$ respectively,

$$\begin{aligned}
|||\mathbf{v}_h|||^2 :=& \|\eta^{\frac{1}{2}}\mathbf{v}_h\|^2_{0,\Omega} + \|\nu^{\frac{1}{2}}\nabla\mathbf{v}_h\|^2_{0,\Omega} + \|\nu^{\frac{1}{2}}\mathbf{v}_h\|^2_{+\frac{1}{2},h,\partial\Omega} + \|\mathbf{v}_h \cdot \mathbf{n}\|^2_{+\frac{1}{2},h,\partial\Omega} \\
&+ \|\big(\kappa_\Gamma \tfrac{|[\![\nu]\!]|}{2\{\nu\}}\big)^{\frac{1}{2}} \{\mathbf{v}_h\}^w \cdot \mathbf{t}_\Gamma\|^2_{0,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}} [\![\mathbf{v}_h]\!]\|^2_{+\frac{1}{2},h,\Gamma} + \|[\![\mathbf{v}_h]\!] \cdot \mathbf{n}_\Gamma\|^2_{+\frac{1}{2},h,\Gamma}, \\
|||(\mathbf{v}_h, q_h)|||^2 :=& |||\mathbf{v}_h|||^2 + \|\nabla \cdot \mathbf{v}_h\|^2_{0,\Omega} + \|q_h\|^2_{0,\Omega} + \|[\![q_h]\!]\|^2_{-\frac{1}{2},h,\Omega}.
\end{aligned}$$

Owing to the assumption $\nu_i + \eta_i \geq \mu > 0$ and exploiting Poincaré–Friedrichs inequalities (see [6]), we obtain that $|||\mathbf{v}_h|||^2 \geq C\|\mathbf{v}_h\|^2_{0,\Omega}$, where the constant C is bounded for any admissible value of ν_i, η_i. We will also make use of the following inverse inequalities (see [25]) that hold true for all $K \in \mathcal{T}_{h,i},\ i = 1, \dots, N$ and for all $v_h \in Y_h^k := \{v_h \in L^2(\Omega) : v_h|_K \in \mathbb{P}^k(K),\ \forall K \in \mathcal{T}_h\}$, provided that the mesh is shape regular,

$$\big(\tfrac{h_E}{k}\big)^{\frac{1}{2}} \|v_h\|_{0,E} \lesssim \|v_h\|_{0,K}, \quad \big(\tfrac{h_K}{k}\big) \|\nabla v_h\|_{0,K} \lesssim \|v_h\|_{0,K} \text{ with } k > 0. \tag{10}$$

These inverse inequalities justify the scaling with respect to h_E and r_E we have applied into (2)-(7). Here and in the sequel, the symbol $\lesssim$ denotes an inequality involving a positive constant C independent of the size of the mesh, h, and of the viscosity, ν. From now on, we incorporate into the constant C also the dependence on r_E and r_K. This notation will substantially simplify the technical aspects of the proofs, but some interesting details as the dependence of the estimates on the polynomial order, the number and the shape of the subregions will be inevitably lost.

Before proceeding, we provide suitable definitions for $\mathbf{V}_i$ and Q_i. Starting from Darcy's equations, namely the inviscid case, the natural solution space is $\mathbf{V}_i := \mathbf{H}_{div}(\Omega_i)$, $Q_i := H^1(\Omega_i)$. Setting $\nu_i = 0$, the bilinear forms (2)-(7) are well defined for any couple $(\mathbf{v}_i, q_i) \in \mathbf{V}_i \times Q_i$. Unfortunately, this is not the case for Stokes' problem with minimal regularity assumptions, for which we expect $\mathbf{u}_i \in [H^1(\Omega_i)]^2$, $p_i \in L^2(\Omega_i)$. This situation may be improved increasing the regularity of the domain (a convex polygon) and of the forcing terms. In order to ensure that the bilinear forms (2)-(7) make sense for the exact solution of the viscous problems, we require $\mathbf{V}_i := [H^{\frac{3}{2}+\epsilon}(\Omega_i)]^2$, $Q_i := H^{\frac{1}{2}+\epsilon}(\Omega_i)$, for any

$\epsilon > 0$. However, we observe that these additional regularity requirements may not be satisfied for heterogeneous problems in presence of non planar or multiple interfaces. For elliptic problems this drawback may be overcome by means of the approach proposed in [16], but the extension of this technique to the case of Stokes equations goes beyond the scope of this work.

The first step to analyze the method proposed here consists in observing that it is strongly consistent by construction and that the bilinear form $\mathcal{C}(\cdot,\cdot)$ is bounded and positive. These properties are made precise in the following lemmas.

Lemma 1 (Consistency) *Let* $(\mathbf{u}, p) \in \mathbf{W}$ *be the weak solution of* (1), *and let* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ *be the solution of* (9). *Then, we have,*

$$\mathcal{C}((\mathbf{u}-\mathbf{u}_h, p-p_h),(\mathbf{v}_h, q_h)) = 0, \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{W}_h. \tag{11}$$

Proof. First of all, we observe that (11) is equivalent to

$$\mathcal{C}((\mathbf{u}, p),(\mathbf{v}_h, q_h)) = \mathcal{G}(\mathbf{v}_h, q_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{W}_h. \tag{12}$$

By virtue of the regularity of $(\mathbf{u}, p)$, we observe that $j_p(p, q_h) = 0$ and since it is the solution in the weak sense of (1) we have $j_u(\mathbf{u}, \mathbf{v}_h) = 0$ and $b(\mathbf{u}, q_h) = 0$. As a result of that we obtain,

$$\mathcal{C}((\mathbf{u}, p),(\mathbf{v}_h, q_h)) = a(\mathbf{u}, \mathbf{v}_h) + b(p, \mathbf{v}_h) \\ + \int_{\partial\Omega} \sigma(\mathbf{u}, p)\mathbf{n}\cdot \mathbf{v}_h + \int_\Gamma \{\sigma(\mathbf{u}, p)\mathbf{n}_\Gamma\}_w \cdot [\![\mathbf{v}_h]\!] + \int_\Gamma \kappa_\Gamma \tfrac{|[\![\nu]\!]|}{2\{\nu\}} \{\mathbf{u}\cdot \mathbf{t}_\Gamma\}^w \{\mathbf{v}_h \cdot \mathbf{t}_\Gamma\}^w. \tag{13}$$

Furthermore, by means of Green's formula we obtain,

$$\begin{aligned} & a(\mathbf{u}, \mathbf{v}_h) + b(p, \mathbf{v}_h) + \int_{\partial\Omega} \sigma(\mathbf{u}, p)\mathbf{n}\cdot \mathbf{v}_h \\ &= \int_\Omega \big[\nu\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}_h) + \eta \mathbf{u}\cdot \mathbf{v}_h - p\nabla\cdot \mathbf{v}_h\big] + \int_{\partial\Omega} \sigma(\mathbf{u}, p)\mathbf{n}\cdot \mathbf{v}_h \\ &= \int_\Omega \mathbf{f}\cdot \mathbf{v}_h - \int_\Gamma [\![\sigma(\mathbf{u}, p)\mathbf{n}_\Gamma \cdot \mathbf{v}_h]\!]. \end{aligned} \tag{14}$$

Thanks to the algebraic identity $[\![ab]\!] = \{a\}_w[\![b]\!] + [\![a]\!]\{b\}^w$ and to the interface condition $[\![\sigma(\mathbf{u}, p)\mathbf{n}_\Gamma]\!] = \kappa_\Gamma \frac{|[\![\nu]\!]|}{2\{\nu\}} \mathbf{t}_\Gamma \{\mathbf{u}\cdot \mathbf{t}_\Gamma\}^w$ the last term of (14) is equivalent to,

$$\int_\Gamma [\![\sigma(\mathbf{u}, p)\mathbf{n}_\Gamma \cdot \mathbf{v}_h]\!] = \int_\Gamma \{\sigma(\mathbf{u}, p)\mathbf{n}_\Gamma\}_w \cdot [\![\mathbf{v}_h]\!] + \int_\Gamma \kappa_\Gamma \tfrac{|[\![\nu]\!]|}{2\{\nu\}} \{\mathbf{u}\cdot \mathbf{t}_\Gamma\}^w \{\mathbf{v}_h \cdot \mathbf{t}_\Gamma\}^w. \tag{15}$$

Finally, we substitute (14) and (15) into (13) and we obtain (12). □

Lemma 2 (Boundedness) *The bilinear form* $\mathcal{C}(\cdot,\cdot)$ *satisfies,*

$$\mathcal{C}((\mathbf{u}_h, p_h),(\mathbf{v}_h, q_h)) \lesssim |||\mathbf{u}_h, p_h|||\ |||\mathbf{v}_h, q_h|||, \ \forall\ (\mathbf{u}_h, p_h),\ (\mathbf{v}_h, q_h) \in \mathbf{W}_h. \tag{16}$$

Proof. Concerning the bilinear forms $a(\mathbf{u}_h, \mathbf{v}_h)$ and $c(\mathbf{u}_h, \mathbf{v}_h)$ we have the following estimates,

$$\int_\Omega \nu\epsilon(\mathbf{u}_h) : \epsilon(\mathbf{v}_h) + \eta\mathbf{u}_h\cdot\mathbf{v}_h + \int_\Gamma \kappa_\Gamma \frac{|[\![\nu]\!]|}{2\{\nu\}}\{\mathbf{u}_h\cdot\mathbf{t}_\Gamma\}^w\{\mathbf{v}_h\cdot\mathbf{t}_\Gamma\}^w + \int_{\mathcal{G}_h} \gamma_{u,i}\{\nu\}_w h_E^{-1}[\![\mathbf{u}_h]\!]\cdot[\![\mathbf{v}_h]\!]$$
$$+\int_{\mathcal{G}_h} \gamma_{u,i} h_E^{-1}[\![\mathbf{u}_h\cdot\mathbf{n}_\Gamma]\!][\![\mathbf{v}_h\cdot\mathbf{n}_\Gamma]\!] + \int_{\mathcal{B}_h} \gamma_{u,i} h_E^{-1}(\mathbf{u}_h\cdot\mathbf{n})(\mathbf{v}_h\cdot\mathbf{n}) + \int_{\mathcal{B}_h} \gamma_{u,i} h_E^{-1}(\mathbf{u}_h\cdot\mathbf{n})(\mathbf{v}_h\cdot\mathbf{n})$$
$$\leq |||\mathbf{u}_h||| \, |||\mathbf{v}_h|||.$$

By virtue of Cauchy-Schwarz inequality and (10) we obtain,

$$\int_\Gamma \{\nu\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_w \cdot [\![\mathbf{v}_h]\!] \lesssim \|\nu^{\frac{1}{2}}\nabla\mathbf{u}_h\|_{0,\Omega}\|\{\nu\}_w^{\frac{1}{2}}[\![\mathbf{v}_h]\!]\|_{+\frac{1}{2},h,\Gamma}.$$

The corresponding symmetric boundary terms can be treated analogously. The bilinear forms $b(p_h, \mathbf{v}_h)$ and $d(p_h, \mathbf{v}_h)$ are treated similarly,

$$b(p_h, \mathbf{v}_h) \lesssim \|p_h\|_{0,\Omega}\big(\|\nabla\cdot\mathbf{v}_h\|_{0,\Omega} + \|\mathbf{v}_h\cdot\mathbf{n}\|_{+\frac{1}{2},h,\partial\Omega}\big)$$
$$d(p_h, \mathbf{v}_h) \lesssim \|p_h\|_{0,\Omega}\|[\![\mathbf{v}_h\cdot\mathbf{n}_\Gamma]\!]\|_{+\frac{1}{2},h,\Gamma}.$$

We obtain (16) by means of a suitable combination of the previous estimates. □

Lemma 3 (Positivity) *For γ_u large enough, there exists a positive constant C_{pos} such that,*

$$\mathcal{C}((\mathbf{v}_h, q_h), (\mathbf{v}_h, q_h)) \geq C_{pos}|||\mathbf{v}_h|||^2, \ \forall(\mathbf{v}_h, q_h) \in \mathbf{W}_h. \tag{17}$$

Proof. We observe that,

$$\mathcal{C}((\mathbf{v}_h, q_h), (\mathbf{v}_h, q_h)) = a(\mathbf{v}_h, \mathbf{v}_h) + c(\mathbf{v}_h, \mathbf{v}_h) + j_u(\mathbf{v}_h, \mathbf{v}_h) + j_p(q_h, q_h). \tag{18}$$

We have to treat the interface and boundary terms, which appear in the definition of $a(\mathbf{v}_h, \mathbf{v}_h)$ and $c(\mathbf{v}_h, \mathbf{v}_h)$. We focus on the interface terms of $c(\mathbf{v}_h, \mathbf{v}_h)$, the boundary terms in $a(\mathbf{v}_h, \mathbf{v}_h)$ will be treated analogously. Precisely, we have,

$$\int_\Gamma \{\nu\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\}_w \cdot [\![\mathbf{v}_h]\!] = \int_\Gamma \{\nu\}_w\{\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\} \cdot [\![\mathbf{v}_h]\!]$$
$$\lesssim \epsilon\|\{\nu\}_w^{\frac{1}{2}}\{\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\}\|^2_{-\frac{1}{2},h,\Gamma} + \frac{1}{\epsilon}\|\{\nu\}_w^{\frac{1}{2}}[\![\mathbf{v}_h]\!]\|^2_{+\frac{1}{2},h,\Gamma}$$

Owing to (10) we observe that, $\|\{\nu\}_w^{\frac{1}{2}}\{\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\}\|^2_{-\frac{1}{2},h,\Gamma} \lesssim \|\nu^{\frac{1}{2}}\nabla\mathbf{v}_h\|^2_{0,\Omega}$. By virtue of the previous inequalities we obtain,

$$a(\mathbf{v}_h, \mathbf{v}_h) + c(\mathbf{v}_h, \mathbf{v}_h) \gtrsim \|\eta^{\frac{1}{2}}\mathbf{v}_h\|^2_{0,\Omega} + (1 - C\epsilon)\|\nu^{\frac{1}{2}}\nabla\mathbf{v}_h\|^2_{0,\Omega}$$
$$+ \|(\kappa_\Gamma\frac{|[\![\nu]\!]|}{2\{\nu\}})^{\frac{1}{2}}\{\mathbf{v}_h\}^w\cdot\mathbf{t}_\Gamma\|^2_{0,\Gamma} + \Big(\gamma_u - \frac{C}{\epsilon}\Big)\Big[\|\nu^{\frac{1}{2}}\mathbf{v}_h\|^2_{+\frac{1}{2},h,\partial\Omega} + \|\{\nu\}_w^{\frac{1}{2}}\mathbf{v}_h\|^2_{+\frac{1}{2},h,\Gamma}\Big]. \tag{19}$$

The result follows replacing (19) in (18) and choosing suitably small ϵ and suitably large γ_u. □

To analyze the stability of our numerical method we follow the approach of Boland and Nicolaides (see [4] and also [15], Chapter II, Section 1.4) in order to split the verification of the inf-sup condition into a local condition on each subdomain and a global condition on a suitable subspace of $\mathbf{W}_h$. To this aim, we introduce $\tilde{Q}_{h,i} = Q_{h,i} \cap L^2_0(\Omega_i)$ and by consequence $Q_{h,i} = \tilde{Q}_{h,i} \oplus \mathbb{R}$ where $p_{h,i} = \tilde{p}_{h,i} + \bar{p}_{h,i}$ is the corresponding splitting of the pressure. Finally, let be $\bar{Q}_h$ the space of constant functions on each subdomain Ω_i that satisfy $\sum_{i=1}^{N} \int_{\Omega_i} \bar{q}_i = 0$. We aim to prove a local inf-sup condition on $\mathbf{V}_{h,i} \times \tilde{Q}_{h,i}$, and a global one, relative to the subspace $\mathbf{V}_h \times \bar{Q}_h$. To this aim, we introduce the following lemmas.

Lemma 4 (Stabilized local inf-sup condition) *For all $\tilde{p}_{h,i} \in \tilde{Q}_{h,i}$ there exists $\tilde{\mathbf{v}}_{p,h,i} \in \mathbf{V}_{h,i} \cap [H^1_0(\Omega_i)]^d$ such that,*

$$b_i(\tilde{p}_{h,i}, \tilde{\mathbf{v}}_{p,h,i}) + d_i(\tilde{p}_{h,i}, \tilde{\mathbf{v}}_{p,h,i}) \gtrsim \|\tilde{p}_{h,i}\|^2_{0,\Omega_i} - C \|[\![\tilde{p}_{h,i}]\!]\|^2_{-\frac{1}{2},h,\Omega_i}, \tag{20}$$

$$\|\tilde{\mathbf{v}}_{p,h,i}\|_{1,\Omega_i} \lesssim \|\tilde{p}_{h,i}\|_{0,\Omega_i}. \tag{21}$$

Lemma 4 We observe that, by means of the surjectivity of the divergence operator from $[H^1_0(\Omega_i)]^d$ to $\tilde{Q}_{h,i}$ for each subdomain Ω_i and for any $\tilde{p}_{h,i} \in \tilde{Q}_{h,i}$ there exists $\tilde{\mathbf{v}}_{p,i} \in [H^1_0(\Omega_i)]^d$ such that,

$$\nabla \cdot \tilde{\mathbf{v}}_{p,i} = -\tilde{p}_{h,i}, \quad \|\tilde{\mathbf{v}}_{p,i}\|_{1,\Omega_i} \lesssim \|\tilde{p}_{h,i}\|_{0,\Omega_i}. \tag{22}$$

Then, we define $\pi^{H^1_0}_{h,i} : [H^1_0(\Omega_i)]^d \to \mathbf{V}_{h,i} \cap [H^1_0(\Omega_i)]^d$ as the H^1_0-conformal L^2-projector. Thanks to the H^1-stability of L^2-projection on finite element spaces (see [5]) we have,

$$\|\pi^{H^1_0}_{h,i} \mathbf{v}\|_{1,\Omega_i} \lesssim \|\mathbf{v}\|_{1,\Omega_i}, \quad \forall \mathbf{v} \in [H^1_0(\Omega_i)]^d, \tag{23}$$

and by means of a classical duality argument we obtain,

$$\|\pi^{H^1_0}_{h,i} \mathbf{v} - \mathbf{v}\|_{0,K} \lesssim h_K |\mathbf{v}|_{1,K} \quad \forall K \in T_h, i = 1, \ldots, N, \quad \forall \mathbf{v} \in [H^1_0(\Omega_i)]^d. \tag{24}$$

Then, we set $\tilde{\mathbf{v}}_{p,h,i} = \pi^{H^1_0}_{h,i} \tilde{\mathbf{v}}_{p,i}$. Since $\tilde{\mathbf{v}}_{p,h,i} \in [H^1_0(\Omega_i)]^d$ we have $d_i(\tilde{p}_{h,i}, \tilde{\mathbf{v}}_{p,h,i}) = 0$.

Case 1, $r = 1$: we proceed as in [10], Theorem 2. By means of (10), (22), (24), applying integration by parts and observing that $\nabla \tilde{p}_{h,i}|_K = 0$ we obtain,

$$\begin{aligned} b_i(\tilde{p}_{h,i}, \tilde{\mathbf{v}}_{p,h,i}) &= \|\tilde{p}_{h,i}\|^2_{0,\Omega_i} - \int_{\mathcal{F}_{h,i}} [\![\tilde{p}_{h,i}]\!] \{\tilde{\mathbf{v}}_{p,h,i} \cdot \mathbf{n}_E - \tilde{\mathbf{v}}_{p,i} \cdot \mathbf{n}_E\} \\ &\gtrsim \|\tilde{p}_{h,i}\|^2_{0,\Omega_i} - \frac{1}{\epsilon} \|[\![\tilde{p}_{h,i}]\!]\|^2_{-\frac{1}{2},h,\Omega_i} - \epsilon \int_{\mathcal{F}_{h,i}} h_E^{-1} \{\tilde{\mathbf{v}}_{p,h,i} \cdot \mathbf{n}_E - \tilde{\mathbf{v}}_{p,i} \cdot \mathbf{n}_E\}^2 \\ &\gtrsim (1-\epsilon) \|\tilde{p}_{h,i}\|^2_{0,\Omega_i} - \frac{1}{\epsilon} \|[\![\tilde{p}_{h,i}]\!]\|^2_{-\frac{1}{2},h,\Omega_i}. \end{aligned} \tag{25}$$

Equation (20) follows by choosing sufficiently small ϵ. Moreover, (21) follows from the combination of (22) and the H^1 stability of the L^2 projection (23).
Case 2, $r > 1$**:** in this case we follow [8]. Note that for $r > 1$ the space $\mathbf{V}_{h,i} \times (Q_{h,i} \cap H^1(\Omega_i))$ satisfies the inf-sup condition (20) without any stabilization term (see [7] and references therein). If we let $\pi_{h,i}^{H^1} \tilde{p}_{h,i}$ denote the H^1-conformal L^2-projection of $\tilde{p}_{h,i}$ onto $Q_{h,i}$ it follows that

$$\|\tilde{p}_{h,i}\|_{0,\Omega_i}^2 = \|\pi_{h,i}^{H^1} \tilde{p}_{h,i}\|_{0,\Omega_i}^2 + \|(I - \pi_{h,i}^{H^1})\tilde{p}_{h,i}\|_{0,\Omega_i}^2.$$

Since the continuous space is inf-sup stable, there is a $\tilde{\mathbf{v}}_{p,h,i} \in \mathbf{V}_{h,i}$ such that,

$$\|\tilde{p}_{h,i}\|_{0,\Omega_i}^2 \leq b_i(\pi_{h,i}^{H^1} \tilde{p}_{h,i}, \tilde{\mathbf{v}}_{p,h,i}) + \|(I - \pi_{h,i}^{H^1})\tilde{p}_{h,i}\|_{0,\Omega_i}^2.$$

We proceed by adding and subtracting $\tilde{p}_{h,i}$ in the form $b_i(\cdot,\cdot)$ and by applying Cauchy-Schwarz inequality to obtain,

$$\|\tilde{p}_{h,i}\|_{0,\Omega_i}^2 \leq b_i(\tilde{p}_{h,i}, \tilde{\mathbf{v}}_{p,h,i}) + \|(I - \pi_{h,i}^{H^1})\tilde{p}_{h,i}\|_{0,\Omega_i} \|\tilde{\mathbf{v}}_{p,h,i}\|_{1,\Omega_i} + \|(I - \pi_{h,i}^{H^1})\tilde{p}_{h,i}\|_{0,\Omega_i}^2.$$

We conclude using $\|\tilde{\mathbf{v}}_{p,h,i}\|_{1,\Omega_i} \leq \|\pi_{h,i}^{H^1} \tilde{p}_{h,i}\|_{0,\Omega_i} \leq \|\tilde{p}_{h,i}\|_{0,\Omega_i}$, the arithmetic/geometric inequality and the following discrete interpolation result, which is proved exploiting the so called Oswald interpolation operator (see for instance [14]),

$$\|(I - \pi_{h,i}^{H^1})\tilde{p}_{h,i}\|_{0,\Omega_i}^2 = \inf_{q_h \in Q_{h,i} \cap H^1(\Omega_i)} \|\tilde{p}_{h,i} - q_h\|_{0,\Omega_i}^2 \lesssim \|[\![\tilde{p}_h]\!]\|_{-\frac{1}{2},h,\Omega_i}^2.$$

□

Lemma 5 (Auxiliary functions) *For all* $i = 1,\ldots,N$, $j \in \mathcal{N}_i$ *there exist functions* $w_{\Gamma_{ij}}^{(i)} \in V_{h,i} \cap H_0^1(\Omega)$, $w_{\Gamma_{ij}}^{(j)} \in V_{h,j} \cap H_0^1(\Omega)$ *and* $w_{\Gamma_{ij}} = w_{\Gamma_{ij}}^{(i)} + w_{\Gamma_{ij}}^{(j)} \in \big(V_{h,i} \oplus V_{h,j}\big) \cap H_0^1(\Omega)$ *such that,*

$$\int_{\Gamma_{ij}} w_{\Gamma_{ij}}^{(k)} = 1,\ \|w_{\Gamma_{ij}}^{(k)}\|_{1,\Omega_k} \lesssim 1,\ k = i,j,\ \|[\![w_{\Gamma_{ij}}]\!]\|_{\frac{1}{2},h,\Gamma_{ij}} \lesssim 1. \quad (26)$$

Lemma 5. For the sake of simplicity, we assume that each interface Γ_{ij} is planar with outward unit normal vector $\mathbf{n}_\Gamma$. Since $\Omega \subset \mathbb{R}^2$ we conclude that Γ_{ij} is an open set in $\mathbb{R}$. Let us denote with $\mathcal{G}_{h,i}$ the trace mesh of $\mathcal{T}_{h,i}$ on Γ_{ij}. Let x_i^M be the node of $\mathcal{G}_{h,i}$ closest to the midpoint of the interface Γ_{ij} and let be x_i^L and x_i^R respectively the leftmost and the rightmost nodes of $\mathcal{G}_{h,i}$ with respect to x_i^M. Let us denote with $h_{E,i} = \max_{E \in \mathcal{G}_{h,i}} h_E$, then for any admissible combination of i and j we have $|x_i^* - x_j^*| \leq \max[h_{E,i}, h_{E,j}]$ for $* = L, M, R$. Let us denote $w_{\Gamma_{ij}}^{(i)}|_{\partial\Omega_i}$ the continuous function which is affine on the intervals $[x_i^L, x_i^M]$ and $[x_i^M, x_i^R]$ and is zero for $x \leq x_i^L$, $x \geq x_i^R$ and $x \in \partial\Omega_i \setminus \Gamma_{ij}$. Moreover, we assume $\int_{\Gamma_{ij}} w_{\Gamma_{ij}}^{(i)} dx = 1$, consequently $(26)_a$ is identically satisfied. Let us now extend $w_{\Gamma_{ij}}^{(i)}|_{\partial\Omega_i}$ on the whole Ω_i by means of the finite element (harmonic) lifting $\mathcal{R}_{h,i}$.

More precisely, we define $w^{(i)}_{\Gamma_{ij}} = \mathcal{R}_{h,i}(w^{(i)}_{\Gamma_{ij}}|_{\partial\Omega_i})$. By the stability of the operator $\mathcal{R}_{h,i}$ we obtain $(26)_b$, namely $\|w^{(i)}_{\Gamma_{ij}}\|_{1,\Omega_i} \lesssim 1$. We proceed analogously for $w^{(j)}_{\Gamma_{ij}}$. Finally, $(26)_c$ holds by virtue of the assumption that $\mathcal{G}_{h,ij}$ is non-degenerate, and of the bound $|x^*_i - x^*_j| \leq \max[h_{E,i}, h_{E,j}]$ for $* = L, M, R$. □

Lemma 6 (Global inf-sup condition on subspaces) *For all* $\bar{p}_h \in \bar{Q}_h \subset Q_h$ *there exists* $\bar{\mathbf{v}}_{p,h} \in \mathbf{V}_h$ *such that,*

$$\mathcal{B}(\bar{p}_h, \bar{\mathbf{v}}_{p,h}) \gtrsim \|\bar{p}_h\|^2_{0,\Omega}, \quad \|\bar{\mathbf{v}}_{p,h}\|_{1,\Omega} \lesssim \|\bar{p}_h\|_{0,\Omega}, \tag{27}$$

$$\int_\Gamma [\![\bar{\mathbf{v}}_{p,h} \cdot \mathbf{n}_\Gamma]\!] = 0, \quad \|[\![\bar{\mathbf{v}}_{p,h} \cdot \mathbf{n}_\Gamma]\!]\|_{\frac{1}{2},h,\Gamma} \lesssim \|\bar{p}_h\|_{0,\Omega}. \tag{28}$$

Lemma 6. Let us define $\bar{\mathbf{v}}_{p,h,ij} := -\bar{p}_{h,i}|\Omega_i| w_{\Gamma_{ij}} \mathbf{n}_i$ and $\bar{\mathbf{v}}_{p,h} := \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} \bar{\mathbf{v}}_{p,h,ij}$. Owing to $(26)_a$ we have,

$$\int_{\Gamma_{ij}} [\![\bar{\mathbf{v}}_{p,h,ij} \cdot \mathbf{n}_i]\!] = -\bar{p}_{h,i}|\Omega_i| \Big(\int_{\Gamma_{ij}} w^{(i)}_{\Gamma_{ij}} - \int_{\Gamma_{ij}} w^{(j)}_{\Gamma_{ij}} \Big) = 0.$$

One immediately verifies that $\int_\Gamma [\![\bar{\mathbf{v}}_{p,h} \cdot \mathbf{n}_\Gamma]\!] = 0$. Then, we address equation $(27)_a$,

$$\begin{aligned}
\mathcal{B}(\bar{p}_h, \bar{\mathbf{v}}_{p,h,ij}) &= -\int_\Omega \bar{p}_h \nabla\cdot\bar{\mathbf{v}}_{p,h,ij} + \int_\Gamma \{\bar{p}_h\}_w [\![\bar{\mathbf{v}}_{p,h,ij} \cdot \mathbf{n}_\Gamma]\!] \\
&= -\,\bar{p}_{h,i} \int_{\Gamma_{ij}} \bar{\mathbf{v}}_{p,h,ij} \cdot \mathbf{n}_\Gamma - \bar{p}_{h,j} \int_{\Gamma_{ij}} \bar{\mathbf{v}}_{p,h,ij} \cdot \mathbf{n}_\Gamma \\
&= \bar{p}^2_{h,i}|\Omega_i| - \bar{p}_{h,i}\bar{p}_{h,j}|\Omega_i| = \bar{p}_{h,i}|\Omega_i|(\bar{p}_{h,i} - \bar{p}_{h,j}).
\end{aligned}$$

Then, we can write,

$$\mathcal{B}(\bar{p}_h, \bar{\mathbf{v}}_{p,h}) = \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} \mathcal{B}(\bar{p}_h, \bar{\mathbf{v}}_{p,h,ij}) = \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} \bar{p}_{h,i}|\Omega_i|(\bar{p}_{h,i} - \bar{p}_{h,j})$$

that can be reinterpreted at the algebraic level as $b(\bar{p}_h, \bar{\mathbf{v}}_{p,h}) = \bar{\mathbf{p}}_h^T B \bar{\mathbf{p}}_h$ where $\bar{\mathbf{p}}_h := [\bar{p}_{h,i}]_{1\leq i\leq N}$ and the matrix $B = [b_{ij}]_{1\leq i,j\leq N}$ is defined as follows,

$$b_{ij} := |\Omega_i|\text{card}(\mathcal{N}_i), \text{ if } j = i; \quad b_{ij} := -|\Omega_i|, \text{ if } j \in \mathcal{N}_i; \quad b_{ij} := 0, \text{ otherwise.}$$

We also introduce the matrix $D = \text{diag}(|\Omega_1|, \ldots, |\Omega_N|)$ and we observe that $\bar{\mathbf{p}}_h^T D \bar{\mathbf{p}}_h = \|\bar{p}_h\|^2_{0,\Omega}$. Then, the argument proposed in [2] (see Theorem 4.3), shows that the quantity $\gamma := \min_{\bar{p}_h\in\bar{Q}_h} (\bar{\mathbf{p}}_h^T B \bar{\mathbf{p}}_h)(\bar{\mathbf{p}}_h^T D \bar{\mathbf{p}}_h)^{-1}$ is positive and it does not depend on h: this proves $(27)_a$. Inequality $(27)_b$, can be derived as follows,

$$\|\bar{\mathbf{v}}_{p,h}\|^2_{1,\Omega} \lesssim \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} \|\bar{\mathbf{v}}_{p,h,ij}\|^2_{1,\Omega} \lesssim \sum_{i=1}^N \bar{p}^2_{h,i}|\Omega_i| \lesssim \|\bar{p}_h\|^2_{0,\Omega}.$$

We finally prove $(27)_b$. For $i = 1, \dots, N$ and $j \in \mathcal{N}_i$ we have that,

$$\|[\![\bar{\mathbf{v}}_{p,h} \cdot \mathbf{n}_\Gamma]\!]\|^2_{\frac{1}{2},h,\Gamma} \lesssim \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \|[\![\bar{\mathbf{v}}_{p,h,ij} \cdot \mathbf{n}_\Gamma]\!]\|^2_{\frac{1}{2},h,\Gamma_{ij}} = \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \bar{p}^2_{h,i}|\Omega_i| \lesssim \|\bar{p}_h\|^2_{0,\Omega}.$$

□

Then, we introduce the global inf-sup condition for the bilinear form $\mathcal{B}(p_h, \mathbf{v}_h)$.

Lemma 7 (Stabilized global inf-sup condition) *For all $p_h \in Q_h$ there exist $\mathbf{v}_{p,h} \in \mathbf{V}_h$ such that,*

$$\mathcal{B}(p_h, \mathbf{v}_{p,h}) \gtrsim \|p_h\|^2_{0,\Omega} - C\|[\![p_h]\!]\|^2_{-\frac{1}{2},h,\Omega}, \tag{29}$$

$$\|\mathbf{v}_{p,h}\|_{1,\Omega} \lesssim \|p_h\|_{0,\Omega}, \quad |||\mathbf{v}_{p,h}||| \lesssim \|p_h\|_{0,\Omega}. \tag{30}$$

We notice that the constants in $(30)_b$ may depend on ν, but does not blow up in the limit case $\nu \to 0$.

Lemma 7. Let be $\tilde{\mathbf{v}}_{p,h} = \sum_{i=1}^{N} \tilde{\mathbf{v}}_{p,h,i}$ and $\tilde{p}_h = \sum_{i=1}^{N} \tilde{p}_{h,i}$. The proof consists in choosing $\mathbf{v}_{p,h} = \tilde{\mathbf{v}}_{p,h} + \delta\bar{\mathbf{v}}_{p,h}$, being $\tilde{\mathbf{v}}_{p,h}$, $\bar{\mathbf{v}}_{p,h}$ as in Lemmas 4 and 6 respectively, and following the argument of Boland and Nicolaides (see [15], Chapter II, Section 1.4). We observe that

$$\mathcal{B}(p_h, \mathbf{v}_{p,h}) = \mathcal{B}(\tilde{p}_h, \tilde{\mathbf{v}}_{p,h}) + \delta\mathcal{B}(\bar{p}_h, \bar{\mathbf{v}}_{p,h}) + \mathcal{B}(\bar{p}_h, \tilde{\mathbf{v}}_{p,h}) + \delta\mathcal{B}(\tilde{p}_h, \bar{\mathbf{v}}_{p,h}). \tag{31}$$

Since $\tilde{\mathbf{v}}_{p,h,i} \in [H^1_0(\Omega_i)]^d$ and by virtue of Lemma 6 we obtain that $\mathcal{B}(\bar{p}_h, \tilde{\mathbf{v}}_{p,h})$ vanishes because

$$\int_{\Omega_i} \bar{p}_{h,i}\nabla\cdot\tilde{\mathbf{v}}_{p,h,i} = \bar{p}_{h,i}\int_{\partial\Omega_i} \tilde{\mathbf{v}}_{p,h,i}\cdot\mathbf{n}_i = 0, \int_{\Gamma_{ij}} \{\bar{p}_h\}_w[\![\tilde{\mathbf{v}}_{p,h}\cdot\mathbf{n}_i]\!] = \{\bar{p}_h\}_w \int_{\Gamma_{ij}} [\![\tilde{\mathbf{v}}_{p,h}\cdot\mathbf{n}_i]\!] = 0,$$

while $\mathcal{B}(\tilde{p}_h, \bar{\mathbf{v}}_{p,h})$ can be estimated as follows,

$$\begin{aligned} \mathcal{B}(\tilde{p}_h, \bar{\mathbf{v}}_{p,h}) &= \int_\Omega \tilde{p}_h\nabla\cdot\bar{\mathbf{v}}_{p,h} + \int_\Gamma \{\tilde{p}_h\}_w[\![\bar{\mathbf{v}}_{p,h}\cdot\mathbf{n}_\Gamma]\!] \\ &\lesssim \|\tilde{p}_h\|^2_{0,\Omega} + \|\bar{\mathbf{v}}_{p,h}\|^2_{1,\Omega} + \Big[\epsilon\|[\![\bar{\mathbf{v}}_{p,h}\cdot\mathbf{n}_\Gamma]\!]\|^2_{\frac{1}{2},h,\Gamma} + \frac{1}{\epsilon}\|\{\tilde{p}_h\}_w\|^2_{-\frac{1}{2},h,\Gamma}\Big] \\ &\lesssim \Big(1 + \frac{C}{\epsilon}\Big)\|\tilde{p}_h\|^2_{0,\Omega} + \Big(1 + C\epsilon\Big)\|\bar{p}_h\|^2_{0,\Omega}. \end{aligned} \tag{32}$$

By replacing (32), (27) and (20) into (31), and suitably choosing δ and ϵ we obtain (29). Equation (30) follows from the combination of (21) and $(27)_b$. □

The well-posedness of problem (9) is a consequence of the following result.

Theorem 4.1 (Stability) *For all* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$*, there exists a suitable constant* $C_{stab} > 0$ *independent of* h *and* ν *such that,*

$$\sup_{(\mathbf{v}_h, q_h)\in \mathbf{W}_h\setminus\{0\}} \frac{\mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))}{|||\mathbf{v}_h, q_h|||} \geq C_{stab}|||\mathbf{u}_h, p_h|||. \tag{33}$$

Proof. Owing to the property $\nabla \cdot \mathbf{V}_{h,i} \subset Q_{h,i}$, we choose $(\mathbf{v}_h, q_h) = (\mathbf{u}_h + \delta_1 \mathbf{v}_{p,h}, p_h + \delta_2 \nabla \cdot \mathbf{u}_h)$ with δ_1, $\delta_2 > 0$, being $\mathbf{v}_{p,h}$ as in Lemma 7. We exploit the bilinearity of $\mathcal{C}(\cdot, \cdot)$ to obtain,

$$\begin{aligned}\mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) &= \mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) \\ &\quad + \delta_1 \mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{v}_{p,h}, 0)) + \delta_2 \mathcal{C}((\mathbf{u}_h, p_h), (0, \nabla \cdot \mathbf{u}_h)).\end{aligned} \tag{34}$$

We recall that the first term on the right hand side of (34) can be estimated as,

$$\mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)) \geq C_{pos}|||\mathbf{u}_h|||^2 + j_u(\mathbf{u}_h, \mathbf{u}_h) + j_p(p_h, p_h), \tag{35}$$

while the second term of (34) can be rewritten as,

$$\mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{v}_{p,h}, 0)) = \mathcal{A}(\mathbf{u}_h, \mathbf{v}_{p,h}) + j_u(\mathbf{u}_h, \mathbf{v}_{p,h}) + \mathcal{B}(p_h, \mathbf{v}_{p,h}). \tag{36}$$

Exploiting Lemma 2, inequality $(30)_b$ and the arithmetic/geometric inequality we obtain,

$$\mathcal{A}(\mathbf{u}_h, \mathbf{v}_{p,h}) + j_u(\mathbf{u}_h, \mathbf{v}_{p,h}) \lesssim \epsilon_1 \|p_h\|^2_{0,\Omega} + \frac{1}{\epsilon_1}|||\mathbf{u}_h|||^2. \tag{37}$$

We finally observe that the third term on the right hand side of (34) is equivalent to,

$$\begin{aligned}&\mathcal{C}((\mathbf{u}_h, p_h), (0, \nabla \cdot \mathbf{u}_h)) \\ &= \|\nabla \cdot \mathbf{u}_h\|^2_{0,\Omega} + \int_{\mathcal{F}_h} [\![p_h]\!][\![\nabla \cdot \mathbf{u}_h]\!] - \int_{\Gamma} \{\nabla \cdot \mathbf{u}_h\}_w [\![\mathbf{u}_h \cdot \mathbf{n}_\Gamma]\!] - \int_{\partial\Omega} (\nabla \cdot \mathbf{u}_h)\mathbf{u}_h \cdot \mathbf{n}. \\ &\gtrsim (1 - C_2\epsilon_2)\|\nabla \cdot \mathbf{u}_h\|^2_{0,\Omega} - \frac{1}{\epsilon_2}\Big[\|[\![p_h]\!]\|^2_{-\frac{1}{2},h,\Omega} + \|[\![\mathbf{u}_h \cdot \mathbf{n}_\Gamma]\!]\|^2_{\frac{1}{2},h,\Gamma} + \|\mathbf{u}_h \cdot \mathbf{n}\|^2_{\frac{1}{2},h,\partial\Omega}\Big].\end{aligned} \tag{38}$$

Then, combining (34), (36), (37), (38) and (29) we obtain (33) as follows,

$$\begin{aligned}\mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) &\gtrsim \Big(1 - C_1\frac{\delta_1}{\epsilon_1}\Big)|||\mathbf{u}_h|||^2 + \delta_2\Big(1 - C_2\epsilon_2\Big)\|\nabla \cdot \mathbf{u}_h\|^2_{0,\Omega} \\ &\quad + \Big(\gamma_u - C_2\frac{\delta_2}{\epsilon_2}\Big)\Big(\|\mathbf{u}_h \cdot \mathbf{n}\|_{\frac{1}{2},h,\partial\Omega} + \|[\![\mathbf{u}_h \cdot \mathbf{n}_\Gamma]\!]\|_{\frac{1}{2},h,\Gamma}\Big) \\ &\quad + \delta_1\Big(1 - C_1\epsilon_1\Big)\|p_h\|^2_{0,\Omega} + \Big(\gamma_p - \delta_1 C_1 - \gamma_p^2 C_2\frac{\delta_2}{\epsilon_2}\Big)\|[\![p_h]\!]\|^2_{-\frac{1}{2},h,\Omega} \\ &\gtrsim |||\mathbf{u}_h, p_h|||,\end{aligned} \tag{39}$$

which is satisfied by choosing sufficiently small $\delta_{1,2}$, $\epsilon_{1,2}$ and γ_p, γ_u large enough. □

Remark 4.2 The previous analysis shows that it is possible to generalize the numerical scheme to different families of finite elements, in particular for the approximation of the pressure. Indeed, Theorem 4.1 depends on the space $Q_{h,i}$ only through the local stabilized inf-sup condition, i.e. Lemma 4, and through estimate (38) that leads to the control of $\|\nabla \cdot \mathbf{u}_h\|_{0,\Omega}$. As a result of that, with suitable modifications to Lemma 4, our coupling method can be also applied to H^1-conformal (possibly stabilized) elements for the pressure, but this would compromise the H_{div} stability.

Now, we aim to study the convergence of the $(\mathbf{u}_h, p_h)$ to $(\mathbf{u}, p)$ when $h \to 0$. First of all, we introduce the broken Sobolev space $H^{\mathbf{s}}(\Omega) := \bigoplus_{i=1}^N H^{s_i}(\Omega_i)$ equipped with the broken norm $\|v\|^2_{\cup\Omega_i,\mathbf{s}} := \sum_{i=1}^N \|v_i\|^2_{s_i,\Omega_i}$ and the seminorm $|v|^2_{\cup\Omega_i,\mathbf{s}} := \sum_{i=1}^N |v_i|^2_{s_i,\Omega_i}$.

Then, we recall the H^1-conformal L^2-projector $\pi^{H^1}_{h,i} : [H^1(\Omega_i)]^2 \to \mathbf{V}_{h,i}$ and the nonconformal L^2-projector $\pi^{L^2}_{h,i} : H^1(\Omega_i) \to Q_{h,i}$. Reminding that $\mathcal{T}_{h,i}$ $i = 1, \ldots, N$ are a family of shape-regular and quasi-uniform triangulations and assuming $(\mathbf{v}, q) \in [H^{r_i+1}(\Omega_i)]^2 \times H^{r_i}(\Omega_i)$, we have the following estimates for $\pi^{H^1}_{h,i}$ and $\pi^{L^2}_{h,i}$ (see [14]),

$$\|\mathbf{v} - \pi^{H^1}_{h,i}\mathbf{v}\|_{m,\Omega_i} \lesssim h_i^{r_i+1-m}|\mathbf{v}|_{r_i+1,\Omega_i},\ m = 0, 1, \quad \|q - \pi^{L^2}_{h,i}q\|_{0,\Omega_i} \lesssim h_i^{r_i}|q|_{r_i,\Omega_i}, \tag{40}$$

$$\|[\![\pi^{H^1}_{h,i}\mathbf{v}]\!]\|_{\frac{1}{2},h,\Gamma_{ij}} \lesssim \sum_{k=i,j} h_k^{r_k}|\mathbf{v}|_{r_k+1,\Omega_k}, \quad \|[\![\pi^{L^2}_{h,i}q]\!]\|_{-\frac{1}{2},h,\Omega_i} \lesssim h_i^{r_i}|q|_{r_i,\Omega_i}, \tag{41}$$

where we remind that the local finite element polynomial order for velocities is $r_i > 0$ while for pressures is $r_i - 1$.

Lemma 8 (Approximability) *For any* $\mathbf{v}, q \in [H^{\mathbf{r}+1}(\Omega)]^2 \times H^{\mathbf{r}}(\Omega)$ *we have,*

$$|||(\mathbf{v} - \pi^{H^1}_{h,i}\mathbf{v}, q - \pi^{L^2}_{h,i}q)||| \lesssim \sum_{i=1}^N h_i^{r_i}\left(|\mathbf{v}|_{r_i+1,\Omega_i} + |q|_{r_i,\Omega_i}\right) \lesssim h^r(|\mathbf{v}|_{\cup\Omega,\mathbf{r}+1} + |q|_{\cup\Omega,\mathbf{r}}).$$

Proof. The result follows exploiting (40)-(41) to provide estimates for each term that appears in the definition of $|||\mathbf{v}, q|||$. □

We are now in position to study the convergence property of the scheme (8), which is summarized in the following theorem.

Theorem 4.3 (Convergence) *Let* $(\mathbf{u}, p) \in \mathbf{W}$ *be the weak solution of* (1), *and let* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ *be the solution of* (8). *Then, the following a-priori error estimate holds true,*

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \lesssim \inf_{(\mathbf{z}_h, r_h)\in\mathbf{W}_h} |||(\mathbf{u} - \mathbf{z}_h, p - r_h)|||. \tag{42}$$

Under the additional regularity assumption $(\mathbf{u}, p) \in [H^{\mathbf{r}+1}(\Omega)]^2 \times H^{\mathbf{r}}(\Omega)$ *we have,*

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \lesssim \sum_{i=1}^{N} h_i^{r_i}\Big(|\mathbf{u}|_{r_i+1,\Omega_i} + |p|_{r_i,\Omega_i}\Big) \lesssim h^r(|\mathbf{u}|_{\cup\Omega,r+1} + |p|_{\cup\Omega,r}). \tag{43}$$

Proof. The proof of (42) is a straightforward consequence of the stability of method (8). Let us decompose the error $(\mathbf{u}-\mathbf{u}_h, p-p_h)$ in two parts

$$\mathbf{u}-\mathbf{u}_h := \mathbf{e}_\pi+\mathbf{e}_h := (\mathbf{u}-\mathbf{z}_h)+(\mathbf{z}_h-\mathbf{u}_h), \quad p-p_h := y_\pi+y_h := (p-r_h)+(r_h-p_h).$$

Then, it follows that

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \leq |||(\mathbf{e}_\pi, y_\pi)||| + |||(\mathbf{e}_h, y_h)|||. \tag{44}$$

Using stability, consistency and boundedness, i.e. Theorem 4.1, Lemma 1 and 2, we get

$$\begin{aligned}|||\mathbf{e}_h, y_h||| &\lesssim \sup_{(\mathbf{v}_h,q_h)\in\mathbf{W}_h\setminus\{0\}} \frac{\mathcal{C}((\mathbf{e}_h, y_h),(\mathbf{v}_h,q_h))}{|||\mathbf{v}_h,q_h|||}\\ &= \sup_{(\mathbf{v}_h,q_h)\in\mathbf{W}_h\setminus\{0\}} \frac{\mathcal{C}((\mathbf{e}_\pi, y_\pi),(\mathbf{v}_h,q_h))}{|||\mathbf{v}_h,q_h|||} \lesssim |||\mathbf{e}_\pi, y_\pi)|||. \end{aligned}\tag{45}$$

Then, combining (44), (45) and we obtain (42). Estimate (43) is recovered choosing $\mathbf{z}_{h,i} := \pi_{h,i}^{H^1}\mathbf{u}$ and $r_{h,i} := \pi_{h,i}^{L^2}p$ and exploiting Lemma 8. □

5 An iterative splitting method

In this section we set up an iterative splitting method aiming to separate the solution of the global discrete problem into subproblems on each subdomain Ω_i. Clearly, to recover the global solution we need to solve the local problems on Ω_i repeatedly, by means of a suitable iterative method. To this purpose, we extend the method proposed in [11] by Burman and Zunino for advection-diffusion-reaction equations. Alternative approaches based on Steklov-Poincaré or Robin-Robin decomposition operators are addressed in [12, 13].

To start with, we introduce the following bilinear forms (for the symmetric case $\varsigma = 1$), which represent the *local* counterparts of (4), (5) and (6) for the iterative splitting method,

$$\begin{aligned}\tilde{c}_i^c((\mathbf{u}_{h,i};\mathbf{u}_{h,\mathbf{j}}),\mathbf{v}_{h,i}) := \sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}} \Big[&-(w_i\nu_i\epsilon(\mathbf{u}_{h,i})\mathbf{n}_i + w_j\nu_j\epsilon(\mathbf{u}_{h,j})\mathbf{n}_i)\cdot\mathbf{v}_{h,i}\\ &- w_i\nu_i\epsilon(\mathbf{v}_{h,i})\mathbf{n}_i\cdot(\mathbf{u}_{h,i}-\mathbf{u}_{h,j})\Big],\end{aligned}\tag{46}$$

$$\tilde{c}_i^p((\mathbf{u}_{h,i};\mathbf{u}_{h,\mathbf{j}}),\mathbf{v}_{h,i}) := \sum_{j\in\mathcal{N}_i}\int_{\mathcal{G}_{h,ij}} \gamma_{u,i}\{\nu\}_w h_E^{-1}\,(\mathbf{u}_{h,i}-\mathbf{u}_{h,j})\cdot\mathbf{v}_{h,i}, \tag{47}$$

$$\tilde{c}_i^t((\mathbf{u}_{h,i};\mathbf{u}_{h,\mathbf{j}}),\mathbf{v}_{h,i}) := \sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}} \kappa_\Gamma \tfrac{[\![\nu]\!]}{2\{\nu\}}\big(w_j\mathbf{u}_{h,i}+w_i\mathbf{u}_{h,j}\big)\cdot\mathbf{t}_\Gamma\big(w_j\mathbf{v}_{h,i}\cdot\mathbf{t}_\Gamma\big), \tag{48}$$

$$\tilde{j}_{u,i}((\mathbf{u}_{h,i};\mathbf{u}_{h,\mathbf{j}}),\mathbf{v}_{h,i}) := \sum_{j\in\mathcal{N}_i}\int_{\mathcal{G}_{h,ij}} \gamma_{u,i} h_E^{-1} (\mathbf{u}_{h,i}-\mathbf{u}_{h,j})\cdot\mathbf{n}_i \mathbf{v}_{h,i}\cdot\mathbf{n}_i + \int_{\mathcal{B}_h}\gamma_{u,i}h_E^{-1}\mathbf{u}_{h,i}\cdot\mathbf{n}_i\mathbf{v}_{h,i}\cdot\mathbf{n}_i, \tag{49}$$

$$\tilde{d}_i^p((p_{h,i};p_{h,\mathbf{j}}),\mathbf{v}_{h,i}) := \sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}} (w_i p_{h,i} + w_j p_{h,j})\,\mathbf{v}_{h,i}\cdot\mathbf{n}_i, \tag{50}$$

$$\tilde{d}_i^u((\mathbf{u}_{h,i};\mathbf{u}_{h,\mathbf{j}}),q_{h,i}) := \sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}} w_i q_{h,i}\,(\mathbf{u}_{h,i}-\mathbf{u}_{h,j})\cdot\mathbf{n}_i, \tag{51}$$

where the normal vectors and the jumps refer to each subregion Ω_i, and $\mathbf{j}$ is a multi index such that $\mathbf{j} := \{j \in \mathcal{N}_i\}$. Furthermore we set,

$$\tilde{c}_i((\mathbf{u}_{h,i};\mathbf{u}_{h,\mathbf{j}}),\mathbf{v}_{h,i}) := \tilde{c}_i^c((\mathbf{u}_{h,i};\mathbf{u}_{h,\mathbf{j}}),\mathbf{v}_{h,i}) + \tilde{c}_i^p((\mathbf{u}_{h,i};\mathbf{u}_{h,\mathbf{j}}),\mathbf{v}_{h,i}) + \tilde{c}_i^t((\mathbf{u}_{h,i};\mathbf{u}_{h,\mathbf{j}}),\mathbf{v}_{h,i}).$$

We propose the following iterative method: for $i = 1,\ldots,N$, given $[\mathbf{u}_{h,i}^{(k-1)}, p_{h,i}^{(k-1)}]$, $[\mathbf{u}_{h,\mathbf{j}}^{(k-1)}, p_{h,\mathbf{j}}^{(k-1)}]$, find $[\mathbf{u}_{h,i}^{(k)}, p_{h,i}^{(k)}] \in \mathbf{V}_{h,i}\times Q_{h,i}$ such that,

$$\begin{aligned} & a_i(\mathbf{u}_{h,i}^{(k)},\mathbf{v}_{h,i}) + \tilde{c}_i((\mathbf{u}_{h,i}^{(k)};\mathbf{u}_{h,\mathbf{j}}^{(k-1)}),\mathbf{v}_{h,i}) + \tilde{j}_{u,i}((\mathbf{u}_{h,i}^{(k)};\mathbf{u}_{h,\mathbf{j}}^{(k-1)}),\mathbf{v}_{h,i}) \\ & + b_i(p_{h,i}^{(k)},\mathbf{v}_{h,i}) + \tilde{d}_i^p((p_{h,i}^{(k)};p_{h,\mathbf{j}}^{(k-1)}),\mathbf{v}_{h,i}) + s_{\sigma_u,i}((\mathbf{u}_{h,i}^{(k)};\mathbf{u}_{h,i}^{(k-1)}),\mathbf{v}_{h,i}) \\ & = \mathcal{F}(\mathbf{v}_{h,i}),\ \forall \mathbf{v}_{h,i}\in\mathbf{V}_{h,i}, \end{aligned} \tag{52}$$

$$\begin{aligned} & b_i(q_{h,i},\mathbf{u}_{h,i}^{(k)}) + \tilde{d}_i^u((\mathbf{u}_{h,i}^{(k)};\mathbf{u}_{h,\mathbf{j}}^{(k-1)}),q_{h,i}) - j_{p,i}(p_{h,i}^{(k)},q_{h,i}) \\ & - s_{\sigma_p,i}((p_{h,i}^{(k)};p_{h,i}^{(k-1)}),q_{h,i}) = 0,\ \forall q_{h,i}\in Q_{h,i}. \end{aligned} \tag{53}$$

We notice that in (52)-(53) we have introduced the new terms,

$$\begin{aligned} s_{\sigma_u,i}((\mathbf{u}_{h,i}^{(k)};\mathbf{u}_{h,i}^{(k-1)}),\mathbf{v}_{h,i}) &:= \sum_{j\in\mathcal{N}_i}\int_{\mathcal{G}_{h,ij}} \sigma_{u,i}h_E^{-1}\Big[(\mathbf{u}_{h,i}^{(k)}-\mathbf{u}_{h,i}^{(k-1)})\cdot\mathbf{n}_i(\mathbf{v}_{h,i}\cdot\mathbf{n}_i) \\ &\qquad + \{\nu\}_w(\mathbf{u}_{h,i}^{(k)}-\mathbf{u}_{h,i}^{(k-1)})\cdot\mathbf{v}_{h,i}\Big] \\ s_{\sigma_p,i}((p_{h,i}^{(k)};p_{h,i}^{(k-1)}),q_{h,i}) &:= \sum_{j\in\mathcal{N}_i}\int_{\mathcal{G}_{h,ij}} \sigma_{p,i}h_E^{-1}(p_{h,i}^{(k)}-p_{h,i}^{(k-1)})q_{h,i}, \end{aligned}$$

with $\sigma_{u,i} := \sigma_u r_i^2$ and $\sigma_{p,i} := \sigma_p r_i^2$ given $\sigma_u, \sigma_p > 0$, responsible to provide a suitable amount of relaxation in order to ensure the convergence of the method. Each of the subproblems of (52)-(53) is substantially equivalent to the multidomain problem (8) with $N = 1$ subdomains. Proceeding as in Theorem 4.1 it can be proven that such problems are well-posed. In particular, we observe that in each subproblem we look for a pressure $p_{h,i}^{(k)} \in Q_{h,i}$ without any constraint on its mean value, because the interface conditions and relaxation terms on Γ_{ij} remove the indetermination of the pressure with respect to constants and the mean value of the pressure depends on the initial guess $p_h^{(0)}$. Anyway, it is possible to satisfy the constraint $\int_\Omega p_h = 0$ by scaling the pressure either at the end of each iteration or at the end of the whole iterative process. Finally, problem (52)-(53) can be also rewritten as follows: find $(\mathbf{u}_h^{(k)},\ p_h^{(k)}) \in \mathbf{W}_h = \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned}\widetilde{\mathcal{C}}((\mathbf{u}_h^{(k)}, p_h^{(k)}), (\mathbf{v}_h, q_h)) + \mathcal{S}_{\sigma_u}((\mathbf{u}_h^{(k)}; \mathbf{u}_h^{(k-1)}), \mathbf{v}_h) + \mathcal{S}_{\sigma_p}((p_h^{(k)}; p_h^{(k-1)}), q_h) \\ = \mathcal{G}(\mathbf{v}_h, q_h) - \mathcal{R}((\mathbf{u}_h^{(k-1)}, p_h^{(k-1)}), (\mathbf{v}_h, q_h)),\ \forall (\mathbf{v}_h, q_h) \in \mathbf{W}_h, \quad (54)\end{aligned}$$

where

$$\mathcal{S}_{\sigma_u}((\mathbf{u}_h^{(k)}; \mathbf{u}_h^{(k-1)}), \mathbf{v}_h) := \sum_{i=1}^N s_{\sigma_u,i}((\mathbf{u}_{h,i}^{(k)}; \mathbf{u}_{h,i}^{(k-1)}), \mathbf{v}_{h,i}),$$
$$\mathcal{S}_{\sigma_p}((p_h^{(k)}; p_h^{(k-1)}), q_h) := \sum_{i=1}^N s_{\sigma_p,i}((p_{h,i}^{(k)}; p_{h,i}^{(k-1)}), q_{h,i}),$$

$$\begin{aligned}\widetilde{\mathcal{C}}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) := \sum_{i=1}^N \Big[a_i(\mathbf{u}_{h,i}, \mathbf{v}_{h,i}) + \tilde{c}_i((\mathbf{u}_{h,i}; 0), \mathbf{v}_{h,i}) + \tilde{j}_{u,i}((\mathbf{u}_{h,i}; 0), \mathbf{v}_{h,i}) \\ + b_i(p_{h,i}, \mathbf{v}_{h,i}) + \tilde{d}_i^p((p_{h,i}; 0), \mathbf{v}_{h,i}) - b_i(\mathbf{u}_{h,i}, q_{h,i}) - \tilde{d}_i^u((\mathbf{u}_{h,i}; 0), q_{h,i}) + j_{p,i}(p_{h,i}, q_{h,i})\Big],\end{aligned}$$

denote the relaxation and the local discrete operators respectively, and

$$\begin{aligned}\mathcal{R}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) := \sum_{i=1}^N \Big[\tilde{c}_i((0; \mathbf{u}_{h,\mathbf{j}}), \mathbf{v}_{h,i}) + \tilde{j}_{u,i}((0; \mathbf{u}_{h,\mathbf{j}}), \mathbf{v}_{h,i}) \\ + \tilde{d}_i^p((0; p_{h,\mathbf{j}}), \mathbf{v}_{h,i}) - \tilde{d}_i^u((0; \mathbf{u}_{h,\mathbf{j}}), q_{h,i})\Big],\end{aligned}$$

can be regarded as the *iteration residual*. From equation (54), it is straightforward to verify that the iterative method is consistent with (8), namely the following identity is satisfied,

$$\mathcal{C}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \widetilde{\mathcal{C}}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) + \mathcal{R}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{W}_h.$$

6 Convergence analysis of the iterative splitting method

In order to analyze the convergence of the iterative method, we reformulate problem (54) as a problem for the splitting error, that is given by $\mathbf{w}_h^{(k)} := \mathbf{u}_h - \mathbf{u}_h^{(k)}$, $\rho_h^{(k)} := p_h - p_h^{(k)}$, where $[\mathbf{u}_h, p_h]$ satisfies (9) and $[\mathbf{u}_h^{(k)}, p_h^{(k)}]$ is the solution of (54) at the iteration k. Subtracting equation (54) from (9) we obtain,

$$\begin{aligned}\mathcal{C}((\mathbf{w}_h^{(k)}, \rho_h^{(k)}), (\mathbf{v}_h, q_h)) + \mathcal{S}_{\sigma_u}((\mathbf{w}_h^{(k)}; \mathbf{w}_h^{(k-1)}), \mathbf{v}_h) + \mathcal{S}_{\sigma_p}((\rho_h^{(k)}; \rho_h^{(k-1)}), q_h) \\ = \mathcal{R}((\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}, \rho_h^{(k)} - \rho_h^{(k-1)}), (\mathbf{v}_h, q_h)), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{W}_h. \quad (55)\end{aligned}$$

Before proceeding, let us introduce $\mathcal{T}(\mathbf{w}_h^{(k)}, \rho_h^{(k)}; \mathbf{w}_h^{(k-1)}, \rho_h^{(k-1)})$ a generic term that is telescopic with respect to the summation over k. More precisely, we require that

$$\sum_{k=1}^{M} \mathcal{T}(\mathbf{w}_h^{(k)}, \rho_h^{(k)}; \mathbf{w}_h^{(k-1)}, \rho_h^{(k-1)}) = \mathcal{T}(\mathbf{w}_h^{M}, \rho_h^{M}; \mathbf{w}_h^{0}, \rho_h^{0}).$$

Our strategy to prove the convergence of the iterative splitting method consists to choose $(\mathbf{v}_h, q_h) = (\mathbf{w}_h^{(k)}, \rho_h^{(k)})$ in (55) and to prove an upper bound for $\mathcal{R}((\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}, \rho_h^{(k)} - \rho_h^{(k-1)}), (\mathbf{w}_h^{(k)}, \rho_h^{(k)}))$ with respect to terms that are either telescopic, and thus collected into $\mathcal{T}(\mathbf{w}_h^{(k)}, \rho_h^{(k)}; \mathbf{w}_h^{(k-1)}, \rho_h^{(k-1)})$, or can be controlled by means of either $\mathcal{C}((\mathbf{w}_h^{(k)}, \rho_h^{(k)}), (\mathbf{w}_h^{(k)}, \rho_h^{(k)}))$, or the relaxation terms $\mathcal{S}_{\sigma_u}((\mathbf{w}_h^{(k)}; \mathbf{w}_h^{(k-1)}), \mathbf{w}_h^{(k)})$, $\mathcal{S}_{\sigma_p}((\rho_h^{(k)}; \rho_h^{(k-1)}), \rho_h^{(k)})$. Then, we aim to show that

$$\sum_{k=1}^{\infty} |||\mathbf{w}_h^{(k)}, \rho_h^{(k)}||| \le C|||\mathbf{w}_h^{(0)}, {\rho_h}^{(0)}|||,$$

which ensures that $|||\mathbf{w}_h^{(k)}, \rho_h^{(k)}||| \to 0$ with a geometric rate of convergence. To this aim, we proceed by steps, corresponding to the following lemmas.

Lemma 9 *There exists a constant $C_{\mathcal{R}}$, which satisfies $0 < 2C_{\mathcal{R}} < C_{pos}$, and two constants $C_{\mathcal{S}}^u$, $C_{\mathcal{S}}^p > 0$ such that*

$$\begin{aligned}
&\mathcal{R}((\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}, \rho_h^{(k)} - \rho_h^{(k-1)}), (\mathbf{w}_h^{(k)}, \rho_h^{(k)}))\\
&\le C_{\mathcal{R}}\Big[|||\mathbf{w}_h^{(k)}|||^2 + |||\mathbf{w}_h^{(k-1)}|||^2\Big] + C_{\mathcal{S}}^p\|\rho_h^{(k)} - \rho_h^{(k-1)}\|^2_{\frac{1}{2},h,\Gamma}\\
&+ C_{\mathcal{S}}^u\Big[\|(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}) \cdot \mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)})\|^2_{\frac{1}{2},h,\Gamma}\Big]\\
&+ \mathcal{T}(\mathbf{w}_h^{(k)}, \rho_h^{(k)}; \mathbf{w}_h^{(k-1)}, \rho_h^{(k-1)}) - \mathcal{S}_{\gamma_u}((\mathbf{w}_h^{(k)}; \mathbf{w}_h^{(k-1)}), \mathbf{w}_h^{(k)}).
\end{aligned} \tag{56}$$

Proof. First of all, we consider the contribution of (46). Summing and subtracting
$\{\nu\}_w\epsilon(\mathbf{w}_{h,i}^{(k-1)})\mathbf{n}_i \cdot \mathbf{w}_{h,j}^{(k-1)}$, the expression of $\tilde{c}_i^c((0; \mathbf{w}_{h,\mathbf{j}}^{(k)} - \mathbf{w}_{h,\mathbf{j}}^{(k-1)}), \mathbf{w}_{h,i}^{(k)})$ can be

rearranged and estimated as follows,

$$\begin{aligned}
&\sum_{i=1}^{N} \tilde{c}_i^c((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{w}_{h,i}^{(k)})\\
&=\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}\frac{1}{2}\Big[-\{\nu\}_w\epsilon(\mathbf{w}_{h,j}^{(k)}-\mathbf{w}_{h,j}^{(k-1)})\mathbf{n}_i\cdot\big(\mathbf{w}_{h,i}^{(k)}-\mathbf{w}_{h,i}^{(k-1)}\big)\\
&\quad+\{\nu\}_w\epsilon(\mathbf{w}_{h,j}^{(k-1)})\mathbf{n}_i\cdot\mathbf{w}_{h,i}^{(k-1)}-\{\nu\}_w\epsilon(\mathbf{w}_{h,j}^{(k)})\mathbf{n}_i\cdot\mathbf{w}_{h,i}^{(k)}\\
&\quad+\{\nu\}_w\epsilon(\mathbf{w}_{h,j}^{(k)})\mathbf{n}_j\cdot\mathbf{w}_{h,i}^{(k-1)}-\{\nu\}_w\epsilon(\mathbf{w}_{h,i}^{(k)})\mathbf{n}_i\cdot\mathbf{w}_{h,j}^{(k-1)}\Big]\\
&\lesssim\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}\frac{1}{2}\Big[\{\nu\}_w\epsilon(\mathbf{w}_{h,j}^{(k-1)})\mathbf{n}_i\cdot\mathbf{w}_{h,i}^{(k-1)}-\{\nu\}_w\epsilon(\mathbf{w}_{h,j}^{(k)})\mathbf{n}_i\cdot\mathbf{w}_{h,i}^{(k)}\Big]\\
&\quad+\epsilon\big(\|\nu^{\frac{1}{2}}\nabla\mathbf{w}_h^{(k)}\|_{0,\Omega}^2+\|\nu^{\frac{1}{2}}\nabla\mathbf{w}_h^{(k-1)}\|_{0,\Omega}^2\big)+\frac{1}{\epsilon}\|\{\nu\}_w\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\|_{\frac{1}{2},h,\Gamma}^2,
\end{aligned}\tag{57}$$

where the first term of the second row of (57) cancels out during the summation over k and thus it is cast into $\mathcal{T}(\mathbf{w}_h^{(k)},\rho_h^{(k)};\mathbf{w}_h^{(k-1)},\rho_h^{(k-1)})$.

Let us now consider the contribution of the penalty terms (47) and (49). We proceed as follows,

$$\begin{aligned}
&\sum_{i=1}^{N}\tilde{c}_i^p((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{w}_{h,i}^{(k)})+\tilde{j}_{u,i}((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{w}_{h,i}^{(k)})\\
&=-\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Big[\int_{\mathcal{G}_{h,ij}}\gamma_{u,j}\{\nu\}_w h_E^{-1}\Big(\big(\mathbf{w}_{h,j}^{(k)}-\mathbf{w}_{h,j}^{(k-1)}\big)\cdot\big(\mathbf{w}_{h,i}^{(k)}-\mathbf{w}_{h,j}^{(k)}\big)+\big(\mathbf{w}_{h,j}^{(k)}-\mathbf{w}_{h,j}^{(k-1)}\big)\cdot\mathbf{w}_{h,j}^{(k)}\Big)\\
&\quad+\int_{\mathcal{G}_{h,ij}}\gamma_{u,j}h_E^{-1}\Big(\big(\mathbf{w}_{h,j}^{(k)}-\mathbf{w}_{h,j}^{(k-1)}\big)\cdot\mathbf{n}_i\big(\mathbf{w}_{h,i}^{(k)}-\mathbf{w}_{h,j}^{(k)}\big)\cdot\mathbf{n}_i+\big(\mathbf{w}_{h,j}^{(k)}-\mathbf{w}_{h,j}^{(k-1)}\big)\cdot\mathbf{n}_i\big(\mathbf{w}_{h,j}^{(k)}\cdot\mathbf{n}_i\big)\Big)\Big]\\
&\lesssim\frac{\gamma_u}{\epsilon}\|\{\nu\}_w^{\frac{1}{2}}\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\|_{\frac{1}{2},h,\Gamma}^2+\frac{\gamma_u}{\epsilon}\|\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\cdot\mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2\\
&\quad+\epsilon\gamma_u\|[\![\mathbf{w}_h^{(k)}]\!]\cdot\mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2+\epsilon\gamma_u\|\{\nu\}_w^{\frac{1}{2}}[\![\mathbf{w}_h^{(k)}]\!]\|_{\frac{1}{2},h,\Gamma}^2-\mathcal{S}_{\gamma_u}((\mathbf{w}_h^{(k)};\mathbf{w}_h^{(k-1)}),\mathbf{w}_h^{(k)}).
\end{aligned}\tag{58}$$

For the coupling terms corresponding to the Beavers-Joseph-Saffman condition in the general case $\nu_i \neq \nu_j$ with $\nu_i, \nu_j > 0$ we proceed as follows,

$$\begin{aligned}
&\sum_{i=1}^{N}\tilde{c}_i^t((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{w}_{h,i}^{(k)})=\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}\kappa_\Gamma\tfrac{|[\![\nu]\!]|}{8\{\nu\}^2}\{\nu\}_w\big(\mathbf{w}_{h,j}^{(k)}-\mathbf{w}_{h,j}^{(k-1)}\big)\cdot\mathbf{t}_\Gamma\mathbf{w}_{h,i}^{(k)}\cdot\mathbf{t}_\Gamma\\
&\lesssim\epsilon|||\mathbf{w}_h^{(k)}|||^2+\frac{1}{\epsilon}\|\{\nu\}_w^{\frac{1}{2}}\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\|_{\frac{1}{2},h,\Gamma}^2.
\end{aligned}\tag{59}$$

Finally, we consider the terms of the iteration residual that depend on the splitting error of the pressure, namely (50) and (51), which can be rearranged as follows,

$$
\begin{aligned}
&\sum_{i=1}^{N}\Big[\tilde{d}_i^p((0;\rho_{h,\mathbf{j}}^{(k)}-\rho_{h,\mathbf{j}}^{(k-1)}),\mathbf{w}_{h,i}^{(k)})-\tilde{d}_i^u((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\rho_{h,i}^{(k)})\Big]\\
=&\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}\Big[w_j\big(\rho_{h,j}^{(k)}-\rho_{h,j}^{(k-1)}\big)\mathbf{w}_{h,i}^{(k)}\cdot\mathbf{n}_i+w_i\rho_{h,i}^{(k)}\big(\mathbf{w}_{h,j}^{(k)}-\mathbf{w}_{h,j}^{(k-1)}\big)\cdot\mathbf{n}_i\Big]\\
=&\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}\Big[w_i\rho_{h,i}^{(k)}\mathbf{w}_{h,j}^{(k-1)}\cdot\mathbf{n}_j-w_i\rho_{h,i}^{(k-1)}\mathbf{w}_{h,j}^{(k)}\cdot\mathbf{n}_j\Big]\\
=&\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}\Big[w_i\big(\rho_{h,i}^{(k)}-\rho_{h,i}^{(k-1)}\big)\big(\mathbf{w}_{h,j}^{(k)}+\mathbf{w}_{h,j}^{(k-1)}\big)\cdot\mathbf{n}_j\\
&+\Big(w_i\rho_{h,i}^{(k-1)}\mathbf{w}_{h,j}^{(k-1)}\cdot\mathbf{n}_j-w_i\rho_{h,i}^{(k)}\mathbf{w}_{h,j}^{(k)}\cdot\mathbf{n}_j\Big)\Big],
\end{aligned}
\tag{60}
$$

where for the last row we have exploited the algebraic identity

$$
a^{(k)}b^{(k-1)}-a^{(k-1)}b^{(k)}=\big(a^{(k)}-a^{(k-1)}\big)\big(b^{(k)}+b^{(k-1)}\big)+a^{(k-1)}b^{(k-1)}-a^{(k)}b^{(k)}.
$$

The first term of the last row can be estimated as follows,

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}w_i\big(\rho_{h,i}^{(k)}-\rho_{h,i}^{(k-1)}\big)\big(\mathbf{w}_{h,j}^{(k)}+\mathbf{w}_{h,j}^{(k-1)}\big)\cdot\mathbf{n}_j\\
\lesssim&\epsilon\big(\|\mathbf{w}_h^{(k)}\cdot\mathbf{n}_\Gamma\|_{-\frac{1}{2},h,\Gamma}^2+\|\mathbf{w}_h^{(k-1)}\cdot\mathbf{n}_\Gamma\|_{-\frac{1}{2},h,\Gamma}^2\big)+\frac{1}{\epsilon}\|\rho_h^{(k)}-\rho_h^{(k-1)}\|_{\frac{1}{2},h,\Gamma}^2\\
\lesssim&\epsilon\big(|||\mathbf{w}_h^{(k)}|||^2+|||\mathbf{w}_h^{(k-1)}|||^2\big)+\frac{1}{\epsilon}\|\rho_h^{(k)}-\rho_h^{(k-1)}\|_{\frac{1}{2},h,\Gamma}^2,
\end{aligned}
\tag{61}
$$

while the second term of the last row in (60) is telescopic and contributes to $\mathcal{T}(\cdot)$.

Equation (56), with constants $C_{\mathcal{S}}^u$ and $C_{\mathcal{S}}^p$ large enough, depending on ϵ and γ_u in the previous inequalities, directly follows from the combination of (57), (58), (59) and (60)-(61) together with the following expression for the telescopic terms,

$$
\begin{aligned}
\mathcal{T}(\mathbf{w}_h^{(k)},\rho_h^{(k)};\mathbf{w}_h^{(k-1)},\rho_h^{(k-1)}):=&\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Big[\int_{\Gamma_{ij}}\Big(w_i\rho_{h,i}^{(k-1)}\mathbf{w}_{h,j}^{(k-1)}\cdot\mathbf{n}_j-w_i\rho_{h,i}^{(k)}\mathbf{w}_{h,j}^{(k)}\cdot\mathbf{n}_j\Big)\\
&+\int_{\Gamma_{ij}}\Big(\tfrac{1}{2}\{\nu\}_w\epsilon(\mathbf{w}_{h,j}^{(k-1)})\mathbf{n}_i\cdot\mathbf{w}_{h,i}^{(k-1)}-\tfrac{1}{2}\{\nu\}_w\epsilon(\mathbf{w}_{h,j}^{(k)})\mathbf{n}_i\cdot\mathbf{w}_{h,i}^{(k)}\Big)\Big].
\end{aligned}
\tag{62}
$$

□

Lemma 10 *There exist two constants* $0 < C_{\mathcal{T}_1}$ *and* $0 < C_{\mathcal{T}_2} < \big(C_{pos} - 2C_{\mathcal{R}}\big)$ *such that,*

$$\begin{aligned}
&\sum_{k=1}^{M} \mathcal{T}(\mathbf{w}_h^{(k)}, \rho_h^{(k)}; \mathbf{w}_h^{(k-1)}, \rho_h^{(k-1)}) = \mathcal{T}(\mathbf{w}_h^{(M)}, {\rho_h}^{(M)}; \mathbf{w}_h^{(0)}, {\rho_h}^{(0)}) \\
&\leq C_{\mathcal{T}_1}\Big[\|\mathbf{w}_h^{0}\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}\mathbf{w}_h^{0}\|^2_{\frac{1}{2},h,\Gamma} + \|{\rho_h}^{(0)}\|^2_{\frac{1}{2},h,\Gamma}\Big] + C_{\mathcal{T}_2}|||\mathbf{w}_h^{(0)}|||^2 \\
&+ C_{\mathcal{T}_1}\Big[\|\mathbf{w}_h^{M}\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}\mathbf{w}_h^{M}\|^2_{\frac{1}{2},h,\Gamma} + \|{\rho_h}^{(M)}\|^2_{\frac{1}{2},h,\Gamma}\Big] + C_{\mathcal{T}_2}|||\mathbf{w}_h^{(M)}|||^2.
\end{aligned} \tag{63}$$

Proof. The result directly follows from the estimates,

$$\int_{\Gamma_{ij}} \{\nu\}_w \epsilon(\mathbf{w}_{h,j})\mathbf{n}_i\cdot\mathbf{w}_{h,i} \lesssim \epsilon\|\nu_j^{\frac{1}{2}}\nabla\mathbf{w}_{h,j}\|^2_{0,\Omega_i} + \frac{1}{\epsilon}\|\{\nu\}_w^{\frac{1}{2}}\mathbf{w}_{h,i}\|^2_{\frac{1}{2},h,\Gamma_{ij}},$$

$$\int_{\Gamma_{ij}} w_i\rho_{h,i}\mathbf{w}_{h,j}\cdot\mathbf{n}_j \lesssim \epsilon\|\mathbf{w}_{h,j}\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma_{ij}} + \frac{1}{\epsilon}\|\rho_{h,i}\|^2_{\frac{1}{2},h,\Gamma_{ij}}.$$

□

Theorem 6.1 *Provided that the relaxation parameters* σ_u *and* σ_p *are large enough, the iterative method* (54) *is convergent. More precisely,*

$$\begin{aligned}
\sum_{k=1}^{\infty}\Big[&|||\mathbf{w}_h^{(k)}|||^2 + \|(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)})\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)})\|^2_{\frac{1}{2},h,\Gamma} \\
&+ \|[\![\rho_h^{(k)}]\!]\|^2_{-\frac{1}{2},h,\Omega} + \|\rho_h^{(k)} - \rho_h^{(k-1)}\|^2_{\frac{1}{2},h,\Gamma}\Big] \lesssim |||(\mathbf{w}_h^{(0)}, \rho_h^{(0)})|||^2.
\end{aligned} \tag{64}$$

Proof. First, we observe that

$$\begin{aligned}
&\mathcal{S}_{\sigma_u}((\mathbf{w}_h^{(k)}; \mathbf{w}_h^{(k-1)}), \mathbf{w}_h^{(k)}) = \tfrac{\sigma_u}{2}\Big[\|(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)})\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)})\|^2_{\frac{1}{2},h,\Gamma} \\
&+ \|\mathbf{w}_h^{(k)}\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} - \|\mathbf{w}_h^{(k-1)}\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}\mathbf{w}_h^{(k)}\|^2_{\frac{1}{2},h,\Gamma} - \|\{\nu\}_w^{\frac{1}{2}}\mathbf{w}_h^{(k-1)}\|^2_{\frac{1}{2},h,\Gamma}\Big], \\
&\mathcal{S}_{\sigma_p}((\rho_h^{(k)}; \rho_h^{(k-1)}), \rho_h^{(k)}) = \tfrac{\sigma_p}{2}\Big[\|\rho_h^{(k)} - \rho_h^{(k-1)}\|^2_{\frac{1}{2},h,\Gamma} + \|\rho_h^{(k)}\|^2_{\frac{1}{2},h,\Gamma} - \|\rho_h^{(k-1)}\|^2_{\frac{1}{2},h,\Gamma}\Big].
\end{aligned}$$

that is a straightforward consequence of $(a^{(k)} - a^{(k-1)})a^{(k)} = \frac{1}{2}\big[(a^{(k)} - a^{(k-1)})^2 + (a^{(k)})^2 - (a^{(k-1)})^2\big]$. Then, we consider equation (55) and observe that by means of the positivity of the bilinear form $\mathcal{C}((\cdot,\cdot),(\cdot,\cdot))$ (see Lemma 3) combined with

(56) and (55) we obtain,

$$
\begin{aligned}
&(C_{pos} - 2C_{\mathcal{R}})|||\mathbf{w}_h^{(k)}||| + \gamma_p \|[\![\rho_h^{(k)}]\!]\|^2_{-\frac{1}{2},h,\Omega} + \left(\tfrac{\sigma_p}{2} - C_{\mathcal{S}}^p\right)\|\rho_h^{(k)} - \rho_h^{(k-1)}\|^2_{\frac{1}{2},h,\Gamma} \\
&+ \left(\tfrac{\sigma_u}{2} + \tfrac{\gamma_u}{2} - C_{\mathcal{S}}^u\right)\Big[\|(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}) \cdot \mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)})\|^2_{\frac{1}{2},h,\Gamma}\Big] \\
&\leq C_{\mathcal{R}}\Big[|||\mathbf{w}_h^{(k-1)}|||^2 - |||\mathbf{w}_h^{(k)}|||^2\Big] + \mathcal{T}(\mathbf{w}_h^{(k)}, \rho_h^{(k)}; \mathbf{w}_h^{(k-1)}, \rho_h^{(k-1)}) \\
&+ \tfrac{1}{2}(\sigma_u + \gamma_u)\Big[\|\mathbf{w}_h^{(k-1)} \cdot \mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}\mathbf{w}_h^{(k-1)}\|^2_{\frac{1}{2},h,\Gamma}\Big] + \tfrac{\sigma_p}{2}\|\rho_h^{(k-1)}\|^2_{\frac{1}{2},h,\Gamma} \\
&- \tfrac{1}{2}(\sigma_u + \gamma_u)\Big[\|\mathbf{w}_h^{(k)} \cdot \mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}\mathbf{w}_h^{(k)}\|^2_{\frac{1}{2},h,\Gamma}\Big] - \tfrac{\sigma_p}{2}\|\rho_h^{(k)}\|^2_{\frac{1}{2},h,\Gamma}.
\end{aligned}
$$

Then, summing up on k and applying Lemma 10 we conclude that,

$$
\begin{aligned}
&\left(C_{pos} - 2C_{\mathcal{R}} - C_{\mathcal{T}_2}\right)\sum_{k=0}^{M}\Big[|||\mathbf{w}_h^{(k)}|||^2 + \gamma_p\|[\![\rho_h^{(k)}]\!]\|^2_{-\frac{1}{2},h,\Omega} + \left(\tfrac{\sigma_p}{2} - C_{\mathcal{S}}^p\right)\|\rho_h^{(k)} - \rho_h^{(k-1)}\|^2_{\frac{1}{2},h,\Gamma} \\
&+ \left(\tfrac{\sigma_u}{2} + \tfrac{\gamma_u}{2} - C_{\mathcal{S}}^u\right)\Big(\|(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}) \cdot \mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)})\|^2_{\frac{1}{2},h,\Gamma}\Big)\Big] \\
&+ \left(\tfrac{\sigma_u}{2} + \tfrac{\gamma_u}{2} - C_{\mathcal{T}_1}\right)\Big[\|\mathbf{w}_h^M \cdot \mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}\mathbf{w}_h^M\|^2_{\frac{1}{2},h,\Gamma}\Big] + \left(\tfrac{\sigma_p}{2} - C_{\mathcal{T}_1}\right)\|\rho_h{}^M\|^2_{\frac{1}{2},h,\Gamma} \\
&\leq \left(C_{\mathcal{R}} + C_{\mathcal{T}_2}\right)|||\mathbf{w}_h^0|||^2 + \left(\tfrac{\sigma_p}{2} + C_{\mathcal{T}_1}\right)\|\rho_h{}^0\|^2_{\frac{1}{2},h,\Gamma} \\
&+ \left(\tfrac{\sigma_u}{2} + \tfrac{\gamma_u}{2} + C_{\mathcal{T}_1}\right)\Big[\|\mathbf{w}_h^0 \cdot \mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\{\nu\}_w^{\frac{1}{2}}\mathbf{w}_h^0\|^2_{\frac{1}{2},h,\Gamma}\Big],
\end{aligned}
$$

which directly implies (64) by choosing σ_u and σ_p large enough and taking the limit for $M \to \infty$. □

Theorem 6.1 is only partially satisfactory since it proves that the iterative method generates a sequence of velocity approximations that converges to the solution of the global problem (8). In order to recover the convergence of the pressure, we exploit the *inf-sup* stability of the discrete problem, as stated by the following result and in the corresponding proof.

Theorem 6.2 *Under the assumptions of Theorem 6.1 we have,*

$$
\sum_{k=1}^{\infty}\|\rho_h^{(k)}\|^2_{0,\Omega} \lesssim |||(\mathbf{w}_h^{(0)}, \rho_h^{(0)})|||^2. \tag{65}
$$

Proof. Let $\mathbf{z}_{p,h}^{(k)} \in \mathbf{W}_h$ be the function corresponding to $\rho_h^{(k)} \in Q_h$ with respect to Lemma 7. Let us choose $(\mathbf{v}_h, q_h) = (\mathbf{z}_{p,h}^{(k)}, 0)$ into equation (55),

$$
\begin{aligned}
\mathcal{C}((\mathbf{w}_h^{(k)}, \rho_h^{(k)}), (\mathbf{z}_{p,h}^{(k)}, 0)) &+ \mathcal{S}_{\sigma_u}((\mathbf{w}_h^{(k)}; \mathbf{w}_h^{(k-1)}), \mathbf{z}_{p,h}^{(k)}) \\
&= \mathcal{R}((\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}, \rho_h^{(k)} - \rho_h^{(k-1)}), (\mathbf{z}_{p,h}^{(k)}, 0))
\end{aligned} \tag{66}
$$

We proceed similarly to the proof of Theorem 4.1 and we observe that,

$$\mathcal{C}((\mathbf{w}_h^{(k)},\rho_h^{(k)}),(\mathbf{z}_{p,h}^{(k)},0)) = \mathcal{A}((\mathbf{w}_h^{(k)},\rho_h^{(k)}),(\mathbf{z}_{p,h}^{(k)},0)) + j_u(\mathbf{w}_h^{(k)},\mathbf{z}_{p,h}^{(k)}) + \mathcal{B}(\rho_h^{(k)},\mathbf{z}_{p,h}^{(k)}),$$

where owing to Lemma 2 and Lemma 7 we get,

$$\mathcal{A}((\mathbf{w}_h^{(k)},\rho_h^{(k)}),(\mathbf{z}_{p,h}^{(k)},0)) \lesssim \frac{1}{\epsilon}|||\mathbf{w}_h^{(k)}|||^2 + \epsilon\|\rho_h^{(k)}\|_{0,\Omega}^2, \tag{67}$$

$$\mathcal{B}(\rho_h^{(k)},\mathbf{z}_{p,h}^{(k)}) \gtrsim \|\rho_h^{(k)}\|_{0,\Omega}^2 - C\|[\![\rho_h^{(k)}]\!]\|_{-\frac{1}{2},h,\Omega}^2. \tag{68}$$

Furthermore, by virtue of (30), we obtain the following estimates,

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}\{\nu\}_w\big(\mathbf{z}_{p,h,i}^{(k)}\big)^2 \lesssim \|\mathbf{z}_{p,h}^{(k)}\|_{1,\Omega}^2 \lesssim \|\rho_h^{(k)}\|_{0,\Omega}^2,$$

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}\big(\mathbf{z}_{p,h,i}^{(k)}\cdot\mathbf{n}_i\big)^2 \lesssim \|\mathbf{z}_{p,h}^{(k)}\|_{1,\Omega}^2 \lesssim \|\rho_h^{(k)}\|_{0,\Omega}^2,$$

where the constants of the previous inequalities may depend on ν, but remain bounded in the limit case $\nu\to 0$. Then, the relaxation term can be estimated as follows,

$$\begin{aligned}
&\mathcal{S}_{\sigma_u}((\mathbf{w}_h^{(k)};\mathbf{w}_h^{(k-1)}),\mathbf{z}_{p,h}^{(k)}) \qquad (69)\\
&= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} h_E^{-1}\sigma_u\Big[(\mathbf{w}_{h,i}^{(k)}-\mathbf{w}_{h,i}^{(k-1)})\cdot\mathbf{n}_i(\mathbf{z}_{p,h,i}^{(k)}\cdot\mathbf{n}_i) + \{\nu\}_w(\mathbf{w}_{h,i}^{(k)}-\mathbf{w}_{h,i}^{(k-1)})\cdot\mathbf{z}_{p,h,i}^{(k)}\Big]\\
&\lesssim \epsilon\sigma_u h^{-1}\|\rho_h^{(k)}\|_{0,\Omega}^2 + \frac{\sigma_u}{\epsilon}\Big[\|\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\cdot\mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2 + \|\{\nu\}_w^{\frac{1}{2}}\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\|_{\frac{1}{2},h,\Gamma}^2\Big],
\end{aligned}$$

where we have put into evidence the global mesh size, h, because the mesh is assumed to be quasi-uniform. The right hand side of (66) is given by,

$$\begin{aligned}
&\mathcal{R}((\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)},\rho_h^{(k)}-\rho_h^{(k-1)}),(\mathbf{z}_{p,h}^{(k)},0))\\
&= \sum_{i=1}^{N}\Big[\tilde{c}_i^c((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h,i}^{(k)}) + \tilde{c}_i^t((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h}^{(k)})\\
&+ \tilde{c}_i^p((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h}^{(k)}) + \tilde{j}_{u,i}((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h}^{(k)}) + \tilde{d}_i^p((0;\rho_{h,\mathbf{j}}^{(k)}-\rho_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h}^{(k)})\Big].
\end{aligned}$$

To estimate the terms on the right hand side we proceed as follows,

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\tilde{c}_i^c((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h,i}^{(k)})\\
&= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{\Gamma_{ij}}\frac{1}{2}\Big[-\{\nu\}_w\epsilon(\mathbf{w}_{h,j}^{(k)}-\mathbf{w}_{h,j}^{(k-1)})\mathbf{n}_i\cdot\mathbf{z}_{p,h,i}^{(k)} + \{\nu\}_w\epsilon(\mathbf{z}_{p,h,i}^{(k)})\mathbf{n}_i\cdot\big(\mathbf{w}_{h,j}^{(k)}-\mathbf{w}_{h,j}^{(k-1)}\big)\Big]\\
&\lesssim \epsilon h^{-1}\|\rho_h^{(k)}\|_{0,\Omega}^2 + \frac{1}{\epsilon}\Big[|||\mathbf{w}_h^{(k)}|||^2 + |||\mathbf{w}_h^{(k-1)}|||^2 + \|\{\nu\}_w^{\frac{1}{2}}\big(\mathbf{w}_{h,i}^{(k)}-\mathbf{w}_{h,i}^{(k-1)}\big)\|_{\frac{1}{2},h,\Gamma}^2\Big].
\end{aligned}$$

Proceeding as in (59), we obtain,

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\tilde{c}_i^t((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h,i}^{(k)})\lesssim\epsilon\|\rho_h^{(k)}\|_{0,\Omega}^2+\frac{1}{\epsilon}\|\{\nu\}_w^{\frac{1}{2}}\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\|_{\frac{1}{2},h,\Gamma}^2,$$

while for the penalty terms we have,

$$\sum_{i=1}^{N}\tilde{c}_i^p((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h,i}^{(k)})+\tilde{j}_{u,i}((0;\mathbf{w}_{h,\mathbf{j}}^{(k)}-\mathbf{w}_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h,i}^{(k)})$$
$$\lesssim\epsilon h^{-1}\|\rho_h^{(k)}\|_{0,\Omega}^2+\frac{\gamma_u}{\epsilon}\|\{\nu\}_w^{\frac{1}{2}}\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\|_{\frac{1}{2},h,\Gamma}^2+\frac{\gamma_u}{\epsilon}\|\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\cdot\mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2.$$

Finally, the last term of $\mathcal{R}((\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)},\rho_h^{(k)}-\rho_h^{(k-1)}),(\mathbf{z}_{p,h}^{(k)},0))$ is treated as follows,

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\tilde{d}_i^p((0;\rho_{h,\mathbf{j}}^{(k)}-\rho_{h,\mathbf{j}}^{(k-1)}),\mathbf{z}_{p,h}^{(k)})\lesssim\epsilon\|\mathbf{z}_{p,h}^{(k)}\cdot\mathbf{n}_\Gamma\|_{-\frac{1}{2},h,\Gamma}^2+\frac{1}{\epsilon}\|\rho_h^{(k)}-\rho_h^{(k-1)}\|_{+\frac{1}{2},h,\Gamma}$$
$$\lesssim\epsilon|||\mathbf{z}_{p,h}^{(k)}|||^2+\frac{1}{\epsilon}\|\rho_h^{(k)}-\rho_h^{(k-1)}\|_{\frac{1}{2},h,\Gamma}\lesssim\epsilon\|\rho_h^{(k)}\|_{0,\Omega}^2+\frac{1}{\epsilon}\|\rho_h^{(k)}-\rho_h^{(k-1)}\|_{\frac{1}{2},h,\Gamma}.$$

Combining the previous inequalities with $(30)_b$, we obtain the following upper bound,

$$\begin{aligned}\mathcal{R}((\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)},\rho_h^{(k)}-\rho_h^{(k-1)}),(\mathbf{z}_{p,h}^{(k)},0))&\lesssim\epsilon(1+h^{-1})\|\rho_h^{(k)}\|_{0,\Omega}^2\\&+\frac{1}{\epsilon}\Big[|||\mathbf{w}_h^{(k)}|||^2+|||\mathbf{w}_h^{(k-1)}|||^2+\|\rho_h^{(k)}-\rho_h^{(k-1)}\|_{\frac{1}{2},h,\Gamma}^2\\+\|\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\cdot\mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2&+\|\{\nu\}_w^{\frac{1}{2}}\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\|_{\frac{1}{2},h,\Gamma}^2\Big].\end{aligned}\quad(70)$$

Then, replacing (67)-(70) into (66) and exploiting (56) we obtain,

$$\begin{aligned}&\big(1-\epsilon(1+h^{-1})\big)\|\rho_h^{(k)}\|_{0,\Omega}^2\\&\lesssim C\|[\![\rho_h^{(k)}]\!]\|_{-\frac{1}{2},h,\Omega}^2+\frac{1}{\epsilon}\Big[\|\rho_h^{(k)}-\rho_h^{(k-1)}\|_{\frac{1}{2},h,\Gamma}^2+|||\mathbf{w}_h^{(k)}|||^2+|||\mathbf{w}_h^{(k-1)}|||^2\\&\qquad+\|\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\cdot\mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2+\|\{\nu\}_w^{\frac{1}{2}}\big(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)}\big)\|_{\frac{1}{2},h,\Gamma}^2\Big].\end{aligned}$$

Summing up over k we easily obtain,

$$\begin{aligned}\sum_{k=1}^{M}\|\rho_h^{(k)}\|_{0,\Omega}^2&\lesssim\sum_{k=1}^{M}\Big[\|[\![\rho_h^{(k)}]\!]\|_{-\frac{1}{2},h,\Omega}^2+\|\rho_h^{(k)}-\rho_h^{(k-1)}\|_{\frac{1}{2},h,\Gamma}^2+\|(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)})\cdot\mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2\\&+\|\{\nu\}_w^{\frac{1}{2}}(\mathbf{w}_h^{(k)}-\mathbf{w}_h^{(k-1)})\|_{\frac{1}{2},h,\Gamma}^2+2|||\mathbf{w}_h^{(k)}|||^2\Big]+|||\mathbf{w}_h^{(0)}|||^2\lesssim|||(\mathbf{w}_h^{(0)},\rho_h^{(0)})|||^2.\end{aligned}\quad(71)$$

Taking the limit $M\to\infty$ and applying (64) the result follows immediately, provided that ϵ is small enough. We notice that (65) would not hold true without the control on the pressure terms given by (64). □

To prove that the iterative splitting method converges in the norm $|||(\mathbf{v}_h, q_h)|||$, we finally address the divergence of the velocity.

Theorem 6.3 *Under the assumptions of Theorem 6.1 we have,*

$$\sum_{k=1}^{\infty} \|\nabla \cdot \mathbf{w}_h^{(k)}\|_{0,\Omega}^2 \lesssim |||(\mathbf{w}_h^{(0)}, \rho_h^{(0)})|||^2. \tag{72}$$

Proof. Let us choose $(\mathbf{v}_h, q_h) = (0, \nabla \cdot \mathbf{w}_h^{(k)})$ into equation (55),

$$\begin{aligned}\mathcal{C}((\mathbf{w}_h^{(k)}, \rho_h^{(k)}), (0, \nabla \cdot \mathbf{w}_h^{(k)})) + \mathcal{S}_{\sigma_p}((\rho_h^{(k)}; \rho_h^{(k-1)}), \nabla \cdot \mathbf{w}_h^{(k)})& \\ = \mathcal{R}((\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}, \rho_h^{(k)} - \rho_h^{(k-1)}), (0, \nabla \cdot \mathbf{w}_h^{(k)})).&\end{aligned} \tag{73}$$

We proceed providing suitable estimates for each term of the previous equality. First, mimicking (38) we obtain,

$$\begin{aligned}\mathcal{C}((\mathbf{w}_h^{(k)}, \rho_h^{(k)}), (0, \nabla \cdot \mathbf{w}_h^{(k)})) \gtrsim (1 - C\epsilon)\|\nabla \cdot \mathbf{w}_h^{(k)}\|_{0,\Omega}^2& \\ - \frac{1}{\epsilon}\Big[\|[\![\rho_h^{(k)}]\!]\|_{-\frac{1}{2},h,\Omega}^2 + \|[\![\mathbf{w}_h^{(k)} \cdot \mathbf{n}_\Gamma]\!]\|_{\frac{1}{2},h,\Gamma}^2 + \|\mathbf{w}_h^{(k)} \cdot \mathbf{n}\|_{\frac{1}{2},h,\partial\Omega}^2\Big].&\end{aligned} \tag{74}$$

Concerning the second term on the left hand side we obtain,

$$\begin{aligned}\mathcal{S}_{\sigma_p}((\rho_h^{(k)}; \rho_h^{(k-1)}), \nabla \cdot \mathbf{w}_h^{(k)}) &= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \int_{\mathcal{G}_{h,ij}} \sigma_{p,i} h_E^{-1} (\rho_{h,i}^{(k)} - \rho_{h,i}^{(k-1)}) \nabla \cdot \mathbf{w}_{h,i}^{(k)} \\ &\lesssim h^{-2}\epsilon\|\nabla \cdot \mathbf{w}_h^{(k)}\|_{0,\Omega}^2 + \frac{1}{\epsilon}\|\rho_h^{(k)} - \rho_h^{(k-1)}\|_{\frac{1}{2},h,\Gamma}^2.\end{aligned} \tag{75}$$

Finally, for the right hand side we have,

$$\begin{aligned}&\mathcal{R}((\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}, \rho_h^{(k)} - \rho_h^{(k-1)}), (0, \nabla \cdot \mathbf{w}_h^{(k)})) \\ &= -\sum_{i=1}^{N} \tilde{d}_i^u((0; \mathbf{w}_{h,\mathbf{j}}^{(k)} - \mathbf{w}_{h,\mathbf{j}}^{(k-1)}), \nabla \cdot \mathbf{w}_h^{(k)}) \\ &= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \int_{\Gamma_{ij}} (\mathbf{w}_{h,\mathbf{j}}^{(k)} - \mathbf{w}_{h,\mathbf{j}}^{(k-1)}) \cdot \mathbf{n}_i w_i \nabla \cdot \mathbf{w}_{h,i}^{(k)} \\ &\lesssim \epsilon\|\nabla \cdot \mathbf{w}_h^{(k)}\|_{0,\Omega}^2 + \frac{1}{\epsilon}\|(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}) \cdot \mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2.\end{aligned} \tag{76}$$

Combining (73)-(76), we easily obtain,

$$\begin{aligned}\big(1 - C\epsilon(2 + h^{-2})\big)\|\nabla \cdot \mathbf{w}_h^{(k)}\|_{0,\Omega}^2 \lesssim \frac{1}{\epsilon}\Big[|||\mathbf{w}_h^{(k)}|||^2& \\ + \|(\mathbf{w}_h^{(k)} - \mathbf{w}_h^{(k-1)}) \cdot \mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2 + \|\rho_h^{(k)} - \rho_h^{(k-1)}\|_{\frac{1}{2},h,\Gamma}^2\Big].&\end{aligned}$$

To conclude, we sum up from $k = 1, \ldots, M$ and take the limit for $M \to \infty$. Choosing ϵ small enough, the result easily follows owing to (64). □

Finally, combining Theorems 6.1, 6.2 and 6.3, it is straightforward to prove the following result.

Lemma 1 *Under the assumptions of Theorem 6.1 we have,*

$$\sum_{k=1}^{\infty} |||(\mathbf{w}_h^{(k)}, \rho_h^{(k)})|||^2 \lesssim |||(\mathbf{w}_h^{(0)}, \rho_h^{(0)})|||^2.$$

7 Numerical results

We present convergence tests for the approximate solutions computed through the iterative splitting method introduced in section 6. To validate the method for different couplings, we consider three cases (see [8]): a Stokes-Stokes problem (P_{SS}) a Darcy-Darcy problem (P_{DD}) and a Darcy-Stokes problem (P_{DS}). They all are bi-domain problems in $\mathbb{R}^2$, such that an exact solution $(\mathbf{u}, p)$ is known, as summarized below:

(P_{SS}) $\Omega_1 = [0, \frac{2}{3}] \times [0, 1]$, $\Omega_2 = [\frac{2}{3}, 1] \times [0, 1]$, $\nu_1 = 1$, $\nu_2 = 1$, $\eta_1 = 0$, $\eta_2 = 0$,

$$\mathbf{u} = (20xy^3, 5x^4 - 5y^4), \quad p = 60x^2y - 20y^3 - 5.$$

(P_{DD}) $\Omega_1 = [0, \frac{2}{3}] \times [0, 1]$, $\Omega_2 = [\frac{2}{3}, 1] \times [0, 1]$, $\nu_1 = 0$, $\nu_2 = 0$, $\eta_1 = 1$, $\eta_2 = 1$,

$$\mathbf{u} = -2\pi(\cos 2\pi x \sin 2\pi y, \sin 2\pi x \cos 2\pi y), \quad p = \sin 2\pi x \sin 2\pi y.$$

(P_{DS}) $\Omega_1 = [0, 1] \times [0, 1]$, $\Omega_2 = [2, 3] \times [0, 1]$, $\nu_1 = 0$, $\nu_2 = 1$, $\eta_1 = 1$, $\eta_2 = 0$,

$$\mathbf{u} = \begin{cases} (1 - 2x + x^2 + y - y^2, x + 2y - 2xy - 2) & \text{on } \Omega_1, \\ (y - y^2, 0) & \text{on } \Omega_2, \end{cases}$$

$$p = \begin{cases} \frac{1}{3}(y - y^2 - xy + xy^2 - x + x^2 - \frac{1}{3}x^3) + \frac{29}{18} & \text{on } \Omega_1, \\ -2x + \frac{59}{18} & \text{on } \Omega_2. \end{cases}$$

For each test case, non-homogeneous boundary data for the velocity are provided by the their known analytical expressions and the Beavers-Joseph-Saffman term is set to zero. We also point out that the solution of P_{DD} is not divergence-free. Accordingly, we include a source term equal to $\nabla \cdot \mathbf{u}$ in the divergence equation. We apply our method for different meshes and for $r = 1, 2$. As regards the user-defined parameters, numerical experiments show that small values of σ_u and σ_p are sufficient to ensure the convergence of the iterative splitting algorithm, while γ_u and γ_p are chosen of the order of unity, see [10, 8, 9, 11, 26]. In particular, we set $\gamma_u = 2$, $\sigma_u = \sigma_p = 2 \times 10^{-3}$ and $\gamma_p = 2 \times 10^{-1}$ for P_{SS} and P_{DS} while $\gamma_p = 2$ for P_{DD}. According to Corollary 1, the iterative splitting algorithm is complemented with the following stopping test $I_h^k := |||(\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}, p^{(k)} - p^{(k-1)}||| \leq tol$, the tolerance being $tol = 10^{-8}$ in all cases.

First, we study the error $E_h := |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||$ of the approximate solution with respect to the exact one. As reported in Table 1, in all cases the error E_h shows a convergence rate which is very close to the expected value, according to Theorem 4.3. In agreement with the analysis pursued in [11], these data also confirm that the asymptotic rate of convergence of the iterative splitting method decreases with respect to h. Since the convergence rate is not constant during the first phase of the iterative process, the asymptotic rate is quantified by the number of iterations necessary to reduce the error from 10^{-7} to 10^{-8}.

Table 1: Approximation error and convergence rate of the iterative method.

P_{SS}	$r = 1$			$r = 2$		
h	E_h	It. $\to 10^{-8}$	It. $10^{-7} \to 10^{-8}$	E_h	It. $\to 10^{-8}$	It. $10^{-7} \to 10^{-8}$
1/8	7.24136	163	19	0.234778	167	19
1/16	2.82873	109	13	0.051305	181	23
1/32	1.29432	111	14	0.012628	353	308
Rate	1.18			2.15		

P_{DD}	$r = 1$			$r = 2$		
h	E_h	It. $\to 10^{-8}$	It. $10^{-7} \to 10^{-8}$	E_h	It. $\to 10^{-8}$	It. $10^{-7} \to 10^{-8}$
1/8	11.3186	367	62	1.83164	1393	294
1/16	5.60897	482	81	0.419515	4091	1194
1/32	2.77365	1209	229	0.101841	10615	4218
Rate	1.01			2.12		

P_{DS}	$r = 1$			$r = 2$		
h	E_h	It. $\to 10^{-8}$	It. $10^{-7} \to 10^{-8}$	E_h	It. $\to 10^{-8}$	It. $10^{-7} \to 10^{-8}$
1/8	0.281895	100	19	0.0012469	155	25
1/16	0.130674	95	14	0.0003184	283	45
1/32	0.063188	169	25	0.0000805	519	85
Rate	1.10			1.95		

Second, we aim to test the method on a more realistic case than problems (P_{SS}), (P_{DD}), (P_{DS}). To this purpose we simulate the fluid flow through a heterogeneous material. We consider a 10 m square as in Fig. 1, featuring both open channels (Stokes' flow) and porous blocks (Darcy's and Brinkman's flows). In particular, the block associated to Brinkman's flow is more permeable than the Darcy's one. As boundary conditions we consider Dirichlet velocity data, with a nonzero parabolic inflow/outflow in the Stokes flow channels only. The data used for the simulation are reported in the caption of Fig. 1. In particular, the fluid viscosity is close to that of water, and the values for η correspond to soil hydraulic conductivities in the range of $0.02 - 0.002$ ms^{-1}, characteristic of gravel/sand. The solution was computed using $N = 9$ (square) subdomains Ω_i, $i = 1, .., N$, a mesh size $h = 0.4$ m (that is about 24 times smaller than the size

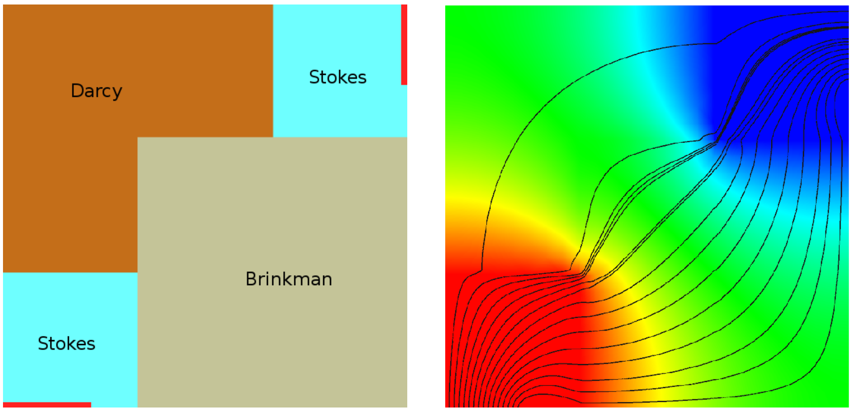

Figure 1: Flow in a heterogeneous material. On the left: different zones of the heterogeneous material have different values of ν and η (Stokes: $\nu = 10^{-5}$ $\mathrm{m^2 s^{-1}}$, $\eta = 0$ $\mathrm{s^{-1}}$; Brinkman: $\nu = 10^{-5}$ $\mathrm{m^2 s^{-1}}$, $\eta = 50$ $\mathrm{s^{-1}}$; Darcy: $\nu = 10^{-5}$ $\mathrm{m^2 s^{-1}}$, $\eta = 500$ $\mathrm{s^{-1}}$). The Beavers-Joseph-Saffman friction coefficient is $\kappa_\Gamma = 1$ Pa / ($\mathrm{ms^{-1}}$). The length of the square is 10 m, inflow and outflow boundaries are highlighted in red and the peak of the parabolic velocity profile is is 5 cm $\mathrm{s^{-1}}$. On the right: a superposition of the pressure and the pathlines of the flow.

of the domain Ω), and $r = 2$ ($\mathbb{P}^2$ finite elements for the velocity). As shown by Fig. 1, most of the injected fluid passes through the more permeable block to reach the outlet. We also notice that the pressure contours follow the block structure of the different materials. According to [18] this confirms the correct behavior of the method.

8 Concluding remarks

Exploiting weighted interior penalty terms, we have proposed a unified mixed stabilized method for incompressible, possibly heterogeneous flow problems. We have proved stability and optimal order error estimates for smooth solutions. Finally, we have shown that the multidomain discrete solution can be computed by means of a Jacobi-type parallel iterative procedure that is convergent for any possible coupling between subdomains. The theoretical predictions have been verified on a series of numerical examples.

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