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A WEIGHTED REDUCED BASIS METHOD FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH RANDOM INPUT DATA

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Abstract: In this work we propose and analyze a weighted reduced basis method to solve elliptic partial differential equation (PDE) with random input data. The PDE is first transformed into a weighted parametric elliptic problem depending on a finite number of parameters. Distinctive importance at different values of the parameters are taken into account by assigning different weight to the samples in the greedy sampling procedure. A priori convergence analysis is carried out by constructive approximation of the exact solution with respect to the weighted parameters. Numerical examples are provided for the assessment of the advantages of the proposed method over the reduced basis method and stochastic collocation method in both univariate and multivariate stochastic problems.

Keywords: weighted reduced basis method, stochastic partial differential equation, uncertainty quantification, stochastic collocation method, Kolmogorov N-width, exponential convergence

1 Introduction

When modelling complex physical system, uncertainties inevitably arise from various sources, e.g. computational geometries, physical parameters, external forces, initial or boundary conditions, and may significantly impact on the computational results. When these uncertainties are incorporated into the underlying physical system, we are facing stochastic problems or uncertainty quantification. Various computational methods have been developed depending on the structure of the stochastic problem, including perturbation, Monte Carlo, stochastic Galerkin, stochastic collocation, reduced basis, generalized spectral decomposition methods [21, 41, 1, 33, 7].

The perturbation method [25] based on Taylor expansion was developed for the random functions with only small fluctuation around a deterministic expectation. This method is only applicable when dealing with small uncertainties and suffer from inevitable errors and extremely complicated structure for high order expansions. The most commonly used "brute-force" Monte-Carlo method [20] as well as its multiple versions, e.g. quasi Monte Carlo [30], multi-level Monte Carlo [23], converge very slowly and become prohibitive for achieving accurate results.

Stochastic Galerkin method, originated from spectral expansion of the random functions on some polynomial chaos, for instance Hermite polynomials of independent random variables, applies the Galerkin approaches to approximate the solution in both stochastic and deterministic space [21, 2]. It enjoys fast convergence provided the solution is regular [14, 13]. However, it yields a very large

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algebraic system, leading to the challenge of designing efficient solvers with appropriate preconditioners [19].

Stochastic collocation method was developed from the non-intrusive deterministic collocation method [36, 40, 1]. In principle, it employs multivariate polynomial interpolations for the integral in the variational formulation of the stochastic system with respect to probability space rather than the Galerkin approximation in the spectral polynomial space. Due to the heavy computation of a deterministic system at each collocation point in high dimensional space, isotropic or anisotropic sparse grids with suitable cubature rules [31, 32] were analyzed and applied to reduce the computation load. This method is preferred for more practical applications because it features the advantages of both direct computation as Monte Carlo method and fast convergence as stochastic Galerkin method [3].

In principle, to solve a stochastic problem we need to solve one deterministic problem at many different realizations of the random inputs in order to evaluate the quantity of interest depending on the stochastic solutions. However, the solutions are "not far from" each other in practice. Therefore, instead of projecting the solutions on some prescribed bases, such as polynomial chaos for stochastic Galerkin method [41], we can project the solution on some space generated by a few precomputed solutions, which leads to the development of reduced basis method. The reduced basis method has been proposed to solve primarily parametric systems [37, 34] and applied to stochastic problems lately [7, 6, 10]. In the later context, it regards the random variables as parameters and select the most representative points in the parameter space by greedy sampling based on a posteriori error estimation. The essential idea for deterministic and stochastic reduced basis method is to separate the whole procedure into an offline stage and an online stage. During the former, the large computational ingredients are computed and stored once and for all, including sampling parameters, assembling matrices and vectors, solving and collecting snapshots of solutions, etc. In the online stage, only the parameter related elements are left to be computed and a small Galerkin approximation problem needs to be solved [34]. Both reduced basis method and stochastic collocation method use precomputed solutions as approximation/construction bases. However the former employs a posteriori error estimation for the construction, and thus is more efficient provided that a posteriori error estimation is easy to compute. Comparison of convergence property as well as computational cost for offline construction and online evaluation between the reduced basis method and stochastic collocation method was investigated in [10].

At our best knowledge, the reduced basis method is currently only used for stochastic problems with uniformly distributed random inputs or parameter space with Lebesgue measure [6, 10]. In order to deal with more general stochastic problems with other distributed random inputs, we propose and analyze a new version of reduced basis method and name it "weighted reduced basis method". The basic idea is to suitably assign a larger weight to samples that are more important or have a higher probability to occur than the others according to either the probability distribution function or some other available weight function depending on the specific application at hand. The benefit is to lighten the reduced space construction using a smaller number of bases without lowering the numerical accuracy.

A priori convergence analysis for reduced basis method by greedy algorithm has been carried out in previous works [28, 8, 5, 26] under various assumptions. More specifically, exponential convergence rate for single-parameter elliptic PDE was obtained in [28] by exploring an eigenvalue problem; algebraic or exponential convergence rate for greedy algorithm in multidimensional problem was achieved implicitly depending on the convergence rate of Kolmogorov N-width in [8] and improved in [5]; exponential convergence rate was also recently obtained in [39] through direct expansion of the solution on a series of invertible elliptic operators. In this work, we carry out a priori convergence analysis of our weighted reduced basis method based on constructive spectral approximation for analytic functions, which is different from [28, 8, 5, 26].

The paper is organized as follows: an elliptic PDE with random input data is set up with appropriate assumptions on both the random coefficient and forcing term in section 2. Section 3 is devoted to the development of the weighted reduced basis method consisting of greedy algorithm, a posteriori error estimate as well as offline-online computational decomposition, which is followed by a priori convergence analysis in section 4. Numerical examples for both one dimensional problem and multiple dimensional problem are presented as verification of the efficiency and convergence properties in section 5. Some brief concluding remarks are drawn in the last section 6.

2 Problem setting

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where Ω is a set of outcomes $\omega \in \Omega$, $\mathcal{F}$ is σ-algebra of events and $P : \mathcal{F} \to [0,1]$ with $P(\Omega) = 1$ assigns probability to the events. Let D be a convex, open and bounded physical domain in $\mathbb{R}^d$ $(d = 1,2,3)$ with Lipschitz continuous boundary ∂D. We consider the following stochastic elliptic problem: find $u : \bar{D} \times \Omega \to \mathbb{R}$ such that it holds almost surely

$$
\begin{aligned}
-\nabla \cdot (a(\cdot,\omega)\nabla u(\cdot,\omega)) = f(\cdot,\omega) \quad \text{in } D, \\
u(\cdot,\omega) = 0 \quad \text{on } \partial D,
\end{aligned}
\tag{2.1}
$$

where $f : D\times\Omega \to \mathbb{R}$ is a random force term and $a : D\times\Omega \to \mathbb{R}$ is a random coefficient; a homogeneous Dirichlet boundary condition is prescribed on the whole boundary ∂D for simplicity. We consider the following assumptions for the random functions $f(\cdot,\omega)$ and $a(\cdot,\omega)$:

Assumption 1 *The random forcing term $f(\cdot,\omega)$ is square integrable with respect to P, i.e.*

$$
\int_{\Omega\times D} f^2(x,\omega)dxdP(\omega) < \infty. \tag{2.2}
$$

Assumption 2 *The random coefficient $a(\cdot,\omega)$ is assumed to be uniformly bounded from below and from above, i.e. there exist constants $0 < a_{min} < a_{max} < \infty$ such that*

$$
P(\omega \in \Omega : a_{min} < a(x,\omega) < a_{max} \quad \forall x \in \bar{D}) = 1. \tag{2.3}
$$

We introduce the Hilbert space $V := L^2_P(\Omega) \otimes H^1_0(D)$ and equip it with the following norm

$$
||v||_V = ||v||_{L^2_P(\Omega)\otimes H^1_0(D)} = \left(\int_{\Omega\times D} |\nabla v|^2 dxdP\right)^{1/2} < \infty. \tag{2.4}
$$

The weak formulation of problem (2.1) is stated as: find $u \in V$ such that

$$
\int_{\Omega\times D} a\nabla u \cdot \nabla v dxdP = \int_{\Omega\times D} fvdxdP \quad \forall v \in V. \tag{2.5}
$$

The existence of a unique solution to problem (2.5) is guaranteed by Lax-Milgram theorem [36] under Assumption 1 and Assumption 2 and the stability inequality holds for the solution straightforwardly

$$
||u||_V \le \frac{C_P}{a_{min}}||f||_V, \tag{2.6}
$$

where the constant C_P comes from the Poincaré inequality $||v||_{L^2(D)} \le C_P||\nabla v||_{L^2(D)} \quad \forall v \in H^1_0(D)$.

The uncertainty of the random functions $a(\cdot,\omega)$ and $f(\cdot,\omega)$, in many practical applications, can be approximately projected to a series of finite dimensional random variables via statistical techniques. For instance, finite linear regression models are widely used to approximate various random fields [15]; under the assumption that the second moment of $a(\cdot,\omega)$ exists, we can apply Karhunen-Loève expansion [38] to the covariance kernel and truncate it up to a finite number of linear terms, etc. For this consideration, we make further assumption to the random functions $a(\cdot,\omega)$ and $f(\cdot,\omega)$ as follows:

Assumption 3 *The random coefficient $a(\cdot,\omega)$ and forcing term $f(\cdot,\omega)$ are linear combinations of a number of random variables $Y(\omega) = (Y_1(\omega),\dots,Y_K(\omega)) : \Omega \to \mathbb{R}^K$ as follows*

$$
a(x,Y) = a_0(x) + \sum_{n=1}^{K} a_k(x)Y_k(\omega) \quad \text{and} \quad f(x,Y) = f_0(x) + \sum_{n=1}^{K} f_k(x)Y_k(\omega) \tag{2.7}
$$

where $a_k \in L^\infty(D)$ and $f_k \in L^2(D)$ for $0 \le k \le K$. More specifically, $\{Y_k\}_{k=1}^K$ are real valued *random variables with joint probability density function $\rho(y)$, being $y = Y(\omega) \in \mathbb{R}$. By denoting $\Gamma_k \equiv Y_k(\Omega), k = 1, \dots, K$ and $\Gamma = \Pi_{k=1}^K \Gamma_k$, we can also view y as a weighted parameter in the parametric domain Γ endowed with the measure $\rho(y)dy$.*

Remark 2.1 *When the random variables $Y_k^a, 1 \le k \le K_a$ for a and $Y_k^f, 1 \le k \le K_f$ for f are not the same, we collect them as $Y = (Y_1^a, \dots, Y_{K_a}^a, Y_1^f, \dots, Y_{K_f}^f)$ and reorder them as $(Y_1, \cdots, Y_K)$ with $K = K_a + K_f$.*

Remark 2.2 *In the more general case that the random function $a(x, Y)$ does not depend on Y linearly, for instance*

$$a(x, Y) = a_0(x) + \exp\left(\sum_{n=1}^{K} a_k(x) Y_k(\omega)\right), \tag{2.8}$$

one can employ the empirical interpolation method [4, 12] to approximate (2.8) with finite affine terms in the form

$$a(x, Y) \approx a_0(x) + \sum_{n=1}^{K'} a'_{k'}(x) \Theta_{k'}(Y(\omega)), \tag{2.9}$$

where $\Theta_{k'}(\cdot), 1 \le k' \le K'$ are functions of Y and can be transformed to random variables $Z_{k'} = \Theta_{k'}(Y(\omega)), 1 \le k' \le K$, resulting in a new random vector $Z = (Z_1, \dots, Z_{K'})$ and $a(x, Z)$ still satisfies Assumption 3.

Under the above assumptions, the weighted parametric weak formulation of the stochastic elliptic problem reads: find $u(y) \in H_0^1(D)$ such that the following equation holds for all $y \in \Gamma$

$$A(u, v; y) = F(v; y) \quad \forall v \in H_0^1(D), \tag{2.10}$$

where $A(\cdot, \cdot; y)$ and $F(\cdot; y)$ are parametrized bilinear and linear forms featuring the following expansion

$$A(u, v; y) = A_0(u, v) + \sum_{k=1}^{K} A_k(u, v) y_k \quad \text{and} \quad F(v; y) = (f_0, v) + \sum_{k=1}^{K} (f_k, v) y_k, \tag{2.11}$$

with the deterministic bilinear forms $A_k(u, v)$ given by $A_k(u, v) := (a_k \nabla u, \nabla v), k = 0, 1, \dots, K$. Because of assumption (2.3) the bilinear form is coercive and continuous, thus the existence of a unique parametric solution $u(y) \in H_0^1(D)$ for $\forall y \in \Gamma$ to problem (2.10) is guaranteed by Lax-Milgram theorem [36]. More often, we are interested in a related quantity $s(u; y)$ as output, e.g. the linear functional $F(u; y)$, as well as its statistics, e.g. the expectation $\mathbb{E}[s]$, which is defined as

$$\mathbb{E}[s] = \int_\Gamma s(u; y) \rho(y) dy. \tag{2.12}$$

3 Weighted reduced basis method

The basic idea behind weighted reduced basis method is to assign different weight in the construction of reduced basis space at different values of parameter $y \in \Gamma$ according to a prescribed weight function $w(y)$. The objective is that when the parameter y has distinctive weight $w(y)$ at different values $y \in \Gamma$, e.g. stochastic problems with random inputs obeying probability distribution far from uniform type, the weighted approach can considerably attenuate the computational effort for large scale computational problems. The general paradigm of weighted reduced basis method is formulated by following closely the reduced basis method in [34, 37, 10]:

Given any approximation space $X^{\mathcal{N}}$ (e.g. finite element/spectral approximation space) of dimension $\mathcal{N}$ for the solution $u^{\mathcal{N}}$ of problem (2.10), a training set of parameter samples $\Xi_{train} \subset \Gamma$ as well as a prescribed maximum dimension $N_{max} \ll \mathcal{N}$, we build the N dimensional (Lagrange) reduced basis space $X_N^{\mathcal{N}}$ for $N = 1, \dots, N_{max}$ in a hierarchical way by taking into account the weight of parameter at different values until satisfying certain tolerance requirement. The reduced basis space $X_N^{\mathcal{N}}$ is spanned by the "snapshots" based on suitably chosen samples $S_N = \{y^1, \dots, y^N\}$ from the training set Ξ_{train}

$$X_N^{\mathcal{N}} = \text{span}\{u^{\mathcal{N}}(y^n), 1 \le n \le N\}. \tag{3.1}$$

Note that $X_1^{\mathcal{N}} \subset X_2^{\mathcal{N}} \subset \cdots \subset X_{N_{max}}^{\mathcal{N}}$. In order to evaluate $s(u; y)$ at any new parameter $y \in \Gamma$, we first seek the solution $u_N^{\mathcal{N}} \in X_N^{\mathcal{N}} \subset X^{\mathcal{N}}$ in the reduced basis space $X_N^{\mathcal{N}}$ by solving a reduced system

$$A(u_N^{\mathcal{N}}, v; y) = F(v; y) \quad \forall v \in X_N^{\mathcal{N}} \tag{3.2}$$

and then approximate $s(u; y)$ by $s(u_N^{\mathcal{N}}; y)$. Moreover, we can also compute the statistics of the output, e.g. expectation $\mathbb{E}[s_N^{\mathcal{N}}]$, by numerical quadrature formula (Gauss or Clenshaw-Curtis quadrature [32])

$$\mathbb{E}[s_N^{\mathcal{N}}] \approx \sum_{m=1}^{M} s(u_N^{\mathcal{N}}; y^m) w(y^m), \tag{3.3}$$

where y^m and $w(y^m), m = 1, \dots, M$ are the K dimensional quadrature abscissas and weights with respect to the probability density function, which can be chosen based on different schemes, e.g. full tensor product quadrature or sparse grid quadrature [32]. Note that the weights $w(y^m), m = 1, \dots, M$ may be distinct to each other depending on both the quadrature formula and the probability density function, so that the solution $u_N^{\mathcal{N}}(y^m)$ is expected to be more accurate where $w(y^m)$ is significantly larger than the other realization of the parameter $y \in \Gamma$.

Accurate computation of the solution $u_N^{\mathcal{N}}$ and the output $s_N^{\mathcal{N}}$ depends crucially on the construction of the reduced basis approximation space. More specifically, how to take different weight of the solution into consideration, how to cheaply and accurately select the most representative samples in order to hierarchically build the reduced basis space as well as how to efficiently evaluate the solution and output based on the way of construction of the approximation space play a key role in the weighted reduced basis method. We address these issues in the following three aspects: the weighted greedy algorithm, the a posteriori error estimate and the offline-online computational decomposition.

3.1 Weighted greedy algorithm

The weighted greedy algorithm essentially deals with the $L^\infty(\Gamma; X_w)$ (X_w is a weighted subspace of X to be specified) optimization problem in a greedy way [37], seeking a new parameter $y^N \in \Gamma$ such that

$$y^N = \arg\sup_{y \in \Gamma} ||u^{\mathcal{N}}(y) - P_N u^{\mathcal{N}}(y)||_{X_w}, \tag{3.4}$$

where $P_N : X^{\mathcal{N}} \to X_N^{\mathcal{N}}$ is the Galerkin projection operator. By solving the infinite dimensional problem (3.4) we would locate the least matching point $y^N \in \Gamma$ in $|| \cdot ||_{X_w}$ norm. A computable (finite dimensional) greedy algorithm rely on twofolds: i), replace the parameter domain Γ by a finite training set $\Xi_{train} \subset \Gamma$ with cardinality $|\Xi_{train}| = n_{train} < \infty$; ii), replace the mismatching term $||u^{\mathcal{N}}(y) - P_N u^{\mathcal{N}}(y)||_{X_w}$ by a cheap weighted posteriori error bound $\triangle_N^w$ that should be as sharp as possible, i.e.

$$c_N \triangle_N^w(y) \le ||u^{\mathcal{N}}(y) - P_N u^{\mathcal{N}}(y)||_{X_w} \le C_N \triangle_N^w(y) \tag{3.5}$$

where C_N/c_N is close to 1. We leave the computation of a posteriori error bound to the next section and present the weighted greedy algorithm in the following procedure, see Algorithm 1.

We note that for the sake of efficient computation of Galerkin projection and offline-online decomposition in practice, we normalize the snapshots by Gram-Schmidt process to get the orthonormal basis of $\{\zeta_1^{\mathcal{N}}, \dots, \zeta_N^{\mathcal{N}}\}$ such that $(\zeta_m^{\mathcal{N}}, \zeta_n^{\mathcal{N}})_X = \delta_{mn}$, $1 \le m, n \le N$ and construct $X_N^{\mathcal{N}} = \text{span}\{\zeta_1^{\mathcal{N}}, \dots, \zeta_N^{\mathcal{N}}\}$.

Another algorithm that might be used for the sampling procedure is proper orthogonal decomposition, POD for short [37], which is rather expensive in dealing with $L^2(\Xi_{train}; X)$ optimization and thus more suitable for low dimensional problems. We remark that for both the greedy algorithm and the POD algorithm, an original training set Ξ_{train} is needed. Two criteria ought be followed for

Algorithm 1 A weighted greedy algorithm for the construction of reduced basis approximation space

procedure INITIALIZATION:
  sample the training set $\Xi_{train} \subset \Gamma$ according to probability density function ρ;
  specify a tolerance ε_{tol} as stopping criteria of the algorithm;
  define the maximum number of reduced bases N_{max};
  choose the first sample $y^1 \in \Xi_{train}$ and build the sample space $S_1 = \{y^1\}$;
  solve the problem (2.10) at y^1, construct the reduced basis space $X_1^{\mathcal{N}} = \text{span}\{u^{\mathcal{N}}(y^1)\}$;
end procedure
procedure CONSTRUCTION:
  for $N = 2, \dots, N_{max}$ **do**
    compute a weighted posteriori error bound $\triangle_{N-1}^w(y)$ for $\forall y \in \Xi_{train}$;
    choose the parameter to maximize $\triangle_{N-1}^w$, i.e. $y^N = \arg\max_{y \in \Xi_{train}} \triangle_{N-1}^w(y)$;
    if $\triangle_{N-1}^w(y^N) \leq \varepsilon_{tol}$ **then**
      $N_{max} = N - 1$;
    end if
    solve problem (2.10) at y^N to obtain $u^{\mathcal{N}}(y^N)$;
    augment the sample space $S_N = S_{N-1} \cup \{y^N\}$;
    augment the reduced basis space $X_N^{\mathcal{N}} = X_{N-1}^{\mathcal{N}} \oplus \text{span}\{u^{\mathcal{N}}(y^N)\}$;
  end for
end procedure

its choice: 1, it should be cheap without too many ineffectual samples in order to avoid too much computation with little gain; 2, it should be sufficient to capture the most representative snapshots so as to build an accurate reduced basis space.

Adaptive approaches for building the training set have also been well explored by starting from a small number of samples to more samples in the space Γ adaptively, see [42] for details.

3.2 A posteriori error bound

The efficiency and reliability of the reduced basis approximation by weighted greedy algorithm relies critically on the availability of an inexpensive, sharp and weighted a posteriori error bound $\triangle_N^w$. For every $y \in \Gamma$, let $R(v; y) \in (X^{\mathcal{N}})'$ be the residual in the dual space of $X^{\mathcal{N}}$, which is defined as

$$R(v; y) := F(v; y) - A(u_N^{\mathcal{N}}(y), v; y) \quad \forall v \in X^{\mathcal{N}}. \tag{3.6}$$

By Riesz representation theorem [36], we have a unique function $\hat{e}(y) \in X^{\mathcal{N}}$ such that

$$(\hat{e}(y), v)_{X^{\mathcal{N}}} = R(v; y) \quad \forall v \in X^{\mathcal{N}} \text{ and } ||\hat{e}(y)||_{X^{\mathcal{N}}} = ||R(\cdot; y)||_{(X^{\mathcal{N}})'} \tag{3.7}$$

where the $X^{\mathcal{N}}$-norm could be specified as, e.g. $||v||_{X^{\mathcal{N}}} = A(v, v; \bar{y})$ at some reference value $\bar{y} \in \Gamma$, which is equivalent to $H_0^1(D)$ norm. Define the error between the "truth" solution and the reduced basis solution as $e(y) := u^{\mathcal{N}}(y) - u_N^{\mathcal{N}}(y)$; by (2.10), (3.2) and (3.6) we have the equation

$$A(e(y), v; y) = R(v; y) \quad \forall v \in X^{\mathcal{N}}. \tag{3.8}$$

By choosing $v = e(y)$ in (3.8), recalling the coercivity constant $\alpha(y)$ with the definition of its lower bound $\alpha_{LB}(y) \leq \alpha(y)$ of the bilinear form $A(\cdot, \cdot; y)$, and using Cauchy-Schwarz inequality, we have

$$\alpha_{LB}(y)||e(y)||^2_{X^{\mathcal{N}}} \leq A(e(y), e(y); y) = R(e(y); y) \leq ||R(\cdot, y)||_{(X^{\mathcal{N}})'}||e(y)||_{X^{\mathcal{N}}} = ||\hat{e}(y)||_{X^{\mathcal{N}}}||e(y)||_{X^{\mathcal{N}}}, \tag{3.9}$$

so that we can define a weighted posteriori error bound $\triangle_N^w(y)$ for the solution $u_N^{\mathcal{N}}(y), y \in \Gamma$ as

$$\triangle_N^w(y) := ||\hat{e}(y)||_{X_w}/\alpha_{LB}(y) \tag{3.10}$$

and obtain immediately the relation $||u^{\mathcal{N}}(y) - u_N^{\mathcal{N}}(y)||_{X_w} \leq \triangle_N^w(y)$ from (3.9). As for output $s(u)$,

$$|s(u^{\mathcal{N}}) - s(u_N^{\mathcal{N}})| \leq ||s||_{(X^{\mathcal{N}})'} ||u^{\mathcal{N}}(y) - u_N^{\mathcal{N}}(y)||_{X^{\mathcal{N}}}, \tag{3.11}$$

where $||s||_{(X^{\mathcal{N}})'}$ is a constant independent of y, the same error bound can also be used in the greedy algorithm when considering the output $s_N^{\mathcal{N}}$. The efficient computation of a sharp and accurate a posteriori error bound thus relies on the computation of a lower bound of the coercivity constant $\alpha_{LB}(y)$ as well as the value $||\hat{e}(y)||_{X_w}$ for any given $y \in \Gamma$. For the former, we apply the successive constraint linear optimization method [24] to compute a lower bound $\alpha_{LB}(y)$ close to the "truth" value $\alpha(y)$. For the latter, we turn to an offline-online computational decomposition procedure.

3.3 Offline-online computational decomposition

The evaluation of the expectation $\mathbb{E}[s_N^{\mathcal{N}}]$ and the weighted a posteriori error estimator $\triangle_N^w$ requires to compute the output $s_N^{\mathcal{N}}$ and the solution $u_N^{\mathcal{N}}$ many times. Similar situations can be encountered for other applications in the context of many query (optimal design, control) and real time computational problems. One of the key ingredients that make reduced basis method stand out in this ground is the offline-online computational decomposition, which becomes possible due to the affine or linear assumption such as that made in (2.7). To start, we express the reduced basis solution in the form

$$u_N^{\mathcal{N}}(y) = \sum_{m=1}^{N} u_{Nm}^{\mathcal{N}}(y)\zeta_m^{\mathcal{N}}. \tag{3.12}$$

Upon replacing it in (3.2) and choosing $v = \zeta_n^{\mathcal{N}}, 1 \leq n \leq N$, we obtain

$$\sum_{m=1}^{N} \left(A_0(\zeta_m^{\mathcal{N}}, \zeta_n^{\mathcal{N}}) + \sum_{k=1}^{K} y_k A_k(\zeta_m^{\mathcal{N}}, \zeta_n^{\mathcal{N}}) \right) u_{Nm}^{\mathcal{N}}(y) = (f_0, \zeta_n^{\mathcal{N}}) + \sum_{k=1}^{K} (f_k, \zeta_n^{\mathcal{N}}) y_k \quad 1 \leq n \leq N. \tag{3.13}$$

From (3.13) we can see that the quantities $A_k(\zeta_m^{\mathcal{N}}, \zeta_n^{\mathcal{N}}), 0 \leq k \leq K, 1 \leq m, n \leq N_{max}$ and $(f_k, \zeta_n^{\mathcal{N}}), 0 \leq k \leq K, 1 \leq n \leq N_{max}$ are independent of y, we may thus precompute and store them in the offline procedure. In the online procedure, we only need to assemble the stiffness matrix in (3.13) and solve the resulting $N \times N$ stiffness system with much less computational effort compared to solve a full $\mathcal{N} \times \mathcal{N}$ stiffness system. As for the computation of the error bound $\triangle_N(y)$, we need to evaluate $||\hat{e}(y)||_{X^{\mathcal{N}}}$ at y chosen in the course of sampling procedure. We expand the residual (3.6) as

$$R(v; y) = F(v; y) - A(u_N^{\mathcal{N}}, v; y) = \sum_{k=0}^{K} (f_k, v) y_k - \sum_{n=1}^{N} u_{Nn}^{\mathcal{N}} \left(\sum_{k=0}^{K} A_k(\zeta_n^{\mathcal{N}}, v) y_k \right), \text{ where } y_0 = 1. \tag{3.14}$$

Set $(\mathcal{C}_k, v)_{X^{\mathcal{N}}} = (f_k, v)$ and $(\mathcal{L}_n^k, v)_{X^{\mathcal{N}}} = -A_k(\zeta_n^{\mathcal{N}}, v) \, \forall v \in X_N^{\mathcal{N}}, 1 \leq n \leq N, 0 \leq k \leq K$, where $\mathcal{C}_k$ and $\mathcal{L}_n^k$ are the representatives in $X^{\mathcal{N}}$ of f_k and $\zeta_n^{\mathcal{N}}$, respectively, whose existence is secured by the Riesz representation theorem. By recalling $(\hat{e}(y), v)_{X^{\mathcal{N}}} = R(v; y)$, we obtain

$$\begin{aligned} ||\hat{e}(y)||_{X^{\mathcal{N}}}^2 &= \sum_{k=0}^{K} y_k \left(\sum_{k'=0}^{K} y_{k'} (\mathcal{C}_k, \mathcal{C}_{k'})_{X^{\mathcal{N}}} \right) \\ &\quad + \sum_{k=0}^{K} \sum_{n=1}^{N} y_k u_{Nn}^{\mathcal{N}}(y) \left(\sum_{k'=0}^{K} y_{k'} 2(\mathcal{C}_{k'}, \mathcal{L}_n^k)_{X^{\mathcal{N}}} + \sum_{k'=0}^{K} \sum_{n'=1}^{N} y_{k'} u_{Nn'}^{\mathcal{N}}(y) (\mathcal{L}_n^k, \mathcal{L}_{n'}^{k'})_{X^{\mathcal{N}}} \right). \end{aligned} \tag{3.15}$$

Therefore, we can compute and store $(\mathcal{C}_k, \mathcal{C}_{k'})_{X^{\mathcal{N}}}, (\mathcal{C}_{k'}, \mathcal{L}_n^k)_{X^{\mathcal{N}}}, (\mathcal{L}_n^k, \mathcal{L}_{n'}^{k'})_{X^{\mathcal{N}}}, 1 \leq n, n' \leq N_{max}, 0 \leq k, k' \leq K$ in the offline procedure, and evaluate $||\hat{e}(y)||_{X^{\mathcal{N}}}$ in the online procedure by assembling (3.15) with $O((K+1)^2 N^2)$ scalar products, which is far efficient provided that $O((K+1)^2 N^2) \ll \mathcal{N}$. As for the weighted error bound $||\hat{e}(y)||_{X_w}$, one option is to simply use $||\hat{e}(y)||_{X_w} = w(y) ||\hat{e}(y)||_{X^{\mathcal{N}}}$.

4 A priori convergence analysis

Without loss of generality, we work in the space X rather than in the discretization space $X^{\mathcal{N}}$ (the stochastic convergence results hold the same for both spaces) and define the Hilbert space $C_w^0(\Gamma; X)$ equipped with the following norm

$$||v||_{C_w^0(\Gamma;X)} = \max_{y\in\Gamma}(w(y)||v(y)||_X) \tag{4.1}$$

for any positive continuous bounded weight function $w : \Gamma \to \mathbb{R}_+$. Because of Assumption 3, the linear coefficient a and forcing term f satisfy $a \in C^0(\Gamma; L^\infty(D))$ and $f \in C_w^0(\Gamma; L^2(D))$.

Theorem 4.1 *Under the Assumption 1-3, the reduced basis approximation to the solution $P_N u$ of the problem (2.10) enjoys the following exponential convergence (the complex region $\Sigma(\Gamma;\tau)$ will be specified later)*

$$||u - P_N u||_{C_w^0(\Gamma;X)} \leq C_w e^{-rN} \max_{z\in\Sigma(\Gamma;\tau)} ||u(z)||_X \tag{4.2}$$

where the constant C_w depends on the weight w and is independent of N, and the rate r is defined as

$$1 < r = log\left(\frac{2\tau}{|\Gamma|} + \sqrt{1 + \frac{4\tau^2}{|\Gamma|^2}}\right). \tag{4.3}$$

Remark 4.1 *The convergence rate stated above does not depend on the specific problem (2.1). In fact, as long as $u = u(y)$ is an analytic function, the exponential convergence rate (4.2) holds for reduced basis approximation as demonstrated in the proof of this theorem later, which provides the same a priori convergence property for problems other than the elliptic problem (2.1) under linear or affine assumptions (2.7) as studied in [28, 26].*

Remark 4.2 *The exponential convergence result (4.2) holds for the case of a single parameter in a bounded parameter domain $|\Gamma| < \infty$. Extension to a single parameter in unbounded domain, e.g. normal distributed random variable, requires that the data a and f feature a fast decrease at the parameter far away from the origin (in particular, belonging to a Schwartz space), and the constructive approximation by spectral expansion on Chebyshev polynomials is replaced by that on Hermite polynomials, see [1].*

Remark 4.3 *A straightforward extension to multidimensional case (e.g. K parameters) leads to non-optimal convergence rate with e^{-rN} replaced by $e^{-r_K N^{1/K}}$. However, when K becomes large, it severely deteriorates the convergence rate. An improved convergence rate $e^{-r' N^{\beta/(\beta+1)}}$ was achieved [5] provided that the Kolmogorov N-width by the optimal N dimensional approximation decays as e^{-rN^β} (although the Kolmogorov N-width is not available in general, some estimation is possible, see [26]). In fact, a direct bound of the reduced basis approximation error σ_N in terms of the Kolmogorov N-width d_N was proven in [5] to be $\sigma_N \leq 2^{N+1} d_N/\sqrt{3}$, and improved recently, see [16] for details.*

In order to prove Theorem 4.1, we need the analytic regularity of the solution of problem (2.10) with respect to the parameter $y \in \Gamma$, which is studied through the following three lemmas.

Lemma 4.2 *Under Assumption 1-3, the solution to problem (2.10) satisfies $u \in C_w^0(\Gamma; H_0^1(D))$ for any positive continuous weight $w : \Gamma \to \mathbb{R}_+$. Moreover, if u and $\tilde{u}$ are two weak solutions of problem (2.10) associated with data a, f and $\tilde{a}, \tilde{f}$ respectively, we have the stability estimate*

$$||u - \tilde{u}||_{C_w^0(\Gamma;H_0^1(D))} \leq \frac{C_P}{a_{min}}||f - \tilde{f}||_{C_w^0(\Gamma;L^2(D))} + \frac{C_P}{a_{min}^2}||\tilde{f}||_{C_w^0(\Gamma;L^2(D))}||a - \tilde{a}||_{C^0(\Gamma;L^\infty(D))} \tag{4.4}$$

Proof We rewrite (2.10) explicitly as: $\forall y \in \Gamma$

$$\int_D a(x,y)\nabla u(x,y)\cdot\nabla v(x)dx = \int_D f(x,y)v(x)dx \quad \forall v \in H_0^1(D). \tag{4.5}$$

A similar problem holds for $\tilde{f}$ and $\tilde{a}$. By subtraction we obtain the difference equation:

$$\int_D a\nabla(u-\tilde{u})\cdot\nabla v dx = \int_D (f-\tilde{f})v dx + \int_D (\tilde{a}-a)\nabla\tilde{u}\cdot\nabla v dx. \tag{4.6}$$

By taking $v = u-\tilde{u}$, applying Cauchy-Schwarz and Poincaré inequalities and using Assumption 2 we have

$$a_{min}||u-\tilde{u}||^2_{H_0^1(D)} \leq C_P||f-\tilde{f}||_{L^2(D)}||u-\tilde{u}||_{H_0^1(D)} + ||\tilde{u}||_{H_0^1(D)}||u-\tilde{u}||_{H_0^1(D)}||a-\tilde{a}||_{L^\infty(D)}. \tag{4.7}$$

so that the following stability estimate holds for $\forall y \in \Gamma$ by the fact $||\tilde{u}||_{H_0^1(D)} \leq (C_P/a_{min})||\tilde{f}||_{L^2(D)}$:

$$||u(y)-\tilde{u}(y)||_{H_0^1(D)} \leq \frac{C_P}{a_{min}}||f(y)-\tilde{f}(y)||_{L^2(D)} + \frac{C_P}{a^2_{min}}||\tilde{f}(y)||_{L^2(D)}||a(y)-\tilde{a}(y)||_{L^\infty(D)}. \tag{4.8}$$

Setting $\tilde{a}(y) = a(y+\delta y)$ and $\tilde{f}(y) = f(y+\delta y)$ such that $y+\delta y \in \Gamma$, we have by the fact $a \in C^0(\Gamma; L^\infty(D))$ and $f \in C^0_w(\Gamma; L^2(D))$ that $\tilde{u}(y) = u(y+\delta y) \to u(y)$ in $H_0^1(D)$ when $\delta y \to 0$. Therefore, the solution is continuous with respect to the parameter $y \in \Gamma$, i.e. $u \in C^0(\Gamma; H_0^1(D))$. An immediate consequence is that any positive continuous weight function $w : \Gamma \to \mathbb{R}_+$, we have

$$||u||_{C^0_w(\Gamma;H_0^1(D))} \leq \frac{C_P}{a_{min}}||f||_{C^0_w(\Gamma;L^2(D))} \tag{4.9}$$

Hence, $u \in C^0_w(\Gamma; H_0^1(D))$ and (4.4) holds straightforwardly from (4.8) by taking the weight function w into consideration. □

A direct application of Lemma 4.2 leads to the following lemma for the existence of partial derivatives of the solution with respect to the parameter $y \in \Gamma$ as well as their bound in $H_0^1(D)$.

Lemma 4.3 *For any $y \in \Gamma$, there exists a unique $\partial_y^\nu u(y)$ in $H_0^1(D)$ provided that the assumptions in Lemma 4.2 as well as Assumption 1-3 are satisfied for any $y \in \Gamma$ and $\nu = (\nu_1, \dots, \nu_K) \in \Lambda$, where $\Lambda \subset \mathbb{N}^K$ is a multiple index set. Moreover, we have the following estimate*

$$||\partial_y^\nu u||_{H_0^1(D)} \leq B|\nu|!\alpha^\nu + \frac{C_P}{a_{min}}|\nu|! \sum_{k:\nu_k\neq 0}\left(\alpha^{\nu-e_k}||f_k||_{L^2(D)}\right) \tag{4.10}$$

where

$$B = \frac{C_P}{a_{min}}||f(y)||_{L^2(D)}, \quad |\nu|! = (\nu_1+\cdots+\nu_K)!, \quad \alpha^\nu = \prod_{k=1}^K \alpha_k^{\nu_k}, \quad \alpha_k = \frac{||a_k||_{L^\infty(D)}}{a_{min}} \tag{4.11}$$

Proof First of all, when $|\nu| = 0$, we have the existence of a unique solution $u \in H_0^1(D)$ and the stability estimate from (4.9) only in physical space, leading to (4.10) as

$$||\partial_y^\nu u(y)||_{H_0^1(D)} = ||u(y)||_{H_0^1(D)} \leq \frac{C_P}{a_{min}}||f(y)||_{L^2(D)} = B. \tag{4.12}$$

For $|\nu| \geq 1$, we expect the general recursive equation for $\partial_y^\nu u$ as (write $a(y)$ in short for $a(x,y)$, etc.)

$$\int_D a(y)\nabla\partial_y^\nu u(y)\cdot\nabla v = -\sum_{k:\nu_k\neq 0}\nu_k\int_D a_k\nabla\partial_y^{\nu-e_k}u(y)\cdot\nabla v + \sum_{k:\nu=e_k}\int_D f_k v, \tag{4.13}$$

where e_k is a K dimensional vector with the kth element as 1 and all the other elements as 0. Indeed, suppose that $|\tilde{\nu}| = |\nu| - 1$ and that $\tilde{\nu} = \nu - e_j$ for some $j = 1,\dots,K$, by hypothesis (4.13) holds for $\tilde{\nu}$, then we claim that it also holds for ν. To see that, take the derivative of (4.13) with respect to y_j by replacing ν as $\tilde{\nu} = \nu - e_j$, we have

$$\begin{aligned}\int_D a(y)\nabla\partial_y^\nu u(y)\cdot\nabla v + \int_D a_j\nabla\partial_y^{\nu-e_j}u(y)\cdot\nabla v &= -\sum_{k\neq j:\nu_k\neq 0}\nu_k\int_D a_k\nabla\partial_y^{\nu-e_k}u(y)\cdot\nabla v \\ -(\nu_j-1)\int_D a_j\nabla\partial_y^{\nu-e_j}u(y)\cdot\nabla v &+ \sum_{k:\nu=e_k}\int_D f_k v,\end{aligned} \tag{4.14}$$

which can be simplified by summing up the same term of integral to end up with the equation (4.13). Upon replacing v by $\partial_y^\nu u(y)$ in (4.13) and multiplying the weight function w, we have by Assumption 2 as well as Cauchy-Schwarz and Poincaré inequalities the following estimate

$$||\partial_y^\nu u(y)||_{H_0^1(D)} \le \sum_{k:\nu_k\neq 0}\nu_k\alpha_k||\partial_y^{\nu-e_k}u(y)||_{H_0^1(D)} + \frac{C_P}{a_{min}}\sum_{k:\nu=e_k}||f_k||_{L^2(D)}. \tag{4.15}$$

Observe that when $|\nu| = 1$, there must be some $k = 1,\dots,K$ such that $\nu = e_k$, the estimate (4.15) becomes

$$||\partial_y^\nu u(y)||_{H_0^1(D)} = ||\partial_{y_k}u(y)||_{H_0^1(D)} \le B\alpha_k + \frac{C_P}{a_{\min}}||f_k||_{L^2(D)}, \tag{4.16}$$

which is the same as in (4.10). Meanwhile, if we take $\tilde{a}(y) = a(y - he_k)$, $\tilde{f}(y) = f(y - he_k)$ and $\tilde{u}(y) = u(y - he_k)$ in (4.6) and set $v_h = (u(y) - u(y - he_k))/h$, then (4.6) becomes

$$\int_D a(y)\nabla v_h(y)\nabla v_h(y) = \int_D f_k v_h - \int_D a_k\nabla u(y - he_k)\cdot\nabla v_h, \tag{4.17}$$

which results in a unique solution $v_h \in V$. Taking the limit $h \to 0$, we have by continuity that $u(y - he_k) \to u(y)$ so that v_0 satisfies the recursive equation for $\partial_y^\nu u$ in (4.13). Therefore, $\partial_y^\nu u$ exists and is a unique solution to (4.13) for $\nu = e_k$. By induction it exists for general $\nu \in \Lambda$ and satisfies the recursive estimate (4.15). If $|\nu| > 1$, the estimate (4.15) becomes

$$||\partial_y^\nu u(y)||_{H_0^1(D)} \le \sum_{k:\nu_k\neq 0}\nu_k\alpha_k||\partial_y^{\nu-e_k}u(y)||_{H_0^1(D)}. \tag{4.18}$$

Suppose for any $|\tilde{\nu}| < |\nu|$, the general bound (4.10) holds, then we have

$$\begin{aligned}||\partial_y^\nu u(y)||_{H_0^1(D)} &\le \sum_{k:\nu_k\neq 0}\nu_k\alpha_k||\partial_y^{\nu-e_j}u(y)||_{H_0^1(D)} \\ &\le \sum_{j:\nu_j\neq 0}\nu_j\alpha_j\left(B(|\nu|-1)!\alpha^{\nu-e_j} + \frac{C_P}{a_{\min}}(|\nu|-1)!\sum_{k:\nu_k\neq 0}\left(\alpha^{\nu-e_j-e_k}||f_k||_{L^2(D)}\right)\right) \\ &= B\left(\sum_{j:\nu_j\neq 0}\nu_j\right)(|\nu|-1)!\alpha^\nu + \frac{C_P}{a_{\min}}\left(\sum_{j:\nu_j\neq 0}\nu_j\right)(|\nu|-1)!\sum_{k:\nu_k\neq 0}\left(\alpha^{\nu-e_k}||f_k||_{L^2(D)}\right) \\ &= B|\nu|!\alpha^\nu + \frac{C_P}{a_{min}}|\nu|!\sum_{k:\nu_k\neq 0}\left(\alpha^{\nu-e_k}||f_k||_{L^2(D)}\right) \equiv C_{a,f}|\nu|!\alpha^\nu,\end{aligned} \tag{4.19}$$

where the constant $C_{a,f}$ is

$$C_{a,f} = B + C_P \sum_{k:\nu_k \neq 0, ||a_k||_{L^\infty(D)} \neq 0} \frac{||f_k||_{L^2(D)}}{||a_k||_{L^\infty(D)}}. \tag{4.20}$$

The proof can now be achieved by an induction argument. $\square$

An analytic extension of the solution u in a certain region Σ such that $\Gamma \subset \Sigma$ is a consequence of the regularity result in Lemma 4.3 provided suitable conditions, as stated in the following lemma.

Lemma 4.4 *Holding all the assumptions in Lemma 4.3, and defining*

$$\Sigma = \left\{ z \in \mathbb{C} : \exists y \in \Gamma \ s.t. \ \alpha \cdot |z - y| = \sum_{k=1}^{K} \alpha_k |z_k - y_k| < 1 \right\}, \tag{4.21}$$

we have the existence of an analytic extension of the stochastic solution u in the complex region Σ and we define $\Sigma(\Gamma; \tau) := \{z \in \mathbb{C} : dist(z, \Gamma) \leq \tau\} \subset \Sigma$ for the largest possible vector $\tau = (\tau_1, \dots, \tau_K)$.

Proof By Taylor expansion of $u(z)$ about $y \in \Gamma$ in the complex domain we obtain

$$u(z) = \sum_{\nu} \frac{\partial_y^\nu u(y)}{\nu!} (z - y)^\nu, \tag{4.22}$$

with $\nu! = \nu_1! \cdots \nu_K!$. Thanks to the regularity result in Lemma 4.3, we obtain

$$\begin{aligned}
\left|\left| \sum_{\nu} \frac{\partial_y^\nu u(y)}{\nu!} (z - y)^\nu \right|\right|_{H_0^1(D)} &\leq \sum_{\nu} \frac{|z - y|^\nu}{\nu!} ||\partial_y^\nu u(y)||_{H_0^1(D)} \\
&\leq C_{a,f} \sum_{n \geq 0 : |\nu| = n} \frac{|\nu|!}{\nu!} (\alpha \cdot |z - y|)^\nu \\
&= C_{a,f} \sum_{n \geq 0} \left(\sum_{k=1}^{K} \alpha_k |z_k - y_k| \right)^n \\
&= \frac{C_{a,f}}{1 - \sum_{k=1}^{K} \alpha_k |z_k - y_k|},
\end{aligned} \tag{4.23}$$

where the second inequality is due to Lemma 4.3 and the first equality comes from the generalized Newton binomial formula. In the complex region defined in (4.21), we obtain that the function $u(z)$ admits a Taylor expansion around $y \in \Gamma$ so that the solution u can be analytically extended to the complex region (4.21). $\square$

To prove the exponential convergence of the weighted reduced basis method for problem (2.10), we bound the error by another type of constructive spectral approximation, or more specifically, extension of Chebyshev polynomial approximation for analytic functions (see [17], Chapter 7). The idea has also been used in the proof of exponential convergence property of stochastic collocation method [1].

Proof of Theorem (4.1): First, we note that the results obtained in the above lemmas in $H_0^1(D)$ norm are still valid in the equivalent $X = A(v, v; \bar{y})$ norm. For any analytic function $u : [-1, 1] \to X$, their exists a spectral expansion on the standard Chebyshev polynomials $c_k : [-1, 1] \to \mathbb{R}$ and $|c_n| \leq 1, n = 0, 1, \dots$ in the form

$$u(t) = \frac{u_0}{2} + \sum_{n=1}^{\infty} u_n c_n(t). \tag{4.24}$$

The nth Chebyshev coefficient satisfies [17]

$$u_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\cos(t))\cos(nt)dt, \quad ||u_n||_X \leq 2\varrho^{-n} \max_{z\in D_\varrho} ||u(z)||_X, \quad n = 0, 1, \ldots, \tag{4.25}$$

where the elliptic disc D_ϱ is bounded by the ellipse E_ϱ with foci ± 1 and the sum of the half-axes ϱ. Therefore, the error of the truncated Chebyshev polynomial approximation is bounded by

$$||u - \Pi_N u||_X \leq \sum_{n\geq N+1} ||u_n||_X \leq \frac{2}{\varrho - 1} e^{-rN} \max_{z\in D_\varrho} ||u(z)||_X, \tag{4.26}$$

being the constant $r = \log(\varrho)$. For any function $u : \Gamma \to X$, we map the parameter domain $\Gamma \to [-1, 1]$ and obtain the same estimate (4.26) with the largest value of r given in (4.3) inside the rescaled complex region $\Sigma([-1, 1], 2\tau/|\Gamma|)$. It's left to prove that the reduced basis approximation error is bounded by the above truncated error. In fact, since the reduced basis approximation adopts Galerkin projection in the reduced basis space X_N, we have by *Cea*'s lemma [36] that

$$||u - P_N u||_X \leq C_1 \inf_{v\in X_N} ||u - v||_X, \tag{4.27}$$

where the constant C_1 is independent of N. Moreover, for any function $u \in \mathcal{P}_N(\Gamma) \otimes X$, a tensor product of polynomial space of polynomials with total degree no more than N and X, we have that $P_{N+1}u = u$ [9, 34], so that the following estimate holds for some constant C_2 independent of N

$$\inf_{v\in X_{N+1}} ||u - v||_X \leq C_2 \inf_{v\in \mathcal{P}_N(\Gamma)\otimes X} ||u - v||_X. \tag{4.28}$$

By the fact that Chebyshev polynomial $c_k \in \mathcal{P}_N([-1, 1]), k = 0, 1, \ldots, N$, we have

$$\inf_{v\in \mathcal{P}_N(\Gamma)\otimes X} ||u - v||_X \leq ||u - \Pi_N u||_X. \tag{4.29}$$

A combination of (4.26), (4.27), (4.28) and (4.29) leads to the following bound for the reduced basis approximation error with $C = 2C_1C_2e^r/(\varrho - 1)$

$$||u - P_N u||_X \leq Ce^{-rN} \max_{z\in D_\varrho} ||u(z)||_X. \tag{4.30}$$

Since the reduced basis approximation $P_N u$ satisfies the linear system (3.13), which can be written in the compact form as

$$A(P_N u, v; y) = F(v; y) \quad \forall v \in X_N, \tag{4.31}$$

we obtain the same regularity for $P_N u$ as for the solution u to system (2.10) with respect to the parameter y. In particular, $P_N u \in C^0_w(\Gamma; X)$, so that $u - P_N u \in C^0_w(\Gamma; X)$. Multiplying both sides of (4.30) by the weight function w and taking the maximum value over the parameter domain Γ, we have by noting $D_\varrho \subset \Xi(\Gamma; \tau)$ that the exponential convergence (4.2) holds.

□

A direct consequence of Theorem 4.1 for the convergence of $s(P_N u)$ and $\mathbb{E}[s(P_N u)]$ is as follows:

Corollary 4.5 *Suppose that the assumptions in Theorem 4.1 are satisfied, we have*

$$||s(u) - s(P_N u)||_{C^0_w(\Gamma)} \leq ||s||_{X'}||u - P_N u||_{C^0_w(\Gamma;X)} \leq C_w||s||_{X'}e^{-rN} \max_{z\in\Xi(\Gamma;\tau)} ||u(z)||_X, \tag{4.32}$$

and

$$|\mathbb{E}[s(u)] - \mathbb{E}[s(P_N u)]| \approx \left|\sum_{m=1}^{M} (s(u; y^m) - s(P_N; y^m))w(y^m)\right| \leq M||s(u) - s(P_N u)||_{C^0_w(\Gamma)}. \tag{4.33}$$

5 Numerical examples

In this section, we present several numerical examples to illustrate the efficiency of the weighted reduced basis method compared to the reduced basis method and the stochastic collocation method. The output of interest is defined as the integral of the solution over the physical domain D

$$s(y) = \int_D u(x, y) dx. \tag{5.1}$$

We define the following two errors as criteria of different numerical methods

$$||s - s_N||_{C^0_w(\Gamma)} \quad \text{and} \quad |\mathbb{E}[s] - \mathbb{E}[s_N]|, \tag{5.2}$$

where s_N is the approximated value of s obtained using N bases for (weighted) reduced basis method or N collocation points for stochastic collocation method. In particular, we use the weight function in one dimension as the probability density function of the random variable obeying $Beta(\alpha, \beta)$ distribution with shape parameter α and β providing distinctive property of the weight, defined as

$$w(y; \alpha, \beta) = \frac{1}{2\text{Beta}(\alpha, \beta)}(1 + y)^{\alpha-1}(1 - y)^{\beta-1} \quad y \in [-1, 1], \tag{5.3}$$

where $\text{Beta}(\alpha, \beta)$ is a constant (beta function) chosen so that $w(\cdot; \alpha, \beta)$ is a probability density function. In our numerical experiments, we use the Gauss-Jacobi quadrature formula to compute the expectation (5.2) with the solution at the abscissas evaluated by the reduced basis methods. As for the stochastic collocation method, we use the Gauss-Jacobi abscissas as the collocation points, which is more accurate than other choices, especially when the weight function is more concentrated. We specify the detailed setting of the weighted reduced basis method in the following subsections. The physical domain is a square $D = (-1, 1)^2$ and homogeneous Dirichlet boundary conditions are prescribed on the entire boundary ∂D.

5.1 One dimensional problem

We set the stochastic coefficient $a(x, \omega)$ in problem (2.1) as

$$a(x, \omega) = \frac{1}{10}(1.1 + \sin(2\pi x_1)Y(\omega)), \tag{5.4}$$

with random variable $Y \sim Beta(\alpha, \beta)$ with $(\alpha, \beta) = (1, 1)$, $(10, 10)$ and $(100, 100)$, respectively. The left of Figure 5.1 depicts the shape of weight at different locations. The forcing term is the deterministic value $f = 1$ for simplicity. We use a tolerance at the same value $\varepsilon = 1 \times 10^{-15}$ for three different weight functions to stop the greedy algorithm. $n_{train} = 1000$ samples are uniformly selected to construct the reduced basis space. Another 1000 samples are used to test the accuracy of different methods. The exponential convergence of the error $||s - s_N||_{C^0_w(\Gamma)}$ in logarithmic scale for three different weight functions is displayed on the right of Figure 5.1 for weighted reduced basis method. The maximum number of bases $N_{max} = 16, 11, 6$ built at the training samples with selection order are visualized by the marker size on the left of Figure 5.1; they are quite different for different weight functions. From the location and selecting order of the samples on the left of Figure 5.1, we can tell that the weight function plays an important role in choosing the most representative bases.

In the comparison of the convergence property of the reduced basis method, the weighted reduced basis method as well as the stochastic collocation method, we select the weight function of $Beta(10, 10)$ and compute the two errors defined in (5.2) with the results shown in Figure 5.2. It's evident that the weighted reduced basis method outperforms the reduced basis method in both norms, and these two methods are more accurate than the stochastic collocation method in $|| \cdot ||_{C^0_w(\Gamma)}$ norm. As for the expectation, the weighted reduced basis method is the best and the reduced basis method does not beat the stochastic collocation method due to the fact that it doesn't take the weight into account.

However, as demonstrated in [10], the computation of both reduced basis methods is more expensive

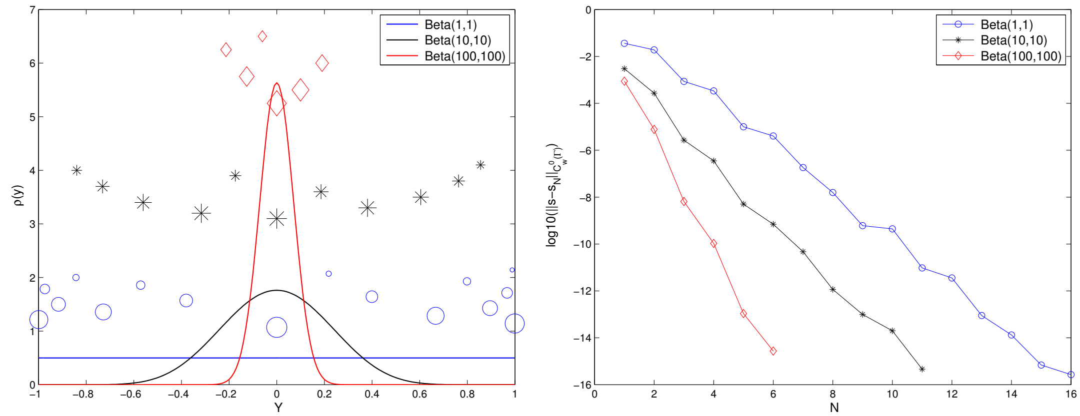

Figure 5.1: Left: Probability density function of $\mathrm{Beta}(\alpha, \beta)$ distribution with different α, β and samples selected by weighted reduced basis approximation in order, the bigger the size the earlier it has been selected; Right: convergence of the error $\log_{10}\left(||s - s_N||_{C^0_w(\Gamma)}\right)$ by weighted reduced basis method.

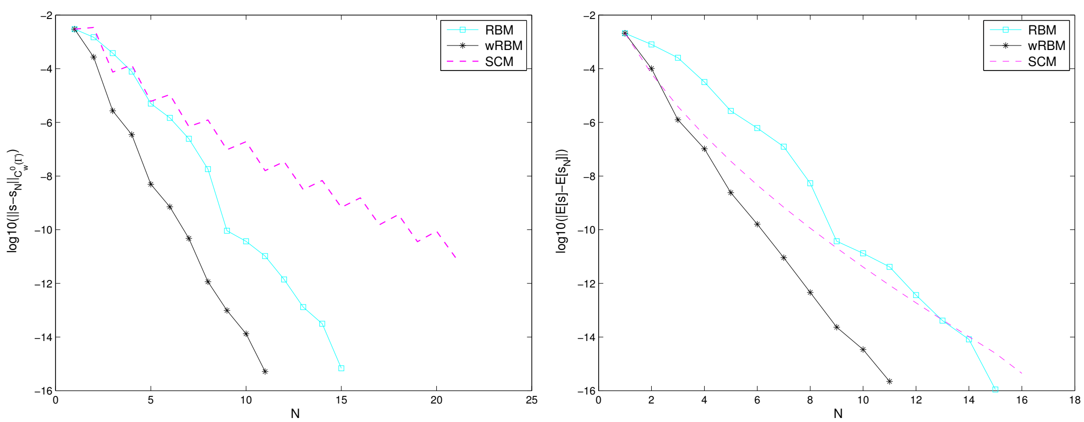

Figure 5.2: Left: convergence of the error $\log_{10}\left(||s - s_N||_{C^0_w(\Gamma)}\right)$ by reduced basis method (RBM), weighted reduced basis method (wRBM) and stochastic collocation method (SCM); Right: convergence of the error $\log_{10}(|E[s] - E[s_N]|)$ by RBM, wRBM and SCM, both with $K = 1, \mathrm{Beta}(10, 10)$.

than that of the stochastic collocation method because of the offline construction with a large number of training samples, especially for the problem requiring low computational effort in one deterministic solving. Similar numerical examples for some other weight functions are presented in the appendix for expository convenience.

5.2 Multiple dimensional problem

For the test of multiple dimensional problem, we specify the coefficient $a(x, \omega)$ as

$$a(x, \omega) = \frac{1}{10}\left(4 + \left(\frac{\sqrt{\pi}L}{2}\right)^{1/2} y_1(\omega) + \sum_{n=1}^{2} \sqrt{\lambda_n}\left(\sin(n\pi x_1) y_{2n}(\omega) + \cos(n\pi x_1) y_{2n+1}(\omega)\right)\right), \quad (5.5)$$

where $y_k, 1 \leq k \leq 5$ obeying $Beta(100, 100)$, $L = 1/4$ and $\lambda_1 \approx 0.3798, \lambda_2 = 0.2391$, which comes from a truncation of Karhunen-Loève expansion [32]. A sufficient number of $n_{train} = 10000$ samples (in fact $n_{train} = 1000$ provides almost the same result in this example) obeying independent and identically distributed $y_k \sim Beta(100, 100), 1 \leq k \leq 5$ are taken within the parameter domain $\Gamma = [-1, 1]^5$ to construct the reduced basis space and another 1000 samples following the same distribution are taken independently to test different methods. We compare the performance of the weighted reduced basis method, the reduced basis method and a sparse grid collocation method, with results displayed in Figure 5.3. The two reduced basis methods are obviously more efficient in both norms (5.2) with the weighted type providing faster convergence: the number of bases constructed for the weighted reduced basis method ($N_{max} = 15$) is half that necessary to the reduced basis method ($N_{max} = 30$).

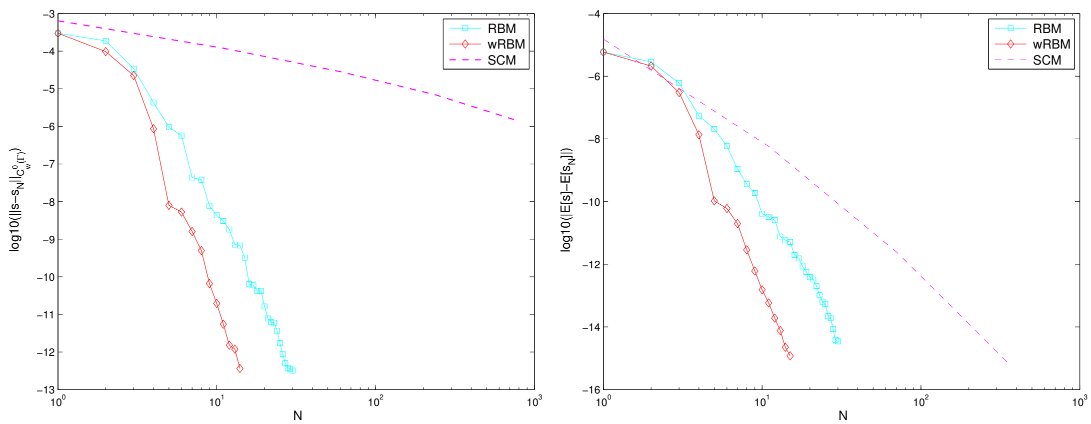

Figure 5.3: Left: convergence of the error $\log_{10}\left(||s - s_N||_{C^0_w(\Gamma)}\right)$; Right: convergence of the error $\log_{10}\left(|E[s] - E[s_N]|\right)$, computed by RBM, wRBM and SCM, both with $K = 5, \mathrm{Beta}(100, 100)$.

As for the computational effort, the stochastic collocation method with sparse grid depends critically on the dimension [32] while the reduced basis methods are near the best approximation in the sense that it considerably alleviate the "curse-of-dimensionality" for analytic problem and save the computational effort significantly for high dimensional problems, especially those with expensive cost for one deterministic solving. The weighted reduced basis method uses less bases than the conventional reduced basis method in both offline construction and online evaluation and thus costs less computational effort, particularly for high concentrated weight function as shown in the above examples. For detailed comparison of computational cost for reduced basis method and stochastic collocation method in various conditions, notably for large scale and high dimensional problems, see [10].

6 Concluding remarks

We proposed a weighted reduced basis method to deal with parametric elliptic problems with distinctive weight or importance at different values of the parameters. This method is particularly useful in solving stochastic problems with random variables obeying various probability distributions. Analytic regularity of the stochastic solution with respect to random variables was obtained under certain assumptions for the random input data, based on which an exponential convergence property of this method was studied by constructive approximation of general functions with analytic dependence on the parameters. The computational efficiency of the proposed method in comparison with the reduced basis method as well as the (sparse grid) stochastic collocation method was demonstrated numerically for both univariate and multivariate stochastic elliptic problems.

There are a few potential limitations we would like to warn the reader on: firstly, the performance of the weighted reduced basis method for low regularity problems is to be investigated, possibly improved by combination of "hp"-adaptive reduced basis method [18]. Secondly, efficient empirical interpolation method [4, 12] needs to be applied in order to use the weighted reduced basis method to solve non-linear stochastic problems or linear stochastic problems with non-affine random inputs exhibiting various probability structure. Finally, we would like to mention that application of the weighted reduced basis method to more general problems, e.g. parabolic problems [22], fluid dynamics [35], multi-physical problems [27], stochastic optimization problems [11], inverse problems [29], as well as more general stochastic problems with various probability structures are ongoing research.

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7 Appendix

To illustrate more about the efficiency of the weighted reduced basis method, we present the following numerical examples with some widely used weight functions other than those considered in section 5:

1. weight function as truncated probability density function of normal distributed random variable:

$$a(x,\omega) = \frac{1}{10}(3.1+\sin(2\pi x_1)Y(\omega)\mathbb{I}(|Y|\leq 3)), Y\sim \text{Normal}(\mu,\sigma); w(y) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right);$$

2. weight function as truncated probability density function of gamma distributed random variable:

$$a(x,\omega) = \frac{1}{10}(10.1+\sin(2\pi x_1)Y(\omega)\mathbb{I}(Y\leq 10)), Y\sim \text{Gamma}(k,\gamma); w(y) = \frac{1}{\gamma^k\Gamma(k)}y^{k-1}\exp(-\frac{y}{\gamma});$$

3. weight function as truncated probability density function of Poisson distributed random variable:

$$a(x,\omega) = \frac{1}{10}(100.1+\sin(2\pi x_1)Y(\omega)\mathbb{I}(Y\leq 100)), Y\sim \text{Poisson}(\lambda); w(y) = \frac{\lambda^y e^{-\lambda}}{y!}.$$

The selected samples for different weight functions and error of $\log_{10}(||s-s_N||_{C^0_w(\Gamma)})$ are displayed in Figure 7.1, 7.2 and 7.3, respectively, from which we can observe that the samples are effectively chosen according to the weight functions. Consequently, both the offline construction and the online evaluation become more efficient by the weighted reduced basis method than the conventional one.

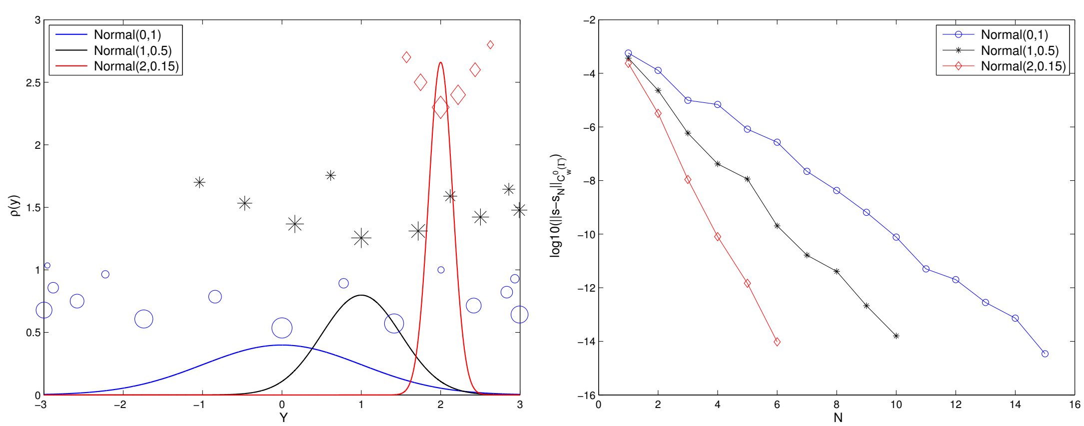

Figure 7.1: Left: Probability density function of $Y\sim \text{Normal}(\mu,\sigma)$ with different μ,σ and samples selected by weighted reduced basis approximation in order, the bigger the size the earlier it has been selected; Right: convergence of the error $\log_{10}\left(||s-s_N||_{C^0_w(\Gamma)}\right)$ by weighted reduced basis method.

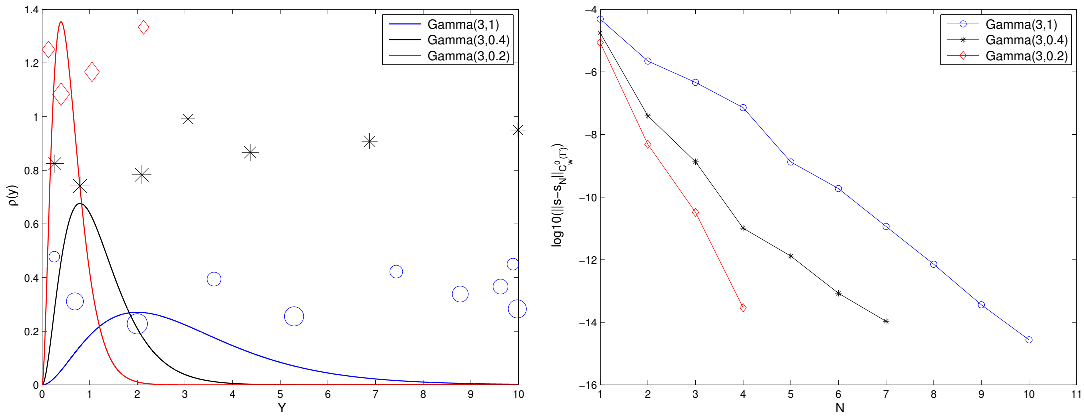

Figure 7.2: Left: Probability density function of $Y \sim \mathrm{Gamma}(k, \gamma)$ with different γ and samples selected by weighted reduced basis approximation in order, the bigger the size the earlier it has been selected; Right: convergence of the error $\log_{10}\left(||s - s_N||_{C^0_w(\Gamma)}\right)$ by weighted reduced basis method.

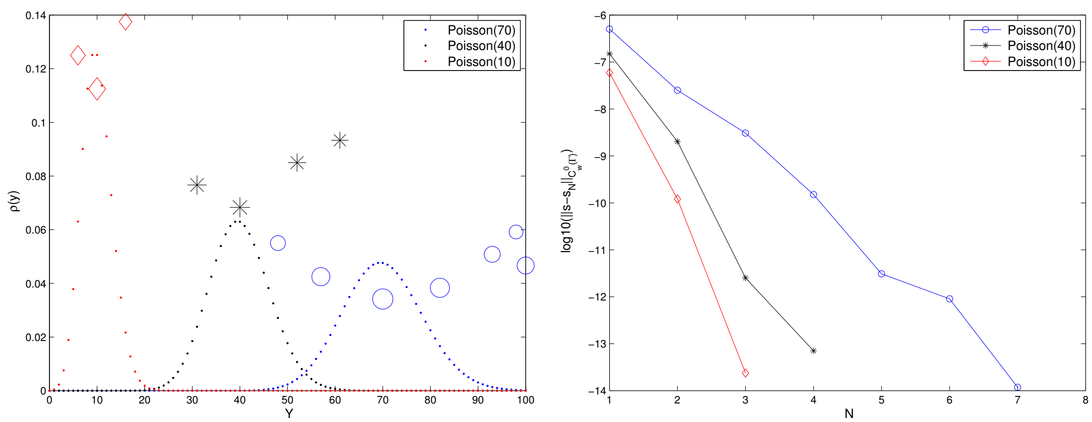

Figure 7.3: Left: Probability density function of $Y \sim \mathrm{Poisson}(\lambda)$ with different λ and samples selected by weighted reduced basis approximation in order, the bigger the size the earlier it has been selected; Right: convergence of the error $\log_{10}\left(||s - s_N||_{C^0_w(\Gamma)}\right)$ by weighted reduced basis method.

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