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Giulia Garegnani, Giorgio Rosatti,
Luca Bonaventura

MOX, Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano, Via Bonardi 9 - 20133 Milano (Italy)

mox@mate.polimi.it http://mox.polimi.it

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Giulia Garegnani[(1)], Giorgio Rosatti[(1)]

Luca Bonaventura[(2)]

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[(1)] CUDAM – Centro Universitario per la Difesa Idrologica dell'Ambiente Montano,
Dipartimento di Ingegneria Civile e Ambientale, Università degli Studi di Trento
Mesiano di Povo, 38050 (TN), Italy
giulia.garegnani@ing.unitn.it, giorgio.rosatti@ing.unitn.it
[(2)] MOX – Modelling and Scientific Computing,
Dipartimento di Matematica "F. Brioschi", Politecnico di Milano
Via Bonardi 9, 20133 Milano, Italy
luca.bonaventura@polimi.it

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Abstract

We analyze the most widely used formulations of sediment transport modelling in the framework of one dimensional channel flow, thereby assessing the impact of typical simplifying assumptions, such as low sediment concentration and decoupling of hydrodynamics and bed evolution. As a result, the importance of using a quasi two-phase formulation is highlighted. Starting from the a quasi two-phase model equations, under the hypothesis of quasi-steady free surface and mixture flows, we derive a simplified equation for the bed evolution, that is also valid in the large sediment concentration regime. The solution of such a simplified equation provides a useful benchmark for numerical methods aimed at computing approximate solutions of the quasi two-phase system. Finally, we propose and evaluate a highly efficient and accurate semi-implicit and semi-Lagrangian numerical method for quasi two-phase mobile bed system.

1 Introduction

Sediment transport modelling plays a key role in realistic river hydraulic simulations. In order to simulate the sediment transport process, a variety of mathematical models have been proposed. These models differ in key aspects of their mathematical formulation, such as the use of simplified conservation equations versus equations for the sediment-liquid mixture, the definition and use of parameterized sediment transport functions, capacity or non capacity models and mobile-bed resistance. In the most widely used models, applicable to large rivers with relatively small bed slopes, the inertia and concentration associated to the solid phase are assumed to be negligible, so that the momentum equation for the solid mass is disregarded and the solid mass flux is assumed to be in local equilibrium with the liquid mass flux. More specifically, the ratio between liquid and solid discharges is assumed to be computable on the basis of various parameters that characterize the flow and the transport regime. This approach is widely used in modelling subcritical river flow (see e.g. [2]). The resulting equations have been analyzed in [10], [21], [28] in the simplified case of sediment transport in a rectangular channel. Modern numerical methods to solve these equations have been proposed e.g. in [12], [16].

In this paper, we present a detailed analysis of the most widely used models in the idealized framework of one dimensional flow in a channel with rectangular cross section. More specifically, different formulations for sediment transport modelling will be discussed, in which a total or partial decoupling of the liquid and sediment motion is introduced, relying on the simplifying assumption of low sediment concentration. The importance of using a quasi two-phase formulation will be highlighted both by theoretical analysis and numerical experiments.

Starting from the quasi two-phase model equations, a simplified equation for the bed evolution is then derived in the case of quasi-steady free surface and mixture flows. The solution of such a simplified equation can be computed very accurately by the method of characteristics and provides a useful benchmark for numerical methods aimed at computing approximate solutions of the quasi two-phase system. Similar equations have been derived in [16] in the limit of low sediment concentration, but the present derivation also holds for large sediment concentration regimes, in which the bed evolution and hydrodynamics are fully coupled and strong nonlinearities of the typical sediment transport formulae play a much larger role. As a consequence, a more interesting benchmark problem is obtained, that allows to compare the performance of different numerical approaches in a physically more relevant context.

Finally, we propose a highly efficient and accurate semi-implicit and semi-Lagrangian numerical method for the quasi two-phase mobile bed system, based on the numerical method introduced in [24] for the fixed bed case. The proposed method is linearly unconditionally stable and allows to employ much longer time-steps than standard explicit discretizations, while still producing accurate solutions. This property is extremely important for realistic applications to morphodynamic problems, where numerical simulation of very long time intervals is necessary to study the long term impact of erosion and deposition processes. The quality of the numerical solutions obtained by the proposed method is assessed by a number of numerical experiments presented in section 7, also on the basis of the newly introduced benchmark.

Extensions of the present results to section averaged models that allow for arbitrary section shapes and applications to natural rivers will be presented in forthcoming papers, again following the approach introduced in [24].

2 A quasi two-phase sediment transport model

Morphodynamics and hydrodynamics in alluvial rivers can be described starting from the equations of conservation of mass and momentum. In the case of strong interaction between flow and morphodynamics, the aggradation and degradation processes are strictly connected with the hydrodynamics. The conservation equations for mass and momentum are then written for the water-sediment mixture. In these quasi two-phase models QTP (see e.g. [6], [21], [26], [29], [32]), the effects of the sediment concentration on the bulk-fluid density and on the temporal variation of the bed elevation are considered. The model equations were analyzed in the case of a rectangular channel by [22].

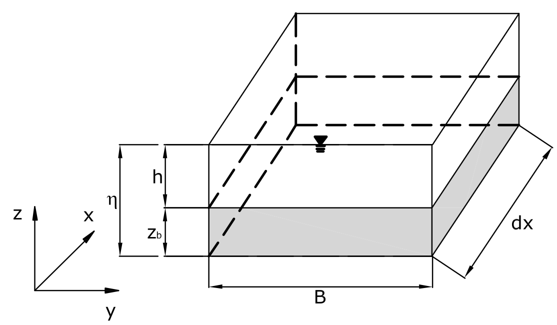


Figure 1: Control volumes for determination of shallow water equations with mobile-bed.

The mass and momentum conservation equations will be written for the control volume V', see figure (1). It is to be remarked that the momentum equation for a quasi two-phase model should represent the momentum conservation law for the mixture of sediment and fluid that is actually moving with the flow. The unknown quantities of the model equations will be the depth h, the flow velocity u and the bed level z_b. Following e.g. [31], the concentration c is defined as the volume of sediment present in the water-sediment volume V, see figure (1) and figure (2):

$$c = \frac{V_s}{V_s + V_w}$$

The volume occupied by water is then $(1-c)\,h$, while the volume occupied by the sediment is ch. In the control volume $V' - V$, the solid concentration c_b is defined as the ratio between the solid volume and the total volume, so that the volume fractions of $V' - V$ occupied by water and sediment are $(1-c_b)\,z_b$ and $c_b z_b$, respectively.

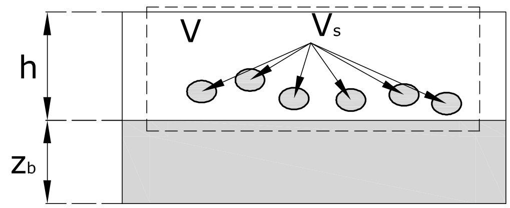


Figure 2: Sketch of the water and sediment mixture.

The water-sediment discharge is the volume rate of water flow, including any suspended sediment that is transported per unit time through a given cross-sectional area. It is therefore given by $q = uh$, where h the water layer thickness and u is the mixture mean velocity. The solid and the liquid discharge are then

$$q_s = cuh, \quad q_w = (1-c)\,uh,$$

respectively. The fundamental equations of motion for a fluid system are the continuity equation for the liquid phase, 1, the momentum equation for the mixture, 2, and the continuity equation for the solid phase, written here in the case of a channel with rectangular cross section and frictionless walls3:

$$\frac{\partial}{\partial t}\left[(1-c)\,h\right] + \frac{\partial}{\partial x}\left[(1-c)\,uh\right] + (1-c_b)\,\frac{\partial z_b}{\partial t} = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\left[(1+c\Delta)\,uh\right] + \frac{\partial}{\partial x}\left[(1+c\Delta)\,u^2 h\right] + g\frac{\partial}{\partial x}\left[(1+c\Delta)\,\frac{h^2}{2}\right] +$$

$$+gh\,(1+c\Delta)\,\frac{\partial z_b}{\partial x} = -\frac{\bar{\tau}_0}{\rho}, \tag{2}$$

$$\frac{\partial\,(ch)}{\partial t} + c_b\frac{\partial z_b}{\partial t} + \frac{\partial (cuh)}{\partial x} = 0, \tag{3}$$

Here, x is the axial coordinate along the channel and g denotes as usual the gravity acceleration, while $\bar{\tau}_0$ is the shear stress and $\Delta = (\rho_s - \rho_w)/\rho_w$ is the submerged relative density of the sediment, where ρ_s, ρ_w denote the sediment and water densities, respectively. The friction term $\bar{\tau}_0/\rho$ and the concentration c are expressed through empirical closure formulae

$$c = c(u, h, z_b, ...) \tag{4}$$

$$\frac{\bar{\tau}_0}{\rho} = -\gamma\,(u, h, d, ...)\,uh. \tag{5}$$

In the fixed bed case, a closure formula for the friction term based on a rigorous derivation from the three dimensional Reynolds averaged equations has been proposed in [11]. Various choices are possible for the empirical relation defining the solid concentration c, such as the Meyer-Peter Müller or monomial formula. By summing the two conservation equations (1) and (3), the equation for the water-sediment mass is obtained, which is most often used in numerical models, so that the model equations (also referred to in the following as complete model) read:

$$\frac{\partial h}{\partial t} + \frac{\partial z_b}{\partial t} + \frac{\partial}{\partial x}(uh) = 0. \tag{6}$$

$$\frac{\partial}{\partial t}\left[(1+c\Delta)\, uh\right] + \frac{\partial}{\partial x}\left[(1+c\Delta)\, u^2 h\right] + g\frac{\partial}{\partial x}\left[(1+c\Delta)\,\frac{h^2}{2}\right] +$$

$$+gh\,(1+c\Delta)\,\frac{\partial z_b}{\partial x} = -\gamma\,(u,h,d,...)\, uh, \tag{7}$$

$$\frac{\partial\,(ch)}{\partial t} + c_b\frac{\partial z_b}{\partial t} + \frac{\partial(cuh)}{\partial x} = 0, \tag{8}$$

On the other hand, in the most widely used models, the inertia and concentration associated to the solid phase are assumed to be negligible, so that the momentum equation for the solid mass is disregarded. The conservation equations for the solid and liquid mass are simplified under the assumption of low sediment concentration. The formulation is the essentially mono-phase (EMP). This approach is widely used in modelling river flow (see e.g. [2]). The resulting equations have been analyzed in [10], [21], [28] in the simplified case of sediment transport in a rectangular channel. However, different simplifications of equations (6)-(8) have been proposed by different authors (see e.g.[1], [4], [13], [16], [17], [20], [30]) under the same assumptions. Consequently, these mathematical models should be analyzed carefully in order to understand whether their (often implicit) assumptions concerning the interactions between these processes are coherent with the targeted application regimes.

3 Non-dimensional analysis of the quasi two-phase equations

A non-dimensional analysis of equations (6)-(8) will now be carried out, in order to clarify the importance of each term with respect the variation of solid concentration in the water column and the coupling of the hydrodynamics to the bed evolution. It should be remarked that two different time scales are employed for the hydrodynamics and bed evolution, since, in general, bed variations occur on a time scale significantly longer than that of typical hydrodynamic phenomena (see e.g. [19]). The scale for the axial coordinate x, instead, is the same for both the hydrodynamics and morphodynamics quantities, since, in the regimes of interest for fluvial morphodynamics, the typical length scale of the bed profile patterns has the same magnitude of the length scales of hydrodynamic profiles. The following scaling factors are considered:

$$t = \frac{l_0}{u_0}t^* \text{ for the free hydrodynamic quantities;}$$

$$t = \frac{l_0}{\epsilon u_0}\tau^* \text{ for the bed evolution;}$$

$$x = l_0 x^*; \quad h = h_0 h^*; \quad z_b = h_0 z_b^*;$$

$$u = u_0 u^*; \quad q = h_0 u_0 q^*; \; q_s = c_0 h_0 u_0 q^*;$$

$$c = \frac{q_s}{q} = c_0 c^*; \quad \gamma = \frac{u_0}{l_0}\gamma^*.$$

Here, h_0 and u_0 are the uniform flow depth and velocity (see e.g. [27]) and l_0 is the length of the hydrodynamic wave; $\epsilon = \tau^*/t^*$ denotes the ratio between the morphodynamic and the hydrodynamic time scales, respectively, and $Fr_0 = u_0/\sqrt{gh_0}$ will denote the Froude number.

After rescaling system (6)-(8) and performing some obvious algebraic manipulations, the following equations are obtained:

$$\frac{\partial h^*}{\partial t^*} + \epsilon \frac{\partial z_b^*}{\partial \tau^*} + \frac{\partial}{\partial x^*}(u^* h^*) = 0, \tag{9}$$

$$\frac{\partial}{\partial t^*}\left[(1 + c_0 c^* \Delta) u^* h^*\right] + \frac{\partial}{\partial x^*}\left[(1 + c_0 c^* \Delta) h^* u^{*2}\right] + \tag{10}$$

$$+\frac{1}{Fr_0^2}\frac{\partial}{\partial x^*}\left[(1 + c_0 c^* \Delta)\frac{h^{*2}}{2}\right] + \frac{1}{Fr_0^2} h^* (1 + c_0 c^* \Delta)\frac{\partial z_b^*}{\partial x} = -\gamma^* u^* h^*,$$

$$\epsilon c_0 \frac{\partial}{\partial \tau^*}(c^* h^*) + \epsilon c_b \frac{\partial z_b^*}{\partial \tau^*} + c_0 \frac{\partial}{\partial x^*}(c^* u^* h^*) = 0. \tag{11}$$

Empirical evidence (see e.g. the ASCE committee report [8]) suggests that topography changes are mainly due to the spatial variation in the sediment flux. For this to be true, the second and third term of the solid mass continuity equation should have the same magnitude, which implies the assumption:

$$\epsilon = \frac{c_0}{c_b} \tag{12}$$

Since, in general, the typical concentration c_0 of the sediment in the water column h is much smaller than the sediment concentration c_b in the river bed, the morphodynamic time scale is much longer than the hydrodynamic time scale. However, if the concentration c_0 increases, the two time scales become comparable. Using this definition (12), the non-dimensional model equations can finally be rewritten as

$$\frac{\partial h^*}{\partial t^*} + \epsilon \frac{\partial z_b^*}{\partial \tau^*} + \frac{\partial}{\partial x^*}(u^* h^*) = 0, \tag{13}$$

$$\frac{\partial}{\partial t^*}\left[(1 + \epsilon c_b \Delta c^*) u^* h^*\right] + \frac{\partial}{\partial x^*}\left[(1 + \epsilon c_b \Delta c^*) h^* u^{*2}\right] + \tag{14}$$

$$+\frac{1}{Fr_0^2}\frac{\partial}{\partial x^*}\left[(1 + \epsilon c_b \Delta c^*)\frac{h^{*2}}{2}\right] + \frac{1}{Fr_0^2} h^* (1 + \epsilon c_b \Delta c^*)\frac{\partial z_b^*}{\partial x} = -\gamma^* u^* h^*,$$

$$\epsilon \frac{\partial}{\partial \tau^*}(c^* h^*) + \frac{\partial z_b^*}{\partial \tau^*} + \frac{\partial}{\partial x^*}(c^* u^* h^*) = 0. \tag{15}$$

Several simplified equation sets can be derived from (6)-(8), see e.g. the proposals and discussion in [5], [9], [33]. We will now present these simplified models in the context of the non-dimensional equations (13)-(15). Some of the approximations underlying these models can be rigorously justified on the basis of the hypothesis of small sediment concentration $\epsilon \ll 1$.

A first common approximation in the $\epsilon \ll 1$ regime is to neglect the terms proportional to the sediment concentration in the momentum equation, thus obtaining

$$\frac{\partial}{\partial t^*}(u^* h^*) + \frac{\partial}{\partial x^*}\left(h^* u^{*2} + \frac{h^{*2}}{2Fr_0^2}\right) + \frac{h^*}{Fr_0^2}\frac{\partial z_b^*}{\partial x} = -\gamma^* u^* h^*. \tag{16}$$

In this way, the momentum equation reduces to that of the equation for clear water and the contribution of the suspended sediment concentration to the momentum flux is neglected. This formulation has been used, among others, in [1], [4], [13], [16], [17], [20], [30] and it was employed also in the model formulation comparison of [33]. This approximation was justified in [33] by assuming that the momentum flux contribution due to the sediment was small with respect to the usually large uncertainties in the data, that are usually accounted for by tuning of the friction term.

Other approximations consist in neglecting the terms proportional to ϵ in either the mixture or the sediment continuity equation. Models using the simplified sediment mass equation

$$\frac{\partial z_b^*}{\partial \tau^*} + \frac{\partial}{\partial x^*}(c^* u^* h^*) = 0, \tag{17}$$

where the sediment storage in the water column has been neglected, have been used e.g. in [33]. Alternatively, the bed variation in the continuity equation for total mass can be neglected, yielding

$$\frac{\partial h^*}{\partial t^*} + \frac{\partial}{\partial x^*}(u^* h^*) = 0. \tag{18}$$

This equation set has been used e.g. in [13]. Our main criticism of all these partially two-phase formulations, in which only some of the $O(\epsilon)$ terms are dropped, is that they all introduce inconsistencies between the mass and momentum conservation equations. If the terms multiplied by ϵ are not neglected in either the continuity equation for the total mass, eq. (13), or the solid mass, eq. (15), also the momentum equation eq. (14) should be the complete one.

Finally, if all terms proportional to ϵ are dropped consistently, one obtains the simplified, EMP formulation

$$\frac{\partial h^*}{\partial t^*} + \frac{\partial}{\partial x^*}(u^* h^*) = 0, \tag{19}$$

$$\frac{\partial}{\partial t^*}(u^* h^*) + \frac{\partial}{\partial x^*}\left(h^* u^{*2} + \frac{h^{*2}}{2Fr_0^2}\right) + \frac{h^*}{Fr_0^2}\frac{\partial z_b^*}{\partial x} = -\gamma^* u^* h^*, \tag{20}$$

$$\frac{\partial z_b^*}{\partial \tau^*} + \frac{\partial}{\partial x^*}(c^* u^* h^*) = 0. \tag{21}$$

The EMP formulation is also widely used, see e.g. [16], [18], [20], [21], [25], [30] and the well known HEC-RAS modelling software [4].

4 Eigenvalues analysis

The range of validity of some of the previous approximations has been discussed in [22] on the basis of the behavior of the eigenvalues of the hyperbolic part of system (13)-(15). The eigenvalue analysis will be extended here to the non-dimensional, QTP system. For simplicity, the symbol $*$ is dropped in the following.

Neglecting friction, the non-dimensional model equations can be rewritten in matrix notation as:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \mathbf{H}\frac{\partial z}{\partial x} = 0, \tag{22}$$

where

$$\mathbf{U}=\left[\begin{array}{c} h+z_b \\ uh\left(1+c_b\epsilon\Delta c\right) \\ ch+\epsilon^{-1}z_b \end{array}\right],\mathbf{F}=\left[\begin{array}{c} uh \\ \left(1+\epsilon c_b\Delta c\right)\left(u^2h+\dfrac{h^2}{2Fr_0^2}\right) \\ cuh \end{array}\right],$$

$$\mathbf{H}=\left[\begin{array}{c} 0 \\ \dfrac{h}{Fr_0^2}\left(1+c_b\epsilon\Delta c\right) \\ 0 \end{array}\right]. \tag{23}$$

Notice that also the bed evolution has been expressed in terms of the free surface time variable t. System (22) can then be rewritten in quasi-linear form as

$$\mathbf{B}\left(\mathbf{W}\right)\frac{\partial\mathbf{W}}{\partial t}+\mathbf{A}\left(\mathbf{W}\right)\frac{\partial\mathbf{W}}{\partial x}=0, \tag{24}$$

where

$$\mathbf{W}=\left[\begin{array}{c} h \\ u \\ z_b \end{array}\right],\quad \mathbf{B}=\left[\begin{array}{lll} 1 & 0 & 1 \\ u\left(\alpha+\xi h\dfrac{\partial c}{\partial h}\right) & h\left(\alpha+\xi u\dfrac{\partial c}{\partial u}\right) & uh\xi\dfrac{\partial c}{\partial z} \\ \epsilon\left(c+h\dfrac{\partial c}{\partial h}\right) & \epsilon h\dfrac{\partial c}{\partial u} & 1+\epsilon h\dfrac{\partial c}{\partial z} \end{array}\right]$$

$$\mathbf{A}=\left[\begin{array}{lll} u & h & 0 \\ \alpha\left(u^2+\dfrac{h}{Fr_0^2}\right)+\xi hr\dfrac{\partial c}{\partial h} & 2uh\alpha+\xi hr\dfrac{\partial c}{\partial u} & \alpha\dfrac{h}{Fr_0^2}+\xi hr\dfrac{\partial c}{\partial z} \\ \epsilon u\left(c+h\dfrac{\partial c}{\partial h}\right) & \epsilon h\left(c+u\dfrac{\partial c}{\partial u}\right) & \epsilon hu\dfrac{\partial c}{\partial z} \end{array}\right]$$

where

$$\xi=\epsilon c_b\Delta,\quad \alpha=1+\xi c,\quad r=u^2+\frac{h}{2Fr_0^2}.$$

The eigenvalues of system (24) can be computed by imposing the condition $\det\left(\mathbf{A}-\lambda\mathbf{B}\right)=0$, which yields the the characteristic polynomial

$$\alpha_3\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+\alpha_0=0, \tag{25}$$

where

$$\alpha_0=\epsilon hu\alpha b_1, \tag{26}$$

$$\alpha_1=\alpha\left(-u^2+\frac{h}{Fr_0^2}\right)+\epsilon\alpha b_2+\xi rb_3\left(1-\epsilon c\right), \tag{27}$$

$$\alpha_2=u\alpha\left(2+\epsilon b_4\right)+\xi b_5\left(1-\epsilon c\right), \tag{28}$$

$$\alpha_3=-\alpha\left(1-\epsilon b_6\right)-\xi u\frac{\partial c}{\partial u}\left(1-\epsilon c\right), \tag{29}$$

with

$$
\begin{aligned}
b_1 &= \frac{h}{Fr_0^2}\left(h\frac{\partial c}{\partial h}-u\frac{\partial c}{\partial u}-h\frac{\partial c}{\partial z}\right)+u^2\frac{\partial c}{\partial z},\\
b_2 &= -\frac{h}{Fr_0^2}\left(c+h\frac{\partial c}{\partial h}-u\frac{\partial c}{\partial u}-h\frac{\partial c}{\partial z}\right)+u^2\left(c+2h\frac{\partial c}{\partial h}-u\frac{\partial c}{\partial u}-3h\frac{\partial c}{\partial z}\right),\\
b_3 &= h\frac{\partial c}{\partial h}-u\frac{\partial c}{\partial u},\\
b_4 &= -2c-3h\frac{\partial c}{\partial h}+2u\frac{\partial c}{\partial u}+3h\frac{\partial c}{\partial z},\\
b_5 &= \left(r+u^2\right)\frac{\partial c}{\partial u}-hu\frac{\partial c}{\partial h},\\
b_6 &= c+h\frac{\partial c}{\partial h}-u\frac{\partial c}{\partial u}-h\frac{\partial c}{\partial z}.
\end{aligned}
$$

In the case of fixed bed, equation (25) reduces to:

$$\widehat{\alpha}_3\lambda^3+\widehat{\alpha}_2\lambda^2+\widehat{\alpha}_1\lambda=0, \tag{30}$$

with

$$\widehat{\alpha}_1=\left(-u^2+\frac{h}{Fr_0^2}\right), \tag{31}$$

$$\widehat{\alpha}_2=2u, \tag{32}$$

$$\widehat{\alpha}_3=-1. \tag{33}$$

The solutions of equation (30) are the two well-known relative celerities of the free-water surface:

$$\lambda_1=u+\frac{\sqrt{h}}{Fr_0}, \tag{34}$$

$$\lambda_2=u-\frac{\sqrt{h}}{Fr_0}. \tag{35}$$

In this case, the eigenvalue λ_3 associated with the propagation of the bed disturbance is equal to zero.

Instead, in the case of mobile-bed flow, the three solutions, fig. (3), depend on the choice of the closure relation, since the solid concentration c should be expressed as a function of the other unknowns by a closure formula. In this case, following e.g. [26], we have chosen the closure formula:

$$c=\beta Fr^2. \tag{36}$$

The eigenvalues of the system have been compared with the eigenvalues of the EMP system derived e.g. in [2] and [10]. In the case of the EMP model, the quasi-linear system (24) has the form:

$$\mathbf{W}=\begin{bmatrix} h\\ u\\ z_b\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}1&0&0\\ u&h&0\\ 0&0&1\end{bmatrix}$$

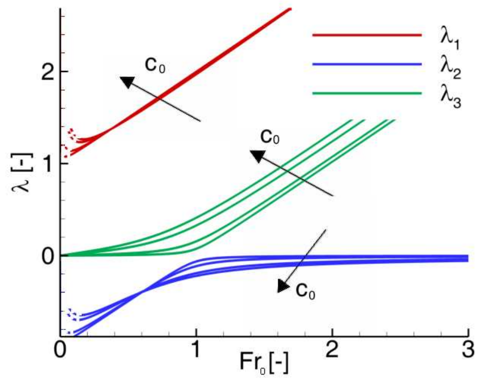


Figure 3: Eigenvalues for the QTP system. Sediment concentration c_0 increasing from 0.000002 to 0.045. The lines are dashed for small values of the Froude number, because high concentrations with small Froude number are not physically meaningful.

$$\mathbf{A} = \begin{bmatrix} u & h & 0 \\ \left(u^2 + \dfrac{h}{Fr_0^2}\right) & 2uh & \dfrac{h}{Fr_0^2} \\ \epsilon u \left(c + h\dfrac{\partial c}{\partial h}\right) & \epsilon h \left(c + u\dfrac{\partial c}{\partial u}\right) & \epsilon hu \dfrac{\partial c}{\partial z} \end{bmatrix}$$

and the coefficients of the characteristic polynomial are

$$\alpha_0 = \epsilon hub_1, \tag{37}$$

$$\alpha_1 = \left(\frac{h}{Fr_0^2} - u^2\right) + \epsilon h \left[\frac{1}{Fr_0^2}\left(c + u\frac{\partial c}{\partial u}\right) - 2u^2 \frac{\partial c}{\partial z}\right], \tag{38}$$

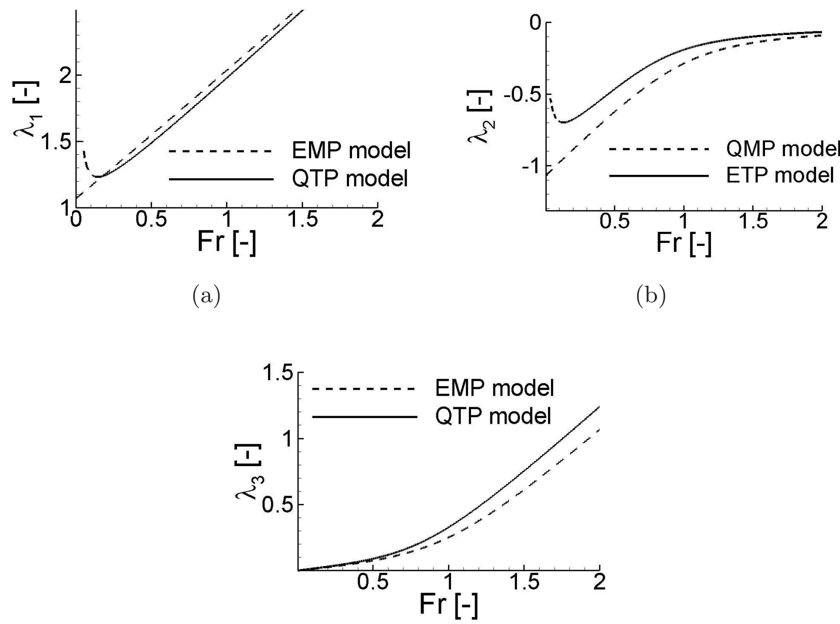
$$\alpha_2 = 2u \left(1 + \epsilon \frac{h}{2}\frac{\partial c}{\partial z}\right), \tag{39}$$

$$\alpha_3 = -1. \tag{40}$$

The eigenvalues of this system, as a function of the Froude number Fr_0 and the sediment concentration c_0, have been compared with the eigenvalues of the QTP model as shown in figure (4). It can be seen that the differences between the respective eigenvalues become significant for non negligible values of the sediment concentration.

5 A simplified equation for bed evolution

In this section, an evolution equation for the unknown z_b describing bed evolution will be derived under the hypothesis of low Froude number and of approximately steady free surface and discharge profiles. Similar equations have been proposed also in [16] for the low sediment concentration case. Our derivation, however, holds independently of the mixture-bed concentration ratio, thus providing a much more interesting and severe benchmark for numerical methods.



(a) (b) (c)

Figure 4: Comparison between the QTP and the EMP system eigenvalues with $c_0 = 0.03$.

We start considering the non-dimensional equations (13)-(15), where the stars have been dropped again for simplicity, and the hydrodynamic time scale has been used in the continuity equation for the total mass:

$$\frac{\partial h}{\partial t} + \frac{\partial z_b}{\partial t} + \frac{\partial}{\partial x}(uh) = 0, \tag{41}$$

$$Fr_0^2 \frac{\partial}{\partial t}\left[(1 + \epsilon c_b \Delta c)\, uh\right] + Fr_0^2 \frac{\partial}{\partial x}\left[(1 + \epsilon c_b \Delta c)\, hu^2\right] \tag{42}$$
$$+ \frac{\partial}{\partial x}\left[(1 + \epsilon c_b \Delta c)\frac{h^2}{2}\right] + h\,(1 + \epsilon c_b \Delta c)\frac{\partial z_b}{\partial x} = -Fr_0^2 \gamma u h,$$

$$\epsilon \frac{\partial}{\partial \tau}(ch) + \frac{\partial z_b}{\partial \tau} + \frac{\partial}{\partial x}(cuh) = 0. \tag{43}$$

In order to derive a simplified equation for bed evolution, it is convenient to express some terms in these equations as functions of the free surface height $\eta = h + z_b$ and of the mixture discharge $q = hu$. Furthermore, we will look for solutions of this system in the case of quasi-steady flow such that:

$$\eta = \bar{\eta}(x) + \delta \eta'(x,t), \qquad q = \bar{q}(x) + \delta q'(x,t).$$

The mean flow quantities $\bar{\eta}, \bar{q}$ are time independent and

$$\left|\eta'\right| \approx \left|\frac{\partial \eta'}{\partial t}\right| \approx \left|\frac{\partial \eta'}{\partial x}\right| \approx \left|q'\right| \approx \left|\frac{\partial q'}{\partial t}\right| \approx \left|\frac{\partial q'}{\partial x}\right| \approx \delta,$$

where now δ is a small parameter that is independent of $\epsilon = c_0/c_b$, which is not required to be small. In particular, without losing any generality, it can be assumed that the sediment concentration formula is a function $c = c(\eta, z_b, q)$. This implies that the derivatives of c with respect to time and space have to be computed by the chain rule as

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial c}{\partial z_b}\frac{\partial z_b}{\partial x} + \frac{\partial c}{\partial q}\frac{\partial q}{\partial x}, \tag{44}$$

$$\frac{\partial c}{\partial \tau} = \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial \tau} + \frac{\partial c}{\partial z_b}\frac{\partial z_b}{\partial \tau} + \frac{\partial c}{\partial q}\frac{\partial q}{\partial \tau} \tag{45}$$

Rewriting now also equation (43) in terms of η and z_b, neglecting terms of order δ and expressing the free surface gradients in terms of z_b, one obtains

$$\epsilon h \bar{c}_z \frac{\partial z_b}{\partial \tau} - \epsilon \bar{c}\frac{\partial z_b}{\partial \tau} + \frac{\partial z_b}{\partial \tau} + \bar{q}\bar{c}_z \frac{\partial z_b}{\partial x} = 0$$

where now $\bar{c} = c(\bar{\eta}, z_b, \bar{q})$and

$$\bar{c}_z = \frac{\partial c}{\partial z_b}(\bar{\eta}, z_b, \bar{q}).$$

The approximate evolution equation for the bed profile is

$$\frac{\partial z_b}{\partial \tau} + \frac{\bar{q}\bar{c}_z}{1 + \epsilon h \bar{c}_z - \epsilon \bar{c}}\frac{\partial z_b}{\partial x} = 0. \tag{46}$$

This nonlinear equation can be solved very accurately by the method of characteristics, thus providing a convenient benchmark for numerical methods for sediment transport. It is to be remarked that the whole derivation is independent of ϵ ratio, so that this equation can be considered to yield a good first approximation for the bed evolution also in regimes of quasi-steady flow with high sediment concentration. If the concentration decrease and the ϵ ratio is small, the simplified solution used by [16] for the EMP model is still obtained.

6 A semi-implicit and semi-Lagrangian numerical method for the QTP system

In order to introduce an efficient semi-implicit and semi-Lagrangian technique to the discretization of the momentum equation, the complete set of non-dimensional model equations (13) -(15) is rewritten using η, q as unknowns and reformulating the momentum equation in non conservative form, so as to obtain:

$$\frac{\partial \eta}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{47}$$

$$\frac{D}{Dt}[(1 + c\Delta)q] + gh\,(1 + c\Delta)\,\frac{\partial \eta}{\partial x} = -\alpha q - g\Delta\frac{h^2}{2}\frac{\partial c}{\partial x} \tag{48}$$

$$\frac{\partial (ch)}{\partial t} + c_b\frac{\partial z_b}{\partial t} + \frac{\partial\,(cq)}{\partial x} = 0. \tag{49}$$

The effective friction coefficient is now defined as

$$\alpha = \gamma + (1 + c\Delta)\frac{\partial u}{\partial x},$$

where γ is expressed for example by the Gauckler-Strickler formula:

$$\gamma = g \frac{|u|}{h^{4/3} k_s^2},$$

where k_s is the Strickler friction coefficient.

Following the approach proposed in [24] for the fixed bed open channel equations, the computational domain is then discretized by a staggered computational grid, where the fluid thickness h and the free surface elevation η are defined at the integer nodes x_i, $i = 1, \ldots, N$, while the discharge q is defined at the half integer nodes $x_{i+1/2} = (x_i + x_{i+1})/2$. The node distribution is arbitrary and the node spacings are defined as $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ and $\Delta x_{i+1/2} = x_{i+1} - x_i$, respectively.

The continuity equation for the fluid is discretized in space by a simple finite volume approach and in time by the θ method, so as to obtain

$$\eta_i^{n+1} - \eta_i^n + \vartheta \Delta t \left(q_{i+1/2}^{n+1} - q_{i-1/2}^{n+1} \right) + (1 - \vartheta) \Delta t \left(q_{i+1/2}^n - q_{i-1/2}^n \right) = 0 \quad (50)$$

where as usual $\theta \in [1/2, 1]$ for stability.

In the momentum equation, the material derivative is approximated in a semi-Lagrangian fashion (see e.g. [24]) as

$$\left. \frac{Dq}{Dt} \right|_{i+1/2} = \frac{q_{i+1/2}^{n+1} - q_*^n}{\Delta t} \quad (51)$$

where q_*^n is the discharge value at the foot of the trajectory, determined by backward integration of $dx/dt = u^n(x)$, where u^n is the velocity field at time level n. Details of the methods that can be used in to approximate the trajectory can be found e.g. in [23]. Also the free surface gradient in the momentum equation is discretized in time by the θ method, while the the terms involving the sediment concentration value are approximated by using their values at the previous time step. As a result, one obtains for the momentum equation the discrete form

$$\begin{aligned} &\left(1 + c_{i+1/2}^n \Delta\right) q_{i+1/2}^{n+1} + \vartheta \Delta t g h_{i+1/2}^n \left(1 + c_{i+1/2}^n \Delta\right) \frac{\eta_{i+1}^{n+1} - \eta_i^{n+1}}{\Delta x_{i+1/2}} \\ &+ \vartheta \Delta t \gamma_{i+1/2}^n q_{i+1/2}^{n+1} = \mathcal{F}_{i+1/2}^n \end{aligned} \quad (52)$$

where all the explicit terms have been summarized as $\mathcal{F}_{i+1/2}^n$, that is defined by:

$$\begin{aligned} \mathcal{F}_{i+1/2}^n = &\left[\left(1 + c_{i+1/2}^n \Delta\right) q_{i+1/2}^n \right. \\ &- (1 - \vartheta) \Delta t g h_{i+1/2}^n \left(1 + c_{i+1/2}^n \Delta\right) \frac{\eta_{i+1}^n - \eta_i^n}{\Delta x_{i+1/2}} \\ &\left. - \frac{g \Delta (h_{i+1/2}^n)^2}{2} \frac{c_{i+1}^n - c_i^n}{\Delta x_{i+1/2}} - (1 - \vartheta) \Delta t \alpha_{i+1/2}^n q_{i+1/2}^n \right]_* \end{aligned}$$

and $*$ denotes quantities interpolated at the foot of the trajectory.

The sediment continuity equation is also discretized by a finite volume approach. As in the momentum equation, the concentration values are kept constant within each timestep, which is consistent with the local equilibrium hypotheses justifying the use of a transport closure formula. Using the fact that $h = \eta - z_b$, the resulting discrete equation is

$$\begin{aligned} & c_i^n \eta_i^{n+1} - c_i^{n-1} \eta_i^n - \left[c_i^n (z_b)_i^{n+1} - c_i^n (z_b)_i^n \right] \\ & + c_b \left[(z_b)_i^{n+1} - (z_b)_i^n \right] + \vartheta \Delta t \left(c_{i+1/2}^n q_{i+1/2}^{n+1} - c_{i-1/2}^n q_{i-1/2}^{n+1} \right) \\ & + (1-\vartheta) \Delta t \left(c_{i+1/2}^n q_{i+1/2}^n - c_{i-1/2}^n q_{i-1/2}^n \right) = 0 \,. \end{aligned} \tag{53}$$

Equations (50)-(53) constitute a system of equations whose unknowns are the discrete values $\eta_i^{n+1}, q_{i+1/2}^{n+1}, (z_b)_i^{n+1}$. It can be solved conveniently by the following steps:

- firstly, an expression for $q_{i+1/2}^{n+1}$ in terms of η_i^{n+1} is derived from equation (52), thus obtaining

$$\begin{aligned} q_{i+1/2}^{n+1} = & \frac{\mathcal{F}_{i+1/2}^n}{(1 + c_{1+1/2}^n \Delta + \vartheta \Delta t \alpha_{i+1/2}^n)} \\ & - \frac{\vartheta \Delta t}{\Delta x_{i+1/2}} \frac{g h_{i+1/2}^n \left(1 + c_{i+1/2}^n \Delta \right)}{(1 + c_{1+1/2}^n \Delta + \vartheta \Delta t \alpha_{i+1/2}^n)} \left(\eta_{i+1}^{n+1} - \eta_i^{n+1} \right) . \end{aligned} \tag{54}$$

- equation (54) is then substituted in equation(50), to yield a linear system for the η_i^{n+1} unknowns, whose matrix is tridiagonal, symmetric and positive definite and can be easily solved by a fast direct method;
- the discrete free surface values η_i^{n+1} is used in equation (54) in order to compute the discharges values $q_{i+1/2}^{n+1}$;
- through a closure relation, the solid concentration $c_{i+1/2}^{n+1}$ is calculated in function of $q_{i+1/2}^{n+1}$ and η_i^{n+1};
- the discrete free surface values η_i^{n+1}, the discharge values $q_{i+1/2}^{n+1}$ and the solid concentrations $c_{i+1/2}^{n+1}$ are used in equation (53) in order to compute the updated bed elevation values.

The method conserves the fluid and sediment mass. Although no explicit stability analysis is available in the nonlinear case, it is linearly unconditionally stable and the use of high order interpolation in the semi-Lagrangian scheme greatly reduces numerical diffusion. The analogous method for the fixed bed case, introduced in [24], has been employed at large Courant number in a wide range of numerical tests and practical applications. As it will be shown in the next section by numerical experiment, the same holds for this mobile bed extension.

An essentially mono-phase model is also proposed and the numerical solutions of both systems are compared in the next section. The coupling terms are neglected in the momentum equation, eq. (48), and in the conservation equation for the solid mass, eq. (49) while, in order to have a system with the free surface as unknown, the temporal derivative of the bed level in equation (47) is taken into account. We obtain the following system:

$$\frac{\partial \eta}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{55}$$

$$\frac{Dq}{Dt} + gh\frac{\partial \eta}{\partial x} = -\alpha q \tag{56}$$

$$c_b \frac{\partial z_b}{\partial t} + \frac{\partial (cq)}{\partial x} = 0. \tag{57}$$

The solution procedure is the same as proposed for the QTP system. Consequently, equation (54) is rewritten consistently with the non-dimensional analysis as:

$$\begin{aligned} q^{n+1}_{i+1/2} = & \frac{\mathcal{F}^n_{i+1/2}}{(1+\vartheta\Delta t\alpha^n_{i+1/2})} \\ & -\frac{\vartheta\Delta t}{\Delta x_{i+1/2}} \frac{gh^n_{i+1/2}}{(1+\vartheta\Delta t\alpha^n_{i+1/2})}\left(\eta^{n+1}_{i+1} - \eta^{n+1}_{i}\right). \end{aligned} \tag{58}$$

The definitions of the coefficient α and the explicit terms $\mathcal{F}$ are thus different :

$$\alpha = \gamma + \frac{\partial u}{\partial x},$$

$$\mathcal{F}^n_{i+1/2} = \left[q^n_{i+1/2} - (1-\vartheta)\Delta t g h^n_{i+1/2}\frac{\eta^n_{i+1} - \eta^n_i}{\Delta x_{i+1/2}} - (1-\vartheta)\Delta t \alpha^n_{i+1/2} q^n_{i+1/2}\right]_*$$

Finally, the continuity equation for the solid mass is:

$$\begin{aligned} & c_b\left[(z_b)^{n+1}_i - (z_b)^n_i\right] + \vartheta\Delta t\left(c^n_{i+1/2}q^{n+1}_{i+1/2} - c^n_{i-1/2}q^{n+1}_{i-1/2}\right) \\ & +(1-\vartheta)\Delta t\left(c^n_{i+1/2}q^n_{i+1/2} - c^n_{i-1/2}q^n_{i-1/2}\right) = 0\,. \end{aligned} \tag{59}$$

The resulting algorithm is the same for the two models. In several methods (see e.g. [8], [16], [33]) the momentum equation and the continuity equation for the mixture are firstly solved under the hypothesis of fixed bed flow, where the bed profile is given by the values at the previous time step. This procedure solution is denoted as asynchronous. In the algorithm proposed here, the only approximation is the explicit discretization of the depth h, the term γ and the concentration c in the momentum equations, eq. (52). This solution procedure can then be described as quasi-synchronous.

7 Numerical results

Selected numerical results obtained with the methods presented in section 6 will be presented here. We will concentrate exclusively on the movable bed case, since a complete set of tests in various regimes of fixed bed flows are presented in [24]. A comparison of numerical results with experimental data in the movable bed case is carried out instead in [15]. More specifically, we will focus here on the comparison of the two possible approaches, employing either the essentially mono-phase model (EMP) or the quasi two-phase model (QTP) model solved by a quasi synchronous procedure (CM), in order to highlight the significant differences arising in the case of high sediment concentration. In all the following test cases the θ parameter is set to 0.6 and the time discretization step Δt it is chosen so as to yield an assigned value of the maximum Courant number based on the velocity $C_{vel} = max\,|u|\,\Delta t/\Delta x$ and of the maximum value of the Courant number based on the celerity $C_{cel} = max(|u| + \sqrt{gh})\Delta t/\Delta x$. The method performs well at Courant numbers larger than one, as expected, thus displaying a major efficiency of this solution procedure with respect to explicit schemes.

7.1 Comparison with solutions of simplified equation

In the first test, the numerical solution of both the QTP and EMP systems has been compared with the solution of the simplified equation (46). A frictionless channel with rectangular cross section and unit width is considered. The initial bathymetry is given by:

$$z_b\,(x, 0) = \begin{cases} 0.2\sin^2\left(\dfrac{x-200}{400}\pi\right) & \text{if } 220m < x < 400m \\ 0 & \text{otherwise} \end{cases}$$

The channel is $5000\,m$ long and discretized with 2000 cells. The efficiency of the scheme allows the use of a large time step ($\Delta t = 32\,s$) that corresponds to Courant number $C_{vel} =$ equal to 1.3. The maximum Courant number based on the celerity C_{cel} is equal to 288. In fact, the scheme is linearly stable as shown in [7, 14] and no significant stability restrictions limit the choice of the time step, especially in the case of subcritical flow. The solid concentration is written as a function of the Froude number $Fr = \sqrt{gh}/U$ as $c = \beta Fr^2$, (see e.g. [26]). The coefficient values for the closure formula and the initial and boundary conditions are reported in table (1).

test case	η $[m]$	q $[m^3 s^{-1}]$	Fr $[-]$	β $[-]$	ϵ $[-]$
a)	50	5	0.0045	200	0.006
b)	50	5	0.0045	2000	0.06

Table 1: Initial and boundary condition

The flow can be considered quasi-steady. If the Froude number tends to zero, the simplified model for the bed evolution is valid. Indeed, if ϵ is negligible, test case a) in table 1, equation (46) reduces to the simplified equation proposed in [16]. The numerical solution of both the models, QTP and EMP, strictly agree, as shown in figure 5 ,with the simplified solution, eq. (46), for very small values of the ration ϵ.

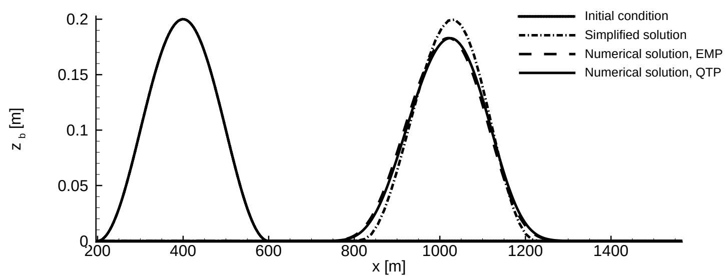


Figure 5: Comparison between the approximate and the numerical solutions in the case a), $\epsilon = 0.006$, at time $t = 300000s$.

In the case b), the solution of the simplified equation for non negligible values of the ϵ ratio agree with the numerical solution of the QTP model, but not with the numerical solution of the EMP model, see fig. (6). Since now $\epsilon = 0.062$, if the two numerical solutions are compared, a significant difference in the propagation velocity of the bed perturbation arises. The shape of the

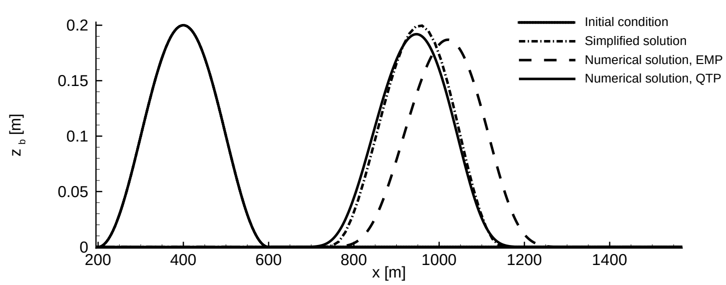


Figure 6: Comparison between the approximate and the numerical solutions in the case b), $\epsilon = 0.06$, at time $t = 30000s$.

numerical solution is slightly different from the solution of the simplified model because of the simplification in eq. (22). Nevertheless, the numerical solution of the QTP system shows a lower propagation velocity of the bed perturbation with respect the model with $\epsilon = 0$, which justifies the conclusion that if the parameter ϵ is not negligible, a QTP model should be employed.

7.2 Impact of model simplifications

A relation for the concentration c can be obtained by employing some classical expression for the transport capacity, for example the Meyer-Peter and Müller formula [26]:

$$c = Fr^2\beta, \tag{60}$$

with

$$\beta = \frac{8}{\Delta}\left(\frac{g}{k_s^2 h^{1/3}} - \frac{g\Delta d}{U^2}\theta_c\right)^{3/2}, \tag{61}$$

$$\theta_c = 0.22 D_*^{-1} + 0.06 \exp\left(-17.77 D_*^{-1}\right), \tag{62}$$

$$D_* = d\left(\frac{g\Delta}{\nu^2}\right)^{1/3}. \tag{63}$$

Here, θ_c is the critical Shields parameter calculated through the Brownlie formula (see equation 62 in [3]), d is the diameter of the sediment and k_s is the Strickler coefficient. The test case consists of a channel with unit width and two changes in the bed slope i, Figure 7:

$$i = \begin{cases} 0.02 & x < 1000\,m, \\ 0.04 & 1000\,m \leq x \leq 1300\,m, \\ 0.02 & x > 1300\,m. \end{cases}$$

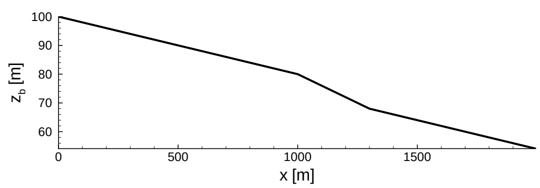


Figure 7: Initial bed elevation.

The friction coefficient is constant along the channel and depends on the diameter of the sediment through the Meyer-Peter Müller formula as:

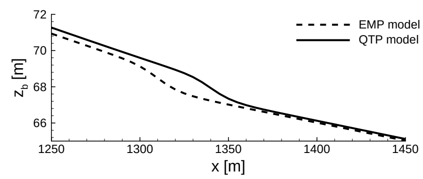
$$k_s = \frac{26}{d^{1/6}} \tag{64}$$

Three different sediment diameters have been considered, in order to achieve different solid concentration values. In table (2) the initial and boundary condition for the three test cases are reported.

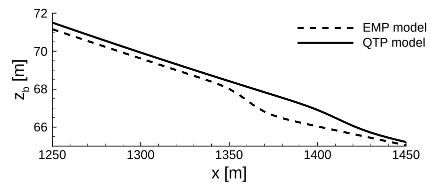
Test Cases	Initial Condition				Upstream b.c.	Downstream b.c.
	d $[m]$	ϵ $[-]$	h $[m]$	q $[m^2 s]$	q $[m^2 s]$	η $[m]$
a)	0.0007	0.003	8	40.9	40.9	68
b)	0.001	0.002	8	38.6	38.6	68
c)	0.006	0.0002	8	28.6	28.6	68

Table 2: Initial and boundary conditions for test

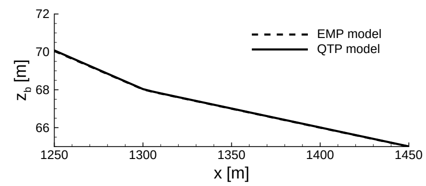
The channel is discretized with 401 cells and the time step is $\Delta t = 1.2\ s$, thus yielding Courant numbers $C_{vel} = 1.2$ and $C_{cel} = 3.6$. In agreement with the non-dimensional analysis, by decreasing the diameter d of the sediment and, consequently, increasing the sediment concentration c, the differences in the two models become more relevant and the two solutions, shown in figure (8), demonstrate the importance of the coupling terms.



(a)



(b)



(c)

Figure 8: Comparison between the QTP and the EMP model of the bed evolution (time = $450s$) in the case a) $c_0 = 0.002$ and $\epsilon = 0.003$, b) $c_0 = 0.0012$ and $\epsilon = 0.002$, c) $c_0 = 0.00012$ and $\epsilon = 0.0002$

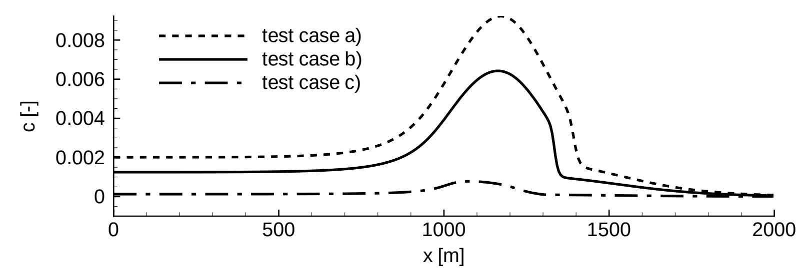


Figure 9: Solid concentration at the time 450s for the three test cases.

The stationary state of the two systems is generally different, even if the same initial and boundary conditions are imposed. The difference is mainly due to the absence of the term $(1 + c\Delta)$ in equation (56) with respect to equation (48). For the QTP model the stationary profile is given by

$$gh\,(1 + c\Delta)\,\frac{\partial z_b}{\partial x} = -\alpha q \tag{65}$$

If the boundary conditions are constant in time, the flow becomes stationary and the bed level is in equilibrium with the flow. As a result, the bed profiles computed by QTP model has a slope coherent with equation (65). Instead, the solution of the EMP model differs according to the non-dimensional analysis from the numerical solution of the QTP system where the concentrations are higher and in the transient phase, Figure 9. In conclusion, the use of the simplified equations can lead to significantly different results where the bed level changes, the flow is unsteady and the concentration c increases.

8 Conclusions and future developments

We have carried out a detailed analysis of the most widely used models for mobile bed river dynamics, in the idealized framework of one dimensional flow in a channel with rectangular cross section. More specifically, quasi two-phase model and essentially mono-phase approaches have been compared, showing that several partly mono-phase formulation are inconsistent with a rigorous scaling analysis and to which extent they can be justified on the basis of the simplifying assumption of low sediment concentration.

Starting from the quasi two-phase model equations, a simplified equation for the bed evolution has been derived in the case of quasi-steady free surface and mixture flows. The solution of such a simplified equation can be computed very accurately by the method of characteristics and provides a useful benchmark for numerical methods aimed at computing approximate solutions of the quasi two phase system. Since the present derivation also holds for large sediment concentration regimes, in which the bed evolution and hydrodynamics are fully coupled and strong nonlinearities of the typical sediment transport formulae play a much larger role, this benchmark problem allows to compare the performance of different numerical approaches in a physically more relevant context.

Finally, a highly efficient and accurate semi-implicit and semi-Lagrangian numerical method for the quasi two-phase mobile bed system is proposed, based on the numerical method proposed in [24] for the fixed bed case. The proposed method is linearly unconditionally stable and allows to employ much longer time-steps than standard explicit discretizations, while still producing accurate solutions. This property is extremely important for realistic applications to morphodynamic problems, where numerical simulation of very long time intervals is necessary to study the long term impact of erosion and deposition processes. The quality of the numerical solutions obtained by the proposed method is assessed by a number of numerical experiments presented in section 7, also employing the newly introduced benchmark. Numerical results show that, with the proposed numerical method, the quasi two-phase models can be used without any loss in efficiency with respect to the essentially mono-phase or fixed bed case. Extensions of the present results to section averaged models that allow for arbitrary section shapes, again following the approach proposed in [24], will be presented in forthcoming papers, along with applications to realistic problems in natural rivers.

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