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Abstract

In this paper we study the performance of a W-cycle multigrid algorithm for high order Discontinuous Galerkin discretizations of the Poisson problem. We recover the well known uniformity of the rate of convergence with respect to the mesh size and the number of levels and study the dependence on the polyonomial order p employed. The theoretical estimates are verified by two- and three-dimensional numerical tests.

1 Introduction

Multigrid algorithms, at present, are recognized to be among the most powerful tools to solve a variety of problems and, thanks to their efficiency, they are widely used in practice. In the framework of Discontinuous Galerkin (DG) schemes for elliptic problems, one of the first contribution is due to Gopalakrishnan and Kanschat [15], who studied a variable V-cycle preconditioner for the h-version of the Symmetric Interior Penalty (SIP) method of [3]. Applying the abstract multigrid theory for non-inherited bilinear forms developed in [6], they prove, under mild regularity assumptions, that the condition number of the preconditioned system is bounded uniformly with respect to the mesh size and the number of levels. In the following years, the topic has been further developed focusing on different multigrid schemes: exploiting the additive theory described in [7, 8], Brenner and Zhao [11] analyzed V-cycle, F-cycle and W-cycle multigrid schemes for SIP discretizations, obtaining a uniform bound for the error propagation operator provided

the number of smoothing steps is large enough. The extension to graded meshes can be found in [10], and the application of the previous multigrid schemes to other symmetric DG discretizations is addressed in [9]. All the previously cited works focus on low order, *i.e.*, linear, DG methods. With regards to high-order DG discretizations, *p*- and *hp*-multigrid schemes are successfully employed for the numerical solution of many different kinds of problems, from Poisson, to Euler and Navier-Stokes equations, see *e.g.* [13, 18, 20, 19, 22, 4]. However, at the best of our knowledge, a theoretical convergence analysis, highlighting the role of the polynomial approximation degree in the convergence estimates of the method, is still lacking. In [2] a complete convergence analysis of a W-cycle algorithm for a wide class of high order DG schemes is provided. More precisely, in [2] it is shown that the W-cycle algorithm converges uniformly with respect the granularity of the mesh and the number of levels; the dependence of the rate of convergence on the polynomial order is also carefully tracked. In this paper, we recall the analysis of the W-cycle *hp*-multigrid schemes described in [2], but focusing, for the sake of simplicity, on the SIP [3] and the LDG [12] approximations and address the performance of the method on two- and three-dimensional test cases. The paper is organized as follows. In Section 2 we briefy introduce the SIP and LDG discretizations of the Poisson equation and the corresponding error estimates [21, 17, 23, 2]. The W-cycle multigrid method is described in Section 4 and the main theoretical results are summarized: we retrieve the uniform convergence with respect to the granularity of the mesh and the number of levels, and show the dependence of the rate of convergence on the polynomial order. The theoretical estimates are then supported by numerical experiments of Section 5, where the W-cycle method is tested on two- and three-dimensional problems.

2 Model problem and notation

As a model problem, we consider the *hp*-DG discretization of the following problem: find $u \in V = H^{s+1}(\Omega) \cap H^1_0(\Omega)$, $s > 1$, such that

$$\int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega f v \; dx \qquad \forall v \in V. \tag{1}$$

Here $\Omega \in \mathbb{R}^d$, $d = 2, 3$, is a polygonal/polyhedral domain and f a given function in $H^{s-1}(\Omega)$.

Let $\mathcal{T}_K$ denote a quasi-uniform partition of Ω into shape-regular elements T. We suppose that $\mathcal{T}_K$ has been obtained by $K-1$ successive uniform refinements, applying recursively the red-green algorithm to an initial (coarse) quasi-uniform partition $\mathcal{T}_1$. For the resulting sequence of nested grids $\mathcal{T}_k$, $k = 1, \ldots, K$, we assume that each element $T \in \mathcal{T}_k$ is an affine image, trough the operator F_T, of a reference element $\widehat{T}$, that can be either the open unit d-hypercube or the unit d-simplex in $\mathbb{R}^d$, $d = 2, 3$. We then define the mesh size $h_k = \max_{T \in \mathcal{T}_k} h_T$, being h_T the diameter of $T \in \mathcal{T}_k$, $k = 1, \ldots, K$. To each level $k = 1, \ldots, K$ we assign a polynomial approximation order $p_k \geq 1$ and define

$$V_k = \{v \in L^2(\Omega) : v \circ \mathsf{F}_T \in \mathbb{M}^{p_k}(\widehat{T}) \quad \forall T \in \mathcal{T}_k\},$$

where $\mathbb{M}^{p_k}(\widehat{T})$ is either the space of all tensor-product polynomials on $\widehat{T}$ of degree p_k in each coordinate direction, if $\widehat{T}$ is the reference d-hypercube or the space of polynomials of total degree at most p_k if $\widehat{T}$ is the reference d-simplex.

For each internal face $\overline{F} = \overline{\partial T}^+ \cap \overline{\partial T}^-$, being $T^\pm$ two adjacent elements with outward unit normal vectors $\mathbf{n}^\pm$, we define jumps and averages as

$$\begin{aligned} [\![\boldsymbol{\tau}]\!] &= \boldsymbol{\tau}^+ \cdot \mathbf{n}^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^-, \qquad & \{\!\{\boldsymbol{\tau}\}\!\} &= \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}, \\ [\![v]\!] &= v^+ \mathbf{n}^+ + v^- \mathbf{n}^-, & \{\!\{v\}\!\} &= \frac{v^+ + v^-}{2}, \end{aligned}$$

with $\boldsymbol{\tau}^\pm$ and $v^\pm$ traces on $\partial T^\pm$, taken within the interior of $T^\pm$, of the (regular enough) functions $\boldsymbol{\tau}$ and v, respectively. For any boundary face $\overline{F} = \overline{\partial T} \cap \overline{\partial\Omega}$, we set $[\![v]\!] = v\mathbf{n}_T$, $\{\!\{\boldsymbol{\tau}\}\!\} = \boldsymbol{\tau}|_T$. We denote by $\mathcal{F}_k^I$ and $\mathcal{F}_k^B$ the sets of interior and boundary faces, respectively, of $\mathcal{T}_k$, $k = 1, \ldots, K$.

Finally, we define the lifting operators $\mathcal{R}_k : [L^2(\mathcal{F}_k)]^d \to [V_k]^d$ and $\mathcal{L}_k : L^2(\mathcal{F}_k^I) \to [V_k]^d$, defined as

$$\begin{aligned} \int_\Omega \mathcal{R}_k(\boldsymbol{\tau}) \cdot \boldsymbol{\eta}\, dx &= -\sum_{F \in \mathcal{F}_k} \int_F \boldsymbol{\tau} \cdot \{\!\{\boldsymbol{\eta}\}\!\}\, ds \quad \forall \boldsymbol{\eta} \in [V_k]^d, \quad k = 1, \ldots, K, \\ \int_\Omega \mathcal{L}_k(v) \cdot \boldsymbol{\eta}\, dx &= -\sum_{F \in \mathcal{F}_k^I} \int_F v [\![\boldsymbol{\eta}]\!]\, ds \qquad \forall \boldsymbol{\eta} \in [V_k]^d, \quad k = 1, \ldots, K, \end{aligned}$$

where, here and in the following, the space $L^p(\mathcal{F}_k)$ is the space of facewise functions in L^p, $1 \le p \le \infty$.

3 Discontinuous Galerkin formulation

On the finest level we are interested in solving the following problem: find $u_K \in V_K$ such that

$$\mathcal{A}_K(u_K, v_K) = \int_\Omega f v_K\, dx \quad \forall v_K \in V_K, \tag{2}$$

with $\mathcal{A}_K(\cdot,\cdot) : V_K \times V_K \to \mathbb{R}$ defined as

$$\begin{aligned} \mathcal{A}_K(u,v) = &\sum_{T \in \mathcal{T}_K} \int_T \nabla u \cdot \nabla v\, dx + \sum_{T \in \mathcal{T}_K} \int_T \nabla u \cdot (\mathcal{R}_K([\![v]\!]) + \mathcal{L}_K(\boldsymbol{\beta} \cdot [\![v]\!]))\, dx \\ &+ \sum_{T \in \mathcal{T}_K} \int_T (\mathcal{R}_K([\![u]\!]) + \mathcal{L}_K(\boldsymbol{\beta} \cdot [\![u]\!])) \cdot \nabla v\, dx + \sum_{F \in \mathcal{F}_K} \int_F \sigma_K [\![u]\!] \cdot [\![v]\!]\, ds \\ &+ \theta \int_\Omega (\mathcal{R}_K([\![u]\!]) + \mathcal{L}_K(\boldsymbol{\beta} \cdot [\![u]\!])) \cdot (\mathcal{R}_K([\![v]\!]) + \mathcal{L}_K(\boldsymbol{\beta} \cdot [\![v]\!]))\, dx. \end{aligned} \tag{3}$$

Choosing suitably the parameter $\theta \in [0,1]$, $\boldsymbol{\beta} \in \mathbb{R}^d$ and the stabilization function $\sigma_K \in L^\infty(\mathcal{F}_K)$ we can describe different DG schemes. In this paper we focus on the

SIP [3] and LDG [12] methods, which are characterized by the following choices. For the SIP method, we choose $\theta = 0$ and $\boldsymbol{\beta} = \mathbf{0}$, while for the LDG method $\theta = 1$ and $\boldsymbol{\beta}$ is a uniformly bounded (and possibly null) vector. For all the schemes, the penalization term $\sigma_K \in L^\infty(\mathcal{F}_K)$ is defined as

$$\sigma_K|_F = \frac{\alpha_K p_K^2}{\min(h_{T^+}, h_{T^-})}, \quad F \in \mathcal{F}_K^I, \qquad \sigma_K|_F = \frac{\alpha_K p_K^2}{h_T} \quad F \in \mathcal{F}_K^B, \tag{4}$$

with $\alpha_K \in \mathbb{R}^+$ and $\overline{F} = \overline{\partial T}^+ \cap \overline{\partial T}^-$.
It can be shown [21, 1, 23, 17] that the bilinear form $\mathcal{A}_K(\cdot,\cdot)$ is continuous and coercive with respect to the following DG norm

$$\|v\|_{\mathrm{DG},K}^2 = \sum_{T\in\mathcal{T}_K} \|\nabla v\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}_K} \|\sigma_K^{1/2}[\![v]\!]\|_{L^2(F)}^2.$$

Moreover, the following error estimates holds [21, 17, 23].

Theorem 1. *Let u be the exact solution of problem* (1) *such that $u \in H^{s+1}(\mathcal{T}_K)$, $s \geq 1$, and let $u_K \in V_K$ be the DG solution of problem* (2)*. Then,*

$$\|u - u_K\|_{\mathrm{DG},K} \lesssim \frac{h_K^{\min(p_K,s)}}{p_K^{s-\mu/2}} \|u\|_{H^{s+1}(\mathcal{T}_K)},$$
$$\|u - u_K\|_{L^2(\Omega)} \lesssim \frac{h_K^{\min(p_K,s)+1}}{p_K^{s+1-\mu}} \|u\|_{H^{s+1}(\mathcal{T}_K)},$$

with $\mu = 0$ whenever a continuous interpolant can be built, cf. [23], or the projector of [14] can be employed; $\mu = 1$ otherwise.

4 Multigrid W-cycle methods

A key point in the definition of the W-cycle algorithm is construction of proper sublevels. As supposed in Section 2, the finest level $\mathcal{T}_K$ is obtained by a sequence of refinements of a starting grid $\mathcal{T}_1$. Thanks to the red-green algorithm, the resulting set of nested grids $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \cdots \subseteq \mathcal{T}_K$ is such that $h_k = h_1 2^{1-k}$. Assuming also that $p_{k-1} \leq p_k$, $k = 2, \ldots, K$, the associated discontinuous spaces are nested, *i.e.*, $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_K$. We assume a local bounded variation on the polynomial order among levels, *i.e.*,

$$p_k \lesssim p_{k-1}, \quad k = 2, \ldots, K,$$

where, here and in the following, we use the symbol $\lesssim$ to mean that the inequality holds up to a positive constant.
On V_k, $k = 1, \ldots, K$, we introduce the mesh-dependent inner product

$$(u, v)_k = h_k^d \sum_{i=1}^{\dim(V_k)} u_i v_i \quad \forall u, v, \in V_k, \quad u_i, v_j \in \mathbb{R},\ i, j = 1, \ldots, \dim(V_k), \tag{5}$$

being u_i and v_i the coefficients of the expansion of u and v with respect to a set of basis functions, orthonormal in the $L^2(\widehat{T})$-inner product. Equation (2) can be then written in terms of the following linear system of equations

$$A_K u_K = f_K,$$

where the operators $A_K : V_K \to V_K'$ and $f_K \in V_K'$ are defined as

$$(A_K u, v)_K = \mathcal{A}_K(u, v), \quad (f_K, v)_K = \int_\Omega f v \, dx \quad \forall u, v \in V_K,$$

being V_K' the dual of V_K. The subproblems on the coarse levels $k = 1, \dots, K-1$, are defined by considering the bilinear forms $\mathcal{A}_k(\cdot,\cdot)$, defined as in (3) but on level V_k

$$\begin{aligned}\mathcal{A}_k(u, v) = &\sum_{T\in\mathcal{T}_k} \int_T \nabla u \cdot \nabla v \, dx + \sum_{T\in\mathcal{T}_k} \int_T \nabla u \cdot (\mathcal{R}_k(\llbracket v \rrbracket) + \mathcal{L}_k(\boldsymbol{\beta} \cdot \llbracket v \rrbracket)) \, dx \\ &+ \sum_{T\in\mathcal{T}_k} \int_T (\mathcal{R}_k(\llbracket u \rrbracket) + \mathcal{L}_k(\boldsymbol{\beta} \cdot \llbracket u \rrbracket)) \cdot \nabla v \, dx + \sum_{F\in\mathcal{F}_k} \int_F \sigma_k \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds \\ &+ \theta \sum_{T\in\mathcal{T}_k} \int_T (\mathcal{R}_k(\llbracket u \rrbracket) + \mathcal{L}_k(\boldsymbol{\beta} \cdot \llbracket u \rrbracket)) \cdot (\mathcal{R}_k(\llbracket v \rrbracket) + \mathcal{L}_k(\boldsymbol{\beta} \cdot \llbracket v \rrbracket)) \, dx,\end{aligned}$$

where $\sigma_k \in L^\infty(\mathcal{F}_k)$ is defined according to (4), but defined on level k. We then set

$$(A_k u, v)_k = \mathcal{A}_k(u, v) \qquad \forall u, v \in V_k.$$

One of the fundamental ingredients to build a multigrid algorithm are the intergrid transfer operators (*prolongation* and *restriction*). The prolongation operator $R_{k-1}^k : V_{k-1} \to V_k$ is the natural injection, while the restriction operator $R_k^{k-1} : V_k \to V_{k-1}$ is the adjoint with respect to the discrete inner product (5). We next define the projection operator $P_k^{k-1} : V_k \to V_{k-1}$ as

$$\mathcal{A}_{k-1}(P_k^{k-1} v, w) = \mathcal{A}_k(v, R_{k-1}^k w) \qquad \forall v \in V_k, w \in V_{k-1}.$$

The second ingredient is the smoothing scheme. In our case we consider a Richardson iteration, and denote by B_k its associated operator $B_k = \Lambda_k \mathrm{I}_k$, where I_k is the identity operator and $\Lambda_k \in \mathbb{R}$ is an upper bound for the spectral readius of A_k, such that

$$\Lambda_k \lesssim \frac{p_k^4}{h_k^2},$$

cf. [1, Lemma 2.6]. We now consider the following problem

$$A_k z = g, \quad k = 1, \dots, K,$$

with a given $g \in V_k'$. Let $z_0 \in V_k$ denote the initial guess. If we apply the W-cycle k-th level iteration to the above linear system with m_1 pre- and m_2 post-smoothing steps, we obtain the approximate solution $\mathsf{MG}_{\mathcal{W}}(k, g, z_0, m_1, m_2)$. For $k = 1$ (coarsest level), the solution is computed with a direct method, that is

$$\mathsf{MG}_{\mathcal{W}}(1, g, z_0, m_1, m_2) = A_1^{-1} g,$$

Algorithm 1 Multigrid W-cycle scheme

Pre-smoothing:
for $i = 1, \dots, m_1$ **do**
$z^{(i)} = z^{(i-1)} + B_k^{-1}(g - A_k z^{(i-1)})$;
end for

Coarse grid correction:
$r_{k-1} = R_k^{k-1}(g - A_k z^{(m_1)})$;
$\overline{e}_{k-1} = \mathsf{MG}_{\mathcal{W}}(k-1, r_{k-1}, 0, m_1, m_2)$;
$e_{k-1} = \mathsf{MG}_{\mathcal{W}}(k-1, r_{k-1}, \overline{e}_{k-1}, m_1, m_2)$;
$z^{(m_1+1)} = z^{(m_1)} + R_{k-1}^k e_{k-1}$;

Post-smoothing:
for $i = m_1 + 2, \dots, m_1 + m_2 + 1$ **do**
$z^{(i)} = z^{(i-1)} + B_k^{-1}(g - A_k z^{(i-1)})$;
end for

$\mathsf{MG}_{\mathcal{W}}(k, g, z_0, m_1, m_2) = z^{(m_1+m_2+1)}$.

while for $k > 1$ $\mathsf{MG}_{\mathcal{W}}(k, g, z_0, m_1, m_2)$ is defined recursively as shown in Algorithm 1. We next recall the convergence results shown in [2]. We introduce the error propagation operator associated to the W-cycle scheme described in Algortihm 1

$$\mathbb{E}_{k,m_1,m_2} v = \begin{cases} 0 & k = 1, \\ G_k^{m_2}(\mathrm{I}_k - R_{k-1}^k(\mathrm{I}_k - \mathbb{E}_{k-1,m_1,m_2}^2)P_k^{k-1})G_k^{m_1} v & k > 1, \end{cases}$$

cf. [16, 5], where $G_k = \mathrm{I}_k - B_k^{-1} A_k$, and define the discrete norms $\|\!| \cdot |\!\|_{s,k}$, $s \in \mathbb{R}$,

$$|\!| v |\!|_{s,k} = \sqrt{(A_k^s v, v)_k}, \qquad v \in V_k, \quad k = 1, \dots, K.$$

We observe that

$$|\!| v |\!|_{1,k}^2 = \sqrt{(A_k v, v)_k} = \mathcal{A}_k(v, v) \quad \forall v \in V_k.$$

Following the classical approach in the multigrid theoretical framework [16, 5], we first recall the following *smoothing property* (Lemma 2) and *approximation property* (Lemma 3), which are a key ingredient for the analysis of multigrid methods. The proofs of Lemma 2 and Lemma 3 can be found in [2].

Lemma 2 (Smoothing property)**.** *For any $v \in V_k$, $k = 1, \dots, K$, it holds*

$$|\!| G_k^m v |\!|_{s,k} \lesssim p_k^{2(s-t)} h_k^{t-s} (1+m)^{(t-s)/2} |\!| v |\!|_{t,k}, \quad 0 \le t \le s \le 2.$$

Lemma 3 (Approximation property)**.** *Let μ be defined as in Theorem 1. Then,*

$$|\!| (\mathrm{I}_k - R_{k-1}^k P_k^{k-1}) v |\!|_{0,k} \lesssim \frac{h_{k-1}^2}{p_{k-1}^{2-\mu}} |\!| v |\!|_{2,k} \quad \forall v \in V_k, k = 2, \dots, K.$$

A key ingredient for the analysis of our W-cycle multigrid scheme is the convergence of the corresponding two-level method, whose error propagation operator is given by

$$\mathbb{E}^{2\text{lvl}}_{k,m_1,m_2} = G^{m_2}_k(\mathrm{I}_k - R^k_{k-1}P^{k-1}_k)G^{m_1}_k.$$

The following convergence result holds, cf. [2] for the proof.

Theorem 4. *There exists a positive constant $C_{2\text{lvl}}$ independent of the mesh size, the polynomial approximation degree and the level k, such that*

$$|||\mathbb{E}^{2\text{lvl}}_{k,m_1,m_2} v|||_{1,k} \leq C_{2\text{lvl}} \frac{p_k^{2+\mu}}{(1+m_1)^{1/2}(1+m_2)^{1/2}} |||v|||_{1,k} \quad v \in V_k$$

with μ defined as in Theorem 1. Therefore, the two-level method converges provided the number of pre-smoothing and post-smoothing steps is chosen sufficiently large.

Using the previous results we then have that the W-cycle multigrid scheme converges uniformly with respect to the mesh size and the number of levels, and the rate of convergence depends on the polynomial order, cf. [2] for the proof.

Theorem 5. *There exist a constant $\widehat{\mathsf{C}} > C_{2\text{lvl}}$ and an integer $\widehat{\mathsf{m}}_k$ independent of the mesh size, but dependent on the polynomial approximation degree, such that*

$$|||\mathbb{E}_{k,m_1,m_2} v|||_{1,k} \leq \widehat{\mathsf{C}} \frac{p_k^{2+\mu}}{(1+m_1)^{1/2}(1+m_2)^{1/2}} |||v|||_{1,k} \quad \forall v \in V_k, \tag{6}$$

provided $m_1 + m_2 \geq \widehat{\mathsf{m}}_k$, with

$$\widehat{\mathsf{m}}_k^{1/2} \geq p_{k-1}^{2+\mu} \frac{C^2_{\text{stab}} \widehat{\mathsf{C}}^2}{\widehat{\mathsf{C}} - C_{2\text{lvl}}}.$$

5 Numerical results

In this section we verify the theoretical estimates given in Theorem 4 and Theorem 5 in the case of h- and p-multigrid schemes. We compute the convergence factor as

$$\rho = \exp\left(\frac{1}{N} \ln \frac{\|\mathbf{r}_N\|_2}{\|\mathbf{r}_0\|_2}\right),$$

with N denoting the iteration counts needed to achieve convergence up to a relative tolerance of 10^{-8} and $\mathbf{r}_N$ and $\mathbf{r}_0$ denoting the final and initial residuals, respectively. We first consider a two-dimensional example. We build the sequence of nested meshes of $\Omega = (0,1)^2$ by applying the red-green algorithm to a starting coarse mesh made of structured triangular and Cartesian elements ($h_1 = 0.25$). In Table 1 and Table 2 we report the convergence factors obtained for the h-multigrid scheme as a function of the number of smoothing steps $m = m_1 = m_2$ and the number of levels. The W-cycle is applied to SIP and LDG discretizations ($\alpha = 10$, $p = 1, 2$) on a sequence of triangular structured grids and Cartesian grids, respectively. The symbol "-" indicates that the maximum number of 10000 iterations has been reached without convergence.

Table 1: 2D test case. Convergence factors ρ of h-multigrid as a function of m and the number of levels ($\alpha = 10$, $p = 1$).

	SIP. Triangular grids.				LDG. Cartesian grids.			
	$k=2$	$k=3$	$k=4$	$k=5$	$k=2$	$k=3$	$k=4$	$k=5$
$m=1$	0.8717	0.8777	0.8791	0.8798	-	-	-	-
$m=2$	0.7658	0.7752	0.7774	0.7784	-	-	-	-
$m=3$	0.6778	0.6890	0.6917	0.6924	0.8948	0.9142	0.9018	0.9034
$m=4$	0.6042	0.6158	0.6182	0.6185	0.8636	0.8786	0.8723	0.8734
$m=6$	0.4946	0.5079	0.5035	0.5014	0.8071	0.8211	0.8203	0.8210
$m=8$	0.4257	0.4456	0.4375	0.4292	0.7647	0.7832	0.7816	0.7811
$m=10$	0.3782	0.4045	0.3963	0.3893	0.7424	0.7585	0.7571	0.7532
$m=12$	0.3439	0.3697	0.3663	0.3558	0.7227	0.7364	0.7352	0.7309
$m=14$	0.3143	0.3378	0.3352	0.3287	0.7038	0.7139	0.7145	0.7098
$m=16$	0.2906	0.3121	0.3097	0.3038	0.6851	0.6916	0.6943	0.6897
$m=18$	0.2719	0.2890	0.2865	0.2812	0.6675	0.6712	0.6739	0.6694
$m=20$	0.2535	0.2721	0.2657	0.2609	0.6502	0.6503	0.6546	0.6505

Table 2: 2D test case. Convergence factors ρ of h-multigrid as a function of m and the number of levels ($\alpha = 10$, $p = 2$).

	SIP. Triangular grids.				LDG. Cartesian grids.			
	$k=2$	$k=3$	$k=4$	$k=5$	$k=2$	$k=3$	$k=4$	$k=5$
$m=1$	0.9611	0.9669	0.9660	0.8798	0.9795	0.9801	0.9796	0.9783
$m=2$	0.9278	0.9364	0.9349	0.7784	0.9608	0.9586	0.9605	0.9577
$m=3$	0.8993	0.9093	0.9073	0.6924	0.9459	0.9431	0.9438	0.9401
$m=4$	0.8741	0.8848	0.8824	0.6185	0.9339	0.9316	0.9304	0.9264
$m=5$	0.8513	0.8625	0.8595	0.5548	0.9235	0.9217	0.9195	0.9155
$m=6$	0.8302	0.8417	0.8384	0.5014	0.9142	0.9128	0.9102	0.9061
$m=8$	0.7931	0.8043	0.8000	0.4292	0.8973	0.8967	0.8937	0.8895
$m=10$	0.7613	0.7710	0.7663	0.3893	0.8817	0.8819	0.8788	0.8740
$m=12$	0.7336	0.7416	0.7364	0.3558	0.8668	0.8676	0.8643	0.8592
$m=14$	0.7086	0.7149	0.7096	0.3287	0.8525	0.8537	0.8502	0.8444
$m=16$	0.6853	0.6910	0.6856	0.3038	0.8386	0.8401	0.8362	0.8301
$m=18$	0.6635	0.6688	0.6634	0.2812	0.8251	0.8267	0.8228	0.8162
$m=20$	0.6431	0.6483	0.6430	0.2609	0.8121	0.8138	0.8092	0.8024

As expected from Theorem 5, the convergence factor is independent of the number of levels k and decreases increasing m.

Table 3 shows the iteration counts and convergence factor (between parenthesis) of h-multigrid as a function of the polynomial approximation degree p and the number of levels k, for both SIP and LDG methods. We also provide a comparison with the iteration counts of the Conjugate Gradient (CG) algorithm. We observe that the multigrid algorithm converges faster than CG and, in accordance to (6), convergence deteriorates increasing p. In Tables 4 we show the iteration counts and the convergence factors (between parenthesis) as a function of m and k, for the p-multigrid algorithm. In this case we fix the grid and from each level to the coarser one we decrease the polynomial order in such a way that $p_{k-1} = p_k - 1$. We observe that uniformity with respect to the number of levels can be appreciated only asymptotically since the ratio p_k/p_{k-1} in our case in not constant. The data in Table 5 confirm that the convergence factor increases with p according to estimate (6), nevertheless the multigrid algorithm clearly outperform the CG method.

Table 3: 2D test case. Iteration counts and convergence factors (between parenthesis) of h-multigrid as a function of p and the number of levels k ($\alpha = 10$, $m = 6$).

	SIP. Triangular grids.			LDG. Cartesian grids.		
	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$
$p = 1$	27 (0.50)	28 (0.51)	27 (0.50)	86 (0.81)	94 (0.82)	93 (0.82)
$p = 2$	99 (0.83)	107 (0.84)	105 (0.84)	206 (0.91)	202 (0.91)	196 (0.91)
$p = 3$	204 (0.91)	219 (0.92)	199 (0.91)	280 (0.94)	278 (0.94)	239 (0.93)
$p = 4$	362 (0.95)	321 (0.94)	263 (0.93)	426 (0.96)	370 (0.95)	338 (0.95)
$p = 5$	427 (0.96)	352 (0.95)	319 (0.94)	564 (0.97)	368 (0.95)	485 (0.96)
$p = 6$	374 (0.95)	447 (0.96)	420 (0.96)	750 (0.98)	499 (0.96)	638 (0.97)
	CG iteration counts					
$p = 1$	133	273	549	130	269	638
$p = 2$	355	706	1421	276	656	1405
$p = 3$	630	1110	2239	484	1122	2393
$p = 4$	872	1788	3631	738	1706	3608
$p = 5$	1273	2597	5263	1051	2402	5049
$p = 6$	1749	3597	7314	1409	3195	6728

Table 4: 2D test case. Convergence factors ρ of p-multigrid as a function of m and the number of levels ($\alpha = 10$, $p = 5$).

	SIP. Triangular grids.			LDG. Cartesian grids.		
	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$
$m = 1$	345 (0.95)	361 (0.95)	553 (0.97)	-	-	-
$m = 2$	201 (0.91)	197 (0.91)	308 (0.94)	343 (0.95)	376 (0.95)	522 (0.97)
$m = 4$	117 (0.85)	113 (0.85)	177 (0.90)	196 (0.91)	208 (0.92)	310 (0.94)
$m = 6$	96 (0.83)	95 (0.82)	127 (0.86)	139 (0.88)	153 (0.89)	229 (0.92)
$m = 8$	83 (0.80)	82 (0.80)	100 (0.83)	133 (0.87)	133 (0.87)	183 (0.90)
$m = 10$	73 (0.78)	72 (0.77)	83 (0.80)	121 (0.86)	119 (0.86)	152 (0.89)
$m = 12$	65 (0.75)	65 (0.75)	72 (0.77)	109 (0.84)	108 (0.84)	131 (0.87)
$m = 14$	59 (0.73)	59 (0.73)	63 (0.75)	99 (0.83)	98 (0.83)	114 (0.85)
$m = 16$	54 (0.71)	54 (0.71)	56 (0.72)	91 (0.82)	90 (0.81)	102 (0.83)
$m = 18$	50 (0.69)	50 (0.69)	51 (0.70)	83 (0.80)	83 (0.80)	92 (0.82)
$m = 20$	46 (0.67)	46 (0.67)	47 (0.67)	77 (0.79)	77 (0.79)	84 (0.80)
	CG iteration counts: 2597			CG iteration counts: 2392		

As a final test, we present a three-dimensional example. We have employed an h-multigrid scheme to solve the linear system arising from a SIP discretization of a diffusion equation on $\Omega = (0, 1)^3$ employing a tetrahedral mesh. The staring mesh size is $h_1 = 0.25$, $\alpha = 10$ and $p = 1, 2, 3$. The results are reported in Table 6. We observe that the performance of our method are in agreement with the theoretical results of Theorem 5.

Table 5: 2D test case. Iteration counts and convergence factors (between parenthesis) of p-multigrid as a function of p and the number of levels ($\alpha = 10$, $m = 6$).

	SIP. Triangular grids.			
	$k = 2$	$k = 3$	$k = 4$	CG
$p = 2$	38 (0.62)	-	-	706
$p = 3$	70 (0.77)	70 (0.77)	-	1110
$p = 4$	80 (0.79)	83 (0.80)	93 (0.82)	1795
$p = 5$	96 (0.83)	95 (0.82)	127 (0.87)	2597
$p = 6$	122 (0.86)	120 (0.86)	126 (0.86)	3597
	LDG. Cartesian grids.			
	$k = 2$	$k = 3$	$k = 4$	CG
$p = 2$	101 (0.83)	-	-	652
$p = 3$	160 (0.89)	167 (0.90)	-	1127
$p = 4$	126 (0.86)	159 (0.89)	178 (0.90)	1701
$p = 5$	139 (0.88)	153 (0.89)	229 (0.92)	2392
$p = 6$	189 (0.91)	189 (0.91)	176 (0.90)	3201

Table 6: 3D test case. Convergence factors ρ of h-multigrid as a function of m and the number of levels ($\alpha = 10$, $p = 1, 2, 3$).

	$p = 1$			$p = 2$			$p = 3$	
	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$
$m = 1$	-	-	-	0.9745	0.9746	0.9722	0.9869	0.9867
$m = 2$	-	0.9544	0.8700	0.9501	0.9500	0.9452	0.9741	0.9737
$m = 3$	0.8321	0.7697	0.7509	0.9271	0.9266	0.9198	0.9617	0.9612
$m = 4$	0.7141	0.7078	0.6853	0.9052	0.9042	0.8957	0.9498	0.9491
$m = 6$	0.6125	0.6022	0.5756	0.8647	0.8625	0.8511	0.9267	0.9262
$m = 10$	0.4633	0.4440	0.4135	0.7935	0.7883	0.7734	0.8840	0.8841
$m = 14$	0.3632	0.3344	0.3055	0.7318	0.7246	0.7081	0.8453	0.8456
$m = 18$	0.2910	0.2652	0.2356	0.6786	0.6687	0.6513	0.8100	0.8103
$m = 20$	0.2633	0.2456	0.2125	0.6544	0.6437	0.6273	0.7933	0.7934

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