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On the simultaneous refinement of the zeros of H-palindromic polynomials

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Abstract

In this paper we propose a variation of the Ehrlich–Aberth method for the simultaneous refinement of the zeros of H-palindromic polynomials.

1 Introduction

The design of efficient numerical methods for solving structured generalized eigenvalue problems has attracted a growing interest in recent years due to application demands. Some interesting examples have been included in the MATLAB toolbox NLEVP [2]. In this paper we are specifically concerned with polynomial H-palindromic eigenvalue problems of the form

$$P(\lambda)\boldsymbol{x} = \left(\sum_{i=0}^{k} A_i \lambda^i\right)\boldsymbol{x} = \mathbf{0}, \quad A_{k-i}^H = A_i \in \mathbb{C}^{n\times n},\ i = 0,\dots,k. \qquad (1)$$

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The structure in the coefficient matrices of (1) induces symmetries in the spectrum of the matrix polynomial. If λ is an eigenvalue then $1/\bar{\lambda}$ is also an eigenvalue and this pairing holds even for the zero eigenvalue its counterpart being an infinite eigenvalue.

Some variants of the explicit and implicit QR eigenvalue algorithm have been devised for dealing with H-palindromic eigenvalue problems [9]. Since the corresponding structured Schur forms exist under additional conditions these algorithms are restricted to certain subclasses. Some methods for computing a structured Schur form from an unstructured one have been also proposed which can be used to post-process the output of the customary QR and QZ algorithms [11]. However these refinement techniques are subjected to the same restrictions and can suffer from numerical difficulties near exceptional eigenvalue configurations.

This paper is concerned with the computation of a structured approximation of the spectrum of a polynomial H-palindromic eigenvalue problem by means of a root-finding method. Our contribution is much in the spirit of the refinement techniques proposed in [11]. The approach taken here consists in finding the structured approximation by using a zerofinding algorithm applied for the refinement of an unstructured approximation providing a set of initial guesses. Since the focus is on the computation of the spectrum rather than of the Schur form, our approach can virtually circumvent restrictions due to the occurrence of exceptional eigenvalues.

Structure preserving rootfinders have been proposed in [10] and [5] for dealing with real and T-palindromic polynomials, respectively. Both algorithms rely upon the computation of quadratic factors associated with the desired pairing of the zeros. These factors are simultaneously approximated by using some modification of the Ehrlich–Aberth process [1, 3]. The method presented in [10] makes use of Bairstow's scheme for refining the coefficients of the quadratic factor. The scheme reduces to the Newton–Raphson iteration applied to the nonlinear system defined from the coefficients in the remainder generated from the synthetic division algorithm applied to the polynomial and its approximated quadratic factor.

The goal of this paper is to devise a similar strategy for covering with H-palindromic polynomials. The derivation of the resulting algorithm is treated in Section 2. First we observe that up to a suitable normalization a quadratic H-palindromic polynomial can be determined by two real parameters. Then we exploit the properties of a certain polynomial Diophantine equation involving the given H-palindromic polynomial of degree n and two approximated factors of degree $n-2$, and 2, respectively. It is shown that under some mild assumptions the solution consists of a a real T-palindromic polynomial of degree 2. In this way by applying the Newton–Raphson iteration to the nonlinear system given from setting the coefficients of this polynomial equal to zero we obtain a method for the refinement of the quadratic factor.

Results of numerical experiments to test our algorithm are given in Section 3. Several examples of quadratic eigenvalue problems with spectral symmetry $(\lambda, 1/\bar{\lambda})$ are considered. An initial structured approximation of the spectrum is generated from the output returned by the *polyeig* function in MATLAB and then refined by applying our simultaneous refinement procedure. Numerical tests indicate that the proposed approach is numerically robust and computationally efficient.

2 The Derivation of the Algorithm

For a given polynomial $p(z) \in \mathbb{C}[z]$ of degree n and for a given $j \in \mathbb{N}$ with $j \geq n$, the *j-reversal* polynomial $q(z)$ is defined by

$$q(z) = (\mathrm{rev}_j\ p)(z) \colon = z^j p(1/z).$$

A nonzero polynomial $p(z)$ is H-palindromic if $(\mathrm{rev}_j\ p)(z) = \bar{p}(z)$ for a certain $j \geq \deg p$. Analogously, the polynomial $p(z)$ is T-palindromic if $(\mathrm{rev}_j\ p)(z) = p(z)$ for a certain $j \geq \deg p$. The natural number j is uniquely defined and it is referred to as the *grade of palindromicity* of $p(z)$ as well as the *grade* of $p(z)$ for short [12].

A palindromic polynomial admits an irreducible factorization in terms of palindromic factors. Specifically, let $p(z) \in \mathbb{C}[z]$ be a H-palindromic polynomial of even grade n. Then it can be factored in the form [12]

$$p(z) = cz^k \prod_{j=1}^{2m} (a_j + \bar{a}_j z) \prod_{j=1}^{\ell} (z + b_j)(\bar{b}_j z + 1),$$

where $c \in \mathbb{R}$, $a_j, b_j \in \mathbb{C}$, $|b_j| \neq 1$. This factorization can be rewritten into a more compact way by grouping the zeros at the origin with their reciprocals at infinity

$$p(z) = c \prod_{j=1}^{2m} (a_j + \bar{a}_j z) \prod_{j=1}^{\ell+k} (z + b_j)(\bar{b}_j z + 1),$$

where $b_{\ell+1} = \ldots = b_{\ell+k} = 0$.

A root–finding algorithm suitably designed for H-palindromic polynomials aims to compute such a structured factorization. The following result is at the basis of the derivation of our method. It generically describes the properties of a certain Diophantine equation associated with a given H-palindromic polynomial.

Theorem 1 *Let $s(z) \in \mathbb{C}[z]$ and $q(z) = a + bz + \bar{a}z^2 \in \mathbb{C}[z]$ be two nonzero H-palindromic polynomials of grade $n_s = 2(m-1)$ and $n_q = 2$, respectively, such that $s(z)$ and $q(z)$ are relatively prime and, moreover, $a \in \mathbb{C} \setminus \mathbb{R}$. For any H-palindromic polynomial $p(z)$ of grade $n_p = 2m$ there exist uniquely*

determined a H-palindromic polynomial $t(z) \in \mathbb{C}[z]$ *of grade* $n_t = 2(m-1)$ *and a T-palindromic polynomial* $r(z) = r_0 + r_1 z + r_0 z^2 \in \mathbb{R}[z]$ *such that*

$$p(z) = t(z)q(z) + r(z)s(z). \tag{2}$$

Proof. Define $x := z + z^{-1}$ and $w := z - z^{-1}$. Observe that $x^2 - w^2 = 4$. Let us introduce the functions $c_j(x) = z^j + z^{-j}$, $j \geq 0$. Such functions are monic real Chebyshev-like polynomials of degree j satisfying the relations

$$c_j(x)c_k(x) = c_{j+k}(x) + c_{|j-k|}(x), \quad j, k, \geq 0.$$

They provide a convenient basis to represent H-palindromic polynomials. If $f(z) \in \mathbb{C}[z]$ is a H-palindromic polynomial of even grade n_f then

$$z^{-n_f/2} f(z) = f_{n_f/2} + \sum_{j=1}^{n_f/2} \mathrm{Re}(f_{n_f/2-j})(z^j + z^{-j}) + \mathrm{i}\,\mathrm{Im}(f_{n_f/2-j})(z^j - z^{-j}),$$

which gives

$$z^{-n_f/2} f(z) = f_R(x) + \mathrm{i}w f_I(x), \quad f_R(x), f_I(x) \in \mathbb{R}[z] \tag{3}$$

$$\mathrm{grade}(f_R(x)) \leq n_f/2, \ \mathrm{grade}(f_I(x)) \leq n_f/2 - 1, \tag{4}$$

where we have used the identity

$$z^j - z^{-j} = w \left(\frac{1 + (-1)^{j+1}}{2} + \sum_{\ell=1}^{\lceil j/2 \rceil} c_{j-2\ell+1}(x) \right), \quad j \geq 1.$$

By replacing the polynomials in (2) with their decompositions (3) we obtain

$$\begin{aligned} &p_R(x) + \mathrm{i}w p_I(x) = \\ &(b + \mathrm{Re}(a)x + \mathrm{i}\,\mathrm{Im}(a)w)(t_R(x) + \mathrm{i}w t_I(x)) + (r_1 + r_0 x)(s_R(x) + \mathrm{i}w s_I(x)), \end{aligned}$$

which is equivalent to the $(2m+1) \times (2m+1)$ real linear system

$$\begin{bmatrix} p_R(x) \\ p_I(x) \end{bmatrix} = \begin{bmatrix} b + \mathrm{Re}(a)x & \mathrm{Im}(a)(x^2 - 4) \\ \mathrm{Im}(a) & b + \mathrm{Re}(a)x \end{bmatrix} \begin{bmatrix} t_R(x) \\ t_I(x) \end{bmatrix} + (r_1 + r_0 t) \begin{bmatrix} s_R(x) \\ s_I(x) \end{bmatrix}. \tag{5}$$

Under the assumptions on the input polynomials $s(z)$ and $q(z)$ it is easily found that if $p(z)$ is the zero polynomial then the unique solution of (2) and, hence, of (5) is $t_R(x) = t_I(x) = 0$ and $r_0 = r_1 = 0$. This means that the coefficient matrix of (5), seen as a linear system whose variables are the coefficients of $t_R(x)$, $t_I(x)$ and r_0, r_1, is nonsingular and, therefore, the coefficients of $t_R(t)$ and $t_I(t)$ together with the scalars r_0 and r_1 are uniquely determined. □

Once two approximated factors $s(z)$ and $q(z)$ of a H-palindromic polynomial $p(z)$ are known the relations (2) and (5) could be exploited in order

to design a refinement scheme similar to the Bairstow method [7] for the computation of a quadratic factor of a real polynomial. This amounts to perform one step of the Newton-Raphson method applied for the solution of the nonlinear system

$$\begin{cases} r_0(a/|a|, b/|a|) = 0 \\ r_1(a/|a|, b/|a|) = 0. \end{cases}$$

The use of the variables $a/|a|$ and $b/|a|$ corresponds with a certain normalization of the sought quadratic factor $q(z)$ and different choices are eligible, too.

The main drawback of such an approach is that the coefficients of some expansion of $p(z)$ should be available and this is unpractical in the case where $p(z)$ is implicitly defined as the determinant of a certain matrix polynomial. To circumvent this problem we elaborate further on the relation (2). Let us recall that the zeros (ξ, η) of the quadratic polynomial

$$q(z) = a + bz + \bar{a}z^2 = \rho(e^{\mathrm{i}\theta} + gz + e^{-\mathrm{i}\theta}z^2), \quad \rho, g, \theta \in \mathbb{R},\ \rho \geq 0,$$

belong to $\Gamma \cup (\mathbb{T} \times \mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$ and $\Gamma = \{(z, 1/\bar{z}) \colon z \in \mathbb{C} \cup \{\infty\}, |z| \neq 1\}$. By a straightforward calculation there follows that if $|g| \leq 2$ then $(\xi, \eta) \in \mathbb{T} \times \mathbb{T}$; otherwise, if $|g| > 2$ then $(\xi, \eta) \in \Gamma$. The computation of (ξ, η) from the parameters θ and g involves square roots of possibly complex arguments. To avoid difficulties with complex differentiation of the square root function it is useful to express the coefficient g in term of an additional real parameter ψ as follows:

$$g = \begin{cases} \psi + \psi^{-1}, \text{ if } |\psi| > 1, \\ 2\cos((1-\psi)\pi/2), \text{ elsewhere.} \end{cases}$$

In this way we find that

$$(\xi, \eta) = (-e^{\mathrm{i}\theta}/\psi, -e^{\mathrm{i}\theta}\psi), \quad \text{for } |\psi| > 1,$$

and

$$(\xi, \eta) = (-e^{\mathrm{i}\theta}e^{\mathrm{i}\pi(1-\psi)/2}, -e^{\mathrm{i}\theta}e^{-\mathrm{i}\pi(1-\psi)/2}), \quad \text{for } |\psi| \leq 1.$$

In addition, since $g = g(\psi)$ is monotonically increasing one can define the inverse function $\psi = \psi(g)$.

Under the assumptions of Theorem 1 it follows that $a \neq 0$ and therefore $\xi \neq 0, \infty$ and $\eta \neq 0, \infty$. By evaluating (2) at the roots of $q(z)$ we obtain that

$$\begin{bmatrix} 1+\xi^2 & \xi \\ 1+\eta^2 & \eta \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \dfrac{p(\xi)}{s(\xi)} \\ \dfrac{p(\eta)}{s(\eta)} \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \xi^{-1} & \\ & \eta^{-1} \end{bmatrix} \begin{bmatrix} 1+\xi^2 & \xi \\ 1+\eta^2 & \eta \end{bmatrix} = \begin{bmatrix} \frac{1}{\xi}+\xi+\frac{1}{\eta}+\eta & 2 \\ \frac{1}{\eta}+\eta-\frac{1}{\xi}-\xi & 0 \end{bmatrix},$$

the linear system can equivalently be expressed as

$$\begin{bmatrix} \frac{1}{\xi}+\xi+\frac{1}{\eta}+\eta & 2 \\ \frac{1}{\eta}+\eta-\frac{1}{\xi}-\xi & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \frac{p(\xi)}{\xi s(\xi)}+\frac{p(\eta)}{\eta s(\eta)} \\ \frac{p(\eta)}{\eta s(\eta)}-\frac{p(\xi)}{\xi s(\xi)} \end{bmatrix}.$$

This formally yields

$$\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \frac{1}{\frac{1}{\eta}+\eta-\frac{1}{\xi}-\xi} A(\xi,\eta) \begin{bmatrix} \frac{p(\xi)}{\xi s(\xi)}+\frac{p(\eta)}{\eta s(\eta)} \\ \frac{p(\eta)}{\eta s(\eta)}-\frac{p(\xi)}{\xi s(\xi)} \end{bmatrix},$$

where

$$A(\xi,\eta) = \begin{bmatrix} 0 & 1 \\ \frac{1}{2}\left(\frac{1}{\eta}+\eta-\frac{1}{\xi}-\xi\right) & -\frac{1}{2}\left(\frac{1}{\xi}+\xi+\frac{1}{\eta}+\eta\right) \end{bmatrix}.$$

If $(\xi,\eta) \in \Gamma$ then $\xi = \xi(\theta,\psi) \neq \eta = \eta(\theta,\psi)$ and, moreover, under the assumptions of Theorem 1 it holds

$$-\eta^{-1}\xi^{-1} \det \begin{bmatrix} 1+\xi^2 & \xi \\ 1+\eta^2 & \eta \end{bmatrix} = \frac{1}{\eta}+\eta-\frac{1}{\xi}-\xi \neq 0.$$

Thus one deduces that

$$\begin{bmatrix} \frac{p(\xi)}{\xi s(\xi)}+\frac{p(\eta)}{\eta s(\eta)} \\ \frac{p(\eta)}{\eta s(\eta)}-\frac{p(\xi)}{\xi s(\xi)} \end{bmatrix} = \mathbf{0} \;\Rightarrow\; \begin{bmatrix} r_0(\theta,\psi) \\ r_1(\theta,\psi) \end{bmatrix} = \mathbf{0}.$$

If, otherwise, $(\xi,\eta) \in \mathbb{T}\times\mathbb{T}$, then ξ and η may virtually coincide so that we obtain

$$\frac{1}{\frac{1}{\eta}+\eta-\frac{1}{\xi}-\xi} \begin{bmatrix} \frac{p(\xi)}{\xi s(\xi)}+\frac{p(\eta)}{\eta s(\eta)} \\ \frac{p(\eta)}{\eta s(\eta)}-\frac{p(\xi)}{\xi s(\xi)} \end{bmatrix} = \mathbf{0} \;\Rightarrow\; \begin{bmatrix} r_0(\theta,\psi) \\ r_1(\theta,\psi) \end{bmatrix} = \mathbf{0}.$$

Hence, we propose to refine the coefficients of the approximated factor $q(z)$ of $p(z)$ by performing one step of the Newton-Raphson method applied

for the solution of the nonlinear system $F(\theta,\psi)=\mathbf{0}$, where $F\colon \mathbb{R}\times\mathbb{R}\to \mathbb{R}\times$ (i)$\mathbb{R}$ is the function defined by

$$F(\theta,\psi)=\begin{cases} \begin{bmatrix} \dfrac{p(\xi)}{\xi s(\xi)}+\dfrac{p(\eta)}{\eta s(\eta)} \\ \dfrac{p(\eta)}{\eta s(\eta)}-\dfrac{p(\xi)}{\xi s(\xi)} \end{bmatrix}, & \text{if } |\psi|>1, \\ \left(\dfrac{1}{\eta}+\eta-\dfrac{1}{\xi}-\xi\right)^{-1}\begin{bmatrix} \dfrac{p(\xi)}{\xi s(\xi)}+\dfrac{p(\eta)}{\eta s(\eta)} \\ \dfrac{p(\eta)}{\eta s(\eta)}-\dfrac{p(\xi)}{\xi s(\xi)} \end{bmatrix}, & \text{elsewhere.} \end{cases}$$

From a computational viewpoint it is worth noting that the function $F(\theta,\psi)$ can be expressed as

$$F(\theta,\psi)=f_\psi(\theta,\psi)\cdot A\cdot\begin{bmatrix} \dfrac{p(\xi)}{\xi s(\xi)} \\ \dfrac{p(\eta)}{\eta s(\eta)} \end{bmatrix}=f_\psi(\theta,\psi)\cdot A\cdot\begin{bmatrix} g(\xi) \\ g(\eta) \end{bmatrix},$$

where $f_\psi(\theta,\psi)$ is a scalar function, $A\in\mathbb{R}^2\times\mathbb{R}^2$ is a nonsingular constant matrix and $g(z)\colon = p(z)/(zs(z))$. This representation is exploited for the efficient computation of the Newton-Raphson iteration. Indeed, for the Jacobian matrix $J(\theta,\psi)$ we have

$$J(\theta,\psi)=J_1(\theta,\psi)+J_2(\theta,\psi),$$

where

$$J_1(\theta,\psi)=A\begin{bmatrix} g(\xi) & \\ & g(\eta) \end{bmatrix}\boldsymbol{e}\boldsymbol{e}^T\begin{bmatrix} \dfrac{\partial f_\psi}{\partial\theta}(\theta,\psi) & \\ & \dfrac{\partial f_\psi}{\partial\psi}(\theta,\psi) \end{bmatrix},\ \boldsymbol{e}=[1,1]^T,$$

and

$$J_2(\theta,\psi)=f_\psi(\theta,\psi)A\begin{bmatrix} g'(\xi) & \\ & g'(\eta) \end{bmatrix}\begin{bmatrix} \dfrac{\partial\xi}{\partial\theta}(\theta,\psi) & \dfrac{\partial\xi}{\partial\psi}(\theta,\psi) \\ \dfrac{\partial\eta}{\partial\theta}(\theta,\psi) & \dfrac{\partial\eta}{\partial\psi}(\theta,\psi) \end{bmatrix}.$$

This leads to the following scheme for computing the Newton-Raphson correction $N(\theta,\psi)\colon = -J^{-1}(\theta,\psi)F(\theta,\psi)$:

$$K(\theta,\psi)N(\theta,\psi)=-f_\psi(\theta,\psi)\boldsymbol{e},$$

where $K=K(\theta,\psi)$ is given by

$$K=\boldsymbol{e}\begin{bmatrix} \dfrac{\partial f_\psi}{\partial\theta}(\theta,\psi) \\ \dfrac{\partial f_\psi}{\partial\psi}(\theta,\psi) \end{bmatrix}+f_\psi(\theta,\psi)\begin{bmatrix} \dfrac{g'(\xi)}{g(\xi)} & \\ & \dfrac{g'(\eta)}{g(\eta)} \end{bmatrix}\begin{bmatrix} \dfrac{\partial\xi}{\partial\theta}(\theta,\psi) & \dfrac{\partial\xi}{\partial\psi}(\theta,\psi) \\ \dfrac{\partial\eta}{\partial\theta}(\theta,\psi) & \dfrac{\partial\eta}{\partial\psi}(\theta,\psi) \end{bmatrix}.$$

In this way most of computation amounts to evaluating the Newton correction of the function g which in turn reduces to computing the Newton corrections of the functions $p(z)$ and $s(z)$. The Newton correction of $p(z)$ can be determined by using the celebrated Jacobi formula for the derivative of a determinant [6]. For the function $s(z)$ it is noticed that under the process of simultaneous refinement this function is known in factored form and therefore its Newton correction reduces to a summation of fractions.

3 Numerical Results

The method proposed in the previous section has been implemented in MATLAB and then tested for the refinement of the zeros of H-palindromic polynomials given as $p(z) := \det(P(z))$, where $P(z) = P_0 + P_1 z + P_2 z^2$, $P_j \in \mathbb{C}^{n\times n}$, is a quadratic matrix polynomial satisfying the spectral symmetry $(\lambda, 1/\bar{\lambda})$. The software is available upon request by the authors.

The resulting computational process performs as follows:

1. An initial approximation $\boldsymbol{\xi} = (\xi_j)$ of the generalized eigenvalues λ_j, $1 \le j \le 2n$, of $P(z)$ is computed by using the MATLAB function *polyeig* applied to $P(z)$.

2. The list $\boldsymbol{\xi}$ is matched against the list $\boldsymbol{\nu} = (1/\bar{\xi}_j)$ to produce two different lists $\boldsymbol{\gamma}_{\rm out}$ and $\boldsymbol{\gamma}_{\rm in}$ of approximations located, respectively, outside and on the unit circle in the complex plane.

3. The elements of $\boldsymbol{\gamma}_{\rm out}$ and $\boldsymbol{\gamma}_{\rm in}$ are used to determine the initial approximations $(\theta_j^{(0)}, \psi_j^{(0)})$, $1 \le j \le n$, of the quadratic factors of $p(z)$.

4. Finally, these latter pairs provide the starting guesses of the simultaneous refinement scheme:

$$\begin{cases} g_j^{(k)}(z) = \dfrac{p(z)}{z\prod_{\ell=1,\ell\neq j}^{n}(e^{{\rm i}\theta_\ell^{(k)}} + g_\ell^{(k)} z + e^{-{\rm i}\theta_\ell^{(k)}} z^2)}; \\ K(\theta_j^{(k)}, \psi_j^{(k)}) N(\theta_j^{(k)}, \psi_j^{(k)}) = -f_{\psi_j^{(k)}}(\theta_j^{(k)}, \psi_j^{(k)})\boldsymbol{e}; \quad j = 1, \ldots, n \\ \theta_j^{(k+1)} = \theta_j^{(k)} + (N(\theta_j^{(k)}, \psi_j^{(k)}))_1 (\mathrm{mod}\ 2\pi); \\ \psi_j^{(k+1)} = \psi_j^{(k)} + (N(\theta_j^{(k)}, \psi_j^{(k)}))_2; \end{cases} \tag{6}$$

The iteration is stopped when the norm of the correction $\| N(\theta_j^{(k)}, \psi_j^{(k)}) \|$ is sufficiently small or $\min\{\mathcal{K}(P(\eta)), \mathcal{K}(P(\xi))\}$ is sufficiently large, where (η, ξ) are the roots of $e^{{\rm i}\theta_j^{(k)}} + g_j^{(k)} z + e^{-{\rm i}\theta_j^{(k)}} z^2$ and $\mathcal{K}(P(z))$ is some estimate of the condition number of $P(z)$. As observed in the previous section one internal iteration in (6) can be carried out at the cost of evaluating two Newton corrections of the corresponding function $g_j^{(k)}(z)$. This task is accomplished

in $O(n^2)$ flops by using some customary linearization of the quadratic matrix polynomial. Within the same bound we also get the estimates for the condition numbers.

Extensive numerical experiments have been conducted to check the accuracy and the robustness of the proposed approach. The test suite has been built to put in evidence the numerical behavior of our refinement method even in the cases not covered in Theorem 1. The test listing includes the following examples:

1. The quadratic eigenvalue problem $p(z) := \det(P(z)) = \det(z^2E + zF + P\bar{E}P) = 0$, where $E, F \in \mathbb{C}^{n\times n}$, $P \in \mathbb{R}^{n\times n}$ is a certain permutation matrix, $F = P\bar{F}P$ and the matrices E and F are defined as $E = I_m \otimes A_2$ and $F = (I_m \otimes (A_0 - iA_1)) + ((A_0 + iA_1) \otimes I_m)$ with $A_0, A_1, A_2 \in \mathbb{R}^{m\times m}$,

$$A_0 = \frac{(m+1)^2}{\pi^2}\left(\begin{pmatrix} -2 & 1 & \\ 1 & \ddots & 1 \\ & 1 & -2 \end{pmatrix} + \operatorname{diag}\left(a_0 + b_0 \sin\left(\frac{j\pi}{m+1}\right)\right)_{j=1:m}\right)$$

$$A_1 = \operatorname{diag}\left(a_1 + b_1 \frac{j\pi}{m+1}\left(1 - e^{-\pi(1-j/(m+1))}\right)\right)_{j=1:m}$$

$$A_2 = \operatorname{diag}\left(a_2 + b_2 \frac{j\pi^2}{m+1}\left(1 - j/(m+1)\right)\right)_{j=1:m}.$$

   This problem arises in the stability analysis of a partial differential equation with delays [13] and is considered in [4] for testing a structured variant of the QZ algorithm named PCP-Schur method, PCP being the acronym of P-conjugate-P-palindromic polynomial. Due to the property $z^2P \cdot \bar{P}(1/z) \cdot P = P(z)$ it follows that $P(z)$ has the spectral symmetry $(\lambda, 1/\bar{\lambda})$ and the PCP-Schur method is designed to maintain such pairing of the generalized eigenvalues.

2. The second example is also related to the stability analysis of delay differential equations (DDE) of the form $x'(t) = A_0x(t) + A_1x(t - r)$, $A_0, A_1 \in \mathbb{R}^{m\times m}$. The associated quadratic eigenvalue problem is $p(z) := \det(P(z)) = \det(z^2E + zF + P\bar{E}P) = 0$, where $E = I_m \otimes A_1$, $F = I_m \otimes A_0 + A_0 \otimes I_m$ and $G = A_1 \otimes I_m$. The matrix polynomial $P(z)$ has the PCP property so that $p(z)$ is H-palindromic.

3. The quadratic eigenvalue problem $p(z) := \det(P(z)) = \det(C + z^2C^H)$, where $C = A + \mathrm{i}B$, $A, B \in \mathbb{C}^{n\times n}$ and A, B are Hermitian matrices. The problem is introduced in [8] with the aim of checking the definiteness of the Hermitian matrix pencil (A, B), i.e.,

$$\min_{\boldsymbol{z}\in\mathbb{C}^n, \|\boldsymbol{z}\|_2=1} \sqrt{(z^HA\boldsymbol{z})^2 + (\boldsymbol{z}^HB\boldsymbol{z})^2} > 0.$$

Specifically, it is shown that if C is nonsingular then a necessary condition for the definiteness is that $P(z)$ has $2n$ generalized eigenvalues of unit modulus.

4. This is a modification of the *Sign1* problem in [2], that is, $q(z) := \det(Q(z)) = \det(E+FZ+Gz^2)$, $E, F, G \in \mathbb{R}^{81\times 81}$ Hermitian matrices. The spectrum of $P(z)$ is approximately located on the unit circle with two large clusters around ± 1 and two eigenvalues close $\pm$i. Here we deal with the eigenvalue problem for the quadratic matrix polynomial $P(z) = (1-z)^2 Q\left(\mathrm{i}\frac{z+1}{1-z}\right)$. Due to the role played by the Moebius transformation there follows that $P(z)$ fulfills the spectral symmetry $(\lambda, 1/\bar{\lambda})$ with eigenvalues located around the imaginary axis. Most of them are clustered near $\pm$i and one eigenvalue is very large in magnitude.

5. The last example is $p(z) := \det(P(z)) = \det(T+\gamma zI+z^2T^H)$, where $\gamma \in \mathbb{R}$ and $T \in \mathbb{C}^{100\times 100}$ is the symmetric tridiagonal Toeplitz matrix with diagonal and subdiagonal entries equal to 0 and i, respectively. It turns out that also $P(z)$ is a complex symmetric Toeplitz matrix and, therefore, there are explicit formulae for the eigenvalues and the determinant. If γ is large then the zeros of $p(z)$ are located on the imaginary axis and are greatly varying in magnitude.

Table 1 illustrates our results for the first set of test problems. We show the considered values of the parameters a_i, b_i, $0 \le i \le 2$, the size m of the coefficient matrices A_i, the number n of the sought eigenvalues, the average number m_{eval} of trace evaluations per eigenvalue performed by our algorithm and the numbers $\#_{Bair}$ and $\#_{PCP}$ of eigenvalues located on the unit circle found by our algorithm and by the PCP-Schur method in [4], respectively. The first and the second test are taken from [4]. The third example is designed to test the algorithm under the occurrence of zero-infinite pairs whereas the last two tests are taught to investigate the case of varying coefficients and/or relatively many eigenvalues on the unit circle.

For the second set of test problems we consider small size matrices specifically obtained to put in evidence some numerical difficulties arising with the PCP-Schur method. More specifically, this algorithm employs some shifting techniques based on the use of suitable Cayley (Moebius) transformations and numerical problems can be expected in the case where some eigenvalues are close to the zero of the denominator. In order to describe these problems we can set $m = 2$ and

$$A_0 = \begin{bmatrix} -5 & -3 \\ -4 & -4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -4 & -1 \\ -9 & -5 \end{bmatrix}.$$

A straightforward calculation says that there are two eigenvalues equal to -1. The Figure 1 gives a plot of the eigenvalues computed by *polyeig*, the

$\begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$	m	n	m_{eval}	$\#_{Bair}$	$\#_{PCP}$
$\begin{pmatrix} 2 & 0.3 \\ -2 & 0.2 \\ -2 & -0.3 \end{pmatrix}$	10	200	1	4	4
	20	800	1.7	4	4
	30	1800	1.8	4	4
	40	3200	1.9	4	4
$\begin{pmatrix} 0.1 & 1.1 \\ 1.7 & -1.5 \\ 0.2 & -0.2 \end{pmatrix}$	10	200	1.2	2	2
	15	450	1.4	2	2
	20	800	1.5	2	2
$\begin{pmatrix} 0.1 & 1.1 \\ 1.7 & -1.5 \\ 0.2 & -0.2\pi^2 m/(m+1)^2 \end{pmatrix}$	10	200	1.3	2	2
	15	450	1.5	2	2
	20	800	1.7	2	2
$\begin{pmatrix} -10.7 & -5.9 \\ 12.8 & -19.4 \\ -18.3 & -13.3 \end{pmatrix}$	10	200	1	10	10
	15	450	1	10	10
	20	800	1	10	10
$\begin{pmatrix} -400 & 19500 \\ 500 & 1500 \\ 9500 & -300 \end{pmatrix}$	10	200	1.8	4	4
	15	450	1.9	8	8
	20	800	1.9	12	12

Table 1: Numerical comparisons between our method and the PCP-Schur method on the first set of test problems

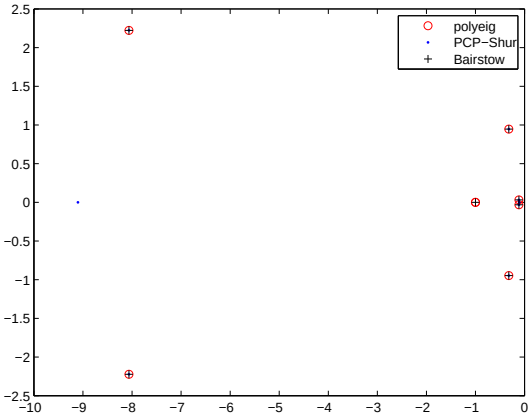

Figure 1: Comparison of eigenvalue plots for a small DDE problem

polyeig	-1.000000001625851e+00	-9.999999983741500e-01
PCP-Schur	-1.0e+00 + 9.228578184151992e-12i	-1.0e+00
Bairstow	-1.0e+00 + 2.449293598294706e-16i	-1.0e+00

Table 2: Accuracy for the approximations of the eigenvalues clustered at -1

PCP-Schur method and our refinement approach based on the Bairstow scheme. It is seen that the PCP-Schur method completely fails to detect the cluster in -1 and one spurious eigenvalue arises near -9.

Another interesting case is found by modifying the matrix A_1 as $A_1 = A_1 + \sqrt{\text{eps}}[1, 1; 0, 0]$. In this case the PCP-Schur method computes eigenvalues close to -1 but the approximations are not very accurate. In table 2 we indicate the approximations of -1 returned by *polyeig*, the PCP-Schur method and our algorithm.

In Figure 2 we check the definiteness of the matrix pair (A, B) where $A = (|i - j|)$ is the Fiedler matrix of order 40 and $B = U^T \cdot U$, U unit upper triangular with $u_{i,j} = -1$ for $j > i$, is the Moler matrix of order 40. The magnitudes of the generalized eigenvalues returned by *polyeig* are approximately between $1 - 2.0e - 12$ and $1 + 2.0e - 12$. Our algorithm determines 80 eigenvalues on the unit circle with $m_{eval} = 1$.

More difficult tests are considered in Figure 3 and Figure 4 where we apply our algorithm for testing the definiteness of the matrix pair (A, B), $A, B \in \mathbb{R}^{n\times n}$ with $n = 10, 20$, respectively, where A is the Fiedler matrix and B is the modified Moler matrix defined by $B = U^T \cdot U + \text{eps}^{3/4} \boldsymbol{e}_1 \boldsymbol{e}_1^T$, $U = (u_{i,j})$ strictly upper triangular matrix with $u_{i,j} = -1$ for $j > i$, and

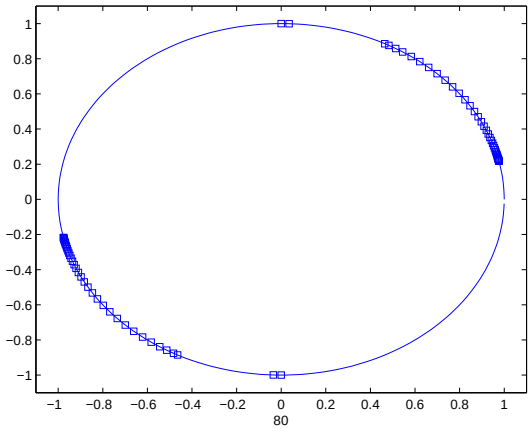

Figure 2: Eigenvalues of the Fiedler–Moler matrix pair (A, B) of order 40

eps denotes the machine precision. When $n = 10$ the magnitudes of the generalized eigenvalues computed by *polyeig* lie approximately in the interval $[1-2.0e-10, 1+2.0e-10]$ whereas for $n = 20$ the range is $[1-6.0e-10, 1+6.0e-10]$. Again in both cases our procedure determines the correct number of eigenvalues on the unit circle at the cost of one iteration per eigenvalue ($m_{eval} = 1$).

Concerning the modified *Sign1* problem in Figure 5 it is shown the finite spectrum of $P(z)$ computed by *polyeig*. The error of the matching algorithm applied to the list of initial approximations is about $1.0e-9$.

In Figure 6 and 7 we plot the finite eigenvalues computed after the refinement procedure that are located on the unit circle and on the imaginary axis, respectively. The error returned by the matching algorithm is of order of the machine precision.

The last test problem is interesting since the eigenvalues can be explicitly characterized as the set of the solutions of $n = 100$ quadratic equations. If $\gamma = 10^8$ the eigenvalues are located approximately in the interval $3.0\mathrm{i}[1.0-e-10, 1.0e+09]$ and its symmetric counterpart with respect to the origin. The list returned by *polyeig* includes many "infinite" eigenvalues. Our procedure reconstruct a correct distribution but in this case it needs many iterations ($m_{eval} = 23$) to resolve the two clusters. At the end the error returned by the matching procedure applied to the list of final approximations and of corrected values is $6.1e-07$.

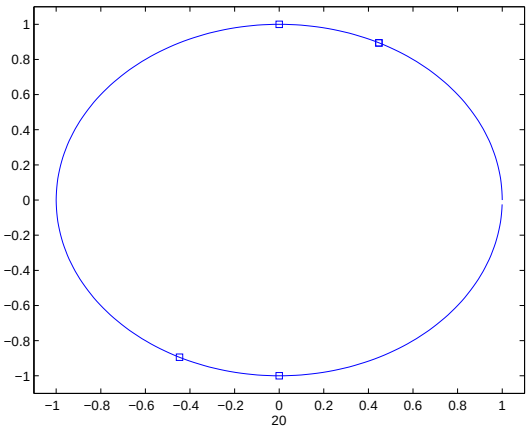

Figure 3: Eigenvalues of the Fiedler–(modified)Moler matrix pair (A, B) for $n = 10$

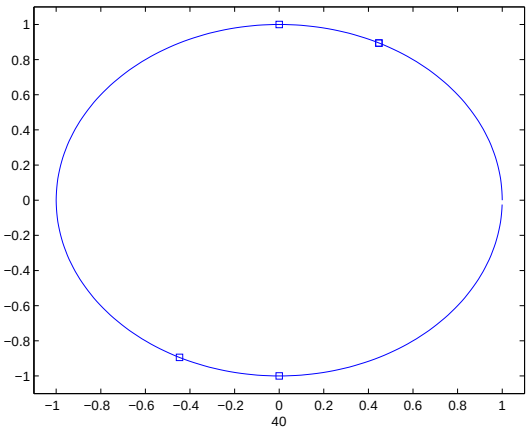

Figure 4: Eigenvalues of the Fiedler–(modified)Moler matrix pair (A, B) for $n = 20$

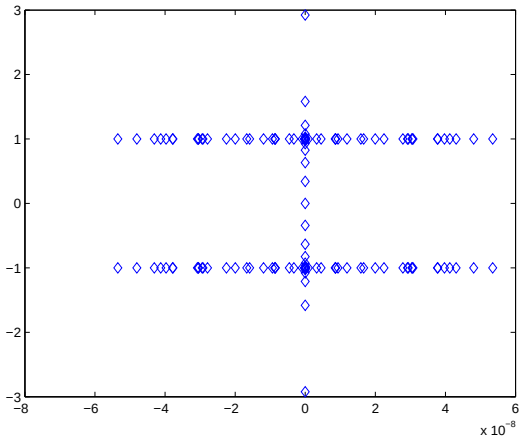

Figure 5: Finite spectrum of the modified *Sign1* problem

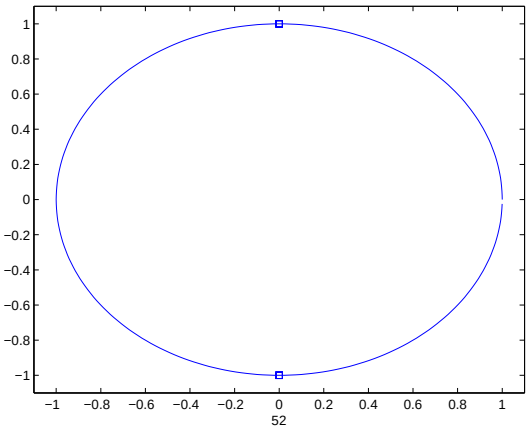

Figure 6: Finite spectrum of the modified *Sign1* problem on the unit circle

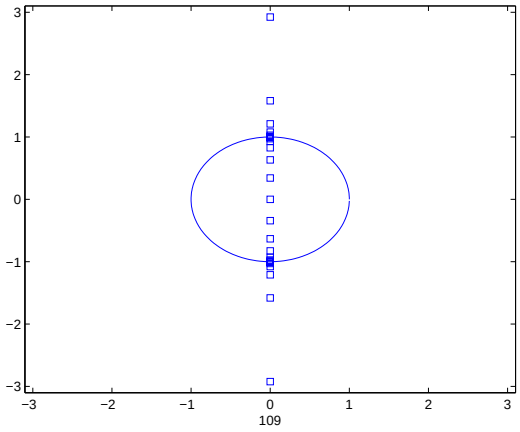

Figure 7: Finite spectrum of the modified *Sign1* problem on the imaginary axis

4 Conclusion

In this paper we have presented a refinement technique for the spectrum of a H-palindromic eigenvalue problem. Our approach incorporates the Bairstow scheme for the approximation of a quadratic factor of a polynomial into the Aberth process for the simultaneous refinement of polynomial roots. Numerical experiments show that the resulting method is computationally efficient and numerically robust. The number of refinement steps generally depends on the quality of initial approximations and possibly can increase in the case of poor starting guesses. In particular the method would benefit of a preliminary deflation of zero–infinite pairs. The design of polynomial method for this task is on ongoing research.

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