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AN OPTIMAL ADAPTIVE FICTITIOUS DOMAIN METHOD

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Abstract. We consider a Fictitious Domain formulation of an elliptic partial differential equation and approximate the resulting saddle-point system using an inexact preconditioned Uzawa iterative algorithm. Each iteration entails the approximation of an elliptic problems performed using adaptive finite element methods. We prove that the overall method converges with the best possible rate and illustrate numerically our theoretical findings.

1. Introduction

In many engineering applications the efficient numerical solution of partial differential equations on deformable or complex geometries is of paramount importance. In this respect, one crucial issue is the construction of the computational grid. To face this problem, one can basically resort to two different types of approaches. In the first approach, a mesh is constructed on a sufficiently accurate approximation of the exact physical domain (see, e.g., isoparametric finite elements [Cia02], isogeometric analysis [CHB09], or Arbitrary Lagrangian-Eulerian formulation [DGH82, HAC97, HLZ81]), while in the second approach one embeds the physical domain into a simpler computational mesh whose elements can intersect the boundary of the given domain. Clearly, the mesh generation process is extremely simplified in the second approach, while the imposition of boundary conditions requires extra work. Among the huge variety of methods sharing the philosophy of the second approach, let us mention here the Immersed Boundary methods (see, e.g., [Pes02]), the Penalty Methods (see, e.g., [Bab73]), the Fictitious Domain/Embedding Domain Methods (see, e.g., [BW90, BG03]) and the cut element method (see, e.g. [BH10, BH12]).

Following up on our earlier work [BBV16], we consider the Fictitious Domain Method with Lagrange multiplier introduced in [Glo94, GG95] (see also [Bab72] for the pioneering work inspiring this approach). In this approach, the physical domain $\widehat{\Omega}$ with boundary γ is embedded into a simpler and larger domain Ω (the fictitious domain), the right-hand side is extended to the fictitious domain and the boundary conditions on γ are appended through the use of a Lagrange multiplier. The Fictitious Domain Method gives rise to a saddle point problem whose exact primary solution restricted to $\widehat{\Omega}$ corresponds to the solution of the original problem.

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Even for smooth data, generally the solution of this saddle point problem is non-smooth. Indeed, when posed on a non-smooth, non-convex domain, generally already the solution of the original PDE will be non-smooth. Depending on the extension of the data, the solution of the extended problem might even be more singular (cf. [Mom06]). To achieve nevertheless the best possible convergence rate allowed by the polynomial orders of the applied trial spaces, we will apply an adaptive solution method.

Convergence and optimality of adaptive methods has been demonstrated for elliptic problems, but much less is known for saddle point problems. In this work, we focus on the case of having a two-dimensional domain and the application of piecewise constant trial spaces for the Lagrange multiplier λ and continuous piecewise linears for the primary variable u. At the end of this paper, we will discuss to which extent our findings generalize to more dimensions and higher order spaces. We solve the saddle-point problem with a nested inexact preconditioned Uzawa iteration. For sufficiently smooth data, it holds that $\lambda \in L_2(\gamma)$. Therefore, in view of the orders of the trial spaces there is no (qualitative) benefit in applying locally refined partitions on γ for the approximation of λ. The arising 'inner' elliptic problems for u will be solved with an adaptive finite element method (afem). A complication is that the forcing functional for these problems involves a weighted integral on γ meaning that the data is not in $L_2(\Omega)$. We apply the recently developed afem from [CDN12] that allows for data in $H^{-1}(\Omega)$. Since the Schur complement operator of our saddle point problem is an operator of order -1, the Uzawa iteration requires a preconditioner. We will apply a biorthogonal wavelet preconditioner. The overall method will be proven to converge with the best possible rate.

The outline of the paper is as follows. In Sect. 2 we recall the Fictitious Domain Method. In Sect. 3–6, we consider the solution of an abstract, infinite dimensional saddle point problem by the Uzawa iteration. We discuss the reduction of the saddle-point problem to its Schur complement (Sect. 3), preconditioning of this Schur complement (Sect. 4), a posteriori error estimation (Sect. 5), and the inexact preconditioned Uzawa iteration combined with a nested iteration technique (Sect. 6). In Sect. 7, we consider the afem from [CDN12] for solving Poisson's problem with $H^{-1}(\Omega)$ data. We show convergence and optimality of a variant that avoids an inner loop for reducing data oscillation. In Sect. 8, we apply this afem for solving the 'inner' elliptic problems in Uzawa, and show that the overall method converges with the best possible rate. In Sect. 9, we report on numerical experiments obtained with our adaptive Fictitious Domain solver. General space dimensions and/or higher order approximations will be discussed in Sect. 10. Finally, in the appendix, we construct a wavelet preconditioner for the Schur complement for our fictitious domain application.

In this work, by $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

For normed linear spaces $\mathbb{A}$ and $\mathbb{B}$, $\mathcal{L}(\mathbb{A}, \mathbb{B})$ will denote the space of bounded linear mappings $\mathbb{A} \to \mathbb{B}$ endowed with the operator norm $\|\cdot\|_{\mathcal{L}(\mathbb{A},\mathbb{B})}$. The subset of invertible operators in $\mathcal{L}(\mathbb{A}, \mathbb{B})$ with inverses in $\mathcal{L}(\mathbb{B}, \mathbb{A})$ will be denoted as $\mathcal{L}\mathrm{is}(\mathbb{A}, \mathbb{B})$.

2. Fictitious domain method

On a *two-dimensional* domain $\widehat{\Omega} \subset \mathbb{R}^2$ with Lipschitz continuous boundary γ, and $\widehat{f} \in L_2(\widehat{\Omega}) \hookrightarrow H^{-1}(\widehat{\Omega})$, $g \in H^1(\gamma) \hookrightarrow H^{\frac{1}{2}}(\gamma)$, we consider the Poisson problem

$$\left\{ \begin{array}{rcll} -\Delta \widehat{u} & = & \widehat{f} & \text{on } \widehat{\Omega}, \\ \widehat{u} & = & g & \text{on } \gamma. \end{array} \right. \tag{2.1}$$

On a Lipschitz $\Omega \subset \mathbb{R}^2$ with $\widehat{\Omega} \Subset \Omega$, $f \in L_2(\Omega)$ being an L_2-bounded extension of $\widehat{f}$, and the bilinear forms $a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx$, $b(v,\lambda) := -\int_\gamma v\lambda \, ds$, we consider the problem of finding $(u,\lambda) \in H_0^1(\Omega) \times H^{-\frac{1}{2}}(\gamma)$ such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \int_\Omega fv \, dx \quad (v \in H_0^1(\Omega)), \\ b(u,\mu) &= -\int_\gamma g\mu \, ds \quad (\mu \in H^{-\frac{1}{2}}(\gamma)). \end{aligned} \tag{2.2}$$

It is well-known that this saddle-point defines a boundedly invertible mapping between $H_0^1(\Omega) \times H^{-\frac{1}{2}}(\gamma)$ and its dual, the main ingredient being the fact that $\inf\{\|v\|_{H^1(\Omega)} \colon v|_\gamma = \mu\}$ defines an equivalent norm on $H^{\frac{1}{2}}(\gamma) = (H^{-\frac{1}{2}}(\gamma))'$.

Setting $\breve{\Omega} := \Omega \setminus \overline{\widehat{\Omega}}$, and applying integration-by-parts to both terms in $a(u,v) = \int_{\widehat{\Omega}} \nabla u \cdot \nabla v \, dx + \int_{\breve{\Omega}} \nabla u \cdot \nabla v \, dx$, one infers that $u|_{\widehat{\Omega}} = \widehat{u}$, being the solution of (2.1), that $\breve{u} := u|_{\breve{\Omega}}$ solves $-\Delta \breve{u} = f$ on $\breve{\Omega}$, $\breve{u} = g$ on γ, and $\breve{u} = 0$ on $\partial\Omega$, and finally that $\lambda = \frac{\partial \breve{u}}{\partial \vec{n}}|_\gamma - \frac{\partial \widehat{u}}{\partial \vec{n}}|_\gamma$, where $\vec{n}$ is the normal to γ exterior to $\widehat{\Omega}$.

Since these Poisson problems on both Lipschitz domains $\widehat{\Omega}$ and $\breve{\Omega}$ have forcing terms in L_2 and Dirichlet boundary data in H^1, [Neč67, Ch. 5, Thm. 1.1]

$$\lambda \in L_2(\gamma), \text{ with } \|\lambda\|_{L_2(\gamma)} \lesssim \|f\|_{L_2(\Omega)} + \|g\|_{H^1(\gamma)}. \tag{2.3}$$

We are going to approximate the solution (u,λ) of (2.2) by functions from finite element spaces, where we consider the lowest order case by taking *continuous piecewise linears* for the approximation for u, and *piecewise constants* for the approximation for λ.

Taking into account the two-dimensional domain and the orders of the finite element spaces, the error measured in $H^1(\Omega)$-norm of the best approximation for u can be expected to be generally at best of order $N^{-\frac{1}{2}}$, where N denotes the dimension of the finite element space on Ω. Moreover, under the mild Besov smoothness condition $u \in B^2_{\tau,q}(\Omega)$ for any $\tau > 1$, $q > 0$, errors of order $N^{-\frac{1}{2}}$ will be obtained with appropriately locally refined partitions as we will generate them with an adaptive finite element method. In view of (2.3), the error measured in $H^{-\frac{1}{2}}(\gamma)$-norm of the best approximation for λ from the space of piecewise constants w.r.t. a *quasi-uniform* partition of γ into N pieces is of order $N^{-\frac{1}{2}}$. Since apparently no overall (qualitative) advantage can be obtained from the application of locally refined partitions on γ, we will consider a sequence of *uniform* dyadically refined partitions on γ.

3. Saddle point problem

The variational problem that arises from the fictitious domain method is an example of a saddle point problem, that in this and the following three sections will be studied in an abstract setting.

Let $\mathbb{U}$ and $\mathbb{A}$ Hilbert spaces. For a bilinear, bounded, symmetric, and coercive $a : \mathbb{U}\times\mathbb{U} \to \mathbb{R}$, a bilinear and bounded $b : \mathbb{U}\times\mathbb{A} \to \mathbb{R}$ with $\inf_{0\neq\mu\in\mathbb{A}} \sup_{0\neq w\in\mathbb{U}} \frac{b(w,\mu)}{\|w\|_{\mathbb{U}}\|\mu\|_{\mathbb{A}}} > 0$ ('inf-sup' condition), given $(f, g) \in \mathbb{U}' \times \mathbb{A}'$ we consider the problem of finding $\boxed{(u, \lambda) \in \mathbb{U} \times \mathbb{A}}$ that satisfies

$$a(u, v) + b(v, \lambda) + b(u, \mu) = f(v) - g(\mu) \quad ((v, \mu) \in \mathbb{U} \times \mathbb{A}). \tag{3.1}$$

It is well-known that under aforementioned conditions on a and b,

$$(u, \lambda) \mapsto ((v, \mu) \mapsto a(u, v) + b(v, \lambda) + b(u, \mu)) \in \mathcal{L}\mathrm{is}(\mathbb{U} \times \mathbb{A}, (\mathbb{U} \times \mathbb{A})').$$

With $A \in \mathcal{L}\mathrm{is}(\mathbb{U}, \mathbb{U}')$, $B \in \mathcal{L}(\mathbb{U}, \mathbb{A}')$ defined by $(Au)(v) = a(u, v)$, $(Bu)(\lambda) = b(u, \lambda)$, equivalent formulations of (3.1) are given by

$$\begin{bmatrix} A & B' \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix},$$

and

$$\begin{bmatrix} A & B' \\ 0 & S \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ BA^{-1}f + g \end{bmatrix},$$

where $S := BA^{-1}B' \in \mathcal{L}(\mathbb{A}, \mathbb{A}')$ is the *Schur complement* operator. Obviously $S = S'$, and furthermore, as demonstrated by the next lemma, S is coercive (so in particular $S \in \mathcal{L}\mathrm{is}(\mathbb{A}, \mathbb{A}')$).

Lemma 3.1. *It holds that* $(S\mu)(\mu) = \sup_{0\neq v\in\mathbb{U}} \frac{b(v,\mu)^2}{a(v,v)} \eqsim \|\mu\|^2_{\mathbb{A}}$ $(\mu \in \mathbb{A})$.

Proof. Let $R_{\mathbb{U}} : \mathbb{U} \to \mathbb{U}'$ denote the Riesz map defined by $(R_{\mathbb{U}}v)(w) = \langle w, v\rangle_{\mathbb{U}}$. Writing $\tilde{B}' = R_{\mathbb{U}}^{-1}B'$, $\tilde{A} = R_{\mathbb{U}}^{-1}A$, we have

$$\begin{aligned}\sup_{0\neq v\in\mathbb{U}} \frac{b(v, \mu)^2}{a(v, v)} &= \sup_{0\neq v\in\mathbb{U}} \frac{(B'\mu)(v)^2}{(Av)(v)} = \sup_{0\neq v\in\mathbb{U}} \frac{\langle v, \tilde{B}'\mu\rangle^2_{\mathbb{U}}}{\langle v, \tilde{A}v\rangle_{\mathbb{U}}} = \sup_{0\neq w\in\mathbb{U}} \frac{\langle w, \tilde{A}^{-\frac{1}{2}}\tilde{B}'\mu\rangle^2_{\mathbb{U}}}{\langle w, w\rangle_{\mathbb{U}}} \\ &= \langle \tilde{A}^{-\frac{1}{2}}\tilde{B}'\mu, \tilde{A}^{-\frac{1}{2}}\tilde{B}'\mu\rangle_{\mathbb{U}} = \langle A^{-1}B'\mu, R_{\mathbb{U}}B'\mu\rangle_{\mathbb{U}} = (S\mu)(\mu) \quad (\mu \in \mathbb{A}).\end{aligned}$$

The second statement follows from the coercivity of a, the boundedness of b, and the inf-sup condition. ☐

As we reserved (u, λ) to denote the exact solution of the saddle point problem, next we fix some more notations that we use throughout this paper. For a finite dimensional (or more generally, closed) subspace $\mathbb{A}_\sigma \subset \mathbb{A}$, where σ runs over a collection $\mathcal{S}$, for $\chi \in \mathbb{A}$ let $\boxed{\chi_\sigma \in \mathbb{A}_\sigma}$ denote its *Galerkin approximation* defined by

$$(S\chi_\sigma)(\mu) = (S\chi)(\mu) \quad (\mu \in \mathbb{A}_\sigma). \tag{3.2}$$

This χ_σ is the best approximation to χ from $\mathbb{A}_\sigma$ w.r.t. to the 'energy-norm' $\mu \mapsto \sqrt{(S\mu)(\mu)}$. For a finite dimensional (or more generally, closed) subspace $\mathbb{U}_\tau \subset \mathbb{U}$, where τ runs over a collection $\mathcal{T}$, for $w \in \mathbb{U}$ let $\boxed{w_\tau \in \mathbb{U}_\tau}$ denote its *Galerkin approximation* defined by

$$a(w_\tau, v) = a(w, v) \quad (v \in \mathbb{U}_\tau), \tag{3.3}$$

being the best approximation to w from $\mathbb{U}_\tau$ w.r.t. $v \mapsto \sqrt{a(v, v)}$.

Given a $\chi \in \mathbb{A}$, let $\boxed{u^\chi \in \mathbb{U}}$ denote the solution of

$$a(u^\chi, v) = f(v) - b(v, \chi) \quad (v \in \mathbb{U}), \tag{3.4}$$

i.e., $u^\chi = A^{-1}(f - B'\chi)$. Note that $u^\lambda = u$. Finally, let $\boxed{(u^{\lambda_\sigma}, \lambda_\sigma) \in \mathbb{U} \times \mathbb{A}_\sigma}$ denote the solution of the *semi-discrete saddle point problem*

$$a(u^{\lambda_\sigma}, v) + b(v, \lambda_\sigma) + b(u^{\lambda_\sigma}, \mu) = f(v) - g(\mu) \quad ((v, \mu) \in \mathbb{U} \times \mathbb{A}_\sigma). \tag{3.5}$$

Note that well-posedness of the original saddle-point problem implies this for the semi-discrete one, uniform in $\sigma \in \mathcal{S}$. In other words,

$$(u, \lambda) \mapsto ((v, \mu) \mapsto a(u, v) + b(v, \lambda) + b(u, \mu)) \in \mathcal{L}\mathrm{is}(\mathbb{U} \times \mathbb{A}_\sigma, (\mathbb{U} \times \mathbb{A}_\sigma)'),$$

with both the norm of the operator and that of its inverse being uniformly bounded. Furthermore, the notations are consistent in the sense that the second component of the solution $(u^{\lambda_\sigma}, \lambda_\sigma)$ of (3.5) satisfies $(S\lambda_\sigma)(\mu) = (S\lambda)(\mu)$ $(\mu \in \mathbb{A}_\sigma)$ (cf. (3.2)), and that the first component satisfies $a(u^{\lambda_\sigma}, v) = f(v) - b(v, \lambda_\sigma)$ $(v \in \mathbb{U})$ (cf. (3.4)).

Finally, we note that well-posedness of any *fully* discrete saddle-point problem, i.e. a Ladyzhenskaya-Babuška-Brezzi (LBB) condition, will never enter our considerations.

4. Preconditioned Uzawa iteration

With $I_\sigma : \mathbb{A}_\sigma \to \mathbb{A}$ being the trivial embedding, and $I'_\sigma : \mathbb{A}' \to \mathbb{A}'_\sigma$ its adjoint, the Galerkin approximation $\lambda_\sigma \in \mathbb{A}_\sigma$ for λ solves

$$S_\sigma \lambda_\sigma = I'_\sigma(BA^{-1}f + g), \quad \text{where } S_\sigma := I'_\sigma S I_\sigma \in \mathcal{L}\mathrm{is}(\mathbb{A}_\sigma, \mathbb{A}'_\sigma). \tag{4.1}$$

At some occasions, I_σ will be omitted from the notation. To solve this system iteratively, we need a (uniform) 'preconditioner': Let $M_\sigma \in \mathcal{L}\mathrm{is}(\mathbb{A}_\sigma, \mathbb{A}'_\sigma)$ be such that $M_\sigma = M'_\sigma$, and, for some constants $r, R > 0$

$$r\|\mu\|^2_{\mathbb{A}} \le (M_\sigma \mu)(\mu) \le R\|\mu\|^2_{\mathbb{A}} \quad (\mu \in \mathbb{A}_\sigma,\, \sigma \in \mathcal{S}). \tag{4.2}$$

W.r.t. the scalar product $(\mu, \chi) \mapsto (M_\sigma\mu)(\chi)$ on $\mathbb{A}_\sigma \times \mathbb{A}_\sigma$, the operator $M_\sigma^{-1} S_\sigma : \mathbb{A}_\sigma \to \mathbb{A}_\sigma$ is symmetric, coercive, and uniformly boundedly invertible.

For solving (4.1), we consider the *damped, preconditioned Richardson iteration* that, for given $\lambda_\sigma^{(0)} \in \mathbb{A}_\sigma$, produces $(\lambda_\sigma^{(j)})_{j \ge 0} \subset \mathbb{A}_\sigma$ defined by

$$\begin{aligned} \lambda_\sigma^{(j+1)} : &= \lambda_\sigma^{(j)} + \beta M_\sigma^{-1} I'_\sigma(BA^{-1}f + g - SI_\sigma\lambda_\sigma^{(j)}) \\ &= \lambda_\sigma^{(j)} + \beta M_\sigma^{-1} I'_\sigma(Bu^{\lambda_\sigma^{(j)}} + g) \end{aligned} \tag{4.3}$$

(cf. (3.4)), in the latter form known as the (damped) *preconditioned Uzawa iteration*. Taking a constant $\beta \in \left(0, \frac{2}{\sup_{\sigma\in\mathcal{S}} \rho(M_\sigma^{-1}S_\sigma)}\right)$, the error measured in the norm on $\mathbb{A}_\sigma$ associated to either S_σ or M_σ is reduced by at least the factor

$$\rho := \sup_{\sigma\in\mathcal{S}} \rho(I - \beta M_\sigma^{-1} S_\sigma) < 1 \tag{4.4}$$

in each step. With the optimal choice

$$\beta = \frac{2}{\sup_{\sigma\in\mathcal{S}} \rho(M_\sigma^{-1}S_\sigma) + (\sup_{\sigma\in\mathcal{S}} \rho(M_\sigma S_\sigma^{-1}))^{-1}}, \tag{4.5}$$

it holds that $\rho = \frac{\kappa-1}{\kappa+1}$ where $\kappa := \sup_{\sigma\in\mathcal{S}} \rho(M_\sigma^{-1}S_\sigma) \sup_{\sigma\in\mathcal{S}} \rho(M_\sigma S_\sigma^{-1})$.=

Let Φ_σ be a basis for $\mathbb{A}_\sigma$. We set $\mathcal{F}_\sigma : \mathbb{R}^{\dim \mathbb{A}_\sigma} \to \mathbb{A}_\sigma : \mathbf{c} \mapsto \mathbf{c}^\top \Phi_\sigma$, so that, equipping $\mathbb{R}^{\dim \mathbb{A}_\sigma} \eqsim (\mathbb{R}^{\dim \mathbb{A}_\sigma})'$ with the standard Euclidean scalar product $\langle\, ,\, \rangle$, its adjoint $\mathcal{F}'_\sigma : \mathbb{A}'_\sigma \to \mathbb{R}^{\dim \mathbb{A}_\sigma}$ is the mapping $f \mapsto f(\Phi_\sigma)$. Setting $\boldsymbol{\lambda}_\sigma^{(j)} := \mathcal{F}_\sigma^{-1}\lambda_\sigma^{(j)}$, in coordinates (4.3) reads as

$$\begin{aligned}\boldsymbol{\lambda}_\sigma^{(j+1)} &= \boldsymbol{\lambda}_\sigma^{(j)} + \beta(\mathcal{F}'_\sigma M_\sigma \mathcal{F}_\sigma)^{-1}\mathcal{F}'_\sigma I'_\sigma(Bu^{\lambda_\sigma^{(j)}} + g) \\ &= \boldsymbol{\lambda}_\sigma^{(j)} + \beta \mathbf{M}_\sigma^{-1}(Bu^{\lambda_\sigma^{(j)}} + g)(\Phi_\sigma).\end{aligned}$$

with *preconditioner* $\mathbf{M}_\sigma := \mathcal{F}'_\sigma M_\sigma \mathcal{F}_\sigma$.

Example 4.1. With $R_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}'$ being the Riesz map defined by $(R_{\mathbb{A}} q)(r) = \langle r, q\rangle_{\mathbb{A}}$, the Riesz map $R_{\mathbb{A}_\sigma} : \mathbb{A}_\sigma \to \mathbb{A}'_\sigma$ is given by $I'_\sigma R I_\sigma$. For the choice $M_\sigma = R_{\mathbb{A}_\sigma}$ (which obviously satisfies (4.2)), for $\chi \in \mathbb{A}$, $\mu \in \mathbb{A}_\sigma$ we have

$$\langle M_\sigma^{-1} I'_\sigma R_{\mathbb{A}} \chi, \mu\rangle_{\mathbb{A}} = (I'_\sigma R_{\mathbb{A}}\chi)(\mu) = (R_{\mathbb{A}}\chi)(\mu) = \langle \chi, \mu\rangle_{\mathbb{A}},$$

or $M_\sigma^{-1} I'_\sigma R_{\mathbb{A}} = Q_\sigma$, being the $\mathbb{A}$-orthogonal projector onto $\mathbb{A}_\sigma$. So with this choice of M_σ, the second line in (4.3) reads as

$$\lambda_\sigma^{(j+1)} := \lambda_\sigma^{(j)} + \beta Q_\sigma R_{\mathbb{A}}^{-1}(Bu^{(j)} + g).$$

This choice of M_σ seems only practically feasible when $\mathbb{A}$ is an L_2-space.

In the setting of a stationary Stokes problem, it holds that $\mathbb{U} = H_0^1(\Omega)^n$, $\mathbb{A} = L_2(\Omega)/\mathbb{R}$, and $R_{\mathbb{A}}^{-1}B = \operatorname{div}$. So with $M_\sigma = R_{\mathbb{A}_\sigma}$, and writing $R_{\mathbb{A}}^{-1}g$ simply as g, the second line in (4.3) reads as $\lambda_\sigma^{(j+1)} := \lambda_\sigma^{(j)} + \beta Q_\sigma(\operatorname{div} u^{(j)} + g)$. From $\|\operatorname{div} \cdot\|_{L_2(\Omega)} \leq \|\nabla \cdot\|_{L_2(\Omega)^{n^2}}$ on $\mathbb{U}$, one infers that in this case one can take $\beta = 1$, see [NP04].

Example 4.2. In the case of the fictitious domain method introduced in Sect. 2, we have $\mathbb{A} = H^{-\frac{1}{2}}(\gamma)$ so that a non-trivial preconditioner is required. For the situation that $\{0\} = \mathbb{A}_{\sigma_0} \subset \mathbb{A}_{\sigma_1} \subset \cdots \subset \mathbb{A}$ is a sequence of spaces of piecewise constant functions w.r.t. to a sequence uniform dyadically refined partitions $\sigma_1 \prec \sigma_2 \prec \cdots$ of γ, with $\sigma_1 = \sigma_\perp$ being some fixed 'bottom' partition, in the appendix we describe a multilevel preconditioner that satisfies (4.2) uniformly in $\sigma \in \mathcal{S} := (\sigma_i)_{i\in\mathbb{N}}$.

Relevant references for Uzawa iterations in possibly infinite dimensional settings include [BPV97, DDU02, BMN02, Bac06]. At some places in the literature, $\mathbb{A}$ is (implicitly) identified with its dual using the Riesz map. Although appropriate for L_2 type spaces, it may obscure the need for a preconditioner in other cases.

5. A posteriori error estimation

Having available a preconditioner M_σ and a basis Φ_σ for $\mathbb{A}_\sigma$, here we use them to derive some first results on a posteriori error estimation.

Proposition 5.1. *For $\sigma \in \mathcal{S}$, let $\chi \in \mathbb{A}_\sigma$ and $\tilde{u} \in \mathbb{U}$ be approximations to λ_σ and u^χ, respectively. Then it holds that*

$$\begin{aligned}&\|\lambda_\sigma - \chi\|_{\mathbb{A}} \eqsim \|u^{\lambda_\sigma} - u^\chi\|_{\mathbb{U}} \eqsim \sup_{0\neq\mu\in\mathbb{A}_\sigma} \frac{b(u^\chi,\mu)+g(\mu)}{\|\mu\|_{\mathbb{A}}}, \\ &\left| \sup_{0\neq\mu\in\mathbb{A}_\sigma} \frac{b(u^\chi,\mu)+g(\mu)}{\|\mu\|_{\mathbb{A}}} - \sqrt{\langle \mathbf{M}_\sigma^{-1}\mathbf{r}, \mathbf{r}\rangle} \right| \lesssim \|u^\chi - \tilde{u}\|_{\mathbb{U}},\end{aligned} \tag{5.1}$$

where $\mathbf{r} := (B\tilde{u} + g)(\Phi_\sigma)$, and

$$\begin{aligned}&\|\lambda - \lambda_\sigma\|_{\mathbb{A}} \eqsim \|u - u^{\lambda_\sigma}\|_{\mathbb{U}} \eqsim \|Bu^{\lambda_\sigma} + g\|_{\mathbb{A}'}, \\ &\big|\|Bu^{\lambda_\sigma} + g\|_{\mathbb{A}'} - \|(I-P_\sigma)(B\tilde{u}+g)\|_{\mathbb{A}'}\big| \lesssim \|I - P_\sigma\|_{\mathcal{L}(\mathbb{A}',\mathbb{A}')}\|u^{\lambda_\sigma} - \tilde{u}\|_{\mathbb{U}}\end{aligned} \tag{5.2}$$

for any $P_\sigma \in \mathcal{L}(\mathbb{A}', \mathbb{A}')$ with $\ker P_\sigma \supset \mathbb{A}_\sigma^\circ := \{f \in \mathbb{A}' \colon f(\mathbb{A}_\sigma) = 0\}$.

Proof. The validity of the first $\eqsim$-symbol in the first line of (5.1) follows from

$$\sup_{0\neq v\in\mathbb{U}}\frac{a(u^{\lambda_\sigma}-u^\chi,v)}{\|v\|_{\mathbb{U}}}=\sup_{0\neq v\in\mathbb{U}}\frac{b(\chi-\lambda_\sigma,v)}{\|v\|_{\mathbb{U}}},$$

the boundedness and coercivity of a, and the boundedness and 'inf-sup condition' satisfied by b. The well-posedness, uniform in $\sigma\in\mathcal{S}$, of the semi-discrete saddle-point problem shows that

$$\begin{aligned}\|\lambda_\sigma-\chi\|_{\mathbb{A}}+\|u^{\lambda_\sigma}-u^\chi\|_{\mathbb{U}}&\eqsim\sup_{0\neq(v,\mu)\in\mathbb{U}\times\mathbb{A}_\sigma}\frac{a(u^{\lambda_\sigma}-u^\chi,v)+b(v,\lambda_\sigma-\chi)+b(u^{\lambda_\sigma}-u^\chi,\mu)}{\|v\|_{\mathbb{U}}+\|\mu\|_{\mathbb{A}}}\\&=\sup_{0\neq\mu\in\mathbb{A}_\sigma}\frac{g(\mu)+b(u^\chi,\mu)}{\|\mu\|_{\mathbb{A}}}.\end{aligned}$$

The boundedness of b shows that

$$\left|\sup_{0\neq\mu\in\mathbb{A}_\sigma}\frac{g(\mu)+b(u^\chi,\mu)}{\|\mu\|_{\mathbb{A}}}-\sup_{0\neq\mu\in\mathbb{A}_\sigma}\frac{g(\mu)+b(\tilde u,\mu)}{\|\mu\|_{\mathbb{A}}}\right|\lesssim\|u^\chi-\tilde u\|_{\mathbb{U}}.$$

The proof of (5.1) is completed by

$$\begin{aligned}\sup_{0\neq\mu\in\mathbb{A}_\sigma}\frac{g(\mu)+b(\tilde u,\mu)}{\|\mu\|_{\mathbb{A}}}&\eqsim\sup_{0\neq\mu\in\mathbb{A}_\sigma}\frac{(B\tilde u+g)(\mu)}{(M_\sigma\mu)(\mu)^{\frac12}}\\&\overset{\mu=\mathcal{F}_\sigma\mathbf{m}}{=}\sup_{0\neq\mathbf{m}\in\mathbb{R}^{\dim\mathbb{A}_\sigma}}\frac{\langle\mathbf{r},\mathbf{m}\rangle}{\langle\mathbf{M}_\sigma\mathbf{m},\mathbf{m}\rangle}=\|\mathbf{M}_\sigma^{-\frac12}\mathbf{r}\|.\end{aligned}$$

Using the same arguments one infers the first $\eqsim$-symbol in the first line of (5.2) and

$$\|\lambda-\lambda_\sigma\|_{\mathbb{A}}+\|u-u^{\lambda_\sigma}\|_{\mathbb{U}}\eqsim\|Bu^{\lambda_\sigma}+g\|_{\mathbb{A}'}=\|(I-P_\sigma)(Bu^{\lambda_\sigma}+g)\|_{\mathbb{A}'},$$

where we used that $b(u^{\lambda_\sigma},\mu)+g(\mu)=0$ for all $\mu\in\Lambda_\sigma$ is equivalent to $Bu^{\lambda_\sigma}+g\in\mathbb{A}_\sigma^\circ$. Now the second line of (5.2) is obvious. $\square$

The a posteriori estimators from Proposition 5.1 read as $\langle\mathbf{M}_\sigma^{-1}\mathbf{r},\mathbf{r}\rangle$, where $\mathbf{r}:=(B\tilde u+g)(\Phi_\sigma)$, and $\|(I-P_\sigma)(B\tilde u+g)\|_{\mathbb{A}'}$ where $\tilde u\in\mathbb{U}$ with $\tilde u\approx u^{\lambda_\sigma}$ or $\tilde u\approx u^\chi$. To arrive at implementable estimators, $\tilde u$ will be a Galerkin approximation to the solutions u^{λ_σ} or u^χ of the 'inner' elliptic problem. Now to evaluate the upper bounds in the second lines of (5.1) and (5.2), for our fictitious domain application an a posteriori error estimator for $\|u^\chi-\tilde u\|_{\mathbb{U}}$ (modulo 'data oscillation') will be given in Sect. 7.2.

6. Nested inexact preconditioned Uzawa iteration

Returning to the preconditioned Uzawa iteration (4.3), in order to arrive at an implementable method we will allow for $u^{\lambda_\sigma^{(j)}}$ to be replaced by an approximation. Furthermore, eventually aiming at a method of optimal computational complexity, we will combine the preconditioned Uzawa iteration with the concept of *nested iteration*: Let $\{0\}=\mathbb{A}_{\sigma_0}\subset\mathbb{A}_{\sigma_1}\subset\cdots\subset\mathbb{A}$ be such that for some constants $\zeta>1$, $L=L(f,g)>0$ (with $L(\xi f,\xi g)=|\xi|L(f,g)$), it holds that

$$\|\lambda-\lambda_{\sigma_i}\|_{\mathbb{A}}\leq L\zeta^{-i}. \tag{6.1}$$

We consider the *nested inexact preconditioned Uzawa iteration* that, with $\lambda_{\sigma_0}^{(K)} = \lambda_{\sigma_0} = 0$, for $i = 1, 2, \cdots$ produces $(\lambda_{\sigma_i}^{(j)})_{0 \leq j \leq K}$ defined by

$$\lambda_{\sigma_i}^{(j)} = \begin{cases} \lambda_{\sigma_{i-1}}^{(K)} & j = 0, \\ \lambda_{\sigma_i}^{(j-1)} + \beta M_{\sigma_i}^{-1} I'_{\sigma_i}(Bu^{(i,j-1)} + g) & 1 \leq j \leq K, \end{cases}$$

where $u^{(i,j-1)} \in \mathbb{U}$ is such that

$$\|u^{\lambda_{\sigma_i}^{(j-1)}} - u^{(i,j-1)}\|_{\mathbb{U}} \leq L\zeta^{-i}. \tag{6.2}$$

In the next two sections, such $u^{(i,j-1)}$ will be found as Galerkin approximations w.r.t. adaptively generated partitions.

Lemma 6.1. *With β and ρ from (4.4), given a constant $M > \frac{\beta\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{A}')}}{(1-\rho)r}$ let $K = K(M)$ be a sufficiently large constant such that $\frac{1}{\sqrt{r}}\big[\rho^K((1+\zeta) + M\zeta) + \frac{1}{1-\rho}\frac{\beta}{\sqrt{r}}\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{A}')}\big] \leq M$. Then, assuming (6.1) and (6.2), we have*

$$\|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(j)}\|_{\mathbb{A}} \leq \tfrac{1}{\sqrt{r}}\big[\rho^j((1+\zeta) + M\zeta) + \tfrac{1}{1-\rho}\tfrac{\beta}{\sqrt{r}}\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{A}')}\big]L\zeta^{-i} \lesssim L\zeta^{-i}$$

$(i \geq 0,\ 0 \leq j \leq K)$, and so in particular,

$$\|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(K)}\|_{\mathbb{A}} \leq ML\zeta^{-i} \qquad (i \geq 0).$$

Furthermore,

$$\|u - u^{(i,j)}\|_{\mathbb{U}} \leq \|A^{-1}B\|_{\mathcal{L}(\mathbb{A},\mathbb{U})}\big(\|\lambda - \lambda_{\sigma_i}\|_{\mathbb{A}} + \|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(j)}\|_{\mathbb{A}}\big) + L\zeta^{-i} \lesssim L\zeta^{-i}.$$

Proof. For $i \geq 1$, define $\|\mu\|_{\sigma_i} := (M_{\sigma_i}\mu)(\mu)^{\frac{1}{2}}$ $(\mu \in \Lambda_{\sigma_i})$. Then, for $0 \leq j \leq K-1$,

$$\|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(j+1)}\|_{\sigma_i} \leq \rho\|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(j)}\|_{\sigma_i} + \tfrac{\beta}{\sqrt{r}}\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{A}')}L\zeta^{-i},$$

where to arrive at the last term we used (6.2) and that the norm on $\mathbb{A}'_{\sigma_i}$ dual to $\|\cdot\|_{\sigma_i}$ is at most a factor $1/\sqrt{r}$ larger that the norm on $\mathbb{A}'_{\sigma_i}$ dual to $\|\cdot\|_{\mathbb{A}}$. By (6.1) and induction, we have

$$\begin{aligned} \|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(0)}\|_{\sigma_i} &= \|\lambda_{\sigma_i} - \lambda_{\sigma_{i-1}}^{(K)}\|_{\sigma_i} \\ &\leq \sqrt{R}(\|\lambda_{\sigma_i} - \lambda\|_{\mathbb{A}} + \|\lambda - \lambda_{\sigma_{i-1}}\|_{\mathbb{A}} + \|\lambda_{\sigma_{i-1}} - \lambda_{\sigma_{i-1}}^{(K)}\|_{\mathbb{A}}) \\ &\leq \sqrt{R}((1+\zeta) + M\zeta)L\zeta^{-i}, \end{aligned}$$

and so for $0 \leq j \leq K$,

$$\begin{aligned} \|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(j)}\|_{\mathbb{A}} &\leq \tfrac{1}{\sqrt{r}}\|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(j)}\|_{\sigma_i} \\ &\leq \tfrac{1}{\sqrt{r}}\big[\rho^j((1+\zeta) + M\zeta) + \tfrac{1}{1-\rho}\tfrac{\beta}{\sqrt{r}}\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{A}')}\big]L\zeta^{-i}, \end{aligned} \tag{6.3}$$

which completes the proof of the first two statements by definition of M.

The second statement follows from

$$\begin{aligned} \|u - u^{(i,j)}\|_{\mathbb{U}} &\leq \|u - u^{\lambda_{\sigma_i}^{(j)}}\|_{\mathbb{U}} + \|u^{\lambda_{\sigma_i}^{(j)}} - u^{(i,j)}\|_{\mathbb{U}} \\ &\leq \|A^{-1}B\|_{\mathcal{L}(\mathbb{A},\mathbb{U})}\big(\|\lambda - \lambda_{\sigma_i}\|_{\mathbb{A}} + \|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(j)}\|_{\mathbb{A}}\big) + L\zeta^{-i} \end{aligned}$$

together with (6.1) and (6.3). □

7. Inner elliptic solver

Inside the nested inexact preconditioned Uzawa iteration, we need to find a sufficiently accurate approximation $u^{(i,j-1)}$ for $u^{\lambda_{\sigma_i}^{(j-1)}}$, cf. (6.2). This $u^{\lambda_{\sigma_i}^{(j-1)}}$ is the solution in $\mathbb{U}$ of the elliptic problem $a(u^\chi, v) = f(v) - b(v, \chi)$ $(v \in \mathbb{U})$, cf. (3.4), with χ reading as $\lambda_{\sigma_i}^{(j-1)}$. In the application of the fictitious domain method, this problem reads as solving $u^\chi \in H_0^1(\Omega)$ that satisfies

$$\int_\Omega \nabla u^\chi \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_\gamma \chi v \, ds \quad (v \in H_0^1(\Omega)). \tag{7.1}$$

Recall that $\Omega \subset \mathbb{R}^2$, $\gamma \subset \Omega$ is a Lipschitz curve, and $f \in L_2(\Omega)$. For the moment, we consider this problem for some arbitrary, but fixed $\chi \in L_2(\Omega)$. The discussion how to deal with the fact that $\chi = \lambda_{\sigma_i}^{(j-1)}$ varies with i and j will be postponed to Sect. 8.

For solving (7.1) we will apply an adaptive linear finite element method. The adaptive triangulations will be generated by newest vertex bisection.

7.1. Newest vertex bisection.

We recall some properties of newest vertex bisection. Proofs can be found on several places in the literature, e.g. in [BDD04, Ste07]. Let $\tau_\perp$ be a fixed conforming 'bottom' triangulation of Ω. Let the assignment of the newest vertices in $\tau_\perp$ be such that if for $T, T' \in \tau_\perp$ the edge $T \cap T'$ is opposite to the newest vertex in T, then it is opposite to the newest vertex in T'. In [BDD04], it was shown that such an assignment always exists.

The infinite family of triangulations that can be created from $\tau_\perp$ by newest vertex bisection is *uniformly shape regular* (only dependent on $\tau_\perp$). The subset of this family of triangulations that additionally is *conforming* will be denoted as $\mathcal{T}$. For $\tau, \tau^* \in \mathcal{T}$, we write $\tau \preceq \tau^*$ $(\tau \prec \tau^*)$ if τ^* is a (strict) refinement of τ. For $\tau, \tau^* \in \mathcal{T}$, we will denote the smallest common refinement of τ and τ^* as $\tau \oplus \tau^*$. It is a triangulation in $\mathcal{T}$, and

$$\#\tau \oplus \tau^* \leq \#\tau + \#\tau^* - \#\tau_\perp.$$

For any collection ω of triangles, let $\mathcal{N}(\omega)$ the set of vertices of $T \in \omega$. For $\tau \in \mathcal{T}$ and $z \in \mathcal{N}(\tau)$, let $\phi_z = \phi_{\tau,z}$ denote the continuous piecewise linear function w.r.t. τ that satisfies $\phi_z(z') = \delta_{zz'}$ $(z' \in \mathcal{N}(\tau))$. We denote by $\Gamma(\tau)$ the set of all edges of τ that are not on $\partial\Omega$. We set $\omega_z = \omega_{\tau,z} := \operatorname{supp} \phi_z$, and let $\Gamma(\omega_z)$ denote the collection of edges of τ that are not on $\partial\omega_z$.

For $\tau \in \mathcal{T}$ and $\mathcal{M} \subset \mathcal{N}(\tau)$, we let

$$\texttt{refine}(\tau, \mathcal{M})$$

denote the procedure that produces the smallest triangulation in $\mathcal{T}$ in which for any $z \in \mathcal{M}$ any $\tau \ni T \subset \omega_z$ has been replaced by at least four subtriangles. The following theorem is an easy consequence of [BDD04, Thm. 2.4].

Theorem 7.1. *Let $(\tau_k)_{k \geq 0}$ defined by $\tau_0 = \tau_\perp$ and $\tau_{k+1} := \texttt{refine}(\tau_k, \mathcal{M}_k)$ for some $\mathcal{M}_k \subset \mathcal{N}(\tau_k)$. Then*

$$\#\tau_k - \#\tau_\perp \lesssim \sum_{j=0}^{k-1} \#\mathcal{M}_j.$$

7.2. **A posteriori error estimation for the 'inner' elliptic problem.** Standard a posteriori error estimation for the Poisson problem requires the forcing function to be in $L_2(\Omega)$. Our problem (7.1) does not satisfy this condition because of its second forcing term. We will therefore use results from [CDN12] about a posteriori error estimation for general forcing functions in $H^{-1}(\Omega)$, and their implementable specializations to forcing functions of types $v \mapsto \int_\Omega hv\,dx$ and $v \mapsto \int_\gamma hv\,ds$ where, for some $p > 1$, $h \in L_p(\Omega)$ or $h \in L_p(\gamma)$, respectively. In view of our application, however, for simplicity we consider the case $p = 2$ only.

For $\tau \in \mathcal{T}$, we set $\mathbb{U}_\tau := \{w \in H^1_0(\Omega) \colon w|_T \in \mathcal{P}_1(T)\}$. We let

$$\texttt{solve}(\tau, f, \chi)$$

denote the procedure that computes the Galerkin approximation u^χ_τ from $\mathbb{U}_\tau$ to the solution u^χ of (7.1) . For $U \in \mathbb{U}_\tau$, $z \in \mathcal{N}(\tau)$, we set

$$\begin{aligned}
j(U, \tau, z) &:= \Big(\sum_{e \in \Gamma(\omega_{\tau,z})} |e|^2 [\![\nabla U \cdot \mathbf{n}_e]\!]^2 \Big)^{\frac{1}{2}},\\
d_\Omega(f, \tau, z) &:= \Big(|\omega_{\tau,z}| \int_\Omega |f|^2 \phi_{\tau,z}\, dx \Big)^{\frac{1}{2}},\\
d_\gamma(\chi, \tau, z) &:= \Big(|\omega_{\tau,z}|^{\frac{1}{2}} \int_\gamma |\chi|^2 \phi_{\tau,z}\, ds \Big)^{\frac{1}{2}},\\
e(U, f, \chi, \tau, z) &:= \Big(j(U, \tau, z)^2 + d_\Omega(f, \tau, z)^2 + d_\gamma(\chi, \tau, z)^2 \Big)^{\frac{1}{2}},
\end{aligned}$$

where $[\![\nabla U \cdot \mathbf{n}_e]\!]$ denotes the jump in the normal derivative of U over e, $|e| :=$ meas(e), and $|\omega_{\tau,z}| := \max_{\tau \ni T \subset \omega_z} \text{meas}(T)$. For $\mathcal{M} \subset \mathcal{N}(\tau)$ we set

$$\begin{aligned}
J(U, \tau, \mathcal{M}) &:= \big(\sum_{z \in \mathcal{M}} j(U, \tau, z)^2 \big)^{\frac{1}{2}}\\
\mathcal{D}_\Omega(f, \tau, \mathcal{M}) &:= \big(\sum_{z \in \mathcal{M}} d_\Omega(f, \tau, z)^2 \big)^{\frac{1}{2}},\\
\mathcal{D}_\gamma(\chi, \tau, \mathcal{M}) &:= \big(\sum_{z \in \mathcal{M}} d_\gamma(\chi, \tau, z)^2 \big)^{\frac{1}{2}},\\
\mathcal{D}(f, \chi, \tau, \mathcal{M}) &:= \big(\mathcal{D}_\Omega(f, \tau, \mathcal{M})^2 + \mathcal{D}_\gamma(\chi, \tau, \mathcal{M})^2 \big)^{\frac{1}{2}},\\
\mathcal{E}(U, f, \chi, \tau, \mathcal{M}) &:= \big(\sum_{z \in \mathcal{M}} e(U, f, \chi, \tau, z)^2 \big)^{\frac{1}{2}}. \qquad (7.2)
\end{aligned}$$

In the last five notations, we will sometimes drop the argument $\mathcal{M}$ from the left hand side in case it is equal to $\mathcal{N}(\tau)$. In the last notation, sometimes we drop the argument U at both sides in case it is equal to u^χ_τ.

Finally, we set

$$\text{Err}(f, \chi, \tau) := \big(|u^\chi - u^\chi_\tau|^2_{H^1(\Omega)} + \mathcal{D}(f, \chi, \tau)^2 \big)^{\frac{1}{2}},$$

which is sometimes called the *total error*.

Remark 7.2. Since neighboring triangles in $\tau \in \mathcal{T}$ have uniformly comparable sizes, and the valence of any $z \in \mathcal{N}(\tau)$ is uniformly bounded, it holds that $|\omega_{\tau,z}| \eqsim$ meas$(\omega_{\tau,z})$. In [CDN12] the last expression is taken as the definition of $|\omega_{\tau,z}|$. We have chosen for the current definition of $|\omega_{\tau,z}|$ because of its property that for $\mathcal{M} \subset$

$\mathcal{N}(\tau)$, $\mathcal{T} \ni \tau^* \succeq \texttt{refine}(\tau, \mathcal{M})$, $z \in \mathcal{M}$, and $z^* \in \mathcal{N}(\tau^*)$ with $\omega_{\tau^*,z^*} \subset \omega_{\tau,z}$, it holds that $|\omega_{\tau^*,z^*}| \leq \frac{1}{4}|\omega_{\tau,z}|$, which will be used to demonstrate Lemma 7.8. (In contrast, note that under these premises, for $z \in \partial\Omega$ it is possible that $\text{meas}(\omega_{\tau^*,z^*}) = \text{meas}(\omega_{\tau,z})$).

Given $\tau \in \mathcal{T}$, $U \in \mathbb{U}_\tau$, $f \in L_2(\Omega)$, and $\chi \in L_2(\gamma)$, we let

$$\texttt{estimate}(U, f, \chi, \tau)$$

denote the procedure that computes $(e(U, f, \chi, \tau, z))_{z \in \mathcal{N}(\tau)}$.

In view of (7.1) setting $h(v) := \int_\Omega f v\, dx + \int_\gamma \chi v\, ds$, from applications of Sobolev's embedding theorem and Poincaré's inequality one may infer that

$$\|h\|_{H^{-1}(\omega_z)} := \sup_{0 \neq v \in H_0^1(\omega_z)} \frac{h(v)}{|\nabla v|_{H^1(\omega_z)}} \lesssim \Big(d_\Omega(f, \tau, z)^2 + d_\gamma(\chi, \tau, z)^2\Big)^{\frac{1}{2}} \tag{7.3}$$

(cf. [CDN12, Sect. 7.1]).

With the forcing term in (7.1) reading as an *arbitrary* $h \in H^{-1}(\Omega)$, and denoting the resulting solution simply by u, the following two lemmas were shown in [CDN12]:

Lemma 7.3 ([CDN12, Lemma 3.2], localized upper bound). *For $\tau \preceq \tau^* \in \mathcal{T}$, it holds that*

$$|u_{\tau^*} - u_\tau|_{H^1(\Omega)} \lesssim \Big(\sum_{z \in \mathcal{N}(\tau \setminus \tau^*)} j(u_\tau, \tau, z)^2 + \|h\|^2_{H^{-1}(\omega_z)}\Big)^{\frac{1}{2}},$$

and so in particular

$$|u - u_\tau|_{H^1(\Omega)} \lesssim \Big(\sum_{z \in \mathcal{N}(\tau)} j(u_\tau, \tau, z)^2 + \|h\|^2_{H^{-1}(\omega_z)}\Big)^{\frac{1}{2}}.$$

Lemma 7.4 ([CDN12, Lemma 3.3], local lower bound). *For $\tau \in \mathcal{T}$, $z \in \mathcal{N}(\tau)$, $U \in \mathbb{U}_\tau$, it holds that*

$$j(U, \tau, z) \lesssim |u - U|_{H^1(\omega_z)} + \|h\|_{H^{-1}(\omega_z)}.$$

Returning to our specific $h(v) = \int_\Omega f v\, dx + \int_\gamma \chi v\, ds$, from (7.3) and the previous two lemmas we infer the following two results:

Lemma 7.5 (localized upper bound). *There exists a constant C_{upp} such that for $\tau \preceq \tau^* \in \mathcal{T}$, it holds that*

$$|u^\chi_{\tau^*} - u^\chi_\tau|_{H^1(\Omega)} \leq C_{\text{upp}} \mathcal{E}(f, \chi, \tau, \mathcal{N}(\tau \setminus \tau^*)),$$

and so in particular,

$$|u^\chi - u^\chi_\tau|_{H^1(\Omega)} \leq C_{\text{upp}} \mathcal{E}(f, \chi, \tau).$$

Lemma 7.6 (global lower and upper bounds). *There exists a constant $c_{\text{low}} > 0$ such that for $\tau \in \mathcal{T}$*

$$c_{\text{low}} \mathcal{E}(f, \chi, \tau) \leq \text{Err}(f, \chi, \tau) \leq \sqrt{(C^2_{\text{upp}} + 1)}\, \mathcal{E}(f, \chi, \tau).$$

7.3. **Contraction property.** Further results about the a posteriori estimator established in [CDN12] will be combined with standard arguments in adaptive finite element theory to show that a weighted sum of the squared error in the Galerkin solution and the squared error estimator contracts when employing bulk chasing.

Whereas the adaptive finite element method investigated in [CDN12] involves an inner loop to reduce data oscillation, this loop will be avoided in our adaptive method.

Lemma 7.7 (stability of the jump estimator)**.** *There exists a constant* $C_{\rm st}$ *such that for* $\tau \in \mathcal{T}$*,* $U, W \in \mathbb{U}_\tau$*, it holds that*

$$|J(U,\tau) - J(W,\tau)| \le C_{\rm st}|U - W|_{H^1(\Omega)}.$$

Proof. Application of triangle inequalities shows that $|J(U,\tau) - J(W,\tau)| \le J(U - W,\tau)$. Now the result follows from an application of Lemma 7.4 with 'h'$= 0$, and thus 'u'$= 0$, and 'U'$=U - W$. □

The next lemma shows reduction of the estimator when employing bulk chasing under the unrealistic assumption that the discrete solution does not change. This assumption will be removed later.

Lemma 7.8. *For* $\tau \in \mathcal{T}$*,* $\mathcal{M} \subset \mathcal{N}(\tau)$*,* $U \in \mathbb{U}_\tau$*, and* $\mathcal{T} \ni \tau^* \succeq \texttt{refine}(\tau, \mathcal{M})$*, it holds that*

$$\mathcal{E}(U, f, \chi, \tau^*)^2 \le \mathcal{E}(U, f, \chi, \tau)^2 - \frac{1}{2}\mathcal{E}(U, f, \chi, \tau, \mathcal{M})^2.$$

Furthermore, for $\mathcal{T} \ni \tau^* \succeq \tau$*, it holds that* $\mathcal{D}(f, \chi, \tau^*) \le \mathcal{D}(f, \chi, \tau)$*.*

Proof. For convenience of the reader we collect the arguments for these statement from the proofs of [CDN12, Lemmas 4.1, 7.1, and Theorem 7.5].

Since the normal derivative of U exhibits jumps only on inter-element boundaries of τ, and the latter belong to exactly two ω_z's for $z \in \mathcal{N}(\tau)$, we have

$$J(U, \tau^*)^2 = 2\sum_{e\in\Gamma(\tau)} \Big(\sum_{\{e^*\in\Gamma(\tau^*):\, e^*\subset e\}} |e^*|^2\Big)[\![\nabla U \cdot \mathbf{n}_e]\!]^2.$$

On the other hand, we have

$$J(U, \tau)^2 = 2\sum_{e\in\Gamma(\tau)} |e|^2[\![\nabla U \cdot \mathbf{n}_e]\!]^2.$$

For any $e \in \Gamma(\tau)$ we have $\sum_{\{e^*\in\Gamma(\tau^*):\, e^*\subset e\}} |e^*|^2 \le |e|^2$. Since for $e \in \Gamma(\omega_z)$ for some $z \in \mathcal{M}$, $\sum_{\{e^*\in\Gamma(\tau^*):\, e^*\subset e\}} |e^*|^2 \le \frac{1}{2}|e|^2$, one infers that

$$J(U, \tau^*)^2 \le \frac{1}{2}J(U, \tau^*, \mathcal{M})^2 + J(U, \tau^*, \mathcal{N}(\tau) \setminus \mathcal{M})^2. \tag{7.4}$$

Next we consider the data oscillation estimators. Since $\phi_{\tau,z} = \sum_{z^*\in\mathcal{N}(\tau^*)} \phi_{\tau,z}(z^*)\phi_{\tau^*,z^*}$, $\sum_{z\in\mathcal{N}(\tau)} \phi_{\tau,z}(z^*) = 1$ for any z^*, $\phi_{\tau,z} \ge 0$, and $\phi_{\tau,z}(z^*) \ne 0$ only if $\omega_{\tau^*,z^*} \subset \omega_{\tau,z}$, we have

$$
\begin{aligned}
\mathcal{D}_\Omega(f,\tau^*)^2 &= \sum_{z^*\in\mathcal{N}(\tau^*)} |\omega_{\tau^*,z^*}| \int_\Omega |f|^2 \phi_{\tau^*,z^*}\,dx \\
&= \sum_{z^*\in\mathcal{N}(\tau^*)} \sum_{z\in\mathcal{N}(\tau)} \phi_{\tau,z}(z^*) |\omega_{\tau^*,z^*}| \int_\Omega |f|^2 \phi_{\tau^*,z^*}\,dx \\
&= \sum_{z\in\mathcal{N}(\tau)} \sum_{\{z^*\in\mathcal{N}(\tau^*)\colon \omega_{\tau^*,z^*}\subset\omega_{\tau,z}\}} \phi_{\tau,z}(z^*) |\omega_{\tau^*,z^*}| \int_\Omega |f|^2 \phi_{\tau^*,z^*}\,dx \\
&\le \frac{1}{4} \sum_{z\in\mathcal{M}} |\omega_{\tau,z}| \int_\Omega |f|^2 \sum_{z^*\in\mathcal{N}(\tau^*)} \phi_{\tau,z}(z^*)\phi_{\tau^*,z^*}\,dx \qquad (7.5)\\
&\quad + \sum_{z\in\mathcal{N}(\tau)\setminus\mathcal{M}} |\omega_{\tau,z}| \int_\Omega |f|^2 \sum_{z^*\in\mathcal{N}(\tau^*)} \phi_{\tau,z}(z^*)\phi_{\tau^*,z^*}\,dx \\
&= \frac{1}{4} \sum_{z\in\mathcal{M}} |\omega_{\tau,z}| \int_\Omega |f|^2 \phi_{\tau,z}\,dx + \sum_{z\in\mathcal{N}(\tau)\setminus\mathcal{M}} |\omega_{\tau,z}| \int_\Omega |f|^2 \phi_{\tau,z}\,dx \\
&= \frac{1}{4}\mathcal{D}_\Omega(f,\tau,\mathcal{M})^2 + \mathcal{D}_\Omega(f,\tau,\mathcal{N}(\tau)\setminus\mathcal{M})^2.
\end{aligned}
$$

Notice that we used our definition of $|\omega_{\tau^*,z^*}|$, see Remark 7.2, to obtain the above inequality.

Since exactly the same arguments show that

$$
\mathcal{D}_\gamma(\chi,\tau^*)^2 \le \frac{1}{2}\mathcal{D}_\gamma(\chi,\tau,\mathcal{M})^2 + \mathcal{D}_\gamma(\chi,\tau,\mathcal{N}(\tau)\setminus\mathcal{M})^2, \tag{7.6}
$$

and combining the latter with (7.4) and (7.5) completes the proof of the first statement.

The second statement is an easy consequence of (7.5) and (7.6) for $\mathcal{M}=\emptyset$. $\square$

For $(e_z)_{z\in\mathcal{N}(\tau)} \subset \mathbb{R}$ and $\theta\in(0,1]$, we let

$$
\mathcal{M} := \mathtt{mark}((e_z)_{z\in\mathcal{N}(\tau)},\theta)
$$

denote the procedure that outputs a smallest $\mathcal{M}\subset\mathcal{N}(\tau)$ that satisfies the *bulk chasing condition* $\sum_{z\in\mathcal{M}} e_z^2 \ge \theta^2 \sum_{z\in\mathcal{N}(\tau)} e_z^2$.

Corollary 7.9 (contraction)**.** *Given a constant $\theta\in(0,1]$, there exists constants $\upsilon>0$ and $\alpha<1$ such that for $\tau\in\mathcal{T}$, $\mathcal{M} := \mathtt{mark}((e(f,\chi,\tau,z)_{z\in\mathcal{N}(\tau)},\theta)$, and $\mathcal{T}\ni\tau^*\succeq\mathtt{refine}(\tau,\mathcal{M})$, it holds that*

$$
|u^\chi - u^\chi_{\tau^*}|^2_{H^1(\Omega)} + \upsilon\mathcal{E}(f,\chi,\tau^*)^2 \le \alpha\Big(|u^\chi - u^\chi_\tau|^2_{H^1(\Omega)} + \upsilon\mathcal{E}(f,\chi,\tau)^2\Big).
$$

Proof. This proof follows the arguments introduced in [CKNS08].

Applications of Lemma 7.7 and that of Young's inequality show that for any $\delta>0$,

$$
\mathcal{E}(f,\chi,\tau^*)^2 \le (1+\delta)\mathcal{E}(u^\chi_\tau,f,\chi,\tau^*)^2 + (1+\delta^{-1})C_{\mathrm{st}}|u^\chi_{\tau^*} - u^\chi_\tau|^2_{H^1(\Omega)}.
$$

Using that $\mathcal{E}(u^\chi_\tau,f,\chi,\tau^*)^2 \le (1-\frac{1}{2}\theta^2)\mathcal{E}(f,\chi,\tau)^2$ by Lemma 7.8, choosing δ such that $(1+\delta)(1-\frac{1}{2}\theta^2) = (1-\frac{1}{4}\theta^2)$, using that

$$
|u^\chi - u^\chi_{\tau^*}|^2_{H^1(\Omega)} = |u^\chi - u^\chi_\tau|^2_{H^1(\Omega)} - |u^\chi_{\tau^*} - u^\chi_\tau|^2_{H^1(\Omega)},
$$

and taking υ such that $\upsilon(1+\delta^{-1})C_{\mathrm{st}}=1$, we find that

$$|u^\chi - u^\chi_{\tau^*}|^2_{H^1(\Omega)} + \upsilon\mathcal{E}(f,\chi,\tau^*)^2 \leq |u^\chi - u^\chi_\tau|^2_{H^1(\Omega)} + \upsilon(1-\frac{1}{4}\theta^2)\mathcal{E}(f,\chi,\tau)^2$$
$$\leq \Big(1-\frac{\theta^2/4}{1+C_{\mathrm{upp}}/\upsilon}\Big)\Big(|u^\chi - u^\chi_\tau|^2_{H^1(\Omega)} + \upsilon\mathcal{E}(f,\chi,\tau)^2\Big)$$

by an application of Lemma 7.5. □

7.4. **Convergence with the best possible rate.** For $s>0$ we define the approximation class $\mathcal{A}^s$ as the collection of $w\in H^1_0(\Omega)$ for which

$$|w|_{\mathcal{A}^s} := \sup_{N\in\mathbb{N}} N^s \min_{\{\tau\in\mathcal{T}\colon \#\tau-\#\tau_\perp\leq N\}} |w-w_\tau|_{H^1(\Omega)} < \infty.$$

Classical estimates show that for $s\leq\frac{1}{2}$, $H^1_0(\Omega)\cap H^{1+2s}(\Omega)\subset\mathcal{A}^s$ where it is sufficient to consider uniform refinements of $\tau_\perp$. Obviously the class $\mathcal{A}^s$ contains many more functions, which is the reason to consider adaptive methods in the first place. As shown in [BDDP02], for $s\in(0,\frac{1}{2}]$, the Besov space $B^{1+2s}_{\tau,q}(\Omega)$ is contained in $\mathcal{A}^s$ for any $q>0$, $\tau>(s+\frac{1}{2})^{-1}$. Although $\mathcal{A}^s$ is non-empty for any $s>0$ as it contains $\mathbb{U}_\tau$ for any $\tau\in\mathcal{T}$, even for $C^\infty(\Omega)$-functions only for $s\leq\frac{1}{2}$ membership in $\mathcal{A}^s$ is guaranteed. For that reason, it is no real restriction to consider only $s\in(0,\frac{1}{2}]$ in the following.

Besides the approximated classes $\mathcal{A}^s$, we need approximation classes for both data terms of the inner elliptic problem (7.1). For $f\in L_2(\Omega)$ and $s>0$, we say that $f\in\mathcal{B}^s_\Omega$ when

$$|f|_{\mathcal{B}^s_\Omega} := \sup_{N\in\mathbb{N}} N^s \min_{\{\tau\in\mathcal{T}\colon \#\tau-\#\tau_\perp\leq N\}} \mathcal{D}_\Omega(f,\tau) < \infty.$$

Similarly, for $\chi\in L_2(\gamma)$, we say that $\chi\in\mathcal{B}^s_\gamma$ when

$$|\chi|_{\mathcal{B}^s_\gamma} := \sup_{N\in\mathbb{N}} N^s \min_{\{\tau\in\mathcal{T}\colon \#\tau-\#\tau_\perp\leq N\}} \mathcal{D}_\gamma(\chi,\tau) < \infty.$$

The approximation classes $\mathcal{B}^s_\Omega$ and $\mathcal{B}^s_\gamma$ should not be confused with Besov spaces.

The next, crucial result shows that the data oscillation terms $\mathcal{D}_\Omega(f,\tau)$ and $\mathcal{D}_\gamma(\chi,\tau)$ can be reduced at rate $\frac{1}{2}$. Knowing this result, standard arguments introduced in [Ste07] will show that the usual adaptive finite element method driven by bulk chasing on the estimator $\mathcal{E}$ converges with the best possible rate $s\in(0,\frac{1}{2}]$.

Theorem 7.10 ([CDN12, Theorems 7.3 and 7.4])**.** *Functions $f\in L_2(\Omega)$ and $\chi\in L_2(\gamma)$ are in $\mathcal{B}^{\frac{1}{2}}_\Omega$ and $\mathcal{B}^{\frac{1}{2}}_\gamma$, respectively, with $|f|_{\mathcal{B}^{\frac{1}{2}}_\Omega}\lesssim\|f\|_{L_2(\Omega)}$ and $|\chi|_{\mathcal{B}^{\frac{1}{2}}_\gamma}\lesssim\|\chi\|_{L_2(\gamma)}$, only dependent on $\tau_\perp$ and, for the second case, the length of γ.*

The next lemma will be the key to bound the minimal number of nodes needed to satisfy the bulk chasing criterion, as it is realized by the routine `mark`. It shows that when τ^* is a sufficiently deep refinement of τ such that its total error is less than or equal to a certain multiple of the total error on τ, then the set of vertices of the triangles that were refined when going from τ to τ^* satisfies the bulk chasing criterion.

Lemma 7.11 (bulk chasing property)**.** *Setting*

$$\theta_* := \tfrac{c_{\mathrm{low}}}{\sqrt{1+C^2_{\mathrm{upp}}}},$$

for $\theta \in (0, \theta_*)$ and any $\mathcal{T} \ni \tau^* \succeq \tau$ with

$$\operatorname{Err}(f,\chi,\tau^*)^2 \leq \left[1 - \tfrac{\theta^2}{\theta_*^2}\right]\operatorname{Err}(f,\chi,\tau)^2, \tag{7.7}$$

it holds that

$$\mathcal{E}(f,\chi,\tau,\mathcal{N}(\tau \setminus \tau^*)) \geq \theta\mathcal{E}(f,\chi,\tau).$$

Proof. Noting that each $T \in \tau$ that contains a $z \in \mathcal{N}(\tau) \setminus \mathcal{N}(\tau \setminus \tau^*)$ is in τ^*, one infers that

$$\mathcal{D}(f,\chi,\tau)^2 \leq \mathcal{D}(f,\chi,\tau,\mathcal{N}(\tau \setminus \tau^*))^2 + \mathcal{D}(f,\chi,\tau^*)^2.$$

Now from lemmas 7.5 and 7.6, and the assumption on τ^*, we obtain that

$$\begin{aligned}
\theta^2(1 + C_{\text{upp}}^2)\mathcal{E}(f,\chi,\tau)^2 &\leq \tfrac{\theta^2}{\theta_*^2}\operatorname{Err}(f,\chi,\tau)^2 \\
&\leq \operatorname{Err}(f,\chi,\tau)^2 - \operatorname{Err}(f,\chi,\tau^*)^2 \\
&\leq |u^\chi_{\tau^*} - u^\chi_\tau|^2_{H^1(\Omega)} + \mathcal{D}(f,\chi,\tau,\mathcal{N}(\tau \setminus \tau^*))^2 \\
&\leq (1 + C_{\text{upp}}^2)\mathcal{E}(f,\chi,\tau,\mathcal{N}(\tau \setminus \tau^*))^2
\end{aligned}$$

being the statement of the lemma. $\square$

Corollary 7.12. *For $\theta \in (0, \theta_*)$, $u^\chi \in \mathcal{A}^s$ for some $s \in (0, \frac{1}{2}]$, $\tau \in \mathcal{T}$, and $\mathcal{M} = \mathtt{mark}(e(f,\chi,\tau,z)_{z \in \mathcal{N}(\tau)}, \theta)$, it holds that*

$$\#\mathcal{M} \lesssim C_s(u^\chi, f, \chi)\operatorname{Err}(f,\chi,\tau)^{-\frac{1}{s}}, \tag{7.8}$$

where

$$C_s(u^\chi, f, \chi) := \left(|u^\chi|^{\frac{1}{s}}_{\mathcal{A}^s} + \|f\|^{\frac{1}{s}}_{L_2(\Omega)} + \|\chi\|^{\frac{1}{s}}_{L_2(\gamma)}\right). \tag{7.9}$$

Proof. Since $u^\chi \in \mathcal{A}^s$, $f \in \mathcal{B}^{\frac{1}{2}}_\Omega$, $\chi \in \mathcal{B}^{\frac{1}{2}}_\gamma$, there exist $\tau_u, \tau_f, \tau_\chi \in \mathcal{T}$ such that

$$\max\Big(|u^\chi - u^\chi_{\tau_u}|_{H^1(\Omega)}, \mathcal{D}_\Omega(f,\tau_f), \mathcal{D}_\gamma(\chi,\tau_\chi)\Big) \leq \sqrt{\tfrac{1}{3}\left[1 - \tfrac{\theta^2}{\theta_*^2}\right]}\operatorname{Err}(f,\chi,\tau) =: \hat{E}, \tag{7.10}$$

and

$$\#\tau_u - \#\tau_\perp \leq |u|^{\frac{1}{s}}_{\mathcal{A}^s}\hat{E}^{-\frac{1}{s}},\ \#\tau_f - \#\tau_\perp \leq |f|^{\frac{1}{2}}_{\mathcal{B}^{\frac{1}{2}}_\Omega}\hat{E}^{-\frac{1}{2}}, \#\tau_\chi - \#\tau_\perp \leq |\chi|^{\frac{1}{2}}_{\mathcal{B}^{\frac{1}{2}}_\gamma}\hat{E}^{-\frac{1}{2}}.$$

Since the left hand sides of the last two inequalities are either 0 or ≥ 1, we also have

$$\#\tau_u - \#\tau_\perp \leq |u|^{\frac{1}{s}}_{\mathcal{A}^s}\hat{E}^{-\frac{1}{s}},\ \#\tau_f - \#\tau_\perp \leq |f|^{\frac{1}{s}}_{\mathcal{B}^{\frac{1}{2}}_\Omega}\hat{E}^{-\frac{1}{s}}, \#\tau_\chi - \#\tau_\perp \leq |\chi|^{\frac{1}{s}}_{\mathcal{B}^{\frac{1}{2}}_\gamma}\hat{E}^{-\frac{1}{s}}. \tag{7.11}$$

From (7.10) and the monotonicity of $\mathcal{D}(f,\chi,\tau)$ and $|u^\chi - u^\chi_\tau|_{H^1(\Omega)}$ as function of τ, it follows that $\tau^* := \tau \oplus \tau_u \oplus \tau_f \oplus \tau_\chi$ satisfies (7.7). In view of the bulk chasing property given by Lemma 7.11, and because $\mathcal{M}$ is a set of minimal cardinality that realizes the bulk chasing criterion, we infer that

$$\begin{aligned}
\#\mathcal{M} \leq \#\mathcal{N}(\tau \setminus \tau^*) \lesssim \#(\tau \setminus \tau^*) &\leq \#\tau^* - \#\tau \\
&\leq \#\tau_u - \#\tau_\perp + \#\tau_f - \#\tau_\perp + \#\tau_\chi - \#\tau_\perp
\end{aligned}$$

where the third inequality is a consequence of the fact that each $T \in \tau \setminus \tau^*$ has been bisected at least once. Now from (7.11), Theorem 7.10, and

$$\hat{E}^{-\frac{1}{s}} = \left(\sqrt{\tfrac{1}{3}\left[1 - \tfrac{\theta^2}{\theta_*^2}\right]}\right)^{-\frac{1}{s}}\operatorname{Err}(f,\chi,\tau)^{-\frac{1}{s}} \eqsim \operatorname{Err}(f,\chi,\tau)^{-\frac{1}{s}}, \tag{7.12}$$

the proof is completed. [1] ☐

The next result guarantees that the nested sequence $(\tau_k)_k$ produced by this adaptive finite element method reduces the total error at the best possible rate.

Theorem 7.13 (convergence with optimal rate). *Let $\theta \in (0, \theta_*)$, and $u^\chi \in \mathcal{A}^s$ for some $s \in (0, \frac{1}{2}]$. Then with τ_k denoting the partition after k iterations of the* `solve` $-$ `estimate` $-$ `mark` $-$ `refine` *loop started with $\tau_0 = \tau_\perp$, it holds that*

$$\#\tau_k - \#\tau_0 \lesssim C_s(u^\chi, f, \chi)\mathrm{Err}(f, \chi, \tau_k)^{-\frac{1}{s}},$$

where $C_s(u^\chi, f, \chi)$ is given by (7.9).

Proof. With $\mathcal{M}_i$ denoting the set of nodes that are marked in $\mathcal{N}(\tau_i)$, applications of Theorem 7.1 and Corollary 7.12 yield

$$\#\tau_k - \#\tau_\perp \lesssim \sum_{i=0}^{k-1} \#\mathcal{M}_i \lesssim C_s(u^\chi, f, \chi) \sum_{i=0}^{k-1} \mathrm{Err}(f, \chi, \tau_i)^{-\frac{1}{s}}.$$

Hence, the equivalence between Err and $\mathcal{E}$ provided by Lemma 7.6 together with the contraction property from Corollary 7.9 imply

$$\begin{aligned}\#\tau_k - \#\tau_\perp \preceq & C_s(u^\chi, f, \chi) \sum_{i=0}^{k-1} \left(\sqrt{|u^\chi - u^\chi_{\tau_i}|^2_{H^1(\Omega)} + \nu\mathcal{E}(f, \chi, \tau_i)^2} \right)^{-\frac{1}{s}} \\ & \eqsim C_s(u^\chi, f, \chi) \left(\sqrt{|u^\chi - u^\chi_{\tau_{k-1}}|^2_{H^1(\Omega)} + \nu\mathcal{E}(f, \chi, \tau_{k-1})^2} \right)^{-\frac{1}{s}}.\end{aligned}$$

Invoking Lemma 7.6 and Lemma 7.8, we arrive at

$$\#\tau_k - \#\tau_\perp \preceq C_s(u^\chi, f, \chi)\mathrm{Err}(f, \chi, \tau_{k-1})^{-\frac{1}{s}} \le C_s(u^\chi, f, \chi)\mathrm{Err}(f, \chi, \tau_k)^{-\frac{1}{s}}. \quad \square$$

8. The adaptive finite element method as an inner solver in Uzawa

We have seen that for $f \in L_2(\Omega)$, and *fixed* $\chi \in L_2(\gamma)$, the adaptive finite element method for solving (7.1) converges with the best possible rate. That is, whenever $u^\chi \in \mathcal{A}^s$ for some $s \in (0, \frac{1}{2}]$, the Galerkin approximations converge to u^χ with rate s. Now we return to the sequence of problems (7.1) where $\chi = \lambda^{(j-1)}_{\sigma_i}$, being the elliptic problems that have to be solved inside the Uzawa iteration. We aim at showing that whenever $u = u^\lambda \in \mathcal{A}^s$, the sequence of all approximations that we generate inside the nested inexact preconditioned Uzawa iteration converge to u with this rate s.

Therefore, it is needed to optimally bound the number of cells selected by any call of `mark` in terms of $|u|_{\mathcal{A}^s}$ (and that of $\|f\|_{L_2(\Omega)}$ and $\|\chi\|_{L_2(\gamma)}$), but not in terms of $|u^\chi|_{\mathcal{A}^s}$. Indeed with χ running over the $\lambda^{(j-1)}_{\sigma_i}$, we do not know whether these $u^\chi \in \mathcal{A}^s$ (with uniformly bounded $\|u^\chi\|_{\mathcal{A}^s}$) for the same value of s for which $u \in \mathcal{A}^s$. In the following we will manage to do so for calls of `mark` (and thus of `refine`, `solve` and `estimate`) that are made when the (total) error in the current Galerkin approximation for u^χ is $\gtrsim |u - u^\chi|_{H^1(\Omega)} \eqsim \|\lambda - \chi\|_{H^{-\frac{1}{2}}(\gamma)}$, cf. (8.1).

[1] Noting that $\left(\sqrt{\frac{1}{3}\left[1 - \frac{\theta^2}{\theta_*^2}\right]}\right)^{-\frac{1}{s}} \to \infty$ if, and only if, $\theta \to \theta_*$ or $s \to 0$, we conclude that the constant 'hidden' in the $\lesssim$-symbol in (7.12), and thus in (7.8), depends on the value of θ or s when they tend to θ_* or 0, respectively. Consequently, this holds true for all results that are going to derived from Corollary 7.12.

In view of the accuracy requirement on these Galerkin solutions inside the nested inexact preconditioned Uzawa iteration (cf. (6.1)-(6.2)), fortunately there is no need for another call of `mark` when this condition is violated. Finally we note that in Lemma 8.5 it will be shown that for χ running over all $\lambda_{\sigma_i}^{(j-1)}$, the norm $\|\chi\|_{L_2(\gamma)}$ will be uniformly bounded.

Lemma 8.1. *Let $\theta \in (0, \theta_*)$, and $u \in \mathcal{A}^s$ for some $s \in (0, \frac{1}{2}]$. Then for $\chi \in L_2(\gamma)$ and $\tau \in \mathcal{T}$ with*

$$\mathrm{Err}(f, \chi, \tau) \gtrsim |u - u^\chi|_{H^1(\Omega)}, \tag{8.1}$$

for $\mathcal{M} = \mathtt{mark}(e(f, \chi, \tau, z)_{z \in \mathcal{N}(\tau)}, \theta)$ it holds that

$$\#\mathcal{M} \lesssim C_s(u, f, \chi)\mathrm{Err}(f, \chi, \tau)^{-\frac{1}{s}}. \tag{8.2}$$

(Without the condition (8.1), $C_s(u, f, \chi)$ in (8.2) would have to be read as the undesirable factor $C_s(u^\chi, f, \chi)$, cf. Lemma 7.12.)

Proof. Since $u \in \mathcal{A}^s$, $f \in \mathcal{B}_\Omega^{\frac{1}{2}}$, $\chi \in \mathcal{B}_\gamma^{\frac{1}{2}}$, there exist $\tau_u, \tau_f, \tau_\chi \in \mathcal{T}$ such that

$$\max\Big(|u - u_{\tau_u}|_{H^1(\Omega)}, \mathcal{D}_\Omega(f, \tau_f), \mathcal{D}_\gamma(\chi, \tau_\chi)\Big) \leq \mathrm{Err}(f, \chi, \tau), \tag{8.3}$$

and

$$\begin{aligned} \#\tau_u - \#\tau_\perp &\leq |u|_{\mathcal{A}^s}^{\frac{1}{s}} \mathrm{Err}(f, \chi, \tau)^{-\frac{1}{s}}, \\ \#\tau_f - \#\tau_\perp &\leq |f|_{\mathcal{B}_\Omega^{\frac{1}{2}}}^{\frac{1}{s}} \mathrm{Err}(f, \chi, \tau)^{-\frac{1}{2}}, \\ \#\tau_\chi - \#\tau_\perp &\leq |\chi|_{\mathcal{B}_\gamma^{\frac{1}{2}}}^{\frac{1}{s}} \mathrm{Err}(f, \chi, \tau)^{-\frac{1}{2}}. \end{aligned} \tag{8.4}$$

Let $\tau^* := \tau_u \oplus \tau_f \oplus \tau_\chi$. Then by $|u^\chi - u^\chi_{\tau^*}|_{H^1(\Omega)} \leq |u - u_{\tau^*}|_{H^1(\Omega)} + |u - u^\chi|_{H^1(\Omega)}$, $|u - u^\chi|_{H^1(\Omega)} \lesssim \mathrm{Err}(f, \chi, \tau)$ by assumption, and $\tau \mapsto \mathcal{D}(\cdot, \cdot, \tau)$ being monotone non-increasing by Lemma 7.8, we have $\mathrm{Err}(f, \chi, \tau^*) \lesssim \mathrm{Err}(f, \chi, \tau)$.

Lemma 7.6 guarantees that

$$\sqrt{|u^\chi - u^\chi_{\tau^*}|^2_{H^1(\Omega)} + \nu \mathcal{E}(f, \chi, \tau^*)^2} \eqsim \mathrm{Err}(f, \chi, \tau^*).$$

Hence, the contraction property (Corollary 7.9) indicates that the left hand side reduces by a constant factor $\alpha < 1$ by each application of the cycle `estimate` $-$ `mark` $-$ `refine` $-$ `solve`. Therefore by applying a fixed, sufficiently large number of those cycles shows that there exists a $\breve{\tau} \in \mathcal{T}$ with $\#\breve{\tau} \lesssim \#\tau^*$ and

$$\mathrm{Err}(f, \chi, \breve{\tau})^2 \leq \big[1 - \tfrac{\theta^2}{\theta_*^2}\big]\mathrm{Err}(f, \chi, \tau)^2.$$

For $\bar{\tau} := \tau \oplus \breve{\tau}$, we have $\bar{\tau} \succeq \tau$ and $\mathrm{Err}(f, \chi, \bar{\tau})^2 \leq \mathrm{Err}(f, \chi, \breve{\tau})^2 \leq \big[1 - \frac{\theta^2}{\theta_*^2}\big]\mathrm{Err}(f, \chi, \tau)^2$, so that from the bulk chasing property given by Lemma 7.11 combined with the minimal cardinality property of the set $\mathcal{M}$, it follows that

$$\begin{aligned} \#\mathcal{M} &\leq \#\mathcal{N}(\tau \setminus \bar{\tau}) \lesssim \#(\tau \setminus \bar{\tau}) \leq \#\bar{\tau} - \#\tau \leq \#\breve{\tau} - \#\tau_\perp \\ &\lesssim \#\tau_u - \#\tau_\perp + \#\tau_f - \#\tau_\perp + \#\tau_\chi - \#\tau_\perp \leq C_s(u, f, \chi)\mathrm{Err}(f, \chi, \tau)^{-\frac{1}{s}}, \end{aligned}$$

by (8.4), and Theorem 7.10. $\square$

Instead of adaptively solving the elliptic problems (7.1) for $\chi = \lambda_{\sigma_i}^{(j-1)}$ for each i and j starting from $\tau_\perp$, we will use the final partition produced for the approximation of $u^{\lambda_{\sigma_i}^{(j)}}$ as the initial partition for the approximation for $u^{\lambda_{\sigma_i}^{(j+1)}}$ when $j < K$, and for $u^{\lambda_{\sigma_{i+1}}^{(0)}}$ otherwise.

We consider the following $\texttt{solve} - \texttt{estimate} - \texttt{mark} - \texttt{refine}$ iteration, that starts from *some* given initial triangulation $\tau_0 \in \mathcal{T}$, thus *not* necessarily equal to $\tau_\perp$, and that is completed by a stopping criterion.

Algorithm 8.2.

$[\tau_k, u^\chi_{\tau_k}] = \texttt{afem}(\tau_0, f, \chi, \varepsilon)$:
  $u^\chi_{\tau_0} = \texttt{solve}(\tau_0, f, \chi)$
  $(e(f,\chi,\tau_0,z))_{z\in\mathcal{N}(\tau_0)} = \texttt{estimate}(u^\chi_{\tau_0}, f, \chi)$
  $k = 0$
  $\texttt{while}\ C_{\rm upp}\mathcal{E}(f,\chi,\tau_k) > \varepsilon\ \texttt{do}$
    $\mathcal{M}_k = \texttt{mark}((e(f,\chi,\tau_k,z))_{z\in\mathcal{N}(\tau_k)}, \theta)$
    $\tau_{k+1} = \texttt{refine}(\tau_k, \mathcal{M}_k)$
    $u^\chi_{\tau_{k+1}} = \texttt{solve}(\tau_{k+1}, f, \chi)$
    $(e(f,\chi,\tau_{k+1},z))_{z\in\mathcal{N}(\tau_{k+1})} = \texttt{estimate}(u^\chi_{\tau_{k+1}}, f, \chi)$
    $k \leftarrow k+1$
  $\texttt{enddo}$

Lemma 8.3. *Let $\theta \in (0,\theta_*)$, $u \in \mathcal{A}^s$ for some $s \in (0,\frac{1}{2}]$, $\chi \in L_2(\gamma)$, $\tau_0 \in \mathcal{T}$, and $\varepsilon > 0$ with*

$$\varepsilon \gtrsim |u - u^\chi|_{H^1(\Omega)}.$$

Let $\tau_0 \prec \cdots \prec \tau_m \subset \mathcal{T}$ denote the sequence of triangulations that is produced by the call $\texttt{afem}(\tau_0, f, \chi, \varepsilon)$, and for $0 \le k \le m-1$, let $\mathcal{M}_k \subset \mathcal{N}(\tau_k)$ denote the sets of nodes that were marked. Then

$$\sum_{k=0}^{m-1} \#\mathcal{M}_k \lesssim C_s(u,f,\chi)\varepsilon^{-1/s},$$

and $|u^\chi - u^\chi_{\tau_m}|_{H^1(\Omega)} \le \varepsilon$, where $C_s(u,f,\chi)$ is given by (7.9).

Proof. The last statement is valid by Lemma 7.5 because the algorithm terminates as a consequence of Corollary 7.9.

For $0 \le k < m$, $\mathrm{Err}(f,\chi,\tau_k) \eqsim C_{\rm upp}\mathcal{E}(f,\chi,\tau_k) > \varepsilon \gtrsim |u - u^\chi|_{H^1(\Omega)}$, where the strict inequality holds for otherwise the algorithm would have stopped at iteration k. By Lemma 8.1, we deduce that $\#\mathcal{M}_k \lesssim C_s(u,f,\chi)\mathrm{Err}(f,\chi,\tau_k)^{-\frac{1}{s}}$. As in the proof of Theorem 7.13, from Lemma 7.6 and Corollary 7.9 we infer that

$$\sum_{k=0}^{m-1} \#\mathcal{M}_k \lesssim C_s(u,f,\chi)\mathrm{Err}(f,\chi,\tau_{m-1})^{-\frac{1}{s}} \le C_s(u,f,\chi)C_{\rm upp}^{\frac{1}{s}}\varepsilon^{-\frac{1}{s}}. \quad \square$$

To use the results that were derived in the abstract setting discussed in Sect. 3, recall that in our fictitious domain setting we have $\mathbb{U} = H^1_0(\Omega)$, $\mathbb{\Lambda} = H^{-\frac{1}{2}}(\gamma)$, and $\{0\} = \mathbb{\Lambda}_{\sigma_0} \subset \mathbb{\Lambda}_{\sigma_1} \subset \cdots \subset \mathbb{\Lambda}$ is the sequence of spaces of piecewise constant functions w.r.t. to uniform dyadically refined partitions of γ starting from the initial partition that underlies $\mathbb{\Lambda}_{\sigma_1}$. Since $\lambda \in L_2(\gamma)$ with $\|\lambda\|_{L_2(\gamma)} \lesssim \|f\|_{L_2(\Omega)} + \|g\|_{H^1(\gamma)}$, (6.1) reads as

$$\|\lambda - \lambda_{\sigma_i}\|_{H^{-\frac{1}{2}}(\gamma)} \le L2^{-i/2},$$

i.e., $\zeta = \sqrt{2}$, and $L = L(f,g) \eqsim \|f\|_{L_2(\Omega)} + \|g\|_{H^1(\gamma)}$.

We are now ready to use the routine `afem` as an inner solver in the nested inexact preconditioned Uzawa iteration. With constants β, M and K as in Lemma 6.1, it reads as follows:

Algorithm 8.4.

```
nested-inexact-preconditioned-Uzawa(f, g)
λ_{σ_0}^{(K)} := 0, τ_{0,K} := τ_⊥
for i = 1, 2, ... do
    λ_{σ_i}^{(0)} := λ_{σ_{i-1}}^{(K)}, τ_{i,0} := τ_{i-1,K}
    for j = 1 to K do
        [τ_{i,j}, u_{τ_{i,j}}^{λ_{σ_i}^{(j-1)}}] := afem(τ_{i,j-1}, f, λ_{σ_i}^{(j-1)}, Lζ^{-i})
        λ_{σ_i}^{(j)} := λ_{σ_i}^{(j-1)} + βM_{σ_i}^{-1} I'_{σ_i}(B u_{τ_{i,j}}^{λ_{σ_i}^{(j-1)}} + g)
    endfor
endfor
```

In order to remove the dependence on $\chi = \lambda_{\sigma_i}^{(j-1)}$ of the upper bounds derived in Lemmas 8.1 and 8.3, we need uniform boundedness of the $\|\lambda_{\sigma_i}^{(j)}\|_{L_2(\gamma)}$:

Lemma 8.5. *For the sequence $((\lambda_{\sigma_i}^{(j)})_{1\le j\le K})_{i\ge 1}$ produced by the above algorithm it holds that $\|\lambda_{\sigma_i}^{(j)}\|_{L_2(\gamma)} \lesssim L = L(f,g)$.*

Proof. With Q_{σ_i} denoting the $L_2(\gamma)$-orthogonal projector onto $\mathbb{A}_{\sigma_i}$, we estimate

$$\begin{aligned}
\|\lambda_{\sigma_i}^{(j)}\|_{L_2(\gamma)} &\le \|\lambda\|_{L_2(\gamma)} + \|\lambda - \lambda_{\sigma_i}^{(j)}\|_{L_2(\gamma)} \\
&\le \|\lambda\|_{L_2(\gamma)} + \|\lambda - Q_{\sigma_i}\lambda\|_{L_2(\gamma)} + \|Q_{\sigma_i}\lambda - \lambda_{\sigma_i}^{(j)}\|_{L_2(\gamma)} \\
&\le 2\|\lambda\|_{L_2(\gamma)} + \|Q_{\sigma_i}\lambda - \lambda_{\sigma_i}^{(j)}\|_{L_2(\gamma)} \\
&\le 2\|\lambda\|_{L_2(\gamma)} + 2^{i/2}\|Q_{\sigma_i}\lambda - \lambda_{\sigma_i}^{(j)}\|_{H^{-\frac{1}{2}}(\gamma)}
\end{aligned}$$

by the application of the inverse inequality $\|\cdot\|_{L_2(\Omega)} \lesssim 2^{i/2}\|\cdot\|_{H^{-\frac{1}{2}}(\Omega)}$ on $\mathbb{A}_{\sigma_i}$ (e.g., see [DFG$^+$04, Thm. 4.6]). The proof is completed by $\|\lambda\|_{L_2(\Omega)} \lesssim L = L(f,g)$ and $\|Q_{\sigma_i}\lambda - \lambda_{\sigma_i}^{(j)}\|_{H^{-\frac{1}{2}}(\gamma)} \le \|(I-Q_{\sigma_i})\lambda\|_{H^{-\frac{1}{2}}(\gamma)} + \|\lambda - \lambda_{\sigma_i}^{(j)}\|_{H^{-\frac{1}{2}}(\gamma)} \lesssim L2^{-i/2}$, for the second term using Lemma 6.1 together with (6.1). $\square$

We are ready to prove that the sequence $((u_{\tau_{i,j}}^{\lambda_{\sigma_i}^{(j-1)}})_{1\le j\le K})_{i\ge 1}$ converges to u with the best possible rate:

Theorem 8.6. *Let $\theta \in (0,\theta_*)$, $u \in \mathcal{A}^s$ for some $s \in (0,\frac{1}{2}]$ and assume that K is sufficiently large. Then for $i \ge 1$,*

$$\frac{\max(\|\lambda - \lambda_{\sigma_i}^{(j)}\|_{H^{-\frac{1}{2}}(\gamma)}, \|u - u_{\tau_{i,j}}^{\lambda_{\sigma_i}^{(j-1)}}\|_{H^1(\Omega)})}{\|f\|_{L_2(\Omega)} + \|g\|_{H^1(\gamma)}} \lesssim 2^{-i/2}, \quad (1 \le j \le K), \tag{8.5}$$

and

$$\#\tau_{i,j} - \#\tau_\perp \lesssim \Big(\Big(\frac{|u|_{\mathcal{A}^s}}{\|f\|_{L_2(\Omega)} + \|g\|_{H^1(\gamma)}}\Big)^{1/s} + 2\Big)\Big(\frac{\|u - u_{\tau_{i,j}}^{\lambda_{\sigma_i}^{(j-1)}}\|_{H^1(\Omega)}}{\|f\|_{L_2(\Omega)} + \|g\|_{H^1(\gamma)}}\Big)^{-1/s}. \tag{8.6}$$

Proof. The first statements follow from (6.1) and Lemma 6.1 with $u^{(i,j)} = u_{\tau_{i,j}}^{\lambda_{\sigma_i}^{(j-1)}}$ and $\zeta = \sqrt{2}$.

With the number of triangulations created inside the $\mathtt{afem}(\tau_{i,j-1}, f, \lambda_{\sigma_i}^{(j-1)}, L2^{-i/2})$ denoted as $m_{i,j-1}$, let $\mathcal{M}_0^{(i,j-1)}, \ldots, \mathcal{M}_{m_{i,j-1}-1}^{(i,j-1)}$ denote the sequence of marked cells that is generated. Since $\|\lambda_{\sigma_i}^{(j-1)}\|_{L_2(\gamma)} \lesssim L$ by Lemma 8.5, and $\|u - u^{\lambda_{\sigma_i}^{(j-1)}}\|_{H^1(\Omega)} \eqsim \|\lambda - \lambda_{\sigma_i}^{(j-1)}\|_{H^{-\frac{1}{2}}(\gamma)} \leq L2^{-i/2}$, Lemma 8.3 shows that

$$\sum_{k=0}^{m_{i,j-1}-1} \#\mathcal{M}_k^{(i,j-1)} \lesssim \big(|u|_{\mathcal{A}^s}^{1/s} + \|f\|_{L_2(\Omega)}^{1/s} + L^{1/s})\big) L^{-1/s}(2^{i/2})^{1/s}.$$

Now an application of Theorem 7.1 shows that

$$\begin{aligned}
\#\tau_{i,j} - \#\tau_\perp &\lesssim \sum_{\breve{j}=1}^{j} \sum_{k=0}^{m^{(i,\breve{j}-1)}-1} \#\mathcal{M}_k^{(i,\breve{j}-1)} + \sum_{\breve{i}=1}^{i-1}\sum_{\breve{j}=1}^{K} \sum_{k=0}^{m^{(\breve{i},\breve{j}-1)}-1} \#\mathcal{M}_k^{(\breve{i},\breve{j}-1)} \\
&\lesssim \Big(\frac{|u|_{\mathcal{A}^s}^{1/s} + \|f\|_{L_2(\Omega)}^{1/s}}{L^{1/s}} + 1)\Big)(2^{i/2})^{1/s} \\
&\lesssim \Big(((L^{-1}|u|_{\mathcal{A}^s})^{1/s} + 2)\Big)(L^{-1}\|u - u_\tau^{\lambda_{\sigma_i}^{(j-1)}}\|_{H^1(\Omega)})^{-1/s}.
\end{aligned} \tag{8.7}$$

□

Remark 8.7. Theorem 8.6 shows that the sequence $((u_{\tau_{i,j}}^{\lambda_{\sigma_i}^{(j-1)}})_{1\leq j\leq K})_{i\geq 1}$ converges to u with the best possible rate, or equivalently, that $\#\tau_{i,j}$ is of the best possible order. The latter even holds true if we read $\#\tau_{i,j}$ as the sum of the cardinality of $\tau_{i,j}$ and that of all preceding ones starting from $\tau_\perp$. This follows from (8.7), $1 \leq j \leq K$, and $\sup_{i\geq 1} \max_{1\leq j\leq K} m_{i,j-1} < \infty$. The latter is a consequence of the fact that the argument $\tau = \tau_{i,j-1}$ in the call $\mathtt{afem}(\tau_{i,j-1}, f, \lambda_{\sigma_i}^{(j-1)}, L2^{-i/2})$ is such that for $j > 1$, $|u^{\lambda_{\sigma_i}^{(j-2)}} - u_\tau^{\lambda_{\sigma_i}^{(j-2)}}|_{H^1(\Omega)} \leq L2^{-i/2}$, and for $j = 0$, $|u^{\lambda_{\sigma_{i-1}}^{(K)}} - u_\tau^{\lambda_{\sigma_{i-1}}^{(K)}}|_{H^1(\Omega)} \leq L2^{-(i-1)/2}$, and so, by the first inequality in (8.5), in both cases $\inf_{U\in\mathbb{U}_\tau} |u^{\lambda_{\sigma_i}^{(j-1)}} - U|_{H^1(\Omega)} \lesssim 2^{-i/2}$. As we have seen, this means that a uniformly bounded number of iterations of $\mathtt{solve} - \mathtt{estimate} - \mathtt{mark} - \mathtt{refine}$ suffices to obtain a Galerkin approximation to $u^{\lambda_{\sigma_i}^{(j-1)}}$ that meets the tolerance $L2^{-i/2}$.

The statement proven in this remark is the first step in a proof of optimal computational complexity of a method in which the exact Galerkin solutions are replaced by inexact ones, following the analysis given in [Ste07].

Remark 8.8. (Cost of subdividing γ). For the overall computational cost of the method, the costs of the repeated updates of the approximate Lagrange multiplier as well as their evaluations when used as right hand sides of the `afem` algorithm need to be accounted for. Both are proportional to the dimension of the spaces $\dim \Lambda_{\sigma_i} \eqsim 2^i$ or equivalently to the cardinality of the underlying mesh $\#\sigma_i$. In view of (8.5), we deduce that $\dim \Lambda_{\sigma_i} \lesssim L^2\|u - u_{\tau_{i,j}}^{\lambda_{\sigma_i}^{(j-1)}}\|_{H^1(\Omega)}^{-2}$, which is smaller than the estimate (8.6) derived for $\#\tau_{i,j}$ ($s \in (0, 1/2]$). The overall computational cost is therefore dominated by the approximation of u in `afem`.

9. Numerical Illustrations

9.1. **A posteriori error estimation.** To assess the performances of Algorithm 8.4, we propose to derive and report the values of a-posteriori estimators for $|u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)}$ and $\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-\frac{1}{2}}(\gamma)}$. Notice that we expect $\lambda_{\sigma_i}^{(K)}$ to be more accurate than $\lambda_{\sigma_i}^{(K-1)}$ but we cannot get a computational estimate for the error in the former.

We start with $u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}$. For any $w \in L_1(\gamma)$, we define $P_{\sigma_i} w$ to be the continuous piecewise linear function defined on each vertex ν as the average of the mean values of w on the elements in σ_i containing ν. Note that when $w \in H^{\frac{1}{2}}(\gamma)$ with $\int_\gamma w(s)\mu(s)\,ds = 0$ for all $\mu \in \mathbb{A}_{\sigma_i}$, these mean values vanish, and so $P_{\sigma_i} w = 0$. As a consequence, P_{σ_i} satisfies the condition required to use (5.2) in Proposition 5.1. For $\Sigma \subset \sigma_i$ a collection of consecutive elements, let $\bar{\Sigma}$ denote Σ augmented with the two neighboring elements in σ_i at both sides of Σ. The Clément type quasi-interpolator P_{σ_i} satisfies $\|P_{\sigma_i} w\|_{L_2(\Sigma)} \lesssim \|w\|_{L_2(\bar{\Sigma})}$ and $\|(I - P_{\sigma_i})w\|_{L_2(\Sigma)} \lesssim 2^{-i}|w|_{H^1(\bar{\Sigma})}$, which implies that $|P_{\sigma_i} w|_{H^{\frac{1}{2}}(\Sigma)} \lesssim |w|_{H^{\frac{1}{2}}(\bar{\Sigma})}$. Using the localization of the $H^{\frac{1}{2}}(\gamma)$-norm of an 'oscillating argument' shown by Faermann in [Fae00, Lemma 2.3], one infers that

(9.1)
$$\begin{aligned}
&\|(I - P_{\sigma_i})w\|^2_{H^{\frac{1}{2}}(\gamma)} \lesssim |(I - P_{\sigma_i})w|^2_{H^{\frac{1}{2}}(\gamma)} \\
&\lesssim \sum_{I \in \sigma_i} |(I - P_{\sigma_i})w|^2_{H^{\frac{1}{2}}(\bar{I})} + 2^{-i}\|(I - P_{\sigma_i})w\|^2_{L_2(I)} \lesssim \sum_{I \in \sigma_i} |w|^2_{H^{\frac{1}{2}}(\bar{\bar{I}})} =: |w|^2_{\frac{1}{2},\sigma_i,\mathrm{loc}}.
\end{aligned}$$

We now invoke formulas (5.2) in Proposition 5.1 to write $|u - u^{\lambda_{\sigma_i}}|_{H^1(\Omega)} \eqsim \|g - u^{\lambda_{\sigma_i}}\|_{H^{\frac{1}{2}}(\gamma)}$, and

$$\big|\|g - u^{\lambda_{\sigma_i}}\|_{H^{\frac{1}{2}}(\gamma)} - \|(I - P_{\sigma_i})(g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}})\|_{H^{\frac{1}{2}}(\gamma)}\big| \lesssim |u^{\lambda_{\sigma_i}} - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)},$$

and so $|u - u^{\lambda_{\sigma_i}}|_{H^1(\Omega)} \lesssim |g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{\frac{1}{2},\sigma_i,\mathrm{loc}} + |u^{\lambda_{\sigma_i}} - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)}$.

Let M_{σ_i} be a preconditioner as in (4.2) and Φ_{σ_i} be a basis for $\mathbb{A}_{\sigma_i}$. Proposition 5.1 with $\mathbf{r} := \langle g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}, \Phi_{\sigma_i}\rangle_{L_2(\gamma)}$, together with Lemma 7.5 leads to

(9.2)
$$\begin{aligned}
|u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} &\leq |u - u^{\lambda_{\sigma_i}}|_{H^1(\Omega)} + |u^{\lambda_{\sigma_i}} - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} \\
&\lesssim |g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{\frac{1}{2},\sigma_i,\mathrm{loc}} + |u^{\lambda_{\sigma_i}} - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} \\
&\leq |g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{\frac{1}{2},\sigma_i,\mathrm{loc}} + |u^{\lambda_{\sigma_i}} - u^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} + |u^{\lambda_{\sigma_i}^{(K-1)}} - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} \\
&\overset{(5.1)}{\lesssim} |g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{\frac{1}{2},\sigma_i,\mathrm{loc}} + \sqrt{\langle \mathbf{M}_{\sigma_i}^{-1}\mathbf{r}, \mathbf{r}\rangle} + |u^{\lambda_{\sigma_i}^{(K-1)}} - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} \\
&\overset{\text{Lem. 7.5}}{\lesssim} \underbrace{|g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{\frac{1}{2},\sigma_i,\mathrm{loc}}}_{E_{\mathrm{outer}}:=} + \underbrace{\sqrt{\langle \mathbf{M}_{\sigma_i}^{-1}\mathbf{r}, \mathbf{r}\rangle}}_{E_{\mathrm{Uzawa}}:=} + \underbrace{\mathcal{E}(u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}, f, \lambda_{\sigma_i}^{(K-1)}, \tau_{i,K})}_{E_{\mathrm{inner}}:=}.
\end{aligned}$$

Notice that when $E_{\mathrm{Uzawa}} + E_{\mathrm{inner}} \lesssim E_{\mathrm{outer}}$, it even holds that

$$|u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} \eqsim E_{\mathrm{outer}} + E_{\mathrm{Uzawa}} + E_{\mathrm{inner}}.$$

Indeed, this follows from the estimate

$$E_{\mathrm{outer}} = |g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{\frac{1}{2},\sigma_i,\mathrm{loc}} \leq 5\|g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}\|_{H^{\frac{1}{2}}(\gamma)} \lesssim |u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} \tag{9.3}$$

and the trace theorem.

Remark 9.1. Concerning the names given to the different terms of the estimator, recall that in Lemma 7.6 we have seen that the inner Galerkin error $|u^{\lambda_{\sigma_i}^{(K-1)}} - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)}$ is equivalent to E_{inner} up to the data oscillation term $\mathcal{D}(f, \lambda_{\sigma_i}^{(K-1)}, \tau_{i,K})$. Furthermore, (5.1) shows that if $E_{\mathrm{inner}}/E_{\mathrm{Uzawa}}$ is sufficiently small, then $\|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-\frac{1}{2}}(\gamma)} \eqsim |u^{\lambda_{\sigma_i}} - u^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} \eqsim E_{\mathrm{Uzawa}}$, which thus is properly called the Uzawa error. Finally, (5.2) shows that if additionally $E_{\mathrm{Uzawa}}/E_{\mathrm{outer}}$ is sufficiently small, then $\|\lambda - \lambda_{\sigma_i}\|_{H^{-\frac{1}{2}}(\gamma)} \eqsim |u - u^{\lambda_\sigma}|_{H^1(\Omega)} \eqsim |(I - P_{\sigma_i})(g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}})|_{H^{\frac{1}{2}}(\gamma)} \lesssim E_{\mathrm{outer}}$, meaning that E_{outer} bounds (up to a multiplicative constant) the outer Galerkin error.

Moving to the estimate of $\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-\frac{1}{2}}(\gamma)}$, the Galerkin orthogonality w.r.t. the energy inner product $(\chi, \mu) \mapsto (S\mu)(\chi)$ yields

$$\begin{aligned}\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-\frac{1}{2}}(\gamma)} &\eqsim \|\lambda - \lambda_{\sigma_i}\|_{H^{-\frac{1}{2}}(\gamma)} + \|\lambda_{\sigma_i} - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-\frac{1}{2}}(\gamma)} \\ &\eqsim |u - u^{\lambda_{\sigma_i}}|_{H^1(\Omega)} + |u^{\lambda_{\sigma_i}} - u^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} \\ &\lesssim E_{\mathrm{outer}} + E_{\mathrm{Uzawa}} + E_{\mathrm{inner}}.\end{aligned}$$

Recalling (9.3), we obtain

$$\begin{aligned}E_{\mathrm{outer}} \lesssim |u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} &\leq |u - u^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} + |u^{\lambda_{\sigma_i}^{(K-1)}} - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_{H^1(\Omega)} \\ &\lesssim \|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-\frac{1}{2}}(\gamma)} + E_{\mathrm{inner}},\end{aligned}$$

and infer that if $E_{\mathrm{Uzawa}} \lesssim E_{\mathrm{outer}}$ and $E_{\mathrm{inner}}/E_{\mathrm{outer}}$ is sufficiently small, then

$$\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-\frac{1}{2}}(\gamma)} \eqsim E_{\mathrm{outer}} + E_{\mathrm{Uzawa}} + E_{\mathrm{inner}}.$$

Remark 9.2. It is tempting to circumvent the somewhat cumbersome computation of the localized Aronszajn-Slobodeckij-norm $\|\,\|_{\frac{1}{2},\sigma_i,\mathrm{loc}}$ by estimating $\|(I - P_{\sigma_i})w\|_{H^{\frac{1}{2}}(\gamma)} \lesssim 2^{-i/2}|w|_{H^1(\gamma)}$ instead of (9.1). This approach, often used in the BEM community, is not appropriate in our context. Indeed, for $w = g - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}|_\gamma$, it yields an estimator greatly overestimating the error and that even does not reduce when the iterations proceed. The reason is that $u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}$ restricted to the boundary γ is piecewise polynomial (possibly oscillating) w.r.t. an irregular partition of the boundary (possibly locally much finer than σ_i).

9.2. **Setting.** We explore the convergence and optimality properties of the nested inexact preconditioned Uzawa algorithm (Algorithm 8.4). We consider the L-shaped domain $\widehat{\Omega} = (-1,1)^2 \setminus (-1,0)^2$, set $g = 0$ and choose $\widehat{f} \in L_2(\widehat{\Omega})$ such that the solution u to (2.1) in polar coordinates (r,ϕ) centered at $(0,0)$ reads

$$\widehat{u}(r,\phi) = h(r) r^{2/3} \sin(2/3(\phi + \pi/2)),$$

where

$$h(r) = \frac{w(3/4-r)}{w(r-1/4)+w(3/4-r)} \quad \text{with} \quad w(r) = \begin{cases} r^2 & \text{if } r > 0 \\ 0 & \text{else.} \end{cases}$$

The fictitious domain formulation (2.2) is obtained by embedding $\widehat{\Omega}$ in the square domain $\Omega = (-1.5, 1.5)^2$ and by letting $\widehat{f}$ to be the zero extension of $f \in L_2(\Omega)$. Note that in that case, the solution (u,λ) of (2.2) satisfies $u|_{\Omega\setminus\widehat{\Omega}} = 0$ and

$$\lambda = \frac{\partial \widehat{u}}{\partial \vec{n}}\Big|_\gamma = \frac{2}{3} h(r)\ r^{-1/3} \in H^s(\gamma),\ s < \frac{1}{6}.$$

Recall that the approximations of u are continuous piecewise linear polynomials w.r.t. locally refined partitions of Ω while the approximations of λ consist of piecewise constant polynomials w.r.t uniform dyadically refined partitions $\sigma_\perp = \sigma_1 \prec \sigma_2 \prec \cdots$ of γ, where $\#\sigma_i = 2^{i+2}$.

9.3. **Performances of the Wavelet Preconditioner.** We start by assessing the efficiency of the wavelet preconditioner $M_{\sigma_i}^{-1}$ developed in Appendix A. It is an approximate inverse of $S_{\sigma_i} \in \mathcal{L}\mathrm{is}(\mathbb{A}_{\sigma_i}, \mathbb{A}'_{\sigma_i})$ and its quality is characterized by a uniform bound on

$$\kappa := \sup_i \rho(M_{\sigma_i}^{-1} S_{\sigma_i}) \sup_i \rho(M_{\sigma_i} S_{\sigma_i}^{-1}) = \sup_i \kappa(M_{\sigma_i}^{-1} S_{\sigma_i}), \tag{9.4}$$

where for an invertible C, $\kappa(C)$ is the spectral condition number defined by $\kappa(C) := \rho(C)\rho(C^{-1})$. The equality in (9.4) follows from the nesting $\mathbb{A}_{\sigma_i} \subset \mathbb{A}_{\sigma_{i+1}}$ and the multi-level character of the preconditioner.

Unfortunately, the exact computation of $\kappa(M_{\sigma_i}^{-1} S_{\sigma_i})$ is impossible because the evaluation of S_{σ_i} requires the inverse of the infinite dimensional $A \in \mathcal{L}\mathrm{is}(\mathbb{U}, \mathbb{U}')$. Instead, we monitor the computable quantity $\kappa(M_{\sigma_i}^{-1} S_{\sigma_i\tau_i})$, where for a partition $\tau_i \in \mathcal{T}$ of Ω, $S_{\sigma_i\tau_i}$ is an approximation of S_{σ_i}. We propose to define $S_{\sigma_i\tau_i} := B_{\sigma_i\tau_i} A_{\tau_i}^{-1} B'_{\sigma_i\tau_i}$, where $B_{\sigma_i\tau_i} \in \mathcal{L}(\mathbb{U}_{\tau_i}, \mathbb{A}'_{\sigma_i})$ and $A_{\tau_i} \in \mathcal{L}\mathrm{is}(\mathbb{U}_{\tau_i}, \mathbb{U}'_{\tau_i})$ are defined by $(B_{\sigma_i\tau_i} w)(\mu) = b(w,\mu)$ ($w \in \mathbb{U}_{\tau_i}$, $\mu \in \mathbb{A}_{\sigma_i}$) and $(A_{\tau_i} w)(v) = a(w,v)$ ($w, v \in \mathbb{U}_{\tau_i}$), respectively. Given σ_i, we know that $S_{\sigma_i\tau_i} \to S_{\sigma_i} \in \mathcal{L}\mathrm{is}(\mathbb{A}_{\sigma_i}, \mathbb{A}'_{\sigma_i})$ when the diameter of the largest element in τ_i tends to zero. Furthermore, $S_{\sigma_i\tau_i}$ is uniformly spectrally equivalent to S_{σ_i} under a uniform LBB condition. To achieve the latter, we perform refinements until the triangles $T \in \tau_i$ intersecting the boundary γ have diameters smaller than 3 times the length of the elements in σ_i, see [GG95]. At this point, we emphasize that the validity of the LBB condition is enforced only to assess the performances of the wavelet preconditioner but is not required for the nested inexact Uzawa algorithm.

The results are collected in Table 1. In the first two columns, we report the number of elements in σ_i and τ_i, while the third and fourth column show the condition numbers of the Schur complement and its preconditioned version, respectively. The last two columns contains the spectral radius of the preconditioned Schur complement and that of its inverse. As predicted, the condition number of the unpreconditioned matrices increases by a factor 2 when the level i of refinement is increased by 1. In contrast, the efficiency of the wavelet preconditioner is confirmed (fourth column) by the nearly constant values of the condition number of the preconditioned Schur complements. The fact that these condition numbers even decrease with an increasing $\#\sigma_i$ is an artifact caused by the replacement of A^{-1} by $A_{\tau_i}^{-1}$.

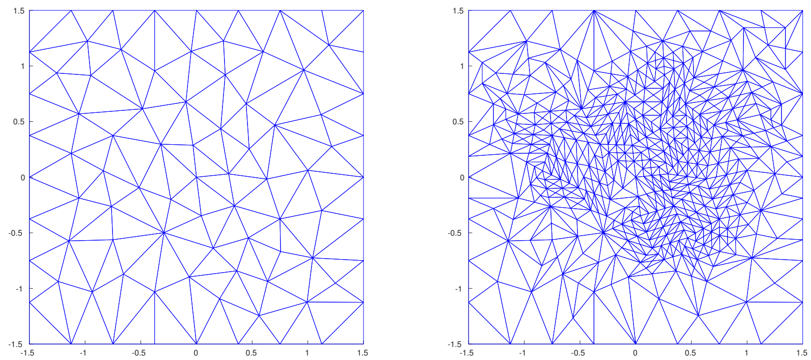
It is worth noting that from the quantities $\rho(M_{\sigma_i}^{-1}S_{\sigma_i\tau_i})$ and $\rho(M_{\sigma_i}S_{\sigma_i\tau_i}^{-1})$ reported in Table 1, it is possible to obtain an estimate for the optimal parameter β defined by (4.5). In fact, we observe that $\rho(M_{\sigma_i}^{-1}S_{\sigma_i\tau_i}) + \rho(M_{\sigma_i}S_{\sigma_i\tau_i}^{-1})^{-1} \approx 0.8$ so from now on we set $\beta = 2/0.8$.

TABLE 1. Spectral condition numbers of the preconditioned and unpreconditioned approximate Schur complement. $\kappa_S = \kappa(S_{\sigma_i\tau_i})$, $\kappa_{M^{-1}S} = \kappa(M_{\sigma_i}^{-1}S_{\sigma_i\tau_i})$, $\rho_S = \rho(S_{\sigma_i})$, $\rho_{M^{-1}S} = \rho(M_{\sigma_i}^{-1}S_{\sigma_i})$, $\rho_{S^{-1}} = \rho(S_{\sigma_i}^{-1})$, $\rho_{MS^{-1}} = \rho(M_{\sigma_i}S_{\sigma_i}^{-1})$.

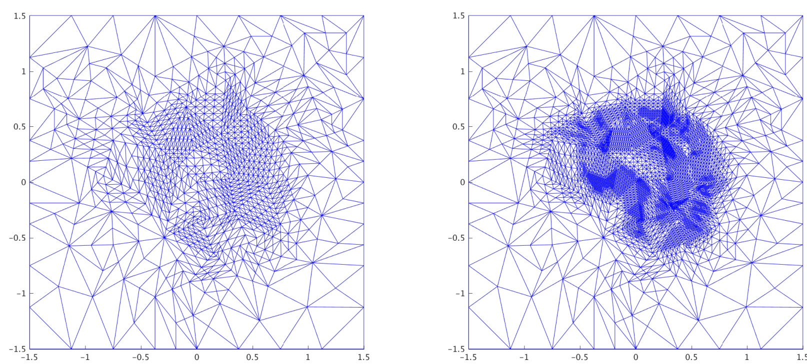
$\#\sigma_i$	$\#\tau_i$	κ_S	$\kappa_{M^{-1}S}$	$\rho_{M^{-1}S}$	$\rho_{MS^{-1}}$
8	1741	6.7095	6.7095	0.5628	11.9217
16	2010	13.5357	6.4439	0.5751	11.2051
24	4770	28.0230	6.0403	0.5867	10.2956
64	11326	57.8325	5.8276	0.5926	9.8335
128	23398	118.4710	5.7412	0.5955	9.6418
256	46134	237.9260	5.6913	0.5968	9.5357
512	85460	488.9130	5.6653	0.5975	9.4824
1024	156092	979.7390	5.6548	0.5978	9.4601

9.4. **Performances of the Nested Inexact Uzawa Algorithm.** We now investigated the performances of the nested inexact preconditioned Uzawa iteration (Algorithm 8.4). The routine `afem` given in Algorithm 8.2 serves as an inner solver in Algorithm 8.4 and is driven by the a posteriori error estimator $\mathcal{E}$, see (7.2). Apart from data oscillation terms, it consists of the square root of the sum of weighted norms of jumps of normal derivatives of the current approximation for u over the edges of the partition of Ω. The numerical observations in [CV99] indicate that, ignoring the data oscillations, $\mathcal{E}$ is approximately a factor $3\sqrt{2}$ larger than the error it estimates (the factor $\sqrt{2}$ stems from the fact that unlike in [CV99] our estimator each jump is counted twice). Therefore, in the following we scale $\mathcal{E}$ by a factor $\sqrt{2}/6$ and set the constant $C_{\text{upp}} = 1$. Note that the same scaling is applied to the quantity E_{inner} defined in (9.2). In addition, we set the constant $L = L(f,g) = \bar{L}\,(\|f\|_{L_2(\Omega)} + \|g\|_{H^1(\gamma)})$ with $\bar{L} = 0.1$, $K = 6$, $\zeta = \sqrt{2}$, $\theta = 0.1$ and recall that β defined in (4.5) is set to $\beta = 2/0.8$ (see Section 9.3).

Figure 1 displays the initial mesh together with the adaptively refined meshes obtained at the first, third and fifth outer iteration of Algorithm 8.4. Figure 2 shows the approximations $u_{\tau_i,6}^{\lambda_{\sigma_i}^{(5)}}$ at the third and sixth outer iterations, while Figure 3(a) provides a comparison between the approximations $\lambda_{\sigma_i}^{(6)}$ and the $L_2(\gamma)$-projection of the exact solution λ onto $\mathbb{A}_{\sigma_1}$ for $i = 1$ and 3. In Figure 3(b)-(c) the traces of the numerical solution on the boundary γ are depicted in red and compared to the (zero) trace of the exact solution.



(a) Initial mesh (left) and mesh at iteration $i = 1$ (right)

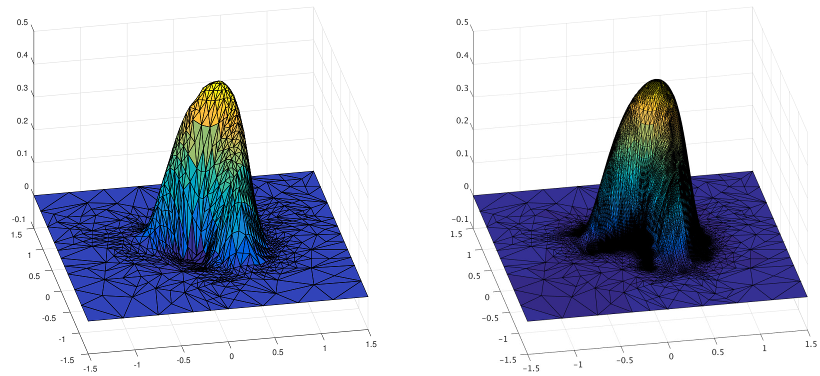


(b) Mesh at iteration $i = 3$ (left) and $i = 5$ (right)

FIGURE 1. Adaptive meshes produced with $K = 6$, $\zeta = \sqrt{2}$, and $\theta = 0.1$.

In Figure 4, for $i = 1, \dots, I := 10$, we report the errors $\|\nabla(u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}})\|_{L_2(\Omega)}$ and $\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-1/2}(\gamma)}$, and compare them to the estimators. We observe a remarkable agreement between the errors and the estimators. In addition, note that E_{outer} and E_{inner} exhibit rates of decay comparable with the ones of the errors, whereas E_{Uzawa} is in all cases much smaller than the other indicators, displaying a plateau whenever $K > 2$ inner iterations are performed. For completeness, we mention that the computation of the norm $\|\cdot\|_{H^{-1/2}(\gamma)}$ is approximated by first building the $L_2(\gamma)$-orthogonal projection μ of the error $\lambda - \lambda_{\sigma_i}^{(K-1)}$ onto $\mathbb{\Lambda}_{\sigma_{I+2}}$ and then employing (4.2) to get $\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-1/2}(\gamma)} \simeq \sqrt{(M_{\sigma_{I+2}}\mu)(\mu)}$.

In Table 2, we report the rates of convergence for the errors $\|\nabla(u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}})\|_{L_2(\Omega)}$ and $\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-1/2}(\gamma)}$ with respect to $\#\tau_{i,K}$ and $\#\sigma_i$, respectively. The rates are computed after excluding the first three iterations of the algorithms. The



(a) $u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}}$ for $i = 3$ (left) and $i = 6$ (right)

FIGURE 2. Solutions produced with $K = 6$, $\zeta = \sqrt{2}$, $\theta = 0.1$.

convergence rate of the $H^1(\Omega)$-error for u is always close to the expected value 0.5 while the convergence rate of the $H^{-1/2}(\gamma)$-error for λ is 0.69. The latter is also in agreement with the theoretical rate $\frac{2}{3}$ expected for $\lambda \in H^s(\gamma)$ with $s < \frac{1}{6}$. Finally, in the last two columns we report the number of elements of $\tau_{I,K}$ and σ_I at the last iteration $I = 10$.

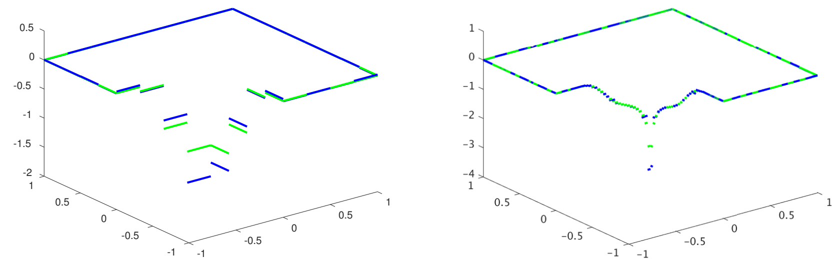
In Table 3, we report the rates of convergence of the estimators E_{outer}, E_{Uzawa} and E_{inner}. The rates observed for E_{outer} are closer to the theoretical value $\frac{2}{3}$ when K increases. The rates obtained for E_{inner} always matches (up to the third significant digit) the theoretical rate expected for $\|\nabla(u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}})\|_{L_2(\Omega)}$. Finally, the low rates exhibited by E_{Uzawa} are explained by the appearance of plateaux for larger values of i when $K \geq 2$ inner iterations are performed.

We conclude this section with one additional table focusing on the behavior of the inner adaptive solver. Recall that in Algorithm 8.4 a fixed number of inner iterations $j = 1, .., K$ is performed within each outer iteration $i = 1, .., I$. Each of these inner iterations lead to bulk mesh refinement (Algorithm 8.2) whenever $C_{\text{upp}}\mathcal{E}(f, \chi, \tau_k) > L\zeta^{-i}$. In Table 4, we report for each outer iteration i, the number of times that the bulk mesh refinement is performed and observe that the refinements are never performed after the second inner iteration.

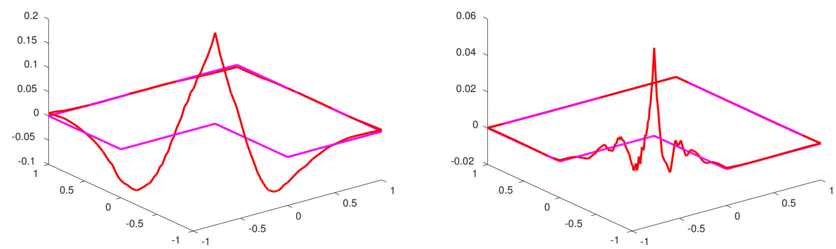
10. General d-dimensional domains and/or higher finite element spaces

So far we considered the case of $d = 2$ space dimensions, and lowest order approximation, i.e., continuous piecewise linears for u, piecewise constants for λ. We now discuss the case of general $d \geq 2$, and general polynomial orders.

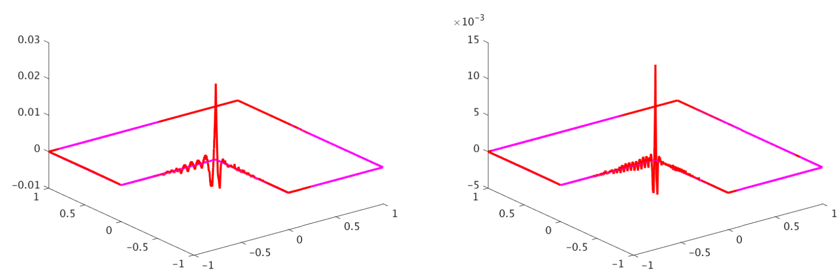
First we address the question for which $s > 0$, membership of u in $\mathcal{A}^s$ can be expected when u is approximated from families of continuous piecewise polynomials of order $p \geq 2$. Since generally $\lambda \neq 0$, the normal derivative of u has a generally non-zero jump over the $(d-1)$-dimensional manifold γ, generally being not-aligned with any mesh. Assuming that apart from this jump, the solution u is smooth,



(a) Comparison between $\lambda^{(K)}_{\sigma_i}$ (blue) and the $L_2(\gamma)$-projection of λ on piecewise constants onto $\mathbb{\Lambda}_{\sigma_i}$ (green) for $i = 3$ (left) and $i = 6$ (right).



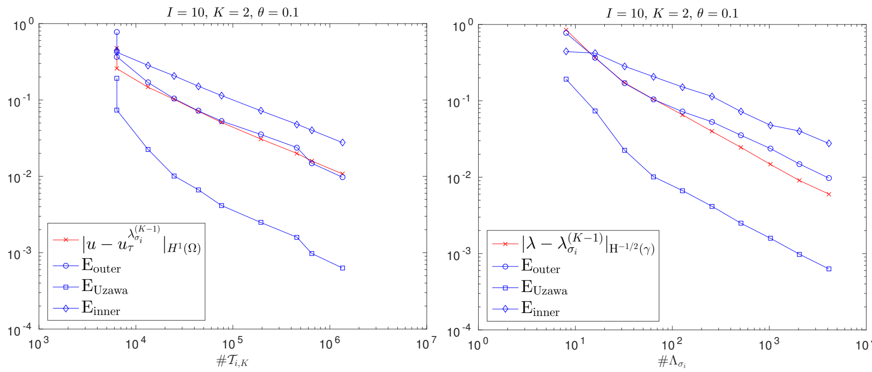
(b) Traces $\gamma u^{\lambda^{(K-1)}_{\sigma_i}}_{\tau_{i,K}}$ (red) and γu (magenta) at $i = 1$ (left) and $i = 3$ (right)



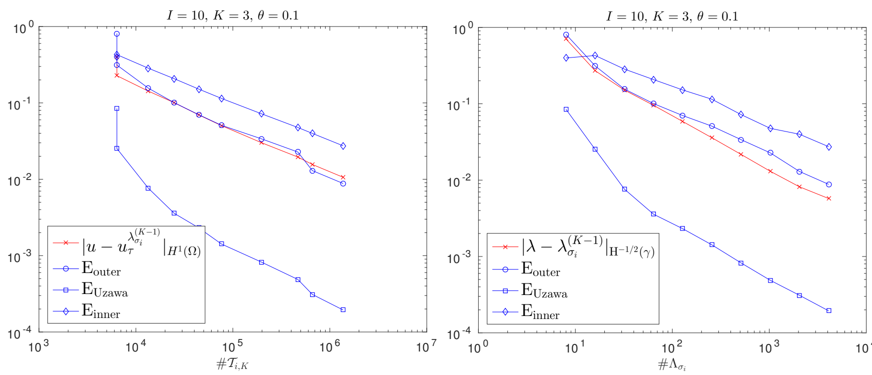
(c) $\gamma u^{\lambda^{(K-1)}_{\sigma_i}}_{\tau_{i,K}}$ (red) and γu (magenta) at $i = 5$ (left) and $i = 6$ (right).

FIGURE 3. Approximations obtained with $K = 6$, $\zeta = \sqrt{2}$, and $\theta = 0.1$.

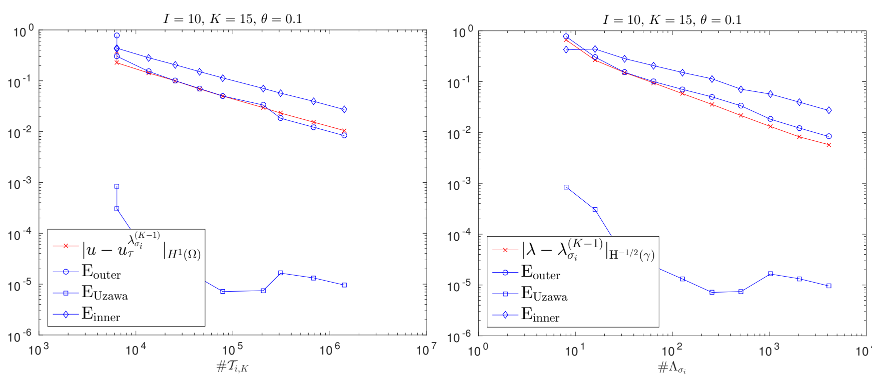
the question of approximability of u in $H^1(\Omega)$ is equivalent to the question of approximability in $L_2(\Omega)$ of a piecewise smooth function, say a piecewise constant one w.r.t. the partition of Ω into $\widehat{\Omega}$ and $\Omega \setminus \overline{\widehat{\Omega}}$, from families of discontinuous polynomials of order $p-1$. Taking cells of diameter h that intersect γ, regardless of the order p the squared $L_2(\Omega)$-norm of the latter approximation error is $\eqsim h^d$ times the number of those cells, being of the order $(1/h)^{d-1}$. We infer that in terms of the total number N of elements in the mesh, which satisfies $N \gtrsim (1/h)^{d-1}$, and with a proper refinement towards γ, even $N \eqsim (1/h)^{d-1}$, it holds that the $L_2(\Omega)$-norm of



(a) Error $\|\nabla(u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}})\|_{L_2(\Omega)}$ and indicators (left); error $\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-1/2}(\gamma)}$ and indicators (right), $K = 2$.



(b) Error $\|\nabla(u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}})\|_{L_2(\Omega)}$ and indicators (left); error $\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-1/2}(\gamma)}$ and indicators (right), $K = 3$.



(c) Error $\|\nabla(u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}})\|_{L_2(\Omega)}$ and indicators (left); error $\|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-1/2}(\gamma)}$ and indicators (right), $K = 15$.

FIGURE 4. Errors and estimators trends ($I = 10$, $\zeta = \sqrt{2}$, $\theta = 0.1$).

this error is $\sqrt{h} \eqsim N^{-\frac{1}{2(d-1)}}$. We conclude that generally at best $u \in \mathcal{A}^{\frac{1}{2(d-1)}}$. In

TABLE 2. Computed rates of convergence of $e_u := \|\nabla(u - u_{\tau_{i,K}}^{\lambda_{\sigma_i}^{(K-1)}})\|_{L_2(\Omega)}$ and $e_\lambda := \|\lambda - \lambda_{\sigma_i}^{(K-1)}\|_{H^{-1/2}(\gamma)}$ w.r.t. $\#\tau_{i,K}$ and $\#\sigma_i$, for different values of K ($I = 10$, $\theta = 0.1$ and $\zeta = \sqrt{2}$).

	e_u	e_λ	$\#\tau_{I,K}$	$\#\sigma_I$
$K = 2$	0.56	0.70	1344310	4096
$K = 3$	0.55	0.69	1372266	4096
$K = 6$	0.56	0.69	1411114	4096
$K = 9$	0.56	0.69	1411274	4096
$K = 15$	0.56	0.69	1411254	4096

TABLE 3. Computed rates of convergence of the error estimators E_{outer}, E_{Uzawa} and E_{inner}, respectively, for different values of K ($I = 10$, $\theta = 0.1$ and $\zeta = \sqrt{2}$).

	E_{outer}	E_{Uzawa}	E_{inner}
$K = 2,$	0.58	0.70	0.50
$K = 3,$	0.59	0.71	0.50
$K = 6,$	0.63	0.67	0.50
$K = 9,$	0.63	0.38	0.50
$K = 15,$	0.63	0.10	0.50

TABLE 4. Number of inner iterations at which bulk mesh refinement is activated, for different values of K ($I = 10$, $\theta = 0.1$ and $\zeta = \sqrt{2}$).

i	1	2	3	4	5	6	7	8	9	10
$K = 2$	2	0	1	1	1	1	1	1	1	1
$K = 3$	2	0	1	1	1	1	2	0	0	1
$K = 6$	2	0	1	1	1	1	1	1	1	1
$K = 9$	2	0	1	1	1	1	1	1	1	1
$K = 15$	2	0	1	1	1	1	1	1	1	1

particular, this means that the approximation with higher order polynomials is of no use.

On the other hand, if the solution $\hat{u}$ of our original PDE posed on $\hat{\Omega}$ is approximated from families of continuous piecewise polynomials of order p w.r.t. (isotropic) partitions of $\hat{\Omega}$, then under appropriate (Besov) smoothness conditions, $\hat{u}$ can be approximated at rate $\frac{p-1}{d}$.

Remark 10.1. Other than for $d = 2$, for $d > 2$ and arbitrary Lipschitz domains these Besov smoothness conditions are not automatically valid for sufficiently smooth data, in which case this rate $\frac{p-1}{d}$ can only be realized by proper anisotropic refinements.

Since for $d > 2$ or $p > 2$, it holds that $\frac{1}{2(d-1)} < \frac{p-1}{d}$, we conclude that for those (d, p) a price to be paid for the application of the Fictitious Domain Method instead of the usual finite element method is that generally it results in a reduced best approximation rate.

Remark 10.2. This deficit of the Fictitious Domain Method might possibly be tackled by considering anisotropic refinements allowing for a more accurate approximation of γ, by enriching the local finite element space on elements that intersect γ, or by constructing an extension of $\widehat{f}$ on $\widehat{\Omega}$ to f on Ω that yields a multiplier λ that is small or preferably zero, and thus avoids the discontinuity in the normal derivative of u over γ.

Knowing that the solution u of the Fictitious Domain Method is at best in $\mathcal{A}^{\frac{1}{2(d-1)}}$, the straightforward generalization to d-dimensions of the adaptive solution method that we have developed for $d = 2$ yields the best possible approximation rate. Indeed, assuming $f \in L_2(\Omega)$ and $g \in H^1(\gamma)$, it holds that $\lambda \in L_2(\gamma)$ and so its approximation in $H^{-\frac{1}{2}}(\gamma)$ by piecewise constants w.r.t. to uniform meshes converges with rate $\frac{1}{2(d-1)}$. A direct generalization of [CDN12, Thms. 7.3-4] from 2 to d dimensions shows that $f \in L_2(\Omega)$ and $\chi \in L_2(\gamma)$ are in the data approximation classes $\mathcal{B}_\Omega^{\frac{1}{d}}$ and $\mathcal{B}_\Omega^{\frac{1}{2(d-1)}}$, respectively (cf. Thm. 7.10). Now the generalization of Thm. 7.13 to d-dimensions shows that whenever $u \in \mathcal{A}^s$ for some $s \in (0, \frac{1}{2(d-1)}]$, the sequence of approximations produced by our nested inexact preconditioned Uzawa algorithm converges with this rate s.

Remark 10.3. Here we assumed that we have an optimal preconditioner for the Schur complement available which currently in the appendix is constructed for $d = 2$. In forthcoming work we will construct such preconditioners of multilevel type whose application requires linear computational complexity in any space dimension d.

Concluding we can say that in any dimension our adaptive method solves the fictitious domain formulation with the best possible rate. On the other hand, for $d > 2$ this rate is generally lower that the best possible rate with which the original PDE can be solved with standard finite elements, i.e., w.r.t. to partitions of the original domain.

Appendix A. An optimal preconditioner for piecewise constant trial spaces w.r.t. the $H^{-\frac{1}{2}}(\gamma)$-norm

As in Example 4.2, let $\mathbb{A} = H^{-\frac{1}{2}}(\gamma)$, and $\{0\} = \mathbb{A}_{\sigma_0} \subset \mathbb{A}_{\sigma_1} \subset \cdots \subset \mathbb{A}$ be a sequence of spaces of piecewise constant functions w.r.t. to a sequence of uniform dyadically refined partitions $\sigma_1 \prec \sigma_2 \prec \cdots$ of γ, with $\sigma_1 = \sigma_\perp$ being some fixed 'bottom' partition. For $i \in \mathbb{N}$, let Φ_i be a basis for $\mathbb{A}_{\sigma_i}$. We aim at constructing $M_{\sigma_i} = M'_{\sigma_i} \in \mathcal{L}(\mathbb{A}_{\sigma_i}, \mathbb{A}'_{\sigma_i})$ with $(M_{\sigma_i}\mu)(\mu) \eqsim \|\mu\|^2_{\mathbb{A}}$ $(\mu \in \mathbb{A}_{\sigma_i})$ such that, with $\mathbf{M}_{\sigma_i} = \mathcal{F}'_{\sigma_i} M_{\sigma_i} \mathcal{F}_{\sigma_i}$, the matrix $\mathbf{M}_{\sigma_i}^{-1}$ can be applied to a vector in linear complexity.

In view of the latter requirement, we consider *multi-level* preconditioners (i.e. wavelet preconditioners, or '(generalized) BPX' preconditioners as proposed in [BPV00]). Other options include 'Calderon preconditioning' as discussed in [SW98, Hip06, BC07].

Usually multilevel preconditioners are based on stability of the $L_2(\gamma)$-orthogonal decomposition of $\mathbb{A}$ associated to the 'multi-resolution analysis' $(\mathbb{A}_{\sigma_i})_i$. With the current choice of $(\mathbb{A}_{\sigma_i})_i$ this stability cannot hold because $\Lambda_{\sigma_i} \not\subset \mathbb{A}' = H^{\frac{1}{2}}(\gamma)$.

Therefore, instead using an orthogonal space decomposition, we (implicitly) will resort on stability of a *bi*orthogonal space decomposition in order to construct a wavelet preconditioner.

Remark A.1. For $\mathbb{A}_{\sigma_i}$ being the space of *continuous* piecewise linears, we could apply e.g. orthogonal (pre-)wavelet preconditioners, the preconditioner from [BPV00] or possibly a preconditioner based on the Dunford Taylor integral representation using the tools developed in [BP15, BP16, BLPXX]. Since however even for piecewise smooth u w.r.t. to the partition $\overline{\Omega} = \overline{\breve{\Omega}} \cup \overline{\hat{\Omega}}$, generally the exact solution λ will not be smooth at corners of γ, the approximation of λ by continuous piecewise polynomials might be less attractive when the domain boundary is not smooth.

Besides Φ_i, let Ψ_i be another basis for $\mathbb{A}_{\sigma_i}$ such that $\mathcal{G}_{\sigma_i} : \mathbb{R}^{\dim \mathbb{A}_{\sigma_i}} \to \mathbb{A}_{\sigma_i} : \mathbf{c} \mapsto \mathbf{c}^\top \Psi_i$ is *uniformly* bounded invertible. Then $M_{\sigma_i} := \mathcal{G}'^{-1}_{\sigma_i} \mathcal{G}^{-1}_{\sigma_i}$ satisfies (4.2) and $\mathbf{M}^{-1}_{\sigma_i} = (\mathcal{F}'_{\sigma_i} M_{\sigma_i} \mathcal{F}_{\sigma_i})^{-1} = \mathcal{F}^{-1}_{\sigma_i} \mathcal{G}_{\sigma_i} (\mathcal{F}^{-1}_{\sigma_i} \mathcal{G}_{\sigma_i})^\top$.

Having for Φ_i and Ψ_i single- and wavelet bases in mind, the matrix $\mathbf{T}_i := \mathcal{F}^{-1}_{\sigma_i} \mathcal{G}_{\sigma_i}$ is known as the wavelet-to-single scale transformation. A wavelet basis Ψ_i is of the form $\Psi^{(1)} \cup \cdots \cup \Psi^{(i)}$, where $\oplus_{k=1}^{\ell} \operatorname{span} \Psi^{(k)} = \mathbb{A}_{\sigma_\ell}$ for $\ell \geq 1$. Let $\mathbf{p}_\ell$ ($\mathbf{q}_\ell$) denote the representation of the embedding $\mathbb{A}_{\sigma_{\ell-1}} \to \mathbb{A}_{\sigma_\ell}$ ($\operatorname{span} \Psi^{(\ell)} \to \mathbb{A}_{\sigma_\ell}$) equipped with $\Phi_{\ell-1}$ and Φ_ℓ, respectively ($\Psi^{(\ell)}$ and Φ_ℓ). Then ordering the wavelets 'level'-wise bottom-to-top, it holds that $\mathbf{T}_i = [\mathbf{p}_i \mathbf{T}_{i-1} \;\; \mathbf{q}_i]$ for $i > 1$, and $\mathbf{T}_1 = \mathbf{q}_1$, so that

$$\mathbf{M}^{-1}_{\sigma_i} = \mathbf{T}_i \mathbf{T}_i^\top = \begin{cases} \mathbf{q}_1 \mathbf{q}_1^\top & \text{when } i = 1, \\ \mathbf{q}_i \mathbf{q}_i^\top + \mathbf{p}_i \mathbf{T}_{i-1} \mathbf{T}_{i-1}^\top \mathbf{p}_i^\top & \text{when } i > 1. \end{cases}$$

In order to describe a suitable wavelet basis, let $\kappa : \mathbb{R}/\mathbb{Z} \to \gamma$ be a continuous parametrization of γ that is piecewise C^1 w.r.t. some 'bottom' partition $\tau_1^{[0,1]}$ of $[0,1] \simeq \mathbb{R}/\mathbb{Z}$. This covers the case of γ being a polygon with 'curved boundaries'. With $\tau_1^{[0,1]} \prec \tau_2^{[0,1]} \prec \cdots$ denoting the nested sequence of partitions of $[0,1]$ created from $\tau_1^{[0,1]}$ by recurrent uniform dyadic refinements, we set $\mathbb{A}_{\sigma_\ell}$ ($\mathbb{A}^{[0,1]}_{\sigma_\ell}$) as the space of piecewise constants w.r.t. $\kappa(\tau_\ell^{[0,1]})$ ($\tau_\ell^{[0,1]}$). We equip $\mathbb{A}^{[0,1]}_{\sigma_\ell}$ with the canonical basis $\Phi_\ell^{[0,1]}$ consisting of functions that are 1 on one element and zero on the others, numbered from left-to-right, giving the first basis function index 0, and let $\Phi_\ell = \kappa(\Phi_\ell^{[0,1]})$. With this choice, we have

$$(\mathbf{p}_\ell \boldsymbol{\mu})_{2n} = (\mathbf{p}_\ell \boldsymbol{\mu})_{2n+1} = \boldsymbol{\mu}_n \quad \text{for } n = 0, \cdots, \dim \mathbb{A}_{\sigma_{\ell-1}} - 1.$$

Assuming compactly supported wavelets, as the matrices $\mathbf{p}_\ell$ the matrices $\mathbf{q}_\ell$ are uniformly sparse, so that the application of our preconditioner is of linear complexity.

To continue, let $\Psi_{[0,1]} = \cup_{k \geq 1} \Psi^{(k)}_{[0,1]}$ be a Riesz basis for $H^{-\frac{1}{2}}(\mathbb{R}/\mathbb{Z})$, such that for $\ell \geq 1$, $\oplus_{k=1}^{\ell} \operatorname{span} \Psi^{(k)}_{[0,1]} = \mathbb{A}^{[0,1]}_{\sigma_\ell}$. The property of $\Psi_{[0,1]}$ being a Riesz basis for $H^{-\frac{1}{2}}(\mathbb{R}/\mathbb{Z})$ is equivalent to the existence of a unique *bi*orthogonal collection $\tilde{\Psi}_{[0,1]}$, i.e. $\langle \Psi_{[0,1]}, \tilde{\Psi}_{[0,1]} \rangle_{L_2(0,1)} = I$, which is a Riesz basis for $H^{\frac{1}{2}}(\mathbb{R}/\mathbb{Z})$. Thanks to κ being continuous and piecewise C^1, the latter implies that $\kappa^{-*} \tilde{\Psi}_{[0,1]} := \tilde{\Psi}_{[0,1]} \circ \kappa^{-1}$ is a Riesz basis for $H^{\frac{1}{2}}(\gamma)$. Using that $\langle u, v \rangle_{L_2(\gamma)} = \int_0^1 u(\kappa(t)) v(\kappa(t)) \omega(t) dt$, where $\omega(t) := \sqrt{D\kappa(t)^\top D\kappa(t)}$, one infers that its unique dual collection is given by $\kappa^{-*}(\Psi_{[0,1]}/\omega)$ which therefore is a Riesz basis for $H^{-\frac{1}{2}}(\gamma)$.

In case κ is piecewise contant w.r.t. the initial partition $\tau_1^{[0,1]}$, as can be realized when γ is a polygon, then we are finished because the wavelets from the latter collection up to 'level' i span exactly $\mathbb{A}_{\sigma_i}$.

For handling the general case, writing $\tau_1^{[0,1]} = \dot{\bigcup}_m [a_m, b_m]$ we make the assumption that κ is such that for each m,

$$\lim_{t\downarrow a_m} \omega(t) = \lim_{t\uparrow b_m} \omega(t) =: \omega_m. \tag{A.1}$$

This assumption can always be satisfied by replacing the initial parametrization $\kappa|_{[a_m,b_m]}$ of the mth patch by $\kappa(\varsigma^{-1}(\iota(\varsigma(\cdot|_{[a_m,b_m]}))))$, with ς being an affine bijection $[a_m, b_m] \to [0,1]$, and ι a suitable diffeomorphism $[0,1] \to [0,1]$. Then the first limit in (A.1) will be multiplied by $\iota'(0)$ and the second one by $\iota'(1)$. Choosing say $\iota(x) = \alpha x^2 + (1-\alpha)x$ for $\alpha \in (-1,1)$, $\frac{\iota'(1)}{\iota'(0)} = \frac{\alpha+1}{\alpha-1}$ can attain any value in $(0,\infty)$ needed to satisfy (A.1).

Now setting $\bar{\omega} \in \mathbb{A}_{\tau_1}^{[0,1]}$ by $\bar{\omega} = \omega_m$ on $[a_m, b_m]$, we infer that

$$\Psi := \kappa^{-*}(\Psi_{[0,1]}/\bar{\omega}) \tag{A.2}$$

is biorthogonal to $\kappa^{-*}(\tilde{\Psi}_{[0,1]}\bar{\omega}/\omega)$. By the continuity of the additional factor $\bar{\omega}/\omega$ and that of its reciprocal, thanks to assumption (A.1), and their piecewise smoothness the latter collection is still a Riesz basis for $H^{\frac{1}{2}}(\gamma)$, meaning that Ψ is a Riesz basis for $H^{-\frac{1}{2}}(\gamma)$. Its wavelets up to 'level' ℓ span the space $\mathbb{A}_{\sigma_\ell}$ as desired.

With the diagonal matrix $\mathbf{D}_\ell$ defined by $(\mathbf{D}_\ell)_{nn} = \omega_m^{-1}$, with $m = m(n)$ such that the nth canonical basis function for $\mathbb{A}_{\sigma_\ell}$ is supported inside the patch ran $\kappa|_{[a_m,b_m]}$, in view of (A.2) we have that $\mathbf{q}_\ell = \mathbf{D}_\ell \mathbf{q}_\ell^{[0,1]}$, and so

$$\mathbf{q}_\ell \mathbf{q}_\ell^\top = \mathbf{D}_\ell \mathbf{q}_\ell^{[0,1]} (\mathbf{q}_\ell^{[0,1]})^\top \mathbf{D}_\ell,$$

where $\mathbf{q}_\ell^{[0,1]}$ is the representation of the embedding $\operatorname{span} \Psi_{[0,1]}^{(\ell)} \to \mathbb{A}_{\sigma_\ell}^{[0,1]}$ equipped with $\Psi_{[0,1]}^{(\ell)}$ and $\Phi_\ell^{[0,1]}$, respectively.

Remains to specify $\mathbf{q}_\ell^{[0,1]} (\mathbf{q}_\ell^{[0,1]})^\top$. By taking for $\Psi_{[0,1]}$ the union of translates and dilates of the 'mother wavelet' of type $(1,3)$ from [CDF92] on levels ≥ 2, properly scaled to adapt to the $H^{-\frac{1}{2}}(\mathbb{R}/\mathbb{Z})$-norm, and complemented with the basis $\Phi_1^{[0,1]}$ for $\mathbb{A}_{\sigma_1}^{[0,1]}$, one obtains $\mathbf{q}_1^{[0,1]} (\mathbf{q}_1^{[0,1]})^\top = I$, and for $\ell > 1$,

$$\begin{aligned}
(\mathbf{q}_\ell^{[0,1]} (\mathbf{q}_\ell^{[0,1]})^\top \boldsymbol{\mu})_{2n} = \tfrac{4^{\ell-1}}{64} (&-\boldsymbol{\mu}_{2n-4} - \boldsymbol{\mu}_{2n-3} + 0\boldsymbol{\mu}_{2n-2} - 16\boldsymbol{\mu}_{2n-1} + 66\boldsymbol{\mu}_{2n} \\
&- 62\boldsymbol{\mu}_{2n+1} + 0\boldsymbol{\mu}_{2n+2} + 16\boldsymbol{\mu}_{2n+3} - \boldsymbol{\mu}_{2n+4} - \boldsymbol{\mu}_{2n+5}), \\
(\mathbf{q}_\ell^{[0,1]} (\mathbf{q}_\ell^{[0,1]})^\top \boldsymbol{\mu})_{2n+1} = \tfrac{4^{\ell-1}}{64} (&-\boldsymbol{\mu}_{2n-4} - \boldsymbol{\mu}_{2n-3} + 16\boldsymbol{\mu}_{2n-2} + 0\boldsymbol{\mu}_{2n-1} - 62\boldsymbol{\mu}_{2n} + 66\boldsymbol{\mu}_{2n+1} \\
&- 16\boldsymbol{\mu}_{2n+2} + 0\boldsymbol{\mu}_{2n+3} - \boldsymbol{\mu}_{2n+4} - \boldsymbol{\mu}_{2n+5}),
\end{aligned}$$

where $0 \leq n \leq \dim \mathbb{A}_{\sigma_{\ell-1}} - 1$, and indices at the right hand side are interpreted modulo $\dim \mathbb{A}_{\sigma_\ell}$.

Alternatively, taking the 4-point mother wavelet from [Osw98], yields

$$\begin{aligned}
(\mathbf{q}_\ell^{[0,1]} (\mathbf{q}_\ell^{[0,1]})^\top \boldsymbol{\mu})_{2n} &= \tfrac{4^{\ell-1}}{9} (-\boldsymbol{\mu}_{2n-3} + 3\boldsymbol{\mu}_{2n-2} - 6\boldsymbol{\mu}_{2n-1} + 10\boldsymbol{\mu}_{2n} - 9\boldsymbol{\mu}_{2n+1} + 3\boldsymbol{\mu}_{2n+2}), \\
(\mathbf{q}_\ell^{[0,1]} (\mathbf{q}_\ell^{[0,1]})^\top \boldsymbol{\mu})_{2n+1} &= \tfrac{4^{\ell-1}}{9} (3\boldsymbol{\mu}_{2n-1} - 9\boldsymbol{\mu}_{2n} + 10\boldsymbol{\mu}_{2n+1} - 6\boldsymbol{\mu}_{2n+2} + 3\boldsymbol{\mu}_{2n+3} - \boldsymbol{\mu}_{2n+4}),
\end{aligned}$$

where again $0 \leq n \leq \dim \mathbb{A}_{\sigma_{\ell-1}} - 1$, and indices at the right hand side are interpreted modulo $\dim \mathbb{A}_{\sigma_j}$. Other than with the construction from [CDF92], the wavelet that is dual to this 4-point wavelet is not compactly supported, which for the present wavelet application is, however, not relevant.

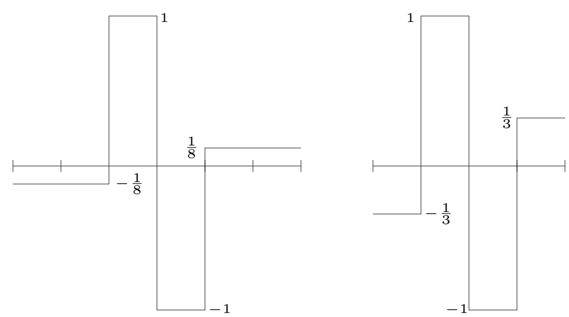


FIGURE 5. Mother wavelets from [CDF92] (left) and [Osw98] (right).

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