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Some properties of BV functions on Carnot groups related to the heat semigroup

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Abstract

In this paper we provide two different characterizations of sets with finite perimeter and functions of bounded variation in Carnot groups, analogous to those which hold in Euclidean spaces, in terms of the short-time behaviour of the heat semigroup. The second one holds under the hypothesis that the reduced boundary of a set of finite perimeter is rectifiable, a result that presently is known in Step 2 Carnot groups.

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1 Introduction

There are various definitions of *variation* of a function, and the related classes BV of *functions of bounded variation*, that make sense in different contexts and are known to be equivalent in wide generality. In the Euclidean frame, i.e., \mathbb{R}^n endowed with the Euclidean metric and the Lebesgue measure, the variation of $f \in L^1(\mathbb{R}^n)$ can be defined as

$$|Df|(\mathbb{R}^n) = \sup \left\{ \int_{\mathbb{R}^n} f \operatorname{div} g \, dx : g \in C_0^1(\mathbb{R}^n, \mathbb{R}^n), \|g\|_{L^\infty(\mathbb{R}^n)} \leq 1 \right\}, \quad (1)$$

and $|Df|(\mathbb{R}^n) < +\infty$ is equivalent to saying that $f \in BV(\mathbb{R}^n)$, i.e., the distributional gradient of f is a (\mathbb{R}^n -valued) finite Radon measure. This approach can be generalized to ambients where a measure and a coherent differential structure, which allows to define a divergence operator, are defined. Alternatively, the variation of f can be defined through a relaxation procedure,

$$|Df|(\mathbb{R}^n) = \inf \left\{ \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} |\nabla f_k| \, dx : f_k \in \operatorname{Lip}(\mathbb{R}^n), f_k \rightarrow f \text{ in } L^1(\mathbb{R}^n) \right\}, \quad (2)$$

and the space BV can be defined, accordingly, as the finiteness domain of the relaxed functional in $L^1(\mathbb{R}^n)$. To generalize (2), no differential structure is needed, apart from

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Lipschitz functions and a suitable substitute of the modulus of the gradient. Definitions (1) and (2) have been extended to several contexts, such as manifolds and metric measure spaces (see e.g. [20], [2]), and in particular Carnot-Carathéodory spaces and Carnot groups, see [10], [6], [17]. However, we point out that the original definition of variation of a function, and in particular of set of finite perimeter, has been given by E. De Giorgi in [13] by a regularization procedure based on the heat kernel. He proved that

$$|Df|(\mathbb{R}^n) = \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} |\nabla T_t f| dx, \quad (3)$$

(where ∇ denotes the gradient with respect to the space variables $x \in \mathbb{R}^n$ and $T_t f(x) = \int_{\mathbb{R}^n} h(t, x - y) f(y) dy$ is the heat semigroup given by the Gauss-Weierstrass kernel h) and that if the above quantity is finite then $f \in BV(\mathbb{R}^n)$. Equality (3) has been recently proved for Riemannian manifolds M in [11], with the only restriction that the Ricci curvature of M is bounded from below, thus generalizing the result in [21], where further bounds on the geometry of M were assumed.

In this paper the frame is that of Carnot groups, where it is known that formulae (1) and (2) (where of course the intrinsic differential structure is used) give equivalent definitions of $|Df|$, see [17]; we show here that (3) holds in a weaker sense, all the quantities involved being again the intrinsic ones. Namely, the two sides of 3 are equivalent, although we don't know whether they are equal (see Theorem 12 for the precise statement).

We think that it is interesting to comment on the proof of (3) in the various situations. Thinking of the left hand side as defined by (1), the \leq inequality follows easily by lower semicontinuity of the total variation and the L^1 convergence of $T_t f$ to f , so that the more difficult part is the \geq inequality. In \mathbb{R}^n it follows from the (trivial) commutation relation $D_i T_t f = T_t D_i f$ between the heat semigroup and the partial derivatives, whereas in the Riemannian case, as treated in [11], it follows from the estimate $\|dT_t f\|_{L^1(M)} \leq e^{K^2 t} |df|(M)$, where d denotes the exterior differential on M , which replaces the gradient in (3), and $-K^2$ is a lower bound for the Ricci curvature of M . In the subriemannian case we are dealing with, the Euclidean commutation does not hold, and we are not able to prove that equality (3) still holds, but only that the semigroup approach identifies the BV class. The proof relies on algebraic properties and Gaussian estimates on the heat kernel and its derivatives which lead to estimate the commutator $[D, T_t]$. Still different arguments are used in the infinite dimensional case of Wiener spaces.

Recently, inspired by [19], another connection between the heat semigroup and BV functions in \mathbb{R}^n has been pointed out in [22], where the following equality is proved:

$$|Df|(\mathbb{R}^n) = \lim_{t \rightarrow 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(t, x - y) |f(x) - f(y)| dy dx, \quad (4)$$

which means that $f \in BV(\mathbb{R}^n)$ if and only if the right hand side is finite, and equality holds. Equality (4) is first proved for characteristic functions and then extended to the general case using the coarea formula. The proof for characteristic functions of sets of finite perimeter, in turn, is based on a blow-up argument on the points of the reduced boundary and uses the rectifiability of the reduced boundary. In this paper we extend (4) to Carnot groups (see Theorem 15) but, accordingly to the preceding comments, our proof depends upon the rectifiability of the reduced boundary, which is presently known to be true only in groups of Step 2, see [18]. Therefore, our proof would immediately generalize to more general groups, if the rectifiability of the reduced boundary were proved; equation (4) assumes a different form in Carnot groups (see Theorem 15 below). This is a consequence of the fact that in general the heat kernel doesn't have the symmetries that the Gaussian kernel has, such as

rotation invariance; however, in the special but important case of groups of Heisenberg type, our result assumes a simpler form (see Remark 16).

We point out that in the Euclidean case both (3) and (4) hold not only in the whole of \mathbb{R}^n , but also in a localized form and with the heat semigroup replaced by the semigroup generated by a general uniformly elliptic operator, under suitable boundary conditions, see [4]. Finally, let us mention that the short-time behavior of the heat semigroup in \mathbb{R}^n has been shown to be useful also to describe further geometric properties of boundaries, see [3].

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2 Preliminaries and Main Results

We collect in the first two subsections the notions on Carnot groups and BV functions needed in the paper. After that, we devote the last subsection to the statement of the main results and some comments.

2.1 Carnot Groups and heat kernels

Basic definitions. Here we recall some basic known facts about Carnot groups, whose precise definition we shall give after introducing some notation. We refer to [7, § 1.4] for the proofs and further details.

Let \mathbb{R}^n be equipped with a Lie group structure by the composition law \circ (which we call *translation*) such that 0 is the identity and $x^{-1} = -x$ (i.e., the group inverse is the Euclidean opposite); let \mathbb{R}^n be also equipped with a family $\{D(\lambda)\}_{\lambda>0}$ of group automorphisms of (\mathbb{R}^n, \circ) (called *dilations*) of the following form

$$D(\lambda) : (x_1, \dots, x_n) \mapsto (\lambda^{\omega_1} x_1, \dots, \lambda^{\omega_n} x_n)$$

where $\omega_1 \leq \omega_2 \leq \dots \leq \omega_n$ are positive integers, with $\omega_1 = \omega_2 = \dots = \omega_q = 1, \omega_i > 1$ for $i > q$ for some $q < n$.

If L_x is the left translation operator acting on functions, $(L_x f)(y) = f(x \circ y)$, we say that a *differential operator* P is *left invariant* if $P(L_x f) = L_x(Pf)$ for every smooth function f . Also, we say that a *differential operator* P is *homogeneous of degree* $\delta > 0$ if

$$P(f(D(\lambda)x)) = \lambda^\delta (Pf)(D(\lambda)x)$$

for every test function f , $\lambda > 0$, $x \in \mathbb{R}^n$, and a function f is *homogeneous of degree* $\delta \in \mathbb{R}$ if

$$f(D(\lambda)x) = \lambda^\delta f(x) \text{ for every } \lambda > 0, x \in \mathbb{R}^n.$$

Now, for $i = 1, 2, \dots, q$, let

$$X_i = \sum_{j=1}^n q_i^j(x) \partial_{x_j} \tag{5}$$

be the unique left invariant vector field (with respect to \circ) which agrees with ∂_{x_i} at the origin, and assume that the Lie algebra generated by X_1, X_2, \dots, X_q coincides with the Lie algebra of \mathbb{G} . Then we say that $\mathbb{G} = (\mathbb{R}^n, \circ, \{D(\lambda)\}_{\lambda>0})$ is a (*homogeneous*) *Carnot group* or a *homogeneous stratified group*.

More explicitly, setting $V_1 = \text{span}\{X_1, X_2, \dots, X_q\}$, $V_{i+1} = [V_1, V_i]$ (the space generated by the commutators $[X, Y]$, with $X \in V_1$, $Y \in V_i$), there is $k \in \mathbb{N}$ such that $\mathbb{R}^n = V_1 \oplus \dots \oplus V_k$, with $V_k \neq \{0\}$ and $V_i = \{0\}$ for $i > k$. The integer k is called the *step* of \mathbb{G} . For every $x \in \mathbb{R}^n$ we identify V_1 at x with the subspace $\mathbb{R}^q \subset \mathbb{R}^n$ whose last $n - q$ coordinates vanish. Note that all vector fields in V_j are j -homogeneous and call

$$Q = \sum_{j=1}^k j \dim V_j = \sum_{j=1}^n \omega_j$$

the *homogeneous dimension* of \mathbb{G} . The operator

$$L = \sum_{j=1}^q X_j^2$$

is called *sublaplacian*; it is left invariant, homogeneous of degree two and, by Hörmander's Theorem, hypoelliptic.

Structure of left- and right-invariant vector fields. It is sometimes useful to consider also the *right-invariant vector fields* X_j^R ($j = 1, \dots, n$), which agree with ∂_{x_j} (and therefore with X_j) at 0; also these X_i^R are homogeneous of degree one for $i \leq q$. (For the following properties of invariant vector fields see [25, pp.606-621] or [8]). As to the *structure of the left (or right) invariant vector fields*, it can be proved that the systems $\{X_i\}$ and $\{X_i^R\}$ have the following “triangular form” with respect to Cartesian derivatives:

$$X_i = \partial_{x_i} + \sum_{k=i+1}^n q_i^k(x) \partial_{x_k} \quad (6)$$

$$X_i^R = \partial_{x_i} + \sum_{k=i+1}^n \bar{q}_i^k(x) \partial_{x_k} \quad (7)$$

where q_i^k, \bar{q}_i^k are polynomials, homogeneous of degree $\omega_k - \omega_i$ (the ω_i 's are the dilation exponents). When the triangular form of the X_i 's is not important, we will keep writing (5), more compactly.

The identities (6)-(7) imply that any Cartesian derivative ∂_{x_k} can be written as a linear combination of the X_i 's (and, analogously, of the X_j^R 's). In particular, any homogeneous differential operator can be rewritten as a linear combination of left invariant (or, similarly, right invariant) homogeneous vector fields, with polynomial coefficients. The above structure of the X_i 's also implies that the formal transpose X_i^* of X_i is just $-X_i$ (as in a standard integration by parts). From the above equations we also find that

$$X_i = \sum_{k=i}^n c_i^k(x) X_k^R$$

where $c_i^k(x)$ are polynomials, homogeneous of degree $\omega_k - \omega_i$. In particular, since $\omega_k - \omega_i < \omega_k$, $c_i^k(x)$ does not depend on x_h for $h \geq k$ and therefore commutes with X_k^R , that is

$$X_i f = \sum_{k=i}^n X_k^R (c_i^k(x) f) \quad (i = 1, \dots, n)$$

for every test function f . The above identity can be sharpened as follows; taking its value at the origin we get:

$$\partial_{x_i} = \sum_{k=i}^q c_i^k \partial_{x_k}$$

since for $k > q$ the $c_i^k(x)$ are homogeneous polynomials of positive degree, and therefore vanish at the origin. Hence $c_i^k = \delta_{ik}$ for $i, k = 1, 2, \dots, q$, that is:

$$X_i f = X_i^R f + \sum_{k=q+1}^n X_k^R (c_i^k(x) f) \quad \text{for } i = 1, \dots, q. \quad (8)$$

Moreover, the operators X_k^R , with $k > q$, can be expressed in terms of the X_1^R, \dots, X_q^R , i.e., for every $k = q+1, \dots, n$ there are constants $\vartheta_{i_1 i_2 \dots i_{\omega_k}}^k$ such that

$$X_k^R = \sum_{1 \leq i_j \leq q} \vartheta_{i_1 i_2 \dots i_{\omega_k}}^k X_{i_1}^R X_{i_2}^R \dots X_{i_{\omega_k}}^R. \quad (9)$$

Convolutions. The *convolution* of two functions in \mathbb{G} is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x \circ y^{-1}) g(y) dy = \int_{\mathbb{R}^n} g(y^{-1} \circ x) f(y) dy,$$

for every couple of functions for which the above integrals make sense. From this definition we see that if P is any left invariant differential operator, then

$$P(f * g) = f * Pg$$

(provided the integrals converge). Note that, if \mathbb{G} is not abelian, we cannot write $f * Pg = Pf * g$. Instead, if X and X^R are, respectively, a left invariant and right invariant vector field which agree at the origin, the following hold (see [25], p.607)

$$(X_i f) * g = f * (X_i^R g); \quad X_i^R (f * g) = (X_i^R f) * g.$$

Explicitly this means

$$\int_{\mathbb{R}^n} X_i^R g(y^{-1} \circ x) f(y) dy = \int_{\mathbb{R}^n} g(y^{-1} \circ x) X_i f(y) dy \quad (10)$$

for any f, g for which the above integrals make sense.

Heat kernels. If $\mathbb{G} = (\mathbb{R}^n, \circ, D(\lambda))$ is a Carnot group, we can naturally define its *parabolic version* setting, in \mathbb{R}^{n+1} :

$$(t, u) \circ_P (s, v) = (t + s, u \circ v); \quad D_P(\lambda)(t, u) = (\lambda^2 t, D(\lambda)u).$$

If we define the *parabolic homogeneous group* $\mathbb{G}_P = (\mathbb{R}^{n+1}, \circ_P, D_P(\lambda))$, its homogeneous dimension is $Q + 2$. Let us now consider the heat operator in \mathbb{G}_P ,

$$\mathcal{H} = \partial_t - \sum_{i=1}^q X_i^2$$

which is translation invariant, homogeneous of degree 2 and hypoelliptic. By a general result due to Folland, see [16], \mathcal{H} possesses a homogeneous fundamental solution $h(t, x)$, usually called *the heat kernel on \mathbb{G}* . The next theorem collects several well-known important facts about h , which are useful in the sequel; the statements (i)-(v) below can be found in [16, Section 1G], while the estimates (11)-(12) are contained in [26, Section IV.4].

Theorem 1 *There exists a function $h(t, x)$ defined in \mathbb{R}^{n+1} with the following properties:*

- (i) $h \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$;
- (ii) $h(\lambda^2 t, D(\lambda)x) = \lambda^{-Q} h(t, x)$ for any $t > 0, x \in \mathbb{R}^n, \lambda > 0$;
- (iii) $h(t, x) = 0$ for any $t < 0, x \in \mathbb{R}^n$;
- (iv) $\int_{\mathbb{R}^n} h(t, x) dx = 1$ for any $t > 0$;
- (v) $h(t, x^{-1}) = h(t, x)$ for any $t > 0, x \in \mathbb{R}^n$.

Let us introduce the heat semigroup, defined as follows for any $f \in L^1(\mathbb{R}^n)$:

$$W_t f(x) = \int_{\mathbb{R}^n} h(t, y^{-1} \circ x) f(y) dy = (f * h)(x).$$

Then, for any $f \in L^1(\mathbb{R}^n)$ and $t > 0$, we have $W_t f \in C^\infty(\mathbb{R}^n)$ and the function $u(t, x) = W_t f(x)$ solves the equation $\mathcal{H}u = 0$ in $(0, \infty) \times \mathbb{R}^n$. Moreover,

$$u(t, x) \rightarrow f(x) \text{ strongly in } L^1(\mathbb{R}^n) \text{ as } t \rightarrow 0.$$

Finally, for every nonnegative integers j, k , for every $x \in \mathbb{R}^n, t > 0$, the following Gaussian bounds hold:

$$\mathbf{c}^{-1} t^{-Q/2} e^{-|x|^2/\mathbf{c}^{-1}t} \leq h(t, x) \leq \mathbf{c} t^{-Q/2} e^{-|x|^2/ct} \quad (11)$$

$$|X_{i_1} \cdots X_{i_j} (\partial_t)^k h(t, x)| \leq \mathbf{c}(j, k) t^{-(Q+j+2k)/2} e^{-|x|^2/ct} \quad (12)$$

for $i_1, i_2, \dots, i_j \in \{1, 2, \dots, q\}$. Here $\mathbf{c} \geq 1$ is a constant only depending on \mathbb{G} , while $\mathbf{c}(j, k)$ depends on \mathbb{G}, j, k .

Remark 2 The bound on the derivatives $X_{i_1} \cdots X_{i_j} \partial_t^k h(t, x)$ still holds, in the same form, if the vector fields X_{i_1}, \dots, X_{i_j} are replaced with any family of j vector fields, homogeneous of degree one; for instance, we will apply this bound to derivatives with respect to the right invariant vector fields X_i^R . Moreover, a homogeneity argument shows that if $a(x)$ is a homogeneous function of degree j' , then

$$|X_{i_1} \cdots X_{i_j} \partial_t^k [a(x) h(t, x)]| \leq c(j, k, a) t^{-(Q+j-j'+2k)/2} e^{-|x|^2/ct}.$$

2.2 BV functions

Let us define the Sobolev spaces $W^{1,p}(\mathbb{G})$, $1 \leq p < \infty$, and the space $BV(\mathbb{G})$ of functions of bounded variation in \mathbb{G} and list their main properties. We remark that the definition of $BV(\mathbb{G})$ goes back to [10] and refer to [1] and to [18] for more information on the Euclidean and the subriemannian case, respectively. We start from the Sobolev case.

Definition 3 For $1 \leq p < \infty$, we say that $f \in L^p(\mathbb{R}^n)$ belongs to the Sobolev space $W^{1,p}(\mathbb{G})$ if there are $f_1, \dots, f_q \in L^p(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x) X_i g(x) dx = - \int_{\mathbb{R}^n} g(x) f_i(x) dx, \quad i = 1, \dots, q,$$

for all $g \in C_0^1(\mathbb{R}^n)$. In this case, we denote f_i as $X_i f$ and set

$$\nabla_X f = (X_1 f, X_2 f, \dots, X_q f).$$

We also let

$$\|\nabla_X f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla_X f| dx = \int_{\mathbb{R}^n} \sqrt{\sum_{i=1}^q |X_i f|^2} dx.$$

We now consider functions of bounded variation.

Definition 4 For $f \in L^1(\mathbb{R}^n)$ we define

$$|D_{\mathbb{G}} f|(\mathbb{R}^n) = \sup \left\{ \int_{\mathbb{R}^n} f(x) \operatorname{div}_{\mathbb{G}} g(x) dx : g \in C_0^1(\mathbb{R}^n, \mathbb{R}^q), \|g\|_{\infty} \leq 1 \right\} \quad (13)$$

where

$$\operatorname{div}_{\mathbb{G}} g(x) = \sum_{i=1}^q X_i g_i(x), \text{ if } g(x) = (g_1(x), \dots, g_q(x))$$

and $\|g\|_{\infty} = \sup_{x \in \mathbb{R}^n} |g(x)|$. We say that $f \in BV(\mathbb{G})$ if $|D_{\mathbb{G}} f|(\mathbb{R}^n)$ is finite.

Remark 5 We point out (see [18, Remarks 2.10, 2.19]) that the total variation depends upon the fixed vector fields X_i , whereas the functional class $BV(G)$ does not.

With the same proof contained e.g. in [1, Prop. 3.6], it is possible to show that if f belongs to $BV(\mathbb{G})$ then its total variation $|D_{\mathbb{G}} f|$ is a finite positive Radon measure and there is a $|D_{\mathbb{G}} f|$ -measurable function $\sigma_f : \mathbb{R}^n \rightarrow \mathbb{R}^q$ such that $|\sigma_f(x)| = 1$ for $|D_{\mathbb{G}} f|$ -a.e. $x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} f(x) \operatorname{div}_{\mathbb{G}} g(x) dx = \int_{\mathbb{R}^n} \langle g, \sigma_f \rangle d|D_{\mathbb{G}} f| \quad (14)$$

for all $g \in C_0^1(\mathbb{G}, \mathbb{R}^q)$ where, here and in the following, we denote by $\langle \cdot, \cdot \rangle$ the usual inner product in \mathbb{R}^q .

We denote by $D_{\mathbb{G}} f$ the vector measure $\sigma_f |D_{\mathbb{G}} f|$, so that $X_i f$ is the measure $(\sigma_f)_i |D_{\mathbb{G}} f|$ and the following integration by parts holds true

$$\int_{\mathbb{R}^n} f(x) X_i g(x) dx = \int_{\mathbb{R}^n} g(x) d(X_i f)(x) \quad (15)$$

for all $g \in C_0^1(\mathbb{G})$. Note also that

$$|X_i f|(\mathbb{R}^n) \leq |D_{\mathbb{G}} f|(\mathbb{R}^n). \quad (16)$$

To visualize the above definitions we note that, whenever f is a smooth function, by the Euclidean divergence theorem we have:

$$\sigma_f = -\frac{\nabla_X f}{|\nabla_X f|}; \quad d|D_{\mathbb{G}} f|(x) = |\nabla_X f|(x) dx;$$

(whenever $|\nabla_X f| \neq 0$) and

$$d(X_i f)(x) = X_i f(x) dx$$

It is clear that $W^{1,1}(\mathbb{G})$ functions are $BV(\mathbb{G})$ functions whose measure gradient is absolutely continuous with respect to Lebesgue measure. Moreover, since it is the supremum of L^1 -continuous functionals, the total variation is $L^1(\mathbb{R}^n)$ lower semicontinuous (see [18, Theorem 2.17]), i.e., $f_k \rightarrow f$ in $L^1(\mathbb{R}^n)$ implies that

$$|D_{\mathbb{G}}f|(\mathbb{R}^n) \leq \liminf_{k \rightarrow \infty} |D_{\mathbb{G}}f_k|(\mathbb{R}^n). \quad (17)$$

Namely:

$$\int_{\mathbb{R}^n} f_k(x) \operatorname{div}_{\mathbb{G}} g(x) dx \leq |D_{\mathbb{G}}f_k|(\mathbb{R}^n), \quad \forall g \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^q), \|g\|_\infty \leq 1.$$

Passing to the liminf as $k \rightarrow \infty$ we get

$$\int_{\mathbb{R}^n} f(x) \operatorname{div}_{\mathbb{G}} g(x) dx \leq \liminf_{k \rightarrow \infty} |D_{\mathbb{G}}f_k|(\mathbb{R}^n)$$

and taking the supremum over all possible g 's we get (17).

Definition 6 (Sets of finite \mathbb{G} -perimeter) *If χ_E is the characteristic function of a measurable set $E \subset \mathbb{R}^n$ and $|D_{\mathbb{G}}\chi_E|$ is finite, we say that E is a set of finite \mathbb{G} -perimeter and we write $P_{\mathbb{G}}(E)$ instead of $|D_{\mathbb{G}}\chi_E|$, $P_{\mathbb{G}}(E, F)$ instead of $|D_{\mathbb{G}}\chi_E|(F)$ for F Borel. Also, we call (generalized inward) \mathbb{G} -normal the q -vector*

$$\nu_E(x) = -\sigma_{\chi_E}(x).$$

Recall that $|\nu_E(x)| = 1$ for $P_{\mathbb{G}}(E)$ -a.e. $x \in \mathbb{R}^n$. In this case (14) takes the form

$$\int_E \operatorname{div}_{\mathbb{G}} g(x) dx = - \int_{\mathbb{R}^n} \langle g, \nu_E \rangle dP_{\mathbb{G}}(E). \quad (18)$$

Remark 7 To help the reader to visualize the above definitions, let us specialize them to the case of a bounded smooth domain E . Let n_E be the Euclidean unit inner normal at ∂E , and consider the q -vector v whose i -th component is defined by

$$v_i = \sum_{j=1}^n q_i^j(x) (n_E)_j(x)$$

(where the q_i^j are the coefficients of the vector fields X_i , defined in (5)). Then

$$\int_E \operatorname{div}_{\mathbb{G}} g(x) dx = - \int_{\partial E} \langle g, v \rangle dH^{n-1}(x)$$

from which we read that in this case

$$\nu_E = \frac{v}{|v|}, \quad dP_{\mathbb{G}}(E) = |v| d(H^{n-1} \llcorner \partial E)(x) \quad (19)$$

(where \llcorner denotes the restriction of the measure) at least at those points of the boundary where $v \neq 0$ (noncharacteristic points).

Identity (10) can be extended to the case $g \in C^1(\mathbb{G})$, $f \in BV(\mathbb{G})$ as follows:

$$\int_{\mathbb{R}^n} X_i^R g(y^{-1} \circ x) f(y) dy = \int_{\mathbb{R}^n} g(y^{-1} \circ x) d(X_i f)(y) \quad (20)$$

where the last integral is made with respect to the measure defined by $X_i f$. To see this, it is enough to take a sequence $f_k \in W^{1,1}(\mathbb{G})$ such that $f_k \rightarrow f$ in $L^1(\mathbb{R}^n)$ and $X_i f_k$ is weakly* convergent to $X_i f$ (the existence of such a sequence is proved in [17]), so that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} X_i^R g(y^{-1} \circ x) f_k(y) dy &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} g(y^{-1} \circ x) X_i f_k(y) dy \\ &= \int_{\mathbb{R}^n} g(y^{-1} \circ x) d(X_i f)(y). \end{aligned}$$

Let us come to some finer properties of BV functions that we need only in the proof of Theorem 15. We refer to [17, Theorem 2.3.5] for a proof of the following statement.

Proposition 8 (Coarea formula) *If $f \in BV(\mathbb{G})$ then for a.e. $\tau \in \mathbb{R}$ the set $E_\tau = \{x \in \mathbb{R}^n : f(x) > \tau\}$ has finite \mathbb{G} -perimeter and*

$$|D_{\mathbb{G}} f|(\mathbb{R}^n) = \int_{-\infty}^{+\infty} |D_{\mathbb{G}} \chi_{E_\tau}|(\mathbb{R}^n) d\tau. \quad (21)$$

Conversely, if $f \in L^1(\mathbb{R}^n)$ and $\int_{-\infty}^{+\infty} |D_{\mathbb{G}} \chi_{E_\tau}|(\mathbb{R}^n) d\tau < +\infty$ then $f \in BV(\mathbb{G})$ and equality (21) holds. Moreover, if $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Borel function, then

$$\int_{\mathbb{R}^n} g(x) d|D_{\mathbb{G}} f|(x) = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} g(x) d|D_{\mathbb{G}} \chi_{E_\tau}|(x) d\tau. \quad (22)$$

Next we have to introduce the reduced boundary. This definition relies on a particular notion of balls in \mathbb{G} . We will denote by $B_\infty(x, r)$ the d_∞ -balls of center x and radius r , where $d_\infty(x, y) = d_\infty(0, y^{-1} \circ x)$ and $d_\infty(0, x)$ is the homogeneous norm in \mathbb{G} which is considered in [18, Theorem 5.1]. We do not recall its explicit analytical definition since we will never need it. The distance d_∞ can be used to define the spherical Hausdorff measures \mathcal{S}_∞^k through the usual Carathéodory construction see e.g. [15, Section 2.10.2].

Definition 9 *Let $E \subset \mathbb{R}^n$ be measurable with finite \mathbb{G} -perimeter. We say that $x \in \partial_{\mathbb{G}}^* E$ (reduced boundary of E) if the following conditions hold:*

(i) $P_{\mathbb{G}}(E, B_\infty(x, r)) > 0$ for all $r > 0$;

(ii) *the following limit exists*

$$\lim_{r \rightarrow 0} \frac{D_{\mathbb{G}} \chi_E(B_\infty(x, r))}{|D_{\mathbb{G}} \chi_E|(B_\infty(x, r))}$$

and equals $\nu_E(x)$.

(iii) *the equality $|\nu_E(x)| = 1$ holds.*

In order to state the next result, let us introduce some notation. For $E \subset \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, $r > 0$ we consider the translated and dilated set E_{r, x_0} defined as

$$E_{r, x_0} = \{x \in \mathbb{R}^n : x_0 \circ D(r)x \in E\} = D(r^{-1})(x_0^{-1} \circ E).$$

For any vector $\nu \in \mathbb{R}^q$ we define the sets

$$S_{\mathbb{G}}^+(\nu) = \{x \in \mathbb{R}^n : \langle \pi x, \nu \rangle \geq 0\}, \quad T_{\mathbb{G}}(\nu) = \{x \in \mathbb{R}^n : \langle \pi x, \nu \rangle = 0\}$$

where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^q$ is the projection which reads the first q components. Hence the set $T_{\mathbb{G}}(\nu)$ is an Euclidean (q -dimensional) “vertical plane” in \mathbb{R}^n ; it is also a subgroup of \mathbb{G} , since on the first q components the Lie group operation is just the Euclidean sum.

We are now in a position to state the following result (see [18, Theorem 3.1]).

Theorem 10 (Blow-up on the reduced boundary) *Let \mathbb{G} be of Step 2. If $E \subset \mathbb{R}^n$ is a set of finite \mathbb{G} -perimeter and $x_0 \in \partial_{\mathbb{G}}^* E$ then*

$$\lim_{r \rightarrow 0} \chi_{E_r, x_0} = \chi_{S_{\mathbb{G}}^+(\nu_E(x_0))} \quad \text{in } L_{\text{loc}}^1(\mathbb{R}^n). \quad (23)$$

(Note that $\nu_E(x_0)$ is pointwise well defined for $x_0 \in \partial_{\mathbb{G}}^* E$, by point (iii) in Definition 9). Moreover, $P_{\mathbb{G}}(E)$ -a.e. $x \in \mathbb{R}^n$ belongs to $\partial_{\mathbb{G}}^* E$.

The last statement in the above theorem allows us to rewrite formula (18) as an integral on the reduced boundary with respect to the $Q - 1$ spherical Hausdorff measure as follows:

$$\int_E \operatorname{div}_{\mathbb{G}} g(x) dx = -\theta_{\infty} \int_{\partial_{\mathbb{G}}^* E} \langle g, \nu_E \rangle d\mathcal{S}_{\infty}^{Q-1}, \quad (24)$$

(here θ_{∞} is a constant depending on \mathbb{G} , see [18, Theorem 3.10]), which looks much closer to the classical divergence theorem. Moreover, the following analogue of (19) holds:

$$P_{\mathbb{G}}(E, \cdot) = \theta_{\infty} \mathcal{S}_{\infty}^{Q-1} \llcorner (\partial_{\mathbb{G}}^* E). \quad (25)$$

In order to apply the L_{loc}^1 convergence of (23) we will need also the following

Remark 11 If $E, \{E_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ are μ -measurable, $\chi_{E_k} \rightarrow \chi_E$ in $L_{\text{loc}}^1(\mathbb{R}^n, \mu)$ and $f \in L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$, then $f\chi_{E_k} \rightarrow f\chi_E$ in $L^1(\mathbb{R}^n)$. In fact, given $\varepsilon > 0$, there is a compact set K such that

$$\int_{\mathbb{R}^n \setminus K} |f| d\mu < \varepsilon,$$

whence

$$\int_{\mathbb{R}^n} |(\chi_{E_k} - \chi_E) f| d\mu \leq \varepsilon + \|f\|_{\infty} \int_K |\chi_{E_k} - \chi_E| d\mu$$

and the last integral tends to 0 as $k \rightarrow \infty$.

2.3 Main Results

We state here the main results of this paper, namely Theorems 12 and 15.

Theorem 12 *Let $f \in L^1(\mathbb{R}^n)$. The quantity*

$$\limsup_{t \rightarrow 0} \|\nabla_X W_t f\|_{L^1(\mathbb{R}^n)}$$

is finite if and only if $f \in BV(\mathbb{G})$. In this case the following holds:

$$\begin{aligned} |D_{\mathbb{G}} f|(\mathbb{R}^n) &\leq \liminf_{t \rightarrow 0} \|\nabla_X W_t f\|_{L^1(\mathbb{R}^n)} \leq \limsup_{t \rightarrow 0} \|\nabla_X W_t f\|_{L^1(\mathbb{R}^n)} \\ &\leq (1 + c) |D_{\mathbb{G}} f|(\mathbb{R}^n), \end{aligned}$$

where $c \geq 0$ is a constant depending only on \mathbb{G} .

Theorem 12 shows that even in Carnot groups Definition 4 is equivalent to the analogue of De Giorgi's original definition (3) given in the Euclidean case, even though we don't know if the limit exists. Section 3 is devoted to the proof of Theorem 12.

The next result requires the additional hypothesis that \mathbb{G} is a Carnot group of Step 2. As we explain in Section 4, the reason is that our proof uses the rectifiability of the reduced boundary of a set of finite perimeter in \mathbb{G} , which is known only in the Step 2 case, see [18]. Our argument works whenever the rectifiability theorem is true, and then would extend immediately to more general cases, where the rectifiability result were extended.

We start by defining the function

$$\phi_{\mathbb{G}}(\nu) = \int_{T_{\mathbb{G}}(\nu)} h(1, x) dx, \quad (26)$$

for vectors $\nu \in \mathbb{R}^q$. Here the integral is taken over $T_{\mathbb{G}}(\nu)$, which is a hyperplane in \mathbb{R}^n ; to simplify notation, we keep denoting by dx the $(n-1)$ -dimensional Lebesgue measure over $T_{\mathbb{G}}(\nu)$.

The function $\phi_{\mathbb{G}}$ is continuous and, using the Gaussian bounds (11), there are constants c_1, c_2 , depending on the group, such that

$$0 < c_1 \leq \phi_{\mathbb{G}}(\nu) \leq c_2. \quad (27)$$

Remark 13 (Rotation invariance of heat kernels) Notice that if the heat kernel is invariant under horizontal rotations (that is, under Euclidean rotations in the space \mathbb{R}^q of the first variables x_1, x_2, \dots, x_q of \mathbb{R}^n), then $\phi_{\mathbb{G}}$ is actually a constant (that is, independent of ν). This happens for instance in all groups of Heisenberg type (briefly called H-type groups), in view of the known formula assigning the heat kernel in that context (for a discussion of H-type groups, see for instance [7, Chap. 18]; for the computation of the heat kernel in this context, see [24]). By the way, we point out that even in the Heisenberg group H^n , where the heat kernel is invariant for horizontal rotations, the sublaplacian is not (unless $n = 1$). On the other hand, from the general formula assigning the heat kernel in Carnot groups of step two, proved by [12] (see also [5]), one can expect the existence of groups of step two (more general than H-type groups) where the heat kernel is not invariant under horizontal rotations and the function $\phi_{\mathbb{G}}$ is not constant.

Let us first state our next result in the case of perimeters.

Theorem 14 *Let \mathbb{G} be of Step 2. If $E \subset \mathbb{R}^n$ is a set of finite \mathbb{G} -perimeter, the following equality holds*

$$\lim_{t \rightarrow 0} \frac{1}{2\theta_{\infty}\sqrt{t}} \int_{E^c} W_t \chi_E(x) dx = \int_{\partial_{\mathbb{G}}^* E} \phi_{\mathbb{G}}(\nu_E) d\mathcal{S}_{\infty}^{Q-1}. \quad (28)$$

Conversely, if

$$\liminf_{t \rightarrow 0} \frac{1}{\sqrt{t}} \int_{E^c} W_t \chi_E(x) dx < +\infty$$

then E has finite perimeter, and (28) holds.

For general BV functions, we have the following.

Theorem 15 *Let \mathbb{G} be of Step 2. Then, if $f \in BV(\mathbb{G})$ the following equality holds:*

$$\int_{\mathbb{R}^n} \phi_{\mathbb{G}}(\sigma_f) d|D_{\mathbb{G}} f| = \lim_{t \rightarrow 0} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| h(t, y^{-1} \circ x) dx dy, \quad (29)$$

where σ_f is defined in (14). Conversely, if the liminf of the quantity on the right hand side is finite, then $f \in BV(\mathbb{G})$ and (29) holds.

Remark 16 Note that, in view of (25), (27) and Remark 13, the right-hand side of (28) is always equivalent to, and in groups of Heisenberg-type is a multiple of, $P_{\mathbb{G}}(E)$; the left-hand side of (29) is always equivalent to, and in groups of Heisenberg-type is a multiple of, $|D_{\mathbb{G}}f|(\mathbb{R}^n)$.

The proofs of the above results will be given in Section 4.

3 Proof of Theorem 12

In this section we give the proof of Theorem 12; first of all, we notice that $W_t f \rightarrow f$ in L^1 as $t \rightarrow 0$; moreover $W_t f \in C^\infty(\mathbb{R}^n)$, hence

$$|D_{\mathbb{G}}W_t f|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |\nabla_X W_t f(x)| dx.$$

Therefore by the lower semicontinuity of the variation, that is (17), the inequality

$$|D_{\mathbb{G}}f|(\mathbb{R}^n) \leq \liminf_{t \rightarrow 0} \int_{\mathbb{R}^n} |\nabla_X W_t f(x)| dx \quad (30)$$

holds, and then it remains to prove only the upper bound. We also notice that inequality (30) implies that if $f \in L^1(\mathbb{R}^n)$ is such that the right hand side of (30) is finite, then $f \in BV(\mathbb{G})$. So, Theorem 12 will follow if we prove that

$$\limsup_{t \rightarrow 0} \int_{\mathbb{R}^n} |\nabla_X W_t f(x)| dx \leq (1+c) |D_{\mathbb{G}}f|(\mathbb{R}^n) \quad (31)$$

for all $f \in BV(\mathbb{G})$.

We start with the following result, which will be useful in the proof of Theorem 12. It is a consequence of the algebraic properties of \mathbb{G} and the Gaussian estimates on the heat kernel h (see section 2.1).

Lemma 17 For $i, j \in \{1, \dots, q\}$, let G_j^i be the kernel defined by the identity:

$$\begin{aligned} \sum_{k=q+1}^n X_k^R(c_i^k(\cdot)h(t, \cdot))(z) &= \sum_{j=1}^q X_j^R \sum_{k=q+1}^n \sum_{1 \leq i_l \leq q} \vartheta_{ji_2 \dots i_{\omega_k}}^k X_{i_2}^R \dots X_{i_{\omega_k}}^R (c_i^k(\cdot)h(t, \cdot))(z) \\ &= \sum_{j=1}^q X_j^R G_j^i(t, z), \end{aligned} \quad (32)$$

where the functions c_i^k and the constants $\vartheta_{i_1 i_2 \dots i_{\omega_k}}^k$ are those introduced in (8) and (9), respectively. They have the following properties:

- (i) $G_j^i(\lambda^2 t, D(\lambda)z) = \lambda^{-Q} G_j^i(t, z)$ for any $\lambda > 0, t > 0, z \in \mathbb{R}^n$;
- (ii) there is a positive constant c , independent of t , such that

$$\int_{\mathbb{R}^n} |G_j^i(t, z)| dz \leq c;$$

- (iii) $\int_{\mathbb{R}^n} G_j^i(t, z) dz = 0$ for any $t > 0$;

(iv) for every $\varepsilon > 0$, $t_0 > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_R} |G_j^i(t, z)| dz < \varepsilon \text{ for any } 0 < t \leq t_0.$$

Proof. (i) holds because $c_i^k(\cdot)$ is homogeneous of degree $\omega_k - \omega_i = \omega_k - 1$ (for $i = 1, 2, \dots, q$), $h(t, \cdot)$ is homogeneous of degree $-Q$, and $X_{i_2}^R \dots X_{i_{\omega_k}}^R$ is a homogeneous differential operator of degree $\omega_k - 1$. For the same reason, the Gaussian estimates on h also imply, by Remark 2,

$$|G_j^i(t, z)| = \left| X_{i_2}^R \dots X_{i_{\omega_k}}^R (c_i^k(\cdot) h(t, \cdot)) (z) \right| \leq \mathbf{c} t^{-Q/2} e^{-|z|^2/\mathbf{c}t} \quad (33)$$

To prove (ii), with the change of variables $z = D(\sqrt{t})w$ and using (i) and (33), we compute:

$$\begin{aligned} \int_{\mathbb{R}^n} |G_j^i(t, z)| dz &= \int_{\mathbb{R}^n} \left| G_j^i(t, D(\sqrt{t})w) \right| t^{Q/2} dw \\ &= \int_{\mathbb{R}^n} |G_j^i(1, w)| dw \leq \int_{\mathbb{R}^n} \mathbf{c} e^{-|w|^2/\mathbf{c}} dw \end{aligned} \quad (34)$$

which is a finite constant. (iii) holds because G_j^i is, by definition, a linear combination of derivatives of the kind

$$X_{i_2}^R \left[X_{i_3}^R \dots X_{i_{\omega_k}}^R (c_i^k(\cdot) h(t, \cdot)) \right] = X_{i_2}^R H(t, \cdot),$$

we know that $(X_i^R)^T = -X_i^R$ (where $()^T$ denotes transposition) and therefore we may deduce that $\int_{\mathbb{R}^n} X_i^R f(x) dx = 0$ for any f for which this integral makes sense. To prove (iv), it suffices to modify the computation (34) as follows:

$$\int_{\mathbb{R}^n \setminus B_R} |G_j^i(t, z)| dz \leq \int_{|w| > R/\sqrt{t}} \mathbf{c} e^{-|w|^2/\mathbf{c}} dw < \varepsilon$$

for R large enough and $t \leq t_0$. ■

The following result contains the main estimate on the commutator between the derivative X_i and the heat semigroup $X_i(W_t f) - W_t(X_i f)$.

Proposition 18 *For any $f \in BV(\mathbb{G})$ and $i \in \{1, 2, \dots, q\}$, $t > 0$, there exists an L^1 function μ_t^i on \mathbb{R}^n such that*

$$X_i(W_t f)(x) = W_t(X_i f)(x) + \mu_t^i(x),$$

where $W_t(X_i f)$ is the function defined by

$$W_t(X_i f)(x) := \int_{\mathbb{R}^n} h(t, y^{-1} \circ x) dX_i f(y).$$

Moreover, for any $t > 0$,

$$\|\mu_t^i\|_{L^1(\mathbb{R}^n)} \leq c q |D_{\mathbb{G}} f|(\mathbb{R}^n), \quad (35)$$

where c is the constant in Lemma 17, (ii).

Proof. Let $f \in BV(\mathbb{G})$ and fix $i \in \{1, 2, \dots, q\}$. By (8) and (20) we have

$$\begin{aligned}
X_i W_t f(x) &= \int_{\mathbb{R}^n} X_i h(t, y^{-1} \circ x) f(y) dy \\
&= \int_{\mathbb{R}^n} X_i^R h(t, y^{-1} \circ x) f(y) dy + \int_{\mathbb{R}^n} \sum_{k=q+1}^n X_k^R (c_i^k(\cdot) h(t, \cdot)) (y^{-1} \circ x) f(y) dy \\
&= \int_{\mathbb{R}^n} X_i^R h(t, y^{-1} \circ x) f(y) dy \\
&\quad + \int_{\mathbb{R}^n} \sum_{k=q+1}^n \sum_{1 \leq i_j \leq q} \vartheta_{i_1 i_2 \dots i_{\omega_k}}^k X_{i_1}^R X_{i_2}^R \dots X_{i_{\omega_k}}^R (c_i^k(\cdot) h(t, \cdot)) (y^{-1} \circ x) f(y) dy \\
&= \int_{\mathbb{R}^n} h(t, y^{-1} \circ x) d(X_i f)(y) \\
&\quad + \sum_{k=q+1}^n \sum_{1 \leq i_j \leq q} \vartheta_{i_1 i_2 \dots i_{\omega_k}}^k \int_{\mathbb{R}^n} X_{i_2}^R \dots X_{i_{\omega_k}}^R (c_i^k(\cdot) h(t, \cdot)) (y^{-1} \circ x) d(X_{i_1} f)(y) \\
&= W_t(X_i f)(x) + \sum_{j=1}^q \int_{\mathbb{R}^n} G_j^i(t, y^{-1} \circ x) d(X_j f)(y),
\end{aligned} \tag{36}$$

where the kernels G_j^i are defined in Lemma 17. Set

$$\mu_t^i(x) = \sum_{j=1}^q \int_{\mathbb{R}^n} G_j^i(t, y^{-1} \circ x) d(X_j f)(y), \tag{37}$$

then by Lemma 17 and (16) we have

$$\|\mu_t^i\|_{L^1(\mathbb{R}^n)} \leq c \sum_{j=1}^q \int_{\mathbb{R}^n} |d(X_j f)|(y) \leq cq |D_{\mathbb{G}} f|(\mathbb{R}^n)$$

that is (35). By (36) and (37), we can write:

$$X_i(W_t f)(x) = W_t(X_i f)(x) + \mu_t^i(x). \tag{38}$$

■

Proof of Theorem 12. Let us conclude with the proof of (31). By Proposition 18 we have (with the supremum taken, accordingly to Definition 4, on all the functions $g \in C_0^1(\mathbb{R}^n, \mathbb{R}^q)$ such that $\|g\|_\infty \leq 1$):

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla_X W_t f(x)| dx &= \sup_g \left\{ \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) X_i(W_t f)(x) dx \right\} \\
&= \sup_g \left\{ \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) \left(\int_{\mathbb{R}^n} h(t, y^{-1} \circ x) dX_i f(y) \right) dx + \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) \mu_t^i(x) dx \right\}
\end{aligned}$$

since $h(t, z^{-1}) = h(t, z)$

$$\begin{aligned}
&= \sup_g \left\{ \sum_{i=1}^q \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} h(t, x^{-1} \circ y) g_i(x) dx \right) dX_i f(y) + \int_{\mathbb{R}^n} \sum_{i=1}^q g_i(x) \mu_t^i(x) dx \right\} \\
&= \sup_g \left\{ \sum_{i=1}^q \int_{\mathbb{R}^n} W_t g_i(y) dX_i f(y) + \int_{\mathbb{R}^n} g_i(x) \mu_t^i(x) dx \right\} \\
&= \sup_g \left\{ \sum_{i=1}^q \int_{\mathbb{R}^n} X_i(W_t g_i)(y) f(y) dy + \int_{\mathbb{R}^n} g_i(x) \mu_t^i(x) dx \right\}
\end{aligned}$$

where in the last identity we have used (15). We now exploit the fact that the total variation of f can be obtained as the supremum on all functions ϕ rapidly decreasing at infinity (as $W_t g$ is, by the Gaussian estimates) verifying the constraint $\|\phi\|_\infty \leq 1$, not necessarily with compact support; therefore the last expression is, by (35)

$$\leq |D_{\mathbb{G}} f| + \|\mu_t^i\|_{L^1(\mathbb{R}^n)} \leq (1 + cq) |D_{\mathbb{G}} f|(\mathbb{R}^n).$$

We have therefore proved that for any $t > 0$,

$$\int_{\mathbb{R}^n} |\nabla_X W_t f(x)| dx \leq (1 + cq) |D_{\mathbb{G}} f|(\mathbb{R}^n).$$

Passing to the limsup as $t \rightarrow 0$ we are done. ■

4 Proof of Theorem 15

Theorem 15, thanks to coarea formula, follows from Theorem 14, so we first prove the latter.

As a preliminary result, we present a characterization of $BV(\mathbb{G})$ functions analogous to that in [9, Proposition 2.3].

Lemma 19 *If $f \in L^1(\mathbb{R}^n)$, $z \in \mathbb{R}^n$ and there is $C > 0$ such that the inequality*

$$\int_{\mathbb{R}^n} |f(x \circ D(t)z) - f(x)| dx \leq Ct \tag{39}$$

holds, then the distributional derivative of f along $Z = \sum_{j=1}^q z_j X_j$ is a finite measure.

Proof. The proof is essentially the same contained in [9], but we repeat it for reader's convenience. We start recalling the following identity (see [7, p. 17]): if we take $\phi \in C_c^1(\mathbb{R}^n)$ and Z is any vector field which at the origin equals z , then

$$Z\phi(x) = \frac{d}{dt} [\phi(x \circ tz)]_{t=0}.$$

We will apply this identity to the vector field $Z = \sum_{i=1}^q z_i X_i$ which at the origin takes the value

$$z^* = (z_1, z_2, \dots, z_q, 0, 0, \dots, 0).$$

Moreover, for any $z \in \mathbb{R}^n$,

$$\frac{d}{dt} [\phi(x \circ D(t)z)]_{t=0} = \frac{d}{dt} [\phi(x \circ tz^*)]_{t=0}$$

hence

$$\frac{d}{dt} [\phi(x \circ D(t)z)]_{/t=0} = \sum_{i=1}^q z_i (X_i \phi)(x).$$

Then, the functional

$$\begin{aligned} T_f \phi &:= \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} \frac{f(x \circ D(t)z) - f(x)}{t} \phi(x) dx \\ &= \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} f(x) \frac{\phi(x \circ D(t)z^{-1}) - \phi(x)}{t} dx = - \int_{\mathbb{R}^n} f(x) Z \phi(x) dx \end{aligned}$$

is well defined and satisfies the condition $|T_f \phi| \leq C \|\phi\|_\infty$. Then, there exists a measure μ_Z such that

$$T_f \phi = \int_{\mathbb{R}^n} \phi(x) d\mu_Z(x), \quad \forall \phi \in C_c^1(\mathbb{R}^n).$$

By the density of $C_c^1(\mathbb{R}^n)$ in $C_c(\mathbb{R}^n)$, we get the conclusion. ■

Proof of Theorem 14. Let us introduce the functions

$$\begin{aligned} g(t) &= \int_{E^c} W_t \chi_E(x) dx \\ F(t) &= g(t^2) = \int_{E^c} W_{t^2} \chi_E(x) dx. \end{aligned}$$

The statement we want to prove is:

$$\lim_{t \rightarrow 0} \frac{g(t)}{\sqrt{t}} = 2\theta_\infty \int_{\partial_{\mathbb{G}}^* E} \phi_{\mathbb{G}}(\nu_E) dS_\infty^{Q-1}.$$

Noting that, for $t \rightarrow 0$, $g(t) \rightarrow \int_{E^c} \chi_E(x) dx = 0$, by De L'Hospital rule we can evaluate $\lim_{t \rightarrow 0} g(t)/\sqrt{t}$ as

$$\lim_{t \rightarrow 0} 2\sqrt{t} g'(t) = \lim_{t \rightarrow 0} 2t g'(t^2) = \lim_{t \rightarrow 0} F'(t).$$

Computing the derivative of F using the divergence theorem (24) and the properties of $h(t, \cdot)$, we get:

$$\begin{aligned} F'(t) &= 2t \int_{E^c} L W_{t^2} \chi_E dx = 2t \int_{E^c} \operatorname{div}_{\mathbb{G}} \nabla_X W_{t^2} \chi_E dx \\ &= 2\theta_\infty t \int_{\partial_{\mathbb{G}}^* E} \langle \nabla_X W_{t^2} \chi_E, \nu_E \rangle dS_\infty^{Q-1} \\ &= 2\theta_\infty t \int_{\partial_{\mathbb{G}}^* E} \langle \nabla_X \left(\int_E h(t^2, y^{-1} \circ x) dy \right), \nu_E(x) \rangle dS_\infty^{Q-1}(x) \end{aligned}$$

Now, taking into account the fact that $h(t, z^{-1}) = h(t, z)$, we have

$$\begin{aligned} \int_E h(t^2, y^{-1} \circ x) dy &= \int_E h(t^2, x^{-1} \circ y) dy = \int_E t^{-Q} h\left(1, D\left(\frac{1}{t}\right)(x^{-1} \circ y)\right) dy \\ &= \int_E t^{-Q} h\left(1, D\left(\frac{1}{t}\right)(x^{-1}) \circ D\left(\frac{1}{t}\right)(y)\right) dy \\ &= \int_{D(\frac{1}{t})E} h\left(1, D\left(\frac{1}{t}\right)(x^{-1}) \circ z\right) dz \\ &= \int_{D(\frac{1}{t})E} h\left(1, z^{-1} \circ D\left(\frac{1}{t}\right)(x)\right) dz \end{aligned}$$

hence

$$\begin{aligned}
F'(t) &= 2\theta_\infty t \int_{\partial_{\mathbb{G}}^* E} \left\langle \frac{1}{t} \int_{D(\frac{1}{t})E} \nabla_X h \left(1, z^{-1} \circ D \left(\frac{1}{t} \right) (x) \right) dz, \nu_E(x) \right\rangle d\mathcal{S}_\infty^{Q-1}(x) \\
&= 2\theta_\infty \int_{\partial_{\mathbb{G}}^* E} \left\langle \int_{D(\frac{1}{t})E} \nabla_X h(1, z^{-1} \circ D \left(\frac{1}{t} \right) (x)) dz, \nu_E(x) \right\rangle d\mathcal{S}_\infty^{Q-1}(x) \\
&= 2\theta_\infty \int_{\partial_{\mathbb{G}}^* E} \left\langle \int_{D(1/t)(x^{-1}E)} \nabla_X h(1, z^{-1}) dz, \nu_E(x) \right\rangle d\mathcal{S}_\infty^{Q-1}(x) \\
&= 2\theta_\infty \sum_{i=1}^q \int_{\partial_{\mathbb{G}}^* E} \nu_E(x)_i \left(\int_{D(1/t)(x^{-1}E)} X_i h(1, z^{-1}) dz \right) d\mathcal{S}_\infty^{Q-1}(x).
\end{aligned}$$

By theorem 10 (blow-up of the reduced boundary), we have the convergence

$$E_{1/t, x^{-1}} = D(1/t)(x^{-1}E) \rightarrow S_{\mathbb{G}}^+(\nu_E(x)), \quad (40)$$

hence, since by the Gaussian estimates $X_i h(1, z^{-1}) \in L^1 \cap L^\infty(\mathbb{R}^n)$, by Remark 11,

$$\begin{aligned}
\lim_{t \rightarrow 0} F'(t) &= 2\theta_\infty \int_{\partial_{\mathbb{G}}^* E} \sum_{i=1}^q \nu_E(x)_i \left(\int_{S_{\mathbb{G}}^+(\nu_E(x))} X_i h(1, z^{-1}) dz \right) d\mathcal{S}_\infty^{Q-1}(x) \\
&= 2\theta_\infty \int_{\partial_{\mathbb{G}}^* E} \sum_{i=1}^q \nu_E(x)_i \left(\int_{S_{\mathbb{G}}^-(\nu_E(x))} X_i h(1, z) dz \right) d\mathcal{S}_\infty^{Q-1}(x)
\end{aligned}$$

with the obvious meaning of the symbol $S_{\mathbb{G}}^-(\nu_E(x))$. Next, we perform a standard integration by parts in the inner integral, exploiting the fact that the (Euclidean) outer normal to the halfspace $S_{\mathbb{G}}^-(\nu_E(x))$ is

$$n_E(x) = (\nu_E(x)_1, \dots, \nu_E(x)_q, 0, \dots, 0)$$

while

$$X_i h(1, z) = \partial_{z_i} h(1, z) + \sum_{j=q+1}^n \partial_{z_j} (q_j^i(z) h(1, z))$$

hence

$$\begin{aligned}
&\int_{\partial_{\mathbb{G}}^* E} \sum_{i=1}^q \nu_E(x)_i \left(\int_{S_{\mathbb{G}}^-(\nu_E(x))} X_i h(1, z) dz \right) d\mathcal{S}_\infty^{Q-1}(x) \\
&= \sum_{i=1}^q (\nu_E(x)_i)^2 \int_{T_{\mathbb{G}}(\nu_E(x))} h(1, z) dz = \phi_{\mathbb{G}}(\nu_E)
\end{aligned}$$

for any $x \in \partial_{\mathbb{G}}^* E$, and we conclude that

$$\lim_{t \rightarrow 0} \frac{1}{2\theta_\infty \sqrt{t}} \int_{E^c} W_t \chi_E(x) dx = \int_{\partial_{\mathbb{G}}^* E} \phi_{\mathbb{G}}(\nu_E) d\mathcal{S}_\infty^{Q-1}. \quad (41)$$

In order to prove the converse, notice that for any z ,

$$\int_{\mathbb{R}^n} \chi_E(x \circ D(\sqrt{t})z) (1 - \chi_E(x)) dx = \frac{1}{2} \int_{\mathbb{R}^n} \left| \chi_E(x \circ D(\sqrt{t})z) - \chi_E(x) \right| dx,$$

and that the Lebesgue measure is right invariant as well. Therefore

$$\begin{aligned}
\int_{E^c} W_t \chi_E(x) \, dx &= t^{-Q/2} \int_{\mathbb{R}^n} \int_E h\left(1, D(1/\sqrt{t})(y^{-1} \circ x)\right) (1 - \chi_E(x)) \, dy dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(1, z) \chi_E(x \circ D(\sqrt{t})z^{-1}) (1 - \chi_E(x)) \, dz dx \\
&= \frac{1}{2} \int_{\mathbb{R}^n} h(1, w^{-1}) \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t})w) - \chi_E(x)| \, dx dw \\
&= \frac{1}{2} \int_{\mathbb{R}^n} h(1, w) \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t})w) - \chi_E(x)| \, dx dw.
\end{aligned}$$

Then, the finiteness of the limit in the left hand side of (41) implies

$$\frac{1}{2} \int_{\mathbb{R}^n} h(1, w) \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t})w) - \chi_E(x)| \, dx dw \leq C\sqrt{t}.$$

On the other hand, by the lower Gaussian estimates on $h(1, w)$ we can also write

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^n} h(1, w) \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t})w) - \chi_E(x)| \, dx dw \\
&\geq \frac{1}{2} \int_{|w| \leq 2} h(1, w) \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t})w) - \chi_E(x)| \, dx dw \\
&\geq c \int_{|w| \leq 2} \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t})w) - \chi_E(x)| \, dx dw
\end{aligned}$$

hence

$$\int_{|w| \leq 2} \int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t})w) - \chi_E(x)| \, dx dw \leq C\sqrt{t}.$$

So, if we define the function

$$\Phi_t(w) = \int_{\mathbb{R}^n} \frac{|\chi_E(x \circ D(\sqrt{t})w) - \chi_E(x)|}{\sqrt{t}} \, dx,$$

by Fatou's Lemma we deduce that

$$\Phi_0(w) = \liminf_{t \rightarrow 0} \Phi_t(w)$$

is integrable on $B = \{|w| \leq 2\}$. So, for any $\varepsilon > 0$, there are a set B_ε of measure less than ε and a constant $C_\varepsilon > 0$ such that for every $w \in B \setminus B_\varepsilon$, $\Phi_0(w) \leq C_\varepsilon$. In particular, we may choose n linearly independent directions w_1, \dots, w_n with $\Phi_0(w_i) \leq C_\varepsilon$, and there is $t_0 > 0$ such that

$$\int_{\mathbb{R}^n} |\chi_E(x \circ D(\sqrt{t})w_i) - \chi_E(x)| \, dx \leq (C_\varepsilon + 1)\sqrt{t}, \quad \forall i = 1, \dots, n, \quad 0 < t \leq t_0.$$

Then by Lemma 19 we can conclude that E has finite perimeter. \blacksquare

We note that the above proof of Theorem 14 simplifies that of the analogous result in the Euclidean setting given in [22].

With an argument similar to that used in the proof of Proposition 8 in [23], it is also possible to prove the following

Proposition 20 *If E has finite perimeter, then*

$$\frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n} |W_t \chi_E - \chi_E| dx \leq c_{\mathbb{G}} P_{\mathbb{G}}(E), \quad t > 0 \quad (42)$$

with

$$c_{\mathbb{G}} = \int_{\mathbb{R}^n} |\nabla_X h(1, z)| dz.$$

Proof. First of all we note that, by (iv) in Theorem 1,

$$\int_{\mathbb{R}^n} (W_t \chi_E(x) - \chi_E(x)) dx = 0.$$

Recalling that $0 \leq W_t \chi_E(x) \leq 1$, we deduce that

$$\begin{aligned} \int_{E^c} W_t \chi_E(x) dx &= \int_{E^c} (W_t \chi_E(x) - \chi_E(x)) dx = \int_{\mathbb{R}^n} (W_t \chi_E(x) - \chi_E(x))^+ dx \\ &= \int_{\mathbb{R}^n} (W_t \chi_E(x) - \chi_E(x))^- dx = \frac{1}{2} \int_{\mathbb{R}^n} |W_t \chi_E(x) - \chi_E(x)| dx \end{aligned} \quad (43)$$

We can also write

$$\int_{E^c} W_t \chi_E(x) dx = \int_{E^c} (W_t \chi_E(x) - \chi_E(x)) dx = \int_0^t ds \int_{E^c} L W_s \chi_E(x) dx.$$

But then by the divergence theorem

$$\begin{aligned} \int_{E^c} L W_t \chi_E(x) dx &= \int_{\partial_{\mathbb{G}}^* E} d_{\mathbb{G}} P(E)(x) \int_{\mathbb{R}^n} \langle \nabla_X h(s, y^{-1} \circ x), \nu_E(x) \rangle \chi_E(y) dy \\ &\leq \int_{\partial_{\mathbb{G}}^* E} d_{\mathbb{G}} P(E)(x) \int_{\mathbb{R}^n} |\nabla_X h(s, y^{-1} \circ x)| dy \end{aligned} \quad (44)$$

By (43) and (44) the conclusion then follows since

$$\begin{aligned} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n} |W_t \chi_E - \chi_E| dx &= \frac{1}{2\sqrt{t}} \int_{E^c} W_t \chi_E(x) dx = \frac{1}{2\sqrt{t}} \int_0^t ds \int_{E^c} L W_s \chi_E(x) dx \\ &\leq \frac{1}{2\sqrt{t}} \int_0^t \left(\int_{\partial_{\mathbb{G}}^* E} dP_{\mathbb{G}}(E)(x) \int_{\mathbb{R}^n} |\nabla_X h(s, y^{-1} \circ x)| dy \right) ds \\ &= \frac{1}{2\sqrt{t}} \int_0^t \left(\int_{\partial_{\mathbb{G}}^* E} dP_{\mathbb{G}}(E)(x) \frac{1}{\sqrt{s}} \int_{\mathbb{R}^n} |\nabla_X h(1, z)| dz \right) ds \\ &\leq c_{\mathbb{G}} P_{\mathbb{G}}(E). \end{aligned}$$

■

Proof of Theorem 15. The derivation of Theorem 15 from Theorem 14 is based on the coarea formula, and is similar to that of Theorem 4.1 of [22]. Assume $f \in BV(\mathbb{G})$ and set $E_\tau = \{f > \tau\}$. By Proposition 8, for a.e. $\tau \in \mathbb{R}$ the set E_τ has finite perimeter, hence by Theorem 14 we can write

$$\lim_{t \rightarrow 0} \frac{1}{2\theta_\infty \sqrt{t}} \int_{E_\tau^c} W_t \chi_{E_\tau}(x) dx = \int_{\partial_{\mathbb{G}}^* E_\tau} \phi_{\mathbb{G}}(\nu_{E_\tau}(x)) dS_\infty^{Q-1}(x)$$

for a.e. τ .

Moreover, using the same arguments as [1, Proposition 3.69] and the continuity of $\phi_{\mathbb{G}}$ it is easily checked that the function $\phi_{\mathbb{G}}(\sigma_f(x))$ is (obviously bounded and) Borel. Comparing f with $f \vee \tau \chi_{E_\tau}$ and using [1, Proposition 3.73] we see that for a.e. $\tau \in \mathbb{R}$ the equality $\sigma_f(x) = \nu_{E_\tau}(x)$ holds for \mathcal{S}_∞^{Q-1} -a.e. $x \in \partial_{\mathbb{G}}^* E_\tau$, whence

$$\phi_{\mathbb{G}}(\sigma_f(x)) = \phi_{\mathbb{G}}(\nu_{E_\tau}(x)) \text{ for a.e. } \tau \in \mathbb{R}, \mathcal{S}_\infty^{Q-1}\text{-a.e. } x \in \partial_{\mathbb{G}}^* E_\tau.$$

We point out that if x belongs to the boundary of two different level sets E_τ, E_σ (which happens whenever f has a jump at x), then the above equality holds for both the levels because $\nu_{E_\tau}(x) = \nu_{E_\sigma}(x)$ for $\sigma, \tau \in [f^-(x), f^+(x)]$. We also notice that, for almost every $x, y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}} |\chi_{E_\tau}(x) - \chi_{E_\tau}(y)| d\tau = |f(x) - f(y)|. \quad (45)$$

With the aid of coarea formula (22) with $g(x) = \phi_{\mathbb{G}}(\sigma_f(x))$, by (43), (45) and using the dominated convergence theorem, we obtain that

$$\begin{aligned} \int_{\mathbb{R}^n} \phi_{\mathbb{G}}(\sigma_f(x)) d|D_{\mathbb{G}} f| &= \theta_\infty \int_{\mathbb{R}} \int_{\partial_{\mathbb{G}}^* E_\tau} \phi_{\mathbb{G}}(\nu_{E_\tau}(x)) d\mathcal{S}_\infty^{Q-1}(x) d\tau \\ &= \int_{\mathbb{R}} \lim_{t \rightarrow 0} \frac{1}{2\sqrt{t}} \int_{E_\tau^c} W_t \chi_{E_\tau}(x) dx d\tau \\ &= \lim_{t \rightarrow 0} \int_{\mathbb{R}} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n} |W_t \chi_{E_\tau} - \chi_{E_\tau}| dx d\tau \\ &= \lim_{t \rightarrow 0} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\chi_{E_\tau}(y) - \chi_{E_\tau}(x)| h(t, y^{-1} \circ x) dx dy d\tau \\ &= \lim_{t \rightarrow 0} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| h(t, y^{-1} \circ x) dx dy. \end{aligned} \quad (46)$$

Conversely, assume that $f \in L^1(\mathbb{R}^n)$ and that

$$\liminf_{t \rightarrow 0} \frac{1}{\sqrt{t}} \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| h(t, y^{-1} \circ x) dx dy$$

is finite. Then,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| h(t, y^{-1} \circ x) dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\chi_{E_\tau}(x) - \chi_{E_\tau}(y)| h(t, y^{-1} \circ x) dx dy d\tau.$$

Since $\int_{E^c} W_t \chi_E \geq 0$, by the above equality and (43) we get

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} \left(\liminf_{t \rightarrow 0} \frac{1}{\sqrt{t}} \int_{E_\tau^c} W_t \chi_{E_\tau} dx \right) d\tau \leq \liminf_{t \rightarrow 0} \int_{\mathbb{R}} \frac{1}{\sqrt{t}} \int_{E_\tau^c} W_t \chi_{E_\tau} dx d\tau \\ &\leq \liminf_{t \rightarrow 0} \frac{1}{2\sqrt{t}} \int_{\mathbb{R}^n \times \mathbb{R}^n} \int_{\mathbb{R}} |\chi_{E_\tau}(x) - \chi_{E_\tau}(y)| h(t, y^{-1} \circ x) dx dy d\tau < +\infty. \end{aligned}$$

In particular, by Theorem 14, for a.e. $\tau \in \mathbb{R}$ the set E_τ has finite perimeter and the limit $\lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} \int_{E_\tau^c} W_t \chi_{E_\tau} dx$ exists. As a consequence, $f \in BV_{\text{loc}}(\mathbb{G})$ and we may use the above

identity (46) to get

$$\begin{aligned} |D_{\mathbb{G}}f|(\mathbb{R}^n) &\leq \frac{1}{\min \phi_{\mathbb{G}}} \int_{\mathbb{R}^n} \phi_{\mathbb{G}}(\sigma_f) d|D_{\mathbb{G}}f| \\ &\leq \frac{1}{\min \phi_{\mathbb{G}}} \liminf_{t \rightarrow 0} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| h(t, y^{-1} \circ x) dx dy < +\infty \end{aligned}$$

that is, $f \in BV(\mathbb{G})$. ■

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