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NEGATIVE DEFINITE FUNCTIONS ON GROUPS WITH POLYNOMIAL GROWTH

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Abstract. The aim of this work is to show that on a locally compact, second countable, compactly generated group G with polynomial growth and homogeneous dimension d_h, there exist a continuous, proper, negative definite function ℓ with polynomial growth dimension d_ℓ arbitrary close to d_h.

1. Introduction and statement of the results

The length function ℓ associated to a finite set of generators of a countable, discrete, finitely generated group Γ, may be used to reveal interesting aspects of the group itself. This is the spirit of metric group theory (see [12]). In a famous result of U. Haagerup [11], for example, the length functions on the free groups $\mathbb{F}_n$ with $n \geq 2$ generators, has been used to prove that, even if these groups are not amenable, they still have an approximation property: there exists a sequence $\varphi_n \in c_0(\mathbb{F}_n)$ of normalized, positive definite functions, converging pointwise to the constant function1. Correspondingly, the trivial representation is weakly contained in a C_0-representation (one whose coefficients vanish at infinity) (see [10]). These properties of $\mathbb{F}_n$ only depend to the fact that the length function is *negative definite and proper*. The class of groups where a length function has the latter properties has been characterized in several independent ways and form the object of intensive investigations (see [2]). On another, somewhat opposite, side, a deeply studied class of groups are those possessing the Kazhdan property (T), i.e. those where *every negative definite function is bounded* (see for example [1]). Finer asymptotic properties of length functions are connected to other fundamental properties of groups. For example, if ℓ has *polynomial growth* (see below for the definition) then Γ is *amenable* and its von Neumann algebra $vN(\Gamma)$ is *hyperfinite* (weak closure on an increasing sequence of matrix algebras).

However, length functions are not necessarily negative definite and one may wonder how far is a given length function from being negative definite.

Reversing the point of view, in the present work we prove that on finitely generated, discrete groups Γ or, more generally, on compactly generated, locally compact, second countable groups G, with polynomial growth, there always exists a continuous, negative definite function with a (polynomial) growth arbitrarily close to the one of Γ or G.

The consideration of negative definite functions and their growth properties are also meaningful in Noncommutative Potential Theory (see [3], [6]) and Quantum Probability (see [4]). As an application of the main results, we provide a natural construction of a noncommutative Dirichlet form on the group von Neumann algebra $\lambda(G)''$ with an upper spectral growth dimension d_S bounded above by the polynomial growth homogeneous dimension d_h of G.

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Let G be a locally compact, second countable, compactly generated group with identity $e \in G$ and let μ be one of its left Haar measures.

The main results of the work are the following.

Theorem 1.1. *Suppose that G has polynomial growth and homogeneous dimension d_h. Then, for all $d > d_h$, there exists a continuous, proper, negative type function ℓ with polynomial growth such that*

$$\mu\{s \in G \,|\, \ell(s) \leq x\} = O(x^d)\,, \quad x \to +\infty. \tag{1.1}$$

Theorem 1.2. *Suppose that G has polynomial growth and homogeneous dimension d_h. Then, there exists a continuous, proper, negative type function ℓ with polynomial growth such that*

$$\forall d > d_h\,, \quad \mu\{s \in G \,|\, \ell(s) \leq x\} = O(x^d) \qquad x \to +\infty. \tag{1.2}$$

As a straightforward application of these results to Noncommutative Potential Theory we have the following corollary. Further investigations will be discussed in [7].

Corollary 1.3. *Let Γ be a discrete, finitely generated group with polynomial growth and homogeneous dimension d_h. Let $\lambda(\Gamma)''$ be the von Neumann algebra generated by the left regular representation and let τ be its trace.*
Then there exists on $L^2(\lambda(\Gamma)'', \tau)$ a noncommutative Dirichlet form $(\mathcal{E}, \mathcal{F})$ with discrete spectrum and upper spectral dimension $d_S \leq d_h$.

Here the *upper spectral dimension d_S* of the Dirichlet form is defined as

$$d_S := \inf\{d > 0 : \limsup_{x \to +\infty} \sharp\{\lambda \in \mathrm{sp}\,(L) : \lambda \leq x\}/x^d < +\infty\}$$

where $\mathrm{sp}(L)$ denotes the spectrum of the self-adjoint, nonnegative operator $(L, \mathrm{dom}(L))$ associated to $(\mathcal{E}, \mathcal{F})$

$$\mathcal{E}[a] = \|\sqrt{L}a\|_2^2 \qquad a \in \mathcal{F} = \mathrm{dom}(L)\,.$$

2. Groups with polynomial growth

Let $K \subset G$ be a compact, symmetric, generator set with non empty interior

$$G = \bigcup_{n=1}^{\infty} K^n\,, \quad K^{-1} = K\,, \quad K^\circ \neq \emptyset\,,$$

where

$$K^{-1} := \{s^{-1} \in G : s \in K\}\,, \qquad K^n := \{s_1 \ldots s_n \in G : s_k \in K\,, k = 1, \ldots, n\}\,.$$

We may always assume that K is a neighborhood of the identity $e \in K^\circ \subset G$.
Let us recall that G has *polynomial growth* if

$$\exists\, c, d > 0 \quad \text{such, that} \quad \mu(K^n) \leq c\,(n+1)^d\,, \qquad n \geq 1 \tag{2.1}$$

or, equivalently, if

$$\frac{1}{\log(n+1)} \log(\mu(K^n)) \leq d + \frac{1}{\log(n+1)} \log(c)\,, \qquad n \geq 1 \tag{2.2}$$

and in that case its *homogeneous dimension* is defined as

$$d_h = \limsup_{n \to \infty} \frac{1}{\log n} \log(\mu(K^n))\,. \tag{2.3}$$

The class of locally compact, second countable, compactly generated groups with polynomial growth includes nilpotent, connected, real Lie groups and finitely generated, nilpotent, countable discrete groups (see [8] and [12] Chapter VII $\mathcal{S}$ 26). For example, the homogeneous dimension of the discrete Heisenberg group is 4 (see [12] Chapter VII $\mathcal{S}$ 21).

Recall also the M. Gromov's characterization [9] of finitely generated, countable discrete groups with polynomial growth as those which have a nilpotent subgroup of finite index (see also [12] Chapter VII $\mathcal{S}$ 29 and [13] for a generalization to locally compact groups).

Remark 2.1. Let us observe that since K is a neighborhood of the identity $e \in G$, we have $K^{n-1} \subset K^{n-1}K^\circ \subset (K^n)^\circ$ so that

$$\bigcup_{k=0}^{\infty}(K^n)^\circ = G$$

and we have an open cover of G. As a consequence

every compact set in G is contained in K^n for some $n \geq 1$.

3. Proof of the Theorems

In the following, we will consider the sequence $\{\alpha_n\}_{n=1}^\infty \subset [0, +\infty)$ defined by

$$\alpha_1 := 0\,, \qquad \frac{\mu(K^n)}{\mu(K^{n-1})} =: 1 + \alpha_n\,, \qquad n \geq 2\,.$$

Equation (2.2) can thus be written as

$$\frac{1}{\log(n+1)}\sum_{k=1}^{n}\log(1+\alpha_k) \leq d + \frac{1}{\log(n+1)}\log(c) \leq d'\,, \qquad n \geq 1\,, \tag{3.1}$$

with $d' := d + \log_2 c$.

3.1. **Some preliminary estimates.** The proof of the theorems relies on the following lemmas.

Lemma 3.1. *For any $\beta \in (0,1)$ the set*

$$E_\beta := \left\{n \in \mathbb{N}^* \,:\, \alpha_n > n^{-\beta}\right\}$$

has vanishing density.

Proof. Let us fix $K > 0$ such that $\log(1 + n^{-\beta}) \geq Kn^{-\beta}$ for all $n \geq 1$. By equation (3.1), we have

$$d' \log(n+1) \geq \sum_{k\in[1,n]\cap E_\beta} \log(1+\alpha_k) \geq Kn^{-\beta}\sharp\{[1,n]\cap E_\beta\}$$

so that

$$\frac{1}{n}\sharp\{[1,n]\cap E_\beta\} \leq \frac{d'}{K}\frac{\log(n+1)}{n^{1-\beta}} \to 0 \quad \text{as} \quad n\to\infty\,.$$

$\square$

Lemma 3.2. *For any $\beta \in (0,1)$ and $\gamma > 1/\beta$, the set*

$$F_{\beta,\gamma} := \left\{n \in \mathbb{N} \,:\, [\,n^\gamma, (n+1)^\gamma\,] \cap \mathbb{N} \subset E_\beta\,\right\}$$

has vanishing density.

Proof. For any $\varepsilon \in (0, 1/2)$ and any integer $N \geq 2$, we have

$$\begin{aligned} d' &\geq \frac{1}{\log(N+1)^\gamma} \sum_{(\varepsilon N)^\gamma \leq k \leq N^\gamma} \log(1+\alpha_k) \\ &\geq \frac{1}{\log(N+1)^\gamma} \sum_{n\in[\varepsilon N, N-1]\cap F_{\beta,\gamma}} \sum_{k\in[n^\gamma,(n+1)^\gamma)} \log(1+\alpha_k) \\ &\geq \frac{K}{\gamma\log(N+1)} \sum_{n\in[\varepsilon N, N-1]\cap F_{\beta,\gamma}} \sum_{k\in[n^\gamma,(n+1)^\gamma)} k^{-\beta} \qquad N \geq 1\,. \end{aligned}$$

For any fixed $n \in [\varepsilon N, N-1]\cap F_{\beta,\gamma}$, there exists $t \in (n, n+1)$ such that $(n+1)^\gamma - n^\gamma = \gamma t^{\gamma-1}$. As $n \geq \varepsilon N$, we have $t/N > n/N \geq \varepsilon$ so that for $n \in [\varepsilon N, N-1]\cap F_{\beta,\gamma}$ the following bound holds true

$$\begin{aligned} \sum_{n^\gamma \leq k < (n+1)^\gamma} k^{-\beta} &\geq N^{-\beta\gamma}\big((n+1)^\gamma - n^\gamma - 2\big) \\ &= N^{-\beta\gamma}\big(\gamma t^{\gamma-1} - 2\big) \\ &> N^{-\beta\gamma}\big(\gamma(\varepsilon N)^{\gamma-1} - 2\big). \end{aligned}$$

Thus we have

$$\begin{aligned} d' &\geq \frac{K}{\gamma\log(N+1)} \sum_{n\in[\varepsilon N, N-1]\cap F_{\beta,\gamma}} \Big(\gamma\varepsilon^{\gamma-1}N^{(1-\beta)\gamma-1} + o(N^{(1-\beta)\gamma-1})\Big) \\ &\geq \frac{K}{\log(N+1)}\Big(\varepsilon^{\gamma-1}N^{(1-\beta)\gamma-1} + o(N^{(1-\beta)\gamma-1})\Big)\sharp\{[\varepsilon N, N-1]\cap F_{\beta,\gamma}\} \end{aligned}$$

and we may deduce

$$\frac{d'}{K\varepsilon^{1-\gamma}}\frac{\log(N+1)}{N^{(1-\beta)\gamma}}\Big(1+o(1)\Big) \geq \frac{\sharp\{[\varepsilon N, N-1]\cap F_{\beta,\gamma}\}}{N}\,.$$

For N big enough, we have

$$\frac{\sharp\{[\varepsilon N, N-1]\cap F_{\beta,\gamma}\}}{N} \leq \varepsilon$$

and finally

$$\frac{\sharp\{[1, N-1]\cap F_{\beta,\gamma}\}}{N} \leq \frac{\sharp\{[1, \varepsilon N]\cap F_{\beta,\gamma}\}}{N} + \frac{\sharp\{[\varepsilon N, N-1]\cap F_{\beta,\gamma}\}}{N} \leq 2\varepsilon\,.$$

□

3.2. **Proofs of the theorems.** To build up a negative type function having the properties required in Theorem 1.2, we begin to construct a suitable sequence of positive definite functions.

Fix $\beta \in (0,1)$ and $\gamma > 1/\beta$. For any fixed $n \notin F_{\beta,\gamma}$, we can chose $k(n) \in [\,n^\gamma, (n+1)^\gamma\,]\backslash\mathbf{E}_\beta$ in such a way that

$$n^\gamma \leq k(n) \leq (n+1)^\gamma\,, \qquad k(n)^{-\beta} \geq \alpha_{k(n)} = \frac{\mu(K^{k(n)+1}) - \mu(K^{k(n)})}{\mu(K^{k(n)})}\,.$$

Denote by $\mathcal{B}(L^2(G,\mu))$ the Banach algebra of all bounded operators on the Hilbert space $L^2(G,\mu)$ and by $\lambda : G \to \mathcal{B}(L^2(G,\mu))$ the left regular unitary representation of G, defined as $(\lambda(s)a)(t) := a(s^{-1}t)$ for $a \in L^2(G,\mu)$ and $s, t \in G$.

Consider now the function $\xi_n := \mu(K^{k(n)})^{-1/2}\chi_{K^{k(n)}} \in L^2(G,\mu)$, with unit norm, and set

$$\omega_n : G \to \mathbb{R} \quad \omega_n(s) := < \xi_n, \lambda(s)\xi_n >_{L^2(G)} = \frac{\mu(sK^{k(n)} \cap K^{k(n)})}{\mu(K^{k(n)})}\,.$$

By construction we have that
• $\omega_n \in C_c(G)$ is a continuous, normalized, positive definite function with compact support. Moreover
• $\omega_n(s) \geq 1 - n^{-\beta\gamma}$ for all $s \in K$.

In fact, by the translation invariance of the Haar measure and since $K^{k(n)} \supset sK^{k(n)-1}$ for $s \in K$, we have

$$\begin{aligned}\omega_n(s) &= \frac{\mu(sK^{k(n)} \cap K^{k(n)})}{\mu(K^{k(n)})} \geq \frac{\mu(sK^{k(n)} \cap sK^{k(n)-1})}{\mu(K^{k(n)})} = \frac{\mu(s(K^{k(n)} \cap K^{k(n)-1})}{\mu(K^{k(n)})}\\ &= \frac{\mu(K^{k(n)} \cap K^{k(n)-1})}{\mu(K^{k(n)})} = \frac{\mu(K^{k(n)-1})}{\mu(K^{k(n)})} = (1+\alpha_{k(n)})^{-1} \geq 1 - \alpha_{k(n)}\\ &\geq 1 - k(n)^{-\beta} \geq 1 - n^{-\beta\gamma}\,.\end{aligned}$$

Lemma 3.3. *For $n \notin F_{\beta,\gamma}$ and any integer $p \geq 1$, we have*

$$\omega_n(s) \geq 1 - pn^{-\beta\gamma} \qquad s \in K^p\,. \tag{3.2}$$

Proof. Consider $s = s_1 \cdots s_p \in K^p$ for some $s_1, \cdots, s_p \in K$ and notice that

$$\begin{aligned}2\big(1-\omega(s)\big) &= ||\xi_n - s\xi_n||^2 = ||\xi_n - s_1\cdots s_p\xi_n||^2\\ &= ||\xi_n - s_1\xi_n + s_1\xi_n - s_1s_2\xi_n + \cdots + s_1\cdots s_{p-1}\xi_n - s_1\cdots s_p\xi_n||^2\\ &\leq \big(||\xi_n - s_1\xi_n|| + ||\xi_n - s_2\xi_n|| + \cdots + ||\xi_n - s_p\xi_n||\big)^2\\ &\leq p^2 \sup_{\sigma\in K} ||\xi_n - \sigma\xi_n||^2\\ &= 2p^2 \sup_{\sigma\in K}\big(1-\omega_n(\sigma)\big) \leq 2p^2n^{-2\beta\gamma}.\end{aligned}$$

□

Proof of Theorem 1.1. By previous lemma, the series $\sum_{n\notin F_{\beta,\gamma}}\big(1-\omega_n(s)\big)$ converges uniformly on K^p for any integer $p \geq 1$. Hence, it converges uniformly on each compact subset of G and

$$\ell : G \to [0,+\infty) \qquad \ell(s) = \sum_{n\notin F_{\beta,\gamma}}\big(1-\omega_n(s)\big)$$

is a continuous, negative definite function. The function ℓ is proper: in fact, if $m \in \mathbb{N}$ is greater or equal to the integer part of $(N+1)^\gamma$, then, for $s \notin K^{2m}$, we have $\omega_n(s) = 0$ for all $n \in [1,N]\backslash F_{\beta,\gamma}$ so that

$$\ell(s) \geq \sharp\{\,[1,N]\backslash F_{\beta,\gamma}\} \qquad s \notin K^{2m}\,. \tag{3.3}$$

According to Lemma 3.2 we can write $\sharp\{\,[1,N]\backslash F_{\beta,\gamma}\} = N(1-\varepsilon_N)$ for $\varepsilon_N \to 0$. The set $\{s\,|\,\ell(s) \leq N(1-\varepsilon_N)\,\}$ is contained in K^{2m}, where m is the integer part of $(N+1)^\gamma + 1$, whose volume $\leq c\,2^d\big(N+1)^\gamma+1\big)^d = O(N^{\gamma d})$ For fixed $\varepsilon > 0$ and N large enough, we have, for $x \in [N-1,N]$,

$$\mu\{\ell \leq x(1-\varepsilon)^{-1}\} \leq c'N^{\gamma d} \leq c''x^{\gamma d}$$

so that $\mu\{\ell \leq x\} \leq c''x^{\gamma d}$. □

Proof of Theorem 1.2. Let us chose a decreasing sequence d_m converging to d_h, and for all m, chose a function ℓ_m such that $\ell_m(x) = O(x^{d_m})$, $x \to +\infty$ and $\sup_K \ell_d(x) = 1$ (by normalization). Hence $\ell \leq k^2$ on K^k. It is enough to set $\ell = \sum_m 2^{-m}\ell_m$: $\{\ell \leq x\} \subset \{\ell_m \leq 2^m x\}$, whose measure is $O(x^{d_m})$.

□

Proof of Corollary 1.3. Recall first that the GNS space $L^2(\lambda(\Gamma)'', \tau)$ can be identified, at the Hilbert space level, with $l^2(\Gamma)$. Consider now the continuous, proper, negative type function ℓ constructed in Theorem 1.2. By [5 $\mathcal{S}$ 10.2], the quadratic form

$$\mathcal{E} : \mathcal{F} \to [0, +\infty) \qquad \mathcal{E}[a] = \sum_{s\in\Gamma} \ell(s)\, |a(s)|^2 \,,$$

defined on the subspace $\mathcal{F} \subset l^2(\Gamma)$ where the sum is finite, is a noncommutative Dirichlet form whose associated nonnegative, self-adjoint operator on $l^2(\Gamma)$ is the multiplication operator by ℓ given by

$$\mathrm{dom}(L) := \{a \in l^2(\Gamma) : \ell \cdot a \in L^2(G, \mu)\} \quad (La)(s) := \ell(s) \cdot a(s) \quad s \in G \,.$$

The stated result thus follows from Theorem 1.2 and the straightforward identification $\mathrm{sp}(L) = \{\ell(s) \in [0, +\infty) : s \in \Gamma\}$.

□

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