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Elastic structures in adhesion interaction

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ELASTIC STRUCTURES IN ADHESION INTERACTION

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Abstract. We study a variational model describing the interaction of two 1-dimensional elastic bodies through an adhesive layer, with the aim of modeling a simplified CFRP structure: e.g. a concrete beam or a medical rehabilitation device glued to a reinforcing polymeric fiber. Different constitutive assumptions for the adhesive layer are investigated: quadratic law and two kinds of softening law. In all cases properties of the equilibrium states of the structural system are analytically deduced. In the case of adhesion with softening, the minimum length of the elastic fiber avoiding debonding failure is estimated in terms of glue carrying capacity and the constitutive parameter of the fiber.

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Introduction

One goal of modern structural engineering relies in understanding and modeling the improvement of several characteristics of structural members, such as the load capacity, ductility and durability. The request of upgrading inadequate or damaged structures stimulates the study of suitable tools to deal with non-conventional design issues such as debonding problems in fiber reinforced elements. Indeed, the application of fiber reinforced polymer (FRP) sheets as an externally bonded reinforcement is generally accepted as an efficient technique to reinforce concrete structures. Though numerical techniques based on classical elastic models are very often considered as a quantitative tool helping the design process, the lack of a qualitative understanding of the mechanical and /or analytical aspects of the phenomena hide the fundamental facts regulating these intriguing mechanical interactions.

It is generally accepted that the essential mechanical behavior of FRP strengthened structural elements relies in the stress transfer between the fiber and the concrete beam through bonding interface (see for instance [4], [6], [11], [12], [23]). Therefore the overall mechanical behavior crucially depends on the interfacial bonding and its governing laws.

Here we pursue an energetic approach by exploiting a unifying perspective for the problems of adhesion and reinforcement of elastic thin structures ([8], [9], [10], [13], [14], [17], [18]) and show that reinforcement reduces to adhesion when the stiffness of one of the structures involved becomes arbitrarily large (for adhesion problems see also [2], [3], [5], [7], [20], [21], [22]).

In this paper, we study a variational model describing two 1-dimensional material bodies (typically a reinforcement fiber and a concrete beam) which interact through an adhesive layer (typically a soft material) under prescribed displacement conditions. Both the reinforcing fiber and the matrix are described as elastic bodies, while different constitutive assumptions on the adhesive layer are investigated.

In Section 1 we assume that the adhesive layer reacts elastically to the slip of the reinforcing fiber with respect to the matrix, while in Section 2 the interfacial bond is modeled with a governing law exhibiting a softening branch for the slip exceeding a maximum value.

The analysis of the solutions suggests that the quadratic interaction described in of Section 1 is not able to capture the essence of adhesion and this fact justifies the assumption of a softening law for the slip constitutive behavior.

In Section 2 we obtain a full characterization of equilibrium states when the matrix has elastic behavior, by studying two different constitutive assumptions: both discontinuous and smooth softening behavior for the adhesive layer. Moreover we deal with the limit case of rigid matrix (concrete beams): the main result are presented in Theorem 2.29, Corollary 2.9 and Remark 2.10 which provide a length estimate of the elastically detached portion of the beam; such estimate depends on the carrying capacity of the glue and the constitutive parameter of the fiber.

1. Adhesion of elastic structures

We consider a system composed by two elastic 1-dimensional structures which are bonded through an adhesive layer. One structure is the fiber, the other one is the matrix and both are parameterized by the variable x in the interval $[0, L]$. The fiber and the matrix are endowed respectively with the elastic energies $E_{\mathrm{f}}, E_{\mathrm{m}}$:

$$E_{\mathrm{f}}(v_{\mathrm{f}})=\frac{1}{2}\int_0^L k_{\mathrm{f}}|v_{\mathrm{f}}'|^2\,dx,\ \ k_{\mathrm{f}}>0;\ \ E_{\mathrm{m}}(v_{\mathrm{m}})=\frac{1}{2}\int_0^L k_{\mathrm{m}}|v_{\mathrm{m}}'|^2\,dx,\ \ k_{\mathrm{m}}>0 \quad (1.1)$$

where the functions $v_{\mathrm{f}}, v_{\mathrm{m}} : [0,1] \to \mathbb{R}$ denote the axial displacements respectively in the fiber and in the matrix respectively, while the strictly positive constants $k_{\mathrm{f}}, k_{\mathrm{m}}$ are the extensional stiffnesses (the Young modulus times the area of the transverse section). The adhesion material layer which bonds together the two elastic structures is energetically represented by the functional

$$E_{\mathrm{ad}}(s) = \frac{1}{2}\int_0^L k_{\mathrm{ad}}|s|^2\,dx, \quad k_{\mathrm{ad}} > 0 \tag{1.2}$$

where k_{ad} is a constitutive parameter characterizing the adhesion material and the slip function s is given by

$$s(x) = v_{\mathrm{f}}(x) - v_{\mathrm{m}}(x). \tag{1.3}$$

The slip s measures the difference of the elastic displacements occurring at the interface separating the two different materials. We introduce the notation $\mathbf{v} = (v_{\mathrm{f}}, v_{\mathrm{m}})$ and examine a total potential energy

$$E = E_{\mathrm{f}} + E_{\mathrm{m}} + E_{\mathrm{ad}}$$

of the following form

$$E(\mathbf{v}) := E(v_{\mathrm{f}}, v_{\mathrm{m}}) = \frac{1}{2}\int_0^L \left(k_{\mathrm{f}}|v_{\mathrm{f}}'|^2 + k_{\mathrm{m}}|v_{\mathrm{m}}'|^2 + k_{\mathrm{ad}}|v_{\mathrm{f}} - v_{\mathrm{m}}|^2\right)\,dx. \tag{1.4}$$

We assume the displacement $d > 0$ is given. Then the admissible configurations belong to the set

$$\mathcal{A} = \{\mathbf{v} = (v_{\mathrm{f}}, v_{\mathrm{m}}) \in H^1((0,L);\mathbb{R}^2),\ \ v_{\mathrm{f}}(L) = d,\ v_{\mathrm{m}}(0) = 0\}. \tag{1.5}$$

The equilibrium states of the system are the solutions the following variational problem:

$$\min\{E(\mathbf{v}) \mid \mathbf{v} \in \mathcal{A}\}. \tag{1.6}$$

The existence of solutions of (1.6) follows by a straight application of the direct methods of the calculus of variations. To this purpose we observe that $E(\mathbf{v}) \geq \left\|v_{\mathrm{f}}'\right\|_{L^2}^2 + \left\|v_{\mathrm{m}}'\right\|_{L^2}^2$ for every $\mathbf{v} \in \mathcal{A}$, hence the functional E enjoys the coercivity property due to Poincaré inequality and the boundary conditions in $\mathcal{A}$. Moreover, the convexity of E with respect to $\mathbf{v}'$ ensures the weak lower semicontinuity in $H^1(0, L)$.

By employing a standard variation argument we get that any minimizer of (1.6) satisfies the following Euler-Lagrange equations

$$\begin{cases} -k_{\mathrm{f}} v_{\mathrm{f}}''(x) = k_{\mathrm{ad}}(v_{\mathrm{m}} - v_{\mathrm{f}}) \quad \forall x \in (0, L) \\ -k_{\mathrm{m}} v_{\mathrm{m}}''(x) = k_{\mathrm{ad}}(v_{\mathrm{f}} - v_{\mathrm{m}}) \quad \forall x \in (0, L) \\ v_{\mathrm{f}}'(0) = 0, \quad v_{\mathrm{f}}(L) = d \\ v_{\mathrm{m}}(0) = 0, \quad v_{\mathrm{m}}'(L) = 0. \end{cases} \tag{1.7}$$

The unique solution of (1.7) is explicitly given by

$$\begin{aligned} v_f(x) &= \mu(c_2 - c_1)x + k(c_2 + c_1) + c_1 e^{\mu x} + c_2 e^{-\mu x} \\ v_m(x) &= \mu(c_2 - c_1)x + k(c_2 + c_1) - k(c_1 e^{\mu x} + c_2 e^{-\mu x}) \end{aligned} \tag{1.8}$$

where we have set

$$\mu = \sqrt{\frac{k_{\mathrm{ad}}}{k_{\mathrm{m}}} \frac{(k+1)}{k}}, \quad k = \frac{k_f}{k_m}. \tag{1.9}$$

and

$$\begin{aligned} c_1 &= d \frac{(k + e^{\mu L})}{1 + 4k e^{\mu L} + k^2 - \mu k L + e^{2\mu L}(1 + k^2 + \mu k L)}, \\ c_2 &= d \frac{e^{\mu L}(1 + k e^{\mu L})}{1 + 4k e^{\mu L} + k^2 - \mu k L + e^{2\mu L}(1 + k^2 + \mu k L)}. \end{aligned} \tag{1.10}$$

Then

$$v_f(x) - v_m(x) = (1 + k)(c_1 e^{\mu x} + c_2 e^{-\mu x}) > 0 \quad \forall x \in [0, L], \tag{1.11}$$

hence v_{f} is strictly convex and v_{m} is strictly concave, moreover $v_{\mathrm{f}} - v_{\mathrm{m}}$ attains its minimum value

$$m = v_{\mathrm{f}}(\bar{x}) - v_{\mathrm{m}}(\bar{x}) = 2(1+k)\sqrt{c_1 c_2}$$

at the point

$$\bar{x} = \frac{1}{2\mu} \ln \frac{c_2}{c_1} \in (0, L).$$

In the limit case when the supporting matrix becomes rigid , say $k_m \to \infty$, we have

$$\bar{x} \to 0, \quad \mu \to \sqrt{\frac{k_{\mathrm{ad}}}{k_{\mathrm{f}}}}, \quad m \to m_\infty := \frac{d}{\cosh\left(L\sqrt{\frac{k_{\mathrm{ad}}}{k_{\mathrm{f}}}}\right)}.$$

Notice that, if k_{ad} is of the same order of k_{f}, then the value m_∞ is smaller than d but it is of the same order of d when $d \to 0$. This last remark may suggests some inadequacy of the constitutive assumption (1.2). Nevertheless in problems arising in structural engineering typically the ratio $k_{\mathrm{ad}}/k_{\mathrm{f}}$ is very large and so for any d the value m_∞ goes to zero much faster than d. It follows that the quadratic interaction described in this section is able to capture the essence of adhesion only in the case $k_{\mathrm{ad}}/k_{\mathrm{f}}$ is very large. For this reason in the next section we examine a different constitutive assumption assumption given by (2.2) and (2.3) (or (2.35) and (2.36)).

Remark 1.1. Notice that, if $k_{\mathrm{f}} = k_{\mathrm{m}}$, then the unique minimizer $\mathbf{v} = (v_{\mathrm{f}}, v_{\mathrm{m}})$ exhibits a graph which is symmetric with respect to the point $(L/2, d/2)$, that is

$$\begin{cases} v_{\mathrm{f}}(x) = \frac{d}{2} - v_{\mathrm{m}}(\frac{L}{2} - x) \\ v_{\mathrm{m}}(x) = \frac{d}{2} - v_{\mathrm{f}}(\frac{L}{2} - x) \end{cases} \tag{1.12}$$

Relationship (1.12) is a consequence of the uniqueness and the following invariance property of the energy:

$$E\left(\frac{d}{2} - v_{\mathrm{m}}(\frac{L}{2} - x), (\frac{d}{2} - v_{\mathrm{f}}(\frac{L}{2} - x)\right) = E(v_{\mathrm{f}}, v_{\mathrm{m}}), \quad \forall (v_{\mathrm{f}}, v_{\mathrm{m}}) \in \mathcal{A}.$$

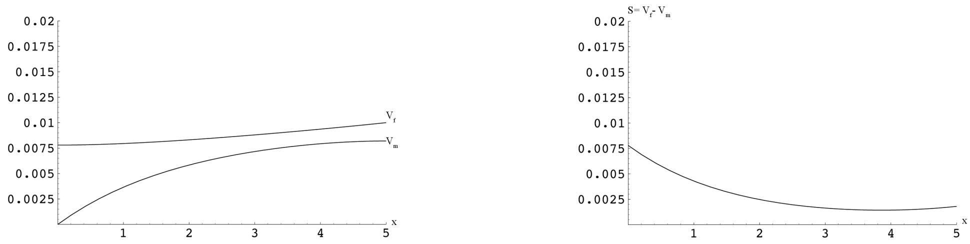

FIGURE 1.1. $k_{\mathrm{f}} = 210000$ MPa; $k_{\mathrm{m}} = 30000$ MPa; $k_{\mathrm{ad}} = 10000$ MPa; $L = 5$ m; $d = 0.01$ m ([1]).

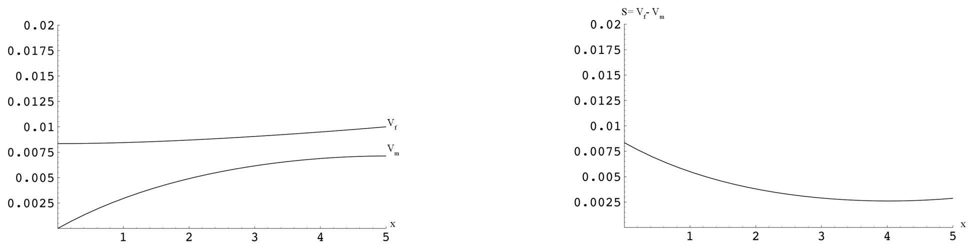

FIGURE 1.2. CFRP plate: $k_{\mathrm{f}} = 165000$ MPa; Concrete: $k_{\mathrm{m}} = 25000$ MPa; Epoxy adhesive: $k_{\mathrm{ad}} = 4500$ MPa; $L = 5$ m; $d = 0.01$ m ([6]).

2. Adhesion with softening

In this section we will examine two constitutive softening laws for the adhesion layer. The first one (discontinuous softening, subsections 2.1, 2.2) presents a sudden transition of the stress which goes to zero discontinuously while the second one (smooth softening, subsection 2.3) exhibits a smooth decay of the stress after a certain value of the slip. About the first one we consider also the limit case when the supporting matrix becomes rigid.

2.1. **Elastic supporting matrix with discontinuous softening glue.**

In this section we assume the interface law governing the constitutive behavior of the glue behaves elastically up to a given value of the slip, while beyond such a threshold the stress discontinuously drops to zero. More precisely, we assume

$$\begin{aligned}&\{\tau : \mathbb{R}_+ \to \mathbb{R}_+ \text{ continuous and weakly increasing in } [0, S_*],\\ &\tau(0) = 0,\ \ \tau(s) > 0\ \ \forall s \in (0, S_*],\ \ \tau(s) = 0\ \ \forall s > S_*.\}\end{aligned} \tag{2.1}$$

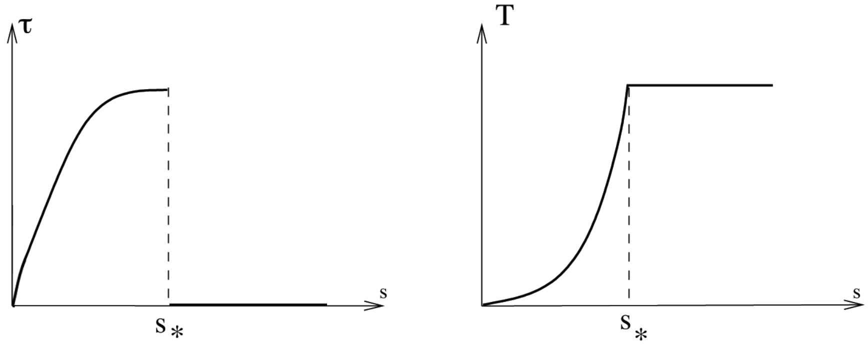

FIGURE 2.1. Energy and constitutive law for discontinuous softening.

The function τ represents the stress-slip constitutive relationship of the adhesive material and we assume for the adhesion energy the following density function defined for every $\sigma \geq 0$:

$$T(\sigma) = G \int_0^\sigma \tau(t)dt, \tag{2.2}$$

where $G > 0$ represents the stiffness of the glue.
Notice that T is continuous and convex in $[0, S_*]$, since $\tau \in BV(0, S_*)$ and $T'' = \tau' \geq 0$ in $(0, S_*)$. We set

$$\widehat{E}_{\mathrm{ad}} = \int_0^L T(|s(x)|)dx, \tag{2.3}$$

where $s(x) = v_{\mathrm{f}}(x) - v_{\mathrm{m}}(x)$.

The total energy of the system takes the form

$$\widehat{E}(\mathbf{v}) := E_{\mathrm{f}}(\mathbf{v}) + E_{\mathrm{m}}(\mathbf{v}) + \widehat{E}_{\mathrm{ad}}(\mathbf{v}), \quad \mathbf{v} = (v_{\mathrm{f}}, v_{\mathrm{m}}) \tag{2.4}$$

where the elastic energies E_{f} and E_{m} are defined in (1.1) and $\widehat{E}_{\mathrm{ad}}$ is defined in (2.3).

Remark 2.1. Note that, if $d > S_*$ then $\mathbf{v}_0 = (d, 0) \in \mathcal{A}$ corresponds to the completely detached state of the two structures.

Motivated by the above remark, we analyze only the nontrivial case occurring when

$$0 < d < S_*. \tag{2.5}$$

All along this subsection we assume (1.1), (2.2)-(2.5) and we select the same admissible set of displacements $\mathcal{A}$ defined in (1.5). The effective role played by (2.5) and the consequences of its failure will be clarified at the end of this section.
The equilibrium states of the system are the solutions the following variational problem.

$$\min\{\widehat{E}(\mathbf{v}) \mid \mathbf{v} \in \mathcal{A}\}. \tag{2.6}$$

Let us note that $\mathbf{v}_0 = (d, 0) \in \mathcal{A}$, i.e. $\mathbf{v}_0$ is an admissible displacement and, by taking into account (2.5), we have the following energy estimate

$$\widehat{E}(\mathbf{v}) \leq \widehat{E}(\mathbf{v}_0) \leq L\,T(S_*), \quad \forall\, \mathbf{v} \in \operatorname{argmin}_{\mathcal{A}} \widehat{E}. \tag{2.7}$$

Lemma 2.2. *Assume (1.1),(1.5), (2.2)-(2.5) and $\mathbf{v} \in$ argmin$_{\mathcal{A}}$ $\widehat{E}$. Then $0 \leq v_{\mathrm{f}} \leq d$ and $0 \leq v_{\mathrm{m}} \leq d$.*

Proof. The thesis follows by observing that for every $(v_{\mathrm{f}}, v_{\mathrm{m}}) \in \mathcal{A}$, by setting $\widetilde{v}_{\mathrm{f}} = \min(v_{\mathrm{f}}, d)$ and $\widetilde{v}_{\mathrm{m}} = \min(v_{\mathrm{m}}, d)$ we have

$$({v_{\mathrm{f}}}^+, {v_{\mathrm{m}}}^+) \in \mathcal{A}, \quad (\widetilde{v}_{\mathrm{f}}, \widetilde{v}_{\mathrm{m}}) \in \mathcal{A}, \tag{2.8}$$

$$T(|{v_{\mathrm{f}}}^+ - {v_{\mathrm{m}}}^+|) \leq T(|v_{\mathrm{f}} - v_{\mathrm{m}}|) \tag{2.9}$$

and the inequality in (2.9) is strict if $|\{x \mid v_{\mathrm{f}}(x) < 0 \text{ or } v_{\mathrm{m}}(x) < 0\}| > 0$,

$$T(|\widetilde{v}_{\mathrm{f}} - \widetilde{v}_{\mathrm{m}}|) \leq T(|v_{\mathrm{f}} - v_{\mathrm{m}}|), \tag{2.10}$$

and the inequality in (2.10) is strict if $|\{x \mid v_{\mathrm{f}}(x) > d \text{ or } v_{\mathrm{m}}(x) > d\}| > 0$. □

Theorem 2.3. *Assume (1.1),(1.5), (2.2)-(2.5). Then problem (2.6) admits a unique solution $\mathbf{v} \in C^2([0, L]; \mathbb{R}^2)$. Moreover $\mathbf{v}$ fulfills $v_{\mathrm{f}}(x) \geq v_{\mathrm{m}}(x)$ $\forall x \in [0, L]$ and the Euler-Lagrange equations*

$$\begin{cases} k_{\mathrm{f}} v_{\mathrm{f}}''(x) = G\tau(v_{\mathrm{f}} - v_{\mathrm{m}}\ ), \ \ x \in (0, L) \\ -k_{\mathrm{m}} v_{\mathrm{m}}''(x) = G\tau(v_{\mathrm{f}} - v_{\mathrm{m}}), \ \ x \in (0, L) \\ v_{\mathrm{f}}'(0) = 0, \quad v_{\mathrm{f}}(L) = d \\ v_{\mathrm{m}}(0) = 0, \quad v_{\mathrm{m}}'(L) = 0. \end{cases} \tag{2.11}$$

Moreover, the following Compliance Identity holds true

$$\widehat{E}(\mathbf{v}) = \int_0^L \left[\frac{G}{2}\tau(v_{\rm f}(x)-v_{\rm m}(x))(v_{\rm f}(x)-v_{\rm m}(x)) + T(v_{\rm f}(x)-v_{\rm m}(x))\right] dx + \\ + k_{\rm f} d\, v_{\rm f}'(L). \tag{2.12}$$

The previous theorem will be proved through a pair of lemmas.

Lemma 2.4. *Assume (1.1),(1.5), (2.2)-(2.5). Then the variational problem (2.6) admits solutions and if* $\mathbf{v} \in$ *argmin*$_{\mathcal{A}}\ \widehat{E}$, *then* $\mathbf{v} \in C^2([0,L];\mathbb{R}^2)$ *and satisfies the following Euler-Lagrange equations in* $(0,L)$

$$\begin{cases} -k_{\rm f} v_{\rm f}''(x) = G\tau(|v_{\rm f}-v_{\rm m}|)\text{sign}(v_{\rm m}-v_{\rm f}), \\ -k_{\rm m} v_{\rm m}''(x) = G\tau(|v_{\rm f}-v_{\rm m}|)\text{sign}(v_{\rm f}-v_{\rm m}), \\ v_{\rm f}'(0)=0, \quad v_{\rm m}'(L)=0 \end{cases} \tag{2.13}$$

together with boundary conditions

$$v_{\rm m}(0)=0, \quad v_{\rm f}(L)=d. \tag{2.14}$$

Proof. Notice that $\widehat{E}(\mathbf{v}) \geq \left\|v_{\rm f}'\right\|_{L^2}^2 + \left\|v_{\rm m}'\right\|_{L^2}^2$ for every $\mathbf{v} \in \mathcal{A}$, hence the functional $\widehat{E}$ enjoys the coercivity property due to Poincaré inequality and the boundary conditions in $\mathcal{A}$. Moreover, the convexity of $\widehat{E}$ with respect to $\mathbf{v}'$ ensures the weak lower semicontinuity in $H^1(0,L)$. Then existence of solutions for the minimization (2.6) follows by a straight application of the direct method of the calculus of variations.

The usual computation shows that the Euler-Lagrange equations (2.13) and the boundary conditions (2.14) are satisfied in $\mathcal{D}'(0,L)$ and by recalling Lemma 2.2 and assumptions on τ we have that $\tau(|v_{\rm f}-v_{\rm m}|)\text{sign}(v_{\rm m}-v_{\rm f})$ is continuous in $[0,L]$ hence $\mathbf{v} \in C^2([0,L];\mathbb{R}^2)$. □

Lemma 2.5. *Assume (1.1),(1.5), (2.2)-(2.5) and* $\mathbf{v} \in$ *argmin*$_{\mathcal{A}}\ \widehat{E}$. *Then*

$$v_{\rm m}(x) \leq v_{\rm f}(x), \quad \forall x \in [0,L] \tag{2.15}$$

and equality may occur only in a (possibly empty) closed interval $[a,b]$ *s.t.*

$$[a,b] \subsetneqq [0,L].$$

Proof. Two alternatives may occur : either $v_m \geq v_f$ in the whole $[0, L]$ or there exists $\bar{x} \in [0, L]$ such that $v_m(\bar{x}) < v_f(\bar{x})$.
In the first case by Lemma 2.2 we get $v_f(0) = v_m(0) = 0,\ v_m(L) = v_f(L) = d$ and by the Euler-Lagrange equations we have that v_m is convex and v_f is concave hence $v_m - v_f$ is convex and therefore $v_m \leq v_f$ that is $v_f \equiv v_m$, which is in contradiction with the boundary conditions in (2.13).
We prove now that if there exists $\bar{x} \in [0, L]$ such that $v_m(\bar{x}) < v_f(\bar{x})$ then $v_m \leq v_f$ in the whole $[0, L]$.
We set $x_1 = \inf\{x \leq \bar{x} :\ \ v_m(x) < v_f(x) \text{ in } (x, \bar{x})\},\quad x_2 = \sup\{x \geq \bar{x} : v_m(x) < v_f(x) \text{ in } (\bar{x}, x)\}$ and we may consider the following cases:

i) $0 < x_1 < x_2 < L$: then by E-L equations we get $v_f - v_m$ convex in $[x_1, x_2]$ and $v_m(x_1) - v_f(x_1) = v_m(x_1) - v_f(x_1) = 0$ that is $v_f \leq v_m$ in $[x_1, x_2]$, a contradiction.

ii) $x_1 = 0 < x_2 < L$: then $v_f(x_2) = v_m(x_2) = 0$ and again either $v_m \geq v_f$ in the whole $[x_2, L]$ or there exists $x_3 \in (x_2, L]$ such that $v_m(x_3) < v_f(x_3)$. Therefore either $v_f \equiv v_m$ in $[x_2, L]$ or $v_f \geq v_m$ in $[0, x_2] \cup [x_3, L]$ and $(v_f - v_m)(x_2) = 0 = (v_f - v_m)(x_3)$. By the minimality of (v_f, v_m) it is readily seen that $v_f = v_m$ in $[x_2, x_3]$ thus proving the thesis in this case. Since the case $0 < x_1 < x_2 = L$ can be handled as the previous one, the proof of (2.15) is achieved. By the boundary conditions in (2.13) equality in (2.15) may occur only in a proper subset of $[0, L]$: a closed interval $[a, b] \subset [0, L]$ with $[a, b] \not\equiv [0, L]$.

□

Proof. (of Theorem 2.3) Assume $\mathbf{v} \in \text{argmin}_{\mathcal{A}} \widehat{E}$.
By we deduce $\mathbf{v} \in C^2([0, L]; \mathbb{R}^2)$ and fulfills Euler -Lagrange equations (2.11). We are left to show the uniqueness which will be obtained through a technique similar to ([19], Theorem 5). We set

$$\psi(\mathbf{p}, \mathbf{q}) = k_{\mathrm{f}}(q_1)^2 + k_{\mathrm{m}}(q_2)^2 + T(p_1 - p_2),\ \forall\, \mathbf{p} = (p_1, p_2),\ \mathbf{q} = (q_1, q_2). \quad (2.16)$$

If there exist two solutions $\mathbf{v} = (v_{\mathrm{f}}, v_{\mathrm{m}}),\quad \mathbf{w} = (w_{\mathrm{f}}, w_{\mathrm{m}})$ with $\mathbf{v} \not\equiv \mathbf{w}$, then by recalling (2.15) and

$$0 \leq s(x) = v_{\mathrm{f}}(x) - v_{\mathrm{m}}(x) \leq d \leq S_* \quad \forall x \in [0, L],$$

the convexity of T in $[0, S_*]$ yields

$$\psi(t\mathbf{v} + (1-t)\mathbf{w}, t\mathbf{v}' + (1-t)\mathbf{w}') - t\psi(\mathbf{v}, \mathbf{v}') - (1-t)\psi(\mathbf{w}, \mathbf{w}') \leq 0 \quad (2.17)$$

hence minimality of $\mathbf{v}$ and $\mathbf{w}$ implies

$$\int_0^L \psi(t\mathbf{v}+(1-t)\mathbf{w}, t\mathbf{v}'+(1-t)\mathbf{w}')\,dx = t\int_0^L \psi(\mathbf{v},\mathbf{v}')dx + (1-t)\int_0^L \psi(\mathbf{w},\mathbf{w}')dx \tag{2.18}$$

and by (2.17)

$$\psi(t\mathbf{v}+(1-t)\mathbf{w}, t\mathbf{v}'+(1-t)\mathbf{w}') - t\psi(\mathbf{v},\mathbf{v}') - (1-t)\psi(\mathbf{w},\mathbf{w}') = 0 \qquad \forall x \in [0,L], \ \ \forall t \in [0,1]. \tag{2.19}$$

By differentiating the previous equality with respect to t we get

$$\begin{aligned} &\psi_p(t\mathbf{v}+(1-t)\mathbf{w}, t\mathbf{v}'+(1-t)\mathbf{w}')\cdot(\mathbf{v}-\mathbf{w}) + \\ &\psi_q(t\mathbf{v}+(1-t)\mathbf{w}, t\mathbf{v}'+(1-t)\mathbf{w}')\cdot(\mathbf{v}'-\mathbf{w}') = \\ &= \psi(\mathbf{v},\mathbf{v}') - \psi(\mathbf{w},\mathbf{w}') \ \ \forall x \in [0,L], \ \ \forall t \in [0,1]. \end{aligned} \tag{2.20}$$

hence by evaluating (2.20) at $t=1$ and $t=0$

$$\begin{aligned} &\psi_p(\mathbf{v},\mathbf{v}'))\cdot(\mathbf{v}-\mathbf{w}) + \psi_q(\mathbf{v},\mathbf{v}')\cdot(\mathbf{v}'-\mathbf{w}') = \\ &\psi_p(\mathbf{w},\mathbf{w}'))\cdot(\mathbf{v}-\mathbf{w}) + \psi_q(\mathbf{w},\mathbf{w}')\cdot(\mathbf{v}'-\mathbf{w}'). \end{aligned} \tag{2.21}$$

Then by taking into account (2.15),(2.16),(2.21) and the fact that τ is not decreasing, we get

$$\begin{aligned} &2k_{\mathrm{f}}(v_{\mathrm{f}}'-w_{\mathrm{f}}')^2 + 2k_{\mathrm{m}}(v_{\mathrm{m}}'-w_{\mathrm{m}}')^2 = \\ &(v_{\mathrm{f}}-v_{\mathrm{m}}-w_{\mathrm{f}}+w_{\mathrm{m}})\{\tau(w_{\mathrm{f}}-w_{\mathrm{m}})-\tau(v_{\mathrm{f}}-v_{\mathrm{m}})\} \le 0 \end{aligned} \tag{2.22}$$

that is $v_{\mathrm{f}}'-w_{\mathrm{f}}' = v_{\mathrm{m}}'-w_{\mathrm{m}}' = 0$. Hence $\mathbf{v}=\mathbf{w}$ in the whole $[0,L]$.
Eventually, by integrating by parts the elastic energies in (1.1) and taking into account (2.11) and (2.13), we get the Compliance Identity (2.40). $\square$

A slight modification of the proof of the previous theorem permits to show also

Theorem 2.6. *Assume (1.1),(1.5), (2.2)-(2.5). If the odd extension of τ is analytic in a neighborhood of the origin then in addition to the thesis of Theorem 2.3 we have*

$$v_{\mathrm{m}}(x) < v_{\mathrm{f}}(x) \quad \forall x \in [0,L]. \tag{2.23}$$

Proof. Since we here all the assumptions of Theorem 2.3 hold true we are only left to prove (2.41).
If there exists $\bar{x}$ such that $v_{\mathrm{f}}(\bar{x}) = v_{\mathrm{m}}(\bar{x})$ then by $\mathbf{v} \in C^2([0, L]; \mathbb{R}^2)$ and Lemma 2.4 we get $v_{\mathrm{f}}'(\bar{x}) = v_{\mathrm{m}}'(\bar{x})$ and by using Euler-Lagrange equations we deduce that $v_{\mathrm{f}}^{(j)}(\bar{x}) = v_{\mathrm{m}}^{(j)}(\bar{x})$ for every $j \geq 0$, hence $v_{\mathrm{f}}(x) \equiv v_{\mathrm{m}}(x)$ in $[0, L]$ by identity principle of analytic functions, so that

$$v_{\mathrm{f}} \equiv v_{\mathrm{m}} \equiv \frac{d}{L} x$$

in contrast with Neumann boundary conditions $v_{\mathrm{f}}'(0) = 0,\ v_{\mathrm{m}}'(L) = 0$. □

2.2. **Rigid supporting matrix with discontinuous softening glue.** In this subsection we face the case of a rigid matrix, as is the case of concrete beams.
When assuming the structural assumptions of the previous subsection (say: (1.1), (1.5), (2.2)-(2.5)), if k_{f} is fixed, $k_{\mathrm{m}} \to +\infty$ and $(v_{\mathrm{f}}, v_{\mathrm{m}})$ is the minimizer of $\widehat{E}$ on $\mathcal{A}$, then $v_{\mathrm{m}} \to 0$ in $H^1(0, L)$.
This claim is a straightforward consequence of estimate (2.7) since we have

$$0 \leq \| v_{\mathrm{m}}' \|^2_{L^2(0,L)} \leq \frac{1}{k_{\mathrm{m}}} \widehat{E}(\mathbf{v}) \leq \frac{1}{k_{\mathrm{m}}}\ L\, T(S_*).$$

Then it is natural to describe the rigid supporting matrix by selecting as admissible axial displacement $\mathbf{v}$ only the ones fulfilling $v_{\mathrm{m}} \equiv 0$. Moreover, in this case we can assume without any restriction, that the length of the matrix is greater than the length L of the fiber. To simplify notation we set $v \equiv v_{\mathrm{f}}$ and so we reduce to study the energy

$$H(v) = \int_0^L \left(\frac{k_{\mathrm{f}}}{2} |v'|^2 + T(|v|) \right) dx. \tag{2.24}$$

Precisely we deal with the problem

$$\min\{H(v) \mid v \in H^1(0, L),\ v(L) = d\}. \tag{2.25}$$

Theorem 2.7. *Assume (2.1),(2.2),(2.5),(2.24). Then:*

i) *The problem (2.25) admits a unique solution $v \in C^2([0, L])$. Moreover $0 \leq v(x) \leq d\ \forall x \in [0, L]$.*

ii) *The solution v of (2.25) fulfills the Euler-Lagrange equation*

$$\begin{cases} k_{\rm f} v''(x) = G\,\tau(v(x)), & x \in (0,L) \\ v'(0) = 0, \quad v(L) = d. \end{cases} \tag{2.26}$$

iii) *The solution v of (2.25) fulfills the following Compliance Identity*

$$H(v) = \int_0^L \left[\frac{G}{2} \tau(v(x)) v(x) + T(v(x)) \right] dx + k_{\rm f} d\, v'(L). \tag{2.27}$$

iv) *The solution v of (2.25) fulfills the following equation (first integral)*

$$T\Big(v(x)\Big) - \frac{k_{\rm f}}{2} (v'(x))^2 \text{ is constant in } [0,L]. \tag{2.28}$$

v) *If v is the solution of (2.25) then either $v(x) > 0\ \forall x \in [0,L]$ or there exists $\xi \in [0,L)$, with $\xi = \xi(d)$, such that $v(x) = 0\ \forall x \in [0,\xi]$ and v is strictly increasing in $[\xi, L]$.*

Proof. Statements i), ii) iii) and the fact that $v \not\equiv 0$ can be proved as in the proof of Theorem 2.3.
iv) Since $H(v) = \int_0^L f(v,v')dx$ we get $\frac{d}{dx}[f - pf_p] = 0$ hence (2.28).
v) The fact that the null set of v, when not empty, is connected and contains $x = 0$ is true since otherwise we could modify v by setting $v = 0$ in the interval $[0,\xi]$ and strictly reduce the energy, in contradiction to the minimality of v. Then, by (2.26) since $\tau(0) = 0$ and $\tau(s) > 0\ \forall s \in (0, S_*)$ we get the thesis. □

Theorem 2.8. *Assume (2.1), (2.2),(2.5)(2.24), v is the solution of (2.25) and set $\xi = \inf\{x \in [0,L] \mid v(x) > 0\}$. Then*

$$L - \xi = \sqrt{\frac{k_f}{2}} \int_{v(\xi)}^d \frac{dt}{\sqrt{T(t) - T(v(\xi))}}. \tag{2.29}$$

Proof. Clearly $v'(\xi) = 0$ and since $v'(x) \geq 0$, by (2.28) we get

$$\sqrt{\frac{k_f}{2}}\, v'(x) = \sqrt{T(v(x)) - T(v(\xi))}, \quad \forall x \in [\xi, L],$$

$$L - x = \sqrt{\frac{k_f}{2}} \int_{v(x)}^d \frac{dt}{\sqrt{T(t) - T(v(\xi))}}, \quad \forall x \in (\xi, L]$$

and now thesis easily follows by letting $x \to \xi$. □

A straightforward consequence of (2.29) is the following statement which, in the same spirit of [1] where an estimate analogous to (2.29) is deduced through heuristic arguments, establishes a sharp estimate on the length of the fiber according to the carrying capacity of the glue and the constitutive parameters of the reinforcing fiber.

Corollary 2.9. *Assume (2.1),(2.2), (2.5),(2.24), v is the solution of (2.25) and set*

$$\overline{L} := \sqrt{\frac{k_{\mathrm{f}}}{2}} \int_0^d \frac{dt}{\sqrt{T(t)}}. \tag{2.30}$$

Then

i) *If $\overline{L} < L$ then $0 < \xi = L - \overline{L}$.*

ii) *If $\overline{L} = L$ then $\xi = 0$ and $v(0) = 0$.*

iii) *If $\overline{L} > L$ then $\xi = 0$ and $v(0) > 0$.*

Proof. We observe that by $T(0) = 0$ and the convexity of T, we get

$$T(y - z) + T(z) \leq T(y), \quad \forall y \in [z, d],\ 0 < z < d,$$

$$\int_z^d \frac{dy}{\sqrt{T(y) - T(z)}} \leq \int_z^d \frac{dy}{\sqrt{T(y - z)}} = \int_0^{d-z} \frac{d\eta}{\sqrt{T(\eta)}} < \int_0^d \frac{d\eta}{\sqrt{T(\eta)}} \tag{2.31}$$

and therefore if $\overline{L} = L$ then by (2.29) and (2.31) we get $\xi = 0,\ v(0) = 0$ and ii) is proved. If $\overline{L} < L$ then either $\xi = 0,\ v(0) > 0$ or $0 < \xi = L - \overline{L}$ but if $\xi = 0,\ v(0) > 0$ then by using (2.31) with $z = v(0)$ we get $L < \overline{L}$, a contradiction. iii) is an obvious consequence of (2.29). □

Remark 2.10. Assume (2.1),(2.2), (2.5),(2.24),(2.30).

- If $L \leq \overline{L}$ then $\xi[d] = 0$.
  Then the mathematical solution v describes a state in which the whole fiber is elastically detached from the matrix: $v(x) > 0$ for every x when $L < \overline{L}$, $v(x) = 0$ iff $x = 0$ when $L = \overline{L}$. This suggests the possibility of structural breakdown. This case includes the case $\overline{L} = +\infty$ which corresponds to the non-integrability at 0 of $T^{-\frac{1}{2}}$.

- If $L > \overline{L}$ then $\xi[d] = L - \overline{L} > 0$.
  Then the mathematical solution v describes a state in which the fiber is only partially elastically detached from the matrix. This suggests that the structure is able to sustain the traction without breakdown. This case may happen only if $T^{-\frac{1}{2}} \in L^1(0, S_*)$, hence it is not compatible with any linear or superlinear stress-slip law for the adhesive material.

An interesting outcome of the Corollary 2.9 is the suggestion to study suitable adhesive property of the glue, say qualitative behavior of τ, in order to make to make $\overline{L}$ as small as possible.

Now we prove that when τ is concave near the origin then the displacement in 0 is negligible with respect to d. This result is in good agreement with the experimental observations (in contrast with the one obtained in Section 1 in the case of quadratic adhesion law), hence it provides reasons in favor of the choice of constitutive assumptions (2.1),(2.2).

To make explicit the dependence of the solution v on the Dirichelet datum d we label the unique v solution of (2.25) by $v[d] = v[d](x)$ and set $\xi[d] = \inf\{x \in [0, L] \mid v[d](x) > 0\}$ for every $d \in (0, S_*)$.

Theorem 2.11. *Assume (2.1),(2.2),(2.5),(2.24) and*

$$\lim_{s \to 0^+} s^{-1}\tau(s) = +\infty. \tag{2.32}$$

Then
either $v[d](0) \equiv 0$ *in a neighborhood of* $d = 0$ *or* $d^{-1}v[d](0) \to 0$ *as* $d \to 0$.

Proof. If $d_n \to 0$ and $v[d_n](0) > 0$ then $\xi[d_n] = 0$ and formula (2.29) holds true, moreover, since $v[d_n](L) = d_n > 0$, $\xi = 0$ and $v[d_n](\xi) = v[d_n](0) > 0$, we have

$$L = \sqrt{\frac{k_{\rm f}}{2G}} \int_{v[d_n](0)}^{d_n} \frac{dt}{\sqrt{\int_{v[d_n](0)}^{t} \tau(s)ds}}. \tag{2.33}$$

By recalling that τ is increasing, and $v[d_n](0) < d_n$ we get

$$\frac{2G}{k_{\rm f}} L^2 \left(\int_{v[d_n](0)}^{d_n} \frac{dt}{\sqrt{(d_n - v[d_n](0))\tau(v[d_n](0))}} \right)^2 \leq \frac{d_n - v[d_n](0)}{\tau(v[d_n](0))},$$

$$2\,G\,\tau(v[d_n](0))\,L^2 \;\leq\; k_f\,d_n$$

which together (2.32) yields the thesis. □

2.3. **Elastic supporting matrix with smooth softening glue.** In this section we assume the stress-slip law governing the constitutive behavior of the glue is continuous with compact support.
The scheme includes the case of unimodal law which behaves elastically up to a given value of the slip and exhibits a softening range beyond such value. Nevertheless we do not make any assumption about the monotonicity region of $\widetilde{\tau}$. More precisely, let

$$\begin{aligned}&\{\widetilde{\tau} : \mathbb{R}_+ \to \mathbb{R}_+ \text{ continuous, } S_* > 0,\\ &\widetilde{\tau}(0) = 0,\ \widetilde{\tau}(s) > 0 \text{ in } [0, S_*],\ \widetilde{\tau}(s) = 0,\, \forall s \geq S_*,\end{aligned} \tag{2.34}$$

$\widetilde{\tau}$ represents the stress-slip constitutive relationship of the adhesive material hence the adhesion energy is now defined as follows:

$$\widetilde{T}(\sigma) = G \int_0^\sigma \widetilde{\tau}(t)dt, \tag{2.35}$$

$$\widetilde{E}_{\mathrm{ad}} = \int_0^L \widetilde{T}(|s(x)|)dx. \tag{2.36}$$

The total energy of the system takes the form

$$\widetilde{E}(\mathbf{v}) := E_{\mathrm{f}}(\mathbf{v}) + E_{\mathrm{m}}(\mathbf{v}) + \widetilde{E}_{\mathrm{ad}}(\mathbf{v}), \quad \mathbf{v} = (v_{\mathrm{f}}, v_{\mathrm{m}}) \tag{2.37}$$

where $v_{\mathrm{f}}, v_{\mathrm{m}}$ are the axial displacements of the fiber and the matrix, the elastic energies E_{f} and E_{m} are defined in (1.1) and $\widetilde{E}_{\mathrm{ad}}$ is defined in (2.36). Referring to (1.5) and (2.37) we face the minimization problem

$$\min\{\widetilde{E}(\mathbf{v}) \mid \mathbf{v} \in \mathcal{A}\}. \tag{2.38}$$

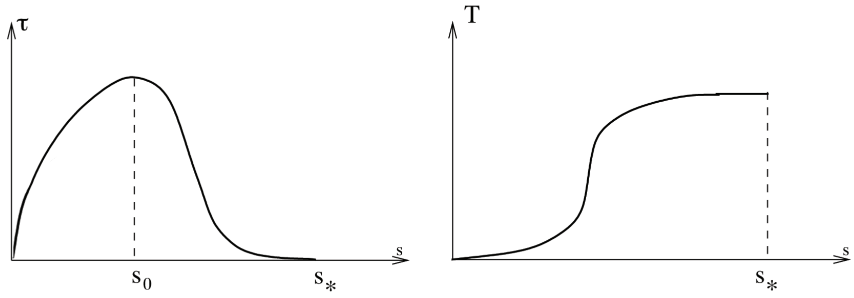

FIGURE 2.2. Energy and constitutive law for smooth softening

Theorem 2.12. *Assume (1.1),(1.5,(2.5), (2.34),(2.35), (2.36),(2.37). Then problem (2.38) admits solution* $\mathbf{v} \in C^2([0,L];\mathbb{R}^2)$ *and satisfy the Euler-Lagrange equations*

$$\begin{cases} k_{\mathrm{f}}\, v_{\mathrm{f}}''(x) = G\,\widetilde{\tau}(v_{\mathrm{f}} - v_{\mathrm{m}}\ ), \;\; x \in (0,L) \\ -k_{\mathrm{m}}\, v_{\mathrm{m}}''(x) = G\,\widetilde{\tau}(v_{\mathrm{f}} - v_{\mathrm{m}}), \;\; x \in (0,L) \\ v_{\mathrm{f}}'(0) = 0, \quad v_{\mathrm{f}}(L) = d \\ v_{\mathrm{m}}(0) = 0, \quad v_{\mathrm{m}}'(L) = 0. \end{cases} \tag{2.39}$$

Moreover, the following Compliance Identity holds true

$$\widetilde{E}(\mathbf{v}) = \int_0^L \left[\frac{G}{2}\widetilde{\tau}(s(x))s(x) + \widetilde{T}(s(x))\right] dx + k_{\mathrm{f}} d\, v_{\mathrm{f}}'(L). \tag{2.40}$$

If, in addition to the previous assumptions the odd extension of $\widetilde{\tau}$ *is analytic in a neighborhood of the origin, then the minimizer* $\mathbf{v}$ *fulfills*

$$v_{\mathrm{m}}(x) < v_{\mathrm{f}}(x), \quad \forall x \in [0,L]. \tag{2.41}$$

Proof. Lemma 2.4 and Lemma 2.5 can be proved in the same way when $\widetilde{\tau}$ is substituted to τ. Then the proofs of existence, (2.39) and (2.40) follow as in Theorem 2.3. The final part of the proof of Theorem 2.3 was based on the convexity of T in $[0, S_*]$, hence it cannot be adapted to $\widetilde{T}$, therefore we cannot deduce uniqueness of the minimizer in general. Nevertheless the inequality (2.41) can be proved exactly as in the proof of Theorem 2.6. □

Remark 2.13. We set $\widetilde{\tau}(S_0) = \max_{s\in[0,S_*]} \widetilde{\tau}(s)$, hence $\dot{\widetilde{\tau}}(s) \geq 0$ if $0 \leq s \leq S_0$ and $\dot{\widetilde{\tau}}(s) \leq 0$ if $S_0 \leq s \leq S_*$. Let $\mathbf{v} \in \mathrm{argmin}\widetilde{E}$, for any $\varphi \in C_0^1([0,L];\mathbb{R}^2)$

we compute the second variation of $\widetilde{E}$, given by (2.4), at $\mathbf{v}$ in the direction of φ, say

$$\delta^2 \widetilde{E}(\mathbf{v}, \varphi) = 2 \int_0^L \Big((k_{\mathrm{f}} + k_{\mathrm{m}}) |\varphi'|^2 + G\dot{\tilde{\tau}} |\varphi|^2 \Big)\, dx. \tag{2.42}$$

By virtue of (2.42) we conjecture instability phenomena may occur since for large values of the slip, i.e. $s > S_0$, $\tau'(s) < 0$ and $\delta^2 \widetilde{E}(\mathbf{v})$ could be negative for some φ.

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