Quasilinear and semilinear singular elliptic equations and systems

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Consider the functional

$$J_r(v) = \frac{1}{2}\int_\Omega (a(x) + |v|^r)|\nabla v|^2 - \int_\Omega f(x)v(x),$$

where

- Ω is a bounded open set in R^N,
- $f \in L^2(\Omega)$ is the datum,
- $0 < \alpha \le a(x) \le \beta$,
- $r > 0$.

It is easy to prove the existence of a minimum u in $W_0^{1,2}(\Omega)$. Moreover the Euler-Lagrange equation (with the lower order term having natural growth w.r.t. the gradient) is, if $r > 1$,

$$u \in W_0^{1,2}(\Omega) : -\mathrm{div}((a(x) + |u|^r)\nabla u) + \frac{r}{2}|u|^{r-2}u|\nabla u|^2 = f(x).$$

Furthermore it is possible to prove the existence [B-Gallouet 93] of a solution $u \in W_0^{1,2}(\Omega)$, even if f belongs only to $L^1(\Omega)$: regularizing effect of the lower order term (for a complete draw about existence and nonexistence with right hand side measure see [Brezis-Nirenberg 97] and [B-Gallouet-Orsina 97]).

On the other hand, if $0 < r < 1$, the Euler-Lagrange equation (for positive f) is not evident; formally we have

$$(1) \quad 0 < u \,\mathrm{in}\, \Omega, \in W_0^{1,2}(\Omega) : -\mathrm{div}((a(x) + |u|^r)\nabla u) + \frac{r}{2}\frac{|\nabla u|^2}{u^{1-r}} = f(x).$$

In the first "hour" of the talk, I will present some existence results for quasilinear elliptic equations (variational or nonvariational) which become singular as the solution approaches zero and whose model is the above Euler-Lagrange equation (see [B2008-60JPP]).

Consider now the functional

$$I_r(v) = \frac{1}{2}\int_\Omega a(x)|\nabla v|^2 - \frac{1}{r}\int_\Omega f(x)|v(x)|^r, \quad 0 < r < 1.$$

Again it is easy to prove the existence of a minimum u in $W_0^{1,2}(\Omega)$, but it is not so easy to write the Euler-Lagrange equation, which, formally, is the semilinear elliptic boundary value problem ($f \geq 0$)

$$(2)\quad \begin{cases} -\mathrm{div}(a(x)\nabla u) = \dfrac{f(x)}{u^{1-r}}\,, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In the second "hour" of the talk, I will present some existence results (in Sobolev spaces depending on r and on the summability of f) for semilinear elliptic equations (variational or nonvariational), which become singular as the solution approaches zero and whose model is the above Euler-Lagrange equation (see [BO2010]).

I will also discuss the relationships between the two problems (1) and (2).

Then, in the third "hour" of the talk, using some of the previous results, I will present some existence results for variational systems of the type

$$\begin{cases} -\Delta u = p\, z^{\theta}\, u^{p-1} & \text{in } \Omega, \\ -\Delta z = \dfrac{\theta\, u^{p}}{z^{1-\theta}} & \text{in } \Omega, \\ u = z = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \theta < 1 < p$ (work in progress with Luigi Orsina).