DPG Method for Convection-Reaction Revisited

Leszek Demkowicz[a], **Nathan V. Roberts**[b]

[a]Oden Institute, The University of Texas at Austin

[b]Sandia National Laboratories

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Literature Review and Goal

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- The DPG method with approximate optimal test functions computed with polynomial (of order $p+1$) enriched test space was proposed in (1). 2D numerical experiments for *constant advection* were shown w/o any proofs.

[1] L. Demkowicz and J. Gopalakrishnan, ``A class of discontinuous Petrov-Galerkin methods. Part II: Optimal test functions,'' *Numer. Meth. Part. D. E.*, vol. 27, pp. 70–105, 2011, See also ICES Report 2009-16

[2] D. Broersen, W. Dahmen, and R. P. Stevenson, ``On the stability of DPG formulations of transport equations,'' *Math. Comp.*, vol. 87, no. 311, pp. 1051–1082, 2018

[3] L. Demkowicz and N. V. Roberts, ``The DPG method for the convection–reaction problem, revisited,'' Oden Institute for Computational Engineering and Sciences, Tech. Rep. 05, 2021

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- Goal of this work: [3] Analyze the stability of the original method.

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Outline

1. The Convection-Reaction Problem
2. Discrete Stability Proof
3. Convergence of Fields and Traces
4. Conclusions

Convection-Reaction Problem

- Convection-reaction problem

$$\begin{cases} b \cdot \nabla u + cu = f & \text{in } \Omega \\ u = g & \text{on } \Gamma_- \end{cases}$$

where advection vector $b \in H(\mathrm{div}, \Omega)$, reaction coefficient c and load f are assumed to be piece-wise smooth, and boundary $\Gamma = \partial\Omega$ is split into three disjoint parts,

$$\Gamma_- := \{x \in \Gamma : b_n(x) < 0\} \quad \Gamma_+ := \{x \in \Gamma : b_n(x) > 0\} \quad \Gamma_0 := \{x \in \Gamma : b_n(x) = 0\}\,.$$

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- Integration by parts

$$\int_\Omega (\underbrace{b \cdot \nabla u + cu}_{=:Au})v = \int_\Omega u(\underbrace{-\operatorname{div}(bv) + cv}_{=:A^* v}) + \int_\Gamma b_n uv\,.$$

Theory of Adjoints within the Closed Operators Theory

- Graph spaces (identical)

$$\begin{cases} H_A(\Omega) := \{u \in L^2(\Omega) : Au \in L^2(\Omega)\} = \{u \in L^2(\Omega) : b \cdot \nabla u \in L^2(\Omega)\} \\ H_{A^*}(\Omega) := \{v \in L^2(\Omega) : A^* v \in L^2(\Omega)\} = H_A(\Omega) \end{cases}$$

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- Density assumption:

$$\overline{C^\infty(\bar{\Omega})}^{H_A} = H_A(\Omega)\,.$$

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$$

- **Density assumption:**

$$
\overline{C^\infty(\bar{\Omega})}^{H_A} = H_A(\Omega)\,.
$$

- **Trace Operator:** There exists a continuous trace operator,

$$
\gamma \; : \; H_A(\Omega) \to L^2_w(\Gamma)
$$

where the weight $w = |b_n|$.

Theory of Adjoints within the Closed Operators Theory

- Domains of operators

$$
\begin{aligned}
D(A) &:= \{u \in H_A(\Omega) : \gamma u = 0 \text{ on } \Gamma_-\} \\
D(A^*) &:= \{v \in H_{A^*}(\Omega) : \gamma v = 0 \text{ on } \Gamma_+\} \,.
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- **Lemma:** Operators $A : D(A) \to L^2(\Omega)$ and $A^* : D(A^*) \to L^2(\Omega)$ are adjoint to each other.

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- **Lemma:** Operators $A : D(A) \to L^2(\Omega)$ and $A^* : D(A^*) \to L^2(\Omega)$ are adjoint to each other.
- **Boundness below assumption:**
  We assume that, with appropriate additional assumptions on coefficients b, c, both operators A, A^* are bounded below.

$$
\begin{aligned}
\|Au\| &\geq \alpha \|u\| \qquad u \in D(A) \\
\|A^*v\| &\geq \alpha \|v\| \qquad v \in D(A^*)
\end{aligned}
$$

Closed Range Theorem for Closed Operators implies that constant α is the same for both operators.

Ultraweak (UW) Variational Formulation

- $$\left\{ \begin{array}{l} u \in L^2(\Omega) \\ (u, A^* v) = l(v) \quad v \in D(A^*) \end{array} \right. \tag{1.1}$$

Ultraweak (UW) Variational Formulation

- $$\left\{\begin{array}{l} u \in L^2(\Omega) \\ (u, A^* v) = l(v) \quad v \in D(A^*) \end{array}\right. \tag{1.1}$$

- **Theorem:** Let A, A^* be bounded below with constant α. Then the bilinear form in (1.1) satisfies the inf-sup condition with

$$\gamma = (\alpha^{-2} + 1)^{-\frac{1}{2}}$$

and the UW formulation is well-posed.
Proof: Let $u \in L^2(\Omega)$. Take solution of the adjoint problem:
$v \in D(A^*)$, $A^* v = u$. Boundedness below implies $\|v\| \leq \alpha^{-1} \|A^* v\|$ and, in turn, $\|v\|_{D(A^*)} \leq (\alpha^{-2} + 1)^{1/2} \|A^* v\|$. This gives:

$$\|u\|^2 = (u, A^* v) = \frac{(u, A^* v)}{\|v\|_V} \|v\|_V \leq \sup_{v \in D(A^*)} \frac{(u, A^* v)}{\|v\|_V} (\alpha^{-2} + 1)^{-1/2} \|u\| \, .$$

Broken UW Variational Formulation

- $$\left\{\begin{array}{l} u \in L^2(\Omega),\ \hat{u} \in \hat{U},\ \hat{u} = u_0 \text{ on } \Gamma_- \\ (u, A_h^* v) + \langle \hat{u}, v \rangle_{\Gamma_h} = l(v) \qquad v \in H_{A^*}(\mathcal{T}_h)\,, \end{array}\right. \tag{1.2}$$

  where $l = (f, \cdot) \in (H_{A^*}(\mathcal{T}_h))'$. Additional boundary integrals can be added to the load.

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- Theorem unpacked: [4]

  Step 1: Test with a conforming $v \in D(A^*)$ to establish stability of the fields:

$$\begin{aligned} \|u\| &\leq \gamma^{-1} \sup_{v\in D(A^*)} \frac{(u,A^* v)}{\|v\|_V} \\ &= \gamma^{-1} \sup_{v\in D(A^*)} \frac{(u,A^* v)+\langle \hat{u},v\rangle_{\Gamma_h}}{\|v\|_V} \leq \gamma^{-1} \sup_{v\in H_{A^*}(\mathcal{T}_h)} \frac{(u,A^* v)+\langle \hat{u},v\rangle_{\Gamma_h}}{\|v\|_V}\,. \end{aligned}$$

  Step 2: Use the stability of fields to establish the stability of traces in the dual norm:

$$\begin{aligned} \sup_{v\in H_{A^*}(\mathcal{T}_h)} \frac{\langle \hat{u},v\rangle_{\Gamma_h}}{\|v\|_V} &= \sup_{v\in H_{A^*}(\mathcal{T}_h)} \frac{(u,A^* v)+\langle \hat{u},v\rangle_{\Gamma_h}-(u,A^* v)}{\|v\|_V} \\ &\leq (1+\gamma^{-1}) \sup_{v\in H_{A^*}(\mathcal{T}_h)} \frac{(u,A^* v)+\langle \hat{u},v\rangle_{\Gamma_h}}{\|v\|_V}\,. \end{aligned}$$

[4] C. Carstensen, L. Demkowicz, and J. Gopalakrishnan, ``Breaking spaces and forms for the DPG method and applications including Maxwell equations,'' *Comput. Math. Appl.*, vol. 72, no. 3, pp. 494–522, 2016

Broken UW Variational Formulation

- *Step 3:* Use the Duality lemma: Let v be the solution of the element variational Neumann problem,

$$\left\{\begin{array}{l} v \in H_{A^*}(K) \\ (A^*v, A^*\delta v)_K + (v, \delta v)_K = \displaystyle\int_{\partial K} b_n \hat{u}\, \delta v \qquad \delta v \in H_{A^*}(K)\,. \end{array}\right.$$

Then $w = -A^*v$ is the solution to the Dirichlet problem,

$$\left\{\begin{array}{l} w \in H_A(K),\; w = \hat{u} \text{ on } \partial K - \partial K_0 \\ (Aw, A\delta w)_K + (w, \delta w)_K = 0 \qquad \delta v \in H_{A^*(K)}\,. \end{array}\right.$$

and,

$$\|w\|_{H_A(\Omega)} = \|v\|_{H_{A^*}(\Omega)}\,.$$

to replace the dual norm for traces with the minimum energy extension norm.

Boundedness Below is Critical. A General Stability Result for A^*

- Assumption: $b(x) = \nabla V(x)$.

[5] L. Demkowicz and N. Heuer, ``Robust DPG method for convection-dominated diffusion problems,'' *SIAM J. Num. Anal*, vol. 51, pp. 2514–2537, 2013, see also ICES Report 2011/13

Boundedness Below is Critical. A General Stability Result for A^*

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- Assumption: $b(x) = \nabla V(x)$.
- Let $v \in H_{A^*}(\Omega)$. Introduce an auxiliary unknown (comp. (5))

$$w(x) := e^{V(x)} v(x) \quad \nabla w = e^V b v + e^V \nabla v \,.$$

Let $f := A^* v$. We have,

$$e^V f = e^V(-b \cdot \nabla v + (c - \operatorname{div} b) v) = -\operatorname{div}(bw) + (|b|^2 + c) w \,.$$

Multiplying both sides with w and integrating over Ω, we obtain,

$$-\int_\Omega \operatorname{div}(bw) w + \int_\Omega (|b|^2 + c) w^2 = \int_\Omega e^V f w \,.$$

The first term is now integrated by parts,

$$-\int_\Omega \operatorname{div}(bw) w = \int_\Omega b \cdot \nabla(\frac{w^2}{2}) - \int_\Gamma b_n w^2 = -\frac{1}{2}\int_\Gamma b_n w^2 - \frac{1}{2}\int_\Omega \operatorname{div} b \, w^2 \,.$$

This gives:

$$-\frac{1}{2}\int_\Gamma b_n w^2 + \int_\Omega \underbrace{(|b|^2 + c - \frac{1}{2}\operatorname{div} b)}_{=:a} w^2 = \int_\Omega e^V f w \,.$$

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Boundedness Below. A More General Stability Result for A^*

Assumption: $a(x) \geq a_{\min} > 0$. Use Young's inequality to estimate the right-hand side,

$$fw \leq \frac{a}{2} w^2 + \frac{e^{2V}}{2a} f^2 \,.$$

This leads to the final estimate,

$$\frac{1}{2} \int_{\Gamma_-} |b_n| w^2 + \frac{1}{2} \int_\Omega a w^2 \leq \int_\Omega \frac{e^{2V}}{2a} f^2 + \frac{1}{2} \int_{\Gamma_+} b_n w^2 \,.$$

In particular, for $v = 0$ on Γ_+, we obtain,

$$\int_\Omega a\, e^{2V} v^2 \leq \int_\Omega \frac{e^{2V}}{a} f^2 \,.$$

If e_1, e_2 are lower and upper bounds for e^{2V}, we obtain,

$$a_{\min}\, e_1 \int_\Omega v^2 \leq \int_\Omega a\, e^{2V} v^2 \leq \frac{e_2}{a_{\min}} \int_\Omega f^2 \,.$$

This gives the final estimate for the boundedness below constant:

$$\frac{e_1}{e_2} a_{\min}^2 \int_\Omega v^2 \leq \int_\Omega f^2 \,.$$

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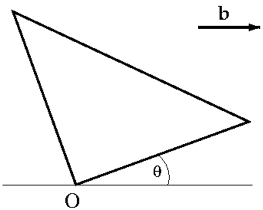
Construction of a Local Fortin Operator Fails

- Without going into details... The existence of the local Fortin operator provides a sufficient but not necessary condition for discrete stability. The required orthogonality conditions *make the construction unique.* Showing boundedness of the Fortin operator reduces to a numerical evaluation of an inf-sup constant α for a rotated master element shown to the right.



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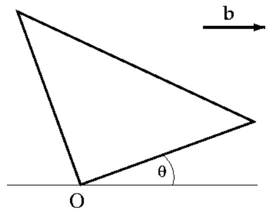
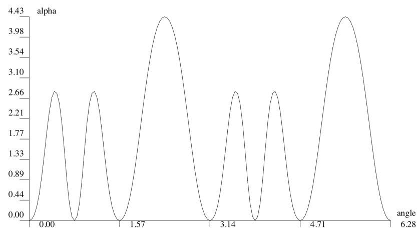
- The figure on the right presents value of α for element of order $p = 3$ and angle θ changing from 0 to 2π. As we can see, whenever one of the triangle edges becomes parallel to the x-axis, the constant degenerates to zero. Evidently, constant α is *not* uniformly (in angle) bounded away from zero. The result does not prove that the DPG method is unstable, it simply reflects the limitation of the local construction of the Fortin operator.





Local Fortin Operator for the Conforming DPG Method (Following [6])

- The element contribution to the bilinear form:

$$(u, -\partial_x v + cv)_K + \langle n_x \hat{u}, v \rangle_{\partial K} = (\partial_x u + cu, v)_K + \langle n_x (\hat{u} - u), v \rangle_{\partial K} \,.$$

Assumptions: $b = \mathrm{const} = (1,0)$, $c = \mathrm{const}$ element-wise, globally bounded.

[6] L. Demkowicz and P. Zanotti, ``Construction of DPG Fortin operators revisited,'' *Comp. and Math. Appl.*, vol. 80, 2261–2271, 2020, Special Issue on Higher Order and Isogeometric Methods

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- Element orthogonality conditions for the Fortin operator:

$$\begin{array}{ll} (\psi, \Pi v - v)_K = 0 & \psi \in \mathcal{P}^{p-1}(K) \\ \langle n_x \phi, \Pi v - v\rangle_{\partial K} = 0 & \phi \in \mathcal{P}^p(K) \,. \end{array} \tag{2.3}$$

  Taking $\psi = -\partial_x \chi$, $\chi \in \mathcal{P}^p(K)$, substituting into $(2.3)_1$, integrating by parts and utilizing $(2.3)_2$, we learn that

$$(\chi, \partial_x(\Pi v - v))_K = 0 \qquad \chi \in \mathcal{P}^p(K) \,. \tag{2.4}$$

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$$(\chi, \partial_x(\Pi v - v))_K = 0 \qquad \chi \in \mathcal{P}^p(K) . \tag{2.4}$$

- This leads to the idea of defining $\partial_x \Pi v$ by L^2-projection,

$$\frac{1}{2} \|\partial_x(\Pi v - v)\|^2_{L^2(K)} \to \min_{\Pi v \in \mathcal{P}^r(K)} .$$

or, equivalently,

$$\begin{cases} \Pi v \in \mathcal{P}^r(K) \\ (\chi, \partial_x(\Pi v - v))_K = 0 \qquad \chi \in \mathcal{P}^{r-1}(K) . \end{cases} \tag{2.5}$$

In order to secure satisfaction of (2.4), we need to assume that $r - 1 \geq p$, i.e., $r \geq p + 1$.

[6] L. Demkowicz and P. Zanotti, ``Construction of DPG Fortin operators revisited,'' *Comp. and Math. Appl.*, vol. 80, 2261–2271, 2020, Special Issue on Higher Order and Isogeometric Methods

Local Fortin Operator for the Conforming DPG Method

- We have immediately,

$$\|\partial_x \Pi v\|_{L^2(K)} \leq \|\partial_x v\|_{L^2(K)} \leq \|-\partial_x v + cv\|_{L^2(K)} + c_{\max}\|v\|_{L^2(K)} \leq \sqrt{1+c_{\max}^2}\|v\|_{H_{A^*}(K)} \, .$$

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- Πv has been defined so far up to polynomials that are independent of x, i.e. the subspace

$$\mathcal{P}_y^r(K) := \operatorname{span}\{1, y, \ldots, y^r\}, \qquad \dim \mathcal{P}_y^r(K) = r+1 \,.$$

We are presented with the task of defining the undefined $\mathcal{P}_y^r(K)$-component of Πv in such a way that we satisfy orthogonality conditions (2.3). It is sufficient to satisfy only condition $(2.3)_1$. Indeed, integration by parts reveals that conditions $(2.3)_1$ and (2.4) imply $(2.3)_2$.

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- For $p \geq 3$, the subspace of bubbles $\mathcal{P}^p_0(K)$ is non-empty. Using $\chi \in \mathcal{P}^p_0(K)$ in (2.5), and integrating by parts, we get,

$$(\partial_x \chi, \Pi v - v)_K = 0 \qquad \chi \in \mathcal{P}^p_0(K) \, .$$

The null space of linear transformation $\partial_x : \mathcal{P}^p_0(K) \to \mathcal{P}^{p-1}(K)$ is trivial which implies that

$$\dim \partial_x(\mathcal{P}^p_0(K)) = \dim \mathcal{P}^p_0(K) = \frac{(p-2)(p-1)}{2} \, .$$

As $\dim \mathcal{P}^{p-1}(K) = \frac{p(p+1)}{2}$, we are missing $\frac{p(p+1)}{2} - \frac{(p-2)(p-1)}{2} = 2p-1$ conditions. This results in the condition for the minimal enriched order r,

$$r+1 \geq 2p-1 \quad \Leftrightarrow \quad r \geq 2p-2 \, .$$

Local Fortin Operator for the Conforming DPG Method

- For $p \leq 2$, the space of bubbles is trivial, so we need to satisfy:

$$r + 1 \geq \dim \mathcal{P}^{p-1}(K) = \frac{p(p+1)}{2}\,.$$

Table below presents the minimum value of enriched order r for different polynomial orders p. As we can see, except for low $p = 1, 2, 3$, the values are very pessimistic. We emphasize that they reflect only the deficiency of the local construction of the Fortin operator.

p	1	2	3	4	5	6
r	2	3	4	6	8	10

Table: Minimal enriched order r resulting from the local construction of Fortin operator for different polynomial orders of discretization.

Local Fortin Operator for the Conforming DPG Method

- We complete now the definition of Πv by requesting the satisfaction of the orthogonality conditions. Consider first the case of $p > 2$ and $r = 2p - 2$. In this case,

$$\dim \mathcal{P}_y^r(K) + \dim(\partial_x \mathcal{P}_0^p(K)) = \dim \mathcal{P}^{p-1}(K)\,.$$

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$$\dim \mathcal{P}_y^r(K) + \dim(\partial_x \mathcal{P}_0^p(K)) = \dim \mathcal{P}^{p-1}(K) \, .$$

- Lemma. Let K be a rotated unit master triangle. There exists a continuous right-inverse of derivative ∂_x,

$$\begin{array}{l} R \, : \, \mathcal{P}^p(K) \to \mathcal{P}^{p+1}(K), \quad \partial_x R\phi = \phi \quad \forall \phi \in \mathcal{P}^p(K) \\ \|R\phi\|_{L^2(K)} \leq \sqrt{2} \|\phi\|_{L^2(K)} \, . \end{array}$$

Local Fortin Operator for the Conforming DPG Method

- We complete now the definition of Πv by requesting the satisfaction of the orthogonality conditions. Consider first the case of $p > 2$ and $r = 2p - 2$. In this case,

$$\dim \mathcal{P}^r_y(K) + \dim(\partial_x \mathcal{P}^p_0(K)) = \dim \mathcal{P}^{p-1}(K)\,.$$

- Lemma. Let K be a rotated unit master triangle. There exists a continuous right-inverse of derivative ∂_x,

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- Let now $v^r = R(\partial_x \Pi v) \in \mathcal{P}^r(K)$. We set up the following system of equations for component $v^r_y \in \mathcal{P}^r_y(K)$.

$$\left\{ \begin{array}{l} v^r_y \in \mathcal{P}^r_y(K) \\ (\psi, v^r_y + v^r - v)_K = 0 \quad \psi \in \mathcal{P}^{p-1}(K)\,. \end{array} \right. \tag{2.6}$$

We introduce the discrete inf-sup constant corresponding to the bilinear form (2.6),

$$\alpha := \inf_{v^r_y \in \mathcal{P}^r_y(K)} \sup_{\psi \in \mathcal{P}^{p-1}(K))} \frac{(\psi, v^r_y)}{\|\psi\|_{L^2(K)}\, \|v^r_y\|_{L^2(K)}} \tag{2.7}$$

Local Fortin Operator for the Conforming DPG Method

- This leads to the L^2-stability bound on the master element,

$$\begin{aligned} \|\hat{v}^r_y\|_{L^2(\hat{K})} &\leq \alpha^{-1}\|\hat{v}^r - \hat{v}\|_{L^2(\hat{K})} \\ &\leq \alpha^{-1}\left(\|\hat{v}^r\|_{L^2(\hat{K})} + \|\hat{v}\|_{L^2(\hat{K})}\right) \end{aligned}$$

and, consequently,

$$\begin{aligned} \|\hat{v}^r_y + \hat{v}^r\|_{L^2(K)} &\leq (\alpha^{-1}+1)\|\hat{v}^r\|_{L^2(\hat{K})} + \alpha^{-1}\|\hat{v}\|_{L^2(\hat{K})} \\ &\leq (\alpha^{-1}+1)\sqrt{2}\|\partial_\xi(\hat{\Pi}\hat{v})\|_{L^2(\hat{K})} + \alpha^{-1}\|\hat{v}\|_{L^2(\hat{K})} \,. \end{aligned}$$

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- A standard scaling argument yields then:

$$\begin{aligned} \|v_y^r + v^r\|_{L^2(\hat{K})} &\leq h\|\hat{v}_y^r + \hat{v}^r\|_{L^2(\hat{K})} \\ &\leq (\alpha^{-1}+1)\sqrt{2}h^2\|\partial_x v\|_{L^2(K)} + \alpha^{-1}\|v\|_{L^2(K)}\,. \end{aligned}$$

Above, as usual, $\hat{v}$ denotes the pullback of v to master element $\hat{K}$. This concludes the proof of boundedness of the Fortin operator in the $H_{A^*}(K)$-norm, with an h-independent continuity constant. The touchy issue with the presented construction is the dependence of inf-sup constant α upon the orientation of the element with respect to the advection field. We will resort now to a numerical experiment to study this dependence.

Numerical Evaluation of the inf-sup Constant α

- Computation of the inf-sup constant α translates into the determination of the smallest eigenvalue for the generalized eigenvalue problem: $B^T G^{-1} B u = \alpha^2 M u$ where

$$G_{ij} = \int_K \psi_i \psi_j \quad B_{jk} = \int_K \psi_j y^k \quad M_{kl} = \int_K y^i y^k \quad i,j = 1, \dim \mathcal{P}^r(K),\ k,l = 1, r+1$$

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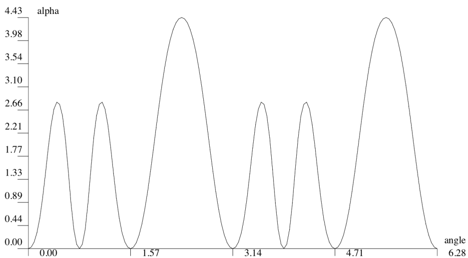
- The figure below presents value of α for element of order $p = 3$ and angle changing from 0 to 2π. As we can see, whenever one of the triangle edges becomes parallel to the x-axis, the constant degenerates to zero. Evidently, constant α is *not* uniformly (in angle) bounded away from zero.



Discrete Stability

We try to emulate the stability analysis for the broken UW formulation at the continuous level.

- Given u_h, find v such that $v \in D(A^*),\ A^* v = u_h$. Then,

$$\|u_h\| = \frac{(u_h, A^* v)}{\|A^* v\|} \leq (1+\alpha^{-2})^{-\frac{1}{2}} \frac{(u_h, A^* v)}{\|v\|_V} \leq (1+\alpha^{-2})^{-\frac{1}{2}} \sup_{v \in V} \frac{(u_h, A^* v)}{\|v\|_V} . \tag{2.8}$$

Challenge: Exact v has to be replaced with a weakly conforming approximation v_h for which $\|A^* v_h\| \geq \alpha \|v_h\|$ and the Fortin condition $(u_h, A^*(v - v_h))$ holds.

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- Idea: Emulate stability proof for the marching DPG method. Divide the domain and the mesh into layers $\Omega_{h,1}, \ldots, \Omega_{h,N}$ defined in a recursive way starting from the outflow boundary:

$$\begin{aligned} \Omega_{h,1} &:= \bigcup \{K \in \mathcal{T}_h \,:\, \partial K_+ \subset \Gamma_+\} \\ \Omega_{h,n} &:= \bigcup \{K \in \mathcal{T}_h \,:\, \partial K_+ \subset \Gamma_+ \cup \Gamma_{h,-,n-1}\}, \quad n = 2, \ldots, N \end{aligned}$$

  where $\Gamma_{h,-,n}$ denotes the inflow part of the boundary of $\Omega_{h,n}$.

Discrete Stability

- Let v_h be an approximation of v. For each element K from the last layer, $K \subset \Omega_{h,N}$,

$$\int_{\partial K_-} e^{2V} |b_n| v_h^2 + \int_K e^{2V} a v_h^2 + \int_K \frac{e^{2V}}{a} |A_h^* v_h|^2 \quad \leq \int_K \frac{2e^{2V}}{a} |A_h^* v_h|^2 + \int_{\partial K_+} e^{2V} b_n v_h^2$$

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- This motivates introducing a constrained minimization problem:

$$\min_{v_h \in \mathcal{P}^{p+1}(K)} \frac{1}{2}\{\int_K \frac{w}{a}(A^*(v_h - v))^2 + \int_{\partial K_+} w b_n (v_h - v)^2\}, \quad w = e^{2V},$$

under the constraints:

$$\begin{aligned} \int_K \delta u_h \, A^*(v_h - v) \quad &= 0 \quad \forall\, \delta u_h \in \mathcal{P}^{p-1}(K)\,, \\ \int_{\partial K_+} w b_n \, \delta w_h (v_h - v) \quad &= 0 \quad \forall\, \delta w_h \in \mathcal{P}_c^{p+1}(\partial K_+)\,. \end{aligned}$$

The first, Fortin's constraint allows for replacing in (2.8) v with v_h. Indeed, it implies that $(u_h, A^* v) = (u_h, A^* v_h)$. The second constraint enforces weak comformity of v_h.

A Local Stability Result (Brezzi's Thm Extended)

- The constrained minimization problem is equivalent to the mixed problem:

$$\left\{\begin{array}{lll}
v_h \in \mathcal{P}^{p+1}(K),\ u_h \in \mathcal{P}^{p-1}(K),\ w_h \in \mathcal{P}_c^{p+1}(\partial K_+) & & \\
\displaystyle\int_K \frac{w}{a} A^* v_h A^* \delta v_h + \int_K u_h A^* \delta v_h + \int_{\partial K_+} w b_n w_h \delta v_h & = \displaystyle\int_K \frac{w}{a} A^* v A^* \delta v_h & \delta v_h \in \mathcal{P}^{p+1}(K) \\
\displaystyle\int_K \delta u_h\, A^* v_h & = \displaystyle\int_K \delta u_h\, A^* v & \delta u_h \in \mathcal{P}^{p-1}(K) \\
\displaystyle\int_{\partial K_+} w b_n\, \delta w_h v_h & = \displaystyle\int_{\partial K_+} w b_n\, \delta w_h v & \delta w_h \in \mathcal{P}_c^{p+1}(\partial K_+)
\end{array}\right.$$

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\displaystyle\int_K \delta u_h\, A^* v_h & = \displaystyle\int_K \delta u_h\, A^* v & \delta u_h \in \mathcal{P}^{p-1}(K) \\
\displaystyle\int_{\partial K_+} w b_n\, \delta w_h v_h & = \displaystyle\int_{\partial K_+} w b_n\, \delta w_h v & \delta w_h \in \mathcal{P}_c^{p+1}(\partial K_+)
\end{array}
\right.
$$

- Let $V_h = \mathcal{P}^{p+1}(K)$, $\|v_h\|_V^2 = \int_K \frac{w}{a}|A^* v|^2 + \int_{\partial K_+} w b_n\, |v|^2$, and,

$$
\begin{aligned}
V_{h,0} &:= \left\{ v_h \in V_h \ :\ \int_{\partial K_+} w b_n\, \delta w_h v_h = 0 \quad \forall\, \delta w_h \in \mathcal{P}_c^{p+1}(\partial K_+) \right\} \\
V_{h,00} &:= \left\{ v_h \in V_{h,0} \ :\ \int_K \delta u_h A^* v_h = 0 \quad \forall\, \delta u_h \in \mathcal{P}^{p-1}(K) \right\}.
\end{aligned}
$$

A Local Stability Result (Brezzi's Thm Extended)

- Introduce norms for the Lagrange multiplier $u_h \in \mathcal{P}_h^{p-1}(K)$, $w_h \in \mathcal{P}_c^{p+1}(\partial K_+)$,

$$\|u_h\|_K^2 := \int_K u_h^2, \quad \|w_h\|_{\partial K_+}^2 := \int_{\partial K_+} wb_n w^2$$

and consider the corresponding inf-sup constants,

$$\alpha_h := \inf_{u_h \in \mathcal{P}^{p-1}(K)} \sup_{v_h \in V_{h,0}} \frac{\int_\Omega u_h A^* v_h}{\|u_h\|_K \, \|v_h\|_V} \quad \beta_h := \inf_{w_h \in \mathcal{P}_c^{p+1}(\partial K_+)} \sup_{v_h \in V_h} \frac{\int_{\partial K_+} wb_n \, w_h v_h}{\|w_h\|_{\partial K_+} \, \|v_h\|_V}$$

A Local Stability Result (Brezzi's Thm Extended)

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- Lemma: The following estimate holds:

$$\int_K \frac{w}{a} |A^* v_h|^2 + \int_{\partial K_+} w b_n \, |v_h|^2 \le (1 + \alpha_h^{-2}) \int_K \frac{w}{a} |A^* v|^2 + \beta_h^{-2} \int_{\partial K_+} w b_n \, |v|^2 \, .$$

Discrete Stability - Cont.

- We obtain the inequality:

$$\int_{\partial K^N_-} e^{2V}|b_n|v_h^2 + \int_{K^N} e^{2V} a v_h^2 + \int_{K^N} \frac{e^{2V}}{a}|A_h^* v_h|^2 \\ \leq (1+\alpha_h^{-2}) \int_{K^N} \frac{2e^{2V}}{a}|\underbrace{A_h^* v}_{=u_h}|^2 + \beta_h^{-2} \int_{\partial K^N_+} e^{2V} b_n v^2 \,.$$

Discrete Stability - Cont.

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$$\leq (1+\alpha_h^{-2}) \int_{K^N} \frac{2e^{2V}}{a}|\underbrace{A_h^* v}_{=u_h}|^2 + \beta_h^{-2} \int_{\partial K^N_+} e^{2V} b_n v^2 \,.$$

- Similarly, for elements $K^{N-1} \subset \Omega_{h,N-1}$, we obtain,

$$\int_{\partial K^{N-1}_-} e^{2V}|b_n|v_h^2 + \int_{K^{N-1}} e^{2V} a v_h^2 + \int_{K^{N-1}} \frac{e^{2V}}{a}|A_h^* v_h|^2$$
$$\leq (1+\alpha_h^{-2}) \int_{K^{N-1}} \frac{2e^{2V}}{a}|\underbrace{A_h^* v}_{=u_h}|^2 + \beta_h^{-2} \underbrace{\int_{\partial K^{N-1}_+} e^{2V} b_n v^2}_{=\int_{\partial K^N_-} e^{2V} b_n (v_h^N)^2} \,.$$

Discrete Stability - Cont.

- We obtain the inequality:

$$\int_{\partial K_-^N} e^{2V}|b_n|v_h^2 + \int_{K^N} e^{2V} a v_h^2 + \int_{K^N} \frac{e^{2V}}{a}|A_h^* v_h|^2 \leq (1+\alpha_h^{-2}) \int_{K^N} \frac{2e^{2V}}{a}|\underbrace{A_h^* v}_{=u_h}|^2 + \beta_h^{-2} \int_{\partial K_+^N} e^{2V} b_n v^2 \, .$$

- Similarly, for elements $K^{N-1} \subset \Omega_{h,N-1}$, we obtain,

$$\int_{\partial K_-^{N-1}} e^{2V}|b_n|v_h^2 + \int_{K^{N-1}} e^{2V} a v_h^2 + \int_{K^{N-1}} \frac{e^{2V}}{a}|A_h^* v_h|^2 \leq (1+\alpha_h^{-2}) \int_{K^{N-1}} \frac{2e^{2V}}{a}|\underbrace{A_h^* v}_{=u_h}|^2 + \beta_h^{-2} \underbrace{\int_{\partial K_+^{N-1}} e^{2V} b_n v^2}_{=\int_{\partial K_-^N} e^{2V} b_n (v_h^N)^2} \, .$$

- We want now to add the two inequalities side-wise and cancel the first term in the first inequality with the last term in the second inequality (a telescoping effect). In order to do so, we have to premultiply the entire first inequality by factor β_h^{-2}. This leads to a multiplicative accumulation of constant β_h^{-2}. The product of such constants can be bounded by a mesh independent constant provided $\beta_h = 1 - O(h)$.

Discrete Stability - cont.

- Conjecture: We postulate the following behavior of stability constants α_h, β_h under the assumption that weights e^{2V} and $2\frac{e^{2V}}{a}$ are uniformly bounded throughout the domain.

$$\beta_h \geq 1 - Ch, \quad \alpha_h > \alpha_0 > 0\,.$$

with a mesh-independent constant $C > 0$.

Discrete Stability - cont.

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$$\beta_h \geq 1 - Ch, \quad \alpha_h > \alpha_0 > 0\,.$$

  with a mesh-independent constant $C > 0$.

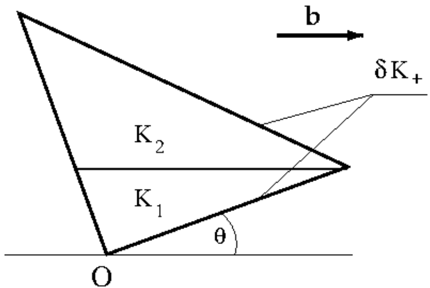
- **Theorem:** Under the conjecture above, the discrete inf-sup condition holds,

$$\sup_{v_h \in V_h^0} \frac{\sum_K \int_K u_h A_h^* v_h}{\|v_h\|_{H_{A^*}}} \geq C\|u_h\|$$

  with a mesh independent constant C. Above, V_h^0 stands for the subspace of weakly conforming broken test functions.

Support of the conjecture with numerical experiments

- **Definition of the enriched test space:** In the case of a single outflow edge $V_h(K) = \mathcal{P}^{p+1}(K)$. In the case of a triangle with two outflow edges, the element is split by the advection vector into two subtriangles, and continuous, piecewise polynomials of order $p+1$ are used.



Construction of the piece-wise polynomial enriched test space.

Support of the conjecture with numerical experiments

- We fix $b = (1,0), C = 1$ and rotate by angle θ a right triangle with size h around the vertex O. Table below presents numerical values of constant α_h for different values of polynomial order p and element size h. All values are the minimum values over rotation angles from the whole range of $\theta \in [0, 2\pi)$. Clearly the inf-sup constant stays uniformly bounded away from zero, and remains of order 1 in the whole range of polynomial orders p and element size h.

p/h	1.0	0.1	0.01	0.001	0.0001
2	0.737	0.712	0.708	0.707	0.707
3	0.657	0.637	0.633	0.633	0.632
4	0.593	0.582	0.578	0.577	0.577
5	0.545	0.539	0.535	0.535	0.535
6	0.508	0.505	0.501	0.500	0.500
7	0.477	0.476	0.472	0.471	0.471
8	0.451	0.451	0.448	0.447	0.447

Table: Minimal (over angles θ) value of inf-sup constant α_h for different values of element size h and polynomial order p, for advection vector $b = (1,0)$ and reaction coefficient $c = 1.0$; weights $a = w = 1$.

Support of the conjecture with numerical experiments

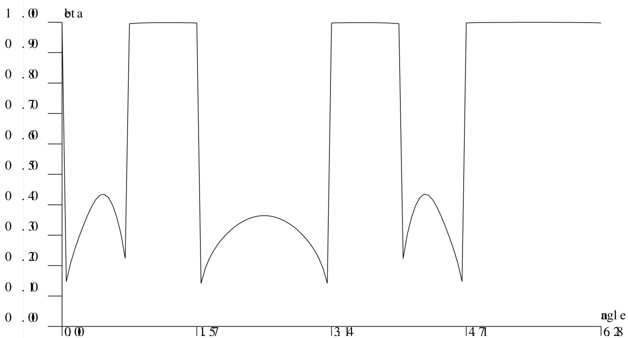
- We fix $b = (1,0), C = 1$ and rotate by angle θ a right triangle with size h around the vertex O. Table below presents the results. The constant stays very close to one, uniformly in the polynomial order, and it converges to one as $h \to 0$.

p/h	1.0	0.1	0.01	0.001	0.0001
2	0.99492667	0.99998878	0.99999998	0.99999999	0.99999999
3	0.99561802	0.99999021	0.99999998	0.99999999	0.99999999
4	0.99597293	0.99999096	0.99999999	0.99999999	0.99999999
5	0.99618684	0.99999143	0.99999999	0.99999999	0.99999999
6	0.99632916	0.99999174	0.99999999	0.99999999	0.99999999
7	0.99643040	0.99999197	0.99999999	0.99999999	0.99999999
8	0.99650598	0.99999214	0.99999999	0.99999999	0.99999999

Table: Composite test space of order $p+1$. Minimal (over angles θ) value of inf-sup constant β_h for different values of element size h and polynomial order p, for advection vector $b = (1,0)$ and reaction coefficient $c = 1.0$; weights $a = w = 1$.

Support of the conjecture with numerical experiments

- The experiment below demonstrates that the simple polynomial test space $\mathcal{P}^{p+1}(K)$ fails to deliver the correct stability constant β_h uniformly in angle θ. The figure below presents results for the case of the rotated master triangle of order $p = 2$, advection vector $b = (1, 0)$ and reaction coefficient $c = 1$. For all triangles with just one outflow edge the inf-sup constant is practically equal one. Unfortunately, the results show a clear degeneration of stability for all triangles with two outflow edges.



Values of the inf-sup constant β_h for the rotated unit triangle of order $p = 2$, $b = (1, 0)$, $c = 1$, and test space constisting of polynomials of order $p + 1$.

Outline

1 The Convection-Reaction Problem

2 Discrete Stability Proof

3 Convergence of Fields and Traces

4 Conclusions

Convergence of Fields

Once we have established the stability for fields,

$$\gamma_h \|u_h\| \le \sup_{v_h \in V_h^0} \frac{(u_h, A^* v_h)}{\|v_h\|_V} \tag{3.9}$$

where V_h^0 is the space of weakly conforming test functions,

$$V_h^0 := \{ v_h \in V_h \;:\; \langle \hat{w}_h, v_h \rangle_{\Gamma_h} = 0 \quad \forall \hat{w}_h \in \hat{U}_h \} \,,$$

we can easily show the convergence of the fields, for both conforming and non-conforming versions of the method.

$$\begin{aligned}
\|u - u_h\| &\le \|u - w_h\| + \|w_h - u_h\| \\
&\le \|u - w_h\| + \gamma_h^{-1} \sup_{v_h \in V_h^0} \frac{(w_h - u_h, A^* v_h)}{\|v_h\|_V} && \text{(condition (3.9))} \\
&= \|u - w_h\| + \gamma_h^{-1} \sup_{v_h \in V_h^0} \frac{(w_h - u_h, A^* v_h) + \langle \hat{w}_h - \hat{u}_h, v_h \rangle_{\Gamma_h}}{\|v_h\|_V} && (\langle \hat{w}_h - \hat{u}_h, v_h \rangle_{\Gamma_h} = 0,\ v_h \in V_h^0) \\
&\le \|u - w_h\| + \gamma_h^{-1} \sup_{v_h \in V_h} \frac{(w_h - u_h, A^* v_h) + \langle \hat{w}_h - \hat{u}_h, v_h \rangle_{\Gamma_h}}{\|v_h\|_V} && \text{(supremum taken over a bigger set)} \\
&\le \|u - w_h\| + \gamma_h^{-1} \sup_{v_h \in V_h} \frac{(w_h - u, A^* v_h) + \langle \hat{w}_h - \hat{u}, v_h \rangle_{\Gamma_h}}{\|v_h\|_V} && \text{(Galerkin orthogonality)} \\
&\le (1 + \gamma_h^{-1}) \|u - w_h\| + \gamma_h^{-1} \sup_{v_h \in V_h} \frac{\langle \hat{w}_h - \hat{u}, v_h \rangle_{\Gamma_h}}{\|v_h\|_V}
\end{aligned}$$

where $w_h, \hat{w}_h$ are arbitrary discrete field and trace.

Convergence of Fields

Note that, for the non-conforming version, the duality pairing has to be understood in the discrete sense,

$$\langle \hat{w}_h, v_h \rangle_{\Gamma_h} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} b_n \hat{w}_h v_h$$

and it makes sense only for discrete test functions v_h. Once we use the Galerkin orthogonality, it is replaced with the actual duality pairing, provided we assume that $\hat{w}_h$ comes from the conforming subspace $\hat{U}_h^c$ of space $\hat{U}_h$ of non-conforming traces. We can follow with the estimate,

$$\sup_{v_h \in V_h} \frac{\langle \hat{w}_h - \hat{u}, v_h \rangle_{\Gamma_h}}{\|v_h\|_V} \leq \sup_{v \in V} \frac{\langle \hat{w}_h - \hat{u}, v \rangle_{\Gamma_h}}{\|v\|_V} = \|\hat{u} - \hat{w}_h\|_E$$

where $\|\cdot\|_E$ is the minimum energy extension norm. This leads to the a-priori error estimate:

$$\|u - u_h\| \leq (1 + \gamma_h^{-1}) \inf_{w_h \in U_h} \|u - w_h\| + \gamma_h^{-1} \inf_{\hat{w}_h \in \hat{U}_h^c} \|\hat{u} - \hat{w}_h\|_E \,.$$

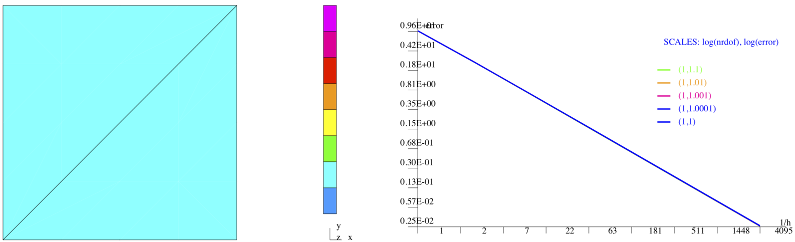
For the non-conforming version, given a sufficient regularity of exact trace $\hat{u}$, we can attempt to estimate the best approximation error in the discrete dual seminorm,

$$\inf_{\hat{w}_h \in \hat{U}_h} \underbrace{\sum_K \sup_{v_h \in V_h(K)} \left(\frac{\int_{\partial K} b_n (\hat{u} - \hat{w}_h) v_h}{\|v_h\|_{V(K)}} \right)^2}_{|\hat{u} - \hat{w}_h|^2_{V_h'}} \,. \tag{3.10}$$

We have more discrete traces $\hat{w}_h$ to approximate with, so the best approximation error should be smaller.

Convergence of Fields

The results above show that the convergence of fields should not be affected by the loss of stability for traces in the minimum energy extension norm discussed next. In order to verify the assertion, we have run an example with a smooth solution $u = 1 + x^3 + y^3$ with a constant advection field $b = (1, 1.1), (1, 1.01), (1, 1.001), (1, 1.0001)$ and the degenerated case $b = (1, 1)$. Note that in the last case, the diagonal edges are excluded. We investigate the convergence on a sequence of globally refined meshes starting with the 2 elements mesh shown below. The convergence curves are sitting literally on top of each other.



Left: initial mesh. Right: h-convergence results for a smooth exact solution and $b = (1, 1.1), (1, 1.01), (1, 1.001), (1, 1.0001), (1, 1)$.

Convergence of Traces

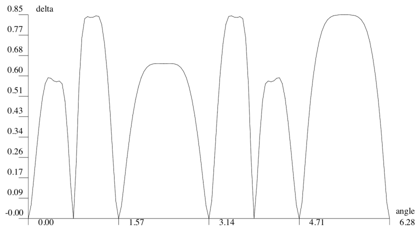
Can we proceed with the Brezzi argument to control traces? The discrete inf-sup constant of interest is defined as follows,

$$\sup_{v \in V_h(K)} \frac{|\int_{\partial K} b_n u v|}{\|v\|_{H_{A^*}(K)}} \geq \delta \|u\|_E \quad u \in \mathcal{P}_c^p(\partial K) \tag{3.11}$$

where

$$\|v\|^2_{H_{A^*}(K)} = \int_K |A^* v|^2 + |v|^2, \qquad \|u\|^2_E = \min_{U|_{\partial K} = u} \int_K |AU|^2 + |U|^2 \,.$$

Figure on the right presents values of constant δ for the unit triangle rotated by an angle $\alpha \in [0, 2\pi]$, $p = 2$,. The minimum energy extensions have been computed with polynomials of order $p + dp, dp = 5$. And the same results hold for element size $h = 0.1, 0.001, 0.0001$. The constant degenerates to zero whenever one of the triangle edges becomes parallel to the advection vector. Clearly, to secure a robust convergence of traces, we have to impose a minimum angle condition on element edges with respect to the advection vector.



Constant δ for a rotated unit triangle, $b = (1, 0)$, $c = 1$ and $p = 2$.

Convergence of Traces

With the inf-sup constant δ_h in place, we can claim the convergence result for the conforming traces. This follows now directly from the Babuška - Brezzi Theorem. We can reason as follows,

$$
\begin{aligned}
\|\hat{u} - \hat{u}_h\|_E &\le \|\hat{u} - \hat{w}_h\|_E + \|\hat{w}_h - \hat{u}_h\|_E \\
&\le \|\hat{u} - \hat{w}_h\|_E + \delta_h^{-1} \sup_{v_h \in V_h} \frac{\langle \hat{w}_h - \hat{u}_h \rangle_{\Gamma_h}}{\|v_h\|_V} \\
&\le \|\hat{u} - \hat{w}_h\|_E + \delta_h^{-1} \sup_{v_h \in V_h} \frac{(w_h - u_h, A_h^* v_h) + \langle \hat{w}_h - \hat{u}_h \rangle_{\Gamma_h} - (w_h - u_h, A_h^* v_h)}{\|v_h\|_V} \\
&\le \|\hat{u} - \hat{w}_h\|_E + \delta_h^{-1} \sup_{v_h \in V_h} \frac{(w_h - u_h, A_h^* v_h) + \langle \hat{w}_h - \hat{u}_h \rangle_{\Gamma_h}}{\|v_h\|_V} + \delta_h^{-1} \|w_h - u_h\| \\
&\le \|\hat{u} - \hat{w}_h\|_E + \delta_h^{-1}(1 + \gamma_h^{-1}) \sup_{v_h \in V_h} \frac{(w_h - u_h, A_h^* v_h) + \langle \hat{w}_h - \hat{u}_h \rangle_{\Gamma_h}}{\|v_h\|_V} \\
&\le \|\hat{u} - \hat{w}_h\|_E + \delta_h^{-1}(1 + \gamma_h^{-1}) \sup_{v_h \in V_h} \frac{(w_h - u, A_h^* v_h) + \langle \hat{w}_h - \hat{u} \rangle_{\Gamma_h}}{\|v_h\|_V} \\
&\le (1 + \delta_h^{-1}(1 + \gamma_h^{-1})) \|\hat{u} - \hat{w}_h\|_E + \delta_h^{-1}(1 + \gamma_h^{-1}) \|u - w_h\| \,.
\end{aligned}
$$

As $w_h, \hat{w}_h$ above are arbitrary functions, we obtain,

$$
\|\hat{u} - \hat{u}_h\|_E \le (1 + \delta_h^{-1}(1 + \gamma_h^{-1})) \inf_{\hat{w}_h} \|\hat{u} - \hat{w}_h\|_E + \delta_h^{-1}(1 + \gamma_h^{-1}) \inf_{w_h} \|u - w_h\| \,.
$$

The result above holds for non-conforming traces as well, provided we replace the minimum energy extension norm with the discrete dual seminorm (3.10). Constant δ_h is then equal one by definition.

Outline

1 The Convection-Reaction Problem

2 Discrete Stability Proof

3 Convergence of Fields and Traces

4 Conclusions

Closing Remarks

- The local construction of Fortin operator (sufficient but not necessary for global stability) fails to show robust (in rotation angle) stability for both polynomial and composite polynomial test spaces.
- The global stability analysis points to the need of using the composite polynomial enriched test space.
- We still have not been able, though, to illustrate the necessity of composite polynomial test space with a numerical example showing a failure of the original DPG method using the polynomial test space only.

So the jury is still out.

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