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IDENTIFYING UNKNOWN TERMS IN HYPERBOLIC AND PARABOLIC EQUATIONS

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ABSTRACT. We recover unknown source terms in nonlinear hyperbolic differential equations and in nonlinear parabolic integro-differential equations in one space variable under the assumption of knowing a first integral (in the hyperbolic case) or the value of the solution at a point inside the domain (in the parabolic case). For this class of problems we prove existence results in classes of smooth solutions. Moreover, for linear hyperbolic and parabolic differential equations in one space variable we recover some characteristic parameters.

1 Introduction

This paper has a twofold character: on one hand, it is intended to be introductory to people working in PDE's, while, on the other hand, it develops some ideas for ODE's [11] related to recovering a central force field accounting for the motion of a given material point along a known trajectory.
The paper is divided into three distinct parts dealing with hyperbolic and parabolic *identification problems.*

The first part deals with nonlinear first- and second-order hyperbolic equations in one space variable, when the right-hand side f is *unknown* and depends, in a semilinear fashion, on the unknown u. The *additional information* necessary to determine f consists of a sort of "first integral" involving u only. In other words, the solution u is required to satisfy an additional zeroth- or first-order equation. In particular, when the equation dealt with is of the second-order and u denotes the displacement from a reference position, the first-order differential equation stands for a prescribed mode of vibrating of the string under consideration. Consequently, we are required to recover a force field f in order that the string may vibrate as prescribed in advance.

On the other part, when a first-order hyperbolic equation is dealt with and a zeroth-order additional information is available, this is equivalent to assuming that the function u is itself *known* and we want to recover an unknown right-hand side f depending on three *independent* variables (t, x, u).

The second part of the paper is concerned with the determination of a right-hand side f in a *quasilinear* parabolic integro-differential equation in one space variable.

The third part is devoted to recovering the (constant) propagation velocity and the conductivity in the model one-dimensional wave- and heat- equations.

Finally, we have collected in the bibliography a list of reference books on Inverse Problems that can be useful for people who need to be acquainted with such fascinating problems.

2 The first hyperbolic identification problem

We consider here the first-order semilinear differential problem

$$D_t u(t,x) - D_x u(t,x) = f(t,x,u(t,x)), \qquad (t,x) \in (0,T) \times (0,l), \quad (2.1)$$

$$u(0,x) = u_0(x), \qquad x \in [0,l], \quad (2.2)$$

$$u(t,l) = g(t), \qquad t \in [0,T]. \quad (2.3)$$

As is well known, problem (2.1)-(2.3) admits a unique solution when $u_0 \in C^1([0,l]), g \in C^1([0,T]), g(0) = u_0(0), f, D_t f, D_u f \in C([0,T] \times [0,l] \times \mathbf{R})$ and $D_u f \in C_b([0,T] \times [0,l] \times \mathbf{R})$, $C_b([0,T] \times [0,l] \times \mathbf{R})$ denoting the Banach space of all functions that are continuous and bounded on $[0,T] \times [0,l] \times \mathbf{R}$.
Suppose now that f itself is *unknown*. To recover f we assume to know a first integral of equation (1.1)

$$h(t,x,u(t,x)) = 0, \qquad (t,x) \in (0,T) \times (0,l), \tag{2.4}$$

where $h \in C^2([0,T] \times [0,l] \times \mathbf{R})$.
Of course, for the solution u to (2.1)-(2.4) to exist the following conditions must be fulfilled:

$$h(0,x,u_0(x)) = 0, \quad x \in [0,l], \quad h(t,l,g(t)) = 0, \quad t \in [0,T]. \tag{2.5}$$

Differentiating both sides in (2.4) with respect to t and x, we find the equations

$$D_t h(t,x,u(t,x)) + D_u h(t,x,u(t,x)) D_t u(t,x) = 0, \qquad (t,x) \in (0,T) \times (0,l), \tag{2.6}$$

$$D_x h(t,x,u(t,x)) + D_u h(t,x,u(t,x)) D_x u(t,x) = 0, \qquad (t,x) \in (0,T) \times (0,l). \tag{2.7}$$

Subtracting member by member (2.6) and (2.7) and using (2.1), we find that f must satisfy the equation

$$D_t h(t,x,u(t,x)) - D_x h(t,x,u(t,x)) + D_u h(t,x,u(t,x)) \cdot f(t,x,u(t,x)) = 0, \qquad (t,x) \in (0,T) \times (0,l). \tag{2.8}$$

Assume now

$$|D_u h(t,x,u)| \geq m > 0, \qquad (t,x,u) \in [0,T] \times [0,l] \times \mathbf{R}. \tag{2.9}$$

Then a solution to (2.8) can be *explicitly* computed in terms of the data by

$$f(t,x,u) = \frac{D_t h(t,x,u) - D_x h(t,x,u)}{D_u h(t,x,u)} := f_0(t,x,u), \qquad (t,x,u) \in [0,T] \times [0,l] \times \mathbf{R}. \tag{2.10}$$

Of course f belongs to $C^1([0,T]\times[0,l]\times\mathbf{R})$, but it is *not* unique, since all the functions

$$f(t,x,u) = f_0(t,x,u)\psi(h(t,x,u)), \qquad (t,x,u) \in [0,T]\times[0,l]\times\mathbf{R}. \quad (2.11)$$

with $\psi \in C(\mathbf{R})$, $\psi(0) = 1$, solve our identification problem (2.1)-(2.4). Moreover, in order to have $D_u f \in C_b([0,T]\times[0,l]\times\mathbf{R})$ we need the assumption

$$D_u h(D_u D_t h - D_u D_x h) - D_u^2 h(D_t h - D_x h) \in C_b([0,T]\times[0,l]\times\mathbf{R}). \quad (2.12)$$

REMARK 2.1 *If we add to condition (2.9) the natural requirements*

$$h(t,x,-\infty) < 0,\ h(t,x,+\infty) > 0, \quad (t,x) \in [0,T]\times[0,l], \quad (2.13)$$

it follows that giving the first integral (2.4) is equivalent to prescribing the solution u in an implicit form.

We could reformulate our identification problem by prescribing u and requiring to recover f from equation (2.1). However, in this case we could *uniquely* determine only the composite function $(t,x) \to f(t,x,u(t,x))$ and not f as a function of three *independent* variables.

Conversely, let u solve problem (2.1)-(2.3) with f being defined by (2.10), where $h \in C^1([0,T]\times[0,l]\times\mathbf{R})$ is a given function satisfying (2.5), (2.9). Let us introduce the function

$$v(t,x) = h(t,x,u(t,x)), \qquad [0,T]\times[0,l]. \quad (2.14)$$

It is easy to check that v satisfies the first-order equation

$$D_t v(t,x) - D_x v(t,x) = 0, \qquad (t,x) \in (0,T)\times(0,l). \quad (2.15)$$

Moreover, from (2.5) we deduce that v satisfies the homogeneous conditions

$$v(0,x) = 0, \quad x \in [0,l], \quad v(t,l) = 0, \quad t \in [0,T]. \quad (2.16)$$

Since all the solutions to (2.15) are of the form

$$v(t,x) = \varphi(t+x), \qquad t \in [0,T],\ x \in [0,l], \tag{2.17}$$

with $\varphi \in C^1([0,T+l])$, we conclude that

$$v(t,x) = 0, \qquad t \in [0,T],\ x \in [0,l]. \tag{2.18}$$

Consequently, u satisfies condition (2.4).

3 The second hyperbolic identification problem

Let us consider the second-order semilinear initial and boundary value problem

$$D_t^2 w(t,x) - D_x^2 w(t,x) = f(t,x,D_t w(t,x) + D_x w(t,x)), \qquad (t,x) \in (0,T)\times(0,l), \tag{3.1}$$

$$w(0,x) = w_0(x), \qquad D_t w(0,x) = w_1(x), \qquad x \in [0,l], \tag{3.2}$$

$$w(t,0) = g_0(t), \quad D_t w(t,l) + D_x w(t,l) = g_1(t), \quad t \in [0,T], \tag{3.3}$$

where function $f \in C([0,T]\times[0,l]\times\mathbf{R})$ has to be determined *explicitly* from (w_0, w_1, g_0, g_1) and the additional information

$$h(t,x,D_t w(t,x) + D_x w(t,x)) = 0, \quad (t,x) \in (0,T)\times(0,l). \tag{3.4}$$

We make the following assumptions:

$$\begin{aligned} &w_0 \in C^2([0,l]), \quad w_1 \in C^1([0,l]), \quad g_0 \in C^2([0,T]),\\ &g_1 \in C^1([0,T]), \quad h \in C^2([0,T]\times[0,l]\times\mathbf{R}), \end{aligned} \tag{3.5}$$

$$|D_u h(t,x,u)| \ge m > 0, \ (t,x,u) \in [0,T]\times[0,l]\times\mathbf{R}, \tag{3.6}$$

$$\begin{aligned} &w_0(0) = g_0(0), \quad w_1(0) = g_0'(0), \quad w_1(l) + w_0'(0) = g_1(0),\\ &h(t,l,g_1(t)) = 0, \qquad t \in [0,T]. \end{aligned} \tag{3.7}$$

It is immediate to observe that the function

$$u(t,x) = D_t w(t,x) + D_x w(t,x), \qquad (t,x) \in (0,T)\times(0,l), \tag{3.8}$$

satisfies equations (2.1)–(2.4) with

$$u_0(x) = w_1(x) + w_0'(x), \quad x \in [0,l], \quad g(t) = g_1(t), \quad t \in [0,T]. \tag{3.9}$$

Then an admissible f is given by

$$f(t,x,u) = \frac{D_t h(t,x,u) - D_x h(t,x,u)}{D_u h(t,x,u)} := f_0(t,x,u), \qquad (t,x,u) \in [0,T] \times [0,l] \times \mathbf{R}. \tag{3.10}$$

Of course, f is *not* unique, as we have already noted in section 2. Once we have determined the pair (u,f), we observe that our original unknown w must satisfy the first-order differential problem

$$D_t w(t,x) + D_x w(t,x) = u(t,x), \ (t,x) \in (0,T) \times (0,l), \tag{3.11}$$

$$w(0,x) = w_0(x), \qquad t \in [0,l], \tag{3.12}$$

$$w(t,0) = g_0(t), \qquad t \in [0,T]. \tag{3.13}$$

Consequently, using the method of characteristics, we obtain the following representation for w:

$$w(t,x) = \begin{cases} w_0(x-t) + \int_0^t u(s, x-t+s)\,ds, & 0 \le t \le x \le l, \\ g_0(t-x) + \int_0^x u(t-x+y, y)\,dy, & 0 \le x \le t \le x+T. \end{cases} \tag{3.14}$$

Our problem is now fully solved.

4 The third hyperbolic identification problem

Let us consider the second-order quasilinear initial and boundary value problem

$$D_t^2 u(t,x) - a(t,x,u(t,x),D_t u(t,x),D_x u(t,x))D_x^2 u(t,x)$$
$$= f(t,x,u(t,x),D_t u(t,x),D_x u(t,x)), \quad (t,x) \in (0,T) \times \mathbf{R}, \tag{4.1}$$

$$u(0,x) = u_0(x), \qquad D_t u(0,x) = u_1(x), \qquad x \in \mathbf{R}, \tag{4.2}$$

where function $f \in C([0,T] \times \mathbf{R}^4)$ has to be determined *explicitly* from (a, u_0, u_1) and the additional information

$$h(t,x,u(t,x),D_t u(t,x),D_x u(t,x)) = 0, \quad (t,x) \in (0,T) \times \mathbf{R}. \tag{4.3}$$

We make the following assumptions:

$$a \in C_b^2([0,T] \times \mathbf{R}^4), \quad a(t,x,u,p,q) \geq \mu > 0, \quad (t,x,u,p,q) \in [0,T] \times \mathbf{R}^4, \tag{4.4}$$

$$u_0 \in C^3(\mathbf{R}), \quad u_1 \in C^2(\mathbf{R}), \quad h \in C^2([0,T] \times \mathbf{R}^4), \tag{4.5}$$

$$h(0,x,u_0(x),u_1(x),u_0'(x)) = 0, \quad x \in \mathbf{R}, \tag{4.6}$$

$$|D_u h(t,x,u,p,q)| \geq m > 0, \quad (t,x,u,p,q) \in [0,T] \times \mathbf{R}^4. \tag{4.7}$$

Differentiating with respect to t and x both sides in (4.3) and omitting variables, we easily deduce the equations

$$D_t h + (D_u h) D_t u + (D_p h) D_t^2 u + (D_q h) D_t D_x u = 0, \quad \text{in } (0,T) \times \mathbf{R}, \tag{4.8}$$

$$D_x h + (D_u h) D_x u + (D_p h) D_t D_x u + (D_q h) D_x^2 u = 0, \quad \text{in } (0,T) \times \mathbf{R}. \tag{4.9}$$

Multiply then equations (4.8) and (4.9) by $D_p h$ and $D_q h$, respectively, and subtract member by member. We find the equation

$$\begin{aligned}(D_p h) D_t h - (D_q h) D_x h + (D_p h)(D_u h) D_t u - (D_q h)(D_u h) D_x u \\ + (D_p h)^2 D_t^2 u - (D_q h)^2 D_x^2 u = 0, \quad \text{in } (0,T) \times \mathbf{R}.\end{aligned} \tag{4.10}$$

Assume now that h is not a general function, but satisfies the first-order equation

$$[D_q h(t,x,u,p,q)]^2 = a(t,x,u,p,q)[D_p h(t,x,u,p,q)]^2, \quad (t,x,u,p,q) \in (0,T) \times \mathbf{R}^4, \tag{4.11}$$

as well as the condition

$$|D_p h(t,x,u,p,q)| \geq m > 0, \quad (t,x,u,p,q) \in (0,T) \times \mathbf{R}^4. \tag{4.12}$$

Then from (4.1) and (4.11) we deduce that f satisfies, in $(0,T)\times\mathbf{R}^4$, the equation

$$(D_ph)D_th-(D_qh)D_xh+(D_ph)(D_uh)D_tu \\ -(D_qh)(D_uh)D_xu+(D_ph)^2f=0. \quad (4.13)$$

As a consequence, by virtue of (4.12) an admissible function f is given, in $(0,T)\times\mathbf{R}^4$, by

$$f=(D_ph)^{-2} \\ \cdot\{-(D_ph)D_th+(D_qh)D_xh-p(D_ph)(D_uh)+q(D_qh)(D_uh)\}. \\ (4.14)$$

Note that, by virtue of assumptions (4.5), (4.16), (4.17) function f belongs to $C^1([0,T];\mathbf{R}^4)$.

Now we should show that any solution to (4.1)-(4.3) with f being defined by (4.14) necessarily satisfies (4.3). Unfortunately, we cannot deal with general functions a. We will limit ourselves to considering functions a with *separated* variables p and q, i.e. functions of the form

$$a(t,x,u,p,q)=[b_1(t,x,u,p)]^2[b_2(t,x,u,q)]^2, \\ (t,x,u,p,q)\in[0,T]\times\mathbf{R}^4, \quad (4.15)$$

where b_1 and b_2 enjoy the properties

$$b_k\in C^1([0,T]\times\mathbf{R}^3), \qquad (t,x,u,z)\in[0,T]\times\mathbf{R}^3,\ k=1,2, \quad (4.16)$$

$$b_k(t,x,u,z)\ge\mu>0, \qquad (t,x,u,z)\in[0,T]\times\mathbf{R}^3,\ k=1,2. \quad (4.17)$$

Moreover, for the sake of simplicity we will limit ourselves to dealing with functions h satisfying the equations

$$D_ph(t,x,u,p,q)=b_1(t,x,u,p)b_2(t,x,u,q)D_qh(t,x,u,p,q), \\ (t,x,u,p,q)\in(0,T)\times\mathbf{R}^4, \quad (4.18)$$

$$h(0,x,u_0(x),u_1(x),u_0'(x))=0, \qquad x\in\mathbf{R}. \quad (4.19)$$

REMARK 4.1 *The procedure described allows to determine an* explicit *expression for f in a very simple way, but, of course, it does not guarantee that our identification problem is solvable. To be sure that our treatment is reasonable we need to show that also the "converse part" works as well, at least in some specific cases.*

REMARK 4.2 *If we solved first the* implicit *problem* (4.3) *under the basic condition* (4.12) *guaranteeing that equation* (4.3) *can be set in a* normal *form with respect to $D_t u$ and under the only initial condition*

$$u(0,x) = u_0(x), \qquad x \in \mathbf{R}, \tag{4.20}$$

we could drop out equation (4.19). *However, the expression of f could not be determined* explicitly *in terms of the data, but only through the formula $f = D_t^2 u - aD_x^2 u$, u being the solution to problem* (4.3), (4.20).
We note that under this point of view the type of equation (4.1) *would play no role. So we could replace operator $f = D_t^2 u - aD_x^2 u$ by* any *nonlinear second-order differential operator $N(t,x,u,D_t u, D_x u, D_t^2 u, D_t D_x u, D_x^2 u)$ and recover f by simply computing such an operator at u. Yet, observe that, in general, u could not be derived in a* closed form *in terms of the data.*
We conclude this remark by observing that in several applications we need an explicit *expression for f, so our previous procedure seems to be appropriate for hyperbolic equations, whenever we are ready to accept some restriction on the form of the first integral h.*

To determine the form of a general solution to the first-order equation (4.18) observe first that the family of functions

$$\begin{aligned} h(t,x,u,p,q,\alpha) = \alpha \int_{u_1(x)}^{p} & b_2(t,x,u,\xi)\,\mathrm{d}\xi \\ & + \alpha \int_{u_0'(x)}^{q} [b_1(t,x,u,\eta)]^{-1}\,\mathrm{d}\eta - \rho(\alpha), \end{aligned} \tag{4.21}$$

depending on the real parameter α and the arbitrary function $\rho : \mathbf{R} \to \mathbf{R}$, defines [22] a *complete integral* of equation (4.19). We want now to eliminate α from (4.21) and the equation $D_\alpha h(t,x,u,p,q,\alpha) = 0$, i.e.

$$\int_{u_1(x)}^{p} b_2(t,x,u,\xi)\,\mathrm{d}\xi + \int_{u_0'(x)}^{q} [b_1(t,x,u,\eta)]^{-1}\,\mathrm{d}\eta - \rho'(\alpha) = 0. \tag{4.22}$$

For this purpose assume that ρ' is invertible and its inverse function ψ belongs to $C^1(\mathbf{R};\mathbf{R})$. Then equation (4.22) yields

$$\alpha = \psi\left(\int_{u_1(x)}^{p} b_2(t,x,u,\xi)\,\mathrm{d}\xi + \int_{u_0'(x)}^{q} [b_1(t,x,u,\xi)]^{-1}\,\mathrm{d}\eta\right) \tag{4.23}$$

Hence, function h we look for is given by

$$\begin{aligned} h(t,x,u,p,q) =& \zeta\left(\int_{u_1(x)}^{p} b_2(t,x,u,\xi)\,\mathrm{d}\xi \right.\\ &\left. + \int_{u_0'(x)}^{q} [b_1(t,x,u,\eta)]^{-1}\,\mathrm{d}\eta\right), \end{aligned} \tag{4.24}$$

where

$$\zeta(r) = r\psi(r) - \rho(\psi(r)), \qquad r \in \mathbf{R}. \tag{4.25}$$

Observe that h satisfies condition (4.19) if and only if

$$\zeta(0) = 0. \tag{4.26}$$

As a consequence, f admits the following representation (cf. (4.14))

$$\begin{aligned} f(t,x,u,p,q) &= [D_p h(t,x,u,p,q)]^{-2} D_q h(t,x,u,p,q) \\ &\cdot \Big\{-b_1(t,x,u,p) b_2(t,x,u,p,q) D_t h(t,x,u,p,q) \\ &+ D_x h(t,x,u,p,q) + [-p b_1(t,x,u,p) b_2(t,x,u,q) + q] \\ &\cdot D_u h(t,x,u,p,q)\Big\} \end{aligned}$$

$$
\begin{aligned}
&= [b_1(t,x,u,p)b_2(t,x,u,q)]^{-2} \\
&\cdot \left\{ \int_{u_1(x)}^{p} [-b_1(t,x,u,p)b_2(t,x,u,q)D_t b_2(t,x,u,\xi) \right. \\
&+ D_x b_2(t,x,u,\xi)]\,\mathrm{d}\xi + \int_{u_0'(x)}^{q} [b_1(t,x,u,p)]^{-2}[b_1(t,x,u,p) \\
&\cdot b_2(t,x,u,q)D_t b_1(t,x,u,\eta) - D_x b_1(t,x,u,\eta)]\,\mathrm{d}\eta \\
&\cdot [-p + q b_1(t,x,u,p)b_2(t,x,u,q)] \int_{u_1(x)}^{p} D_u b_2(t,x,u,\xi)\,\mathrm{d}\xi \\
&+ [p - q b_1(t,x,u,p)b_2(t,x,u,q)] \int_{u_0'(x)}^{q} [b_1(t,x,u,\xi)]^{-2} \\
&\cdot D_u b_1(t,x,u,\eta)\,\mathrm{d}\eta - u_1'(x)b_2(t,x,u,u_1(x)) \\
&\left. - u_0''(x)[b_1(t,x,u,u_0'(x))]^{-1} \right\}.
\end{aligned} \tag{4.27}
$$

Conversely, assume that u is the solution to the *direct* problem (4.1)-(4.3) with f being defined by (4.27).
It is an easy task to check (cf. (4.27)) that the function

$$v(t,x) = h(t,x,u(t,x),D_t u(t,x),D_x u(t,x)), \quad (t,x) \in (0,T) \times \mathbf{R}. \tag{4.28}$$

solves the equations

$$D_t v(t,x) - b_1(t,x,u(t,x),D_t u(t,x))b_2(t,x,u(t,x),D_x u(t,x))$$

$$\cdot D_x v(t,x) = 0, \qquad (t,x) \in (0,T) \times \mathbf{R}, \tag{4.29}$$

$$v(0,x) = 0, \qquad x \in \mathbf{R}. \tag{4.30}$$

We are now in a position to prove the following "converse" result stating that the identification problem (4.1)-(4.3) can be solved provided the *direct* problem (4.1), (4.2) with f defined by (4.27) is solvable in a suitable functional space. In other words, the identification problem is solvable if a specific associated direct problem is.

THEOREM 4.1 *Let problem* (4.1), (4.2) *with* a *and* f *defined by* (4.15) *and* (4.27), *respectively, admit a unique solution* $u \in C^1([0,T];L^2(\mathbf{R})) \cap C([0,T];H^1(\mathbf{R}))$ *such that the function*

$$(t,x) \longrightarrow D_x[b_1(t,x,u(t,x),D_t u(t,x))b_2(t,x,u(t,x),D_x u(t,x))]$$

is bounded on $[0,T]\times\mathbf{R}$. *Then* u *satisfies also equation* (4.3).

PROOF. Multiply both sides in (4.28) by $2v(t,x)$ and integrate with respect to x over $\mathbf{R}$. We easily find the equations

$$\begin{aligned} 0 &= D_t\int_{\mathbf{R}} v(t,x)^2\,\mathrm{d}x - \int_{\mathbf{R}} b_3(t,x)D_x[v(t,x)^2]\,\mathrm{d}x \\ &= D_t\int_{\mathbf{R}} v(t,x)^2\,\mathrm{d}x + \int_{\mathbf{R}} D_x b_3(t,x)v(t,x)^2\,\mathrm{d}x, \qquad t\in[0,T], \end{aligned} \tag{4.31}$$

where we have set

$$b_3(t,x) = b_1(t,x,u(t,x),D_tu(t,x))b_2(t,x,u(t,x),D_xu(t,x)).$$

From our assumption on b_3 and (4.31), (4.29) we easily derive the differential inequality and the initial condition

$$D_t\int_{\mathbf{R}} v(t,x)^2\,\mathrm{d}x \le M\int_{\mathbf{R}} v(t,x)^2\,\mathrm{d}x, \qquad t\in[0,T], \tag{4.32}$$

$$\int_{\mathbf{R}} v(0,x)^2\,\mathrm{d}x = 0. \tag{4.33}$$

Hence, we easily derive the relations

$$\int_{\mathbf{R}} v(t,x)^2\,\mathrm{d}x = 0,\ t\in[0,T] \iff v(t,x)=0,\ (t,x)\in[0,T]\times[0,l]. \tag{4.34}$$

As a consequence, u satisfies equation (4.3). □

5 The first parabolic identification problem

In this section we are going to deal with the problem of identifying a unknown right-hand side *independent of* x in the following nonlinear integro-differential parabolic equation

$$\begin{aligned} &D_tu(t,x) - a\Big(\int_{-\infty}^{+\infty} |D_xu(t,y)|^2\,\mathrm{d}y\Big)D_x^2u(t,x) \\ &= f\Big(t,\int_{-\infty}^{+\infty} |D_xu(t,y)|^2\,\mathrm{d}y\Big), \qquad (t,x)\in(0,T)\times\mathbf{R}, \end{aligned} \tag{5.1}$$

subject to the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbf{R}. \tag{5.2}$$

To recover the unknown function $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ we prescribe the additional information

$$u(t,0) = z_0(t), \qquad t \in [0,T]. \tag{5.3}$$

As far as the data a, u_0 and z_0 are concerned we will assume

$$a \in C_b^1(\mathbf{R}), \qquad a(p) \geq \mu > 0, \qquad p \in \mathbf{R}, \tag{5.4}$$

$$u_0 \in H^{1+\varepsilon}(\mathbf{R}) \text{ for some } \varepsilon \in \mathbf{R}_+, \quad z_0 \in H^1((0,T)), \quad u_0(0) = z_0(0), \tag{5.5}$$

where $H^s(\Omega)$, $s \in \mathbf{R}_+$, denotes the usual Sobolev space related to $L^2(\Omega)$, Ω being an open domain in $\mathbf{R}^n$.

To solve the *identification problem* (5.1)-(5.3) we introduce the auxiliary unknown

$$v(t,x) = D_x u(t,x) \quad \Longleftrightarrow \quad u(t,x) = z_0(t) + \int_0^x v(t,\eta)\,\mathrm{d}\eta. \tag{5.6}$$

Differentiating equation (5.1) with respect to x, we immediately deduce that v solves the following nonlinear parabolic integro-differential Cauchy problem

$$D_t v(t,x) - a\left(\int_{-\infty}^{+\infty} |v(t,y)|^2\,\mathrm{d}y\right) D_x^2 v(t,x) = 0, \qquad (t,x) \in (0,T) \times \mathbf{R}, \tag{5.7}$$

$$v(0,x) = u_0'(x), \qquad x \in \mathbf{R}. \tag{5.8}$$

Taking the Fourier transforms of the left- and right-sides in (5.7), (5.8), we obtain the following Cauchy problem for an *ordinary* differential equation

$$D_t \hat{v}(t,\xi) + \xi^2 a(\|\hat{v}(t,\cdot)\|_{L^2(\mathbf{R})}^2)\hat{v}(t,\xi) = 0, \quad (t,\xi) \in (0,T) \times \mathbf{R}, \tag{5.9}$$

$$\hat{v}(0,\xi) = i\xi\hat{u}_0(\xi), \qquad \xi \in \mathbf{R}. \quad (5.10)$$

It is immediate to check that $\hat{v}$ solves the integral equation

$$\hat{v}(t,\xi) = i\xi\hat{u}_0(\xi)\exp\Big[-\xi^2\int_0^t a(\|\hat{v}(s,\cdot)\|^2_{L^2(\mathbf{R})})\,\mathrm{d}s\Big], \qquad (t,\xi) \in (0,T)\times\mathbf{R}. \quad (5.11)$$

Integrating the squares of both sides of (5.11) over $\mathbf{R}$ and setting

$$\varphi(t) = \|\hat{v}(t,\cdot)\|^2_{L^2(\mathbf{R})}, \qquad t \in (0,T). \quad (5.12)$$

we easily deduce the nonlinear integral equation

$$\varphi(t) = \int_{-\infty}^{+\infty} \xi^2|\hat{u}_0(\xi)|^2 \exp\Big[-2\xi^2\int_0^t a(\varphi(s,\cdot))\,\mathrm{d}s\Big]\,\mathrm{d}\xi =: N(\varphi)(t), \qquad t \in (0,T). \quad (5.13)$$

Observe now that

$$N(\varphi)(t) \le \int_{-\infty}^{+\infty} \xi^2|\hat{u}_0(\xi)|^2\,\mathrm{d}\xi = \|u_0'\|^2_{L^2(\mathbf{R})}, \qquad t \in (0,T),\ \varphi \in C_+([0,T]), \quad (5.14)$$

where $C_+([0,T]) = \{\psi \in C([0,T]) : \psi(t) \ge 0,\ t \in [0,T]\}$.
From (5.13), (5.14) we deduce that N maps $C_+([0,T])$ into the bounded cone of positive functions

$$K = \Big\{\varphi \in C_+([0,T]) : \|\varphi\|_{C_+([0,T])} \le \|u_0'\|^2_{L^2(\mathbf{R})}\Big\}. \quad (5.15)$$

In particular, N maps K into itself.
We now show that N is a contraction mapping on K. For this purpose

observe that

$$
\begin{aligned}
&|N(\varphi_2)(t) - N(\varphi_1)(t)| \\
\le\ & \int_{-\infty}^{+\infty} \xi^2 |\hat{u}_0(\xi)|^2 \Big| \exp\Big[-2\xi^2 \int_0^t a(\varphi_2(s,\cdot))\,\mathrm{d}s \Big] \\
& - \exp\Big[-2\xi^2 \int_0^t a(\varphi_1(s,\cdot))\,\mathrm{d}s \Big] \Big| \,\mathrm{d}\xi \\
\le\ & 2\int_0^t |a(\varphi_2(s)) - a(\varphi_1(s))|\,\mathrm{d}s \int_{-\infty}^{+\infty} \xi^4 |\hat{u}_0(\xi)|^2 \\
& \cdot \exp\Big[-2\xi^2 \min_{j=1,2} \Big(\int_0^t a(\varphi_j(s,\cdot))\,\mathrm{d}s \Big) \Big] \,\mathrm{d}\xi \\
\le\ & 2\|a'\|_{C([0,T])} \int_0^t |\varphi_2(s) - \varphi_1(s)|\,\mathrm{d}s \int_{-\infty}^{+\infty} \xi^{2+2\varepsilon} |\hat{u}_0(\xi)|^2 \xi^{2-2\varepsilon} \\
& \cdot \exp(-2\mu t \xi^2)\,\mathrm{d}\xi \\
\le\ & C(\varepsilon)\|a'\|_{C([0,T])} \int_0^t t^{-1+\varepsilon} |\varphi_2(s) - \varphi_1(s)|\,\mathrm{d}s \\
& \cdot \int_{-\infty}^{+\infty} \xi^{2+2\varepsilon} |\hat{u}_0(\xi)|^2 \,\mathrm{d}\xi \\
\le\ & C(\varepsilon)\|a'\|_{C([0,T])} \|u_0\|^2_{H^{1+\varepsilon}(\mathbf{R})} \int_0^t (t-s)^{-1+\varepsilon} |\varphi_2(s) - \varphi_1(s)|\,\mathrm{d}s, \\
& \qquad t \in (0,T),\ \varphi_1, \varphi_2 \in K. \quad (5.16)
\end{aligned}
$$

From (5.16) we easily deduce the estimate

$$
|N(\varphi_2)(t) - N(\varphi_1)(t)| \le C_1 \int_0^t (t-s)^{-1+\varepsilon} |\varphi_2(s) - \varphi_1(s)|\,\mathrm{d}s, \qquad t \in (0,T),\ \varphi_1, \varphi_2 \in K, \quad (5.17)
$$

where Γ denotes the Euler's gamma function.

By induction we can easily show that the iterates of N satisfy the integral inequalities

$$|N^m(\varphi_2)(t) - N^m(\varphi_1)(t)| \le \frac{[C_1\Gamma(\varepsilon)]^m}{\Gamma(m\varepsilon)}$$
$$\cdot \int_0^t (t-s)^{-1+m\varepsilon}|\varphi_2(s)-\varphi_1(s)|\,\mathrm{d}s, \quad t \in (0,T),\ \varphi_1,\varphi_2 \in K,\ m \in \mathbf{N}. \tag{5.18}$$

In particular, from (5.18) we derive the basic estimate

$$\|N^m(\varphi_2) - N^m(\varphi_1)\|_{C([0,T])}$$
$$\le \frac{[C_1T\Gamma(\varepsilon)]^m}{\Gamma(m\varepsilon+1)}\|\varphi_2 - \varphi_1\|_{C([0,T])},\ \varphi_1,\varphi_2 \in K,\ m \in \mathbf{N}. \tag{5.19}$$

Since $[C_1T\Gamma(\varepsilon)]^m[\Gamma(m\varepsilon+1)]^{-1} \to 0$ as $m \to +\infty$, from a well-known corollary of the Banach-Caccioppoli fixed-point theorem we conclude that equation (5.13), i.e. $\varphi = N(\varphi)$ admits a unique solution in K. Moreover, no other solution can exist in $C_+([0,T])$ owing to inequality (5.14).

From (5.11) we deduce that $\hat{v}$ admits the following representation in terms of φ:

$$\hat{v}(t,\xi) = i\xi\hat{u}_0(\xi)\exp\Big[-\xi^2\int_0^t a(\varphi(s))\,\mathrm{d}s\Big],$$
$$(t,\xi) \in (0,T)\times\mathbf{R}. \tag{5.20}$$

Consequently, coming back to the original function v, we find the representation

$$v(t,x) = \int_{-\infty}^{+\infty} E\Big(\int_0^t a(\varphi(s))\,\mathrm{d}s, x-y\Big)u_0'(y)\,\mathrm{d}y,$$
$$(t,y) \in (0,T)\times\mathbf{R}, \tag{5.21}$$

where E denotes the *fundamental solution* of the heat equation

$$E(t,x) = (4\pi t)^{-1/2}\exp\Big(-\frac{x^2}{4t}\Big). \tag{5.22}$$

Finally, observe that the function u defined in (5.6) satisfies

$$
\begin{aligned}
& D_t u(t,x) - a\Big(\int_{-\infty}^{+\infty} |D_x u(t,y)|^2 \,\mathrm{d}y\Big) D_x^2 u(t,x) \\
= \;& z_0'(t) + \int_0^x D_t v(t,\eta)\,\mathrm{d}\eta - a\Big(\int_{-\infty}^{+\infty} |v(t,y)|^2 \,\mathrm{d}y\Big) D_x v(t,x) \\
= \;& z_0'(t) + \int_0^x \Big[D_t v(t,\eta) - a(\varphi(t)) D_x^2 v(t,\eta)\Big]\,\mathrm{d}\eta \\
& -a(\varphi(t)) D_x v(t,0) \\
= \;& z_0'(t) - a(\varphi(t)) D_x v(t,0), \qquad (t,x) \in (0,T)\times \mathbf{R}. \qquad (5.23)
\end{aligned}
$$

From (5.1) and (5.21) we conclude that f can be chosen as

$$
f(t,p) = z_0'(t) - a(p) D_x v(t,0), \qquad (t,x) \in (0,T)\times \mathbf{R}. \qquad (5.24)
$$

REMARK 5.1 *In the present case we cannot express f explicitly in terms of the data. Of course, f is not unique.*

REMARK 5.2 *Our identification problem can be generalized to that where the right-hand side in* (5.1) *has the more general form*

$$
f\Big(t, \int_{-\infty}^{+\infty} |D_x u(t,y)|^2 \,\mathrm{d}y\Big) g(x),
$$

provided g solves a n-th order linear differential equation with constant coefficients $A(D_x)g(x) = 0$, $x \in \mathbf{R}$. *In this case we must prescribe the following n additional conditions on $x = 0$:*

$$
D_x^j u(t,0) = z_j(t), \qquad t \in (0,T),\; j = 0,\dots,n-1.
$$

The new unknown v is now defined by

$$
\begin{aligned}
v(t,x) = A(D_x)u(t,x) \quad & \Longleftrightarrow \\
& u(t,x) = \sum_{j=0}^{n-1} \frac{1}{j!} z_j(t) x^j + \int_0^x G(x,\xi) v(t,\xi)\,\mathrm{d}\xi,
\end{aligned}
$$

where G denotes the Green function related to the Cauchy problem at $x = 0$ for the differential operator $A(D_x)$.

6 Recovering some basic parameters in the wave- and heat-equations

We consider here the problem of recovering the positive constant c, the speed of wave propagation, related to the vibration of a endless homogeneous string when its initial shape and velocity are known:

$$D_t^2 u(t,x) - c^2 D_x^2 u(t,x) = f(t,x), \qquad (t,x) \in (0,T) \times \mathbf{R}, \quad (6.1)$$

$$u(0,x) = u_0(x), \qquad D_t u(0,x) = u_1(x), \qquad x \in \mathbf{R}. \quad (6.2)$$

A natural information to determine c consists in prescribing, i.e. in measuring, the *displacement* u at a fixed point x_0 at some positive time t_0:

$$u(t_0, x_0) = a. \quad (6.3)$$

For the sake of simplicity we will require that the data satisfy the following assumptions:

H1 $u_0 \in C^2(\mathbf{R})$ and the limits $u_0(-\infty)$ and $u_0(+\infty)$ (exist and) have the same sign, if they are both infinite;

H2 $u_1 \in C^1(\mathbf{R}) \cap L^1(\mathbf{R})$;

H3 $f, D_x f \in C([0,T] \times \mathbf{R})$ and $f \in L^1((0,t_0) \times \mathbf{R})$.

As is well-known, the solution to the Cauchy problem (6.1)-(6.2) is given by the D'Alembert formula

$$\begin{aligned} u(t,x) &= \frac{1}{2}[u_0(x+ct) + u_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\,\mathrm{d}y \\ &\quad + \frac{1}{2c}\int_0^t \mathrm{d}s \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\,\mathrm{d}y, \ (t,x) \in [0,T] \times \mathbf{R}. \end{aligned} \quad (6.4)$$

Consequently, condition (6.3) amounts to solving the following equation for c:

$$\begin{aligned} a &= \frac{1}{2}[u_0(x_0+ct_0) + u_0(x_0-ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} u_1(y)\,\mathrm{d}y \\ &\quad + \frac{1}{2c}\int_0^{t_0} \mathrm{d}s \int_{x_0-c(t_0-s)}^{x_0+c(t_0-s)} f(s,y)\,\mathrm{d}y =: g(c). \end{aligned} \quad (6.5)$$

According to our assumptions we easily derive the following relations:

$$\begin{aligned} g(0) &= u_0(x_0) + t_0 u_1(x_0) + \int_0^{t_0} f(s, x_0)\,\mathrm{d}s, \\ g(+\infty) &= \frac{1}{2}[u_0(-\infty) + u_0(+\infty)]. \end{aligned} \tag{6.6}$$

Therefore, since $g \in C([0, +\infty))$, equation (6.5) is solvable in $\mathbf{R}_+$ if a belongs to the open interval with end-points $g(0)$ and $g(+\infty)$, provided

$$\begin{aligned} g(0) &= u_0(x_0) + t_0 u_1(x_0) + \int_0^{t_0} f(s, x_0)\,\mathrm{d}s \\ &\neq \frac{1}{2}[u_0(-\infty) + u_0(+\infty)] = g(+\infty). \end{aligned} \tag{6.7}$$

To discuss the uniqueness of such solutions, we compute

$$\begin{aligned} g'(c) &= \frac{t_0}{2}[u_0'(x_0 + ct_0) - u_0'(x_0 - ct_0)] \\ &+ \Big\{ -\frac{1}{2c^2} \int_{x_0 - ct_0}^{x_0 + ct_0} u_1(y)\,\mathrm{d}y + \frac{t_0}{2c}[u_1(x_0 + ct_0) + u_1(x_0 - ct_0)] \Big\} \\ &+ \Big\{ -\frac{1}{2c^2} \int_0^{t_0} \mathrm{d}s \int_{x_0 - c(t_0 - s)}^{x_0 + c(t_0 - s)} f(s, y)\,\mathrm{d}y \\ &+ \frac{1}{2c} \int_0^{t_0} (t_0 - s)[f(s, x_0 + ct_0) + f(s, x_0 - ct_0)]\,\mathrm{d}s \Big\} := \sum_{j=1}^{3} g_j(c). \end{aligned} \tag{6.8}$$

We note that g' is strictly positive (negative) on $\mathbf{R}_+$ if the following additional assumptions are satisfied:

H4 u_0' is increasing (decreasing) on $\mathbf{R}$;

H5 u_1 and $x \to f(t, x)$, $t \in [0, T]$, are non-increasing (non-decreasing) on $(-\infty, x_0]$ and non-decreasing (non-increasing) in $[x_0, +\infty)$.

From *H5* we easily derive the inequalities

$$g_2(c) = \frac{1}{2c^2}\int_{x_0}^{x_0+ct_0}[u_1(x_0+ct_0) - u_1(y)]\,\mathrm{d}y + \frac{1}{2c^2}\int_{x_0-ct_0}^{x_0}[u_1(x_0-ct_0) - u_1(y)]\,\mathrm{d}y \geq 0, \qquad c \in \mathbf{R}_+, \tag{6.9}$$

$$g_3(c) = \frac{1}{2c^2}\int_0^{t_0}\mathrm{d}s\int_{x_0}^{x_0+c(t_0-s)}[f(s,x_0+ct_0) - f(s,y)]\,\mathrm{d}y + \frac{1}{2c^2}\int_0^{t_0}\mathrm{d}s\int_{x_0-c(t_0-s)}^{x_0}[f(s,x_0-ct_0) - f(s,y)]\,\mathrm{d}y \geq 0, \qquad c \in \mathbf{R}_+. \tag{6.10}$$

Therefore, according to *H4* and *H5*, g' is strictly positive. As a consequence, under assumptions *H1-H5* c can be uniquely recovered whenever $a \in (g(0), g(+\infty))$.

REMARK 6.1 *Conditions H5 are very strict, since they require that* u_1 *and* any *function* $f(t,\cdot)$, $t \in [0,T]$, *should have a minimum (maximum) at* $x = x_0$. *However, this is not a severe restriction if we assume that* f *is* independent *of* x, *since in this case* g_3 *reduces to the null function. If, in addition,* u_1 *is constant, condition H5 can be dropped.*

We now deal with a similar problem for the one-dimensional heat equation:

$$D_t u(t,x) - cD_x^2 u(t,x) = f(t,x), \qquad (t,x) \in (0,T)\times\mathbf{R}, \tag{6.11}$$

$$u(0,x) = u_0(x), \qquad x \in [0,l]. \tag{6.12}$$

To determine the conductivity coefficient c again we prescribe condition (6.3), i.e. we measure the *temperature* u at a fixed point x_0 at some positive time t_0.
Moreover, for the sake of simplicity we will require that the data satisfy the following assumptions:

H6 $u_0 \in C_b^{2+\alpha}(\mathbf{R})$ for some $\alpha \in (0,1)$ and the limits $u_0(-\infty)$ and $u_0(+\infty)$ exist and are both finite;

H7 u_0' is strictly increasing (decreasing) on $\mathbf{R}$;

H8 $f \in C_b^{\alpha/2,\alpha}([0,T]\times\mathbf{R})$ and the limits $f(t,-\infty)$ and $f(t,+\infty)$ exist and are finite for any $t\in[0,T]$;

H9 $f(\cdot,-\infty), f(\cdot,+\infty) \in L^1((0,t_0))$ and $D_x f \in C_b([0,T]\times\mathbf{R})$;

H10 $x \to D_x f(t,x)$ is non-decreasing (non-increasing) on $\mathbf{R}$ for any $t\in[0,T]$;

As is well known, the solution to the Cauchy problem (6.11)-(6.12) is given by the following formulae, where $(t,x)\in(0,T)\times\mathbf{R}$:

$$\begin{aligned} u(t,x) &= \int_{\mathbf{R}} E(ct, x-y)u_0(y)\,\mathrm{d}y \\ &\quad + \int_0^t \mathrm{d}s \int_{\mathbf{R}} E(c(t-s), x-y) f(s,y)\,\mathrm{d}y \\ &= \pi^{-1/2}\int_{\mathbf{R}} \exp(-y^2) u_0(x - 2t^{1/2}c^{1/2}y)\,\mathrm{d}y \\ &\quad + \pi^{-1/2}\int_0^t \mathrm{d}s \int_{\mathbf{R}} \exp(-y^2) f(s, x-2t^{1/2}c^{1/2}y)\,\mathrm{d}y, \end{aligned} \tag{6.13}$$

where the fundamental kernel E is defined by (5.22).
Consequently, condition (6.3) amounts to solving the following equation for c:

$$\begin{aligned} a &= \pi^{-1/2}\int_{\mathbf{R}} \exp(-y^2) u_0(x_0 - 2t_0^{1/2}c^{1/2}y)\,\mathrm{d}y \\ &\quad + \pi^{-1/2}\int_0^{t_0} \mathrm{d}s \int_{\mathbf{R}} \exp(-y^2) f(s, x_0 - 2t_0^{1/2}c^{1/2}y)\,\mathrm{d}y \\ &=: g(c), \quad c\in\mathbf{R}_+. \end{aligned} \tag{6.14}$$

According to our assumptions we easily derive the following relations:

$$g(0) = u(x_0) + \int_0^{t_0} f(s,x_0)\,\mathrm{d}s, \tag{6.15}$$

$$g(+\infty) = \frac{1}{2}[u_0(-\infty) + u_0(+\infty)] + \frac{1}{2}\int_0^{t_0} [f(s,-\infty) + f(s,+\infty)]\,ds. \tag{6.16}$$

Therefore, since $g \in C([0,+\infty))$, equation (6.14) is solvable in $\mathbf{R}_+$ if a belongs to the open interval with end-points $g(0)$ and $g(+\infty)$, provided

$$\begin{aligned} g(0) = u(x_0) + \int_0^{t_0} f(s,x_0)\,ds \neq \frac{1}{2}[u_0(-\infty) + u_0(+\infty)] \\ + \frac{1}{2}\int_0^{t_0} [f(s,-\infty) + f(s,+\infty)]\,ds = g(+\infty). \end{aligned} \tag{6.17}$$

To discuss the uniqueness of such solutions, for all $c \in \mathbf{R}_+$ we compute

$$\begin{aligned} g'(c) &= -t_0^{1/2}(\pi c)^{-1/2}\Big\{\int_{\mathbf{R}} y \exp(-y^2) u_0'(x_0 - 2t_0^{1/2}c^{1/2}y)\,dy \\ &\quad + \int_0^{t_0} ds \int_{\mathbf{R}} y \exp(-y^2) D_x f(s,x_0 - 2t_0^{1/2}c^{1/2}y)\,dy\Big\}, \\ &= t_0^{1/2}(\pi c)^{-1/2}\Big\{\int_0^{+\infty} y \exp(-y^2)[u_0'(x_0 + 2t_0^{1/2}c^{1/2}y)) \\ &\quad - u_0'(x_0 - 2t_0^{1/2}c^{1/2}y)]\,dy \\ &\quad + \int_0^{t_0} ds \int_0^{+\infty} y \exp(-y^2)[D_x f(s,x_0 + 2t_0^{1/2}c^{1/2}y) \\ &\quad - D_x f(s,x_0 - 2t_0^{1/2}c^{1/2}y)]\,dy\Big\}. \end{aligned} \tag{6.18}$$

We note that, according to *H7* and *H10*, g' is strictly positive (negative) on $\mathbf{R}_+$. As a consequence, under assumptions *H6*-*H10* c can be uniquely recovered whenever $a \in (g(0), g(+\infty))$.

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